Problem 7.1 10 points:

Consider a steady-state, fully developed channel flow. At the wall (y=0), the velocity is zero and pressure is p_w. Show that mean axial pressure gradient is uniform across the flow: $\frac{\partial\langle p\rangle}{\partial x} = \frac{dp_w}{dx}$.

Hint: write the lateral mean-momentum equation and integrate it in y-direction analytically.

Solution:

The lateral mean momentum equation is:

$$\frac{1}{\rho}\frac{\delta\langle p(x,y)\rangle}{\delta y} + \frac{\delta\langle v^2\rangle}{\delta y} = 0$$

Integrate:

$$\int_0^y \frac{1}{\rho}\frac{\delta\langle p(x,y)\rangle}{\delta y'}dy' + \int_0^y \frac{\delta\langle v^2\rangle}{\delta y'}dy' = 0$$

$$\frac{1}{\rho}(\langle p(x,y)\rangle - \langle p_{wall}(x)\rangle) + \langle v^2(y)\rangle - \langle v^2(0)\rangle = 0$$

The velocity at the wall (y=0) is zero.

$$\frac{1}{\rho}(\langle p(x,y)\rangle - \langle p_{wall}(x)\rangle) + \langle v^2(y)\rangle = 0$$

Taking the derivative with x,

$$\frac{1}{\rho}\left(\frac{\delta\langle p(x,y)\rangle}{\delta x} - \frac{\delta\langle p_{wall}(x)\rangle}{\delta x}\right) = 0$$

$$\Longrightarrow \frac{\delta\langle p\rangle}{\delta x} = \frac{\delta\langle p_{wall}\rangle}{\delta x}$$