

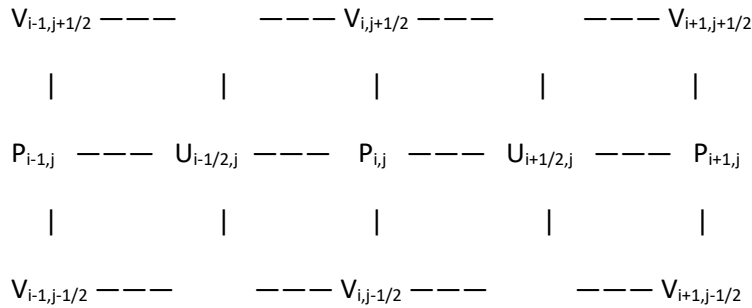
Problem 3.3 20 points:

Assume a 2D grid which is not isotropic, e.g. the resolution in the x direction is k_x and the resolution in the y direction is k_y .

- Derive the **incompressibility condition** (Eq. (2.17) in the notes/text). Show that your result is consistent with notes if you assume $k_x = k_y = k$.
- Derive the **predictor step** (2.18), **projection step** (2.19) and the **pressure equation** (2.20). Show that your result is consistent with notes if you assume $k_x = k_y = k$.
- Derive the **advection term** in the x-direction for the non-isotropic grid. Show that your result is consistent with notes if you assume $k_x = k_y = k$.

Solution:

(a)



According to the pattern of the pressure control volume which centered at $P_{i,j}$

Using divergence theorem and mid-point rule

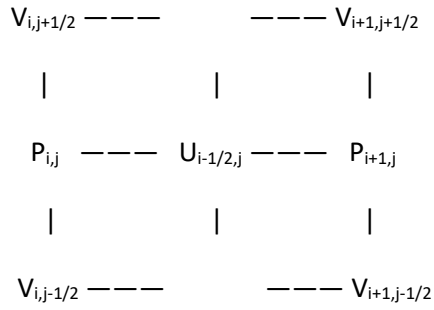
$$k_y \left(u_{i+\frac{1}{2},j}^{n+1} - u_{i-\frac{1}{2},j}^{n+1} \right) + k_x \left(v_{i,j+\frac{1}{2}}^{n+1} - v_{i,j-\frac{1}{2}}^{n+1} \right) = 0$$

If $k_y = k_x = k$

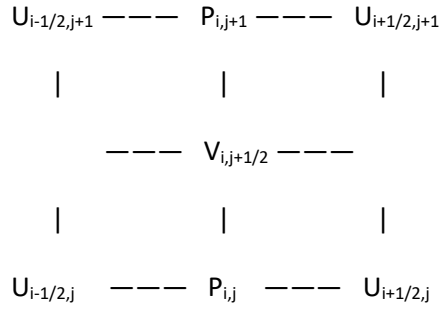
$$\Rightarrow u_{i+\frac{1}{2},j}^{n+1} - u_{i-\frac{1}{2},j}^{n+1} + v_{i,j+\frac{1}{2}}^{n+1} - v_{i,j-\frac{1}{2}}^{n+1} = 0$$

(b)

The pattern of u-velocity control volume:



The pattern of v-velocity control volume



The predictor step:

$$u_{i+\frac{1}{2},j}^* = u_{i+\frac{1}{2},j}^n + \Delta t \left[-(A_x)_{i+\frac{1}{2},j}^n + v(D_x)_{i+\frac{1}{2},j}^n + (f_x)_{i+\frac{1}{2},j} \right]$$

$$v_{i,j+\frac{1}{2}}^* = v_{i,j+\frac{1}{2}}^n + \Delta t \left[-(A_y)_{i,j+\frac{1}{2}}^n + v(D_y)_{i,j+\frac{1}{2}}^n + (f_y)_{i,j+\frac{1}{2}} \right]$$

The projection step:

$$u_{i+\frac{1}{2},j}^{n+1} = u_{i+\frac{1}{2},j}^* - \frac{\Delta t}{\rho} \cdot \frac{1}{k_x} (P_{i+1,j}^{n+1} - P_{i,j}^{n+1})$$

$$v_{i,j+\frac{1}{2}}^{n+1} = v_{i,j+\frac{1}{2}}^* - \frac{\Delta t}{\rho} \cdot \frac{1}{k_y} (P_{i,j+1}^{n+1} - P_{i,j}^{n+1})$$

The pressure equation:

$$u_{i-\frac{1}{2},j}^{n+1} = u_{i-\frac{1}{2},j}^* - \frac{\Delta t}{\rho} \cdot \frac{1}{k_x} (P_{i,j}^{n+1} - P_{i-1,j}^{n+1})$$

$$v_{i,j-\frac{1}{2}}^{n+1} = v_{i,j-\frac{1}{2}}^* - \frac{\Delta t}{\rho} \cdot \frac{1}{k_y} (P_{i,j}^{n+1} - P_{i,j-1}^{n+1})$$

If $k_y = k_x = k$

$$\Rightarrow u_{i-\frac{1}{2},j}^{n+1} = u_{i-\frac{1}{2},j}^* - \frac{\Delta t}{\rho} \cdot \frac{1}{k} (P_{i,j}^{n+1} - P_{i-1,j}^{n+1})$$

$$\Rightarrow v_{i,j-\frac{1}{2}}^{n+1} = v_{i,j-\frac{1}{2}}^* - \frac{\Delta t}{\rho} \cdot \frac{1}{k} (P_{i,j}^{n+1} - P_{i,j-1}^{n+1})$$

