Problem 7.2 20 points:

Derive the exact transport equation for turbulent kinetic energy (TKE), k (Eq. (10.35) in lecture notes).

Use the following steps:

a) Subtract the Reynolds equations (momentum written for mean velocities) from N.S. momentum equations, thus obtain the equation for fluctuating velocity;

b) Obtain a scalar product of fluctuating velocity and the vector-equation you got in part a) and apply Reynolds averaging to the result.

Solution:

(a)

N.S. momentum equations $\frac{DU}{Dt} = \frac{-1}{\rho}\nabla p + \nu\nabla^2 U$ (1)

Reynolds equations $\frac{\bar{D}<U_j>}{Dt} = \nu\nabla^2 < U_j > - \frac{\delta<u_i \cdot u_j>}{\delta x_i} - \frac{1}{\rho}\frac{\delta<p>}{\delta x_j}$ (2)

Subtract (2) from (1):

$$\frac{DU}{Dt} - \frac{\bar{D} < U_j >}{Dt} = \frac{-1}{\rho}\nabla p + \nu\nabla^2 U - \nu\nabla^2 < U_j > + \frac{\delta < u_i \cdot u_j >}{\delta x_i} + \frac{1}{\rho}\frac{\delta < p >}{\delta x_j}$$

Since

$u = U - < U_j >$ and $p' = p - < p >$

$$\frac{DU}{Dt} - \frac{\bar{D} < U_j >}{Dt} = \frac{-1}{\rho}\nabla p + \nu\nabla^2 u_j + \frac{\delta < u_i \cdot u_j >}{\delta x_i} + \frac{1}{\rho}\frac{\delta < p >}{\delta x_j}$$

$$\frac{DU}{Dt} - \frac{\bar{D} < U_j >}{Dt} = \nu\nabla^2 u_j + \frac{\delta < u_i \cdot u_j >}{\delta x_i} - \frac{1}{\rho}\left(\nabla p - \frac{\delta < p >}{\delta x_j}\right)$$

$$\frac{DU}{Dt} - \frac{\bar{D} < U_j >}{Dt} = \nu\nabla^2 u_j + \frac{\delta < u_i \cdot u_j >}{\delta x_i} - \frac{1}{\rho}\frac{\delta p'}{\delta x_j}$$

Since

$$\frac{DU_j}{Dt} = \frac{\delta U_j}{\delta t} + \frac{\delta (U_i \cdot U_j)}{\delta x_i}$$

$$< \frac{DU_j}{Dt} > = \frac{\delta < U_j >}{\delta t} + \frac{\delta < U_i \cdot U_j >}{\delta x_i} = \frac{\delta < U_j >}{\delta t} + < U_i > \frac{\delta < U_j >}{\delta x_i} + \frac{\delta < u_i \cdot u_j >}{\delta x_i}$$

$$\frac{\bar{D} < U_j >}{Dt} =< \frac{DU_j}{Dt} > - \frac{\delta < u_i \cdot u_j >}{\delta x_i}$$

$$\frac{DU_j}{Dt} - \left[< \frac{DU_j}{Dt} > - \frac{\delta < u_i \cdot u_j >}{\delta x_i} \right] = \nu \nabla^2 u_j + \frac{\delta < u_i \cdot u_j >}{\delta x_i} - \frac{1}{\rho} \frac{\delta p'}{\delta x_j}$$

$$\frac{DU_j}{Dt} - \left[\frac{\delta < U_j >}{\delta t} + \frac{\delta < U_i \cdot U_j >}{\delta x_i} - \frac{\delta < u_i \cdot u_j >}{\delta x_i} \right] = \nu \nabla^2 u_j + \frac{\delta < u_i \cdot u_j >}{\delta x_i} - \frac{1}{\rho} \frac{\delta p'}{\delta x_j}$$

$$\frac{DU_j}{Dt} - \frac{\delta < U_j >}{\delta t} - \frac{\delta < U_i \cdot U_j >}{\delta x_i} = \nu \nabla^2 u_j - \frac{1}{\rho} \frac{\delta p'}{\delta x_j}$$

$$\frac{\delta U_j}{\delta t} + \frac{\delta (U_i \cdot U_j)}{\delta x_i} - \frac{\delta < U_j >}{\delta t} - \frac{\delta < U_i \cdot U_j >}{\delta x_i} = \nu \nabla^2 u_j - \frac{1}{\rho} \frac{\delta p'}{\delta x_j}$$

Using $u = U - < U >$

$$\Longrightarrow \frac{\delta u_j}{\delta t} + \frac{\delta}{\delta x_i} (U_i \cdot U_j - < U_i \cdot U_j >) = \nu \nabla^2 u_j - \frac{1}{\rho} \frac{\delta p'}{\delta x_j}$$

Since

$$\begin{aligned} \frac{\delta}{\delta x_i} \left(U_i \cdot U_j - < U_i \cdot U_j > \right) & \\ &= \frac{\delta}{\delta x_i} \left(u_i < U_j > + u_j < U_i > + u_i \cdot u_j - < u_i \cdot u_j > \right) \\ &= u_i \frac{\delta}{\delta x_i} \left(< U_j > \right) + < U_i > \frac{\delta}{\delta x_i} u_j + u_i \frac{\delta}{\delta x_i} u_j - \frac{\delta}{\delta x_i} (< u_i \cdot u_j >) \end{aligned}$$

$$\Longrightarrow \frac{\delta u_j}{\delta t} + u_i \frac{\delta}{\delta x_i} \left(< U_j > \right) + < U_i > \frac{\delta}{\delta x_i} u_j + u_i \frac{\delta}{\delta x_i} u_j - \frac{\delta}{\delta x_i} \left(< u_i \cdot u_j > \right) = \nu \nabla^2 u_j - \frac{1}{\rho} \frac{\delta p'}{\delta x_j}$$

