Positive matching decomposition of graphs

M. Farrokhi D. G.

IASBS
Department of Mathematics

AS-SMS
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Preliminaries

- $[n] = \{1, \ldots, n\}$

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- $[n] = \{1, \ldots, n\}$
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- Each vertex $x \in X_i \mapsto e_i \in \mathbb{R}^c$

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The followings are equivalent:

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- A set of vectors X in $\mathbb{R}^d$ is in general position if

any d-subset of X is linearly independent.

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- $S = \mathbb{R}[x_1, \dots, x_d]$
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Positive Matching Decompositions (PMD)

Definition (Positive matchings[1] of graphs)

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$$M \text{ is positive if } M = \{e \in E \colon \omega(e) > 0\}$$
$$\text{for some weight function } \omega \colon V \to \mathbb{R}$$

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Remark

$\mathbb{R}$ can be replaced with $\mathbb{Z}$

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$M_1, \ldots, M_p$ is a pmd of Γ if

M_i is a positive matching in $\Gamma \setminus M_1 \cup \cdots \cup M_{i-1}$,

for all i with $1 \leq i \leq p$

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$$\mathrm{pmd}(\Gamma) = \min(p : \ M_1, \ldots, M_p \text{ is a pmd of } \Gamma)$$

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- $\mathrm{pmd}(C_n) = 3$ for all $n \geq 3$

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Question

Any characterization of graphs Γ with $\mathrm{pmd}(\Gamma) = 3$?

Theorem (A. Conca and V. Welker, 2019[1])

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Theorem (A. Conca and V. Welker, 2019[1])

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$L_\Gamma^{\mathbb{K}}(d)$ is a radical and complete intersection

$L_\Gamma^{\mathbb{K}}(d)$ is prime if $d \geq \mathrm{pmd}(\Gamma) + 1$

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Theorem (A. Conca and V. Welker, 2019[1])

If Γ is a simple graph of order $n > 1$, then

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Theorem (A. Conca and V. Welker, 2019[1])

If Γ is a simple graph of order $n > 1$, then

(1) $\mathrm{pmd}(\Gamma) \leq 2n - 3$,

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(1) $\mathrm{pmd}(\Gamma) \leq 2n-3$,

(2) $\mathrm{pmd}(\Gamma) \leq n-1$ *if Γ is a bipartite graph,*

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(3) $\mathrm{pmd}(\Gamma) \geq \Delta(\Gamma)$ *with equality if* Γ *is a forest*.

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W is alternathing w.r.t. M if

the edges of W alternate between M and M^c

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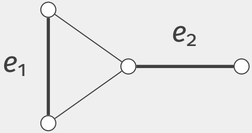

Tree-like Graphs

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$$\mathrm{pmd}(\Gamma') = \max\{\mathrm{pmd}(\Gamma), \Delta(\Gamma')\}.$$

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$$\Delta(\Gamma) \leq \mathrm{pmd}(\Gamma) \leq \Delta(\Gamma) + 1.$$

Definition (Cactus graph)

A connected graph with any two cycles having at most one vertex in common.

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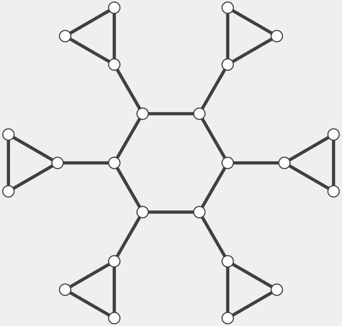
$\mathrm{pmd}(\Gamma) = \Delta(\Gamma)$ *if* Γ *is triangle-free and non-cycle*

Problem

Any characterization of cacti with given pmd?



A cactus with $\mathrm{pmd} = 4$

Multipartite Graphs

Theorem

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- $\mathrm{pmd}(K_n) = 2n - 3$ *for all* $n \geq 2$.

Complete multipartite graphs

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Conjecture

$\mathrm{pmd}(\Gamma) \leq 2\Delta(\Gamma) - 1$ for any graph Γ.

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$$\mathrm{pmd}(\Gamma) \leq \mathrm{pmd}(\Gamma - M) + \Delta(\Gamma) - 1.$$

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Cartesian Product of Graphs

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CARTESIAN PRODUCTS

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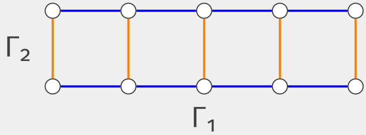

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$\mathrm{pmd}(\Gamma_1 \square \Gamma_2) \leq \min\{\mathrm{pmd}(\Gamma_i)\chi(\Gamma_j) + \mathrm{pmd}(\Gamma_j) \colon \{i, j\} = \{1, 2\}\}.$

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$$\operatorname{pmd}(\square^n \Gamma) \leq ((n-1)\chi(\Gamma) + 1) \cdot \operatorname{pmd}(\Gamma).$$

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Corollary

$\mathrm{pmd}(Q_n) \leq 2n-1.$

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$F_1, \ldots, F_n$ is a forest decomposition of Γ

$$(\Gamma - E(F_1) \cup \cdots \cup E(F_{i-1}))[V(F_i)] \text{ is a forest,}$$

$$\text{for all } i = 1, \ldots, n.$$

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$$\mathrm{pmd}(\Gamma_1 \square \Gamma_2) \leq n \cdot \mathrm{pmd}(\Gamma_2) + \sum_{i=1}^{n} \max\{\Delta(F'_i), \chi(\Gamma_2)\}.$$

