Lattice path enumeration in rectangular area

M. Farrokhi D. G.

Institute for Advanced Studies in Basic Sciences (IASBS)

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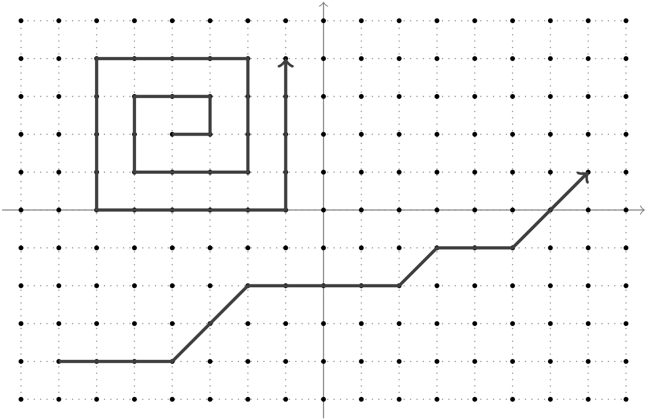

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    - Computer science
    - Random walks
    - Algebra (Group theory, Commutative rings, etc.)
2. Physics
    - Thermodynamic models
    - Phase transitions
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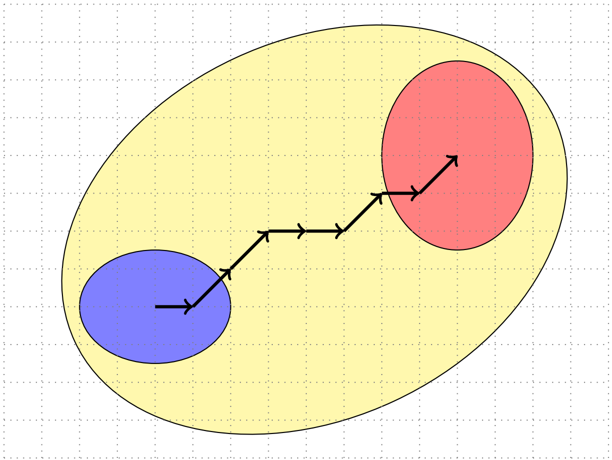
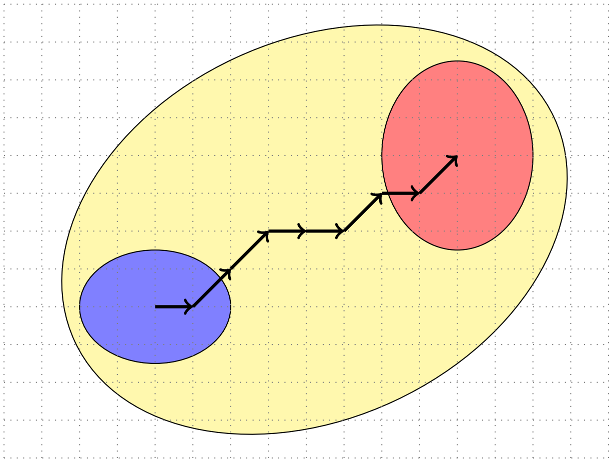


Steps$=\{(1,0)=\longrightarrow, \quad (1,1)=\nearrow, \quad (1,-1)=\searrow\}$

Problem: Count the number of all possible legal paths!



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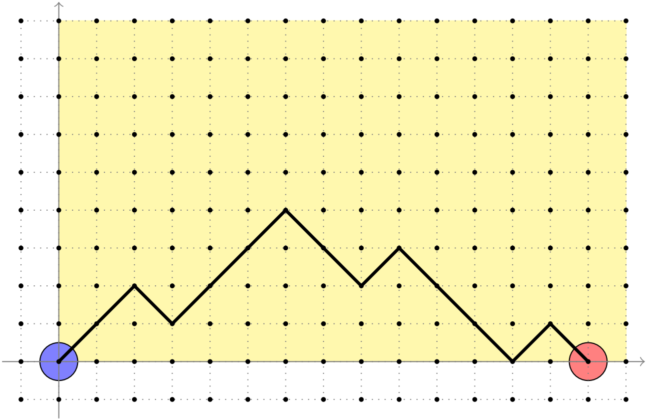

$$S = \{(1,1) = \nearrow, \quad (1,-1) = \searrow\}$$

$$\ell((0,0),(2n,0),S) = \frac{1}{n+1}\binom{2n}{n} \text{ the Catalan number } C_n$$