$$\frac{\delta u_j}{\delta t} + < U_i > \frac{\delta}{\delta x_i} u_j + u_i \frac{\delta}{\delta x_i} u_j = -u_i \frac{\delta}{\delta x_i} \left(< U_j > \right) + \frac{\delta}{\delta x_i} \left(< u_i \cdot u_j > \right) + \nu \nabla^2 u_j - \frac{1}{\rho} \frac{\delta p'}{\delta x_j}$$

$$\frac{\delta u_j}{\delta t} + (< U_i > + u_i) \frac{\delta}{\delta x_i} u_j = -u_i \frac{\delta}{\delta x_i} \left(< U_j > \right) + \frac{\delta}{\delta x_i} \left(< u_i \cdot u_j > \right) + \nu \nabla^2 u_j - \frac{1}{\rho} \frac{\delta p'}{\delta x_j}$$

Since

$$\frac{Du_j}{Dt} = \frac{\delta u_j}{\delta t} + (< U_i > + u_i) \frac{\delta}{\delta x_i} u_j$$

$$\Longrightarrow \frac{Du_j}{Dt} == -u_i \frac{\delta}{\delta x_i} \left(< U_j > \right) + \frac{\delta}{\delta x_i} \left(< u_i \cdot u_j > \right) + \nu \nabla^2 u_j - \frac{1}{\rho} \frac{\delta p'}{\delta x_j}$$

(b)

Multiplying by u_j to the result of part a

$$u_j \cdot \frac{\delta u_j}{\delta t} + (< U_i > + u_i) \cdot u_j \cdot \frac{\delta}{\delta x_i} u_j = -u_i \cdot u_j \cdot \frac{\delta}{\delta x_i}(< U_j >) + u_j \cdot \frac{\delta}{\delta x_i}(< u_i \cdot u_j >) + \nu \nabla^2 u_j u_j - u_j \cdot \frac{1}{\rho} \frac{\delta p'}{\delta x_j}$$

Since

$$u_j \cdot \frac{\delta u_j}{\delta t} = \frac{\delta {u_j}^2}{\delta t} = 2 \cdot u_j \cdot \frac{\delta u_j}{\delta t}$$

$$\Longrightarrow \frac{1}{2} \cdot \frac{\delta {u_j}^2}{\delta t} = u_j \cdot \frac{\delta u_j}{\delta t} = \frac{\delta}{\delta t} \cdot (\frac{1}{2} u_j \cdot u_j)$$

$$\frac{\delta}{\delta t} \cdot \left(\frac{1}{2} u_j \cdot u_j\right) + < U_i > \frac{\delta}{\delta x_i}\left(\frac{1}{2} u_j \cdot u_j\right) + \frac{\delta}{\delta x_i}\left(\frac{1}{2} u_i \cdot u_j \cdot u_j\right)$$
$$= -u_i \cdot u_j \cdot \frac{\delta}{\delta x_i}(< U_j >) + u_j \cdot \frac{\delta}{\delta x_i}(< u_i \cdot u_j >) + \nu \nabla^2 u_j u_j - u_j \cdot \frac{1}{\rho} \frac{\delta p'}{\delta x_j}$$

Taking the average and let $< u_i > = 0$

$$k = < \frac{1}{2} u_j \cdot u_j >$$

$$\frac{\delta k}{\delta t} + < U_i > \frac{\delta k}{\delta x_i} + \frac{\delta}{\delta x_i} < \frac{1}{2} u_i \cdot u_j \cdot u_j >$$
$$= -< u_i \cdot u_j > \frac{\delta}{\delta x_i} < U_j > + \nu < u_j \nabla^2 u_j > - \frac{1}{\rho} \frac{\delta < u_j \cdot p' >}{\delta x_j}$$

Since

$$\nu < u_j \nabla^2 u_j > = 2 \cdot \nu \cdot \frac{\delta}{\delta x_i} < u_j \cdot s_{ij} > - \epsilon$$

$$T_i = < \frac{1}{2} u_i \cdot u_j \cdot u_j > - 2 \cdot \nu \cdot < u_j \cdot s_{ij} > + \frac{< u_j \cdot p' >}{\rho}$$

$$\Longrightarrow \frac{\delta k}{\delta t} + < U_i > \frac{\delta k}{\delta x_i} + \frac{\delta}{\delta x_i} < \frac{1}{2} u_i \cdot u_j \cdot u_j >$$
$$= -< u_i \cdot u_j > \frac{\delta}{\delta x_i} < U_j > + 2 \cdot \nu \cdot \frac{\delta}{\delta x_i} < u_j \cdot s_{ij} > - \epsilon - \frac{1}{\rho} \frac{\delta < u_j \cdot p' >}{\delta x_j}$$

$$\Longrightarrow \frac{\delta k}{\delta t} + < U_i > \frac{\delta k}{\delta x_i} + \frac{\delta}{\delta x_i}\left(< \frac{1}{2} u_i \cdot u_j \cdot u_j > - 2\nu < u_j \cdot s_{ij} > + \frac{< u_j \cdot p' >}{\rho}\right)$$
$$= -< u_i \cdot u_j > \frac{\delta}{\delta x_i} < U_j > - \epsilon$$

Fluctuating flux of energy: $T_i' =< \frac{1}{2} u_i \cdot u_j \cdot u_j > -2\nu < u_j \cdot s_{ij} > + \frac{<u_j \cdot p'>}{\rho}$

Production term: $P = -< u_i \cdot u_j > \frac{\delta}{\delta x_i} < U_j >$

$\Rightarrow \frac{\overline{D}k}{Dt} + \nabla \cdot T_i{}' = P - \varepsilon$