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Cartesian Products

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$\mathcal{P}$-covering number of $\mathcal{S}$ by $\mathcal{F}$ is

$$\mathrm{cov}_{\mathcal{P}}(\mathcal{S}, \mathcal{F}) = \min\{|\mathcal{C}| : \mathcal{C} \text{ is a } \mathcal{P}\text{-cover of } \mathcal{S} \text{ by } \mathcal{F}\}$$

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$\mathcal{E}$ is an acyclic family of edge-sets (AFE) if

$$\cup_{E \in \mathcal{E}} \Gamma[E] \text{ is acyclic.}$$

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$$\mathrm{pmd}(\Gamma_1 \Box \Gamma_2) \leq \mathrm{cov}_{\mathrm{AFE}}(\chi_j * \Gamma_i, \mathcal{M}_i) + p_j.$$

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$$\mathrm{pmd}(\Gamma_1 \square \Gamma_2) \leq \mathrm{pmd}(\Gamma_2) + n \cdot \max\left\{\sum_{i=1}^{n} \Delta(F_i'), \chi(\Gamma_2)\right\}.$$

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Conjecture

There exists a constant $c > 0$ such that

$$\mathrm{pmd}(\Gamma_1) + \mathrm{pmd}(\Gamma_2) \leq \mathrm{pmd}(\Gamma_1 \square \Gamma_2) + c$$

for all graphs Γ_1 and Γ_2.

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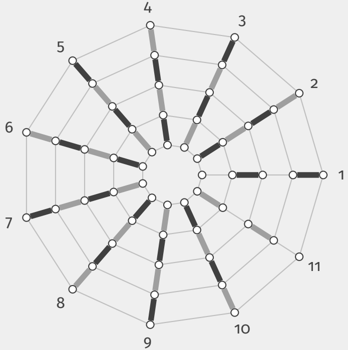
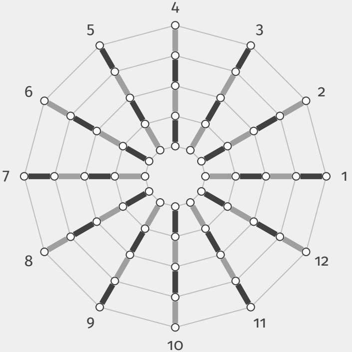
Cartesian Products



n is odd



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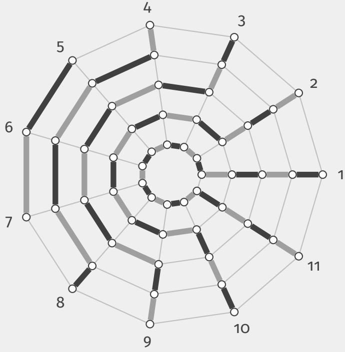
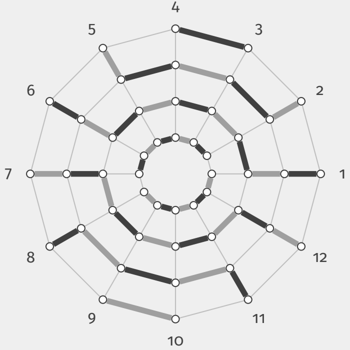


$n > 2m$ is odd



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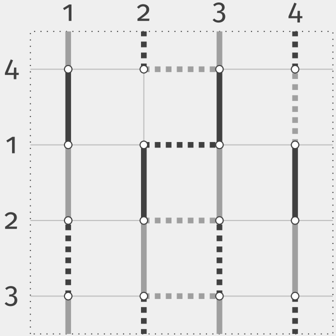

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Cayley Graphs

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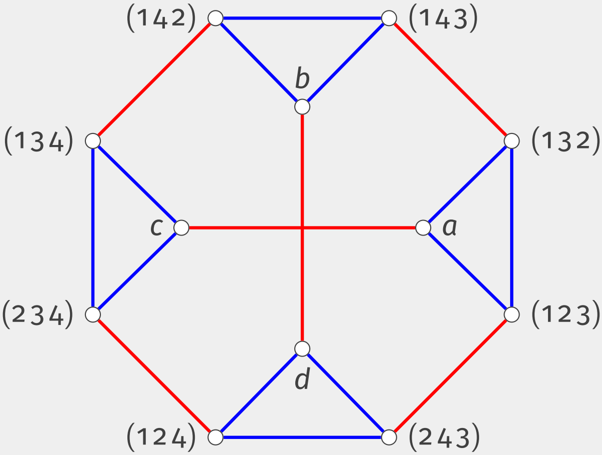
$$\mathrm{Cay}(G, C) = \Big(G, \{\text{edges } \{g, gc\}\}\Big)$$



$\alpha = (1\ 2\ 3)$ and $\beta = (1\ 2)(3\ 4)$

$a = ()$, $b = (1\ 4)(2\ 3)$, $c = (1\ 2)(3\ 4)$, and $d = (1\ 3)(2\ 4)$

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(4) $\Gamma[H \cdot e]$ *is a* ***union of edges*** *iff*

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(3) *if* $I(C) = \mathit{Involutions}(C)$ *and* $H \in \{H_c\}_{c \in C}$ *has min. order, then*

$$\mathrm{pmd}(\Gamma) \leq \left(|C| - \frac{1}{2}|I(C)|\right)[G : H].$$

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- *Merging the last two (suitably chosen) matchings yields*

$$\mathrm{pmd}(Q_n) \leq 2n - 1.$$

Thanks!