Factorization number of finite abelian groups

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Definition

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Connection with subgroup commutativity degree

If G is a finite group, then

$$scd(G) = \frac{1}{|L(G)|^2} \sum_{H \leq G} F_2(H),$$

where $L(G)$ is the set of all subgroups of G.

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5. projective special linear groups $PSL(3, p^n)$ and projective special unitary groups $PSU(3, p^n)$ (Blaum, 1983).
6. cyclic groups $\mathbb{Z}_n$, elementary abelian p-groups $\mathbb{Z}_p^n$, dihedral groups D_{2n}, quasi-dihedral groups QD_{2^n}, generalized quaternion groups Q_{4n} and Modular p-group M_{p^n} (Farrokhi and Saeedi, Submitted).

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3. $S(\mathcal{X}) = \{(y_1, \ldots, y_m) : m \leq n, y_m \leq \cdots \leq y_1 \text{ and } y_i \leq x_i, \text{ for } 1 \leq i \leq m\}$;

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4. if $\mathcal{Y} = (y_1, \ldots, y_m) \in S(\mathcal{X})$, then $T_p(\mathcal{X} : \mathcal{Y})$ is the set of all m-tuples $(g_1, \ldots, g_m) \in G_p(\mathcal{X})^m$ such that $|g_i| = p^{y_i}$ and $\langle g_1, \ldots, g_m \rangle \cong G_p(\mathcal{Y})$, for each $i = 1, \ldots, m$, and

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5. $\mu_i(\mathcal{X}) = i \max\{j : x_j \geq i\} + x_{\max\{j : c_j \geq i\}+1} + \cdots + x_n$,

First formula for factorization number

Lemma

If $\mathcal{A} = (a_1, \ldots, a_n)$ *is a non-increasing sequence of natural numbers and* $\mathcal{B} = (b_1, \ldots, b_m) \in S(\mathcal{A})$*, then*

$$|T_p(\mathcal{A} : \mathcal{B})| = \prod_{i=1}^{m} \left(p^{\mu_{b_i}(\mathcal{A})} - \frac{p^{\mu_{b_i-1}(\mathcal{A}) + \mu_{b_i}(\mathcal{B}_{i-1})}}{p^{\mu_{b_i-1}(\mathcal{B}_{i-1})}} \right).$$

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Corollary

The number of p-subgroups of type $\mathcal{B}$ of a finite abelian p-group of type $\mathcal{A}$ is

$$\binom{\mathcal{A}}{\mathcal{B}}_p = \frac{T_p(\mathcal{A} : \mathcal{B})}{T_p(\mathcal{B} : \mathcal{B})}.$$

First formula for factorization number

Corollary

The number of subgroups of a finite abelian p-group G of type $\mathcal{A}$ is

$$|L(G)| = \sum_{\mathcal{B}\in S(\mathcal{A})} \binom{\mathcal{A}}{\mathcal{B}}_p .$$

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Corollary

The number of subgroups of a finite abelian p-group G of type $\mathcal{A}$ is

$$|L(G)| = \sum_{\mathcal{B} \in S(\mathcal{A})} \binom{\mathcal{A}}{\mathcal{B}}_p.$$

Theorem

If G is a finite abelian p-group of type $\mathcal{A}$, then

$$F_2(G) = |L(G)|^2 - \sum_{\mathcal{B} \in S(\mathcal{A}) \setminus \{\mathcal{A}\}} \binom{\mathcal{A}}{\mathcal{B}}_p F_2(G_p(\mathcal{B})).$$

