Problem 1.4 10 points:

Consider incompressible N.S. equations in Cartesian coordinates. Derive Poisson's equation for pressure by taking the divergence of the momentum equation and then **applying continuity equation** to obtain:

$$\nabla^2 p = -\rho\left[\left(u_i u_j\right)_{,j}\right]_{,i}$$

Note:

The momentum equation for incompressible flow:

$$\frac{\partial u_j}{\partial t} + \frac{\partial\left(u_i u_j\right)}{\partial x_i} = -\frac{1}{\rho}\frac{\partial p}{\partial x_j} + \nu\frac{\partial^2 u_j}{\partial x_i \partial x_i} + \frac{1}{\rho} f_j$$

Solution:

Continuity equation:

$$\frac{\partial u_j}{\partial x_j} = 0$$

Given:

$$\frac{\partial u_j}{\partial t} + \frac{\partial\left(u_i u_j\right)}{\partial x_i} = -\frac{1}{\rho}\frac{\partial p}{\partial x_j} + \nu\frac{\partial^2 u_j}{\partial x_i \partial x_i} + \frac{1}{\rho} f_j$$

Applying divergence on the momentum equation yields:

$$\frac{\partial}{\partial t}\left(\frac{\partial u_j}{\partial x_j}\right) + \frac{\partial}{\partial x_i}\left[\frac{\partial u_i u_j}{\partial x_j}\right] = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x_j^2} + \nu\frac{\partial}{\partial x_j}\frac{\partial^2 u_j}{\partial x_i^2} + \frac{1}{\rho} f_j$$

With the incompressibility and zero body force, the above can be reduced into:

$$\frac{\partial}{\partial x_i}\left[\frac{\partial u_i u_j}{\partial x_j}\right] = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x_j^2}$$

Rearrange and use Einstein's notation:

$$\frac{\partial^2 p}{\partial x_j^2} = \nabla^2 p = -\rho\left[\left(u_i u_j\right)_{,j}\right]_{,i}$$