Homework #7 (3 problems; 40 points)

due: 11:45am on April 17th Friday 2020.

Problem 7.1 [10]	Problem 7.2 [20]	Problem 7.3 [10]	Total [40]

Problem 7.1 10 points:

Consider a steady-state, fully developed channel flow. At the wall (y=0), the velocity is zero and pressure is p_w. Show that mean axial pressure gradient is uniform across the flow: $\frac{\partial\langle p\rangle}{\partial x} = \frac{dp_w}{dx}$.

Hint: write the lateral mean-momentum equation and integrate it in y-direction analytically.

Problem 7.2 20 points:

Derive the exact transport equation for turbulent kinetic energy (TKE), k (Eq. (10.35) in lecture notes).

Use the following steps:

a) Subtract the Reynolds equations (momentum written for mean velocities) from N.S. momentum equations, thus obtain the equation for fluctuating velocity;

b) Obtain a scalar product of fluctuating velocity and the vector-equation you got in part a) and apply Reynolds averaging to the result.

Problem 7.3 10 points:

Show that the transport equation for turbulence dissipation rate (ε, Eq. (10.53) in notes) can be obtained from the equation for TKE (Eq. (10.41)).

Note: the *P* in Eq. (10.41) is not pressure, but the production term in Eq. (10.35)

NE577 Hwk 7.

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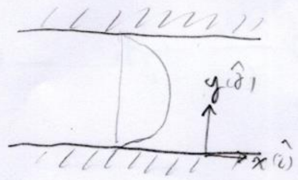
#7-1

S.S. fully-dev. flow. 2D.



given

mean momentum equation on y direction

$$\cancel{\frac{\partial U_y}{\partial t}} + \cancel{(\vec{U}\cdot\nabla)U_y} = -\frac{1}{\rho}\frac{\partial \langle P(x,y)\rangle}{\partial y} + \cancel{\nu\nabla^2 U_y} + \cancel{\frac{\partial\langle u_i' u_j'\rangle}{\partial x}}$$, given velocity $=0$ at $y=0$.

$$\therefore \frac{\partial\langle P(x,y)\rangle}{\partial y} = 0 \rightarrow \langle P(x,y)\rangle \text{ is constant on } y \text{ direction, or } \langle P(x,y)\rangle = \langle P(x)\rangle = \langle P(x)\rangle\Big|_{y=0} = P_w$$

$$\therefore \underline{\frac{\partial\langle P\rangle}{\partial x} = \frac{\partial P_w}{\partial x}}$$

7.2

N-S momentum equ:

$$\frac{\partial u_j}{\partial t} + \frac{\partial (u_i u_j)}{\partial x_i} = -\frac{1}{\rho}\frac{\partial P}{\partial x_j} + \nu \frac{\partial^2 u_j}{\partial x_i \partial x_i} + \frac{1}{\rho} f_i \quad — ①$$

Reynolds equ.

$$\frac{\partial U_j}{\partial t} + \frac{\partial (U_i U_j)}{\partial x_i} = -\frac{1}{\rho}\frac{\partial \langle P\rangle}{\partial x_j} + \nu \frac{\partial^2 U_j}{\partial x_i \partial x_i} - \frac{\partial \langle u_i' u_j'\rangle}{\partial x_i} + \frac{1}{\rho} f_i$$

① - ②

$$\left(\frac{\partial u_j}{\partial t} - \frac{\partial U_j}{\partial t}\right) + \left[\frac{\partial (u_i u_j)}{\partial x_i} - \frac{\partial (U_i U_j)}{\partial x_i}\right] = -\frac{1}{\rho}\left(\frac{\partial P}{\partial x_j} - \frac{\partial \langle P\rangle}{\partial x_j}\right) + \nu\left(\frac{\partial^2 u_j}{\partial x_i \partial x_i} - \frac{\partial^2 U_j}{\partial x_i \partial x_i}\right) + \frac{\partial \langle u_i' u_j'\rangle}{\partial x_i}$$

by definition: $u_j = U_j + u_j'$ where U_j is the averaged velocity, u_j' is fluctuation.

