The Fibonacci Sequence

M. Farrokhi D. G.

IASBS / Department of Mathematics

November 5, 2019

History

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Organizations

The Fibonacci Quarterly

Official Publication of The Fibonacci Association

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Definitions and Basic Properties

Definition

The Fibonacci sequence $\{F_n\}$

$$\ldots, 13, -8, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, 13, \ldots$$

is defined by two two consecutive terms, say $F_0 = 0$ and $F_1 = 1$, and satisfies the recursive relation

$$F_n = F_{n-1} + F_{n-2}$$

for all integers n. The related Lucas sequence $\{L_n\}$

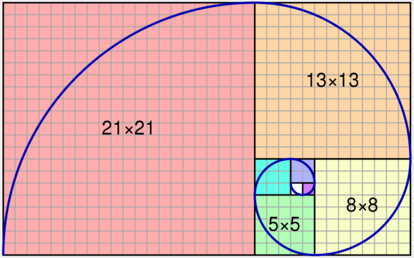
$$\ldots, -11, 7, -4, 3, -1, 2, 1, 3, 4, 7, 11, \ldots$$

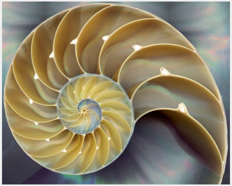
is defined analogously by $L_0 = 2$, $L_1 = 1$, and satisfies the same recursive relation

$$L_n = L_{n-1} + L_{n-2}$$

for all integers n.

Fibonacci numbers in nature





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- $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$
- $\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2}$

- $\gcd(F_m, F_n) = F_{\gcd(m,n)}$

[1]M. Farrokhi D. G., Some remarks on the equation $F_n = kF_m$ in Fibonacci numbers, *J. Integer Seq.* **10**(5) (2007), Article 7.

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Basic Properties: Divisibility

- $\gcd(F_m, F_n) = F_{\gcd(m,n)}$
- $\operatorname{lcm}(F_m, F_n) = F_k$ if and only if $F_k = F_m$ or F_n (MFDG, 2007[1])

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- $p \mid F_{p-\left(\frac{5}{p}\right)}$ for all primes p

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Theorem (Cornelis Gerrit Lekkerkerker, 1952[3] and Edouard Zeckendorf, 1972[4])

Every natural number can be written (uniquely) as a sum of non-consecutive Fibonacci numbers.

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Example

$$100 = 89 + 8 + 3 = F_{11} + F_6 + F_4 = (10000101000)_F.$$

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Let

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}.$$

Then

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n$$

are called the *Binet's formulas* of the Fibonacci and Lucas numbers, respectively.

The number

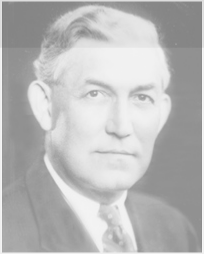
$$\alpha = \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = 1 + \cfrac{1}{1 + \cfrac{1}{1+\cfrac{1}{1+\cdots}}}$$

is called the *golden ratio*.

Why Studying the Fibonacci Sequence?

Theorem (R. D. Carmichael, 1913[5])

Let $n > 1$ be a natural number.



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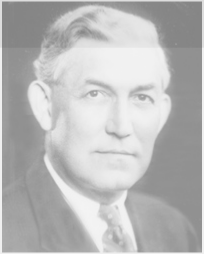

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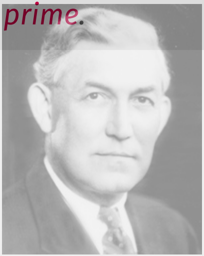

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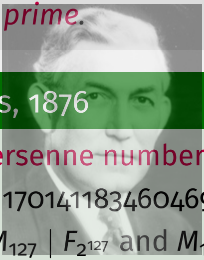
Édouard Lucas, 1876

The 39-digit Mersenne number

$$M_{127} = 170141183460469231731687303715884105727$$

is a prime for $M_{127} \mid F_{2^{127}}$ and $M_{127} \nmid F_{2^{126}}$.



