Problem 1.1 10 points:

Show that material derivative definition (D/Dt) yields compressible mass, momentum and energy conservation equations starting with the partial derivative ($\partial/\partial t$) -based equations in below.

a) mass: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$ (2 points)

b) momentum: $\frac{\partial(\rho u_i)}{\partial t} + \nabla \cdot (\rho u_i \vec{u}) = -P_{,i} + \tau_{ji,i} + \rho f_i$ (4 points)

c) energy: $\frac{\partial}{\partial t}\left[\rho\left(e + \frac{u^2}{2}\right)\right] + \nabla \cdot \left[\rho\left(e + \frac{u^2}{2}\right)\vec{u}\right] = \rho \dot{q} + (kT_{,i})_{,i} - (u_i p)_{,i} + (u_j \tau_{ij})_{,i} + \rho \vec{f} \vec{u}$

(4 points)

Note 1: use chain rule for the derivation of $\nabla \cdot (\rho \vec{u}), \nabla \cdot (\rho u_i \vec{u})$, and $\nabla \cdot \left[\rho\left(e + \frac{u^2}{2}\right)\vec{u}\right]$.

Note 2: use the solution of a) in the derivation of b) and c).

Note 3: $u^2 = u_i u_i$

Solution:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i}$$

a) conservation of mass:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$$

With $\nabla \cdot (\rho \vec{u}) = \vec{u} \cdot \nabla \rho + \rho(\nabla \cdot \vec{u})$

$$\frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho + \rho \nabla \cdot \vec{u} = 0$$

Or:

$$\frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i} + \rho \nabla \cdot \vec{u} = 0$$

$$\frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho + \rho(\nabla \cdot \vec{u}) = 0$$

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \vec{u} = 0$$

b) conservation of momentum:

$$\frac{\partial(\rho u_i)}{\partial t} + \nabla \cdot (\rho u_i \vec{u}) = -P_{,i} + \tau_{ji,i} + \rho f_i$$

Let $RHS = -P_{,i} + \tau_{ji,i} + \rho f_i$

Expand the LHS of the momentum equation with chain rule:

$$\frac{\partial(\rho u_i)}{\partial t} + \nabla \cdot (\rho u_i \vec{u}) = \rho\frac{\partial u_i}{\partial t} + u_i\frac{\partial \rho}{\partial t} + u_i\vec{u} \cdot \nabla\rho + \rho\vec{u} \cdot \nabla u_i + \rho u_i(\nabla \cdot \vec{u})$$

Regroup the terms:

$$u_i\left[\frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla\rho + \rho(\nabla \cdot \vec{u})\right] + \rho\left(\frac{\partial u_i}{\partial t} + \vec{u} \cdot \nabla u_i\right)$$

Based on the results of a) $\frac{\partial \rho}{\partial t} + u_i\frac{\partial \rho}{\partial x_i} + \rho\nabla \cdot \vec{u} = 0$ and incorporate the RHS:

$$\rho\left(\frac{\partial u_i}{\partial t} + \vec{u} \cdot \nabla u_i\right) = RHS$$

$$\rho\frac{Du_i}{Dt} = RHS$$

c) conservation of energy:

$$\frac{\partial}{\partial t}\left[\rho\left(e + \frac{u^2}{2}\right)\right] + \nabla \cdot \left[\rho\left(e + \frac{u^2}{2}\right)\vec{u}\right] = \rho\dot{q} + (kT_{,i})_{,i} - (u_i p)_{,i} + (u_j\tau_{ij})_{,i} + \rho\vec{f}\vec{u}$$

Let $RHS = \dot{q} + (kT_{,i})_{,i} - (u_i p)_{,i} + (u_j\tau_{ij})_{,i} + \rho\vec{f}\vec{u}$

Let $E = e + \frac{u^2}{2}$ be the total specific energy of the fluid.

Expand the LHS of the energy equation with chain rule:

$$\frac{\partial(\rho E)}{\partial t} + \nabla \cdot (\rho E\vec{u}) = \rho\frac{\partial E}{\partial t} + E\frac{\partial \rho}{\partial t} + \rho\vec{u} \cdot \nabla \mathrm{E} + \mathrm{E}\vec{u} \cdot \nabla\rho + \rho E(\nabla \cdot \vec{u})$$

Regroup the terms:

$$E\left[\frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla\rho + \rho(\nabla \cdot \vec{u})\right] + \rho\left(\frac{\partial E}{\partial t} + \vec{u} \cdot \nabla \mathrm{E}\right)$$

Again, the first term is zero with the conservation of mass, thus:

$$\rho\left(\frac{\partial E}{\partial t}+\vec{u}\cdot\nabla\mathrm{E}\right)=RHS$$

$$\rho\frac{DE}{Dt}=RHS$$

$$\rho\frac{D}{Dt}\left(e+\frac{u^2}{2}\right)=RHS$$