1

Level set re-initialization

$\phi = 0$ is accurately advected using this approach, however:

- distance function will not remain a distance function.

$|\nabla \phi|$ remains as close to 1 as possible, especially near the interface.

The process as follows:

a) $\phi_d \equiv \phi$

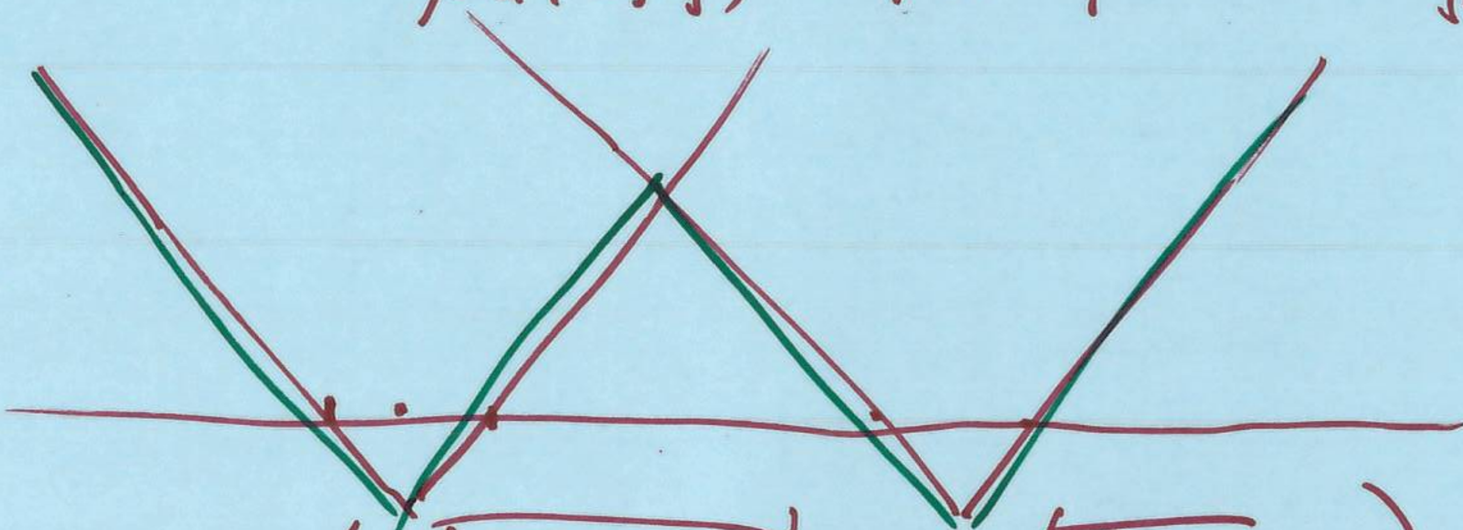
b) $\phi_d : \quad |\underline{\nabla} \phi_d| = 1 \qquad (3.47)$

for values (x, y) within M cells of the zero level set, $|\phi_d| < Mh$

(2) A level-set function that satisfies (3.47) is called a distance function, because $\phi_d(x,y)$ is the signed normal distance to zero l.s. value of ϕ_d.

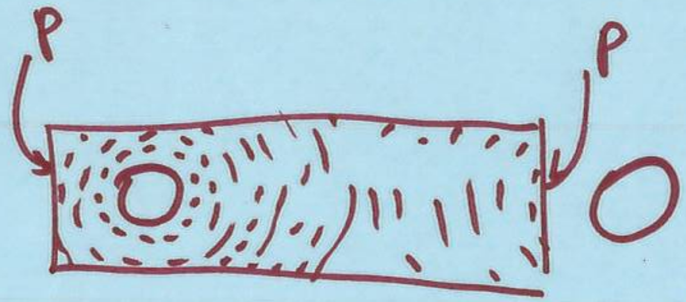
E.g.: $\phi(x,y) = x^2+y^2-1$ ← not a dist. func.

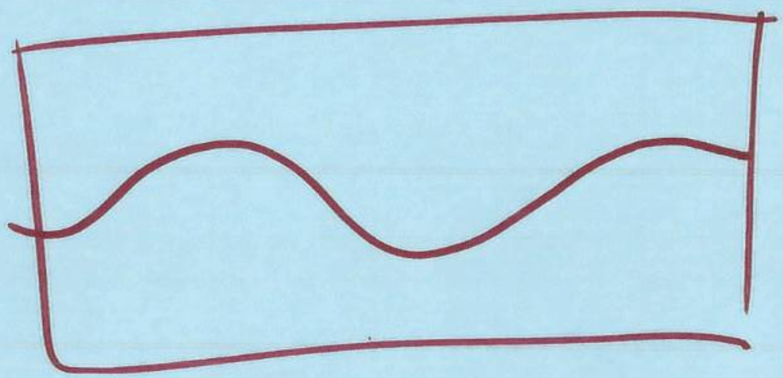
$\phi_d(x,y) = \sqrt{x^2+y^2}-1$ ← is a dist. func.

