Problem 1.2 20 points:

Starting from the **vector** conservation form of the Navier-Stokes momentum equations, **derive** the equations in cylindrical coordinates (r, φ, z). Provide a separate momentum equation in each direction (written for velocities: u_r, u_φ, u_z).

Note 1: The vector conservation form of the N.S. momentum equation for Newtonian fluids:

$$\frac{\partial \vec{u}}{\partial t} + \nabla \cdot (\vec{u}\vec{u}) = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{u} + \vec{f}$$

Note 2: definition of gradient function and material derivative in cylindrical coordinates will be useful. Be **careful** about the terms marked in red.

(1) $$\nabla \cdot = \frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \varphi} + \frac{\partial}{\partial z}$$

(2) $$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$

(3) $$\frac{D}{Dt} \equiv \left(\frac{\partial}{\partial t} + \vec{u} \cdot \nabla \cdot\right)$$

(4) $$\frac{\partial \vec{e_r}}{\partial r} = 0 \qquad \frac{\partial \vec{e_r}}{\partial \varphi} = \vec{e_\varphi} \qquad \frac{\partial \vec{e_r}}{\partial z} = 0$$

$$\frac{\partial \vec{e_\varphi}}{\partial r} = 0 \qquad \frac{\partial \vec{e_\varphi}}{\partial \varphi} = -\vec{e_r} \qquad \frac{\partial \vec{e_\varphi}}{\partial z} = 0$$

$$\frac{\partial \vec{e_z}}{\partial r} = 0 \qquad \frac{\partial \vec{e_z}}{\partial \varphi} = 0 \qquad \frac{\partial \vec{e_z}}{\partial z} = 0$$

Solution:

The flow is incompressible:

$$\nabla \cdot \vec{u} = 0$$

Apply chain rule on the advection term:

$$\nabla \cdot (\vec{u}\vec{u}) = \vec{u}(\nabla \cdot \vec{u}) + \vec{u} \cdot \nabla \vec{u} = \vec{u} \cdot \nabla \vec{u}$$

With the nabla operator in the cylindrical coordinate ($\nabla = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial r}, \frac{\partial}{\partial z}\right)$), the velocity gradient tensor can be written as:

$$\nabla\vec{u} = \begin{pmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{r}\left(\frac{\partial u_r}{\partial \varphi} - u_\varphi\right) & \frac{\partial u_r}{\partial z} \\ \frac{\partial u_\varphi}{\partial r} & \frac{1}{r}\left(\frac{\partial u_\varphi}{\partial \varphi} + u_r\right) & \frac{\partial u_\varphi}{\partial z} \\ \frac{\partial u_z}{\partial z} & \frac{1}{r}\left(\frac{\partial u_z}{\partial \varphi}\right) & \frac{\partial u_z}{\partial z} \end{pmatrix}$$

(one must be very careful about the chain rules)

Then the inner product of velocity vector and the Jacobian:

$$\begin{aligned}\vec{u}\cdot\nabla\vec{u} = &\left[u_r\frac{\partial u_r}{\partial r} + u_\varphi\frac{1}{r}\left(\frac{\partial u_r}{\partial \varphi} - u_\varphi\right) + u_z\frac{\partial u_r}{\partial z}\right]\overrightarrow{e_r} \\ &+ \left[u_r\frac{\partial u_\varphi}{\partial r} + u_\varphi\frac{1}{r}\left(\frac{\partial u_\varphi}{\partial \varphi} + u_r\right) + u_z\frac{\partial u_\varphi}{\partial z}\right]\overrightarrow{e_\varphi} \\ &+ \left[u_r\frac{\partial u_r}{\partial z} + u_\varphi\frac{1}{r}\left(\frac{\partial u_z}{\partial \varphi}\right) + u_z\frac{\partial u_z}{\partial z}\right]\overrightarrow{e_z}\end{aligned}$$

The pressure gradient:

$$-\frac{1}{\rho}\nabla p = -\frac{1}{\rho}\left(\frac{\partial p}{\partial r}\overrightarrow{e_r} + \frac{1}{r}\frac{\partial p}{\partial \varphi}\overrightarrow{e_\varphi} + \frac{\partial u_z}{\partial z}\overrightarrow{e_z}\right)$$

