Homework #3 (4 problems, 75 points)

Due: February 26th Wednesday, 2020.

Problem 3.1 [15]	Problem 3.2 [10]	Problem 3.3 [20]	Problem 3.4 [30]	Total [75]

Problem 3.1 15 points:

Estimate the number of bubbles which can be simulated on a fastest supercomputer in the world (see www.top500.org).

Assume the following parameters:

1) Each computing core can process:

   (a) 2,048 nodal points; (b) 4,096 nodal points; (c) 8,192 nodal points;

   (d) Assume that to compute every 1,000,000 nodal points, about 3.25 TFLOPS of "Rmax" power is required as listed in the Top500.org. Note that 1PFLOPS (peta-**fl**oating-point **o**perations **p**er **s**econd) = 1,000 TFLOPS (tera-) = 1,000,000 GFLOPS (giga-) = 10^{15} FLOPS.

   Note: Rmax in www.top500.org corresponds to the total FLOPS for all the cores.

2) Mesh resolution for each bubble is (a) 10 (b) 30 (c) 50 nodal points across the diameter;

3) The computational domain is a cube with uniform grid resolution in each direction, and domain size is dictated by available computing power;

4) The bubble concentration is (a) 1% (b) 10% and (c) 20%.

5) Only: (a) 10% (b) 30% and (c) 60% (d) 100% of computing cores of the fastest machine are available for the computation.

(i) Provide the **expression** you used to estimate the number of bubbles which can be simulated by the fastest super computer. Use the notations listed in the following table:

#	number of bubbles
nd	nodal points along the diameter of each bubble
Nc	number of cores of the supercomputer
n0	nodal points each computing core can process
vf	bubble concentration
av	availability of computing cores

Note 1: This evaluation is based on a 3D simulation. Apply the sphere volume expression for computing the nodal points for each bubble.

Note 2: the nodal points of the computational domain are the total nodal points that all the available cores could process.

(ii)Compute the nodal points of 1(d), and then evaluate the number of bubbles for the following combinations:

Combination 1: 1(a, b, c, d) – 2(a) – 4(b) – 5(a);

Combination 2: 1(a) – 2(a, b, c) – 4(a) – 5(a);

Combination 3: 1(a) – 2(a) – 4(a, b, c) – 5(a);

Combination 4: 1(a) – 2(a) – 4(b) – 5(a, b, c, d);

(iii) Discuss the results and think of real multiphase flow systems (and their size) which can be simulated today using DNS approach. How this is affected by the choice of the parameters in 1)-5)?

Extra credit (5 points): compute the number of bubbles in combination 5: 1(d) – 2(c) – 4(a) – 5(d). Then repeat the calculation for the fastest machine in top500 in November 2009. Based on the cores and Rmax in both 2009 and 2019, predict 1(d) and the number of bubbles of combination 5 in 2029. Assume that the rate of growth in each decade is exponential, i.e. if the 2019 machine was 10 times faster than 2009, we assume that 2029 machine is 10 times faster than 2019.

Problem 3.2 10 points:

Follow the class notes and **draw** a detailed schematics of control volumes for using staggered grid approach (when pressure, x-velocity (u) and y-velocity (v) are computed at different locations). And derive the approximation of the **y-component** of the advection term $\left(A_y\right)_{i,j+\frac{1}{2}}^{n}$.

Make sure all the steps are explained.

Note 1: the vector form of the advection term: $\underline{A}\left(\underline{u}^n\right) = \frac{1}{\Delta V} \oint_S \underline{u}^n \left(\underline{u}^n \cdot \underline{n}\right) dS$

Note 2: **mark** the left, right, top, bottom **faces** of the control volume $v_{i,j+1/2}$ and the corresponding **normal vectors** of those faces.

Problem 3.3 20 points:

Assume a 2D grid which is not isotropic, i.e. the resolution in the x direction is h_x and the resolution in the y direction is h_y.

a) Derive the **incompressibility condition** for pressure control volume (Eq. (2.17) in the notes/textbook). Then show that your result is consistent with notes if assume $h_x = h_y = h$.

$$u^{n+1}_{i+1/2,j} - u^{n+1}_{i-1/2,j} + v^{n+1}_{i,j+1/2} - v^{n+1}_{i,j-1/2} = 0 \tag{2.17}$$

b) Derive the **predictor step** (2.18), **projection step** (2.19) and the **pressure equation** (2.20). Then show that your result is consistent with notes if assume $h_x = h_y = h$.

$$\begin{aligned} u^{*}_{i+1/2,j} &= u^{n}_{i+1/2,j} + \Delta t\Big(-(A_x)^n_{i+1/2,j} + \nu(D_x)^n_{i+1/2,j} + (f_x)_{i+1/2,j}\Big) \\ v^{*}_{i,j+1/2} &= v^{n}_{i,j+1/2} + \Delta t(-(A_y)^n_{i,j+1/2} + \nu(D_y)^n_{i,j+1/2} + (f_y)_{i,j+1/2}\Big) \end{aligned} \tag{2.18}$$

$$\begin{aligned} u^{n+1}_{i+1/2,j} &= u^{*}_{i+1/2,j} - \frac{1}{\rho}\frac{\Delta t}{h}(p^{n+1}_{i+1,j} - p^{n+1}_{i,j}) \\ v^{n+1}_{i,j+1/2} &= v^{*}_{i,j+1/2} - \frac{1}{\rho}\frac{\Delta t}{h}(p^{n+1}_{i,j+1} - p^{n+1}_{i,j}) \end{aligned} \tag{2.19}$$

$$\begin{aligned} &\frac{p^{n+1}_{i+1,j} + p^{n+1}_{i-1,j} + p^{n+1}_{i,j+1} + p^{n+1}_{i,j-1} - 4p^{n+1}_{i,j}}{h^2} \\ &\qquad = \frac{\rho}{\Delta t}\left(\frac{u^{*}_{i+1/2,j} - u^{*}_{i-1/2,j} + v^{*}_{i,j+1/2} - v^{*}_{i,j-1/2}}{h}\right). \end{aligned} \tag{2.20}$$

