On non-normal graphs of finite groups

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(joint work with A. Erfanian and B. Tolue)

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Definitions

Definition

Let H be a subgroup of a group G. The non-normal graph of G with respect to H, denoted by $\mathfrak{N}_{H,G}$, is a bipartite graph with vertex sets $H \setminus H_G$ and $G \setminus N_G(H)$ as its parts in such a way that two vertices $h \in H \setminus H_G$ and $g \in G \setminus N_G(H)$ are adjacent if $h^g \notin H$.

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Remark

If H is a normal subgroup of G, then $\mathfrak{N}_{H,G}$ is a null graph.

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Let H be a subgroup of a finite group G. The subgroup normality degree of H in G is defined as follows.

$$P_N(H, G) = \frac{|\{(h, g) \in H \times G : h^g \in H\}|}{|H||G|}.$$

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Remark

If H is a subgroup of a finite group G, then

$$|E(\mathfrak{N}_{H,G})| = |H||G|(1 - P_N(H,G)),$$

where $E(\mathfrak{N}_{H,G})$ denotes the set of all edges of $\mathfrak{N}_{H,G}$.

Preliminary results

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- $B(G, H, g) = \{k \in H : k^g \in H\} = H \cap H^{g^{-1}}$,

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Then

1. $N_{\mathfrak{N}_{H,G}}(h) = G \setminus A(G,H,h)$,

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Then

1. $N_{\mathfrak{N}_{H,G}}(h) = G \setminus A(G,H,h)$,
2. $N_{\mathfrak{N}_{H,G}}(g) = H \setminus B(G,H,g)$,
3. $\deg_{\mathfrak{N}_{H,G}}(h) < \deg_{\mathfrak{N}_{H,G}}(g)$ if G is finite.

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2. $H \setminus H_G$ *and* $G \setminus N_G(H)$ *are invariants of* $\mathfrak{N}_{H,G}$,

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Theorem

Let H be a subgroup of a group G. Then

1. $G \setminus N_G(H)$ *is the unique largest independent subset of* $\mathfrak{N}_{H,G}$,
2. *the size of largest dominating subsets of* $\mathfrak{N}_{H,G}$ *are at most* $d(H) + [G : N_G(H)] - 1$.

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Let H be a non-normal subgroup of a group G. Then

1. $\mathfrak{N}_{H,G}$ *has a pendant if and only if* $|H| = 2$ *and* $\mathfrak{N}_{H,G}$ *is a star graph.*
2. *If* $|H| > 2$, *then* $\mathrm{gr}(\mathfrak{N}_{H,G}) = 4$.

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Let H be a non-normal subgroup of a group G. Then $\operatorname{diam}(\mathfrak{N}_{H,G}) \leq 4$.

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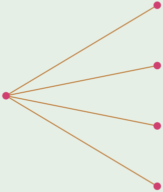
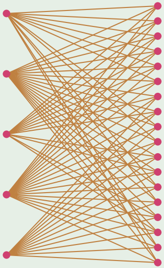
Example

Diamter 2



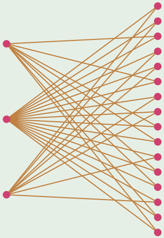
$\mathfrak{N}_{\langle(1\ 2)\rangle,S_3}$

Diamter 3



$\mathfrak{N}_{S_3,S_4}$

Diamter 4



$\mathfrak{N}_{A_4',S_4\times\mathbb{Z}_2}$

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Let H be a subgroup of a group G. Then H is called a TI-subgroup of G if $H \cap H^g = 1$ for all $g \in G \setminus N_G(H)$.

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Theorem

Let H be a non-normal subgroup of a group G. Then $\mathrm{diam}(\mathfrak{N}_{H,G}) = 2$ if and only if $\mathfrak{N}_{H,G}$ is a complete bipartite graph if and only if H/H_G is a TI-subgroup of G/H_G.

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Example

If G is a Frobenius group with complement H, then H is a TI-subgroup of G and hence $\mathrm{diam}(\mathfrak{N}_{H,G}) = 2$.

Diameter

Theorem

Let H be a non-normal subgroup of a group G. Then $\mathrm{diam}(\mathfrak{N}_{H,G}) = 4$ if and only if $G = A(G, H, h) \cup A(G, H, k)$ for some $h, k \in H \setminus H_G$.

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Corollary

If H is cyclic subgroup of a group G such that H/H_G is not a TI-subgroup of G/H_G, then $\operatorname{diam}(\mathfrak{N}_{H,G}) = 3$.

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Corollary

If H is a non-abelian subgroup of order pq of a group G such that H/H_G is not a TI-subgroup of G/H_G, then $\mathrm{diam}(\mathfrak{N}_{H,G}) = 3$.

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Definition[1]

A group G is a Krutik group is $A(G, H, h)$ is a subgroup of G for all subgroups H of G and elements h of H.