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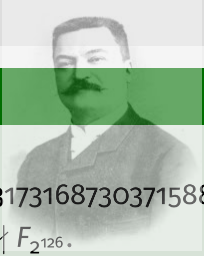

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The *Hosoya index* $Z(\Gamma)$ of a graph Γ is the number of its matchings.

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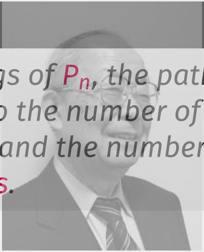
Theorem

The number of matchings of P_n, the path of order n, is equal to $Z(P_n) = F_{n+1}$. This is also the number of partial tilings of the $1 \times n$ rectangle by dominoes, and the number of tilings of the $2 \times n$ rectangle with dominoes.



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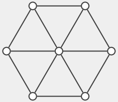
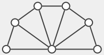
The wheel graph W_n with n spokes contains $L_{2n} - 2$ spanning trees. Also, the fan graph F_n with n triangles contains F_{2n} spanning trees.

Wheel graph W_6   Fan graph F_6

[7] A. J. Q. Hilton, Spanning trees and Fibonacci and Lucas numbers, *Fibonacci Quart.* **12** (1974), 259–262.

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Theorem (Jacob Siehler, 2019[8])

The wheel graph W_n with n spokes contains nF_{2n-1} spanning unicycles.

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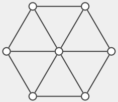

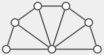

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Determinants of lower Hessenberg matrices

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A square matrix $[a_{ij}]_{n\times n}$ for which $a_{ij} = 0$ for all $j > i + 1$ is called a *lower Hessenberg matrix*.

[9] N. D. Cahill, J. R. D'Errico, D. A. Narayan, and J. Y. Narayan, Fibonacci determinants, *College Math. J.* **33**(3) (2002), 221–225.

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A square matrix $[a_{ij}]_{n\times n}$ for which $a_{ij} = 0$ for all $j > i + 1$ is called a *lower Hessenberg matrix*.

Theorem (N. D. Cahill, J. R. D'Errico, D. A. Narayan, and J. Y. Narayan, 2002[9])

$$\det \begin{pmatrix} 1 & i & 0 & 0 & \cdots & 0 \\ i & 1 & i & 0 & \cdots & 0 \\ 0 & i & 1 & i & \cdots & 0 \\ 0 & 0 & i & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & i \\ 0 & 0 & 0 & \cdots & i & 1 \end{pmatrix} = F_{n+1}$$

[9] N. D. Cahill, J. R. D'Errico, D. A. Narayan, and J. Y. Narayan, Fibonacci determinants, *College Math. J.* **33**(3) (2002), 221–225.

Theorem (Continued)

$$\det\begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & -1 & 0 & \cdots & 0 \\ 1 & 1 & 2 & -1 & \cdots & 0 \\ 1 & 1 & 1 & 2 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & -1 \\ 1 & 1 & 1 & \cdots & 1 & 2 \end{pmatrix} = F_{2n}$$

$$\det\begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & -1 & 0 & \cdots & 0 \\ 1 & 1 & 2 & -1 & \cdots & 0 \\ 1 & 1 & 1 & 2 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & -1 \\ 1 & 1 & 1 & \cdots & 1 & 2 \end{pmatrix} = F_{2n+1}$$

Determinants of lower Hessenberg matrices

Theorem (Continued)

$$\det \begin{pmatrix} 2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 2 & 1 & \cdots & 0 \\ 1 & 1 & 1 & 2 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 1 \\ 1 & 1 & 1 & \cdots & 1 & 2 \end{pmatrix} = F_{n+2}$$

$$\det \begin{pmatrix} 3 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 3 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 3 & -1 & \cdots & 0 \\ 0 & 0 & -1 & 3 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & 0 & \cdots & -1 & 3 \end{pmatrix} = F_{n+1}$$

Theorem (E. Lucas, 1876[10] and J. Wasteels, 1902[11])

The positive integer x is a Fibonacci number if and only if there exists an integer y such that

$$y^2 - xy - x^2 = \pm 1.$$

James P. Jones

[10] E. Lucas, *Nouv. Corresp. Math.* **2** (1876), 201–206.