Combining the projection step and continuity equation,

$$\begin{aligned} k_y \left(u_{i+\frac{1}{2},j}^{n+1} - u_{i-\frac{1}{2},j}^{n+1} \right) + k_x \left(v_{i,j+\frac{1}{2}}^{n+1} - v_{i,j-\frac{1}{2}}^{n+1} \right) \\ = k_y \left[u_{i+\frac{1}{2},j}^* - u_{i-\frac{1}{2},j}^* - \frac{\Delta t}{\rho} \cdot \frac{1}{k_x} (P_{i+1,j}^{n+1} + P_{i-1,j}^{n+1} - 2 \cdot P_{i,j}^{n+1}) \right] \\ + k_x \left[v_{i,j+\frac{1}{2}}^* - v_{i,j-\frac{1}{2}}^* - \frac{\Delta t}{\rho} \cdot \frac{1}{k_y} (P_{i,j+1}^{n+1} + P_{i,j-1}^{n+1} - 2 \cdot P_{i,j}^{n+1}) \right] = 0 \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{k_y}{k_x} (P_{i+1,j}^{n+1} + P_{i-1,j}^{n+1} - 2 \cdot P_{i,j}^{n+1}) + \frac{k_x}{k_y} (P_{i,j+1}^{n+1} + P_{i,j-1}^{n+1} - 2 \cdot P_{i,j}^{n+1}) \\ = \frac{\rho}{\Delta t} \left[k_y \left(u_{i+\frac{1}{2},j}^* - u_{i-\frac{1}{2},j}^* \right) + k_x \left(v_{i,j+\frac{1}{2}}^* - v_{i,j-\frac{1}{2}}^* \right) \right] \end{aligned}$$

If $k_y = k_x = k$

$$\Rightarrow P_{i+1,j}^{n+1} + P_{i-1,j}^{n+1} + P_{i,j+1}^{n+1} + P_{i,j-1}^{n+1} - 4 \cdot P_{i,j}^{n+1} = \frac{\rho \cdot k}{\Delta t} \left[u_{i+\frac{1}{2},j}^* - u_{i-\frac{1}{2},j}^* + v_{i,j+\frac{1}{2}}^* - v_{i,j-\frac{1}{2}}^* \right]$$

(c)

$$\text{Right: } \left(\frac{u_{i+\frac{3}{2},j}^n + u_{i+\frac{1}{2},j}^n}{2} \right)^2 k_y$$

$$\text{Left: } -\left(\frac{u_{i+\frac{1}{2},j}^n + u_{i-\frac{1}{2},j}^n}{2}\right)^2 k_y$$

$$\text{Above: } \left(\frac{u_{i+\frac{1}{2},j+1}^n + u_{i+\frac{1}{2},j}^n}{2}\right)\left(\frac{u_{i+1,j+\frac{1}{2}}^n + u_{i,j+\frac{1}{2}}^n}{2}\right) k_x$$

$$\text{Below: } -\left(\frac{u_{i+\frac{1}{2},j}^n + u_{i+\frac{1}{2},j-1}^n}{2}\right)\left(\frac{u_{i+1,j-\frac{1}{2}}^n + u_{i,j-\frac{1}{2}}^n}{2}\right) k_x$$

$$\underline{A}(\underline{u}^n) = \frac{1}{\Delta V} \oint_S \underline{u}^n (\underline{u}^n \cdot \underline{n}) dS$$

$$\begin{aligned} (A_x)_{i+\frac{1}{2},j}^n &= \frac{1}{k_x} \left[\left(\frac{u_{i+\frac{3}{2},j}^n + u_{i+\frac{1}{2},j}^n}{2} \right)^2 - \left(\frac{u_{i+\frac{1}{2},j}^n + u_{i-\frac{1}{2},j}^n}{2} \right)^2 \right] \\ &\quad + \frac{1}{k_y} \left[\left(\frac{u_{i+\frac{1}{2},j}^n + u_{i+\frac{1}{2},j+1}^n}{2} \right) \left(\frac{v_{i,j+\frac{1}{2}}^n + v_{i+1,j+\frac{1}{2}}^n}{2} \right) \right. \\ &\quad \left. - \left(\frac{u_{i+\frac{1}{2},j}^n + u_{i+\frac{1}{2},j-1}^n}{2} \right) \left(\frac{v_{i+1,j-\frac{1}{2}}^n + v_{i,j-\frac{1}{2}}^n}{2} \right) \right] \end{aligned}$$

$$\text{If } k_y = k_x = k$$

$$\begin{aligned} (A_x)_{i+\frac{1}{2},j}^n &= \frac{1}{k} \left[\left(\frac{u_{i+\frac{3}{2},j}^n + u_{i+\frac{1}{2},j}^n}{2} \right)^2 - \left(\frac{u_{i+\frac{1}{2},j}^n + u_{i-\frac{1}{2},j}^n}{2} \right)^2 \right] \\ &\quad + \frac{1}{k} \left[\left(\frac{u_{i+\frac{1}{2},j}^n + u_{i+\frac{1}{2},j+1}^n}{2} \right) \left(\frac{v_{i,j+\frac{1}{2}}^n + v_{i+1,j+\frac{1}{2}}^n}{2} \right) \right. \\ &\quad \left. - \left(\frac{u_{i+\frac{1}{2},j}^n + u_{i+\frac{1}{2},j-1}^n}{2} \right) \left(\frac{v_{i+1,j-\frac{1}{2}}^n + v_{i,j-\frac{1}{2}}^n}{2} \right) \right] \end{aligned}$$