HW 8 Moatasim Farooque

Problem 8.1

Lateral mean momentum eq

$$\frac{1}{\rho}\frac{\delta\langle P(x,y)\rangle}{\delta y}+\frac{\delta\langle v^2\rangle}{\delta y}=0$$

After integration

$$\int_0^y \frac{1}{\rho}\frac{\delta\langle P(x,y)\rangle}{\delta y'}dy'+\int_0^y\frac{\delta\langle v^2\rangle}{\delta y'}dy'=0$$

$$\frac{1}{\rho}(\langle P(x,y)\rangle-\langle P_{wall}(x)\rangle)+\langle v^2(y)\rangle-\langle v^2(0)\rangle=0$$

The velocity at the wall $(y=0)$ is zero

$$\frac{1}{\rho}(\langle P(x,y)\rangle-\langle P_{wall}(x)\rangle)+\langle v^2(y)\rangle=0$$

Taking the derivative with x

$$\frac{1}{\rho}\left(\frac{\delta\langle P(x,y)\rangle}{\delta x}-\frac{\delta\langle P_{wall}(x)\rangle}{\delta x}\right)=0$$

$$\Rightarrow\frac{\delta\langle P\rangle}{\delta x}=\frac{\delta\langle P_{wall}\rangle}{\delta x}$$

8.2

(a) N.S momentum equations $\frac{DU}{Dt}=-\frac{1}{\rho}\nabla P+\nu\nabla^2U$

Reynolds equations $\frac{\bar{D}\langle U_j\rangle}{Dt}=\nu\nabla^2\langle U_j\rangle-\frac{\delta\langle u_i\cdot u_j\rangle}{\delta x_i}-\frac{1}{\rho}\frac{\delta\langle P\rangle}{\delta x_j}$

Subtract 2 from 1

$$\frac{DU}{Dt}-\frac{\bar{D}\langle U_j\rangle}{Dt}=-\frac{1}{\rho}\nabla p+\nu\nabla^2U-\nu\nabla^2\langle U_j\rangle+\frac{\delta\langle u_i\cdot u_j\rangle}{\delta x_i}+\frac{1}{\rho}\frac{\delta\langle P\rangle}{\delta x_j}$$

Since

$$u = U - \langle U_j \rangle \text{ and } p' = p - \langle p \rangle$$

$$\frac{DU}{Dt} - \frac{\bar{D}\langle U_j \rangle}{Dt} = -\frac{1}{\delta} \nabla p + \nu \nabla^2 U_j + \frac{\delta \langle u_i \cdot u_j \rangle}{\delta x_i} + \frac{1}{\delta} \frac{\delta \langle p \rangle}{\delta x_j}$$

$$\frac{DU}{Dt} - \frac{\bar{D}\langle U_j \rangle}{Dt} = \nu \nabla^2 U_j + \frac{\delta \langle u_i \cdot u_j \rangle}{\delta x_i} - \frac{1}{\delta}\left(\nabla p - \frac{\delta \langle p \rangle}{\delta x_j} \right)$$

$$\frac{DU}{Dt} - \frac{\bar{D}\langle U_j \rangle}{Dt} = \nu \nabla^2 U_j + \frac{\delta \langle u_i \cdot u_j \rangle}{\delta x_i} - \frac{1}{\delta} \frac{\delta p'}{\delta x_j}$$

Since

$$\frac{DU_j}{Dt} = \frac{\delta U_j}{\delta t} + \frac{\delta (U_i \cdot U_j)}{\delta x_i}$$

$$\left\langle \frac{DU_j}{Dt} \right\rangle = \frac{\delta \langle U_j \rangle}{\delta t} + \frac{\delta \langle U_i \cdot U_j \rangle}{\delta x_i} = \frac{\delta \langle U_j \rangle}{\delta t} + \langle U_i \rangle \frac{\delta \langle U_j \rangle}{\delta x_i} + \frac{\delta \langle u_i \cdot u_j \rangle}{\delta x_i}$$

$$\frac{\bar{D}\langle U_j \rangle}{Dt} = \left\langle \frac{DU_j}{Dt} \right\rangle - \frac{\delta \langle u_i \cdot u_j \rangle}{\delta x_i}$$