$$\frac{\partial u_j'}{\partial t} + \left[\frac{\partial (U_i + u_i')(U_j + u_j')}{\partial x_i} - \frac{\partial U_i U_j}{\partial x_i}\right] = -\frac{1}{\rho}\frac{\partial P'}{\partial x_j} + \nu \frac{\partial^2 u_j'}{\partial x_i \partial x_i} + \frac{\partial \langle u_i' u_j'\rangle}{\partial x_i}$$

$$= \frac{\partial u_j'}{\partial t} + \cancel{\frac{\partial U_i U_j}{\partial x_i}} + \frac{\partial U_i u_j'}{\partial x_i} + \frac{\partial u_i' U_j}{\partial x_i} + \frac{\partial u_i' u_j'}{\partial x_i} - \cancel{\frac{\partial U_i U_j}{\partial x_i}} = -\frac{1}{\rho}\frac{\partial P'}{\partial x_j} + \nu \frac{\partial^2 u_j'}{\partial x_i^2} + \frac{\partial \langle u_i' u_j'\rangle}{\partial x_i}$$

take dot product btw this equation and fluctuation on j. u_j'

$$u_j'\frac{\partial u_j'}{\partial t} + u_j'\frac{\partial U_i u_j'}{\partial x_i} + u_j'\frac{\partial u_i' U_j}{\partial x_i} + u_j'\frac{\partial u_i' u_j'}{\partial x_i} = -\frac{1}{\rho}u_j'\frac{\partial P'}{\partial x_j} + u_j'\nu\frac{\partial^2 u_j'}{\partial x_i^2} + u_j'\frac{\partial \langle u_i' u_j'\rangle}{\partial x_i}$$

take Reynolds averaging on every term:

$$\left\langle u_j'\frac{\partial u_j'}{\partial t}\right\rangle + \left\langle u_j'\frac{\partial U_i u_j'}{\partial x_i}\right\rangle + \left\langle u_j'\frac{\partial u_i' U_j}{\partial x_i}\right\rangle + \left\langle u_j'\frac{\partial u_i' u_j'}{\partial x_i}\right\rangle = -\frac{1}{\rho}\left\langle u_j'\frac{\partial P'}{\partial x_j}\right\rangle + \nu\left\langle u_j'\frac{\partial^2 u_j'}{\partial x_i^2}\right\rangle + \left\langle u_j'\frac{\partial \langle u_i' u_j'\rangle}{\partial x_i}\right\rangle$$

using identity: $\left\langle \phi\frac{\partial \phi}{\partial t}\right\rangle = \frac{1}{2}\left\langle\frac{\partial \phi^2}{\partial t}\right\rangle = \frac{1}{2}\frac{\partial}{\partial t}\langle \phi^2\rangle$, likewise to $\frac{\partial}{\partial x_i}$ terms.

$$\langle\langle \phi\rangle \gamma\rangle = \langle\phi\rangle\langle\gamma\rangle$$

$$\langle \phi + \gamma\rangle = \langle\phi\rangle + \langle\gamma\rangle$$

$$\langle\langle \phi\rangle \gamma\rangle = \langle\phi\rangle\langle\gamma\rangle$$

$$\langle u_j' \frac{\partial u_j'}{\partial t} \rangle = \frac{\partial}{\partial t} \frac{1}{2} \langle u_j'^2 \rangle = \frac{\partial}{\partial t} k$$

$$\langle u_j' \frac{\partial U_i u_j'}{\partial x_i} \rangle = U_i \frac{\partial}{\partial x_i} \frac{1}{2} \langle u_j'^2 \rangle = U_i \frac{\partial}{\partial x_i} k$$

$$\langle u_j' \frac{\partial u_i' U_j}{\partial x_i} \rangle = U_j \frac{\partial}{\partial x_i} \langle u_i' u_j' \rangle$$

$$\langle u_j' \frac{\partial u_i' u_j'}{\partial x_i} \rangle = \frac{\partial}{\partial x_i} \frac{1}{2} \langle u_i' u_j'^2 \rangle.$$

$$\left\langle -\frac{1}{\rho} u_j' \frac{\partial p'}{\partial x_j} \right\rangle = -\frac{1}{\rho} \frac{\partial}{\partial x_j} \langle p' u_j' \rangle$$

$$\left\langle u_j' \frac{\partial \langle u_i' u_j' \rangle}{\partial x_i} \right\rangle = \langle u_j' \rangle \frac{\partial \langle u_i' u_j' \rangle}{\partial x_i} = 0 \quad \text{since } \langle u_j' \rangle = 0.$$

$\nu \langle u_j' \frac{\partial^2 u_j'}{\partial x_i \partial x_i} \rangle$ by chain rule of $\nu \left\langle \frac{\partial}{\partial x_i} \left(u_j' \frac{\partial u_j'}{\partial x_i} \right) \right\rangle = \nu \left\langle \frac{\partial}{\partial x_i} (u_j' s_{ij}) \right\rangle$ by definition.

