Groups with perfect Cayley groups

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Definition

A Cayley graph $\mathrm{Cay}(G, S)$ of a group G with respect to a subset $S = S^{-1}$ of $G \setminus \{1\}$ is a graph with vertex set G whose edges are $\{g, sg\}$ for all $g \in G$ and $s \in S$.

Definition

A group G is Cayley $\mathcal{P}$-group (resp. minimal Cayley $\mathcal{P}$-group) if all its Cayley graphs (resp. minimal Cayley graphs) satisfy the property $\mathcal{P}$.

Theorem (Babai and Sós[1]; Godsil and Imrich[2])

For every finite graph Γ, the order of a Cayley Γ-free group is bounded above by $(2+\sqrt{3})|\Gamma|^3$.

[1] L. Babai and V. T. Sós, Sidon sets in groups and induced subgraphs of Cayley graphs, *European J. Combin.* **6** (1985), 101–114.

[2] C. D. Godsil and W. Imrich, Embedding graphs in Cayley graphs, *Graphs Combin.* **3** (1987), 39–43.

[3] L. Babai, Chromatic number and subgraphs of Cayley graphs, *Theory and applications of graphs (Proc. Internat. Conf., Western Mich. Univ., Kalamazoo, Mich., 1976)*, pp. 10–22.

[4] J. Spencer, What's not inside a Cayley graph, *Combinatorica* **3**(2) (1983), 239–241.

Theorem (Babai and Sós[1]; Godsil and Imrich[2])

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Theorem

The following graphs cannot be embedded as induced subgraph in any minimal Cayley graph:

- $K_4 \setminus e$ *(diamond) and* $K_{3,5}$ *(Babai [3]);*
- *an infinite class of graphs (Spencer [4]).*

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[2] C. D. Godsil and W. Imrich, Embedding graphs in Cayley graphs, *Graphs Combin.* **3** (1987), 39–43.

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Conjecture (Lovász, 1969)

All connected Cayley graphs are Hamiltonian, or equivalently all finite groups are Cayley Hamiltonian groups.

[1]M. Jixiang and H. Qiongxiang, Almost all Cayley graphs are Hamiltonian, *Acta Math. Sinica, New Ser.* **12**(2) (1996), 151–155

Conjecture (Lośvasz, 1969)

All connected Cayley graphs are Hamiltonian, or equivalently all finite groups are Cayley Hamiltonian groups.

Theorem (Jixiang and Qiongxiang[1])

Finite groups are almost Cayley Hamiltonian groups.

[1] M. Jixiang and H. Qiongxiang, Almost all Cayley graphs are Hamiltonian, *Acta Math. Sinica, New Ser.* **12**(2) (1996), 151–155

Theorem

The following groups are Cayley Hamiltonian groups:

- *Abelian groups;*
- *Hamiltonian groups (Alspach and Qin [1], 2001);*
- *p-groups (Witte [2], 1986);*
- *Dihedral groups of order divisible by 4 (Alspach, Chen and Dean [3], 2010);*
- *Every group G with $G' = [G, G] \cong C_{p^n}$ (Keating and Witte [4], 1982);*
- *Groups of order kp ($24 \neq k \leq 31$), kp^2 (≤ 4), kp^3 ($k \leq 2$), kpq ($k \leq 5$), and pqr (Kutnar, Marušič, Morris, Morris and Šparl [5], 2012).*

[1] B. Alspach and Y. S. Qin, Hamilton-connected Cayley graphs on Hamiltonian groups, *European J. Combin.* **22** (2001), 777–787.

[2] D. Witte, Cayley digraphs of prime-power order are Hamiltonian, *J. Combin. Theory Ser. B* **40** (1986), 107–112.

[3] B. Alspach, C. C. Chen and M. Dean, Hamilton paths in Cayley graphs on generalized dihedral groups, *Ars Math. Contemp.* **3** (2010), 29–47.

[4] K. Keating and D. Witte, On Hamilton cycles in Cayley graphs in groups with cyclic commutator subgroup, *Cycles in graphs (Burnaby, B.C., 1982)*, North-Holland Mathematics Studies 115 (North-Holland, Amsterdam, 1985) 89–102.

[5] K. Kutnar, D. Marušič, D. Morris, J. Morris and P. Šparl, Hamiltonian cycles in Cayley graphs of small order, *Ars Math Contemp.* **5** (2012), 27–71.

