On the center of automorphism group of a group

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In this talk we assume that $Z(G)$ is cyclic and we obtain the structure of $Z(Aut(G))$.

As a result we will show that $\exp(Z(Aut(G))) \leq \exp(Z(G))$ while $|Z(Aut(G))|$ might be greater that $|Z(G)|$. (exp. QD_{16})

Definitions

Let $Aut_c(G) = C_{Aut(G)}(Inn(G))$ be the group of central automorphisms of G. Then $g^{-1}\theta(g) \in Z(G)$ for each $\theta \in Aut_c(G)$ and $g \in G$. In particular $Z(Aut(G)) \subseteq Aut_c(G)$.

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Definition

Let G be a group with cyclic center $Z(G) = \langle z \rangle$ and $\theta \in Aut_c(G)$. Then $\bar{\theta}$ is the homomorphism from G to $Z(G)$, which sends g to $g^{-1}\theta(g)$ for each $g \in G$. Also α_θ is the smallest nonnegative integer k such that $\bar{\theta}(z) = z^k$.

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Fact

In the above definition α_θ is independent of the choice of z.

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Lemma

Let G be a group with cyclic center. If $\theta \in Aut_c(G)$ and $g \in G$, then

(a) $\bar{\theta}^k(g) = \bar{\theta}(g)^{\alpha_\theta^{k-1}}$;

for each nonnegative integer k.

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(b) $\theta^k(g) = g^{\binom{k}{0}} \bar{\theta}(g)^{\binom{k}{1}} \ldots \bar{\theta}^{k-1}(g)^{\binom{k}{k-1}} \bar{\theta}^k(g)^{\binom{k}{k}}$, *and*

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(c) $\theta^k(g) = g\bar{\theta}(g)^{\frac{1}{\alpha_\theta}\left((1+\alpha_\theta)^k-1\right)}$,

for each nonnegative integer k.

Lemma

Let G be a group with cyclic center of order n. Then for all $\varphi, \psi \in Z(Aut(G))$,

(a) $\alpha_{\varphi\psi} + 1 \overset{n}{\equiv} (\alpha_\varphi + 1)(\alpha_\psi + 1)$, *and*

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(a) $\alpha_{\varphi\psi} + 1 \overset{n}{\equiv} (\alpha_\varphi + 1)(\alpha_\psi + 1)$, *and*

(b) *The map $\alpha^* : Z(Aut(G)) \longrightarrow Aut(Z(G)) \cong U(\mathbb{Z}_n)$ given by $\alpha^*(\varphi) = \alpha_\varphi + 1$ is a homomorphism, where $\alpha_\varphi + 1$ is identified with the automorphism, which sends z to $z^{\alpha_\varphi + 1}$.*

Basic lemmas

Lemma

Let G be a group with cyclic center of finite order $n = p_1^{a_1} \cdots p_m^{a_m}$ and $\varphi \in Z(Aut(G))$. Then

$$|\varphi| | \mathrm{lcm}(d_1, \ldots, d_m),$$

where $d_i = p_i^{a_i}$, when $p_i | \alpha_\varphi$ and $d_i = p_i^{a_i - 1}(p_i - 1)$, when $p_i \nmid \alpha_\varphi$. In particular $\exp(Z(Aut(G)) \leq \exp(Z(G))$.

Main theorem: Groups with finite cyclic center

Theorem

Let G be a group with cyclic center of finite order $n = p_1^{a_1} \cdots p_m^{a_m}$. Then $Z(Aut(G)) \cong A_1 \times \cdots \times A_m$, where the subgroup A_i is isomorphic to

(a) *the trivial group;*

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(c) *the cyclic group of order $p_i^{a_i-1}(p_i - 1)$,*

for $i = 1, 2, \ldots, n$.

Proof: Groups with finite cyclic center

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Let φ_i be the following map

$$\begin{aligned} \varphi_i : G &\longrightarrow G \\ g &\longmapsto g\bar{\varphi}_i(g). \end{aligned}$$

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$$Z(Aut(G)) = A_1 \times \cdots \times A_m,$$

where $\bar{\psi}(g) \in P_i$, for each $\psi \in A_i$.

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If $\alpha_i^* = \alpha^*|_{A_i}$, then

$$\frac{A_i}{\mathrm{Ker}\alpha_i^*} \cong B_i \leq U(P_i) \cong \mathbb{Z}_{p_i^{a_i-1}(p_i-1)}.$$

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Now if $\exp(A_i) \nmid p_i^{a_i}$, then $p_i > 2$ and $\exp(A_i)|p_i^{a_i-1}(p_i-1)$ so that there exists an automorphism $\varphi \in A_i \setminus \{I\}$ such that $|\varphi||p_i - 1$.

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Let $n_i = n/p_i^{a_i}$, $z_i = z^{n_i}$ and $\alpha_\varphi^i = \alpha_\varphi/n_i$.

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Then $P_i = \langle z_i \rangle$, $n_i|\alpha_\varphi$ and $p_i \nmid \alpha_\varphi$.

Hence α_φ^i is an integer coprime to p_i and there exists an integer β such that $1 + \alpha_\varphi^i\beta$ is a primitive root modulo $p_i^{a_i}$.

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In particular $P_i \cap \mathrm{Ker}\varphi_\beta = \langle 1 \rangle$ and $G = P_i\mathrm{Ker}\varphi_\beta \cong P_i \times \mathrm{Ker}\varphi_\beta$.

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Clearly $\psi \in A_i$ is of order 2 if and only if $\alpha_\psi^i \equiv -2 \pmod{p_i^{a_i}}$.

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If $\psi \in A_i$ is an automorphism of order 2 and $\theta \in \mathrm{Ker}\alpha_i^*$, then $\alpha^i_{\psi\theta} \equiv -2 \pmod{p_i^{a_i}}$.

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Hence $\theta = I$ and consequently $\mathrm{Ker}\alpha_i^* = \langle I \rangle$.

Therefore

$$A_i \cong \mathbb{Z}_{p_i^{a_i - 1}(p_i - 1)}.$$

Example

Let $G = \langle a, b : a^p = b^p = [a,b]^p = 1, [a,b]^a = [a,b]^b = [a,b] \rangle$ be the p-group of order p^3 and exponent p, where p is a prime. Then $Z(Aut(G)) = \langle I \rangle$.

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Example

Let $G = \langle a, b : a^{p^2} = b^p = 1, a^b = a^{p+1} \rangle$ be the p-group of order p^3 and exponent p^2, where p is a prime. Then $Z(Aut(G)) \cong \mathbb{Z}_p$.

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Example

If $G = \mathbb{Z}_{p^n}$, then $Z(Aut(G)) \cong \mathbb{Z}_{p^{n-1}(p-1)}$.

Corollary: Nilpotent groups with finite cyclic center

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Let G be a finite nilpotent group with cyclic center of order $n = p_1^{a_1} \cdots p_m^{a_m}$. Then either the Sylow p_i-subgroup of G is cyclic or the subgroup A_i in the previous theorem is isomorphic to

(a) *the trivial group, or*

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(c) *a nontrivial torsion-free abelian group.*

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Let $G = \langle a, b, c : [a, c] = [b, c] = 1 \rangle$ be a group with infinite cyclic center. Then $Z(Aut(G)) = \langle I \rangle$.

Example

If $G = \mathbb{Z}$, then $Z(Aut(G)) \cong \mathbb{Z}_2$.

Open problems

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Problem

Suppose that $\gcd(p_i - 1, |A_i|) \neq 1$. Is it true that $p_i - 1 \| A_i|$ or even A_i has an element of order $p_i - 1$?

Thank You!