$$= \nu \left\langle s_{ij} s_{ij}' + u_j' \frac{\partial s_{ij}}{\partial x_i} \right\rangle = \nu \langle s_{ij}' s_{ij}' \rangle + \nu \left\langle u_j' \frac{\partial s_{ij}}{\partial x_i} \right\rangle$$

$$\therefore \nu \left\langle u_j' \frac{\partial^2 u_j}{\partial x_i \partial x_i} \right\rangle = \nu \frac{\partial}{\partial x_i} \langle u_j' s_{ij} \rangle - \nu \langle s_{ij}' s_{ij}' \rangle$$

$\therefore$ the equation becomes:

$$\frac{\partial k}{\partial t} + U_i \frac{\partial}{\partial x_i} k = \underbrace{- \frac{\partial}{\partial x_i} \frac{1}{2} \langle u_i' u_j'^2 \rangle - \frac{\partial}{\partial x_i} \frac{\langle u_j' p' \rangle}{\rho} + \nu \frac{\partial}{\partial x_i} \langle u_j' s_{ij}' \rangle}_{-\vec{T}'} \underbrace{- U_j \frac{\partial}{\partial x_i} \langle u_i' u_j' \rangle}_{\mathbb{P}} \underbrace{- \nu \langle s_{ij}' s_{ij}' \rangle}_{\varepsilon}$$

by definition

$$\frac{\bar{D}}{\bar{D}t} = \frac{\partial}{\partial t}(\,) + (\vec{U} \cdot \nabla)(\,) \quad ,$$

$$\vec{T}' = \frac{1}{2} \langle u_i' u_j'^2 \rangle + \frac{\langle u_j' p' \rangle}{\rho} - \nu \langle u_j' s_{ij}' \rangle$$

$$\mathbb{P} = - \langle u_i' u_j' \rangle \frac{\partial U_i}{\partial x_j}$$

$$\varepsilon = \nu \langle s_{ij}' s_{ij}' \rangle$$

$$\nabla \cdot = \frac{\partial}{\partial x_i}$$

(div)

$$\frac{\bar{D}k}{\bar{D}t} = -\nabla \cdot \vec{T}' + \mathbb{P} - \varepsilon$$

7.3

eqn (10.41): $\frac{\bar{D}k}{\bar{D}t} = \nabla\cdot\left(\frac{\nu_t}{\sigma_k}\nabla k\right) + P - \varepsilon$, $\begin{cases} P = -\langle u_i' u_j' \rangle \frac{\partial U_i}{\partial x_j} \\ \varepsilon = \nu \langle \frac{\partial^2 u_j'^2}{\partial x_i^2} \rangle \end{cases}$

k has dimension $[L^2 T^{-2}]$, ε is the dissipation rate of k, $[L^2 T^{-3}]$

ε and k are related by: $\varepsilon = \frac{C_D k^{\frac{3}{2}}}{l_m}$, $\frac{\varepsilon}{k} = \frac{C_D k^{\frac{1}{2}}}{l_m}$ $\left[\frac{1}{T}\right]$ to get eqn for ε. multiply 10.41 by $\frac{\varepsilon}{k} = \frac{C_D k^{\frac{1}{2}}}{l_m}$

$$\frac{C_D k^{\frac{1}{2}}}{l_m}\frac{\bar{D}k}{\bar{D}t} = \nabla\cdot\left(\frac{\nu_t}{\sigma_k}\nabla k\right)\cdot\left(\frac{C_D k^{\frac{1}{2}}}{l_m}\right) + \frac{P\varepsilon}{k} - \frac{\varepsilon^2}{k} \to \left[\frac{L^2}{T^4}\right]$$ rearrange

assume $\frac{C_D k^{\frac{1}{2}}}{l_m}\frac{\bar{D}k}{\bar{D}t} = \frac{\bar{D}}{\bar{D}t}\left(\frac{C_D k^{\frac{3}{2}}}{l_m}\right) = \frac{\bar{D}}{\bar{D}t}\varepsilon$

$$\nabla\cdot\left(\frac{\nu_t}{\sigma_k}\nabla k\right)\left(\frac{C_D k^{\frac{1}{2}}}{l_m}\right) = \nabla\cdot\left(\frac{\nu_t}{\frac{\sigma_k l_m}{C_D}}\nabla\left(\frac{C_D k^{\frac{3}{2}}}{l_m}\right)\right) = \left(\nabla\cdot\left(\frac{\nu_t}{\sigma_\varepsilon}\nabla\varepsilon\right)\right)$$

$\to \frac{\bar{D}\varepsilon}{\bar{D}t} = \nabla\cdot\left(\frac{\nu_t}{\sigma_\varepsilon}\nabla\varepsilon\right) + \frac{P\varepsilon}{k} - \frac{\varepsilon^2}{k}$ which has the form of (10.53)