Problem 2.4 10 points:

Derive Stokes' momentum equation from N-S equation by assuming the Re << 1. Specifically demonstrate why the **external force term** cannot be neglected while the **advection term** can.

Note: Stat with N-S equation in below:

$$\frac{\partial u}{\partial t} + (\underline{u} \cdot \nabla)\underline{u} = -\frac{1}{\rho}\nabla p + \upsilon \nabla^2 \underline{u} + f$$

Solution:

Introduce dimensionless variables

(see Section 1.4 in textbook of Prosperetti and Tryggavason for more details)

$$\nabla^2 u = \frac{U}{L^2}\nabla_*^2 u_*$$

$$\frac{1}{Sl}\frac{\partial u_*}{\partial t_*} + (u_* \cdot \nabla_*)u_* = -\frac{\Delta p}{\rho U^2}\nabla_* p_* + \frac{1}{Re}\nabla_*^2 u_* + \frac{fL}{U^2}f_*$$

$Sl = \frac{U_\tau}{L}$ ---- Strouhal number

$Re = \frac{LU}{\upsilon}$ ---- Reynolds number

When the flow is dominated by viscosity, the proper pressure scale is $\Delta p = \frac{\mu U}{L}$

$$\frac{\Delta p}{\rho U^2} = \frac{\mu U}{L} \cdot \frac{1}{\rho U^2} = \frac{1}{Re}$$

$$\Longrightarrow \frac{1}{Sl}\frac{\partial u_*}{\partial t_*} + (u_* \cdot \nabla_*)u_* = -\frac{1}{Re}\nabla_* p_* + \frac{1}{Re}\nabla_*^2 u_* + \frac{fL}{U^2}f_*$$

Multiply Re on both sides $\Longrightarrow \frac{Re}{Sl}\frac{\partial u_*}{\partial t_*} + Re \cdot (u_* \cdot \nabla_*)u_* = -\nabla_* p_* + \nabla_*^2 u_* + \frac{fL^2}{\upsilon U}f_*$

Since Re<<1, $\frac{\mathrm{Re}}{\mathrm{Sl}} \ll 1$ as well, thus time derivation term and advection term on LHS could be both ignored. $\Longrightarrow$ LHS=0

For external force term, since U is extremely small, $\frac{fL^2}{\upsilon U}$ could not be neglected.

LHS=0 $\Longrightarrow 0 = -\nabla \mathrm{p} + \mu \nabla^2 \mathrm{u} + \rho f$ which is the Stokes' momentum equation