$$\frac{DU_j}{Dt} - \left[\left\langle \frac{DU_j}{Dt} \right\rangle - \frac{\delta \langle u_i \cdot u_j \rangle}{\delta x_i} \right] = \nu \nabla^2 U_j + \frac{\delta \langle u_i \cdot u_j \rangle}{\delta x_i} - \frac{1}{\delta} \frac{\delta p'}{\delta x_j}$$

$$\frac{\delta u_j}{\delta t} + (\langle U_i \rangle + u_i)\frac{\delta}{\delta x_i} u_j = -u_i \frac{\delta}{\delta x_i}(\langle U_j \rangle) + \frac{\delta}{\delta x_i}(\langle u_i \cdot u_j \rangle)$$
$$+ \nu \nabla^2 u_j - \frac{1}{\rho}\frac{\delta p'}{\delta x_j}$$

Since

$$\frac{Du_j}{Dt} = \frac{\delta u_j}{\delta t} + (\langle U_i \rangle + u_i)\frac{\delta}{\delta x_i} u_j$$

$$\Rightarrow \frac{Du_j}{Dt} == -u_i \frac{\delta}{\delta x_i}(\langle U_j \rangle) + \frac{\delta}{\delta x_i}(\langle u_i \cdot u_j \rangle) + \nu \nabla^2 u_j - \frac{1}{\rho}\frac{\delta p'}{\delta x_j}$$

(b) Multiplying by u_j to the result of part a

$$u_j \cdot \frac{\delta u_j}{\delta t} + (\langle U_j \rangle + u_i) \cdot u_j \cdot \frac{\delta}{\delta x_i} u_j = -u_i \cdot u_j \cdot \frac{\delta}{\delta x_i}(\langle U_j \rangle)$$
$$+ u_j \cdot \frac{\delta}{\delta x_i}(\langle u_i \cdot u_j \rangle) + \nu \nabla^2 u_j u_j - u_j \cdot \frac{1}{\rho}\frac{\delta p'}{\delta x_j}$$

Since

$$u_j \cdot \frac{\delta u_j}{\delta t} = \frac{\delta u_j^2}{\delta t} = 2 \cdot u_j \cdot \frac{\delta u_j}{\delta t}$$

$$\Rightarrow \frac{1}{2} \cdot \frac{\delta u_j^2}{\delta t} = u_j \cdot \frac{\delta u_j}{\delta t} = \frac{\delta}{\delta t} \cdot \left(\frac{1}{2} u_j \cdot u_j\right)$$

$$\frac{\delta}{\delta t} \cdot \left(\frac{1}{2} u_j \cdot u_j\right) + \langle U_i \rangle \frac{\delta}{\delta x_i}\left(\frac{1}{2} u_j \cdot u_j\right) + \frac{\delta}{\delta x_i}\left(\frac{1}{2} u_i \cdot u_j \cdot u_j\right)$$
$$= -u_i \cdot u_j \cdot \frac{\delta}{\delta x_i}(\langle U_j \rangle) + u_j \cdot \frac{\delta}{\delta x_i}(\langle u_i \cdot u_j \rangle) + \nu \nabla^2 u_j u_j$$
$$- u_j \cdot \frac{1}{\rho}\frac{\delta p'}{\delta x_j}$$

$$\frac{DU_j}{Dt} - \left[\frac{\delta\langle U_j\rangle}{\delta t} + \frac{\delta\langle U_i \cdot U_j\rangle}{\delta x_i} - \frac{\delta\langle U_i \cdot U_j\rangle}{\delta x_i}\right] = \nu\nabla^2 U_j + \frac{\delta\langle U_i \cdot U_j\rangle}{\delta x_i} - \frac{1}{\delta}\frac{\delta p'}{\delta x_j}$$

$$\frac{DU_j}{Dt} \cdot \frac{\delta\langle U_j\rangle}{\delta t} - \frac{\delta\langle U_i \cdot U_j\rangle}{\delta x_i} = \nu\nabla^2 U_j - \frac{1}{\delta}\frac{\delta p'}{\delta x_j}$$

$$\frac{\delta U_j}{\delta t} + \frac{\delta(U_i \cdot U_j)}{\delta x_i} - \frac{\delta\langle U_j\rangle}{\delta t} - \frac{\delta\langle U_i \cdot U_j\rangle}{\delta x_i} = \nu\nabla^2 U_j - \frac{1}{\delta}\frac{\delta p'}{\delta x_j}$$