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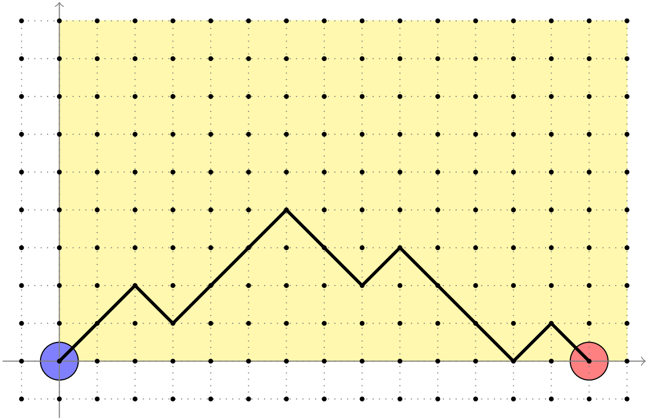

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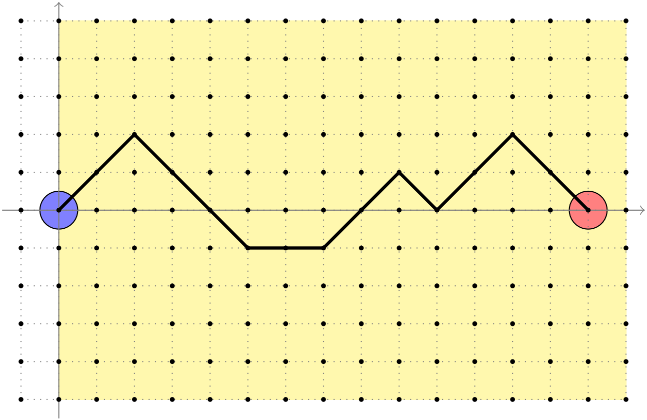

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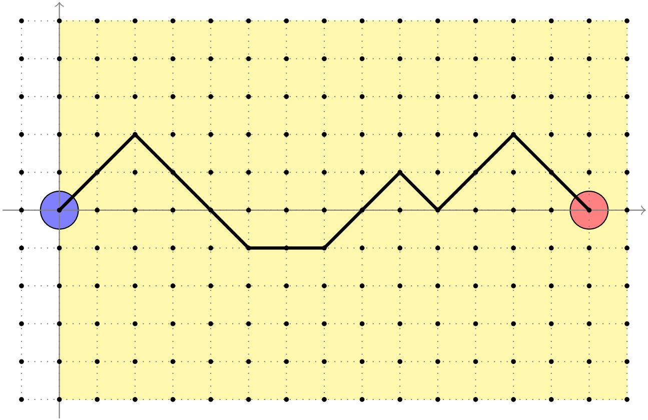

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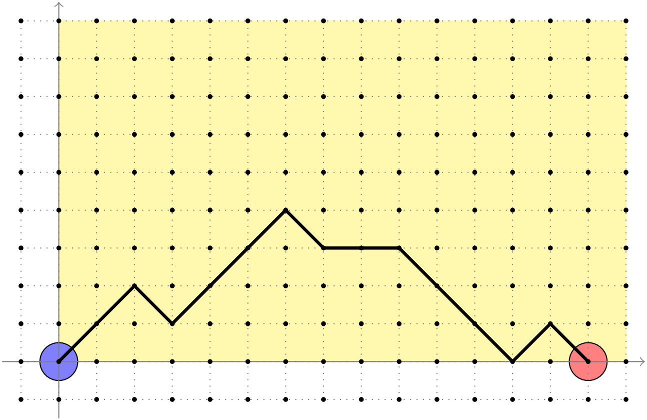

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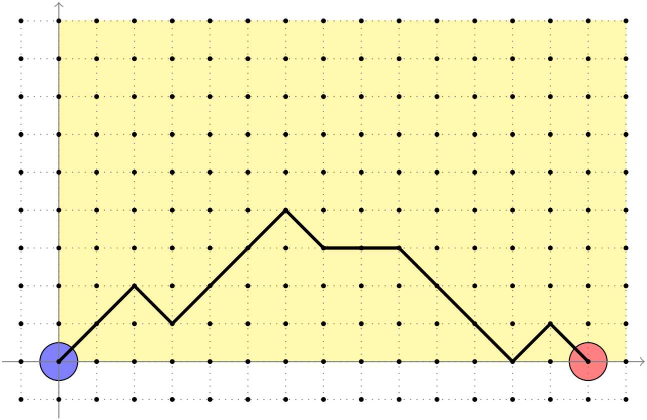