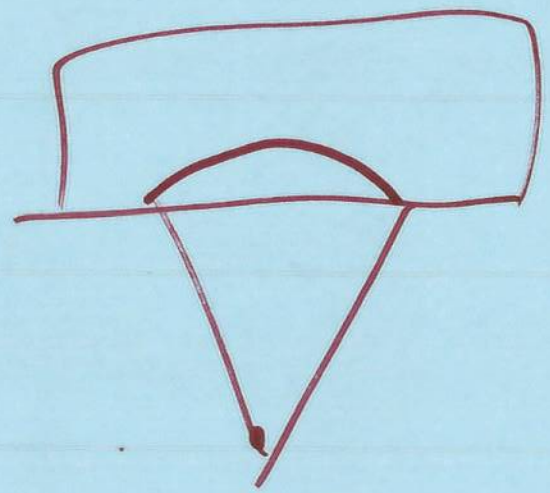


$$\min\left(\sqrt{(x-5)^2+y^2}-1,\ \sqrt{x^2+y^2}-1\right)$$

3 Sussman, Smereka & Osher (1994) introduced an iterative approach to re-initialize a level set.







④

Pros: if ϕ is close to a valid ϕ_d, only a few iterations would be needed.

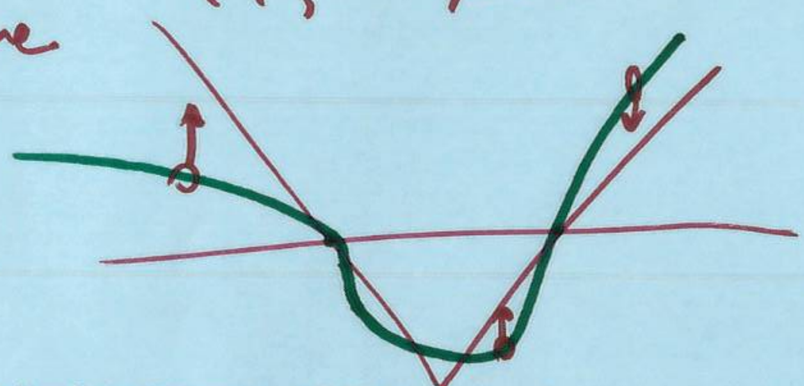
Re-initilization step:

$$\frac{\partial \phi_d}{\partial \tau} = \operatorname{sgn}(\phi)\left(1 - |\underline{\nabla} \phi_d|\right), \qquad (3.48)$$

with i.c.: $\phi_d(\underline{x}, 0) = \phi(\underline{x})$

$$\operatorname{sgn}(\phi) = \begin{cases} -1, & \phi < 0 \\ 0, & \phi = 0 \\ 1, & \phi > 0 \end{cases}$$

τ is the artificial time



5 Nice feature: near front re-init. occurs first.

Re-write (3.48):

$$\frac{\partial \varphi_d}{\partial t} + \underline{W} \cdot \underline{\nabla} \varphi_d = \operatorname{sgn}(\varphi_d) \qquad (3.49)$$

where $\underline{W} = \operatorname{sgn}(\phi) \dfrac{\nabla \varphi_d}{|\nabla \varphi_d|}$.

(3.49) is non-linear hyperbolic equation with $\underline{W}$ pointing outwards from the interface.

$\Rightarrow \varphi_d$ will be re-init. to $|\nabla \phi_d| = 1$ @ interface first.

Not necessary to solve (3.49) to s.s.

For example if $\Delta \tau = \frac{h}{2}$ and $2\varepsilon_e = 2Mh$

then we can stop @ 2M time steps. $\leftarrow$ half interface thickness, see (3.53)

6 Solving re-initialization:

- same R.-K./ENO used for l.s. can be utilized where (3.47) is satisfied $|\phi_d| < Mh$
- (3.49) is re-written as:

$$\frac{\partial \phi_d}{\partial \tau} = L \phi_d$$

- Solve (3.49) for $\tau = 0, \ldots, Mh$ assuming $\Delta\tau = \frac{h}{2}$ (taking 2M t/step).
- $(\phi_d)^n_{i,j}$ are discrete values defined @ $\tau = \tau^n$, $x = x_i$, $y = y_j$
- predictor: $(\phi_d)^*_{i,j} = (\phi_d)^n_{i,j} + \Delta\tau L \phi^n_d$
- corrector: $(\phi_d)^{n+1}_{i,j} = (\phi_d)^n_{i,j} + \frac{\Delta\tau}{2}\left(L\phi^n_d + L\phi^*_d\right)$

7. Discretization:

$$L\phi = \mathrm{sgn}_{M_h}(\phi)\left(1 - \sqrt{\left(\frac{\tilde{D}_x}{h}\right)^2 + \left(\frac{\tilde{D}_y}{h}\right)^2}\right) \quad (3.50)$$

where

$$\tilde{D}_x = \begin{cases} \tilde{D}_x^+, & \mathrm{sgn}(\phi)\, D_x^+(\phi_d)_{i,j} < 0 \text{ and } \\ & \mathrm{sgn}(\phi)\, D_x^-(\phi_d)_{i,j} < -\mathrm{sgn}(\phi)\, D_x^+(\phi_d)_{i,j} \\ \tilde{D}_x^-, & \mathrm{sgn}(\phi)\, D_x^-(\phi_d)_{i,j} > 0 \text{ and } \\ & \mathrm{sgn}(\phi)\, D_x^+(\phi_d)_{i,j} > -\mathrm{sgn}(\phi)\, D_x^-(\phi_d)_{i,j} \\ \frac{1}{2}\left(\tilde{D}_x^+ + \tilde{D}_x^-\right) & \text{otherwise} \end{cases} \quad (3.51)$$