Problem 6.1 10 points:

Derive the value of level-set function on top/bottom cell boundaries, $\phi_{i,j+\frac{1}{2}}$, $\phi_{i,j-\frac{1}{2}}$. Follow the lecture example for the left/right cell boundaries $\phi_{i+\frac{1}{2},j}$ in Eq. (3.44).

$$\phi_{i+\frac{1}{2},j} = \begin{cases} \phi_{i,j} + \frac{1}{2} M\left(D_x^+ \phi_{i,j}, D_x^- \phi_{i,j}\right), & \frac{1}{2}\left(u_{i+1,j} + u_{i,j}\right) > 0 \\ \phi_{i+1,j} - \frac{1}{2} M\left(D_x^+ \phi_{i+1,j}, D_x^- \phi_{i+1,j}\right), & \frac{1}{2}\left(u_{i+1,j} + u_{i,j}\right) < 0. \end{cases} \quad (3.44)$$

Solution:

$D_y^+ \phi_{i,j} = \phi_{i,j+1} - \phi_{i,j}$

$D_y^- \phi_{i,j} = \phi_{i,j} - \phi_{i,j-1}$

Defining M

$$M(a,b) = \begin{cases} a & |a| < |b| \\ b & |b| < |a| \end{cases}$$

Therefore, for top boundary:

$$\phi_{i,j+1/2} = \begin{cases} \phi_{i,j} + \frac{1}{2} M\left(D_y^+ \phi_{i,j}, D_y^- \phi_{i,j}\right) & \frac{1}{2}\left(v_{i,j+1} + v_{i,j}\right) > 0 \\ \phi_{i,j+1} - \frac{1}{2} M\left(D_y^+ \phi_{i,j+1}, D_y^- \phi_{i,j+1}\right) & \frac{1}{2}\left(v_{i,j+1} + v_{i,j}\right) < 0 \end{cases}$$

And for bottom boundary:

$$\phi_{i,j-1/2} = \begin{cases} \phi_{i,j-1} + \frac{1}{2} M\left(D_y^+ \phi_{i,j-1}, D_y^- \phi_{i,j-1}\right) & \frac{1}{2}\left(v_{i,j-1} + v_{i,j}\right) > 0 \\ \phi_{i,j} - \frac{1}{2} M\left(D_y^+ \phi_{i,j}, D_y^- \phi_{i,j}\right) & \frac{1}{2}\left(v_{i,j-1} + v_{i,j}\right) < 0 \end{cases}$$