Homework #1

due: 11:45am, January 27, 2020.

Problem 1.1	Problem 1.2	Problem 1.3	Problem 1.4	Total

Problem 1.1 10 points:

Show that material derivative definition (D/Dt) yields compressible mass, momentum and energy conservation equations starting with the partial derivative ($\partial/\partial t$) based equations in below.

a) mass: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0$

b) momentum: $\frac{\partial(\rho u_i)}{\partial t} + \nabla \cdot (\rho u_i u) = -P_{,i} + \tau_{ji,i} + \rho f_i$

c) energy: $\frac{\partial}{\partial t}\left[\rho\left(e + \frac{u^2}{2}\right)\right] + \nabla \cdot \left[\rho\left(e + \frac{u^2}{2}\right)u_i\right] = \rho\dot{q} + (kT_{,i})_{,i} - (u_i p)_{,i} + (u_j \tau_{ij})_{,i} + \rho\vec{f}\vec{u}$

Note 1: use chain rule for the derivation of $\nabla \cdot (\rho u), \nabla \cdot (\rho u_i u)$, and $\nabla \cdot (\rho u_i u)$.

Note 2: use the conclusion of a) in the derivation of b) and c).

Problem 1.2 20 points:

The following is the **vector** conservation form of the Navier-Stokes momentum equations.

$$\frac{\partial \underline{\boldsymbol{u}}}{\partial t} + \underline{\boldsymbol{\nabla}} \cdot (\underline{\boldsymbol{uu}}) = -\frac{1}{\rho}\underline{\boldsymbol{\nabla}}p + \nu\nabla^2\underline{\boldsymbol{u}} + \underline{\boldsymbol{f}}$$

Derive the convection term $\underline{\boldsymbol{\nabla}} \cdot (\underline{\boldsymbol{uu}})$ and diffusion term $\nabla^2\underline{\boldsymbol{u}}$ in cylindrical coordinates (r, φ, z).

Note: you will need to use the gradient functions in below. Be careful about the terms marked in red.

(1) $$\nabla \cdot = \frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \varphi} + \frac{\partial}{\partial z}$$

(2) $$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$

(3)

$$\frac{\partial \vec{e_r}}{\partial r} = 0 \qquad \frac{\partial \vec{e_r}}{\partial \varphi} = \vec{e_\varphi} \qquad \frac{\partial \vec{e_r}}{\partial z} = 0$$

$$\frac{\partial \vec{e_\varphi}}{\partial r} = 0 \qquad \frac{\partial \vec{e_\varphi}}{\partial \varphi} = -\vec{e_r} \qquad \frac{\partial \vec{e_\varphi}}{\partial z} = 0$$

$$\frac{\partial \vec{e_z}}{\partial r} = 0 \qquad \frac{\partial \vec{e_z}}{\partial \varphi} = 0 \qquad \frac{\partial \vec{e_z}}{\partial z} = 0$$

Problem 1.3 10 points:

Determine the classification of the incompressible 2-D N.S. equations (e.g. whether they are elliptic, parabolic or hyperbolic):

a) in space

b) in time

Problem 1.4 10 points:

Consider incompressible N.S. equations in Cartesian coordinates. Derive Poisson's equation for pressure by taking the divergence of the momentum equation and then **applying continuity equation** to obtain:

$$\nabla^2 p = -\rho\left[\left(u_i u_j\right)_{,j}\right]_{,i}$$

Note:

The momentum equation for incompressible flow:

$$\frac{\partial u_j}{\partial t} + \frac{\partial\left(u_i u_j\right)}{\partial x_i} = -\frac{1}{\rho}\frac{\partial p}{\partial x_j} + \nu\frac{\partial^2 u_j}{\partial x_i \partial x_i} + \frac{1}{\rho} f_j$$