The Laplacian of velocity in the viscous stress term:

$$\begin{aligned}\nabla^2\vec{u} &= \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}\right]\left(u_r\overrightarrow{e_r} + u_\varphi\overrightarrow{e_\varphi} + u_z\overrightarrow{e_z}\right) \\ &= \left\{\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(u_r\overrightarrow{e_r}\right)\right] + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\left(u_r\overrightarrow{e_r}\right) + \frac{\partial^2}{\partial z^2}\left(u_r\overrightarrow{e_r}\right)\right\} \\ &+ \left\{\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(u_\varphi\overrightarrow{e_\varphi}\right)\right] + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\left(u_\varphi\overrightarrow{e_\varphi}\right) + \frac{\partial^2}{\partial z^2}\left(u_\varphi\overrightarrow{e_\varphi}\right)\right\} \\ &+ \left\{\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(u_z\overrightarrow{e_z}\right)\right] + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\left(u_z\overrightarrow{e_z}\right) + \frac{\partial^2}{\partial z^2}\left(u_\varphi\overrightarrow{e_\varphi}\right)\right\}\end{aligned}$$

By applying chain rule and (4):

$$\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}(u_r\overrightarrow{e_r})\right]+\frac{1}{r^2}\frac{\partial^2 u_\varphi}{\partial^2\varphi}(u_r\overrightarrow{e_r})+\frac{\partial^2}{\partial z^2}(u_r\overrightarrow{e_r})\right\}$$

$$=\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial u_r}{\partial r}\overrightarrow{e_r}+\frac{\partial\overrightarrow{e_r}}{\partial r}u_r\right)\right]+\frac{1}{r^2}\left(\frac{\partial^2 u_r}{\partial\varphi^2}\overrightarrow{e_r}+2\frac{\partial u_r}{\partial\varphi}\frac{\partial\overrightarrow{e_r}}{\partial\varphi}+\frac{\partial^2\overrightarrow{e_r}}{\partial\varphi^2}u_r\right)\right.$$
$$\left.+\left(\frac{\partial^2 u_r}{\partial z^2}+2\frac{\partial u_r}{\partial z}\frac{\partial\overrightarrow{e_r}}{\partial z}+\frac{\partial^2\overrightarrow{e_r}}{\partial z^2}\right)\right\}$$
$$=\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial u_r}{\partial r}\right)\right]\overrightarrow{e_r}+\frac{1}{r^2}\left(\frac{\partial^2 u_r}{\partial\varphi^2}\overrightarrow{e_r}+2\frac{\partial u_r}{\partial\varphi}\overrightarrow{e_\varphi}-u_r\overrightarrow{e_r}\right)+\frac{\partial^2 u_r}{\partial z^2}\overrightarrow{e_r}\right\}$$

Similarly:

$$\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}(u_\varphi\overrightarrow{e_\varphi})\right]+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}(u_\varphi\overrightarrow{e_\varphi})+\frac{\partial^2}{\partial z^2}(u_\varphi\overrightarrow{e_\varphi})\right\}$$

$$=\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial u_\varphi}{\partial r}\overrightarrow{e_\varphi}+\frac{\partial\overrightarrow{e_\varphi}}{\partial r}u_\varphi\right)\right]+\frac{1}{r^2}\left(\frac{\partial^2 u_\varphi}{\partial\varphi^2}\overrightarrow{e_\varphi}+2\frac{\partial u_\varphi}{\partial\varphi}\frac{\partial\overrightarrow{e_\varphi}}{\partial\varphi}+\frac{\partial^2\overrightarrow{e_\varphi}}{\partial\varphi^2}u_\varphi\right)\right.$$
$$\left.+\left(\frac{\partial^2 u_\varphi}{\partial z^2}\overrightarrow{e_\varphi}+2\frac{\partial u_\varphi}{\partial z}\frac{\partial\overrightarrow{e_\varphi}}{\partial z}+\frac{\partial^2\overrightarrow{e_\varphi}}{\partial z^2}u_\varphi\right)\right\}$$
$$=\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial u_\varphi}{\partial r}\right)\right]\overrightarrow{e_\varphi}+\frac{1}{r^2}\left(\frac{\partial^2 u_\varphi}{\partial\varphi^2}\overrightarrow{e_\varphi}-2\frac{\partial u_\varphi}{\partial\varphi}\overrightarrow{e_r}-u_\varphi\overrightarrow{e_\varphi}\right)+\frac{\partial^2 u_\varphi}{\partial z^2}\overrightarrow{e_\varphi}\right\}$$