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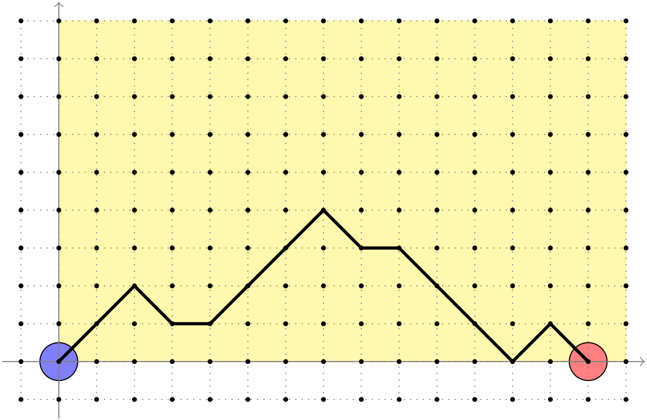

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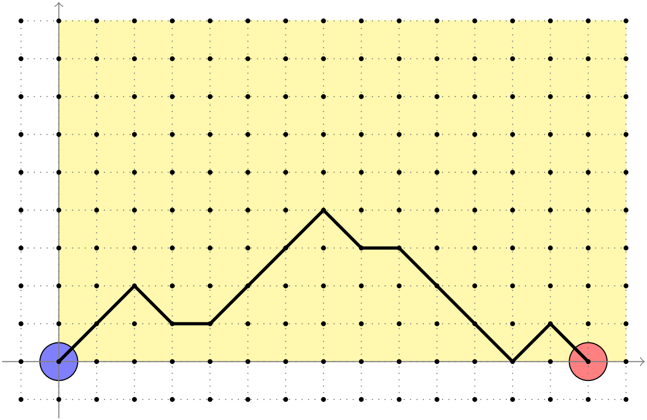

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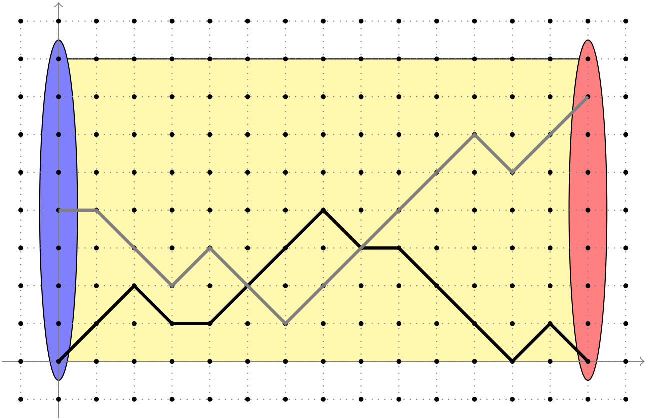
Our paths



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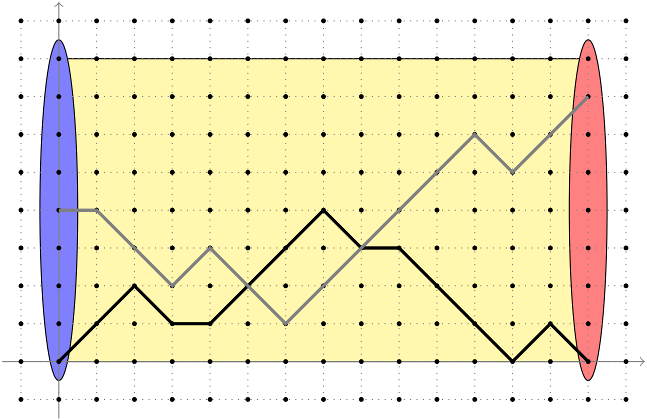

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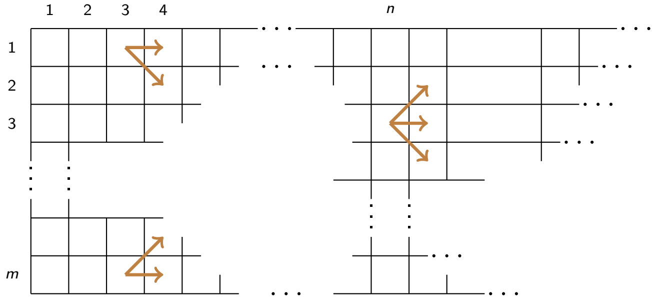
An instant table



