Finite groups with few subgroup normality degrees

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Definitions

Definition

Let G be a finite group and H be a subgroup of G. The subgroup normality degree of H in G is

$$P_N(H, G) = \frac{|\{(h, g) \in H \times G : h^g \in H\}|}{|H||G|}.$$

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Remark

If G is a finite group and H is a subgroup of G, then

$$P_N(H,G) = \frac{1}{n}\left(1 + \sum_{i=1}^{n-1} \frac{1}{[H : H \cap H^{g_i^{-1}}]}\right),$$

where $n = [G : N_G(H)]$ and $\{1, g_1, \ldots, g_{n-1}\}$ is a left transversal to $N_G(H)$ in G.

Definitions

Notation

For every finite group G let

$$\mathcal{P}_N(G) = \{P_N(H,G) : H \leq G\}$$

and

$$\mathcal{P}_N^*(G) = \mathcal{P}_N(G) \setminus \{1\}.$$

Also, let

$$\mathcal{P}_N = \bigcup_{G \in \mathcal{F}} \mathcal{P}_N(G),$$

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Fact

Let G be a finite group. Then $|\mathcal{P}_N(G)| = 1$ if and only if G is a Dedekind group.

Known results

Theorem (Saeedi, Farrokhi and Jafari [1])

Let G be a finite group and H be a subgroup of G. If p is the smallest prime dividing $|H|$, then

$$\frac{1}{[G:H]}+\frac{1}{|H|}-\frac{1}{|G|}\leq P_N(H,G)\leq\frac{1}{p}+\left(1-\frac{1}{p}\right)\frac{1}{[G:N_G(H)]}.$$

[1] Subgroup normality degrees of finite groups I, *Arch. Math.* **96** (2011), 215–224.

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(2) *if $N_G(H)\ntrianglelefteq G$, then $P_N(H,G)\leq \frac{2}{q+1}\leq \frac{2}{3}$.*

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Let G be a non-Abelian finite simple group. Then $\max \mathcal{P}^*_N(G) \leq \frac{8}{15}$ *and the equality holds if* $G \cong A_5$.

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Conjecture (Saeedi, Farrokhi and Jafari [1])

Let $G \ncong A_5$ be a non-Abelian finite simple group. Then $\max \mathcal{P}_N^*(G) \leq \frac{11}{21}$ and the equality holds if and only if $G \cong PSL(2,7)$.

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(2) $\mathcal{P}_N(H) = \mathcal{P}_N(K)$;

(3) $\max \mathcal{P}_N^*(H) = \max \mathcal{P}_N^*(K)$;

(4) $\min \mathcal{P}_N(H) = \min \mathcal{P}_N(K)$.

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Theorem (Farrokhi and Saeedi [1])

$$\mathcal{P}_N \cap \left(\frac{1}{2}, 1\right] = \left\{\frac{1}{2} + \frac{1}{2i}\right\}_{i=1}^{\infty}.$$

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Conjecture (Farrokhi and Saeedi [1])

The values of $\mathcal{P}_N$ in the interval $(\frac{1}{3}, \frac{1}{2}]$ fall into the following seven sequences

$$\left\{\frac{2i+1}{5i+4}\right\}, \left\{\frac{2i+1}{5i+3}\right\}, \left\{\frac{2i+1}{5i+2}\right\}, \left\{\frac{2i+1}{5i+1}\right\}, \left\{\frac{2i+1}{4i+8}\right\}, \left\{\frac{2i+1}{4i+4}\right\}, \left\{\frac{i}{3i-6}\right\}.$$

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Conjecture (Farrokhi and Saeedi [1])

For each natural number n, the set $\mathcal{P}_N \cap (\frac{1}{n+1}, \frac{1}{n}]$ is a union of some finitely many sequences of the form $\{(ai+b)/(ci+d)\}_{i=1}^{\infty}$.

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Theorem (Farrokhi and Saeedi [1])

If G is a finite group such that $\mathcal{P}_N^*(G) \subseteq (0, \frac{1}{2}]$ or $(\frac{3}{10}, 1)$, then G is a solvable group. Moreover both of the intervals are sharp.

