Numbers and their sums of digits (in different bases)

M. Farrokhi D. G.

IASBS

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Definition

Let $B := \{b_i\}_{i\geqslant 0}$ be an increasing sequence of natural numbers with $b_0 = 1$. Then, every natural number n can be expressed uniquely as

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The sequence B is called a base and the number n in base B is represented as

$$(n)_B := (n_m, \ldots, n_1, n_0) \text{ or } \overline{n_m \cdots n_1 n_0}.$$

Zeckendorf's Theorem[1], 1972; Lekkerkerker[2], 1952

Every natural number can be written uniquely as the sum of non-consecutive Fibonacci numbers.

[1]E. Zeckendorf, Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas, *Bull. Soc. R. Sci. Lige* **41** (1972), 179–182.

[2]C. G. Lekkerkerker, Representation of natural numbers as a sum of Fibonacci numbers, *Simon Stevin* **29** (1952), 190–195.

Zeckendorf's Theorem[1], 1972; Lekkerkerker[2], 1952

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Example

If $F = \{F_i\}$ is the sequence of Fibonacci number as a base, then

$$(100)_F = \overline{10000101000}_F.$$

[1] E. Zeckendorf, Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas, *Bull. Soc. R. Sci. Lige* **41** (1972), 179–182.

[2] C. G. Lekkerkerker, Representation of natural numbers as a sum of Fibonacci numbers, *Simon Stevin* **29** (1952), 190–195.

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- Factorials and multinomials p-valuations

Theorem

Let $b \geqslant 2$ be a base, $d \geqslant 1$, and $B = (b_i)_{i \geqslant 0}$, where b_i is the remainder of b^i modulo d. Let $S_b^*(n) = \langle B, (n)_b \rangle$, for all $n \geq 1$. Then

$$n \overset{d}{\equiv} 0 \qquad \text{if and only if} \qquad S_b^*(n) \overset{d}{\equiv} 0.$$

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Example

A number $n = (n_i)_{10}$ is divisible by 7 if and only if

$$\cdots + \underbrace{-2n_{11} + \cdots + n_6} + \underbrace{-2n_5 - 3n_4 - n_3 + 2n_2 + 3n_1 + n_0}$$

is divisible by 7.

Corollary

If $b \geqslant 2$ and $d \mid b-1$, then

$$n \overset{d}{\equiv} 0 \qquad \textit{if and only if} \qquad S_b(n) \overset{d}{\equiv} 0.$$

Definition

Let $n \geqslant 1$ and p be a prime. The p-valuation of n is defined as

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Theorem

For any $n \geqslant 1$ and prime p, we have

$$\nu_p(n!) = \frac{n - S_p(n)}{p - 1}.$$

Kummer's Theorem[1], 1852

The largest power of a prime p dividing a multinomial is given by

$$\nu_p\left(\binom{n}{m_1,\ldots,m_k}\right)=\frac{1}{p-1}\left(\sum_{i=1}^{k}S_p(m_i)-S_p(n)\right)$$

[1]E. Kummer, Über die Ergänzungsstze zu den allgemeinen Reciprocitätsgesetzen, *J. Reine Angew. Math.* **44** (1852), 93–146.

Bush[1], 1940; Ballot[2], 2013

For any fixed base $b \geqslant 2$, we have

$$\frac{S_b(1) + \cdots + S_b(n)}{n} \sim \frac{(b-1)}{2\log b} \log n.$$

[1] L. E. Bush, An asymptotic formula for the average sum of the digits of integers, *Amer. Math. Monthly* **47**(3) (1940), 154–156.

[2] C. Ballot, On Zeckendorf and base b digit sums, *Fibonacci Quart.* **51**(4) (2013), 319–325.

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Ballot[2], 2013

For the Fibonacci base $F := \{F_i\}$, we have

$$\frac{S_F(1) + \cdots + S_F(n)}{n} \sim \frac{(\alpha - 1)}{\sqrt{5} \log \alpha} \log n,$$

where $\alpha = (1 + \sqrt{5})/2$.

[1] L. E. Bush, An asymptotic formula for the average sum of the digits of integers, *Amer. Math. Monthly* **47**(3) (1940), 154–156.

