Finite groups with at most three relative commutativity degrees

R. Barzegar, A. Erfanian and M. Farrokhi D. G.

Ferdowsi University of Mashhad

Sabzevar
31 January - 2 February 2012

Definition

Let G be a finite group. Then the *commutativity degree* of G, denoted by $d(G)$, is the probability that two randomly chosen elements of G commutes. In other words,

$$d(G) = \frac{|\{(x, y) \in G \times G : xy = yx\}|}{|G|^2}.$$

Definition

Let G be a finite group. Then the *commutativity degree* of G, denoted by $d(G)$, is the probability that two randomly chosen elements of G commutes. In other words,

$$d(G) = \frac{|\{(x,y) \in G \times G : xy = yx\}|}{|G|^2}.$$

Theorem (Gustafson, 1973)

Let G be a finite non-abelian group. Then

$$d(G) \leq \frac{5}{8}$$

and the equality holds if and only if $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

The commutativity degree can be generalized in the following form.

The commutativity degree can be generalized in the following form.

Definition

Let G be a finite group and H be a subgroup of G. Then the *relative commutativity degree* of H in G, denoted by $d(H,G)$, is the probability that an element of H commutes with an element of G, i.e.,

$$d(H,G)=\frac{|\{(h,g)\in H\times G : hg=gh\}|}{|H||G|}.$$

The commutativity degree can be generalized in the following form.

Definition

Let G be a finite group and H be a subgroup of G. Then the *relative commutativity degree* of H in G, denoted by $d(H, G)$, is the probability that an element of H commutes with an element of G, i.e.,

$$d(H, G) = \frac{|\{(h, g) \in H \times G : hg = gh\}|}{|H||G|}.$$

Remark

If $H = G$, then $d(H, G) = d(G)$.

Theorem (Erfanian, Lescot and Rezaei, 2007)

Let G be a finite group and H be a subgroup of G. Then

$$d(G) \leq d(H,G) \leq d(H)$$

and if H is not normal in G, then

$$d(G) < d(H,G) < d(H)$$

Introduction

Theorem (Erfanian, Lescot and Rezaei, 2007)

Let G be a finite group and H be a subgroup of G. Then

$$d(G) \leq d(H, G) \leq d(H)$$

and if H is not normal in G, then

$$d(G) < d(H, G) < d(H)$$

Theorem (Erfanian, Lescot and Rezaei, 2007)

Let G be a finite group, H be a subgroup of G and N be a normal subgroup of G. Then

$$d(H, G) \leq d\left(\frac{H}{N}, \frac{G}{N}\right) d(N).$$

Theorem (Erfanian, Lescot and Rezaei, 2007)

Let G be a finite group and H be a subgroup of G.

Theorem (Erfanian, Lescot and Rezaei, 2007)

Let G be a finite group and H be a subgroup of G.

(1) *If $H \subseteq Z(G)$, then $d(H, G) = 1$,*

Theorem (Erfanian, Lescot and Rezaei, 2007)

Let G be a finite group and H be a subgroup of G.

(1) *If $H \subseteq Z(G)$, then $d(H, G) = 1$,*

(2) *If $H \not\subseteq Z(G)$ and H is abelian, then $d(H, G) \leq \frac{3}{4}$,*

Theorem (Erfanian, Lescot and Rezaei, 2007)

Let G be a finite group and H be a subgroup of G.

(1) *If $H \subseteq Z(G)$, then $d(H, G) = 1$,*

(2) *If $H \not\subseteq Z(G)$ and H is abelian, then $d(H, G) \leq \frac{3}{4}$,*

(3) *If $H \not\subseteq Z(G)$ and H is not abelian, then $d(H, G) \leq \frac{5}{8}$.*

Theorem (Erfanian, Lescot and Rezaei, 2007)

Let G be a finite group and H be a subgroup of G.

(1) *If $H \subseteq Z(G)$, then $d(H,G) = 1$,*

(2) *If $H \nsubseteq Z(G)$ and H is abelian, then $d(H,G) \leq \frac{3}{4}$,*

(3) *If $H \nsubseteq Z(G)$ and H is not abelian, then $d(H,G) \leq \frac{5}{8}$.*

Theorem (Erfanian, Lescot and Rezaei, 2007)

Let G be a finite group and H be a subgroup of G.

Theorem (Erfanian, Lescot and Rezaei, 2007)

Let G be a finite group and H be a subgroup of G.

(1) *If $H \subseteq Z(G)$, then $d(H,G) = 1$,*

(2) *If $H \not\subseteq Z(G)$ and H is abelian, then $d(H,G) \leq \frac{3}{4}$,*

(3) *If $H \not\subseteq Z(G)$ and H is not abelian, then $d(H,G) \leq \frac{5}{8}$.*

Theorem (Erfanian, Lescot and Rezaei, 2007)

Let G be a finite group and H be a subgroup of G.