[11] J. Wasteels, *Mathesis* **2**(3) (1902), 60–62.

[12] J. P. Jones, Diophantine representation of the Fibonacci numbers, *Fibonacci Quart.* **13** (1975), 84–88.

Diophantine equations and representations

Theorem (E. Lucas, 1876[10] and J. Wasteels, 1902[11])

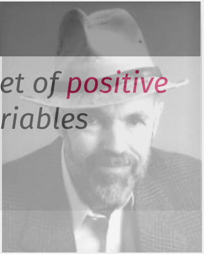
The positive integer x is a Fibonacci number if and only if there exists an integer y such that

$$y^2 - xy - x^2 = \pm 1.$$

Theorem (James P. Jones, 1975[12])

The set of Fibonacci numbers is identical with the set of positive values of a polynomial of the fifth degree in two variables

$$2xy^4 + x^2y^3 - 2x^3y^2 - y^5 - x^4y + 2y.$$



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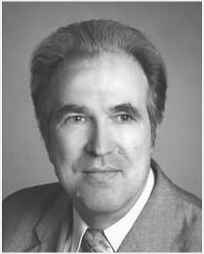
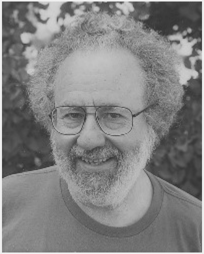
Problem

Find an algorithm to determine whether a given polynomial Diophantine equation with integer coefficients has an integer solution.

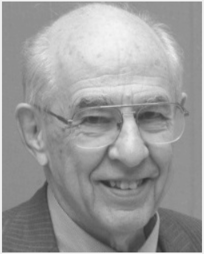


Yuri Matiyasevich

Julia Robinson



Martin Davis



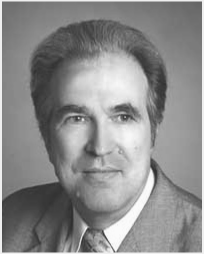
Hilary Putnam

Problem

Find an algorithm to determine whether a given polynomial Diophantine equation with integer coefficients has an integer solution.

Matiyasevich-Robinson-Davis-Putnam's Theorem

Every computably enumerable set is Diophantine.



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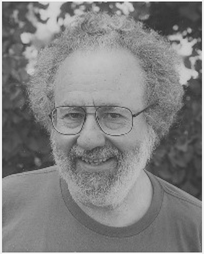
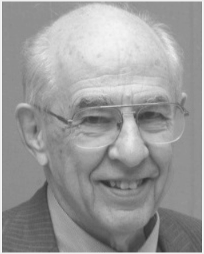
Julia Robinson



Martin Davis



Hilary Putnam

Identities

Theorem (N. D. Cahill, J.R. D'Errico, and J. P. Spence, 2002[13])

$$F_{2n} = \prod_{k=1}^{n-1} \left(3 - 2\cos\frac{\pi k}{n}\right) \quad \textit{and} \quad F_n = i^{n-1}\frac{\sin\left(n\cos^{-1}(-\frac{i}{2})\right)}{\sin\left(\cos^{-1}(-\frac{i}{2})\right)}.$$

[13] N. D. Cahill, J.R. D'Errico, and J. P. Spence, Complex Factorizations of the Fibonacci and Lucas Numbers, *Fibonacci Quart.* **41**(1) (2003), 13–19.

[14] N. D. Cahill and D. A. Narayan, Fibonacci and Lucas numbers as tridiagonal matrix determinants, *Fibonacci Quart.* **42**(3) (2004), 216–221.

Representations by Products

Theorem (N. D. Cahill, J.R. D'Errico, and J. P. Spence, 2002[13])

$$F_{2n} = \prod_{k=1}^{n-1} \left(3 - 2\cos\frac{\pi k}{n}\right) \quad \textit{and} \quad F_n = i^{n-1} \frac{\sin\left(n \cos^{-1}(-\frac{i}{2})\right)}{\sin\left(\cos^{-1}(-\frac{i}{2})\right)}.$$

Theorem (N. D. Cahill and D. A. Narayan, 2004[14])

$$F_{2mn} = F_{2m} \prod_{k=1}^{n-1} \left(L_{2m} - 2\cos\frac{\pi k}{n}\right).$$

[13] N. D. Cahill, J.R. D'Errico, and J. P. Spence, Complex Factorizations of the Fibonacci and Lucas Numbers, *Fibonacci Quart.* **41**(1) (2003), 13–19.

