Problem 5.4 20 points:

Use the code you have created for Problem 4.1. If you have not got nearly full credit (below 18 points) for that problem and did not improve/fix the code, you are allowed to ask TA for the code (laminar solution N.S. solver).

a) Implement the pressure solver using the iterative methods discussed in class. Demonstrate that the solver shows the correct pressure gradient compared to the analytical value of the single phase laminar flow solved in Problem 4.1 a). Submit your code. [15 points]

b) Do you expect the steady state solution to change with initial conditions? To test it, initialize your velocity field to be $0\ m/s$, u_{ave}, u_{max} and compare steady state solutions using 30 pressure cells across the channel. Discuss the result. [5 points]

Solutions:

(a) Pseudo code for the solver:

1. Simulation parameters specification (geometry (L_x, L_y), number of pressure cells (N_x, N_y), fluid properties (ρ, μ, ν), given flow conditions($u_{max}, \frac{dp}{dx}$), max number of iterations ($n_{iter,max}$), Δt, pressure iteration tolerance $\epsilon_{p,tol}$, max number of pressure iteration ($n_{piter,max}$)
2. Data array declaration and initialization (e.g. x-velocity($u\ (N_x + 2, N_y + 2)$), pressure p, reference x-velocity, surface area and normal of the pressure cell boundaries)
3. Initialization of the solution array and convergence metric:

$$u = u_{ini}, L_{2r} = \frac{L_{2,curent}}{L_{2,prev}} = 1.0$$

4. While ($n_{iter} < n_{iter,max}$ && $L_{2r} \leq 1.0$):

   - Enforce B.C. for u^n

   - Predictor step for all cells:

$$u^* = u^n + \Delta t\left(-A_x(u^n) + \nu D_x(u^n) - \frac{1}{\rho}\frac{\partial p}{\partial x}\right)$$

   - Enforce B.C. for u^*

- pressure iteration loop (solve for pressure):

While ($maxval(\epsilon_p) > \epsilon_{p,tol}$ && $n_{piter} < n_{piter,max}$):

for ($i = 1, i < Nx + 2, i + +$): (each column of the domain)

$$RHS = \frac{\rho}{\Delta t}\left(\frac{u^*_{i+\frac{1}{2},j} - u^*_{i-\frac{1}{2},j} + v^*_{i,j+\frac{1}{2}} - v^*_{i,j-\frac{1}{2}}}{h}\right)$$

$$P^{n+1}_{i,j} = \frac{-1}{3}[h^2 * RHS - \left(p^n_{i+1,1} + p^n_{i-1,1} + p^n_{i,2}\right)]$$

$$RHS = \frac{\rho}{\Delta t}\left(\frac{u^*_{i+\frac{1}{2},j} - u^*_{i-\frac{1}{2},j} + v^*_{i,j+\frac{1}{2}} - v^*_{i,j-\frac{1}{2}}}{h}\right)$$

$$P^{n+1}_{i,j} = \frac{-1}{3}[h^2 * RHS - \left(p^n_{i+1,N_y} + p^n_{i-1,N_y} + p^n_{i,N_y-1}\right)]$$

for ($j = 2, j < Nx, j + +$): (rows that are not adjacent to the walls)

$$RHS = \frac{\rho}{\Delta t}\left(\frac{u^*_{i+\frac{1}{2},j} - u^*_{i-\frac{1}{2},j} + v^*_{i,j+\frac{1}{2}} - v^*_{i,j-\frac{1}{2}}}{h}\right)$$

$$P^{n+1}_{i,j} = \frac{-1}{4}[h^2 * RHS - \left(p^n_{i+1,j} + p^n_{i-1,j} + p^n_{i,j-1} + p^n_{i,j+1}\right)]$$

Compute ϵ_p for each pressure cell $\epsilon_p = p^{n+1} - p^n$

Evaluate the maximum ϵ_p overall:

$$\epsilon_{p,max} = maxval\left(\epsilon_p\right)$$

Assign new pressure values to the old one:

$$p^n = p^{n+1}$$

Enforce pressure periodicity for p^{n+1}:

$$p^{n+1}_{0,j} = p^{n+1}_{Nx,j}$$

$$p_{N_x+1,j}^{n+1} = p_{1,j}^{n+1}$$

$$j \in [1, N_y]$$

Plus one to the pressure iteration count:

$$n_{piter} += 1$$

- Corrector step for all cells:

$$u^{n+1} = u^* - \frac{\Delta t}{\rho} \frac{(p_{i+1,j}^{n+1} - p_{i,j}^{n+1})}{h}$$