The $m \times \infty$ table

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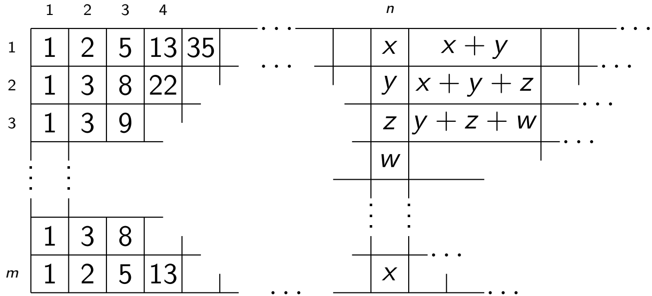
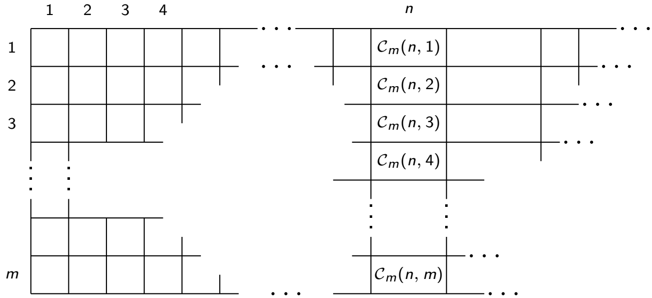


The $m \times \infty$ table

Notations



The $m \times \infty$ table

$$\mathcal{I}_m(n) := \mathcal{C}_m(n,1) + \cdots + \mathcal{C}_m(n,m).$$

1. $\mathcal{I}_1(n) = 1$.
2. $\mathcal{I}_2(n) = 2^n$.
3. $\mathcal{I}_3(n) = Q_{n+1}$, a Pell-Lucas number defined as

$$Q_1 = 1,\ Q_2 = 3,\ \text{and}\ Q_{n+2} = 2Q_n + Q_{n+1}\ \text{for}\ n \geqslant 0.$$

4. $\mathcal{I}_4(n) = 2F_{n+1}$ twice a Fibonacci number. Moreover,

$$\mathcal{C}_4(n,1) = F_{2n-1} \quad \text{and} \quad \mathcal{C}_4(n,2) = F_{2n}.$$

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Theorem

1. *For $m \geqslant n$ we have*

$$\mathcal{I}_{m+1}(n) - \mathcal{I}_m(n) = \sum_{i=0}^{n-1} \mathcal{C}_m(i,1)\mathcal{C}_m(n-i,1),$$

where $C_m(0,1) := 1$.

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for all $a,b\geqslant 1$*.*

Theorem

We have

$$\mathcal{I}_m(n) = m3^{n-1} - 2\sum_{k=1}^{n-1} 3^{n-k-1}\mathcal{C}_m(k,1).$$

If $m \geqslant n$ and $1 \leqslant k \leqslant n$, then

$$\mathcal{C}_m(k,1) = 3^{k-1} - \sum_{i=1}^{k-1} 3^{i-1} M_{k-i-1}.$$

Theorem

Inside the $m \times \infty$ table, a nontrivial linear combination of columns entries is a constant if and only if $m \equiv 1 \pmod 4$, and the equation is given by

$$\alpha_1 \mathcal{C}_m(n,1) + \alpha_3 \mathcal{C}_m(n,3) + \cdots + \alpha_{m-2} \mathcal{C}_m(n,m-2) + \alpha_m \mathcal{C}_m(n,m) = \lambda$$

for all $n \geqslant 1$, where $\lambda \neq 0$ is a fixed number and

$$\alpha_{2i+1} + \alpha_{m-2i} = (-1)^i 2\lambda,$$

for all $i = 0, \ldots, (m-1)/4$.