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Corollary

If G is a finite non-solvable group, then

$$\mathcal{P}_N^*(G) \cap \left(0, \frac{3}{10}\right] \neq \emptyset \text{ and } \mathcal{P}_N^*(G) \cap \left(\frac{1}{2}, 1\right) \neq \emptyset.$$

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Corollary

If G is a finite group with two subgroup normality degrees, then G is solvable.

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Conjecture

If G is a finite group with at most four subgroup normality degrees, then G is solvable with fairly restricted structure.

Preliminary results

Lemma

Let G_1 and G_2 be two finite groups. If $H_1 \leq G_1$ and $H_2 \leq G_2$, then

$$P_N(H_1 \times H_2, G_1 \times G_2) = P_N(H_1, G_1) \times P_N(H_2, G_2).$$

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Lemma

Let G be a group. If $N \leq H \leq G$ such that $N \trianglelefteq G$, then

$$P_N(H, G) = P_N\left(\frac{H}{N}, \frac{G}{N}\right).$$

Lemma

Let G be a finite group. If $|\mathcal{P}_N(G)| = 2$ and $x \in G$ is a p-element, then $\langle x^p \rangle \trianglelefteq G$. In particular $P_N(\langle x \rangle, G) = \frac{1}{p} + \frac{1}{d} - \frac{1}{pd}$, where $d = [G : N_G(\langle x \rangle)]$.

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Lemma

Let G be a finite group. If $|\mathcal{P}_N(G)| = 2$ and $x \in G$ is a p-element such that $\langle x \rangle \ntrianglelefteq G$, then $N_G(\langle x \rangle)/C_G(x)$ is a p-group of order less than $|x|$.

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- $\mathcal{B}$ is a 2-group of order 16 with the following presentation

$$\mathcal{B} = \langle a, b, c, d : a^2 = b^2 = c^2 = 1, d^2 = a, (cd)^2 = b, [a, b] = [a, c] = [b, c] = 1\rangle,$$

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$$\mathcal{C} = \langle a, b, c, d : a^2 = b^2 = c^4 = d^4 = [\{a, b\}, \{c, d\}] = 1, [a, b] = [c, d] = c^2 = d^2\rangle,$$

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- $\mathcal{E}_p$ is the non-abelian p-group of order p^3 and exponent p.

Main results

Theorem

Let G be a finite group. If $|\mathcal{P}_N(G)| = 2$, then either G is a nilpotent group or the hypercentral factor of G is isomorphic to $\mathcal{A}_{pq}$, for some distinct primes p and q.

Main Theorem

Let G be a finite non-nilpotent group. Then $|\mathcal{P}_N(G)| = 2$ if and only if $G = PQ \times R_1 \times \cdots \times R_n$, where $P, Q, R_1, \ldots, R_n$ are Dedekind Sylow $p, q, r_1, \ldots, r_n$-subgroups of G $(p > q)$, respectively, and either

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(1) $G/Z(G) \cong \mathcal{A}_{pq}$ if $Q, R_1, \ldots, R_n$ are Abelian;

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(1) $G/Z(G) \cong \mathcal{A}_{pq}$ if $Q, R_1, \ldots, R_n$ are Abelian;

(2) $G/Z(G) \cong \mathcal{A}_{pq} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ if Q is Abelian and R_i is a Hamiltonian 2-group for some i and R_j is Abelian if $j \neq i$;

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(3) $G/Z(G) \cong \mathcal{A}_{2p} \times \mathbb{Z}_2$ or $\mathcal{A}_{2p} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ if Q is a Hamiltonian 2-group,

where $p, q, r_1, \ldots, r_n$ are distinct primes.

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Lemma

Let G be a finite p-group. If $|\mathcal{P}_N(G)| = 2$ and $H \leq Z_2(G)$ is an elementary Abelian subgroup of order p^2, then $H \cap Z(G) \neq 1$.

