Problem 6.3 10 points:

Derive the $D_x^+ D_x^- (\phi_d)_{i,j}$ term in Eq. (3.52), using the definition in Eq. (3.46). [5 points]

Then derive the $D_y^+ D_y^- (\phi_d)_{i,j}$ term. [5 points]

$$D_x^+ \phi_{i,j} = \phi_{i+1,j} - \phi_{i,j}, \qquad D_x^- \phi_{i,j} = \phi_{i,j} - \phi_{i-1,j}. \tag{3.46}$$

$$\begin{aligned} \widetilde{D}_x^+ &= D_x^+ (\phi_d)_{i,j} - \frac{1}{2} M\left(D_x^+ D_x^- (\phi_d)_{i,j}, D_x^+ D_x^- (\phi_d)_{i+1,j}\right), \\ \widetilde{D}_x^- &= D_x^- (\phi_d)_{i,j} + \frac{1}{2} M\left(D_x^+ D_x^- (\phi_d)_{i,j}, D_x^+ D_x^- (\phi_d)_{i-1,j}\right). \end{aligned} \tag{3.52}$$

Note: This derivation will help with the coding in Problem 6.2.

Solution:

$$D_x^+ D_x^- (\phi_d)_{i,j} = D_x^+ (D_x^- (\phi_d)_{i,j}) = D_x^+ \left(\phi_{i,j} - \phi_{i-1,j}\right)$$

Method 1 (treat $(\phi_{i,j} - \phi_{i-1,j})$ as a whole):

$$D_x^+ \left(\phi_{i,j} - \phi_{i-1,j}\right) = \left(\phi_{i+1,j} - \phi_{i,j}\right) - (\phi_{i,j} - \phi_{i-1,j}) = \phi_{i+1,j} + \phi_{i-1,j} - 2\phi_{i,j}$$

Method 2 (D_x^+ is a linear operator):

$$\begin{aligned} D_x^+ \left(\phi_{i,j} - \phi_{i-1,j}\right) &= D_x^+ \phi_{i,j} - D_x^+ \phi_{i-1,j} = \left(\phi_{i+1,j} - \phi_{i,j}\right) - \left(\phi_{i,j} - \phi_{i-1,j}\right) \\ &= \phi_{i+1,j} + \phi_{i-1,j} - 2\phi_{i,j} \end{aligned}$$

Similarly:

$$D_y^+ D_y^- (\phi_d)_{i,j} = \phi_{i,j+1} + \phi_{i,j-1} - 2\phi_{i,j}$$