Theorem (Abdollahi and Jazaeri[2], 2014; Ahmady, Bell and Mohar[3], 2014)

A finite group is Cayley integral if and only if it is isomorphic to one of the following groups:

(1) *an abelian group of exponent* 1, 2, 3, 4 *or* 6;

(2) S_3;

(3) $C_3 \rtimes C_4 = \langle x, y : x^3 = y^4 = 1, x^y = x^{-1} \rangle$;

(4) $Q_8 \times C_2^n$ $(n \geq 0)$.

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[1] F. Harary and A. J. Schwenk, Which Graphs Have Integral Spectra? in: Lecture Notes in Mathematics, 406, Springer, 1974, 45–51.

[2] A. Abdollahi and M. Jazaeri, Groups all of whose undirected Cayley graphs are integral, *European J. Combin.* **38** (2014), 102–109.

[3] A. Ahmady, J. Bell and B. Mohar, Integral Cayley graphs and groups, *SIAM J. Discrete Math.* **28**(2) (2014), 685–701.

Lemma

Let G be a finite group. A connected Cayley graph $\mathrm{Cay}(G, S)$ is bipartite if and only if $[G : \langle S^2 \rangle] = 2$ and $S \subseteq G \setminus \langle S^2 \rangle$.

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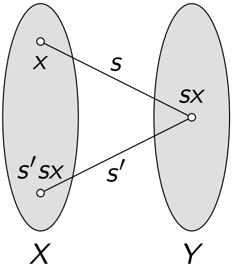

$$\langle S^2\rangle X = X$$

$$\langle S^2\rangle Y = Y$$

$$[G:\langle S^2\rangle]=2$$

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G =2-group:

- $X = \{x_1, \ldots, x_n\}$ (a minimal generating set of G);
- $S = X \cup X^{-1}$;
- $H = \langle \Phi(G), x_1x_2, \ldots, x_1x_n \rangle$;

$\Rightarrow$ $H = \langle S^2 \rangle$ and $S \subseteq G \setminus H$;

$\Rightarrow$ $\mathrm{Cay}(G, S)$ =bipartite.

Definition

- A proper n-coloring of a graph Γ is an assignment of n colors to vertices of Γ such that adjacent vertices have distinct colors;
- The chromatic number $\chi(\Gamma)$ of Γ is the minimum number of colors to color Γ properly;
- The clique number $\omega(\Gamma)$ of Γ is the maximum size of a complete subgraph of Γ;
- A graph Γ is perfect if $\chi(\Gamma') = \omega(\Gamma')$ for every induced subgraph Γ' of Γ.

Theorem (Strong Perfect Graph Theorem[1])

A graph Γ is perfect if and only if neither Γ nor Γ^c has an induced odd cycle of length ≥ 5.

[1] M. Chudnovsky, N. Robertson, P. Seymour and R. Thomas, The strong perfect graph theorem, *Ann. of Math. (2)* **164**(1) (2006), 51–229.

Notation

Suppose G is a finite group, $H \leq G$ and $g \in G$:

- $\overline{G} = G/\Phi(G)$;
- $\overline{H} = H\Phi(G)/\Phi(G)$;
- $\overline{g} = g\Phi(G)$.

[1] M. Farrokhi D. G., Finite groups with a given Frattini factor group, *In preparation*.

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Theorem (MFDG[1])

Let G be a finite solvable group and P be a Sylow p-subgroup of G. If either $\overline{P} \trianglelefteq G$ or $\overline{P}$ is cyclic, then $d(P) = d(\overline{P})$.

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Definition

The Hughes-Thompson subgroup of a group G with respect to a prime p is defined as

$$H_p(G) := \langle g \in G : |g| \neq p \rangle.$$

[1] D. R. Hughes and J. G. Thompson, The H_p-problem and the structure of H_p-groups, *Pacific J. Math.* **9** (1959), 1097–1101.

[2] E. G. Straus and G. Szekeres, On a problem of D. R. Hughes, *Proc. Amer. Math. Soc.* **9**(1) (1958), 157–158.

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Theorem

Let G be a finite group and $p \in \pi(G)$. If

- *G is not a p-group (Hughes and Thompson [1]), or*
- *G is a p-group ($p = 2, 3$) (Straus and Szekeres [2]),*

then

$$H_p(G) = 1, \quad [G : H_p(G)] = p, \quad \text{or} \quad H_p(G) = G.$$

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Definition

Let G be a group and $N \leq H \leq G$ with $N \trianglelefteq G$. Then H/N is a Frattini factor of G if

$$\frac{H}{N} \subseteq \Phi\left(\frac{G}{N}\right).$$

[1] A. Luccini, The largest size of a minimal generating set of a finite group, *Arch. Math. (Basel)* **101** (2013), 1–8.

