Finite groups with a given number of elements of each order

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$\mathbb{Z}_p \rtimes \mathbb{Z}_q$	$\{1,p,q\}$	$\{1,p-1,p(q-1)\}$

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Let G be a finite group whose order is divisible by a number n. Then $\sum_{d\mid n} w_d$ is divisible by n.

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- *If $p \in \omega(G)$ is prime, then $w_p \equiv -1 \pmod{p}$.*
- *If $d \in \omega(G) \setminus \{1\}$, then ω_d is odd if and only if $d = 2$.*

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Freud and Pálfy, 1996

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$\omega(p)$	Not known but contains $\{(p-1)p_1^{\alpha_1}\cdots p_n^{\alpha_n} : p_i^{\alpha_i}\equiv 1 \pmod p, i=1,\ldots,n\}$

A simple fact

If G is a group with $|w^*(G)| = 1$, then $|G| = 1$ or 2.

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(4) $H \times \mathbb{Z}_2$, *where H is a p-group of exponent $p > 2$.*

Preliminary results

Lemma

Let G be a finite group. Then $w(G) = \{1, p, q\}$ if and only if G is a Frobenius group whose kernel is a p-group of exponent p and complements are cyclic q-groups of order q, where p and q are distinct primes.

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Let G be a finite group. Then $w(G) = \{1, p, q, pq\}$ and $w^*(G) = \{1, m, n\}$ if and only if $G/Z(G)$ is a Frobenius group, $Z(G) \cong \mathbb{Z}_2$ and either $G \cong \mathbb{Z}_2 \times (\mathbb{Z}_p^k \rtimes \mathbb{Z}_2)$ or $G \cong \mathbb{Z}_2^k \rtimes \mathbb{Z}_p$.

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(2) *$G = O_{pqp}(G)$ is a 3-step group, $O_{pq}(G) = O_p(G) \rtimes \mathbb{Z}_q$ is a Frobenius group, $G/O_p(G) \cong \mathbb{Z}_q \rtimes \mathbb{Z}_p$ is a Frobenius group, $\exp(P) = p$ and $Q \cong \mathbb{Z}_q$,*

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Example

A p-groups G of exponent p^2 satisfies $|w^*(G)| = 3$.

Thank You for Your Attention!