Using $u = U - \langle U\rangle$

$$\Rightarrow \frac{\delta u_j}{\delta t} + \frac{\delta}{\delta x_i}(U_i \cdot U_j - \langle U_i \cdot U_j\rangle) = \nu\nabla^2 u_j - \frac{1}{\delta}\frac{\delta p'}{\delta x_j}$$

Since

$$\frac{\delta}{\delta x_i}(U_i \cdot U_j - \langle U_i \cdot U_j\rangle) = \frac{\delta}{\delta x_i}(u_i\langle U_j\rangle + u_j\langle U_i\rangle + u_i \cdot u_j - \langle u_j \cdot u_j\rangle)$$

$$= u_i\frac{\delta}{\delta x_i}(\langle U_j\rangle) + \langle U_i\rangle\frac{\delta}{\delta x_i}u_j + u_i\frac{\delta u_j}{\delta x_i} - \frac{\delta}{\delta x_i}(\langle u_i \cdot u_j\rangle)$$

$$\Rightarrow \frac{\delta u_j}{\delta t} + u_i\frac{\delta}{\delta x_i}(\langle U_j\rangle) + \langle U_i\rangle\frac{\delta}{\delta x_i}u_j + u_i\frac{\delta}{\delta x_i}u_j - \frac{\delta}{\delta x_i}(\langle u_i \cdot u_j\rangle)$$

$$= \nu\nabla^2 u_j - \frac{1}{\delta}\frac{\delta p'}{\delta x_j}$$

$$\frac{\delta u_j}{\delta t} + \langle U_i\rangle\frac{\delta}{\delta x_i}u_j + u_i\frac{\delta}{\delta x_i}u_j = -u_i\frac{\delta}{\delta x_i}(\langle U_j\rangle) + \frac{\delta}{\delta x_i}(\langle u_i \cdot u_j\rangle) + \nu\nabla^2 u_j - \frac{1}{\delta}\frac{\delta p'}{\delta x_j}$$

Taking the average and let $\langle u_i \rangle = 0$

$$k = \langle \frac{1}{2} u_j \cdot u_j \rangle$$

$$\frac{\delta k}{\delta t} + \langle U_i \rangle \frac{\delta k}{\delta x_i} + \frac{\delta}{\delta x_i} \langle \frac{1}{2} u_i \cdot u_j \cdot u_j \rangle$$

$$= -\langle u_i \cdot u_j \rangle \frac{\delta}{\delta x_i} \langle U_j \rangle + \nu \langle u_j \nabla^2 u_j \rangle - \frac{1}{\rho} \frac{\delta \langle u_j \cdot p' \rangle}{\delta x_j}$$

Since $\nu \langle u_j \nabla^2 u_j \rangle = 2 \cdot \nu \cdot \frac{\delta}{\delta x_i} \langle u_j \cdot s_{ij} \rangle - \epsilon$

$$T_i = \langle \frac{1}{2} u_i \cdot u_j \cdot u_j \rangle - 2 \cdot \nu \cdot \langle u_j \cdot s_{ij} \rangle + \frac{\langle u_j \cdot p' \rangle}{\rho}$$

$$\Rightarrow \frac{\delta k}{\delta t} + \langle U_i \rangle \frac{\delta k}{\delta x_i} + \frac{\delta}{\delta x_i} \langle \frac{1}{2} u_i \cdot u_j \cdot u_j \rangle$$

$$= -\langle u_i \cdot u_j \rangle \frac{\delta}{\delta x_i} \langle U_j \rangle + 2 \cdot \nu \cdot \frac{\delta}{\delta x_i} \langle u_j \cdot s_{ij} \rangle - \epsilon - \frac{1}{\rho} \frac{\delta \langle u_j \cdot p' \rangle}{\delta x_j}$$

$$\Rightarrow \frac{\delta k}{\delta t} + \langle U_i \rangle \frac{\delta k}{\delta x_i} + \frac{\delta}{\delta x_i} \left(\langle \frac{1}{2} u_i \cdot u_j \cdot u_j \rangle - 2\nu \langle u_j \cdot s_{ij} \rangle + \frac{\langle u_j \cdot p' \rangle}{\rho} \right) = -\langle u_i \cdot u_j \rangle \frac{\delta}{\delta x_i} \langle U_j \rangle - \epsilon$$