[14] N. D. Cahill and D. A. Narayan, Fibonacci and Lucas numbers as tridiagonal matrix determinants, *Fibonacci Quart.* **42**(3) (2004), 216–221.

Product difference identities

- $F_{n+2}F_{n+1}F_{n-3} - F_n^3 = (-1)^n F_{n+3}$ (R. S. Melham, 2003[15])

[15] R. S. Melham, A Fibonacci identity in the spirit of Simson and Gelin-Cesàro, *Fibonacci Quart.* **41**(2) (2003), 142–43.

[16] S. Fairgrieve and H. W. Gould, Product difference Fibonacci identities of Simson, Gelin-Cesaro, Tagiuri and generalizations, *Fibonacci Quart.* **43**(2) (2005), 137–141.

- $F_{n+2}F_{n+1}F_{n-3} - F_n^3 = (-1)^n F_{n+3}$ (R. S. Melham, 2003[15])
- $F_{n+2}F_{n-1}^2 - F_n^3 = (-1)^n F_{n+1}$ (S. Fairgrieve and H. W. Gould, 2005[16])

[15] R. S. Melham, A Fibonacci identity in the spirit of Simson and Gelin-Cesàro, *Fibonacci Quart.* **41**(2) (2003), 142–43.

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- $F_n^4 - F_{n-2}F_{n-1}F_{n+1}F_{n+2} = 1$ (E. Gelin and E. Cesáro, 1880)

[15] R. S. Melham, A Fibonacci identity in the spirit of Simson and Gelin-Cesàro, *Fibonacci Quart.* **41**(2) (2003), 142–43.

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- $F_{n-3}F_{n+1}^3 - F_n^4 = (-1)^n(F_{n-1}F_{n+3} + 2F_n^2)$ (S. Fairgrieve and H. W. Gould, 2005)

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[15] R. S. Melham, A Fibonacci identity in the spirit of Simson and Gelin-Cesàro, *Fibonacci Quart.* **41**(2) (2003), 142–43.

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- $F_n^4 - F_{n-2}F_{n-1}F_{n+1}F_{n+2} = 1$ (E. Gelin and E. Cesáro, 1880)
- $F_{n-3}F_{n+1}^3 - F_n^4 = (-1)^n(F_{n-1}F_{n+3} + 2F_n^2)$ (S. Fairgrieve and H. W. Gould, 2005)
- $F_{n-1}^3 F_{n+1}F_{n+2} - F_n^5 = -F_n + (-1)^n F_{n-1}F_{n+1}F_{n+2}$ (S. Fairgrieve and H. W. Gould, 2005)

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[15] R. S. Melham, A Fibonacci identity in the spirit of Simson and Gelin-Cesàro, *Fibonacci Quart.* **41**(2) (2003), 142–43.

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- $F_{n-3}F_{n+1}^3 - F_n^4 = (-1)^n(F_{n-1}F_{n+3} + 2F_n^2)$ (S. Fairgrieve and H. W. Gould, 2005)
- $F_{n-1}^3 F_{n+1}F_{n+2} - F_n^5 = -F_n + (-1)^n F_{n-1}F_{n+1}F_{n+2}$ (S. Fairgrieve and H. W. Gould, 2005)
- $F_{n-3}F_{n-2}F_{n-1}F_{n+1}F_{n+2}F_{n+3} - F_n^6 = (-1)^n(4F_n^4 - (-1)^n F_n^2 - 4)$ (S. Fairgrieve and H. W. Gould, 2005)

[15] R. S. Melham, A Fibonacci identity in the spirit of Simson and Gelin-Cesàro, *Fibonacci Quart.* **41**(2) (2003), 142–43.

[16] S. Fairgrieve and H. W. Gould, Product difference Fibonacci identities of Simson, Gelin-Cesaro, Tagiuri and generalizations, *Fibonacci Quart.* **43**(2) (2005), 137–141.