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Theorem (Luccini[1])

The largest size of a minimal generating set of a finite solvable group equals the number of non-Frattini factors of a chief series of the group.

[1] A. Luccini, The largest size of a minimal generating set of a finite group, *Arch. Math. (Basel)* **101** (2013), 1–8.

Lemma

Let $G = \langle g \rangle$ *be a cyclic group. Then*

(1) $\mathrm{Cay}(G, \{g^{\pm 2}, g^{\pm 3}\})$ *has an induced* 5*-cycle for* $|g| \geq 10$.

(2) $\mathrm{Cay}(G, \{g^{\pm 1}, g^{\pm 4}\})$ *has an induced* 5*-cycle for* $|g| \geq 8$.

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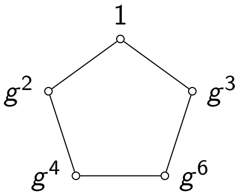
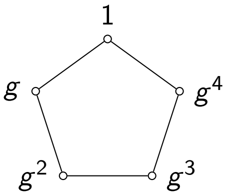


(1)



(2)

Notation

- $A_4 = \langle x, y : x^2 = y^3 = (xy)^3 = 1 \rangle$.
- $E_3 = \langle x, y : x^3 = y^3 = [y, x, y] = [x, y, x] = 1 \rangle$.
- $S_p(G)$ is a Sylow p-subgroup of G.

Theorem

A finite group G is a minimal Cayley perfect group if and only if

- *G is a 2-group, or*
- *G is isomorphic to C_3, C_6, S_3, $C_3 \times C_3$, A_4 or E_3.*

C_3	$C_6,\ S_3$	$C_3 \times C_3$	A_4	E_3
$\clubsuit$	$\clubsuit$	$\clubsuit$	$\clubsuit$	$\clubsuit$
	$\spadesuit$		$\spadesuit$	

All possible Cayley graphs

$G \neq 2$-group is a minimal Cayley perfect group:

$\Rightarrow$ $G \setminus \Phi(G) = \{\text{elements of orders } 2^k, 2^k \cdot 3 \text{ and } 3\}$;

- Q :=Sylow 3-subgroup of G;
- $\exp(Q) > 3$;

$\Rightarrow$ $1 \subset H_3(Q) \subseteq S_3(\Phi(G)) \subset Q$;

$\Rightarrow$ $S_3(\Phi(G)) = H_3(Q)$;

$\Rightarrow$ $d(Q) = d(\overline{Q}) = 1$;

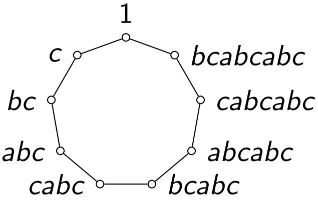
$\Rightarrow$ $Q \cong C_3$ (※);

$\Rightarrow$ $\exp(Q) = 3$.

(1) $G = 3$-group:

- $d(G) \geq 3$;
- $X = \{a, b, c, \ldots\}$ = a m.g.s. of G;

$\Rightarrow$ $\mathrm{Cay}(G, X \cup X^{-1})$ has an induced 9-cycle arising from the relation $(abc)^3 = 1$ (※);



$\Rightarrow$ $d(G) \leq 2$;

$\Rightarrow$ $G \cong C_3$, $C_3 \times C_3$ or E_3.

(2) G is a $\{2,3\}$-group:

- $\mathcal{C} = \{X : X \cong C_6, S_3$ or $A_4\}$;

$\Rightarrow$ $X \in \mathcal{C}$ whenever $|X| \in \{6, 12\}$, $S = S^{-1} \ni s$ of order 3 and $\mathrm{Cay}(X, S) =$perfect;

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Suppose $G \notin \mathcal{C}$ and $|G|$ =minimum:

- l_f =the number of non-Frattini factors in a chief series of G;

$l_f = 2$:

$\Rightarrow$ $\overline{G} \cong C_6$, S_3 or A_4;

(i) $\overline{G} \cong C_6$:

$\Rightarrow$ $G = \langle x \rangle$ is cyclic;
$\Rightarrow$ $\mathrm{Cay}(G, \{x^{\pm 2}, x^{\pm 3}\})$ has an induced 5-cycle as $|G| > 6$ (※).