(1) *If $d(H,G) = \frac{3}{4}$, then $H/H \cap Z(G) \cong \mathbb{Z}_2$,*

Theorem (Erfanian, Lescot and Rezaei, 2007)

Let G be a finite group and H be a subgroup of G.

(1) *If $H \subseteq Z(G)$, then $d(H,G) = 1$,*

(2) *If $H \not\subseteq Z(G)$ and H is abelian, then $d(H,G) \leq \frac{3}{4}$,*

(3) *If $H \not\subseteq Z(G)$ and H is not abelian, then $d(H,G) \leq \frac{5}{8}$.*

Theorem (Erfanian, Lescot and Rezaei, 2007)

Let G be a finite group and H be a subgroup of G.

(1) *If $d(H,G) = \frac{3}{4}$, then $H/H \cap Z(G) \cong \mathbb{Z}_2$,*

(2) *If $d(H,G) = \frac{5}{8}$, then $H/H \cap Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Definition

Let G be a finite group. Then

$$\mathcal{D}(G) = \{d(H, G) : H \leq G\}.$$

Definition

Let G be a finite group. Then

$$\mathcal{D}(G) = \{d(H, G) : H \leq G\}.$$

Corollary

Let G be a finite group and H be a subgroup of G. Then $|\mathcal{D}(G)| = 1$ if and only if G is abelian.

Definition

Let G be a finite group. Then

$$\mathcal{D}(G) = \{d(H, G) : H \leq G\}.$$

Corollary

Let G be a finite group and H be a subgroup of G. Then $|\mathcal{D}(G)| = 1$ if and only if G is abelian.

Question

What can be said about finite groups with few relative commutativity degrees?

Lemma

Let G be a finite group and $H \leq K \leq G$. Then $d(K,G) \leq d(H,G)$ and equality holds if and only if $K = HC_K(g)$, for all $g \in G$.

Lemma

Let G be a finite group and $H \leq K \leq G$. Then $d(K,G) \leq d(H,G)$ and equality holds if and only if $K = HC_K(g)$, for all $g \in G$.

Lemma

Let G be a finite non-abelian group and $x \in G \setminus Z(G)$. Then $d(\langle x \rangle, G) \neq 1, d(G)$.

Some elementary lemmas

Lemma

Let G be a finite group and $H \leq K \leq G$. Then $d(K,G) \leq d(H,G)$ and equality holds if and only if $K = HC_K(g)$, for all $g \in G$.

Lemma

Let G be a finite non-abelian group and $x \in G \setminus Z(G)$. Then $d(\langle x\rangle, G) \neq 1, d(G)$.

Corollary

There is no finite group with two relative commutativity degrees.

Lemma

Let G be a finite non-abelian group and suppose that $\mathcal{D}(G) = \{1, d, d(G)\}$. If H is a subgroup of G such that $d(H, G) = d$, then H is abelian.

Some elementary lemmas

Lemma

Let G be a finite non-abelian group and suppose that $\mathcal{D}(G) = \{1, d, d(G)\}$. If H is a subgroup of G such that $d(H, G) = d$, then H is abelian.

Lemma

Let G be a finite group with $|\mathcal{D}(G)| = 3$. Then $C_G(x)$ is an abelian maximal subgroup of G, for all $x \in G \setminus Z(G)$.

Main theorems

Theorem

Let G be a finite nilpotent group. Then $|\mathcal{D}(G)| = 3$ if and only if $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$. In particular,

$$\mathcal{D}(G) = \left\{1, \frac{2p-1}{p^2}, \frac{p^2+p-1}{p^3}\right\}.$$

Main theorems

Theorem

Let G be a finite nilpotent group. Then $|\mathcal{D}(G)| = 3$ if and only if $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$. In particular,

$$\mathcal{D}(G) = \left\{1, \frac{2p-1}{p^2}, \frac{p^2+p-1}{p^3}\right\}.$$

Theorem

Let G be a finite non-nilpotent group. Then $|\mathcal{D}(G)| = 3$ if and only if $G/Z(G)$ is a non-cyclic group of order pq, where p and q are distinct primes. In particular,

$$\mathcal{D}(G) = \left\{1, \frac{1}{p} + \frac{1}{q} - \frac{1}{pq}, \frac{1}{p} + \frac{1}{q^2} - \frac{1}{pq^2}\right\},$$

whenever $p > q$.

Examples

Theorem

Let $D_{2n} = \langle a, b : a^n = b^2 = 1, a^b = a^{-1} \rangle$ be the dihedral group of order $2n$. Then

$$|\mathcal{D}(G)| = \begin{cases} 2\tau(n) - 1, & n \text{ odd}, \\ 2k\tau(m) - 1, & n = 2^k m,\ k \geq 1 \text{ and } m \text{ odd}. \end{cases}$$

where $\tau(m)$ is the number of divisors of m.