[1] B. A. Krutik, Über einige Eigenschaften der endlichen Gruppen, *Rec. Math. Mat. Sbornik N. S.* **10**(52) (1942), 239–247.

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3. Frobenius groups $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_n$ in which the complements act irreducibly on the kernels are Krutik groups.

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Theorem (Brandl and Deaconescu, 1993[2])

Let G be a Krutik group. Then

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Theorem (Brandl and Deaconescu, 1993[2])

Let G be a Krutik group. Then

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Corollary

If G is a Krutik group, then $\mathrm{diam}(\mathfrak{N}_{H,G}) = 3$ for all non-normal subgroup H of G such that H/H_G is not a TI-subgroup of G/H_G.

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Let H be a non-normal subgroup of a group G. Then $\mathfrak{N}_{H,G}$ is planar *if and only if*

1. $H \cong \mathbb{Z}_2$ *or* $\mathbb{Z}_3$;
2. $H \cong \mathbb{Z}_4$ *or* $\mathbb{Z}_2 \times \mathbb{Z}_2$ *such that* $H \cap Z(G) \neq 1$;

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3. $H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ *and* $H \cap Z(G) = 1$. *If* $H \setminus \{1\} = \{h, k, l\}$, *then*

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3. $H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ *and* $H \cap Z(G) = 1$*. If* $H \setminus \{1\} = \{h, k, l\}$*, then*
    1. *if* h, k, l *are pairwise non-conjugate, then* $|h^G| = |k^G| = |l^G| = 2$*,* $N_G(H) = C_G(H)$ *and* $[G : N_G(H)] = 4$*;*

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    2. *if* h, k *are conjugate and* l *is not conjugate to* h *and* k, *then*
        1. *if* $|l^G| = 2$, *then either* $|h^G| = [G : N_G(H)] = 4$, *or* $|h^G| = 6$ *and* $[G : N_G(H)] = 12$,

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    1. *if* h, k, l *are pairwise non-conjugate, then* $|h^G| = |k^G| = |l^G| = 2$, $N_G(H) = C_G(H)$ *and* $[G : N_G(H)] = 4$;
    2. *if* h, k *are conjugate and* l *is not conjugate to* h *and* k, *then*
        1. *if* $|l^G| = 2$, *then either* $|h^G| = [G : N_G(H)] = 4$, *or* $|h^G| = 6$ *and* $[G : N_G(H)] = 12$,
        2. *if* $|l^G| = 3$, *then* $|h^G| = [G : N_G(H)] = 3$, *or* $|h^G| = 4$ *and* $[G : N_G(H)] = 6$,

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Let H be a non-normal subgroup of a group G. Then $\mathfrak{N}_{H,G}$ is planar if and only if

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3. $H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ *and* $H \cap Z(G) = 1$. *If* $H \setminus \{1\} = \{h, k, l\}$, *then*
   1. *if* h, k, l *are pairwise non-conjugate, then* $|h^G| = |k^G| = |l^G| = 2$, $N_G(H) = C_G(H)$ *and* $[G : N_G(H)] = 4$;
   2. *if* h, k *are conjugate and* l *is not conjugate to* h *and* k, *then*
      1. *if* $|l^G| = 2$, *then either* $|h^G| = [G : N_G(H)] = 4$, *or* $|h^G| = 6$ *and* $[G : N_G(H)] = 12$,
      2. *if* $|l^G| = 3$, *then* $|h^G| = [G : N_G(H)] = 3$, *or* $|h^G| = 4$ *and* $[G : N_G(H)] = 6$,
      3. *if* $|l^G| = 4$, *then* $|h^G| = 4$ *and* $[G : N_G(H)] = 8$.

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Let H be a non-normal subgroup of a group G. Then $\mathfrak{N}_{H,G}$ is planar if and only if

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2. $H \cong \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ such that $H \cap Z(G) \neq 1$;
3. $H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $H \cap Z(G) = 1$. If $H \setminus \{1\} = \{h, k, l\}$, then
    1. if h, k, l are pairwise non-conjugate, then $|h^G| = |k^G| = |l^G| = 2$, $N_G(H) = C_G(H)$ and $[G : N_G(H)] = 4$;
    2. if h, k are conjugate and l is not conjugate to h and k, then
        1. if $|l^G| = 2$, then either $|h^G| = [G : N_G(H)] = 4$, or $|h^G| = 6$ and $[G : N_G(H)] = 12$,
        2. if $|l^G| = 3$, then $|h^G| = [G : N_G(H)] = 3$, or $|h^G| = 4$ and $[G : N_G(H)] = 6$,
        3. if $|l^G| = 4$, then $|h^G| = 4$ and $[G : N_G(H)] = 8$.
    3. if h, k, l are pairwise conjugate, then $|h^G| = [G : N_G(H)] = 7$, $|h^G| = 8$ and $[G : N_G(H)] = 16$, or $|h^G| = 6$ and $[G : N_G(H)] = 4$.

Thank you for your attention!