Theorem (R. S. Melham, 2011[17])

If

$$\Delta_n(x_1,\ldots,x_k) := F_{n+x_1+\cdots+x_k}\prod_{i=1}^{k} F_{n-x_i} - F_{n-x_1-\cdots-x_k}\prod_{i=1}^{k} F_{n+x_i},$$

then

$$\begin{aligned}
\Delta_n(a,b) =& (-1)^{n+a+b}F_aF_bF_{a+b}L_n,\\
\Delta_n(a,b,c) =& (-1)^{n+a+b+c}F_{a+b}F_{b+c}F_{c+a}F_{2n},\\
\Delta_n(a,b,c,d) =& (-1)^nL_n\Big(\Delta_{-1}(a,b,c,d)F_n^2\\
&+(-1)^{a+b+c+d}F_aF_bF_cF_dF_{a+b+c+d}F_{n-1}F_{n+1}\Big),\\
\Delta_n(a,b,c,d,e) =& (-1)^nF_{2n}\Big(\Delta_{-1}(a,b,c,d,e)F_n^2\\
&-\frac{1}{6}[\Delta_{-2}(a,b,c,d,e)+3\Delta_{-1}(a,b,c,d,e)]F_{n-1}F_{n+1}\Big).
\end{aligned}$$

[17] R. S. Melham, On product difference Fibonacci identities, *Integers* **11** (2011), A10, 8 pp.

Theorem (R. S. Melham, 2011[18])

We have

$$F_{n+a+b-c}F_{n-a+c}F_{n-b+c} - F_{n-a-b+c}F_{n+a}F_{n+b} \\ = (-1)^{n+a+b+c}F_{a+b-c}\left(F_cF_{n+a+b-c} + (-1)^cF_{a-c}F_{b-c}L_n\right)$$

and

$$F_{n+a+b+c-d}F_{n-a+d}F_{n-b+d}F_{n-c+d} - F_{n-a-b-c+2d}F_{n+a}F_{n+b}F_{n+c} \\ = (-1)^{n+a+b+c}F_{a+b-d}F_{b+c-d}F_{c+a-d}F_{2n+d}.$$

[18] R. S. Melham, On product difference Fibonacci identities, *Integers* **11** (2011), A10, 8 pp.

Theorem (Various authors [19] [20] [21] [22] [23] [24] [25])

For integers a, b, c,

$$F_{a+b+c} = F_{a+1}F_{b+1}F_{c+1} + F_aF_bF_c - F_{a-1}F_{b-1}F_{c-1}.$$

[19] H. Belbachirand H. Harik, Hakim, Link between Hosoya index and Fibonacci numbers, *Miskolc Math. Notes* **19**(2) (2018), 741–748.

[20] J. L. Brown, Solution to Problem H-4, Fibonacci Quart. **1**(3) (1963), 47–48.

[21] M. Farrokhi D. G., Some remarks on the equation $F_n = kF_m$ in Fibonacci numbers, *J. Integer Seq.* **10**(5) (2007), Article 7.

[22] M. Griffiths, On a trivariate Fibonacci identity, *Math. Gaz.* **101** (2017), 519–522.

[23] R. C. Johnson, Fibonacci numbers and matrices (2009).

[24] R. Knott, Fibonacci and golden ratio formulae (2016).

[25] I. D. Ruggles, Problem H-4, Fibonacci Quart. **1**(1) (1963), p. 47.

Theorem (Hacéne Belbachir and Hakim Harik, 2018[26])

For any positive integers $r_1, \dots, r_s$ ($s \geq 2$), there exists a set Ω_s of s-tuples of $\{-1, 0, 1\}$ such that

$$F_{r_1+\cdots+r_s+1} = \sum_{(\varepsilon_1,\dots,\varepsilon_s)\in\Omega_s} \prod_{i=1}^{s} F_{r_i+\varepsilon_i}.$$

[26]H. Belbachirand H. Harik, Hakim, Link between Hosoya index and Fibonacci numbers, *Miskolc Math. Notes* **19**(2) (2018), 741–748.

