① For inc. flows the mass cons. is the same:

$$\underline{\nabla} \cdot \underline{u} = 0 \qquad (3.9)$$

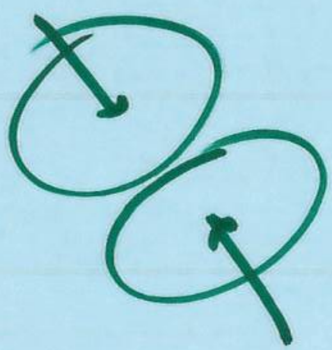
If the density ~~it~~ varies from 1 particle to another, but is constant for each particle as it moves, we need to follow each particle:

$$\frac{D\rho}{Dt} = 0 \qquad (3.10)$$

When the interface is well defined, we only need to find $H$ and then construct density directly from $H$.

(2) Topology changes


Note: "pinching of threads"
less sensitive to numerical treatment.



"rupture of films"
is more compex to model since lengthscale $\ll$ mesh size.

3 Let's update eq. (2.3) for predicted velocity:

$$\underline{u}^* = \underline{u}^n + \Delta t \Big( - \underline{A}_h(\underline{u}^n) + \frac{1}{\rho^n} \underline{D}_h(\underline{u}^n) + \frac{1}{\rho^n_*} \underline{f}_l$$

$$\boxed{? \ \frac{\mu^n}{\rho^n} \underline{D}_h} \qquad + \frac{1}{\rho^n} \underline{F}_\gamma^n \Big) \qquad (3.11)$$

Details on $\underline{F}_\gamma^n$ calculation depend on specific method.

Pressure - Poisson: (modified (2.5)):

$$\nabla_h \frac{1}{\rho^n} \cdot \nabla_h p = \frac{1}{\Delta t} \nabla_h \cdot \underline{u}^* \qquad (3.12)$$

4

# The level-set method

Osher & Sethien (1988)

Sussman, Smereka & Osher (1994)

Pros:
- no special treatment for topological changes
- robust, accurate representation of normals, curvature
- interface thickness control allows for higher-order accuracy for curvature etc.

5

Level set function:

$$\phi(\underline{x},t) = \begin{cases} +d, & \underline{x} \text{ in the liquid} \\ -d, & \underline{x} \text{ in the gas} \end{cases}$$

$d$ represents normal distance to the interface at time $t$

The level set is advected by:

$$\frac{\partial \phi}{\partial t} + \underline{u} \cdot \nabla \phi = 0 \qquad (3.39)$$

where $\underline{u}$ is fluid velocity

⑥ The eq. is derived in such a way that $\phi$ is constant along the particle paths:

$$\frac{d\phi(\underline{x}(t),t)}{dt} = 0 \qquad (3.40)$$

$$\frac{d\underline{x}(t)}{dt} \equiv \underline{u}$$ implies:

$$\frac{d\underline{x}(t)}{dt} \cdot \underline{\nabla}\phi(\underline{x},t) + \frac{\partial\phi(\underline{x},t)}{\partial t} = 0$$

which leads to (3.39)

<u>Note</u>: the eq. is hyperbolic.

7 Often the following scheme is used:

a) re-write (3.39) as

$$\phi_t = \mathcal{L}\phi$$

Assume $\phi^n_{i,j}$ and $\underline{u}^n_{i,j}$ are valid discrete values defined at $t = t^n$, $x = x_i$, $y = y_j$.

We advance to $t = t^{n+1}$ by first defining predicted value:

$$\phi^*_{i,j} = \phi^n_{i,j} + \Delta t \mathcal{L}\phi^n \quad (3.41)$$

and

$$\phi^{n+1}_{i,j} = \phi^n_{i,j} + \frac{\Delta t}{2}\left(\mathcal{L}\phi^n + \mathcal{L}\phi^*\right) \quad (3.42)$$

(8) The operator $L\Phi$ is discretized as:

$$L\Phi = -u_{i,j}\frac{\varphi_{i+\frac{1}{2},j} - \varphi_{i-\frac{1}{2},j}}{h} - v_{i,j}\frac{\varphi_{i,j+\frac{1}{2}} - \varphi_{i,j-\frac{1}{2}}}{h} \quad (3.43)$$

where the value of $\Phi$ at cell boundaries is found from:

$$\varphi_{i+\frac{1}{2},j} = \begin{cases} \varphi_{i,j} + \frac{1}{2}M\left(D_x^+\varphi_{i,j};\, D_x^-\varphi_{i,j}\right); & \frac{1}{2}\left(u_{i+1,j}+u_{i,j}\right) > 0 \\ \varphi_{i+1,j} - \frac{1}{2}M\left(D_x^+\varphi_{i+1,j};\, D_x^-\varphi_{i+1,j}\right); & \frac{1}{2}\left(u_{i+1,j}+u_{i,j}\right) < 0 \end{cases} \quad (3.44)$$

$M$ is a switch defined by:

$$M(a,b) = \begin{cases} a, & |a| < |b| \\ b, & |b| \leq |a| \end{cases} \quad (3.45)$$

9 and the differences are defined as:

~~a, b~~

$$D_x^+\phi_{i,j} = \phi_{i+1,j} - \phi_{i,j} \qquad (3.46)$$

$$D_x^-\phi_{i,j} = \phi_{i,j} - \phi_{i-1,j}$$

Eq. for $\phi_{i,j+\frac{1}{2}}$ is similar.

$\phi = 0$ contour is accurately advected, however distance function off the interface will not remain a proper distance function.