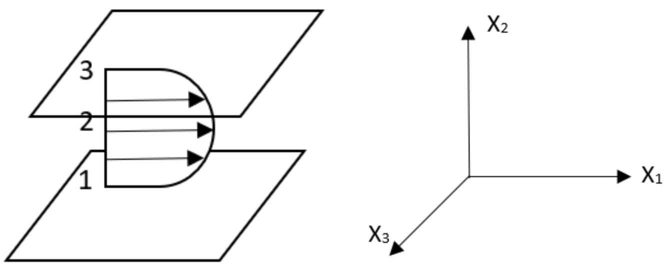
Homework #4 (1 problem, 30 points)

due: March 8th, 2023.

Problem 4.1	Total

Problem 4.1 30 points:

Consider a laminar flow in a channel (between two parallel plates) depicted in below. The streamwise direction is x or x_1 axis, and the centerline between the two plates is $y = 0$ or $x_2 = 0$.



The distance between the plates is $\boldsymbol{L_2 = 0.02\ m}$**.** Use the coordinates for the 3 points in the figure:

Point 1: $x_2 = -0.01\ m$;

Point 2: $x_2 = 0\ m$;

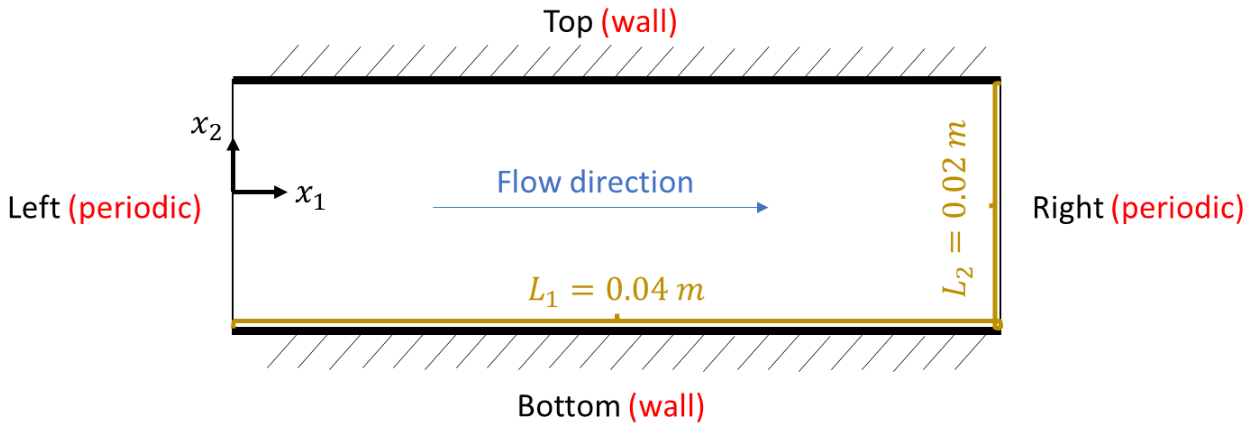
Point 3: $x_2 = 0.01\ m$.

Assume the maximum (peak) velocity is $0.01875\ m/s$ and perform the following analysis:

a) Solve the incompressible N.S. equation $\rho\frac{Du_i}{Dt} = -p_{,i} + \mu\nabla^2 u_i$ for this problem **analytically** at steady state with the body force ignored, given kinematic viscosity $\nu = 10^{-6}\ m^2/s$ and density $\rho = 10^3 kg/m^3$. Compute the **pressure gradient** $\boldsymbol{p_{,1}}$ and **velocity profile** $\boldsymbol{u_1(x_2)}$, and then compute the average velocity $\overline{\boldsymbol{u_1}(\boldsymbol{x_2})}$. [5 points]

Note: pay attention to the units.

b) Develop a staggered grid for this problem using the domain **length of $L_1 = 0.04\ m$**. Assume that grid is 2D, isotropic. Boundaries are denoted by inlet, outlet, top, and bottom. [5 points]



Boundary conditions of this fluid domain are:

Left	Right	Top	Bottom
Periodic		Wall	Wall

c) Using a programming tool (Fortran is encouraged, but not required), create a simple N.S. solver for the problem under consideration. **<u>Submit your code.</u>** [10 points]

<u>Notes:</u>

- Do **NOT** solve for pressure, simply use the pressure distribution obtained from the pressure gradient in a).
- Assume that <u>normal to the wall velocity is zero</u> and do NOT solve for it. Only solve for the x-velocity component.
- Assign initial condition to be a uniform velocity profile equal to the average velocity $\overline{u_1(x_2)}$ from the analytic solution.

d) Perform iteration until convergence/steady state (no further change in velocity is observed) on 3 different mesh resolutions: 30, 50 and 70 pressure cells **across the channel**. Plot and compare the results with the analytical solution obtained in part a). Then use the normalized L^2 norm in below to estimate the solution error for each mesh resolution: [5 points]

- The normalized L^2 norm to estimate the solution error (based on the error of the numerical solution relative to the analytic solution at each cell):

$$L^2 = \sqrt{\frac{1}{N}\sum_{j=1}^{N}(u_{numerical} - u_{exact})^2}$$

e) Do you expect the steady state solution to change with initial conditions? To test it, initialize your velocity field to be $0\ m/s$, peak velocity $0.01875\ m/s$, and $\overline{u_1(x_2)}$. Compare steady state solutions using 30 pressure cells across the channel. **Discuss the result**. [5 points]

Note: only initial condition is changed, boundary conditions should remain the same.