~~Flu.~~ Fluctuating flux of energy $T_i' = \langle \frac{1}{2} u_i \cdot u_j \cdot u_j \rangle - 2\nu \langle u_j \cdot s_{ij} \rangle + \frac{\langle u_j \cdot p' \rangle}{\rho}$

Production term: $P = -\langle u_i \cdot u_j \rangle \frac{\delta}{\delta x_i} \langle U_j \rangle$

$$\Rightarrow \frac{\bar{D} k}{\bar{D} t} + \nabla \cdot T_i' = P - \epsilon$$

$\mathbb{S}$-5)

$\vec{F_D} = \frac{1}{2} C_D A_x \rho_L V_R^2$

$\vec{F_B} = (\rho_L - \rho_g) g V$

a) Using force balance including drag force & buoyancy force

$F_b = (\rho_L - \rho_g) g V$

$F_D = \frac{1}{2} C_D A_x \rho_L V_R^2$

$F_b = F_D \Rightarrow (\rho_L - \rho_g) g V = \frac{1}{2} C_D A_x \rho_L V_R^2$

where A_x is the crosssectional area of the spherical bulb and V is its Volume

$A_x = \frac{\pi D_{dv}^2}{4}$, $V = \frac{\pi D_{dv}^3}{6}$

$$\boxed{V_R = \sqrt{\frac{4(\rho_L - \rho_g) D_{dv} \cdot g}{3 C_D \rho_L}}}$$

Problem 8.4

1) Group1: Mean diameter of 1mm and volume fraction of 3%.
2) Group2: // 1.5mm 1%
3) Group3; // 2.0mm // // // 1%.

$V_{2\phi} = 0.6\, D_{dv} \alpha_{dv} |V_r|$

a)

$$V_r = \sqrt{\frac{4(\rho_L-\rho_g) D_{dv} \cdot g}{3 C_D \rho_L}} \quad (1)$$

Under the condition of room temp and 1 atm

$\rho_L = 999.21 \frac{kg}{m^3}$, $\rho_g = 1.205\, kg/m^3$

Substituting in (1) $D_{dv} = [1\times10^{-3}, 1.5\times10^{-3}, 2.0\times10^{-3}]$ mm

$V_{r_1} = 0.163\, m/s \Rightarrow V_{2\phi_1} = 0.6 D_{dv_1} \cdot \alpha_{dv_1} \cdot V_{r_1} = 2.938\times10^{-6}\, m^2/s$

$V_{r_2} = 0.19988\, m/s \Rightarrow V_{2\phi_2} = 0.6 D_{dv_2} \cdot \alpha_{dv_2} \cdot V_{r_2} = 1.7989\times10^{-6}\, m^2/s$

$V_{r_3} = 0.23\, m/s \Rightarrow V_{2\phi_3} = 0.6 D_{dv_3} \cdot \alpha_{dv_3} \cdot V_{r_3} = 2.7696\times10^{-6}\, m^2/s$

Normalized velocity contributions

$$\frac{V_{2\phi_1}}{V_{cL}} = 2.798, \quad \frac{V_{2\phi_2}}{V_{cL}} = 1.713, \quad \frac{V_{2\phi_3}}{V_{cL}} = 2.6378$$

(b) $\alpha = 5\%$

$$D_{dve} = \frac{3}{5} \cdot D_{dv_1} + \frac{1}{5} D_{dv_2} + \frac{1}{5} \cdot D_{dv_3} = 1.3\,mm$$

$$V_{re} = \sqrt{\frac{4(\rho_L-\rho_g) D_{dve} \cdot g}{3 C_D \rho_L}} = \cancel{0.19\, m/s}\ 0.186\, m/s$$

$V_{2\phi_E} = 0.6\, D_{dve} \cdot \alpha_{dve} \cdot V_{rE} = 7.257\times10^{-6}\, m^2/s$

$$\frac{V_{2\phi_E}}{V_{cL}} = 6.911$$

[Void fraction has the greatest influence on $V_{2\phi}$, since α is high for equivalent group $V_{2\phi}$ will be higher than the others combined]