## Problem 3.3 20 points:

Assume a 2D grid which is not isotropic, e.g. the resolution in the x direction is $k_x$ and the resolution in the y direction is $k_y$.

a) Derive the **incompressibility condition** (Eq. (2.17) in the notes/text). Show that your result is consistent with notes if you assume $k_x = k_y = k$.

b) Derive the **predictor step** (2.18), **projection step** (2.19) and the **pressure equation** (2.20). Show that your result is consistent with notes if you assume $k_x = k_y = k$.

c) Derive the **advection term** in the x-direction for the non-isotropic grid. Show that your result is consistent with notes if you assume $k_x = k_y = k$.

Solution:

(a)

$V_{i-1,j+1/2}$ ——— ——— $V_{i,j+1/2}$ ——— ——— $V_{i+1,j+1/2}$

| | | | |

$P_{i-1,j}$ ——— $U_{i-1/2,j}$ ——— $P_{i,j}$ ——— $U_{i+1/2,j}$ ——— $P_{i+1,j}$

| | | | |

$V_{i-1,j-1/2}$ ——— ——— $V_{i,j-1/2}$ ——— ——— $V_{i+1,j-1/2}$

According to the pattern of the pressure control volume which centered at $P_{i,j}$

Using divergence theorem and mid-point rule

$$k_y\left(u_{i+\frac{1}{2},j}^{n+1} - u_{i-\frac{1}{2},j}^{n+1}\right) + k_x\left(v_{i,j+\frac{1}{2}}^{n+1} - v_{i,j-\frac{1}{2}}^{n+1}\right) = 0$$

If $k_y = k_x = k$

$$\Longrightarrow u_{i+\frac{1}{2},j}^{n+1} - u_{i-\frac{1}{2},j}^{n+1} + v_{i,j+\frac{1}{2}}^{n+1} - v_{i,j-\frac{1}{2}}^{n+1} = 0$$

(b)

The pattern of u-velocity control volume:

$V_{i,j+1/2}$ ——— ——— $V_{i+1,j+1/2}$

| | |

$P_{i,j}$ ——— $U_{i-1/2,j}$ ——— $P_{i+1,j}$

| | |

$V_{i,j-1/2}$ ——— ——— $V_{i+1,j-1/2}$

The pattern of v-velocity control volume

$U_{i-1/2,j+1}$ ——— $P_{i,j+1}$ ——— $U_{i+1/2,j+1}$

| | |

——— $V_{i,j+1/2}$ ———

| | |

$U_{i-1/2,j}$ ——— $P_{i,j}$ ——— $U_{i+1/2,j}$

The predictor step:

$$u^*_{i+\frac{1}{2},j} = u^n_{i+\frac{1}{2},j} + \Delta t\left[-(A_x)^n_{i+\frac{1}{2},j} + \nu(D_x)^n_{i+\frac{1}{2},j} + (f_x)_{i+\frac{1}{2},j}\right]$$

$$v^*_{i,j+\frac{1}{2}} = v^n_{i,j+\frac{1}{2}} + \Delta t\left[-(A_y)^n_{i,j+\frac{1}{2}} + \nu(D_y)^n_{i,j+\frac{1}{2}} + (f_y)_{i,j+\frac{1}{2}}\right]$$

The projection step:

$$u^{n+1}_{i+\frac{1}{2},j} = u^*_{i+\frac{1}{2},j} - \frac{\Delta t}{\rho}\cdot\frac{1}{k_x}\left(P^{n+1}_{i+1,j} - P^{n+1}_{i,j}\right)$$

$$v^{n+1}_{i,j+\frac{1}{2}} = v^*_{i,j+\frac{1}{2}} - \frac{\Delta t}{\rho}\cdot\frac{1}{k_y}\left(P^{n+1}_{i,j+1} - P^{n+1}_{i,j}\right)$$

The pressure equation:

$$u^{n+1}_{i-\frac{1}{2},j} = u^{*}_{i-\frac{1}{2},j} - \frac{\Delta t}{\rho} \cdot \frac{1}{k_x} \left(P^{n+1}_{i,j} - P^{n+1}_{i-1,j}\right)$$

$$v^{n+1}_{i,j-\frac{1}{2}} = v^{*}_{i,j-\frac{1}{2}} - \frac{\Delta t}{\rho} \cdot \frac{1}{k_y} \left(P^{n+1}_{i,j} - P^{n+1}_{i,j-1}\right)$$

If $k_y = k_x = k$

$$\Longrightarrow u^{n+1}_{i-\frac{1}{2},j} = u^{*}_{i-\frac{1}{2},j} - \frac{\Delta t}{\rho} \cdot \frac{1}{k} \left(P^{n+1}_{i,j} - P^{n+1}_{i-1,j}\right)$$

$$\Longrightarrow v^{n+1}_{i,j-\frac{1}{2}} = v^{*}_{i,j-\frac{1}{2}} - \frac{\Delta t}{\rho} \cdot \frac{1}{k} \left(P^{n+1}_{i,j} - P^{n+1}_{i,j-1}\right)$$

Combining the projection step and continuity equation,

$$\begin{aligned} k_y \left(u^{n+1}_{i+\frac{1}{2},j} - u^{n+1}_{i-\frac{1}{2},j}\right) + k_x \left(v^{n+1}_{i,j+\frac{1}{2}} - v^{n+1}_{i,j-\frac{1}{2}}\right) \\ &= k_y \left[u^{*}_{i+\frac{1}{2},j} - u^{*}_{i-\frac{1}{2},j} - \frac{\Delta t}{\rho} \cdot \frac{1}{k_x} \left(P^{n+1}_{i+1,j} + P^{n+1}_{i-1,j} - 2 \cdot P^{n+1}_{i,j}\right)\right] \\ &+ k_x \left[v^{*}_{i,j+\frac{1}{2}} - v^{*}_{i,j-\frac{1}{2}} - \frac{\Delta t}{\rho} \cdot \frac{1}{k_y} \left(P^{n+1}_{i,j+1} + P^{n+1}_{i,j-1} - 2 \cdot P^{n+1}_{i,j}\right)\right] = 0 \end{aligned}$$

$$\begin{aligned} \Longrightarrow \frac{k_y}{k_x} \left(P^{n+1}_{i+1,j} + P^{n+1}_{i-1,j} - 2 \cdot P^{n+1}_{i,j}\right) + \frac{k_x}{k_y} \left(P^{n+1}_{i,j+1} + P^{n+1}_{i,j-1} - 2 \cdot P^{n+1}_{i,j}\right) \\ &= \frac{\rho}{\Delta t} \left[k_y \left(u^{*}_{i+\frac{1}{2},j} - u^{*}_{i-\frac{1}{2},j}\right) + k_x \left(v^{*}_{i,j+\frac{1}{2}} - v^{*}_{i,j-\frac{1}{2}}\right)\right] \end{aligned}$$

If $k_y = k_x = k$

$$\Longrightarrow P^{n+1}_{i+1,j} + P^{n+1}_{i-1,j} + P^{n+1}_{i,j+1} + P^{n+1}_{i,j-1} - 4 \cdot P^{n+1}_{i,j} = \frac{\rho \cdot k}{\Delta t} \left[u^{*}_{i+\frac{1}{2},j} - u^{*}_{i-\frac{1}{2},j} + v^{*}_{i,j+\frac{1}{2}} - v^{*}_{i,j-\frac{1}{2}}\right]$$

(c)

Right: $\left(\frac{u^{n}_{i+\frac{3}{2},j} + u^{n}_{i+\frac{1}{2},j}}{2}\right)^2 k_y$

Left: $-\left(\frac{u^n_{i+\frac{1}{2},j}+u^n_{i-\frac{1}{2},j}}{2}\right)^2 \mathrm{k}_y$

Above: $\left(\frac{u^n_{i+\frac{1}{2},j+1}+u^n_{i+\frac{1}{2},j}}{2}\right)\left(\frac{u^n_{i+1,j+\frac{1}{2}}+u^n_{i,j+\frac{1}{2}}}{2}\right)k_x$

Below: $-\left(\frac{u^n_{i+\frac{1}{2},j}+u^n_{i+\frac{1}{2},j-1}}{2}\right)\left(\frac{u^n_{i+1,j-\frac{1}{2}}+u^n_{i,j-\frac{1}{2}}}{2}\right)k_x$

$$\underline{A}(\underline{u}^n) = \frac{1}{\Delta V}\oint_S \underline{u}^n(\underline{u}^n \cdot \underline{n})\, dS$$

$$(A_x)^n_{i+\frac{1}{2},,j} = \frac{1}{k_x}\left[\left(\frac{u^n_{i+\frac{3}{2},j} + u^n_{i+\frac{1}{2},j}}{2}\right)^2 - \left(\frac{u^n_{i+\frac{1}{2},j} + u^n_{i-\frac{1}{2},j}}{2}\right)^2\right] + \frac{1}{k_y}\left[\left(\frac{u^n_{i+\frac{1}{2},j} + u^n_{i+\frac{1}{2},j+1}}{2}\right)\left(\frac{v^n_{i,j+\frac{1}{2}} + v^n_{i+1,j+\frac{1}{2}}}{2}\right) - \left(\frac{u^n_{i+\frac{1}{2},j} + u^n_{i+\frac{1}{2},j-1}}{2}\right)\left(\frac{v^n_{i+1,j-\frac{1}{2}} + v^n_{i,j-\frac{1}{2}}}{2}\right)\right]$$

If $k_y = k_x = k$

$$(A_x)^n_{i+\frac{1}{2},,j} = \frac{1}{k}\left[\left(\frac{u^n_{i+\frac{3}{2},j} + u^n_{i+\frac{1}{2},j}}{2}\right)^2 - \left(\frac{u^n_{i+\frac{1}{2},j} + u^n_{i-\frac{1}{2},j}}{2}\right)^2\right] + \frac{1}{k}\left[\left(\frac{u^n_{i+\frac{1}{2},j} + u^n_{i+\frac{1}{2},j+1}}{2}\right)\left(\frac{v^n_{i,j+\frac{1}{2}} + v^n_{i+1,j+\frac{1}{2}}}{2}\right) - \left(\frac{u^n_{i+\frac{1}{2},j} + u^n_{i+\frac{1}{2},j-1}}{2}\right)\left(\frac{v^n_{i+1,j-\frac{1}{2}} + v^n_{i,j-\frac{1}{2}}}{2}\right)\right]$$