(a) Assumption: Flow is steady and fully developed → streamwise velocity is only function of x_2 or y.

Hence the governing equation becomes:

$$\frac{dp}{dx} = \mu \frac{d^2u}{dy^2}$$

The physical interpretation of this equation is that the applied pressure gradient will be balanced by the wall shear stress.

Rearrange:

$$\frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{dp}{dx}$$

Integrate wrt y:

$$\int \frac{d^2u}{dy^2} dy = \frac{du}{dy} = \frac{1}{\mu}\frac{dp}{dx} y + C_1$$

Integrate wrt y again:

$$u(y) = \frac{1}{2\mu}\frac{dp}{dx}\, y^2 + C_1 y + C_2$$

where C_1 and C_2 are constants that will be determined by the boundary conditions. Given the assumptions, we have:

$$\begin{cases} u(y = \pm h) = 0, h = 0.01 \\ \frac{du}{dy}|_{y=0} = 0 \end{cases}$$

Then we find that $C_1 = 0, C_2 = -\frac{1}{2\mu}\frac{dp}{dx}h^2$. Hence the analytical solution can be expressed as:

$$u(y) = \frac{1}{2\mu}\frac{dp}{dx}(y^2 - h^2)$$

Given $u_{max} = u(y = 0) = 0.01875\, m/s$, we can determine the pressure gradient on the streamwise direction that drives the flow:

$$u_{max} = u(y = 0) = -\frac{1}{2\mu}\frac{dp}{dx}h^2$$

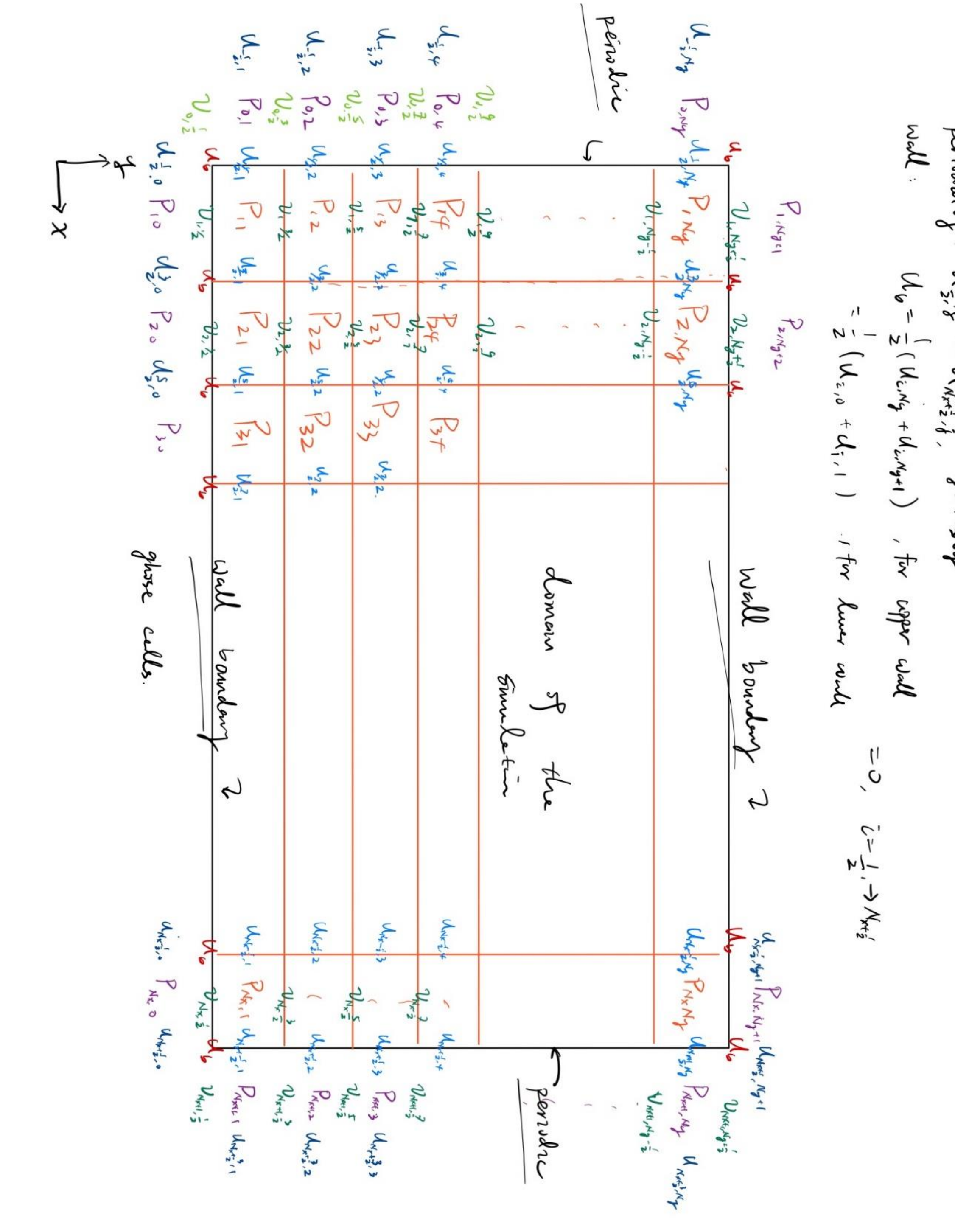
$$\frac{dp}{dx} = -u_{max}\frac{2\mu}{h^2} = 0.375\ Pa/m$$