Examples

Theorem

Let $D_{2n} = \langle a, b : a^n = b^2 = 1, a^b = a^{-1} \rangle$ *be the dihedral group of order* $2n$*. Then*

$$|\mathcal{D}(G)| = \begin{cases} 2\tau(n) - 1, & n \text{ odd}, \\ 2k\tau(m) - 1, & n = 2^k m,\ k \geq 1 \text{ and } m \text{ odd}. \end{cases}$$

where $\tau(m)$ *is the number of divisors of* m*.*

Corollary

$|\mathcal{D}(D_{2n})| = 3$ *if and only if* $n = p$ *or* $2p$*, where* p *is a prime.*

Examples

Definition

Let G_1 and G_2 be two groups and H_1 and H_2 be subgroup of G_1 and G_2, respectively. Suppose that α is an isomorphism from $G_1/Z(G_1)$ to $G_2/Z(G_2)$ such that its restriction to $H_1/H_1 \cap Z(G_1)$ is an isomorphism from $H_1/H_1 \cap Z(G_1)$ to $H_2/H_2 \cap Z(G_2)$ and β is an isomorphism from $[H_1, G_1]$ to $[H_2, G_2]$. Then the pair (α, β) is called a relative isoclinism from (H_1, G_1) to (H_2, G_2) if the following diagram is commutative:

$$\begin{tikzcd} \frac{H_1}{H_1\cap Z(G_1)} \times \frac{G_1}{Z(G_1)} \arrow[r, "\alpha^2"] \arrow[d, "\gamma_1"'] & \frac{H_2}{H_2\cap Z(G_2)} \times \frac{G_2}{Z(G_2)} \arrow[d, "\gamma_2"] \\ {[H_1, G_1]} \arrow[r, "\beta"] & {[H_2, G_2]} \end{tikzcd}$$

Definition

where

$$\gamma_1(h_1(H_1 \cap Z(G_1)), g_1 Z(G_1)) = [h_1, g_1]$$

and

$$\gamma_2(h_2(H_2 \cap Z(G_2)), g_2 Z(G_2)) = [h_2, g_2]$$

for each $h_1, \in H_1$, $h_2 \in H_2$, $g_1 \in G_1$ and $g_2 \in G_2$. If $H_1 = G_1$ and $H_2 = G_2$, then we say that G_1 and G_2 are isoclinic.

Lemma

If G_1 and G_2 are two isoclinic groups and $H_1 \leq G_1$, then there exists a subgroup H_2 of G_2 such that (H_1, G_1) is relative isoclinic to (H_2, G_2).

Lemma

If G_1 and G_2 are two isoclinic groups and $H_1 \leq G_1$, then there exists a subgroup H_2 of G_2 such that (H_1, G_1) is relative isoclinic to (H_2, G_2).

Lemma

If (H_1, G_1) is relative isoclinic to (H_2, G_2), then $d(H_1, G_1) = d(H_2, G_2)$.

Lemma

If G_1 and G_2 are two isoclinic groups and $H_1 \leq G_1$, then there exists a subgroup H_2 of G_2 such that (H_1, G_1) is relative isoclinic to (H_2, G_2).

Lemma

If (H_1, G_1) is relative isoclinic to (H_2, G_2), then $d(H_1, G_1) = d(H_2, G_2)$.

Lemma

If G_1 and G_2 are two isoclinic groups, then $\mathcal{D}(G_1) = \mathcal{D}(G_2)$.

Corollary

If $n = 2^k m$ (m odd) is a natural number, then $|\mathcal{D}(Q_{4n})| = 2(k+1)\tau(m) - 1$.

Corollary

If $n = 2^k m$ (m odd) is a natural number, then $|\mathcal{D}(Q_{4n})| = 2(k+1)\tau(m) - 1$.

Corollary

If $n \geq 3$, then $|\mathcal{D}(QD_{2^n})| = 2n - 3$.

References I

- A. Erfanian, R. Rezaei and P. Lescot, *On the relative commutativity degree of a subgroup of a finite group*, Comm. Algebra, **35**(12) (2007), 4183-4197.
- A. Erfanian and R. Rezaei, *A note on the relative commutativity degree of finite groups*, submitted.
- M. Farrokhi D. G., *Finite groups with two subgroup normality degrees*, submitted.
- W. H. Gustafson, *What is the probability that two group elements commute?*, Amer. Math. Monthly, **80** (1973), 1031-1304.
- P. Lescot, *Isoclinism classes and commutativity degrees of finite groups*, J. Algebra, **177** (1995), 847-869.

References II

F. Saeedi, M. Farrokhi D. G. and S. H. Jafari, *Subgroup normality degrees of finite groups I*, Arch. Math., **96**(3) (2011), 215-224.

Thank You