(ii) $\overline{G} \cong S_3$:

$\Rightarrow$ $G = \langle x, y : x^3 = y^{2^k} = 1, x^y = x^{-1} \rangle$ $(k \geq 1)$;
$\Rightarrow$ $\mathrm{Cay}(G, \{x^{\pm 1}, y^{\pm 1}\})$ has an induced $(2^k + 3)$-cycle determined by $y^{2^k - 2} xyx^{-1} yx = 1$ as $k > 1$ (※).

(iii) $\overline{G} \cong A_4$:

$\Rightarrow$ $\overline{G} = \langle \overline{x}, \overline{y} : \overline{x}^2 = \overline{y}^3 = (\overline{xy})^3 = \overline{1} \rangle$;

$\Rightarrow$ one can assume that $(xy)^3 = 1$ or $x^{y^{-1}} x^y x = 1$;

- $|x| = 2^m > 2$;
- $\langle a, b : a^{2^m} = b^{2^m} = (ab)^{2^m} = 1 \rangle$ is infinite for $m > 1$;

$\Rightarrow$ there is a relation $w = 1$ (of minimum length) in x, x^y independent of $x^{2^m} = (x^y)^{2^m} = (x^y x)^{2^m} = 1$;

- $w = x^{a_1 y} x^{b_1} \cdots x^{a_k y} x^{b_k}$ with $0 < a_i, b_i < 2^m$ and $(a_1, b_1) \neq (1, 1)$;
- $w' = x^{-y^{-1}} x^{-1} x^{(a_1 - 1)y} x^{b_1} x^{a_2 y} x^{b_2} \cdots x^{a_k y} x^{b_k}$;

$\Rightarrow$ either $w = 1$ or $w' = 1$ determines an induced odd cycle in $\mathrm{Cay}(G, \{x^{\pm 1}, y^{\pm 1}\})$ of length ≥ 7 (※).

$l_f \geq 3$:

- Let
$$\Phi(G) = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{l-1} \trianglelefteq G_l = G;$$
- $M = G_{l-1}$;
- G_{n_i}/G_{n_i-1} =non-Frattini factor ($i = 1, \ldots, l_f$);

$\Rightarrow$ $X = \{x_1, \ldots, x_{l_f}\}$ =m.g.s. of G ($x_i \in G_{n_i} \setminus G_{n_i-1}$ is a p-element, $i = 1, \ldots, l_f$);

- $x_i, x_j \in S_2(G)$ or $S_3(G)$ if $x_i, x_j = p$-element ($p = 2, 3$);
- $Y_i = X \setminus \{x_i\}$ for $i = 1, \ldots, l_f$;

$\Rightarrow$ $x_i \in Y_{j_i}$ (containing elements of even and odd orders) for all $i = 1, \ldots, l_f$;

$\Rightarrow$ $\mathrm{Cay}(\langle Y_{j_i}\rangle, Y_i \cup Y_{j_i}^{-1})$ =perfect;

$\Rightarrow$ $\langle Y_{j_i}\rangle \cong C_6$, S_3 or A_4;

$\Rightarrow$ $|x_i|$ is prime for all $i = 1, \ldots, l_f$;

Claim: $l_f = l = 3$:

- $Y_i \ni$ elements of even and odd for $i = 1$ or 2;

$\Rightarrow$ $G = G_{n_2}\langle Y_i\rangle$ implies $G/G_{n_2} \cong \langle Y_i\rangle/(G_{n_2} \cap \langle Y_i\rangle) \cong C_2$ or C_3;

$\Rightarrow$ $n_2 = n_3 - 1$ and $n_3 = l$;

$\Rightarrow$ $l_f = 3$;

$\Rightarrow$ $\Phi(G/G_1) = G_{l-2}/G_1$ and $G/G_1 = \langle G_1x_2, G_1x_3\rangle$;

- $(|x_2|, |x_3|) = (2,3)$ or $(3,2)$;
  - $\Rightarrow$ $\langle x_2, x_3\rangle \in \mathcal{C}$ so that $l = 3$;
- $(|x_2|, |x_3|) = (2,2)$;
  - $\Rightarrow$ G/G_1 =a dihedral group;
    - $\{x_1, x_2x_3, x_3\}$ is a m.g.s. of G;
  - $\Rightarrow$ $(x_2x_3)^2 = 1$;
  - $\Rightarrow$ $G_{l-2} = \langle G_1, (x_2x_3)^2\rangle = G_1$ so that $l = 3$;

- $(|x_2|, |x_3|) = (3, 3)$;
  $\Rightarrow$ $G/G_1 \cong C_3 \times C_3$ or E_3;
  - $G/G_1 \cong E_3$:
    - $[x_1, x_2] = [x_1, x_3] = 1$ implies $G = \langle x_1 \rangle \times \langle x_2, x_3 \rangle$ so that $[x_2, x_3] \in \Phi(G)$ (※);

    $\Rightarrow$ $[x_1, x_3] \neq 1$, say;
    $\Rightarrow$ $\{x_1, x_2x_3, x_3\}$ is a m.g.s. of G if $[x_1, x_2x_3] \neq 1$;
    $\Rightarrow$ $\langle x_1, x_2x_3 \rangle \cong A_4$;
    $\Rightarrow$ $\mathrm{Cay}(G, X \cup X^{-1})$ has an induced 9-cycle determined by $(x_1x_2x_3)^3 = 1$ (※);
    $\Rightarrow$ $[x_1, x_2x_3] = 1$;
    $\Rightarrow$ we may assume $[x_1, x_2] = 1$ $(x_2 \mapsto x_2x_3)$;
    $\Rightarrow$ $[x_1, x_2x_3^{-1}] \neq 1$, $\{x_1, x_2, x_2x_3^{-1}\}$ is a m.g.s. of G, and

$$x_1x_1^{x_3^{-1}x_2^{-1}} = (x_1x_1^{x_3^{-1}})^{x_2^{-1}} = x_1^{x_3x_2^{-1}} = x_1^{(x_2x_3^{-1})^{-1}} = x_1x_1^{x_2x_3^{-1}} = x_1x_1^{x_2^{-1}x_3^{-1}};$$

    $\Rightarrow$ $[x_2, x_3]$ commutes with x_1;
    $\Rightarrow$ $G = \langle x_1, x_1^{x_3} \rangle \rtimes \langle x_2, x_3 \rangle$;
    $\Rightarrow$ $[x_2, x_3] \in \Phi(G)$ (※);
    $\Rightarrow$ $G/G_1 \cong C_3 \times C_3$. Thus $l = 3$;

Structure of $\Phi(G)$:

- x_u =involution;

$\Rightarrow$ gx_u=involution for all $g \in \Phi(G)$;

$\Rightarrow$ $g^{x_u} = g^{-1}$ for all $g \in \Phi(G)$;

$\Rightarrow$ $\Phi(G)$=abelian;

- $\langle x_u, x_v \rangle \cong A_4$ for some x_v;

$\Rightarrow$ $g^{x_u^{x_v^{\pm 1}}} = g^{-1}$ for all $g \in \Phi(G)$ ($x_u \mapsto x_u^{x_v^{\pm 1}}$);

$\Rightarrow$ $g^{-1} = g^{x_u^{x_v^{-1}}} = g^{x_u x_u^{x_v}} = g$;

$\Rightarrow$ $\Phi(G)$ =elementary abelian 2-group.

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$\Rightarrow \mathrm{Cay}(H, Y_3 \cup Y_3^{-1})$ =perfect;

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$H =$ 2-group;

$\Rightarrow$ $[G : M] = 3$, $M = S_2(G)$ and $\Phi(G) = \Phi(M)$;

$\Rightarrow$ $\overline{M}$ =elementary abelian 2-group;

$\Rightarrow$ $\langle Y_i \rangle \cong C_6$ or A_4 for $i = 1, 2$;

- $\langle Y_1 \rangle \cong \langle Y_2 \rangle \cong C_6$:
  - $\Rightarrow$ $M = H$, $G/\Phi(G) \cong C_6 \times C_2$;
  - $\Rightarrow$ G =nilpotent;
    - $M \setminus \Phi(M) \ni$ an element of order ≥ 4 implies $G \setminus \Phi(G) \ni$ an element x of order ≥ 12 (※);
  - $\Rightarrow$ $M \setminus \Phi(M)$ contains only involutions;
  - $\Rightarrow$ $M \cong C_2 \times C_2$ so that $G \cong C_6 \times C_2$ (※);

- $\langle Y_i \rangle \cong A_4$ for some $i = 1, 2$:
  - $\Rightarrow$ $\Phi(G)$ =elementary abelian 2-group;
  - We show that $\langle Y_j \rangle \cong C_6$ for $j \in \{1, 2\} \setminus \{i\}$:
    - $\langle Y_1 \rangle = \langle x_2, x_3 \rangle \cong A_4$;
    - $\Rightarrow$ $\{x_1, x_2, x_2x_3\}$ is a m.g.s of G and $|x_2x_3| = 3$;
    - $\Rightarrow$ $\mathrm{Cay}(G, X \cup X^{-1})$ has an induced 9-cycle given by $(x_1x_2x_3)^3 = 1$ if $[x_1, x_2x_3] \neq 1$(※);
    - $\Rightarrow$ $[x_1, x_2x_3] = 1$;
    - $\Rightarrow$ $[x_1, x_3] = 1$ and $\langle Y_2 \rangle = \langle x_1, x_3 \rangle \cong C_6$ $(x_3 \mapsto x_2x_3)$;
  - $\Rightarrow$ Apply $(x_i, x_j) \mapsto (gx_i, gx_j)$ for $g \in \Phi(G)$ yields

$$(gx_i)^{x_3^{-1}} = (gx_i)(gx_i)^{x_3} \quad \text{and} \quad (gx_j)^{x_3} = (gx_j);$$

  - $\Rightarrow$ $g^{x_3^{-1}} = gg^{x_3} = 1$;
  - $\Rightarrow$ $\Phi(G) = 1$;
  - $\Rightarrow$ $G \cong A_4 \times C_2$;
  - $\Rightarrow$ $\mathrm{Cay}(G, \{a^{\pm 1}, b^{\pm 1}\})$ has an induced 7-cycle arising from $b^{-1}abab^2a^{-1} = 1$ $(a := x_3^{x_i}$ and $b := x_jx_3)$ (※).

$H = $ 3-group:

- $\Rightarrow$ $[G : M] = 2$ and $M = S_3(G)$;
- $\Rightarrow$ $\langle Y_i \rangle \cong C_6$ or S_3 for $i = 1, 2$;
- $\Rightarrow$ $x_i^{x_3} = x_i^{\epsilon_i}$ with $\epsilon_i = \pm 1$ for $i = 1, 2$;
- $\Rightarrow$ $|M| = |H| = 9$ or 27;
- $\Rightarrow$ $\mathrm{Cay}(G, X \cup X^{-1})$ has an induced 7-cycle arising from $x_3 x_2^{\epsilon_2} x_1^{-\epsilon_1} x_3 x_2 x_1 x_2 = 1$ if $[x_1, x_2] = 1$ (※);
- $\Rightarrow$ $\mathrm{Cay}(G, X \cup X^{-1})$ has an induced 11-cycle arising from $x_3 x_2^{\epsilon_2} x_1^{-\epsilon_1} x_3 x_2 x_1 x_2 x_1^{-1} x_2^{-1} x_1 x_2 = 1$ if $[x_1, x_2] \neq 1$ (※).

$H = \{2, 3\}$-group:

$\Rightarrow$ $H \in \mathcal{C}$ so that $H \cong C_6$, S_3 or A_4;

- $|x_2| = |x_3| = 2$:
  - $\Rightarrow$ $[x_2, x_3] = 1$ and $x_1^{x_2}, x_1^{x_3} \in \langle x_1 \rangle$;
  - $\Rightarrow$ $G \cong C_6 \times C_2$ or $S_3 \times C_2$ (※);
- $|x_2| = |x_3| = 3$:

$\Rightarrow$ x_1 commutes with $[x_2, x_3]$ and x_3, say;

- $[x_1, x_2] = 1$ implies $G = \langle x_1 \rangle \times \langle x_2, x_3 \rangle$;
  - $\Rightarrow$ $\mathrm{Cay}(G, X \cup X^{-1})$ has an induced 9-cycle arising from $x_1 x_2 x_3 x_2^{-1} x_1 x_2^{-1} x_3 x_2 x_3 = 1$ (※);
- $[x_1, x_2] \neq 1$;
  - $\Rightarrow$ $\mathrm{Cay}(G, X \cup X^{-1})$ has an induced 9-cycle arising from $(x_1 x_2 x_3)^2 x_1 x_3 x_2 = 1$ if $[x_2, x_3] = 1$ (※);
  - $\Rightarrow$ $\mathrm{Cay}(G, X \cup X^{-1})$ has an induced 13-cycle arising from $(x_1 x_2 x_3)^2 x_2^{-1} x_3^{-1} x_2 x_3 x_1 x_3 x_2 = 1$ if $[x_2, x_3] \neq 1$ (※).

- $|x_2| \neq |x_3|$:
  - $[x_2, x_3] = 1$;
  - $\Rightarrow$ $H = p$-group with $p \in \{2, 3\}$ $(x_2 \longleftrightarrow x_3)$ (※);
  - $\Rightarrow$ $[x_2, x_3] \neq 1$;
  - $[x_1, x_2] = 1$;
  - $\Rightarrow$ $\langle x_2, x_2^{x_3}, x_2^{x_3^{-1}} \rangle \trianglelefteq G$ is elementary abelian (?);
  - $\Rightarrow$ $|x_2| = |x_3|$ $(x_1 \longleftrightarrow x_2)$ (※);
  - $\Rightarrow$ $[x_1, x_2] \neq 1$;
  - $\Rightarrow$ $\{\langle x_1, x_2\rangle, \langle x_2, x_3\rangle\} = \{S_3, A_4\}$;
  - $\Rightarrow$ $\Phi(G)$=elementary abelian 2-group;

- $(|x_1|, |x_2|, |x_3|) = (2, 3, 2)$:
  $\Rightarrow$ $\langle x_1, x_2\rangle \cong A_4$ and $\langle x_2, x_3\rangle \cong S_3$;
  $\Rightarrow$ $|x_1x_3| = 2^m$ ($\langle x_1, x_3\rangle$=dihedral 2-group);
  $\Rightarrow$ $\mathrm{Cay}(G, X \cup X^{-1})$ has an induced $(2^{m+1} + 5)$-cycle arising from $(x_1x_2)^2(x_3x_1)^{2^m-1}x_2x_3x_2^{-1} = 1$ (※);

- $(|x_1|, |x_2|, |x_3|) = (2, 3, 2)$:
  $\Rightarrow$ $\langle x_1, x_2\rangle \cong A_4$ and $\langle x_2, x_3\rangle \cong S_3$;
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- $(|x_1|, |x_2|, |x_3|) = (3, 2, 3)$:
  $\Rightarrow$ $\langle x_1, x_2\rangle \cong S_3$ and $\langle x_2, x_3\rangle \cong A_4$;
  $\Rightarrow$ $|x_1x_3| = 3$;
  - $Q = S_3(G) \ni x_2x_3$;
  - $y \in \Phi(G)\langle x_2, x_2^{x_3}\rangle = S_2(G)$ such that $x_1^y \in Q$;

  $\Rightarrow$ $(x_1x_2x_3)^3 = 1$ ($x_1 \mapsto x_1^y$);
  $\Rightarrow$ $\mathrm{Cay}(G, X \cup X^{-1})$ has an induced 9-cycle (※).

Corollary

Let G be a nontrivial finite group. Then G is a Cayley perfect group if and only if G is isomorphic to one of the groups C_2, C_3, C_4, $C_2 \times C_2$, S_3, C_6, $C_2 \times C_2 \times C_2$, $C_2 \times C_4$, D_8, Q_8 or $C_3 \times C_3$.