Theorem (MFDG, 2008[27])

For $n \geq 1$,

$$\sum_{i=1}^{n-1} F_{n-i}F_i^2 = \frac{1}{2}\sum_{i=1}^{n}(-1)^{n-i}(F_{2i} - F_i) = \binom{F_{n+1}}{2} - \binom{F_n}{2}.$$

Also

$$\sum_{i=1}^{n-1} F_{n-i}F_iF_{i+1} = \binom{F_{n+1}}{2}$$

and

$$\sum_{i=1}^{n-1} F_{n-i}F_{2i} = \binom{F_n}{2} + \binom{F_{n+1}}{2}.$$

[27] M. Farrokhi D. G., An identity generator: Basic commutators, *Electron. J. Combin.* **15**(1) (2008), Note 15.

Theorem (MFDG, 2008[28])

For $n \geq 1$,

$$\begin{aligned}
\sum_{i=1}^{n-1} F_{n-i}F_i^3 &= \frac{1}{2}F_{n-1}F_nF_{n+1} - \binom{F_n+1}{3}, \\
\sum_{i=1}^{n-1} F_{n-i}F_iF_{i+1}^2 &= \binom{F_{n+2}+1}{3} - \frac{1}{2}F_nF_{n+1}F_{n+2}, \\
\sum_{i=1}^{n-1} F_{n-i}F_i^2F_{i+1} &= \binom{F_{n+2}+1}{3} - \binom{F_n+1}{3}, \\
\sum_{i=1}^{n-1} F_{n-i}F_iF_{i+1}F_{i+2} &= \binom{F_{n+2}+1}{3} + \binom{F_{n+1}+1}{3} - \binom{F_n+1}{3} - \frac{1}{2}F_nF_{n+1}F_{n+2}.
\end{aligned}$$

[28] M. Farrokhi D. G., An identity generator: Basic commutators, *Electron. J. Combin.* **15**(1) (2008), Note 15.

Definition

The generalized Fibonacci and Lucas numbers are defined, respectively, by the Binet's formulas, as follows

$$U_n(p,q) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n(p,q) = \alpha^n + \beta^n,$$

where $\alpha = \frac{1}{2}(p + \sqrt{p^2 - 4q})$ and $\beta = \frac{1}{2}(p - \sqrt{p^2 - 4q})$. The numbers $U_n(p,q)$ and $V_n(p,q)$ can be defined recursively by

$$U_n(p,q) = -qU_{n-2}(p,q) + pU_{n-1}(p,q),$$
$$V_n(p,q) = -qV_{n-2}(p,q) + pV_{n-1}(p,q)$$

for all integers n, where $U_0 = 0$, $U_1 = 1$, $V_0 = 2$, and $V_1 = p$.

Theorem (MFDG, 2009[29])

If $\{U_n\}$ is a generalized Fibonacci sequence, then for all natural numbers m,

$$U_{a_1+\cdots+a_m-\binom{m+1}{2}} = \sum_{i=1}^{m} \left\{ {m \atop i} \right\} U_{a_1-i}\cdots U_{a_m-i}, \tag{1}$$

where $a_1, \ldots, a_m$ are integers and

$$\left\{ {m \atop i} \right\} = \left(\prod_{j=1,\ j\neq i}^{m} U_{j-i} \right)^{-1}.$$

[29] M. Farrokhi D. G., Generalization of an identity involving the generalized Fibonacci numbers and its applications, *Integers* **9** (2009), Article 39, 497–513.

Definition

Let m and k be any natural numbers. Then for any $n \geq 1$,

$$S_m(n; p, q; k) := \sum_{a_1+\cdots+a_m=n} U_{ka_1} \cdots U_{ka_m}.$$

[30] S. Vajda, *Fibonacci and Lucas numbers, and the Golden Section: Theory and Applications*, Halsted Press (1989).

[31] R. A. Dunlap, *The Golden Ratio and Fibonacci Numbers*, World Scientific Press, 1997.