[2] C. Ballot, On Zeckendorf and base b digit sums, *Fibonacci Quart.* **51**(4) (2013), 319–325.

Katai[1], 1967; Shiokawa[2], 1974

For any $b \geqslant 2$, we have

$$\frac{S_b(p_1) + \cdots + S_b(p_{\pi(n)})}{n} \sim \frac{b-1}{2\log b}.$$

[1] I. Katai, On the sum of digits of prime numbers, *Ann. Univ. Sci. Budapest Rolando Eotvos nom. Sect. Math.* **10** (1967), 89–93.

[2] I. Shiokawa, On the sum of digits of prime numbers, *Proc. Japan Acad.* **50** (1974), 551–554.

Madritsch and Stoll[1], 2014

Let $b_1, b_2 \geqslant 2$ and $P_1, P_2 \in \mathbb{C}[x]$ be polynomials of degrees with $P_1(\mathbb{N}), P_2(\mathbb{N}) \subseteq \mathbb{N}$. Then

$$\frac{1}{N}\sum_{n=1}^{N}\frac{S_{b_1}(P_1(n))}{S_{b_2}(P_2(n))} \sim \frac{b_1-1}{b_2-1}\left(\frac{\log b_1}{\log b_2}\right)^{-1}\frac{r_1}{r_2}.$$

[1] M. G. Madritsch and T. Stoll, On a second conjecture of Stolarsky: the sum of digits of polynomial values, *Arch. Math. (Basel)* **102**(1) (2014), 49–57.

Mahler[1], 1927

The sequence

$$\left(\frac{1}{N}\sum_{n<N}(-1)^{S_2(n)}(-1)^{S_2(n+k)}\right)_{N\geqslant 1}$$

converges for all $k \in \mathbb{N}$, and its limit is different from zero for infinitely many k.

[1] K. Mahler, The spectrum of an array and its application to the study of the translation properties of a simple class of arithmetical functions. II: On the translation properties of a simple class of arithmetical functions, *J. Math. Phys. Mass. Inst. Techn.* **6** (1927), 158–163.

Newman[1], 1969

The sum

$$\sum_{n\leqslant N}(-1)^{S_2(3n)}$$

is always positive.

[1] D. J. Newman, On the number of binary digits in a multiple of three, *Proc. Amer. Math. Soc.* **21** (1969), 719–721.

Gelfond[1], 1968

Let $s, t, b \geqslant 2$, $s > s' \geqslant 0$, $t > t' \geqslant 0$, and $\gcd(t, b-1) = 1$. Then

$$\#\{0 \leqslant n < N \mid n \overset{s}{\equiv} s',\ S_b(n) \overset{t}{\equiv} t'\} = \frac{N}{st} + g(N),$$

where $g(N) = O_q(N^{\lambda})$ with $\lambda = \frac{1}{2\log q} \log \frac{q \sin(\pi/2m)}{\sin(\pi/2mq)} < 1$.

[1] A. O. Gelfond, Sur les nombres qui ont des propriétés additives et multiplicatives données, *Acta Arith.* **13** (1968), 259–265.

Mauduit and Rivat[2], 2010

Let $b, m \geqslant 2$ and r be integers and $d = \gcd(d, m-1)$. Then

$$\#\{\text{prime } p \leqslant x \mid S_b(p) \overset{m}{\equiv} r\} = \frac{d}{m}\pi(x; r, d) + O_{b,m}(x^{1-\sigma_{b,m}})$$

for some constant $\sigma_{b,m} > 0$.

[1] C. Mauduit and J. Rivat, Sur un problème de Gelfond: la somme des chiffres des nombres premiers, *Ann. of Math. (2)* **171**(3) (2010), 1591–1646.

Drmota and Larcher[1], 2001

Let $b_1, \dots, b_m \geqslant 2$ be pairwise coprime integers. Then the m-dimensional sequence

$$(\{\theta_1 S_{b_1}(n)\}, \dots, \{\theta_m S_{b_m}(n)\})$$

is equidistributed modulo in $[0,1]^m$ if and only if $\theta_1, \dots, \theta_m$ are irrationals.

