Self 2-distance graphs with a forbidden structure

M. Farrokhi D. G.
(Joint work with A. Azimi)

Muroran Institute of Technology

32nd Symposium on Algebraic Combinatorics,
Kanazawa University,
June 23, 2015

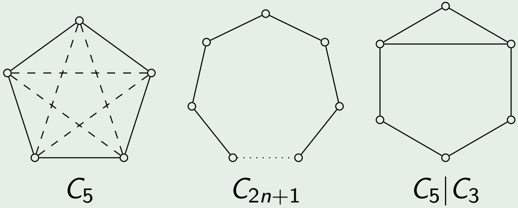
Definition

Let Γ be a graph and n be a natural number. The *n-distance graph* of Γ, denoted by Γ_n, is the graphs with the same vertex set as Γ such that two distinct vertices are adjacent in Γ_n if they are at distance n in Γ.

Definition

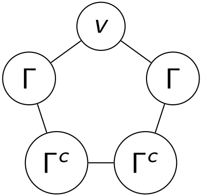
Let Γ be a graph and n be a natural number. The *n-distance graph* of Γ, denoted by Γ_n, is the graphs with the same vertex set as Γ such that two distinct vertices are adjacent in Γ_n if they are at distance n in Γ.

Example



Proposition

Every graph is an induced subgraph of a self 2-distance graph.



Lemma

Let Γ *be a graph. Then* $\mathrm{diam}(\Gamma) = 2$ *if and only if* $\Gamma_2 = \Gamma^c$.

Lemma

Let Γ be a graph. Then $\mathrm{diam}(\Gamma) = 2$ if and only if $\Gamma_2 = \Gamma^c$.

Proposition

Let Γ be a self-complementary graph with diameter two. Then $\Gamma_2 \cong \Gamma$.

Lemma

Let Γ be a graph. Then $\mathrm{diam}(\Gamma) = 2$ if and only if $\Gamma_2 = \Gamma^c$.

Proposition

Let Γ be a self-complementary graph with diameter two. Then $\Gamma_2 \cong \Gamma$.

Lemma

If Γ is a self 2-distance graph which is not an odd cycle, then $\mathrm{gr}(\Gamma) = 3$.

Theorem

Let Γ be a connected self 2-distance graph with no square. Then either Γ is an odd cycle or it is the edged product $C_5|C_3$.

Theorem

Let Γ be a connected self 2-distance graph with no square. Then either Γ is an odd cycle or it is the edged product $C_5|C_3$.

Step 1.

$\Delta(\Gamma) = 3$.

Theorem

Let Γ be a connected self 2-distance graph with no square. Then either Γ is an odd cycle or it is the edged product $C_5|C_3$.

Step 1.

$\Delta(\Gamma) = 3$.

Step 2.

Γ has no $C_5|C_3$ subgraph.

Theorem

Let Γ be a connected self 2-distance graph with no square. Then either Γ is an odd cycle or it is the edged product $C_5|C_3$.

Step 1.

$\Delta(\Gamma) = 3$.

Step 2.

Γ has no $C_5|C_3$ subgraph.

Theorem

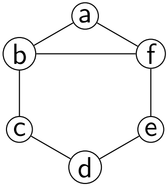
Let Γ be a connected self 2-distance graph with no square. Then either Γ is an odd cycle or it is the edged product $C_5|C_3$.

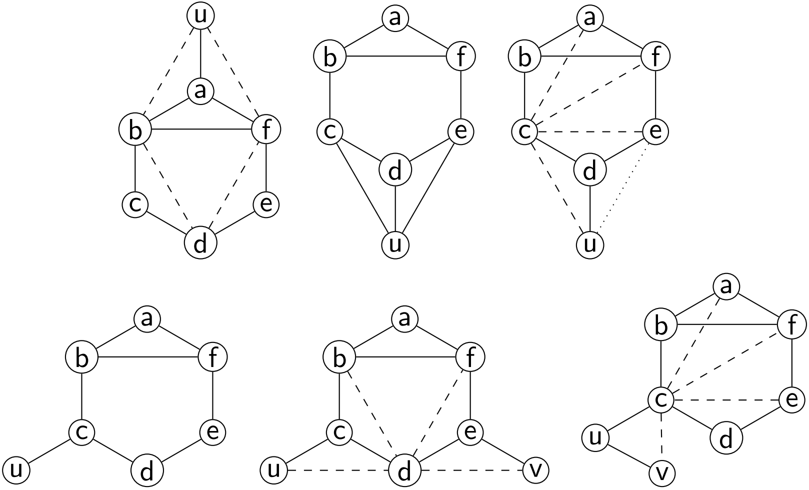
Step 1.

$\Delta(\Gamma) = 3$.

Step 2.

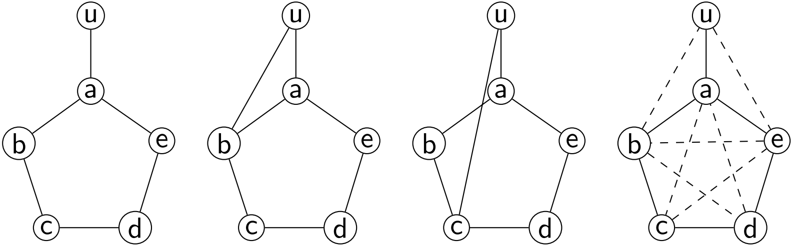
Γ has no $C_5|C_3$ subgraph.




u
a
b
f
c
e
d
a
b
f
c
e
d
u
a
b
f
c
e
d
u
a
b
f
c
e
u
d
a
b
f
c
e
u
d
v
a
b
f
c
e
u
d
v

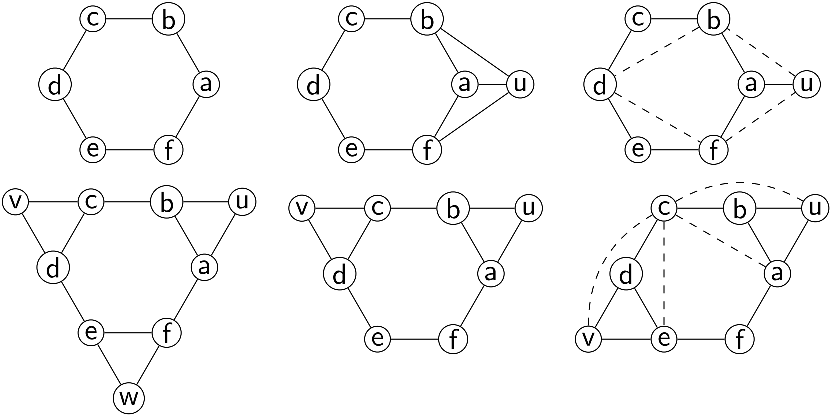
Step 3.

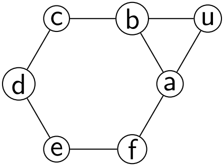
Γ has no pentagon.

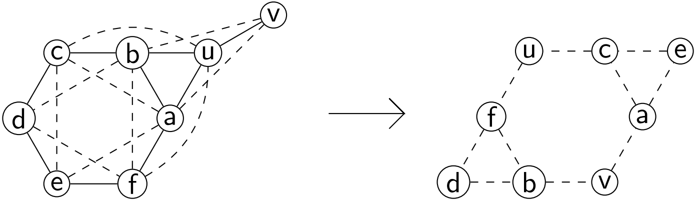


Step 4.

Γ has no hexagon.

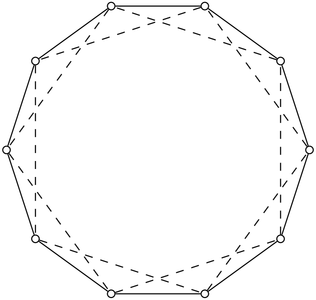


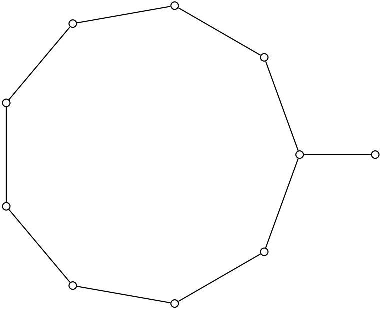

c
b
u
d
a
e
f

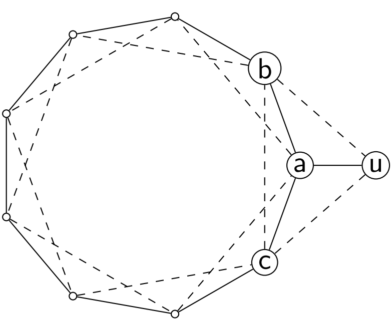

v
c
b
u
d
a
e
f
u
c
e
f
a
d
b
v

Step 5.

Γ has no cycle of length exceeding three.

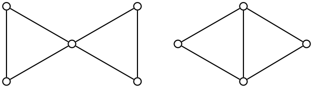





b
a
u
c
d
b
u
a
c

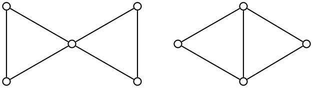
Step 6.

Triangles in Γ have disjoint vertices.



Step 6.

Triangles in Γ have disjoint vertices.



Step 7.

$|E(L(\Gamma))| = |E(\Gamma_2)| + 3\nabla(\Gamma)$.

Step 8.

Γ has exactly three pendants.

Step 8.

Γ has exactly three pendants.

Proof.

Let $\Gamma' = \Gamma(\Delta \mapsto \bullet)$. Then Γ' is a tree.

Step 8.

Γ has exactly three pendants.

Proof.

Let $\Gamma' = \Gamma(\Delta \mapsto \bullet)$. Then Γ' is a tree.

Step 8.

Γ has exactly three pendants.

Proof.

Let $\Gamma' = \Gamma(\Delta \mapsto \bullet)$. Then Γ' is a tree.

Put $v = |V(\Gamma)|$, $v' = |V(\Gamma')|$, $e = |E(\Gamma)|$ and $e' = |E(\Gamma')|$. Then

$$\begin{cases} v' = v - 2\nabla(\Gamma), \\ e' = e - 3\nabla(\Gamma), \\ e' = v' - 1 \end{cases} \Longrightarrow \nabla(\Gamma) = e - v + 1$$

Step 8.

Γ has exactly three pendants.

Proof.

Let $\Gamma' = \Gamma(\Delta \mapsto \bullet)$. Then Γ' is a tree.

Put $v = |V(\Gamma)|$, $v' = |V(\Gamma')|$, $e = |E(\Gamma)|$ and $e' = |E(\Gamma')|$. Then

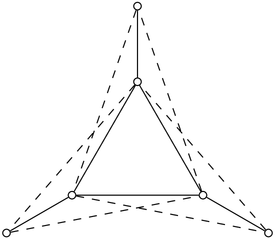
$$\begin{cases} v' = v - 2\nabla(\Gamma), \\ e' = e - 3\nabla(\Gamma), \\ e' = v' - 1 \end{cases} \implies \nabla(\Gamma) = e - v + 1$$

If n_i $(i = 1, 2, 3)$ is the number of vertices of degree i in Γ, then

$$\begin{cases} |V(\Gamma)| = n_1 + n_2 + n_3, \\ |E(\Gamma)| = \frac{1}{2}(n_1 + 2n_2 + 3n_3), \\ |E(L(\Gamma))| = n_2 + 3n_3, \end{cases} \implies n_1 = 3$$

Step 9.

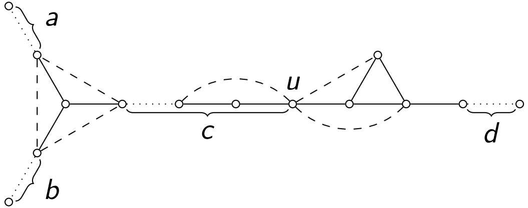
Every triangle of Γ has a vertex of degree 2.



Step 10.

Γ has a triangle with two vertices of degree 2.

Suppose on the contrary. Then Γ must be



for some $a, b, d \geq 1$ and $1 \neq c \geq 0$. Then $d_\Gamma(\text{triangle}, \text{claw}) = c$ and

$$d_{\Gamma_2}(\text{triangle}, \text{claw}) = \begin{cases} \frac{c+4}{2}, & c \text{ is even}, \\ \\ \frac{c-3}{2}, & c \text{ is odd}. \end{cases}$$

Hence, $c = 4$. On the other hand,

$$|E(\Gamma)| = a + b + c + d + 4$$

and

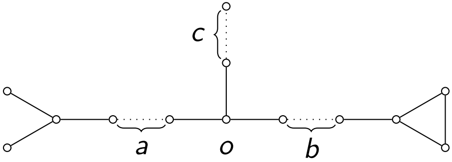
$$|E(\Gamma_2)| = a + b + c + d + 5 - [\frac{1}{d}]$$

when $c \geq 2$. Then $d = 1$ and $a \pm 1, b \mp 1 = 2, 3$, from which it follows that $\Gamma_2 \ncong \Gamma$, a contradiction.

Step 10.

Γ has no triangle with two vertices of degree 2.

For some $a, b \geq 0$ and $c \geq 1$, Γ is isomorphic to



Since

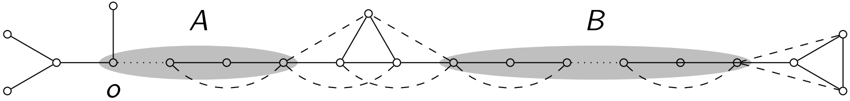
$$|E(\Gamma)| = a + b + c + 9$$

and

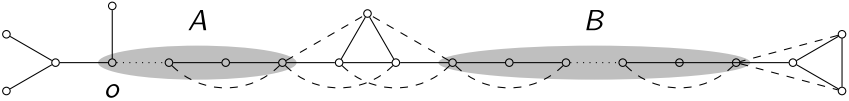
$$|E(\Gamma_2)| = a + b + c + 8 + [\frac{1}{a+1}] + [\frac{1}{b+1}],$$

we have $ab = 0$. Also, $c = 1$.

First assume that $a = 0$.

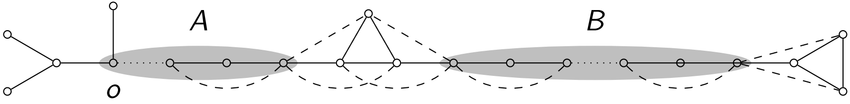


First assume that $a = 0$.



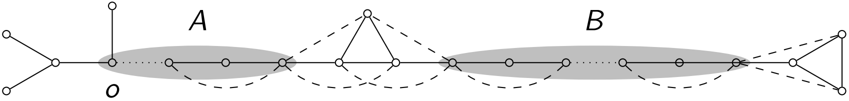
- $|A| \geq 3$ otherwise A has a vertex of degree ≥ 4 in Γ_2.

First assume that $a = 0$.



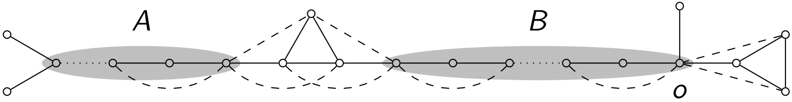
- $|A| \geq 3$ otherwise A has a vertex of degree ≥ 4 in Γ_2.
- $|B| \geq 4$ since triangles in Γ_2 are at distance at least five.

First assume that $a = 0$.

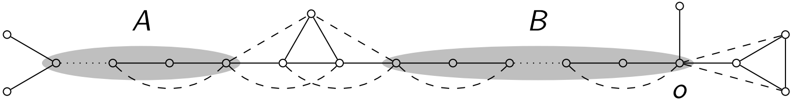


- $|A| \geq 3$ otherwise A has a vertex of degree ≥ 4 in Γ_2.
- $|B| \geq 4$ since triangles in Γ_2 are at distance at least five.
- Γ has three claws, a contradiction.

Finally assume that $b = 0$.

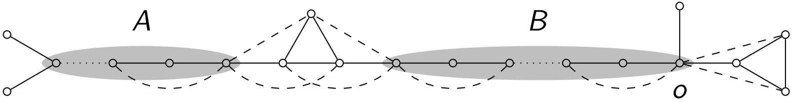


Finally assume that $b = 0$.



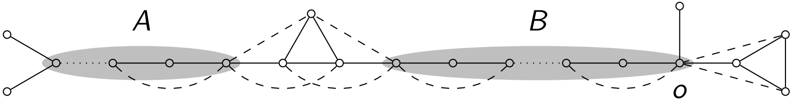
• $|A| \neq 1$ otherwise two induced claws are connected with two triangles with distance zero while it is not true in Γ_2.

Finally assume that $b = 0$.



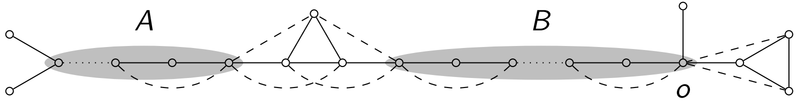
• $|A| \neq 1$ otherwise two induced claws are connected with two triangles with distance zero while it is not true in Γ_2.
• $|A| \geq 3$ otherwise $|A| = 2$ and A has a vertex of degree four in Γ_2.

Finally assume that $b = 0$.



• $|A| \neq 1$ otherwise two induced claws are connected with two triangles with distance zero while it is not true in Γ_2.
• $|A| \geq 3$ otherwise $|A| = 2$ and A has a vertex of degree four in Γ_2.
• $|B| \geq 3$ for otherwise it has a vertex of degree four in Γ_2.

Finally assume that $b = 0$.



- $|A| \neq 1$ otherwise two induced claws are connected with two triangles with distance zero while it is not true in Γ_2.
- $|A| \geq 3$ otherwise $|A| = 2$ and A has a vertex of degree four in Γ_2.
- $|B| \geq 3$ for otherwise it has a vertex of degree four in Γ_2.
- Γ_2 has three induced claws, a contradiction.

Theorem

Let Γ be a self 2-distance graph with disjoint triangles. Then either Γ is an odd cycle or it is the edged product $C_5|C_3$.

Theorem

Let Γ be a self 2-distance graph with disjoint triangles. Then either Γ is an odd cycle or it is the edged product $C_5|C_3$.

Step 1.

$\Delta(\Gamma) = 3$.

Theorem

Let Γ be a self 2-distance graph with disjoint triangles. Then either Γ is an odd cycle or it is the edged product $C_5|C_3$.

Step 1.

$\Delta(\Gamma) = 3$.

Step 2.

If Γ has no $C_5|C_3$ subgraph.

Theorem

Let Γ be a self 2-distance graph with disjoint triangles. Then either Γ is an odd cycle or it is the edged product $C_5|C_3$.

Step 1.

$\Delta(\Gamma) = 3$.

Step 2.

If Γ has no $C_5|C_3$ subgraph.

Theorem

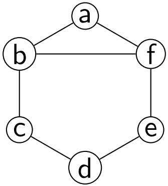
Let Γ be a self 2-distance graph with disjoint triangles. Then either Γ is an odd cycle or it is the edged product $C_5|C_3$.

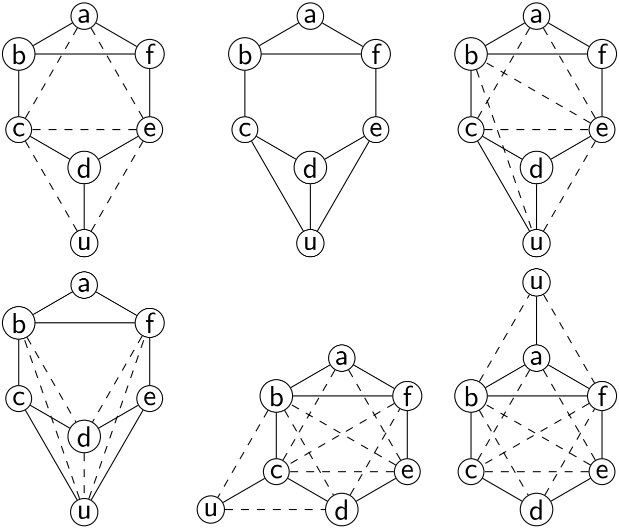
Step 1.

$\Delta(\Gamma) = 3$.

Step 2.

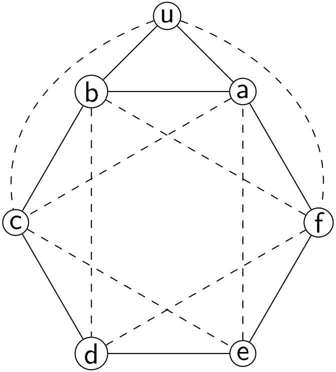
If Γ has no $C_5|C_3$ subgraph.




a
b
f
c
e
d
u

Step 3.

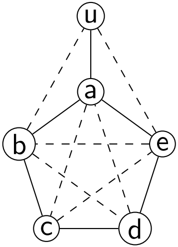
The graph Γ has no hexagon.

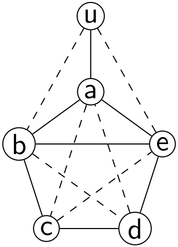


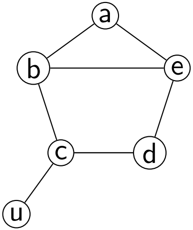
u is adjacent to a, b, d, e in $(\Gamma_2)_2$

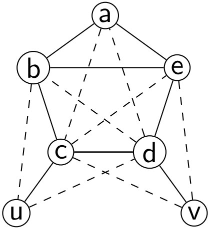
Step 4.

The graph Γ has no pentagon.

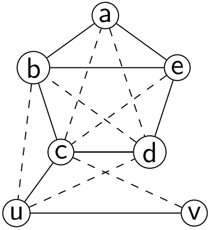






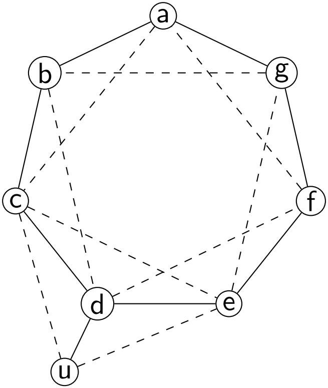


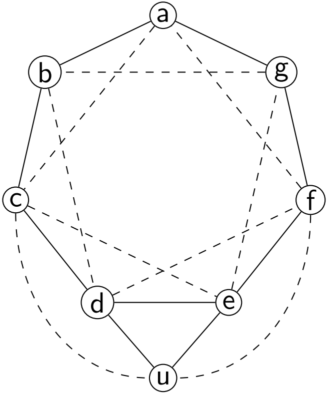
a is adjacent to
b, e, u, v in $(\Gamma_2)_2$



Step 5.

The graph Γ has no heptagon.

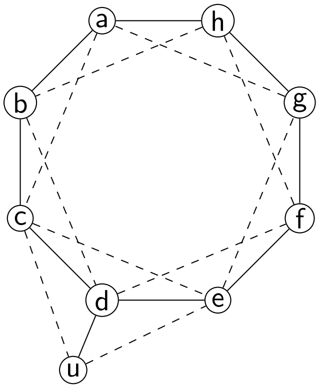


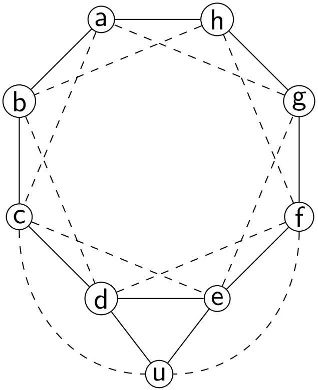


$(\Gamma_2)_2$ has triangles $\{a, e, u\}$ and $\{a, d, u\}$

Step 6.

The graph Γ has no octagon.

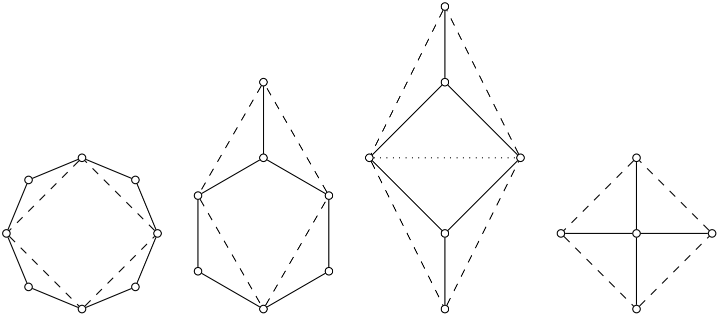




u is adjacent to a, d, e, h in $(\Gamma_2)_2$

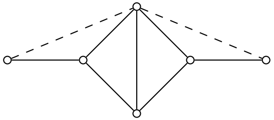
Step 7.

The graph Γ has no square.



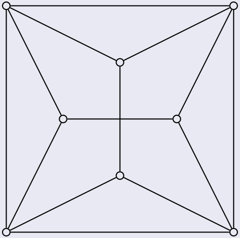
Corollary

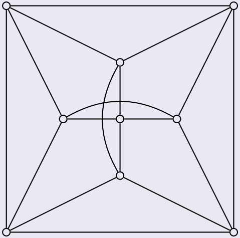
There is no cubic self 2-distance graph.



Theorem

Let Γ be a self 2-distance graph with no diamond as subgraph. Then either Γ is an odd cycle, it is the edged product $C_5|C_3$, or it is isomorphic to one the following graphs:





Definition

A graph Γ with v vertices is strongly regular of degree k if there are integers λ and μ such that every two adjacent vertices have λ common neighbours and every two non-adjacent vertices have μ common neighbours. The numbers (v, k, λ, μ) are the parameters of the corresponding graph.

[1] J. J. Seidel, A survey of two-graphs in *Proc. Int. Coil. Teorie Combinatorie*, **I** (1973), Acad. Naz. Lincei (1976), 481–511.

Definition

A graph Γ with v vertices is strongly regular of degree k if there are integers λ and μ such that every two adjacent vertices have λ common neighbours and every two non-adjacent vertices have μ common neighbours. The numbers (v, k, λ, μ) are the parameters of the corresponding graph.

Theorem (Seidel[1])

Every strongly regular self 2-distance graphs is a self-complimentary graph and has parameters $(4t + 1, 2t, t - 1, t)$ *where the number of vertices is a sum of two squares.*

[1] J. J. Seidel, A survey of two-graphs in *Proc. Int. Coil. Teorie Combinatorie*, **I** (1973), Acad. Naz. Lincei (1976), 481–511.

Conjecture

There are no regular self 2-distance graphs of odd degree.

Conjecture

There are no regular self 2-distance graphs of odd degree.

Conjecture

Every self 2-distance graph is 2-connected.

Thank You for Your Attention!