- Update L_2 for this iteration and compute L_{2r}

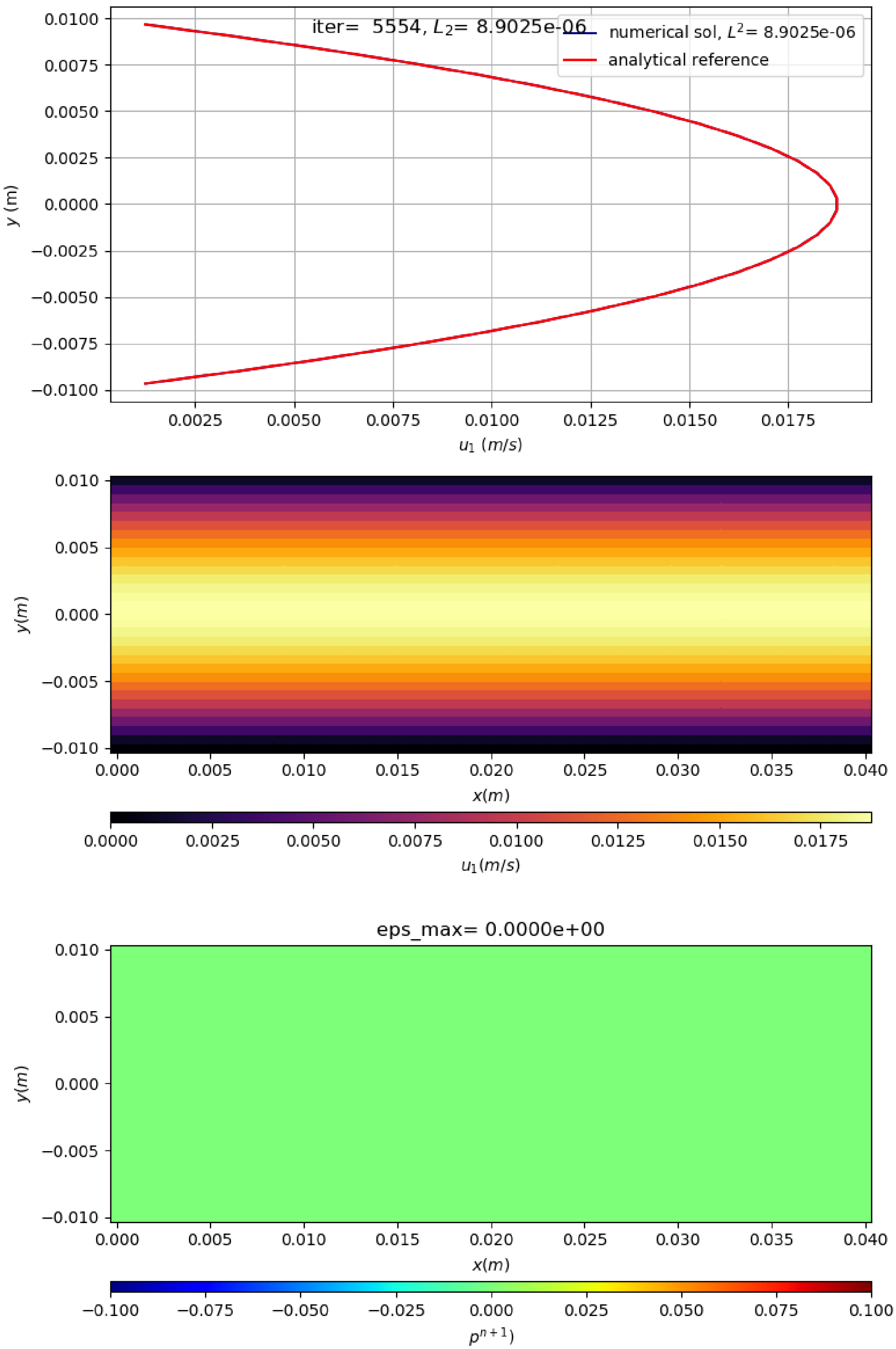
Note:

1. Since periodicity is enforced, the pressure gradient will be applied in the manner similar to the body force.
2. Group the pressure iteration operations into a function for the convenience of debugging

Results:

30 elements across domain height with $\frac{\partial p}{\partial x} = -0.375 \frac{N}{m^3}$:

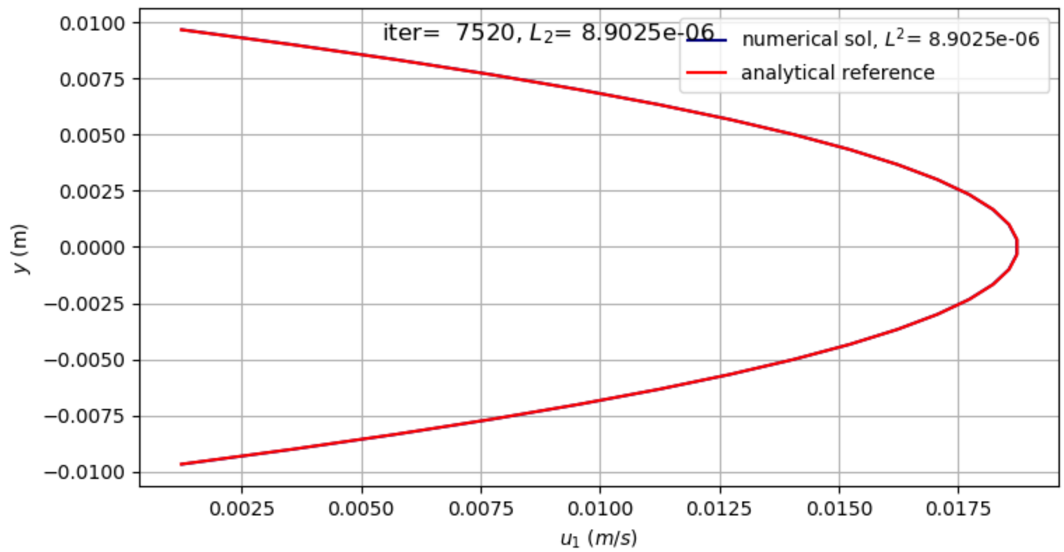
u_{ave} was used as the initial condition (I.C.). The steady state solution converged to the analytical one, and the L_2 norm is identical to that in 4.1 with $N_y = 30$. Since the periodic boundary condition is applied to the streamwise direction, the pressure distribution in the domain should be uniform.

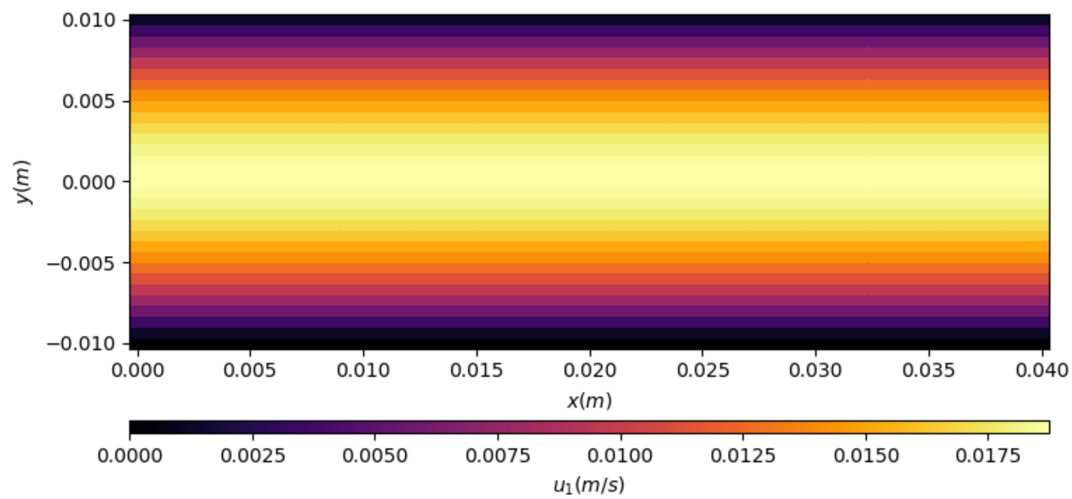

iter= 5554, L_2= 8.9025e-06
numerical sol, L^2= 8.9025e-06
analytical reference
y (m)
u_1 (m/s)
$y(m)$
$x(m)$
$u_1(m/s)$
eps_max= 0.0000e+00
$y(m)$
$x(m)$
$p^{n+1})$