The mean velocity can be found by averaging the velocity profile across the wall ($L_y = 2h$):

$$\frac{\int_{-h}^{h} u(y)dy}{L_y} = \frac{1}{2\mu L_y}\frac{dp}{dx}\left[\frac{y^3}{3} - h^2 y\right]_{-h}^{h} = \frac{1}{2\mu L_y}\frac{dp}{dx}\left(-\frac{2}{3}h^3 - \frac{2}{3}h^3\right) = \frac{1}{4\mu h}\frac{dp}{dx}\left(-\frac{4}{3}h^3\right)$$

$$= \frac{-1}{3\mu}\frac{dp}{dx}h^2 = \frac{2}{3}u_{max} = 0.0125\ m/s$$

(b) Note: the boundary of the pressure cells should align with the domain boundary



(c) Pseudo code for the solver:

1. Simulation parameters specification (geometry (L_x, L_y), number of pressure cells (N_x, N_y), fluid properties (ρ, μ, ν), given flow conditions$(u_{max}, \frac{dp}{dx})$, max number of iterations $(n_{iter,max})$, Δt)
2. Data array initialization (e.g. x-velocity($u, (N_x + 2, N_y + 2)$), reference x-velocity, surface area and normal of the pressure cell boundaries)
3. Initialization of the solution array and convergence metric:

$$u = u_{ini}, L_{2r} = \frac{L_{2,curent}}{L_{2,prev}} = 1.0$$

4. While ($n_{iter} < n_{iter,max}$ && $L_{2r} \leq 1.0$):

   -Predictor step:

$$u^* = u^n + \Delta t(-A_x(u^n) + \nu D_x(u^n))$$

   -Pseudo corrector step (Do not solve for pressure):

$$u^{n+1} = u^* - \frac{\Delta t}{\rho}\frac{dp}{dx}$$

   -Enforce boundary conditions:

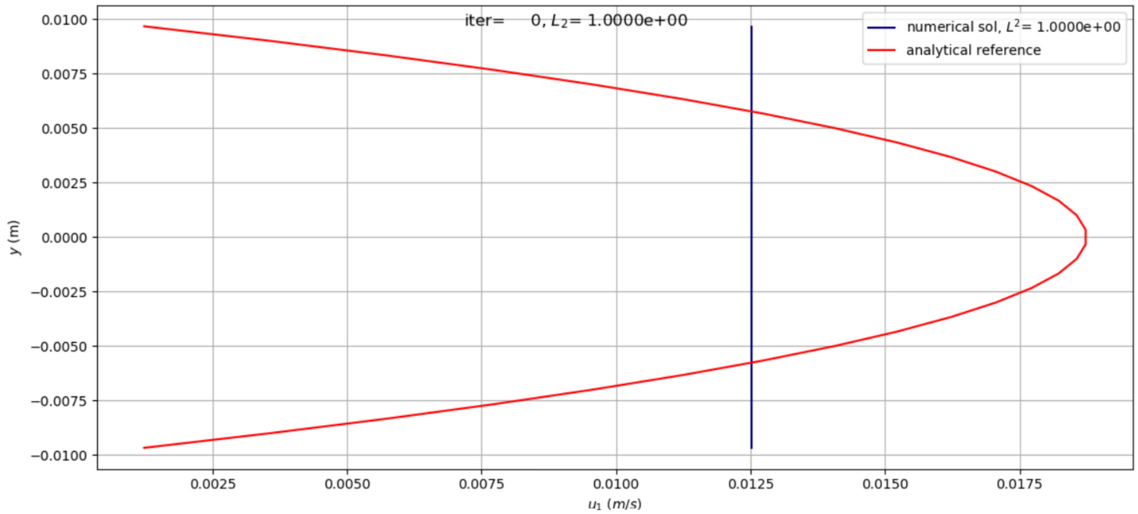
$$\begin{cases} u_{i,N_y+1} = -u_{i,N_y} \\ u_{i,0} = -u_{i,1} \end{cases}, i = \tfrac{1}{2} \rightarrow N_x + \tfrac{1}{2} \text{ for upper and lower walls}$$

$$\begin{cases} u_{N_x+\frac{1}{2},j} = u_{\frac{3}{2},j} \\ u_{-\frac{1}{2},j} = u_{N_x-\frac{1}{2},j} \end{cases}, j = 1 \rightarrow N_y \text{ for the periodicity on the streamwise direction}$$

   - Update L_2 for this iteration and compute L_{2r}

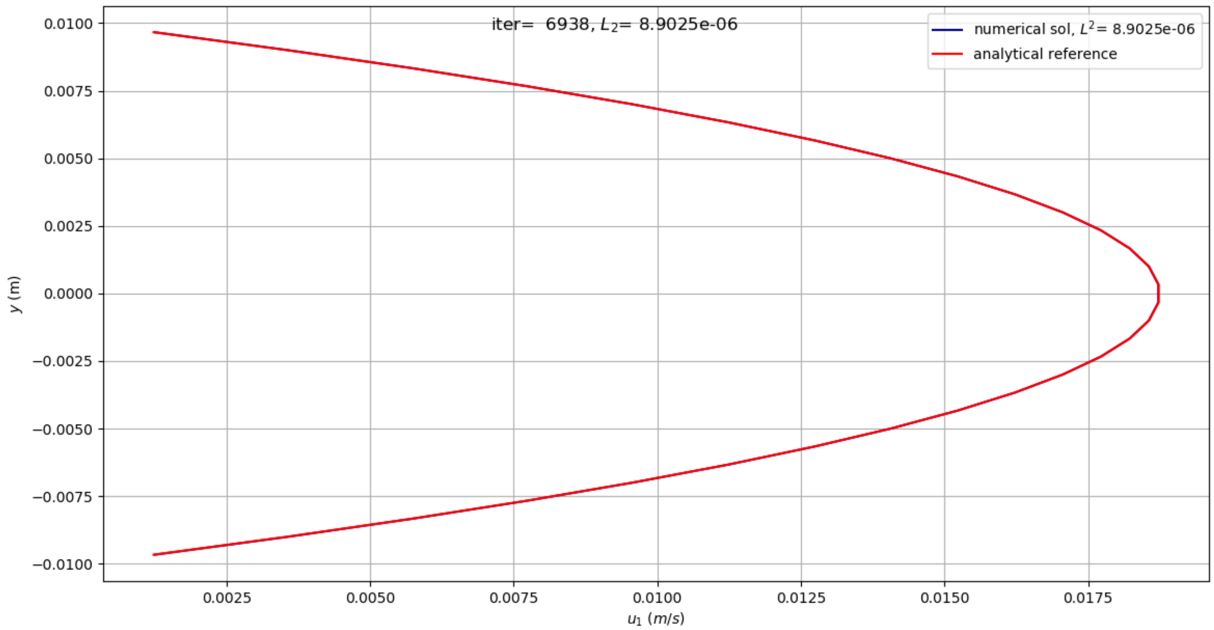
(d) Results of the simulation:

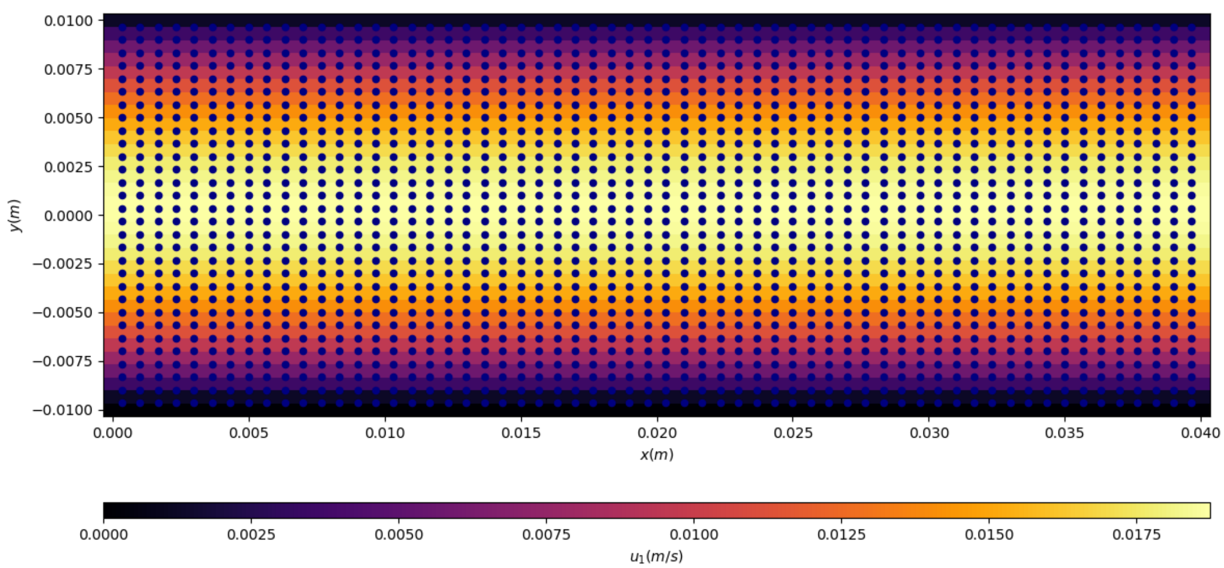
Initial condition is set to be $\overline{u(y)}$:



Results of $N_y = 30$ (navy blue dots are the center of the pressure cells):

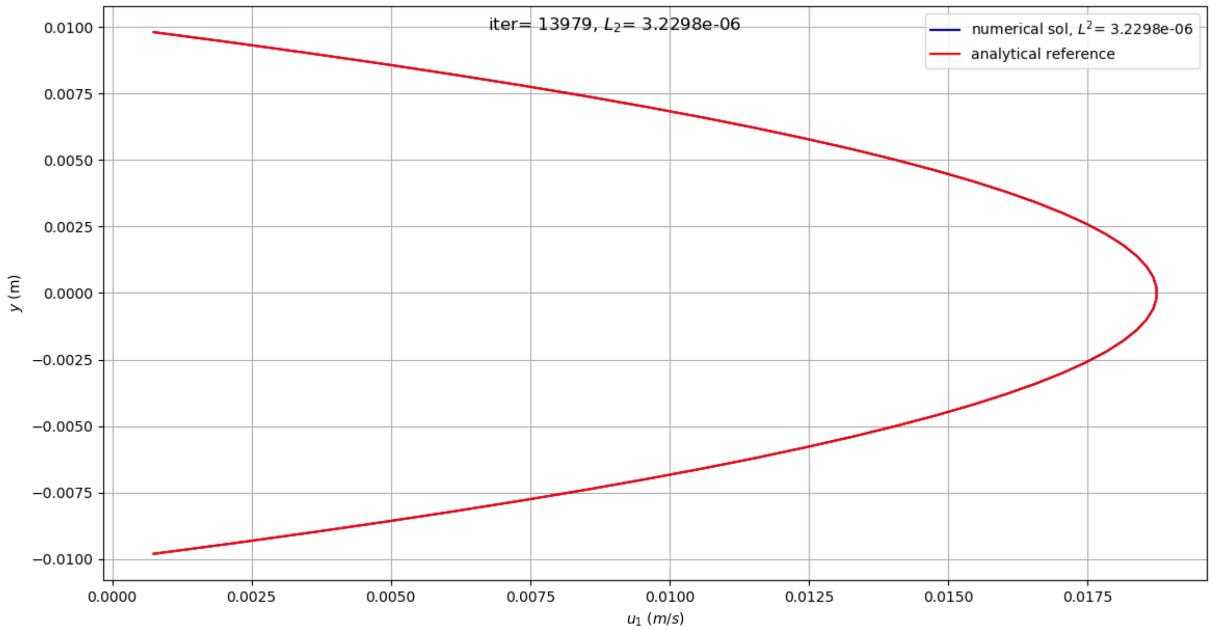
L_2 at convergence: $8.90e - 6$

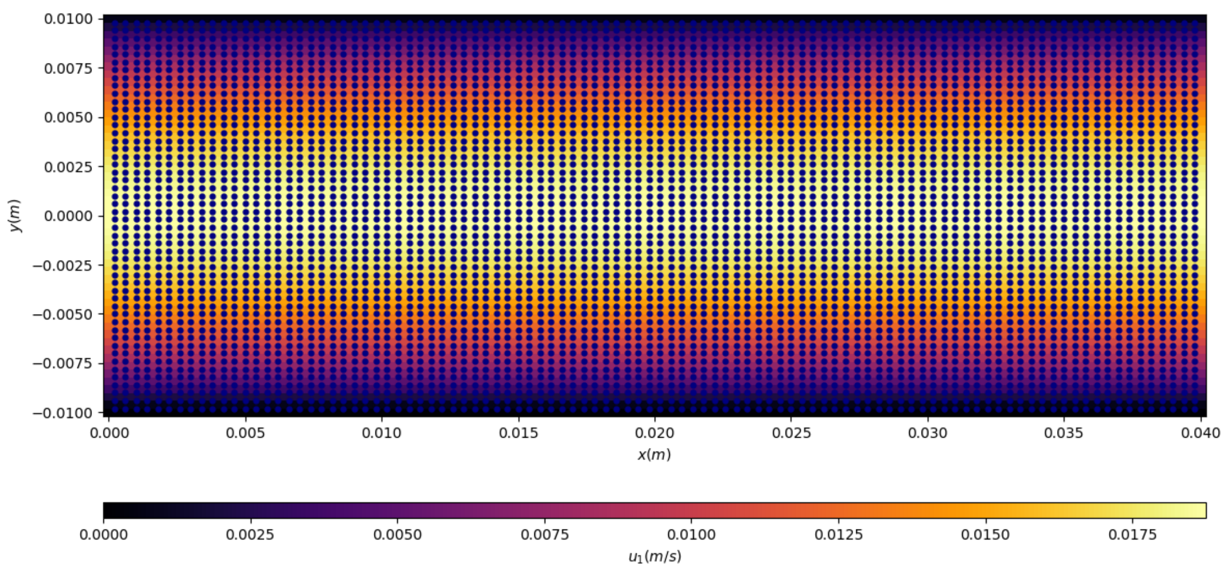




Results of $N_y = 50$:

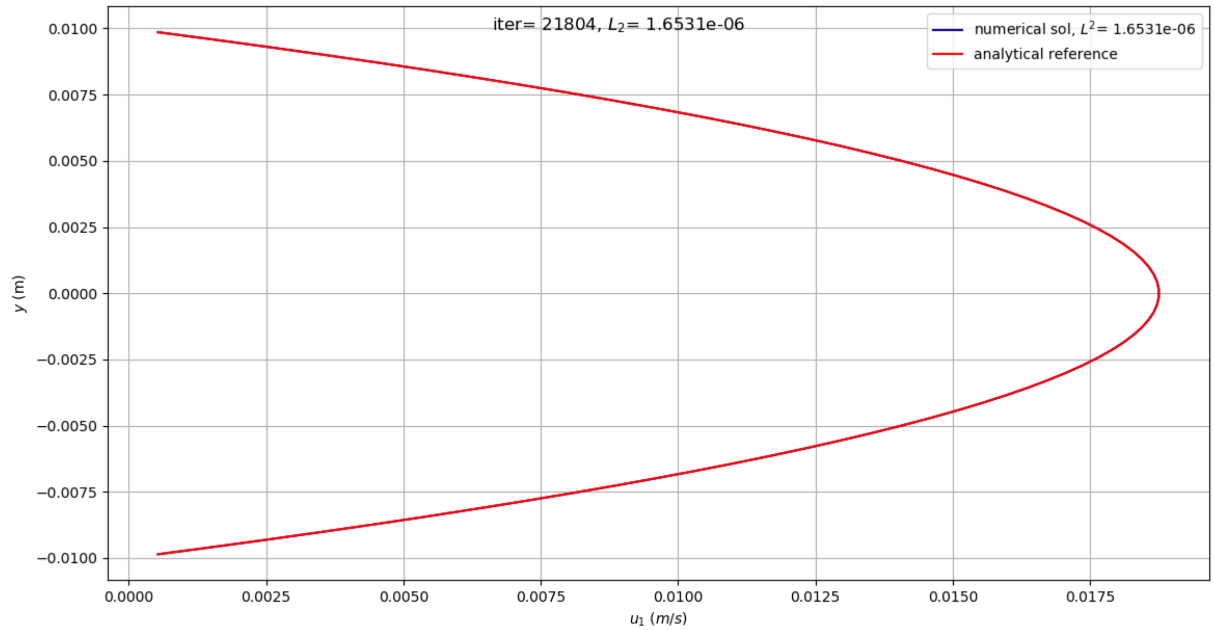
L_2 at convergence: $3.23e-6$

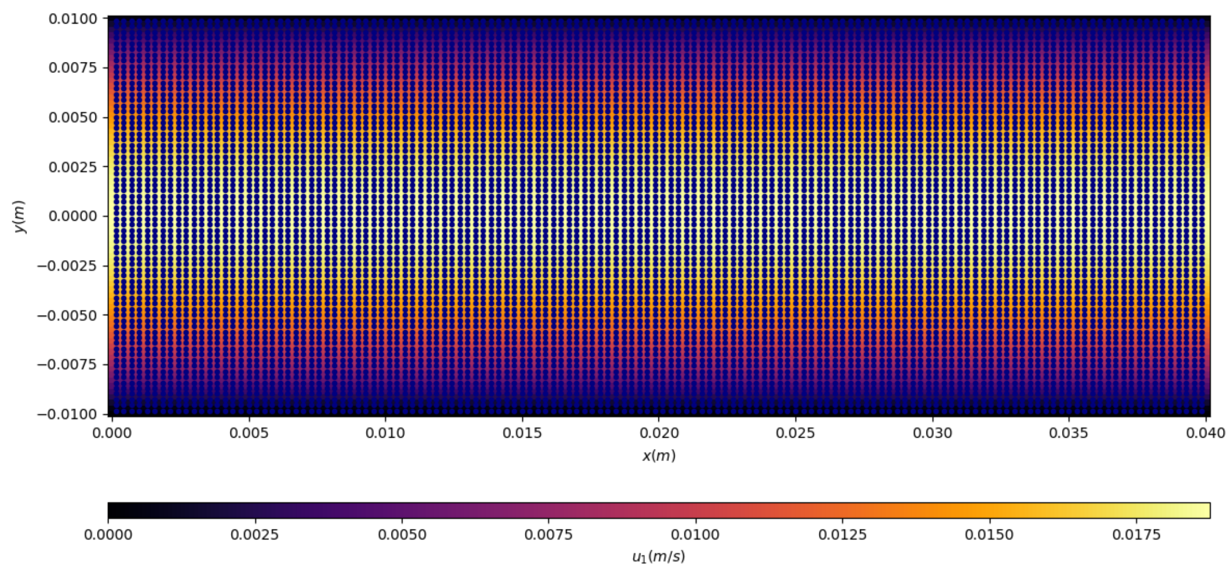




Results of $N_y = 70$:

L_2 at convergence: $1.653e - 6$





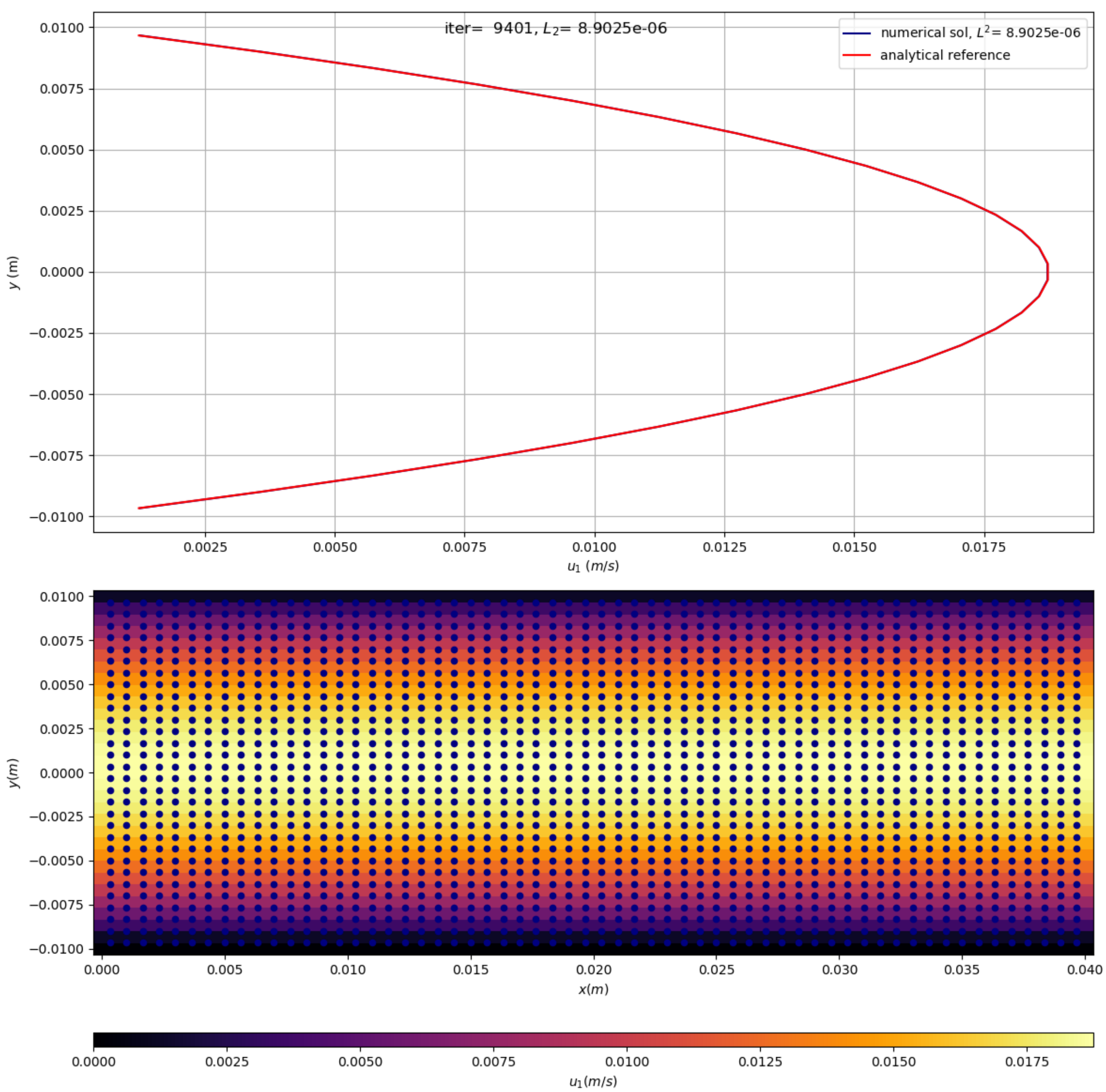
Discussion:

The solution error L_2 at convergence decreased roughly by a factor of 2 as the spatial resolution increased from $N_y = 30$, 50 to 70. The

(e) The resulting velocity distribution does not change with the initial velocity because the physics will lead to the steady state solution. But the initial conditions that is further from the converged one would take slightly more iterations to reach the final state.