- Δ: the difference operator defined as $\Delta a(n) = a(n+1) - a(n)$ for any sequence $\{a(n)\}$ of numbers.
- $\mathcal{M}_m = \mathcal{M}_m(\Delta)$ defines as

$$\mathcal{M}_m(0) = \begin{cases} 2, & m \text{ is odd,} \\ 1, & m \text{ is even,} \end{cases}$$

$\mathcal{M}_m(1) = \Delta - 2 + \mathcal{M}_m(0)$, and

$$\mathcal{M}_m(n+2) = \Delta\mathcal{M}_m(n+1) - \mathcal{M}_m(n)$$

for all $n \geqslant 0$. Set $\mathcal{M}_o := \mathcal{M}_1$ and $\mathcal{M}_e := \mathcal{M}_2$.
- $\mathcal{M}' = \mathcal{M}'(\Delta)$ defines as $\mathcal{M}'(0) = 1$, $\mathcal{M}'(1) = \Delta$, and

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Lemma

Inside the $m \times \infty$ table, we have

$$\mathcal{M}_m(k-b)\mathcal{C}_m(n,a) = \mathcal{M}_m(k-a)\mathcal{C}_m(n,b)$$

for all $a, b = 1, \ldots, k$, where $k = \lceil m/2 \rceil$.

Corollary

Inside the $m \times \infty$ *table, we have*

$$\mathcal{M}_m(k-a)\mathcal{I}_m(n) = 2(\mathcal{M}_m(k-1) + \cdots + \mathcal{M}_m(1) + 1)\mathcal{C}_m(n, a)$$

for all $a = 1, \ldots, k$ *and* $n \geqslant 1$, *where* $k = \lceil m/2 \rceil$.

In particular, for $a = k$, *we have*

$$\mathcal{I}_m(n) = \frac{2}{\mathcal{M}_m(0)}(\mathcal{M}_m(k-1) + \cdots + \mathcal{M}_m(1) + 1)\mathcal{C}_m(n, k)$$

for all $n \geqslant 1$.

Lemma

Inside the $m \times \infty$ table, we have

$$\mathcal{M}'(b-1)\mathcal{C}_m(n,a) = \mathcal{M}'(a-1)\mathcal{C}_m(n,b)$$

for all $a, b = 1, \ldots, k$, where $k = \lceil m/2 \rceil$.

Corollary

Inside the $m \times \infty$ table, we have

$$\mathcal{M}'(a-1)\mathcal{I}_m(n) = 2\left(\frac{\mathcal{M}'(k-1)}{\mathcal{M}_m(0)} + \mathcal{M}'(k-2) + \cdots + \mathcal{M}'(0)\right)\mathcal{C}_m(n,a)$$

for all $a = 1, \ldots, k$ and $n \geqslant 1$, where $k = \lceil m/2 \rceil$.

In particular, for $a = 1$, we have

$$\mathcal{I}_m(n) = 2\left(\frac{\mathcal{M}'(k-1)}{\mathcal{M}_m(0)} + \mathcal{M}'(k-2) + \cdots + \mathcal{M}'(0)\right)\mathcal{C}_m(n,1)$$

for all $n \geqslant 1$.

Let

$$\mathcal{T}_m := \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}$$

be the matrix of the $m \times \infty$ table, and $\mathcal{C}_m(n)$ be the n^{th} column of the $m \times \infty$ table. Then

$$\mathcal{T}_m \mathcal{C}_m(n) = \mathcal{C}_m(n+1) \quad \text{and} \quad \mathcal{I}_m(n) = \mathbf{1}^T \mathcal{T}_m^n \mathbf{1}$$

for all $n \geqslant 1$.

Let

$$\mathcal{O}_1 = [1], \quad \mathcal{O}_2 = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \quad \mathcal{O}_k = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 2 & 1 \end{bmatrix} \quad (k \geqslant 3),$$

and

$$\mathcal{E}_1 = [2], \quad \mathcal{E}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad \mathcal{E}_k = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 2 \end{bmatrix} \quad (k \geqslant 3).$$

Also, let

$$\mathcal{T}^*_m = \begin{cases} \mathcal{O}_{\lceil \frac{m}{2} \rceil}, & m \text{ is odd}, \\ \mathcal{E}_{\lceil \frac{m}{2} \rceil}, & m \text{ is even} \end{cases}$$

be the reduced matrix of the $m \times \infty$ table for all $m \geqslant 1$.

Assume $C^*_m(n)$ is the reduced n^{th} column in the $m \times \infty$ table including entries in the rows $1, \ldots, \lceil m/2 \rceil$.