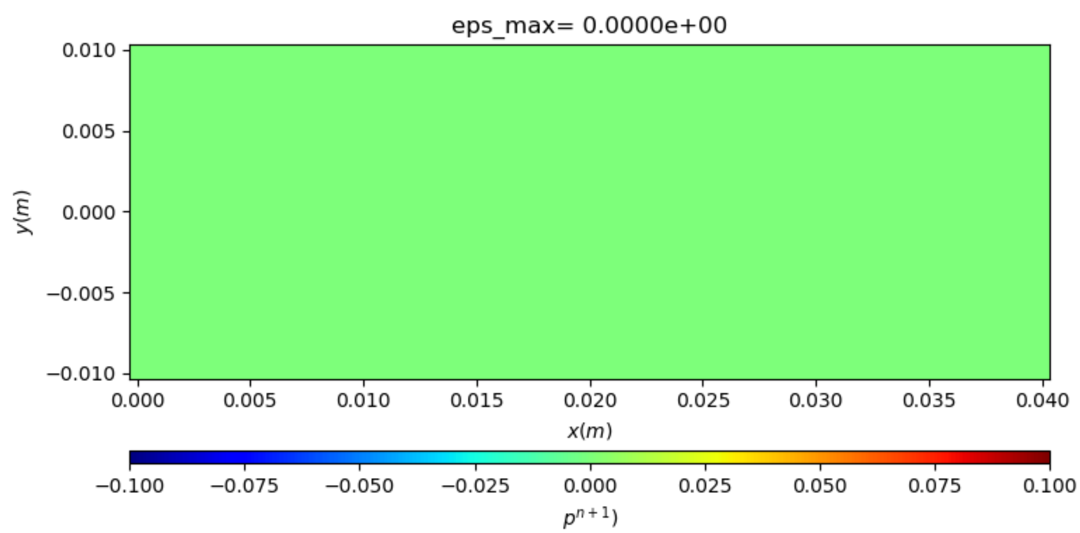
b) Change initial condition of the velocity field and results are identical. The velocity distributions and L_2 norms are also identical to those of Prob 4.1. Again, the numerical solution did not change with initial solution, which fits the expectation that the physics of the NS equation will pull the velocity distribution to the analytical expectation step by step.

Similar convergence behavior is observed across the results of different I.C.: those with I.C.s that are further from the steady state solution will need more time steps to converge.

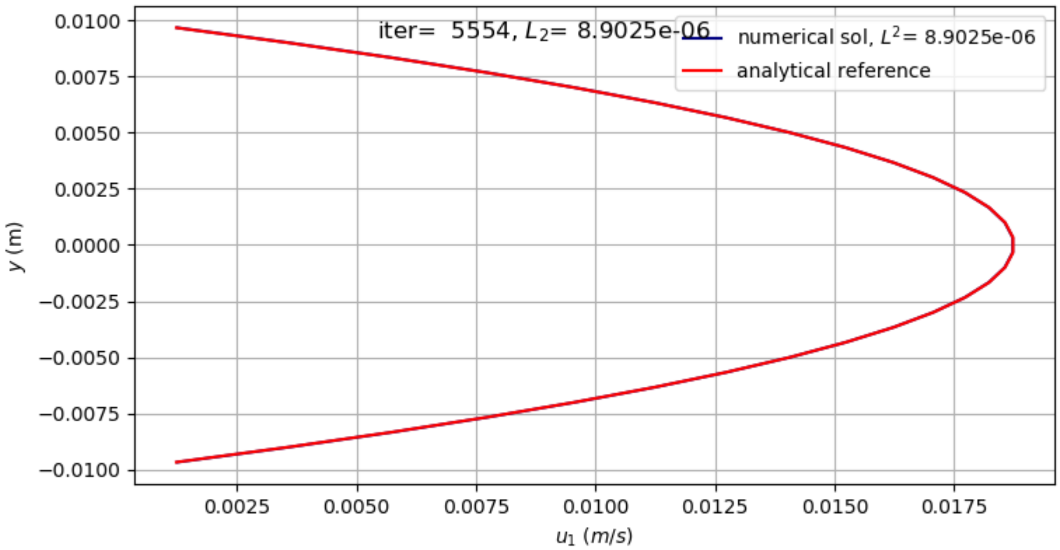
Results with $u = 0\ m/s$ as initial condition:

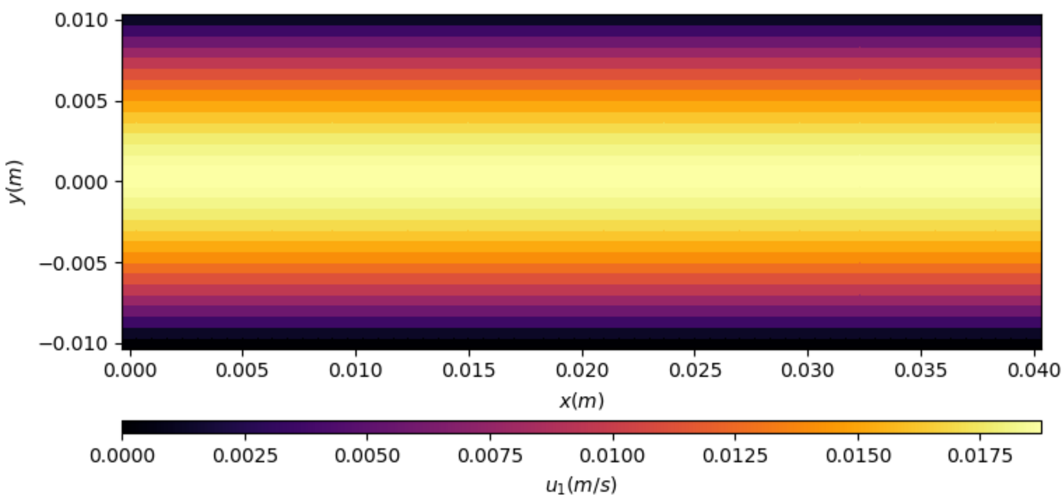


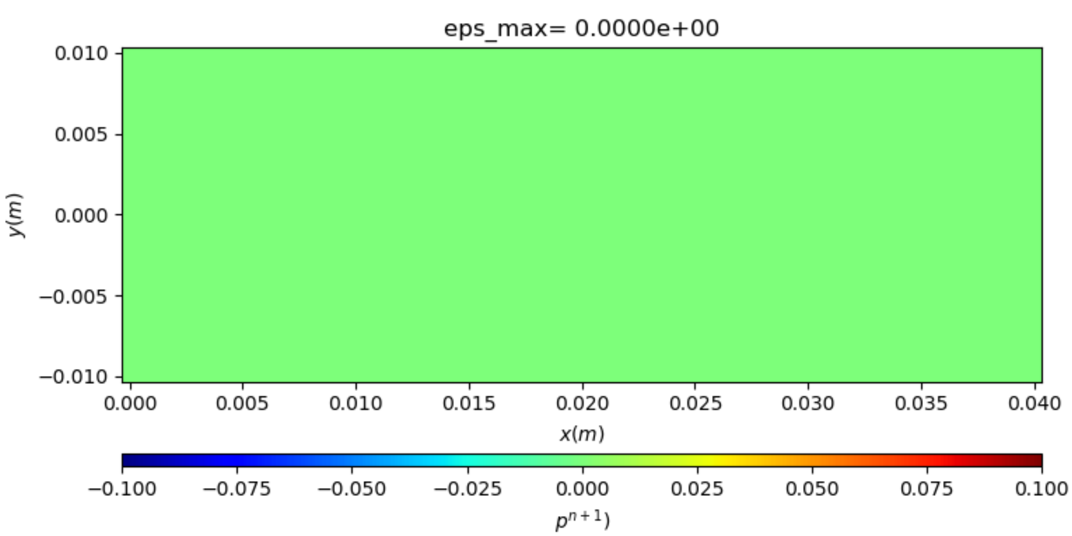




Results with $u = u_{ave}$ as initial condition:







Results with $u = u_{max}$ as initial condition: