Homework #1

due: 12:00 pm, February 3rd, 2023.

Problem 1.1	Problem 1.2	Problem 1.3	Problem 1.4	Problem 1.5	Problem 1.6	Total

Problem 1.1 10 points:

Show that material derivative definition (D/Dt) yields compressible mass, momentum and energy conservation equations starting with the partial derivative ($\partial/\partial t$) -based equations in below.

a) mass: $\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0$ (2 points)

b) momentum: $\frac{\partial(\rho u_i)}{\partial t} + \nabla\cdot(\rho u_i\vec{u}) = -P_{,i} + \tau_{ji,i} + \rho f_i$ (4 points)

c) energy: $\frac{\partial}{\partial t}\left[\rho\left(e+\frac{u^2}{2}\right)\right] + \nabla\cdot\left[\rho\left(e+\frac{u^2}{2}\right)\vec{u}\right] = \rho\dot{q} + (kT_{,i})_{,i} - (u_i p)_{,i} + (u_j\tau_{ij})_{,i} + \rho\vec{f}\vec{u}$

(4 points)

Note 1: use chain rule for the derivation of $\nabla\cdot(\rho\vec{u}), \nabla\cdot(\rho u_i\vec{u})$, and $\nabla\cdot\left[\rho\left(e+\frac{u^2}{2}\right)\vec{u}\right]$.

Note 2: use the solution of a) in the derivation of b) and c).

Note 3: $u^2 = u_i u_i$

Problem 1.2 20 points:

Starting from the **vector** conservation form of the Navier-Stokes momentum equations, **derive** the equations in cylindrical coordinates (r, φ, z). Provide a separate momentum equation in each direction (written for velocities: u_r, u_φ, u_z).

Note 1: The vector conservation form of the N.S. momentum equation for Newtonian fluids:

$$\frac{\partial\vec{u}}{\partial t} + \nabla\cdot(\vec{u}\vec{u}) = -\frac{1}{\rho}\nabla p + \nu\nabla^2\vec{u} + \vec{f}$$

Note 2: definition of gradient function and material derivative in cylindrical coordinates will be useful. Be **careful** about the terms marked in red.

(1) $$\nabla \cdot = \frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \varphi} + \frac{\partial}{\partial z}$$

(2) $$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$

(3) $$\frac{D}{Dt} \equiv \left(\frac{\partial}{\partial t} + \vec{u} \cdot \nabla \cdot\right)$$

(4) $$\frac{\partial \overrightarrow{e_r}}{\partial r} = 0 \qquad \frac{\partial \overrightarrow{e_r}}{\partial \varphi} = \overrightarrow{e_\varphi} \qquad \frac{\partial \overrightarrow{e_r}}{\partial z} = 0$$

$$\frac{\partial \overrightarrow{e_\varphi}}{\partial r} = 0 \qquad \frac{\partial \overrightarrow{e_\varphi}}{\partial \varphi} = -\overrightarrow{e_r} \qquad \frac{\partial \overrightarrow{e_\varphi}}{\partial z} = 0$$

$$\frac{\partial \overrightarrow{e_z}}{\partial r} = 0 \qquad \frac{\partial \overrightarrow{e_z}}{\partial \varphi} = 0 \qquad \frac{\partial \overrightarrow{e_z}}{\partial z} = 0$$

Problem 1.3 10 points:

Determine the classification (e.g. whether they are elliptic, parabolic or hyperbolic) of:

a) heat equation ($\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2},\ a > 0$) (3 points);

b) incompressible 2-D N.S. equations (in space (4 points) and in time (3 points)).

Problem 1.4 10 points:

Consider incompressible N.S. equations in Cartesian coordinates. Derive Poisson's equation for pressure by taking the divergence of the momentum equation and then **applying continuity equation** to obtain:

$$\nabla^2 p = -\rho\left[\left(u_i u_j\right)_{,j}\right]_{,i}$$

Note:

The momentum equation for incompressible flow:

$$\frac{\partial u_j}{\partial t} + \frac{\partial(u_i u_j)}{\partial x_i} = -\frac{1}{\rho}\frac{\partial p}{\partial x_j} + \nu \frac{\partial^2 u_j}{\partial x_i \partial x_i} + \frac{1}{\rho} f_j$$

Problem 1.5 10 points:

Based on Reynolds number dependency of DNS meshing requirements provided in the notes, **derive** and plot:

a) Mesh size dependence on Reynolds number for **3D simulation** (5 points)

b) Time-dependent computational cost dependence on Reynolds number. Assume that the time step is proportional to the mesh spacing ($\Delta t \sim \Delta x$) (5 points)

Assume that mesh size of Re = 10^4 is **one non-dimensional unit**. Computation cost of this simulation is also **one unit** with the same Re. Provide plots for both a) and b) with Re up to 10^6. Use both linear and log-scale representations.

Note 1: in your plot, the spacing on x axis should be $\Delta x = 10^4$, and y=1 corresponds to computational cost of Re = 10^4.

Note 2: hand-drawn plots are not accepted.

Note 3: mesh size is the total number of cells in the domain.

Problem 1.6 10 points:

Using **dimensional analysis derive** the expressions for Kolmogorov's length, time and velocity scales.

Note: the turbulence dissipation rate (ε) has the dimensions of $\left[\frac{m^2}{s^3}\right]$ and kinematic viscosity has the dimensions of $\left[\frac{m^2}{s}\right]$.