Problem 5.2 15 points:

Review the derivation done in class (Eqns. 3.1 – 3.7).

(a) Derive the gradient of the Heaviside function in 2D in terms of 1D delta function. [6 points]

(b) Repeat the derivation in 3D. [5 points]

(c) Express the continuous expressions for dynamic viscosity (μ) and thermal conductivity (k) and their gradients based on the Heaviside function and its 3D gradient. [4 points]

Solution:

(a) For 1D case,

$$H(x) = \int_L \delta(x - x')dL'$$

Take divergence of the Heaviside function and then change the variable from x to x':

$$\nabla H(x) = \int_L \nabla\delta(x - x')dL'$$

$$= -\int_L \nabla'\delta(x - x')dL'$$

For 1-d condition, normal direction is same as the x direction. Introducing local coordinates normal (n):

$$\delta(x - x') = \delta(n)$$

then

$$\nabla H(x) = -\int_L \nabla'\delta(x - x')dL'$$

$$= -\int_L \nabla'\delta(n')dL'$$

$$= -\delta(n)$$

For 2D case,

$$H(x, y) = \int_A \delta(x - x')\delta(y - y')dA'$$

$$\nabla H(x, y) = \int_A \nabla[\delta(x - x')\delta(y - y')]dA'$$

Introducing local coordinates normal (n) and tangent (s) to rewrite the expression:

$$\delta(x - x')\delta(y - y') = \delta(s)\delta(n)$$

$$\nabla H(x, y) = -\int_A \nabla'[\delta(x - x')\delta(y - y')]dA'$$

$$= -\int_s \delta(s')\delta(n') \cdot \hat{n}'dS'$$

$$= -\delta(n)\hat{n}$$

(b) For 3D case,

$$\mathrm{H} = \int_V \delta(x - x')\delta(y - y')\delta(z - z')dV'$$

$$\nabla H = \int_V \nabla[\delta(x - x')\delta(y - y')\delta(z - z')]dV'$$

$$= -\int_V \nabla'[\delta(x - x')\delta(y - y')\delta(z - z')]dV'$$

$$= -\oiint_S \delta(x - x')\delta(y - y')\delta(z - z')\hat{n}'dS'_1 dS'_2$$

$$= -\iint_S \delta(x - x')\delta(y - y')\delta(z - z')\hat{n}'dS'_1 dS'_2$$

Since $\delta(x - x')\delta(y - y')\delta(z - z') = \delta(s_1)\delta(s_2)\delta(n)$

$$\Rightarrow \nabla H = -\iint_S \delta(s'_1)\delta(s'_2)\delta(n')\hat{n}'dS'_1 dS'_2$$

$$= -\delta(n)\hat{n}$$

(c) Assume the dynamic viscosity of each phase is constant

$\mu(x,y,z) = \mu_1 H(x,y,z) + \mu_0[1 - H(x,y,z)]$

the gradient of density is given by

$$
\begin{aligned}
\nabla\mu &= \mu_1 \nabla H - \mu_0 \nabla H \\
&= (\mu_1 - \mu_0)\nabla H \\
&= (\mu_1 - \mu_0)(-\delta(n)\hat{n}) \\
&= \Delta\mu\delta(n)\hat{n}
\end{aligned}
$$

Where $\Delta\mu = \mu_0 - \mu_1$

Assume the thermal conductivity of each phase is constant

$k(x,y,z) = k_1 H(x,y,z) + k_0[1 - H(x,y,z)]$

$$
\begin{aligned}
\nabla k &= k_1 \nabla H - k_0 \nabla H \\
&= (k_1 - k_0)\nabla H \\
&= (k_1 - k_0)(-\delta(n)\hat{n}) \\
&= \Delta k\delta(n)\hat{n}
\end{aligned}
$$

Where $\Delta k = k_0 - k_1$