G	S	5-cycle
$\langle a\rangle \times \langle b\rangle,\ \lvert a\rvert = \lvert b\rvert = 4$	$\{a^{\pm 1}, b^{\pm 1}, a^2b^2\}$	$1, a^{-1}, (ab)^{-1}, ab, a, 1$
$\langle a, b : a^4 = b^4 = 1, [a, b] = a^2\rangle$	$\{a^{\pm 1}, b^{\pm 1}, a^2b^2\}$	$1, a^{-1}, (ba)^{-1}, ab, a, 1$
$\langle a\rangle \times \langle b\rangle \times \langle c\rangle,\ \lvert a\rvert = 4, \lvert b\rvert = \lvert c\rvert = 2$	$\{a^{\pm 1}, b, c, a^2bc\}$	$1, c, ca^{-1}, ba, b, 1$
$\langle a, b, c : a^4 = b^2 = c^2 = 1, a^c = a^{-1}b\rangle$	$\{a^{\pm 1}, b, c\}$	$1, c, ca^{-1}, ba, b, 1$
$\langle a, b, c : b^2 = c^2 = [a, b] = [a, c] = 1,$ $[b, c] = a^2\rangle$	$\{a^{\pm 1}, b, c, a^2bc\}$	$1, a^{-1}, a^{-1}c, ab, a, 1$
$\langle a, b : a^4 = b^2 = 1, a^b = a^{-1}\rangle \times \langle c\rangle,$ $\lvert c\rvert = 2$	$\{a^{\pm 1}, b, c, a^2bc\}$	$1, a^{-1}, a^{-1}c, ab, a, 1$
$\langle a, b : a^4 = 1, a^2 = b^2, a^b = a^{-1}\rangle \times \langle c\rangle,$ $\lvert c\rvert = 2$	$\{a^{\pm 1}, b^{\pm 1}, c, (a^2b^{-1}c)^{\pm 1}\}$	$1, a^{-1}, a^{-1}c, ab, a, 1$
$\langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle,$ $\lvert a\rvert = \lvert b\rvert = \lvert c\rvert = \lvert d\rvert = 2$	$\{a, b, c, d, abcd\}$	$1, a, ab, abc, abcd, 1$
$\langle a, b, c : a^3 = b^3 = [a, c] = [b, c] = 1,$ $c = [a, b]\rangle$	$\{a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, (a^{-1}bc)^{\pm 1}\}$	$1, a, ac, bc^{-1}, b, 1$
$\langle a, b : a^2 = b^3 = (ab)^3 = 1\rangle$	$\{a, b^{\pm 1}, a^b\}$	$1, a^b, b^a, ab, a, 1$

Corollary

Let G be a nontrivial finite group. Then G is a Cayley perfect group if and only if G is isomorphic to one of the groups C_2, C_3, C_4, $C_2 \times C_2$, S_3, C_6, $C_2 \times C_2 \times C_2$, $C_2 \times C_4$, D_8, Q_8 or $C_3 \times C_3$.

G	S	5-cycle
$\langle a\rangle \times \langle b\rangle,\ \lvert a\rvert = \lvert b\rvert = 4$	$\{a^{\pm1}, b^{\pm1}, a^2b^2\}$	$1, a^{-1}, (ab)^{-1}, ab, a, 1$
$\langle a, b : a^4 = b^4 = 1, [a, b] = a^2\rangle$	$\{a^{\pm1}, b^{\pm1}, a^2b^2\}$	$1, a^{-1}, (ba)^{-1}, ab, a, 1$
$\langle a\rangle \times \langle b\rangle \times \langle c\rangle,\ \lvert a\rvert = 4, \lvert b\rvert = \lvert c\rvert = 2$	$\{a^{\pm1}, b, c, a^2bc\}$	$1, c, ca^{-1}, ba, b, 1$
$\langle a, b, c : a^4 = b^2 = c^2 = 1, a^c = a^{-1}b\rangle$	$\{a^{\pm1}, b, c\}$	$1, c, ca^{-1}, ba, b, 1$
$\langle a, b, c : b^2 = c^2 = [a, b] = [a, c] = 1, [b, c] = a^2\rangle$	$\{a^{\pm1}, b, c, a^2bc\}$	$1, a^{-1}, a^{-1}c, ab, a, 1$
$\langle a, b : a^4 = b^2 = 1, a^b = a^{-1}\rangle \times \langle c\rangle, \lvert c\rvert = 2$	$\{a^{\pm1}, b, c, a^2bc\}$	$1, a^{-1}, a^{-1}c, ab, a, 1$
$\langle a, b : a^4 = 1, a^2 = b^2, a^b = a^{-1}\rangle \times \langle c\rangle, \lvert c\rvert = 2$	$\{a^{\pm1}, b^{\pm1}, c, (a^2b^{-1}c)^{\pm1}\}$	$1, a^{-1}, a^{-1}c, ab, a, 1$
$\langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle, \lvert a\rvert = \lvert b\rvert = \lvert c\rvert = \lvert d\rvert = 2$	$\{a, b, c, d, abcd\}$	$1, a, ab, abc, abcd, 1$
$\langle a, b, c : a^3 = b^3 = [a, c] = [b, c] = 1, c = [a, b]\rangle$	$\{a^{\pm1}, b^{\pm1}, c^{\pm1}, (a^{-1}bc)^{\pm1}\}$	$1, a, ac, bc^{-1}, b, 1$
$\langle a, b : a^2 = b^3 = (ab)^3 = 1\rangle$	$\{a, b^{\pm1}, a^b\}$	$1, a^b, b^a, ab, a, 1$

to be continued...

Thank You for Your Attention!