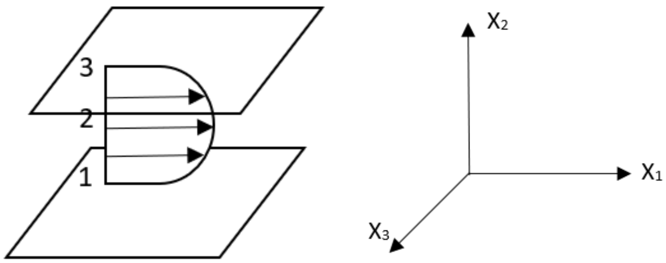
Note: when derive pressure equation, do not forget $u^{n+1}_{i-1/2,j}$ and $v^{n+1}_{i,j-1/2}$ terms.

c) Derive the **advection term** in the x-direction for the non-isotropic grid. Then show that your result is consistent with notes if assume $h_x = h_y = h$.

$$\begin{aligned} (A_x)^n_{i+1/2,j} = \frac{1}{h}\Bigg\{ &\left(\frac{u^n_{i+3/2,j} + u^n_{i+1/2,j}}{2}\right)^2 - \left(\frac{u^n_{i+1/2,j} + u^n_{i-1/2,j}}{2}\right)^2 \\ &+ \left(\frac{u^n_{i+1/2,j+1} + u^n_{i+1/2,j}}{2}\right)\left(\frac{v^n_{i+1,j+1/2} + v^n_{i,j+1/2}}{2}\right) \\ &- \left(\frac{u^n_{i+1/2,j} + u^n_{i+1/2,j-1}}{2}\right)\left(\frac{v^n_{i+1,j-1/2} + v^n_{i,j-1/2}}{2}\right)\Bigg\} \end{aligned}$$

Problem 3.4 30 points:

Consider a laminar flow in a channel (between two parallel plates) depicted in below. The streamwise direction is x or x_1 axis, and the centerline between the two plates is y=0 or x_2=0.



The distance between the plates is 0.01m. Use the coordinates for the 3 points in the figure:

Point 1: x_2= -0.005 m;

Point 2: x_2= 0 m;

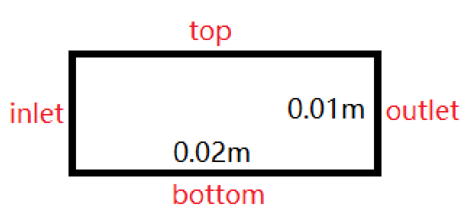
Point 3: x_2= 0.005 m.

Assume the maximum (peak) velocity is 0.03 m/s and perform the following analysis:

a) Solve the incompressible N.S. equation $\rho \frac{Du_i}{Dt} = -p_{,i} + \mu \nabla^2 u_i$ for this problem analytically at steady state with the body force ignored, given kinematic viscosity $\nu = 10^{-6}\ m^2/s$ and density $\rho = 10^3 kg/m^3$. Compute the **pressure gradient $p_{,1}$** and **velocity profile $u_1(x_2)$**, and then compute the average velocity $\overline{u_1(x_2)}$.

Note: pay attention to the units.

b) Develop a staggered grid for this problem using the domain **length** of **0.02 mm**. Assume that grid is 2D, isotropic. Boundaries are denoted by inlet, outlet, top, and bottom.



Boundary conditions of this fluid domain are:

inlet	outlet	top	bottom
Periodical		Wall	Wall

c) Using a programming tool (Fortran is encouraged, but not required), create a simple N.S. solver for the problem under consideration. Submit your code.

Notes:

- Do NOT solve for pressure, simply use the pressure distribution obtained from the pressure gradient in a).
- Assume that <u>normal to the wall velocity is zero</u> and do NOT solve for it. Only solve for the x-velocity component.
- Assign Initial condition to be a uniform velocity profile equal to the average velocity $\overline{u_1(x_2)}$ from the analytic solution.

d) Perform iteration until convergence/steady state (no further change in velocity is observed) on 3 different grid resolutions: 30, 50 and 70 pressure cells across the channel. Plot and compare the results with the analytical solution obtained in part a). Then use the normalized L^2 norm in below to estimate the solution error for each grid resolution:

- The normalized L^2 norm to estimate the solution error (based on the error of the numerical solution relative to the analytic solution at each cell):

$$L^2 = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(u_{numerical} - u_{exact})^2}$$

e) Do you expect the steady state solution to change with initial conditions? To test it, initialize your velocity field to be 0m/s, peak velocity 0.03 m/s, and $\overline{u_1(x_2)}$. Compare steady state solutions using 30 pressure cells across the channel. Discuss the result.

<u>Note:</u> only initial condition is changed, and boundary conditions should remain the same.