[1] M. Drmota and G. Larcher, The sum-of-digits-function and uniform distribution modulo 1, *J. Number Theory* **89**(1) (2001), 65–96.

[2] C. Mauduit and J. Rivat, Sur un problème de Gelfond: la somme des chiffres des nombres premiers, *Ann. of Math. (2)* **171**(3) (2010), 1591–1646.

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Mauduit and Rivat[2], 2010

The sequence $\{\theta S_b(p)\}_{p \in \mathbb{P}}$ is equidistributed in $[0,1]$ if and only if θ is irrational.

[1] M. Drmota and G. Larcher, The sum-of-digits-function and uniform distribution modulo 1, *J. Number Theory* **89**(1) (2001), 65–96.

[2] C. Mauduit and J. Rivat, Sur un problème de Gelfond: la somme des chiffres des nombres premiers, *Ann. of Math. (2)* **171**(3) (2010), 1591–1646.

- $\mathcal{P}_b^k := \{p \in \mathbb{P} : S_b(p), \ldots, S_b^{(k)}(p) \in \mathbb{P}\}$ for all $k \geqslant 1$.

[1] G. Harman, Counting primes whose sum of digits is prime, *J. Integer Seq.* **15**(2) (2012), Article 12.2.2, 7 pp.

- $\mathcal{P}_b^k := \{p \in \mathbb{P} : S_b(p), \ldots, S_b^{(k)}(p) \in \mathbb{P}\}$ for all $k \geqslant 1$.
- $L_k(x) = \begin{cases} \log^{(k)} x, & x > \exp^{(k-1)}(e), \\ 1, & \text{o.w.}, \end{cases}$ for all $k \geqslant 1$.

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Harman[1], 2012

Let $b \geqslant 2$ and $k \geqslant 1$ be integers, and $x \geqslant 2$. Then

$$\sum_{\substack{p<x \\ p\in\mathcal{P}_b^k}} \frac{1}{p} = \left(\frac{b-1}{\varphi(b-1)}\right)^k L_{k+2}(x) + O(1).$$

[1] G. Harman, Counting primes whose sum of digits is prime, *J. Integer Seq.* **15**(2) (2012), Article 12.2.2, 7 pp.

Harman[1], 2012

The expansion, in any given base, of almost all real numbers contains infinitely many primes whose sum of digits are also prime

[1] G. Harman, Primes whose sum of digits is prime and metric number theory, *Bull. Lond. Math. Soc.* **44**(5) (2012), 1042–1049.

Cusick's conjecture, 2012

Whether

$$\mathrm{dens}\{n \geqslant 0 \mid S_2(n+t) \geqslant S_2(n)\} > \frac{1}{2}$$

for all $t \geqslant 0$?

[1]M. Drmota, M. Kauers, and L. Spiegelhofer, On a conjecture of Cusick concerning the sum of digits of n and $n+t$, *SIAM J. Discrete Math.* **30**(2) (2016), 621–649.

Cusick's conjecture, 2012

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Drmota, Kauers, and Spiegelhofer[1], 2016

Cusick's conjecture is true for a set of density one.

[1] M. Drmota, M. Kauers, and L. Spiegelhofer, On a conjecture of Cusick concerning the sum of digits of n and $n+t$, *SIAM J. Discrete Math.* **30**(2) (2016), 621–649.

Dartyge, Luca, and Stănică[1], 2009

For each natural number n, there exist infinitely many numbers k ($k \neq b^t$, $t \geqslant 0$), such that

$$S_b(n) = S_b(kn).$$

[1]C. Dartyge, F. Luca, and P. Stănică, On digit sums of multiples of an integer, *J. Number Theory* **129**(11) (2009), 2820–2830.

Adams-Watters and Ruskey[1], 2009

For any $b \geqslant 2$, we have

$$\sum_{n=1}^{\infty} S_b(n)x^n = \frac{1}{1-x}\sum_{m=0}^{\infty} \frac{x^{b^m} - bx^{m+1} + (b-1)x^{(b+1)b^m}}{(1-x^{b^m})(1-x^{b^{m+1}})}.$$

[1] F. T. Adams-Watters and F. Ruskey, Generating functions for the digital sum and other digit counting sequences, *J. Integer Seq.* **12**(5) (2009), Article 09.5.6, 9 pp.

