On the probability of being a 2-Engel group

A. Erfanian and M. Farrokhi D. G.

Ferdowsi University of Mashhad

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Definition

If G is a finite group, then the *commutativity degree* of G, denoted by $d(G)$, is the probability that two randomly chosen elements $x, y \in G$ commute, i.e.,

$$d(G) = \frac{|\{(x, y) \in G \times G : xy = yx\}|}{|G|^2}.$$

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If G is a finite group, then

$$d(G) = \frac{k(G)}{|G|},$$

where $k(G)$ is the number of conjugacy classes of G.

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6. *if $d(G) > \frac{3}{40}$, then G is solvable and $d(G) = \frac{1}{12}$ if and only if $G \cong A_5 \times B$, where B is abelian (R. M. Guralnick and G. R. Robinson, 2006)*

Definitions

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If G is a finite group, then the *n-Engel degree* of G, denoted by $E_n(G)$, is the probability that two randomly chosen elements $x, y \in G$ satisfy the equation $[y,_n x] = 1$, i.e.,

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Let G be a group and $x \in G$. Then

$$E_G(x) = \{y \in G : [y,x,x] = 1\}.$$

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A finite group G is called 3-metabelian if every subgroup of G that is generated by three elements is metablian. In other words, G is 3-metabelian if and only if $[[x, y], [x, z]] = 1$ for all elements $x, y, z \in G$.

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Known results

Let G be a 3-metabelian group, $x_1, x_2, x_3, x_4 \in G$ and $\pi \in S_4$. Then

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5. $[[x_1, x_2], [x_3, x_4]] = [[x_{\pi_1}, x_{\pi_2}], [x_{\pi_3}, x_{\pi_4}]]$;

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5. $[[x_1, x_2], [x_3, x_4]] = [[x_{\pi_1}, x_{\pi_2}], [x_{\pi_3}, x_{\pi_4}]]$;
6. $[[x_1, x_2], [x_3, x_4]] = [x_1, x_2, x_3, x_4][x_1, x_2, x_4, x_3]^{-1}$.

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Let G be a finite 3-metabelian group with an element x such that $C_G(x)x^G = G$. Then

1. $G = [x, G] \rtimes C_G(x)$;
2. *if* $x \in L(G)$, *then* $x \in Z(G)$.

Main Theorems

Theorem

Let G be a finite 3-metablian group, which is not 2-Engel group and $p = \min \pi(G)$. Then

$$E_2(G) \leq \frac{1}{p} + \left(1 - \frac{1}{p}\right) \frac{|L_2(G)|}{|G|}$$

and if $L_2(G) \leq G$, then

$$E_2(G) \leq \frac{2p-1}{p^2}.$$

Moreover, both of the upper bounds are sharp at any prime p.

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Let G be a finite 3-metablian group, which is not 2-Engel group and $p = \min \pi(G)$. Then

$$E_2(G) \geq d(G) - (p-1)\frac{|Z(G)|}{|G|} + (p-1)\frac{k_G(L(G))}{|G|}$$

and if either G is a p-group, or G' is a cyclic 2-group or a generalized quaternion 2-group, then

$$E_2(G) \geq pd(G) - (p-1)\frac{|Z(G)|}{|G|}.$$

Moreover, both of the lower bounds are sharp at any prime p.

Example

Let $G = D_{2n} = \langle a, b : a^n = b^2 = 1, a^b = a^{-1} \rangle$ be a dihedral group of order $2n$. Then

$$E_2(G) = \begin{cases} \frac{n+1}{2n}, & n \text{ odd}, \\ \frac{n+2}{2n}, & n = 2m, m \text{ odd}, \\ \frac{n+4}{2n}, & n = 4m. \end{cases}$$

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Let $G = Q_{4n} = \langle a, b : a^{2n} = 1, a^n = b^2, a^b = a^{-1} \rangle$ be a generalized quaternion group of order $4n$. Then

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Let $p > 2$ be a prime and

$$G = \langle a, b, c : a^{p^2} = b^p = c^p = 1, [a, b] = a^p, [a, c] = b, [b, c] = 1 \rangle.$$

Then

$$E_2(G) = \frac{2p - 1}{p^2}.$$

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