Second formula for factorization number

Lemma

Let G be an elementary abelian p-group and $X \leq G$. Then the number of subgroups Y of G of order p^n $(n \leq d(G) - d(X))$ such that $X \cap Y = 1$ is

$$p^{nd(X)} \binom{d(G) - d(X)}{n}_p .$$

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Lemma

Let G be an elementary abelian p-group and $X \leq Y \leq G$. Then the number of subgroups Z of G of order $p^{d(G)-d(Y)+n}$ $(n \leq d(Y) - d(X))$ such that $X \cap Z = 1$ and $YZ = G$ is

$$p^{nd(X)+(d(Y)-n)(d(G)-d(Y))} \binom{d(Y) - d(X)}{n}_p.$$

Second formula for factorization number

Theorem

Let G be a finite abelian p-group. Then

$$F_2(G) = \sum_{G^p = AB} \sum_{\substack{0 \leq i,j \leq n \\ n \leq i+j \leq 2n}} \frac{|\Omega_1(G)|^{d(A)+d(B)}|\Omega_1(G^p)|^{i+j}}{|\Omega_1(A)|^{d(A)}|\Omega_1(B)|^{d(B)}} \cdot \frac{p^{(n-i)(n-j)}}{p^{id(A)+jd(B)}} \cdot \binom{n}{i}_p \binom{i}{n-j}_p,$$

where $n = d(\Omega_1(G)) - d(\Omega_1(G^p))$.

Application: Automorphisms

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Let G be a finite abelian p-group of type $\mathcal{A}$. Then

1. $|\mathrm{Aut}(G)| = |T_p(\mathcal{A} : \mathcal{A})|$.
2. *if* $\mathcal{A} = (a_1, \ldots, a_1, \ldots, a_m, \ldots, a_m) = (x_1, \ldots, x_n)$, *where the number of* a_i *is* b_i *and* $a_i > a_{i+1}$, *then*

$$|\mathrm{Aut}(G)| = \prod_{i=1}^{m} \prod_{j=N_{i-1}+1}^{N_i} \left(p^{x_j N_i + x_{N_i+1} + \cdots + x_m} - p^{(x_j - 1)N_{i'} + x_{N_{i'}+1} + \cdots + x_m + j - 1} \right)$$

and in particular

$$|\mathrm{Aut}(G)|_p = \prod_{i=1}^{m} \prod_{j=N_{i-1}+1}^{N_i} p^{(x_j - 1)N_{i'} + x_{N_{i'}+1} + \cdots + x_m + j - 1},$$

where $i' = i + 1 - \mathrm{Sign}(a_{i+1} - a_i + 1)$ *and* $N_i = b_1 + \cdots + b_i$, *for each* $i = 1, \ldots, m$.

Application: Gaussian binomiral coefficients

Definition

Let q be a prime power and n be an integer. Then the numbers $[n]_q$ and $[n]_q!$ are called the q-integer and q-factorial and defined as follows:

$$[n]_q = \frac{q^n - 1}{q - 1} \text{ and } [n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q.$$

The Gaussian binomial coefficient can be defined in terms of q-factorials by

$$\binom{n}{i}_q = \frac{[n]_q!}{[i]_q![n-i]_q!}$$

and as usual the Gaussian polynomial coefficients can be defined by

$$\binom{n}{i_1, \ldots, i_k}_q = \frac{[n]_q!}{[i_1]_q! \cdots [i_k]_q![n - i_1 - \cdots - i_k]_q!},$$

where $0 \leq i_1 + \cdots + i_k \leq n$.

Remark

The Gaussian binomial coefficient $\binom{n}{i}_q$ is the number of subspaces of dimension i in a vector space of dimention n over the field of order q and also appear in the theory of partition of integers.

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The Gaussian binomial coefficient $\binom{n}{i}_q$ is the number of subspaces of dimension i in a vector space of dimention n over the field of order q and also appear in the theory of partition of integers.

Corollary

Let q be a prime power and n be a natural number. Then

$$\left(\sum_{i=0}^{n}\binom{n}{i}_q\right)^2=\sum_{0\leq i+j+k\leq n}\binom{n}{i,j,k}_q q^{j(n-i-j-k)}.$$

In particular

$$4^n=\sum_{0\leq i+j+k\leq n}\binom{n}{i,j,k}.$$

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Thank You