And

$$\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}(u_z\overrightarrow{e_z})\right]+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}(u_z\overrightarrow{e_z})+\frac{\partial^2}{\partial z^2}(u_z\overrightarrow{e_z})\right\}$$

$$=\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial u_z}{\partial r}\overrightarrow{e_z}+\frac{\partial\overrightarrow{e_z}}{\partial r}u_z\right)\right]+\frac{1}{r^2}\left(\frac{\partial^2 u_z}{\partial\varphi^2}\overrightarrow{e_z}+2\frac{\partial u_z}{\partial\varphi}\frac{\partial\overrightarrow{e_z}}{\partial\varphi}+\frac{\partial^2\overrightarrow{e_z}}{\partial\varphi^2}u_z\right)\right.$$
$$\left.+\left(\frac{\partial^2 u_z}{\partial z^2}\overrightarrow{e_z}+2\frac{\partial u_z}{\partial z}\frac{\partial\overrightarrow{e_z}}{\partial z}+\frac{\partial^2\overrightarrow{e_z}}{\partial z^2}u_z\right)\right\}$$
$$=\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial u_z}{\partial r}\right)\right]\overrightarrow{e_z}+\frac{1}{r^2}\left(\frac{\partial^2 u_z}{\partial\varphi^2}\overrightarrow{e_z}\right)+\left(\frac{\partial^2 u_z}{\partial z^2}\overrightarrow{e_z}\right)\right\}$$

Group terms associated with $\overrightarrow{e_r},\overrightarrow{e_\varphi},\overrightarrow{e_z}$ together:

$$\nabla^2\vec{u}=\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial u_r}{\partial r}\right)\right]+\frac{1}{r^2}\left(\frac{\partial^2 u_r}{\partial\varphi^2}-u_r-2\frac{\partial u_\varphi}{\partial\varphi}\right)+\frac{\partial^2 u_r}{\partial z^2}\right\}\overrightarrow{e_r}$$
$$+\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial u_\varphi}{\partial r}\right)\right]+\frac{1}{r^2}\left(\frac{\partial^2 u_\varphi}{\partial\varphi^2}-u_\varphi+2\frac{\partial u_r}{\partial\varphi}\right)+\frac{\partial^2 u_\varphi}{\partial z^2}\right\}\overrightarrow{e_\varphi}$$
$$+\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial u_z}{\partial r}\right)\right]+\frac{1}{r^2}\left(\frac{\partial^2 u_z}{\partial\varphi^2}\right)+\frac{\partial^2 u_z}{\partial z^2}\right\}\overrightarrow{e_z}$$

Finally, the momentum equation in cylindrical coordinates:

$$\frac{\partial u_r}{\partial t} + \left[u_r \frac{\partial u_r}{\partial r} + \frac{u_\varphi}{r}\left(\frac{\partial u_r}{\partial \varphi} - u_\varphi\right) + u_z \frac{\partial u_r}{\partial z}\right]$$
$$= -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial u_r}{\partial r}\right)\right] + \frac{1}{r^2}\left(\frac{\partial^2 u_r}{\partial \varphi^2} - u_r - 2\frac{\partial u_\varphi}{\partial \varphi}\right) + \frac{\partial^2 u_r}{\partial z^2}\right] + f_r$$

$$\frac{\partial u_\varphi}{\partial t} + \left[u_r \frac{\partial u_\varphi}{\partial r} + u_\varphi \frac{1}{r}\left(\frac{\partial u_\varphi}{\partial \varphi} + u_r\right) + u_z \frac{\partial u_\varphi}{\partial z}\right]$$
$$= -\frac{1}{\rho r}\frac{\partial p}{\partial \varphi} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial u_\varphi}{\partial r}\right)\right] + \frac{1}{r^2}\left(\frac{\partial^2 u_\varphi}{\partial \varphi^2} - u_\varphi + 2\frac{\partial u_r}{\partial \varphi}\right) + \frac{\partial^2 u_\varphi}{\partial z^2}\right] + f_\varphi$$

$$\frac{\partial u_z}{\partial t} + \left[u_r \frac{\partial u_r}{\partial z} + u_\varphi \frac{1}{r}\left(\frac{\partial u_z}{\partial \varphi}\right) + u_z \frac{\partial u_z}{\partial z}\right]$$
$$= -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial u_z}{\partial r}\right)\right] + \frac{1}{r^2}\left(\frac{\partial^2 u_z}{\partial \varphi^2}\right) + \frac{\partial^2 u_z}{\partial z^2}\right] + f_z$$