Then

$$\mathcal{T}^*_m C^*_m(n) = C^*_m(n+1)$$

for all $n \geqslant 1$.

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Assume $C_m^*(n)$ is the reduced n^{th} column in the $m \times \infty$ table including entries in the rows $1, \dots, \lceil m/2 \rceil$.

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for all $n \geqslant 1$.

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Lemma

For every $m \geqslant 1$, we have

$$\det \mathcal{T}_m^* = \begin{cases} (-1)^{\left\lfloor \frac{\lceil \frac{m}{2} \rceil + 1}{3} \right\rfloor} 2^{\chi_{3\mathbb{Z}}(\lceil \frac{m}{2} \rceil)}, & m \text{ is odd}, \\ (-1)^{\left\lfloor \frac{\lceil \frac{m}{2} \rceil}{3} \right\rfloor} 2^{\chi_{3\mathbb{Z}+1}(\lceil \frac{m}{2} \rceil)}, & m \text{ is even}, \end{cases}$$

where χ denotes the characteristic function.

Corollary

Let $k = \lceil m/2 \rceil$. Then

$$\det([C_m^*(n+1) \cdots C_m^*(n+k)]) = \det(\mathcal{T}_m^*)^n$$

for all $n \geqslant 0$.

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Theorem

Let $k = \lceil m/2 \rceil$ and $[\alpha_1 \cdots \alpha_k]^T$ be the solution to the matrix equation

$$[C_m^*(1) \cdots C_m^*(k)][\alpha_1 \cdots \alpha_k]^T = [C_m^*(k+1)]$$

inside the $m \times \infty$ table. Then

$$\mathcal{C}_m(n+k, i) = \alpha_1 \mathcal{C}_m(n, i) + \cdots + \alpha_k \mathcal{C}_m(n+k-1, i) \qquad (1)$$

for all $1 \leqslant i \leqslant m$ and $n \geqslant 1$. As a result,

$$\mathcal{I}_m(n+k) = \alpha_1 \mathcal{I}_m(n) + \cdots + \alpha_k \mathcal{I}_m(n+k-1) \qquad (2)$$

for all $n \geqslant 1$. Moreover, these recurrence relations are of the minimum degree.

Corollary

For every $m \geqslant 1$, the following polynomials are equal:

(1) $\det(xI - \mathcal{T}_m^*)$;

(2) $\mathcal{M}_m(k)(x-1)$;

(3) $x^k - [1 \cdots x^{k-1}][C_m^*(1) \cdots C_m^*(k)]^{-1}[C_m^*(k+1)]$,

where $k = \lceil m/2 \rceil$.

Theorem

For any $m \geqslant 1$ and $a, b \geqslant 1$, we have

$$\mathcal{M}_m(a+b) = \mathcal{M}'(a)\mathcal{M}_m(b) - \mathcal{M}'(a-1)\mathcal{M}_m(b-1)$$

and

$$\mathcal{M}'(a+b) = \mathcal{M}'(a)\mathcal{M}'(b) - \mathcal{M}'(a-1)\mathcal{M}'(b-1).$$

Lemma

Let $m \geqslant 1$, $k = \lceil m/2 \rceil$, $0 \leqslant a \leqslant k$, and $0 \leqslant b \leqslant \min\{a, k-a\}$. Then

$$\mathcal{M}_o(b)\mathcal{C}_m(n,a) = \mathcal{C}_m(n,a-b) + \mathcal{C}_m(n,a+b).$$

Theorem

For any $m \geqslant 1$ and $a \geqslant b \geqslant 0$, we have

$$\mathcal{M}_o(b)\mathcal{M}_m(a) = \mathcal{M}_m(a+b) + \mathcal{M}_m(a-b)$$

and

$$\mathcal{M}_o(b)\mathcal{M}'(a) = \mathcal{M}'(a+b) + \mathcal{M}'(a-b).$$

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For any $a, b \geqslant 1$ we have

$$\mathcal{M}_m(ab) = \mathcal{M}'(a-1)\left(\mathcal{M}_o(b)\right)\mathcal{M}_m(b) - \mathcal{M}'(a-2)\left(\mathcal{M}_o(b)\right)\mathcal{M}_m(0)$$

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Theorem (Factorization theorem for $\mathcal{M}_o$)

For all $a, b \geqslant 1$, we have

$$\mathcal{M}_o(ab) = \mathcal{M}_o(a) \circ \mathcal{M}_o(b) = \mathcal{M}_o(b) \circ \mathcal{M}_o(a).$$

As a result, if $n = p_1^{a_1} \dots p_k^{a_k}$ is the canonical factorization of n into distinct primes $p_1, \dots, p_k$, then

$$\mathcal{M}_o(n) = \mathcal{M}_o(p_1)^{a_1} \dots \mathcal{M}_o(p_k)^{a_k},$$

where all the products are the composition of functions.

