Problem 3.1 10 points:

Capturing capillary wave propagation around an interface plays and important role in numerical stability of two-phase flow simulations, and it also poses restrictions on the time step size. For a capillary wave with wave number $k = \frac{2\pi}{\lambda}$, the phase velocity can be expressed as:

$$c_\sigma(k) = \sqrt{\frac{\sigma k}{\rho_1 - \rho_2}}$$

where λ is the wavelength, c_σ is the phase velocity, σ is the surface tension coefficient, and ρ_1, ρ_2 are the density of the fluids at the respective side of the interface ($\rho_1 > \rho_2$). Consider a two-phase flow simulation conducted on a mesh with uniform spacing Δx:

a) Find the expression for the minimum wavelength of a sinusoidal capillary wave (λ_{min}) that can be **unambiguously represented** by the given mesh spacing Δx [2 points] and discuss your reasons [3 points]. (Hint: Nyquist theorem)

b) Based on the result of a), find the expression for the time step size restriction (Δt_{max}) such that the propagation of the capillary wave with wavelength λ_{min} can be captured by the grid. [5 points]

(a) Given the assumption that the capillary wave is sinusoidal, and its wavelength is λ. If one would like to sample a frequency of characteristic frequency f, the sampling rate should be at least twice of the sampled frequency, or $f_{sampling} \geq 2f$, so that the signal can be unambiguously represented (or the aliasing effect can be avoided). Now consider a sinusoidal capillary wave of wavelength λ, the analogy of the Nyquist theorem tells us that the spacing between the two neighboring grid points (treat them as spatial sampling points) should be less then $\frac{\lambda}{2}$ such that the wave can be unambiguously captured. Hence, given a uniform grid with spatial spacing Δx, the minimal wavelength that can be unambiguously represented by the grid is $\lambda_{min} = 2\Delta x$.

(b) The capillary wave of wavelength of λ_{min} is also the fastest one that can be captured by the grid with spacing Δx (since the wave number is the highest, k_{max}). To capture the wave propagation, the CFL number based on the $c_\sigma(k_{max})$ should be lower than one, i.e.:

$$CFL_{c_\sigma} = \frac{c_\sigma(k_{max})\Delta t}{\Delta x} \leq 1$$

Hence the limiting time step size:

$$\Delta t_{max} = \frac{\Delta x}{c_\sigma(k_{max})}$$