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Corollary

A finite p-group G with two subgroup normality degrees has subgroup breadth 1*.*

Theorem (Cutolo, Smith and Wiegold [1], Cooper [3])

If G is a finite group with cyclic-breadth 1, then $\Phi(G) \subseteq Z_2(G)$. In particular, G is of nilpotent class at most 3.

[1] The nilpotency class of p-groups in which subgroups have few conjugates, *J. Algebra* **300**(1) (2006), 160–170.

[2] On finite p-groups with subgroups of breadth 1, *Bull. Aust. Math. Soc.* **80** (2010), 84–98.

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Lemma (Cutolo, Smith and Wiegold [2, Lemma 1])

Let G be a finite p-group of cyclic breadth 1. Then the exponent of $G/Z(G)$ is p, if p is odd or $p = cl(G) = 2$ and is 4, if $p = 2$ and $cl(G) = 3$.

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Lemma (Cutolo, Smith and Wiegold [1, Theorem A])

Let G be a finite p-group of cyclic breadth 1. Then $|G'| \leq p^2$ and consequently $[G : C_G(g)] \leq p^2$ for each $g \in G$.

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Lemma (Cutolo, Smith and Wiegold [1, Theorem B])

Let G be a finite p-group of subgroup breadth 1.

(1) *If $p = 2$, then $|G/Z(G)| \leq 16$.*

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Main results

Nilpotent case

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(1) *If $p = 2$, then $|G/Z(G)| \leq 16$.*

(2) *If $p > 2$, then $|G/Z(G)| \leq p^3$.*

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Main Theorem (continued)

Let G be a finite p-group of nilpotent class 2. Then $|\mathcal{P}_N(G)| = 2$ if and only if

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(1) $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$;

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(1) $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$;

(2) $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, $G' \cong \mathbb{Z}_p \times \mathbb{Z}_p$, G has a unique Abelian maximal subgroup M and $HZ(G) \subset M$ for each subgroup H such that $H \cap G' = 1$; or

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(3) $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $G' \cong \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ and there exists a maximal central subgroup Z of $Z(G)$ such that $G/Z \cong \mathcal{C}$. Moreover,

   (a) if $|G'| = 2$, then $[H : H \cap Z(G)] = 2$ for each $H \in \mathcal{N}(G)$,

Main Theorem (continued)

Let G be a finite p-group of nilpotent class 2. Then $|\mathcal{P}_N(G)| = 2$ if and only if

(1) $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$;

(2) $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, $G' \cong \mathbb{Z}_p \times \mathbb{Z}_p$, G has a unique Abelian maximal subgroup M and $HZ(G) \subset M$ for each subgroup H such that $H \cap G' = 1$; or

(3) $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $G' \cong \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ and there exists a maximal central subgroup Z of $Z(G)$ such that $G/Z \cong \mathcal{C}$. Moreover,

   (a) if $|G'| = 2$, then $[H : H \cap Z(G)] = 2$ for each $H \in \mathcal{N}(G)$,
   (b) if $|G'| = 4$ and $G/\mathcal{Z} \cong \mathcal{D}$ for some $\mathcal{Z} \leq Z(G)$, then $[H : H \cap Z(G)] = 2$ for every $H \in \mathcal{N}(G)$ such that $H \cap G' \neq 1$, and

Main Theorem (continued)

Let G be a finite p-group of nilpotent class 2. Then $|\mathcal{P}_N(G)| = 2$ if and only if

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(2) $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, $G' \cong \mathbb{Z}_p \times \mathbb{Z}_p$, G has a unique Abelian maximal subgroup M and $HZ(G) \subset M$ for each subgroup H such that $H \cap G' = 1$; or

(3) $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $G' \cong \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ and there exists a maximal central subgroup Z of $Z(G)$ such that $G/Z \cong \mathcal{C}$. Moreover,

   (a) if $|G'| = 2$, then $[H : H \cap Z(G)] = 2$ for each $H \in \mathcal{N}(G)$,
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   (c) if $|G'| = 4$ and $G/\mathcal{Z} \ncong \mathcal{D}$ for all $\mathcal{Z} \leq Z(G)$, then $[H : H \cap Z(G)] = 2$ and for each $H \in \mathcal{N}(G)$, either $H \cap G' \neq 1$ or $[H, G] \subseteq (G' \setminus Z) \cup \{1\}$.

Main Theorem (continued)

Let G be a finite p-group of nilpotent class 3 (p odd). Then $|\mathcal{P}_N(G)| = 2$ if and only if $G/Z(G) \cong \mathcal{E}_p$, $C_G(Z_2(G))$ is an Abelian maximal subgroup of G and for each $H \in \mathcal{N}(G)$, either $H \leq C_G(Z_2(G))$ or $\gamma_3(G) \leq H$.

Main Theorem (continued)

Let G be a finite p-group of nilpotent class 3 (p odd). Then $|\mathcal{P}_N(G)| = 2$ if and only if $G/Z(G) \cong \mathcal{E}_p$, $C_G(Z_2(G))$ is an Abelian maximal subgroup of G and for each $H \in \mathcal{N}(G)$, either $H \leq C_G(Z_2(G))$ or $\gamma_3(G) \leq H$.

Main Theorem (continued)

Let G be a finite 2-group of nilpotent class 3. Then $|\mathcal{P}_N(G)| = 2$ if and only if

Main Theorem (continued)

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Main Theorem (continued)

Let G be a finite 2-group of nilpotent class 3. Then $|\mathcal{P}_N(G)| = 2$ if and only if

(1) $G/Z(G) \cong D_8$ and either $H \subseteq Z_2(G)$ or $\gamma_3(G) \leq H$; or

Main Theorem (continued)

Let G be a finite p-group of nilpotent class 3 (p odd). Then $|\mathcal{P}_N(G)| = 2$ if and only if $G/Z(G) \cong \mathcal{E}_p$, $C_G(Z_2(G))$ is an Abelian maximal subgroup of G and for each $H \in \mathcal{N}(G)$, either $H \leq C_G(Z_2(G))$ or $\gamma_3(G) \leq H$.

Main Theorem (continued)

Let G be a finite 2-group of nilpotent class 3. Then $|\mathcal{P}_N(G)| = 2$ if and only if

(1) $G/Z(G) \cong D_8$ and either $H \subseteq Z_2(G)$ or $\gamma_3(G) \leq H$; or

(2) $G/Z(G) \cong \mathcal{B}$ and for each $H \in \mathcal{N}(G)$, either $\gamma_3(G) \leq H$ or

$$\frac{HZ(G)}{Z(G)} = \left\langle \frac{Z_2(G)}{Z(G)} \Big\backslash \left(\frac{G}{Z(G)}\right)^{\{2\}} \right\rangle \text{ and } \frac{N_G(H)}{Z(G)} = \Omega_1\left(\frac{G}{Z(G)}\right).$$

Corollary 1

Let G be a finite group. If $|\mathcal{P}_N(G)| = 2$, then

$$\mathcal{P}_N(G) = \left\{\frac{1}{p} + \frac{1}{q} - \frac{1}{pq}, 1\right\}$$

for some primes p and q.

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for some primes p and q.

Corollary 2

Let G be a finite group. Then $|\mathcal{P}_N(G)| = 2$ if and only if there exist primes p and q such that $[G : N_G(H)] = p$ and $[H : H \cap H^g] = q$ for each $H \in \mathcal{N}(G)$ and $g \in G \setminus N_G(H)$. In particular G is solvable, and it is nilpotent if and only if $p = q$.

Corollary 3

If G is a finite group such that $|\mathcal{P}_N(G)| = 2$, then $\mathcal{P}_N(H) \subseteq \mathcal{P}_N(G)$ for each subgroup H of G.

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If G is a finite nilpotent group, then $\mathcal{P}_N(H) \subseteq \mathcal{P}_N(G)$ for each subgroup H of G.

Conjecture

If G is a finite group, which is neither a Dedekind group nor a minimal non-Dedekind group such that $\mathcal{P}_N^*(H) \cap \mathcal{P}_N^*(G) = \emptyset$ for each proper subgroup H of G, then $G \cong A_5$.

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Thank You