Prime functions

- $\mathcal{F}_{m,p}$ is a function defined on the set of multipliers $\{\mathcal{M}_m(n)\}$ as

$$\mathcal{F}_{m,p}(\mathcal{M}_m(n)) = \mathcal{M}'(p-1)(\mathcal{M}_e(n))\mathcal{M}_m(n) - \mathcal{M}'(p-2)(\mathcal{M}_e(n))\mathcal{M}_m(0).$$

- $\mathcal{F}'_p$ is a function defined on the set of multipliers $\{\mathcal{M}'(n)\}$ as

$$\mathcal{F}'_p(\mathcal{M}'(n)) = \mathcal{M}'(p-1)(\mathcal{M}_e(n))\mathcal{M}'(n) - \mathcal{M}'(p-2)(\mathcal{M}_e(n))\mathcal{M}'(0).$$

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Theorem (Uniform factorization theorem)

Let $n = p_1^{a_1} \dots p_k^{a_k}$ be the canonical factorization of n into distinct primes $p_1, \dots, p_k$. Then

$$\mathcal{M}_m(n) = \mathcal{F}_{m,p_1}^{a_1} \dots \mathcal{F}_{m,p_k}^{a_k} \mathcal{M}_m(1)$$

and

$$\mathcal{M}'(n) = \mathcal{F'}_{p_1}^{a_1} \dots \mathcal{F'}_{p_k}^{a_k} \mathcal{M}'(1),$$

where all the products are the composition of functions.

Lemma

For all $n \geqslant 1$, we have

$$\mathcal{M}_e(n) = \mathcal{M}'(n) - \mathcal{M}'(n-1)$$

and

$$\mathcal{M}_o(n) = \mathcal{M}_e(n) + \mathcal{M}_e(n-1) = \mathcal{M}'(n) - \mathcal{M}'(n-2).$$

Corollary

For all $n \geqslant 1$, we have

$$1 + \mathcal{M}_m(1) + \cdots + \mathcal{M}_m(n) = \mathcal{M}'(n) + (\mathcal{M}_m(0) - 1)\mathcal{M}'(n-1).$$

Corollary

Inside the $m \times \infty$ table, we have

$$\mathcal{I}_m(n) = \frac{2}{\mathcal{M}_m(0)}\Big(\mathcal{M}'(k-1) + (\mathcal{M}_m(0) - 1)\mathcal{M}'(k-2)\Big)\mathcal{C}_m(n,k).$$

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Theorem

For all $n \geqslant 1$, we have

$$\mathcal{M}_o(n) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \left[\binom{n+1-i}{i} - \binom{n-1-i}{i-2} \right] \Delta^{n-2i},$$

$$\mathcal{M}_e(n) = \sum_{i=0}^{n} (-1)^{\lceil \frac{i}{2} \rceil} \binom{n - \lceil \frac{i}{2} \rceil}{\lfloor \frac{i}{2} \rfloor} \Delta^{n-i},$$

$$\mathcal{M}'(n) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \binom{n-i}{i} \Delta^{n-2i}.$$

Relations to other polynomials

- $(1/2)\mathcal{M}_o(n)(2x)$ is the Chebyshev polynomial $T_n(x)$ of the first kind;
- $|\mathcal{M}_e(n)|(x)$ is the Fibonacci polynomial $F_{n+1}(x)$;
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Open problem

Definition

A matrix $A \in GL(n, q)$ is a Singer element if $|A| = q^n - 1$.

Theorem

Singer elements exist in every $GL(n, q)$.

The order of $\mathcal{E}_k \pmod 2$ equals $2^k - 1$ if

$$k = 1, 2, 3, 5, 9, 11, 14, 23, 26, 29, 35, 39, 41, 53, ?, ?, ?, \ldots.$$

Question

Which $\mathcal{E}_k$ are Singer?

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Thank You for Your Attention!