Allouche and Shallit[1], 1988

For any $b \geqslant 2$, we have

$$\sum_{n=1}^{\infty} \frac{S_b(n)}{n(n+1)} = \frac{b}{b-1} \log b.$$

[1] J-P. Allouche and J. Shallit, Sums of digits and the Hurwitz zeta function,

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Allouche and Shallit[1], 1988

For any $b \geqslant 2$, we have

$$\sum_{n=1}^{\infty} S_b(n) \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right) = \frac{b^s - b}{b^s - 1} \zeta(s),$$

where s is a complex number with $\Re(s) > 0$.

[1] J-P. Allouche and J. Shallit, Sums of digits and the Hurwitz zeta function,

Shallit[1], 1985

For every $b \geqslant 2$, we have

$$\prod_{i=0}^{\infty} \frac{1+c_i}{1+d_i} = \frac{1}{\sqrt[k]{k}},$$

where c_i and d_i are such that

$$bi \leqslant c_i \quad \text{and} \quad d_i \leqslant b(i+1)$$

and

$$S_b(c_i) \overset{b}{\equiv} j-1 \quad \text{and} \quad S_b(d_i) \overset{b}{\equiv} j,$$

in which $1 \leqslant j < k$ is a fixed number.

[1] J. O. Shallit, On infinite products associated with sums of digits, *J. Number Theory* **21**(2) (1985), 128–134.

Stewart[1], 1980

For every Fibonacci number F_n, we have

$$S_b(F_n) > C_b \frac{\log n}{\log \log n}$$

where C_b is a positive constant, depending only on b.

[1] C. L. Stewart, On the representation of an integer in two different bases, *J. Reine Angew. Math.* **319** (1980), 63–72

Luca and Shparlinski[1], 2011

For almost all natural numbers n, we have

$$S_b\left(\binom{2n}{n}\right) \text{ and } S_b\left(\frac{1}{n+1}\binom{2n}{n}\right) > \varepsilon(n)\sqrt{\log n},$$

where ε is any function satisfying $\varepsilon(n) \mapsto 0$ is any function.

[1] F. Luca and I. E. Shparlinski, On the g-ary expansions of middle binomial coefficients and Catalan numbers, *Rocky Mountain J. Math.* **41**(4) (2011), 1291–1301

Luca and Shparlinski[1], 2010

For the n^{th}-Apéry number

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

we have

$$S_b(A_n) > C_b \left(\frac{\log n}{\log \log n} \right)^{\frac{1}{4}},$$

where C_b is a positive constant, depending only on b.

[1]F. Luca and I. E. Shparlinski, On the g-ary expansions of Apéry, Motzkin, Schröder and other combinatorial numbers, *Ann. Comb.* **14**(4) (2010), 507–524.

Knopfmacher and Luca[1], 2015

For almost all natural numbers n, we have

$$S_b\left(\sum_{k=0}^{n}\binom{n}{k}^{r_0}\binom{n+k}{k}^{r_1}\cdots\binom{n+km}{k}^{r_m}\right) > C_{b,r_0,\ldots,r_m}\frac{\log n}{\log\log n},$$

where $C_{b,r_0,\ldots,r_m}$ is a positive constant, depending only on $b, r_0, \ldots, r_m$, and $r_0, \ldots, r_m$ are nonnegative integers satisfying $r_0 > 0$ and $(r_0, \ldots, r_m) \neq (1, \ldots, 1)$.

[1] A. Knopfmacher and F. Luca, Digit sums of binomial sums, *J. Number Theory* **132**(2) (2012), 324–331.

Luca[1], 2012

For almost all natural numbers n, we have

$$S_b(P(n)) > \frac{\log n}{7 \log\log n},$$

where $P(n)$ is the partition function.

[1] F. Luca, On the number of nonzero digits of the partition function, *Arch. Math. (Basel)* **98**(3) (2012), 235–240.

Cilleruelo, Luca, Rué, and Zumalacárregui[1], 2013

For almost all natural numbers n, we have

$$S_b(B_n) > \frac{\log n}{60 \log b},$$

where B_n is the n^{th} Bell number.

[1] J. Cilleruelo, F. Luca, J. Rué, and A. Zumalacárregui, On the sum of digits of some sequences of integers, *Cent. Eur. J. Math.* **11**(1) (2013), 188–195.

Sanna[1], 2015

For each integer $n > e^e$, we have

$$S_b(n!) \text{ and } S_b(\mathrm{lcm}[1,\ldots,n]) > C_b \log n \log\log\log n,$$

where C_b is a positive constant, depending only on b.

[1] C. Sanna, On the sum of digits of the factorial, *J. Number Theory* **147** (2015), 836–841.

A word with no subwords of the form *axaxa* is called overlap-free.

[1] J-P. Allouche and J. Shallit, Sums of digits, overlaps, and palindromes, *Discrete Math. Theor. Comput. Sci.* **4**(1) (2000), 1–10.

[2] T. W. Cusick and L. C. Ciungu, Sum of digits sequences modulo m, *Theoret. Comput. Sci.* **412**(35) (2011), 4738–4741.

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Cusick and Ciungu[2], 2011

The sequence $(S_b(n) \mod m)_n$ is ultimately periodic if and only if $m \mid b-1$.

[1] J-P. Allouche and J. Shallit, Sums of digits, overlaps, and palindromes, *Discrete Math. Theor. Comput. Sci.* **4**(1) (2000), 1–10.

[2] T. W. Cusick and L. C. Ciungu, Sum of digits sequences modulo m, *Theoret. Comput. Sci.* **412**(35) (2011), 4738–4741.

b	2	3	4	5	6	7	8	9	10	11
$(23)_b$	10111	212	113	43	35	32	27	25	23	21
$S_b(23)$	4	5	5	7	8	5	9	7	5	3

b	12	13	14	15	16	17	18	19	20	21	22	23
$(23)_b$	$1B$	$1A$	19	18	17	16	15	14	13	12	11	10
$S_b(23)$	12	11	10	9	8	7	6	5	4	3	2	1

b	24	25	26	27	28	29	30	$\cdots$
$(23)_b$	N	N	N	N	N	N	N	$\cdots$
$S_b(23)$	23	23	23	23	23	23	23	$\cdots$

b	2	3	4	5	6	7	8	9	10	11
$(23)_b$	10111	212	113	43	35	32	27	25	23	21
$S_b(23)$	4	5	5	7	8	5	9	7	5	3

b	12	13	14	15	16	17	18	19	20	21	22	23
$(23)_b$	$1B$	$1A$	19	18	17	16	15	14	13	12	11	10
$S_b(23)$	12	11	10	9	8	7	6	5	4	3	2	1

b	24	25	26	27	28	29	30	$\cdots$
$(23)_b$	N	N	N	N	N	N	N	$\cdots$
$S_b(23)$	23	23	23	23	23	23	23	$\cdots$

$$\left\{S_b(n) \mid 2 \leqslant b \leqslant \frac{n}{2}\right\} = \{3, 4, 5, 7, 8, 9\}.$$

Theorem

Let

$$S(n) := \left| \left\{ S_b(n) \mid 2 \leqslant b \leqslant \frac{n}{2} \right\} \right|.$$

for all $n \geqslant 1$. Then

$$\lim_{n \mapsto \infty} \frac{S(n)}{n} =$$

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$$\lim_{n \mapsto \infty} \frac{S(n)}{n} = \sum_{p} \left(\prod_{q<p} \left(1 - \frac{1}{q}\right) \cdot \frac{1}{p(p+1)} \right)$$

Proof of the theorem

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$$S'(n) := \left|\left\{ S_b(n) \mid \sqrt{n} < b \leqslant \frac{n}{2} \right\}\right|$$

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Then

$$\lim_{n \mapsto \infty} \frac{S(n)}{n} = \lim_{n \mapsto \infty} \frac{S'(n)}{n}.$$

Proof of the theorem

Step 2.

Put

$$A_k(n) = \left\{ k(b-1) : \frac{n}{k+1} < b \leqslant \frac{n}{k} \right\}$$

for all $2 \leqslant k \leqslant n/2 - 1$.

Proof of the theorem

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$$A_k(n) = \left\{ k(b-1) : \frac{n}{k+1} < b \leqslant \frac{n}{k} \right\}$$

for all $2 \leqslant k \leqslant n/2 - 1$.

$$\frac{n}{k+1} < b \leqslant \frac{n}{k} \quad \text{if and only if} \quad k = \left[\frac{n}{b} \right].$$

Proof of the theorem

Step 3.

We have

$$S_b(n) = n - \left[\frac{n}{b}\right](b-1)$$

for all b satisfying

$$\sqrt{n} < b \leqslant \frac{n}{2}.$$

Proof of the theorem

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We have

$$S_b(n) = n - \left[\frac{n}{b}\right](b-1)$$

for all b satisfying

$$\sqrt{n} < b \leqslant \frac{n}{2}.$$

Therefore,

$$S'(n) = \left| \bigcup_{k=2}^{\left[\frac{n}{2}\right]-1} A_k(n) \right|.$$

Proof of the theorem

Step 4.

We have

$$A_{k'}(n) \subseteq A_k(n) \text{ if } k \text{ divides } k'.$$

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Hence,

$$\bigcup_{2 \leqslant k \leqslant \frac{n}{2}-1} A_k(n) = \bigcup_{2 \leqslant p \leqslant \frac{n}{2}-1} A_p(n),$$

Proof of the theorem

Step 5.

We have

$$|(a, b] \cap \mathbb{Z}| = [b] - [a]$$

for all $a, b \in \mathbb{R}$.

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Thus,

$$|A_p(n)| = \left[\frac{n}{p}\right] - \left[\frac{n}{p+1}\right]$$

for all $2 \leqslant p \leqslant n/2 - 1$.

Proof of the theorem

Step 6.

Suppose

$$p_1(b_1 - 1) = \cdots = p_m(b_m - 1) = p(b - 1)$$
$$\in A_{p_1}(n) \cap \cdots \cap A_{p_m}(n) \cap A_p(n),$$

where

$$2 \leqslant p_1 < \cdots < p_m < p \leqslant \frac{n}{2} - 1$$

are primes.

Proof of the theorem

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$$\begin{aligned} p_1(b_1-1) = \cdots = p_m(b_m-1) = p(b-1) \\ \in A_{p_1}(n) \cap \cdots \cap A_{p_m}(n) \cap A_p(n), \end{aligned}$$

where

$$2 \leqslant p_1 < \cdots < p_m < p \leqslant \frac{n}{2} - 1$$

are primes.

Then $p_1, \ldots, p_m$ divides $b-1$.

Proof of the theorem

Step 7.

Put

$$b = p_1 \dots p_m t + 1.$$

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$$\frac{n}{p_1 \dots p_m (p+1)} - \frac{1}{p_1 \dots p_m} < t \leqslant \frac{n}{p_1 \dots p_m p} - \frac{1}{p_1 \dots p_m}.$$

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Therefore,

$$|A_{p_1}(n) \cap \cdots \cap A_{p_m}(n) \cap A_p(n)| = \left[\frac{n}{p_1 \dots p_m p} - \frac{1}{p_1 \dots p_m}\right] - \left[\frac{n}{p_1 \dots p_m(p+1)} - \frac{1}{p_1 \dots p_m}\right].$$

Proof of the theorem

Step 8.

We have

$$\lim_{n \mapsto \infty} \frac{[\alpha n + \beta]}{n} = \alpha$$

for all $\alpha, \beta \in \mathbb{R}$.

Proof of the theorem

Step 8.

We have

$$\lim_{n\mapsto\infty}\frac{[\alpha n+\beta]}{n}=\alpha$$

for all $\alpha,\beta\in\mathbb{R}$.

Therefore,

$$\lim_{n\mapsto\infty}\frac{S'(n)}{n}=\sum_{p}\frac{1}{p(p+1)}-\sum_{q<p}\frac{1}{qp(p+1)}+\sum_{r<q<p}\frac{1}{rqp(p+1)}-\cdots$$

$$=\sum_{p}\left(\prod_{q<p}\left(1-\frac{1}{q}\right)\cdot\frac{1}{p(p+1)}\right)\approx 0.2296277628.$$

Thank You for Your Attention!