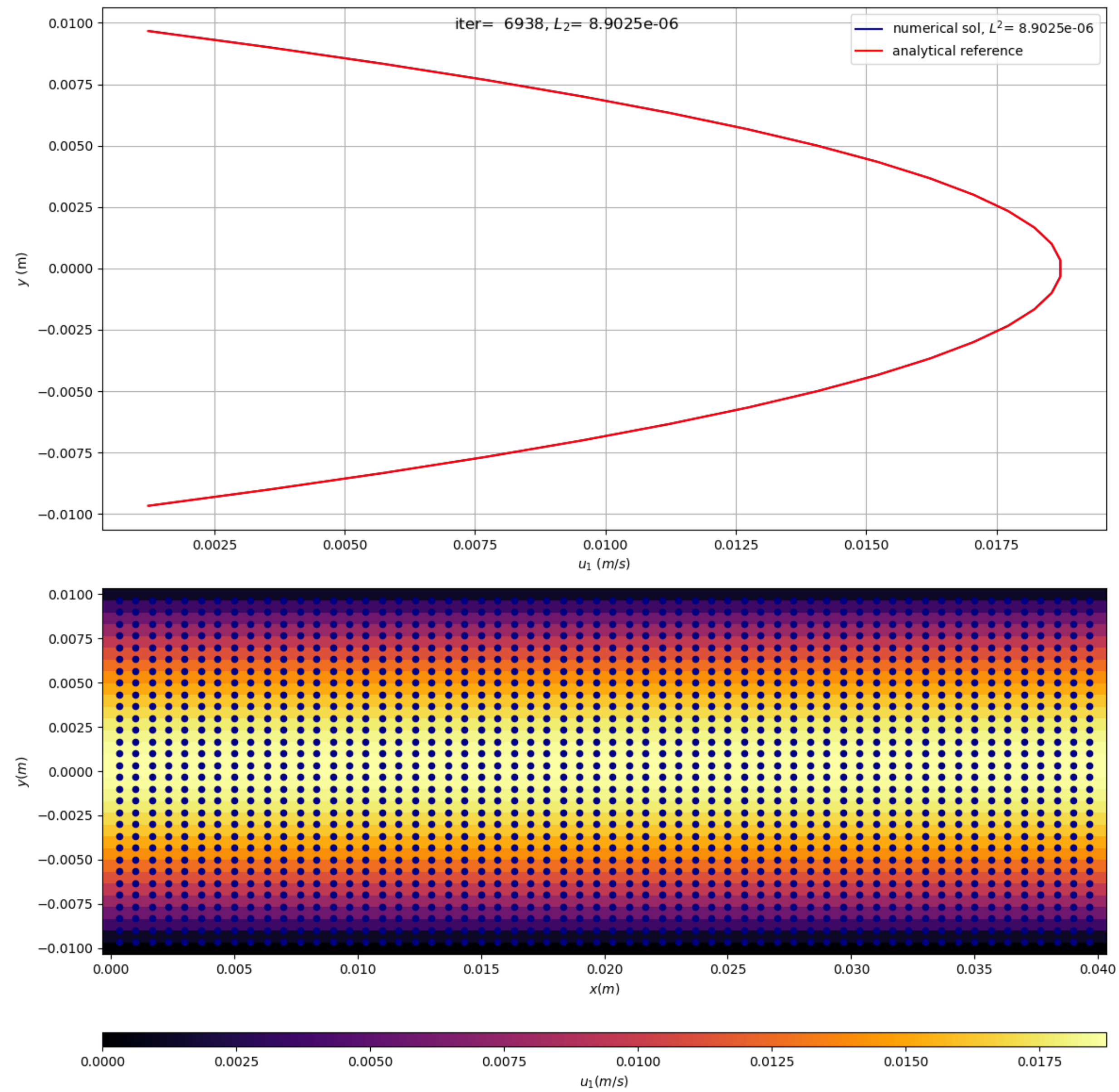
Results of $u(t = 0) = 0.0\ m/s$

L_2 at convergence: $8.9e - 6$, 9401 steps



Results of $u(t=0) = \overline{u(y)}$

L_2 at convergence: $8.9e-6$, 6938 steps



Results of $u(t=0) = u_{max}$

L_2 at convergence: $2.05e-5$, 20,000 iterations (terminated due to max iterations reached)