Definition

Let m and k be any natural numbers. Then for any $n \geq 1$,

$$S_m(n; p, q; k) := \sum_{a_1+\cdots+a_m=n} U_{ka_1} \cdots U_{ka_m}.$$

Theorem (S. Vajda[30] and R. A. Dunlap[31])

$$S_2(n; 1, -1; 1) = \sum_{a+b=n} F_a F_b = \frac{1}{5}\left(nL_n - F_n\right),$$

[30] S. Vajda, *Fibonacci and Lucas numbers, and the Golden Section: Theory and Applications*, Halsted Press (1989).

[31] R. A. Dunlap, *The Golden Ratio and Fibonacci Numbers*, World Scientific Press, 1997.

Sums of products: Example 2

Theorem (Tofiq Mansour, 2005[32])

For $n \geq m$,

$$\sum_{i=0}^{m}\left[(4q^k)^{m-i}\left(\sum_{j=0}^{i}(-1)^j\binom{i}{j}(i+1-j)^m\right)\left(\frac{V_k^2(p,q)-4q^k}{U_k(p,q)}\right)^i S_{i+1}(n+i-m;p,q;k)\right]$$
$$=\sum_{i=1}^{m}\left[\frac{(-1)^{m-1}(2q^k)^{m-i}}{(i-1)!}\left(\sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}(j+1)^{m-1}\right)\left(\sum_{j=0}^{i}v_{m,i,j}U_{(n+i-m-j)k}(p,q)\binom{i}{j}\right)\right],$$

where

$$v_{m,i,j}=(-2q^k)^jV_k^{i-j}(p,q)\prod_{l=1}^{i}(n+i+m-j-l).$$

[32] T. Mansour, Generalizations of some identities involving the Fibonacci numbers, *Fibonacci Quart.* **43** (2005), 307–315.

Theorem (MFDG, 2009[33])

Let m and k be natural numbers. Then for any $n \geq 1$,

$$\frac{S_m(n;p,q;k)}{U_k(p,q)^m} = \frac{(b_m + pa_m)\delta_{m,n+m^2} - a_m\delta_{m,n+m^2+1} + a_m^2 S_{m-1}(n+1;\bar{p},\bar{q};1)}{b_m^2 + pa_m b_m + qb_m^2},$$

where $(\bar{p},\bar{q}) = (V_k(p,q), q^k)$,

$$\delta_{m,n} = \alpha_{m,n} - \sum_{i=1}^{m-1} (a_i\gamma_{m,n,i} + b_i\beta_{m,n,i}),$$

[33] M. Farrokhi D. G., Generalization of an identity involving the generalized Fibonacci numbers and its applications, *Integers* **9** (2009), Article 39, 497–513.

Sums of products: Example 2

Theorem (Continued)

$$\begin{aligned}
\gamma_{m,n,i} =& -qU_m(\bar{p},\bar{q})\sum_{i=1}^{m} U_{i-m-1}(\bar{p},\bar{q})S_{m-1}(n-i;\bar{p},\bar{q};1) \\
&+ U_{m+1}(\bar{p},\bar{q})\sum_{i=1}^{m-1} U_{i-m}(\bar{p},\bar{q})S_{m-1}(n-i;\bar{p},\bar{q};1), \\
\beta_{m,n,i} =& -qU_{m-1}(\bar{p},\bar{q})\sum_{i=1}^{m-1} U_{i-m}(\bar{p},\bar{q})S_{m-1}(n-i;\bar{p},\bar{q};1) \\
&+ U_m(\bar{p},\bar{q})\sum_{i=1}^{m-2} U_{i-m+1}(\bar{p},\bar{q})S_{m-1}(n-i;\bar{p},\bar{q};1),
\end{aligned}$$

Theorem (Continued)

$$
\begin{aligned}
\alpha_{m,n} = &\binom{n-1}{m-1} U_{n-\binom{m+1}{2}}(\bar{p},\bar{q}) \\
&- \sum_{i=1}^{m} \left\{ {m \atop i} \right\} \sum_{j=1}^{m-1} \binom{m}{j} \sum_{k=1}^{n-(i+1)(m-j)} \\
&\left(\sum_{\substack{a_1,\dots,a_j<i \\ a_1+\cdots+a_j=k}} U_{a_1-i}(\bar{p},\bar{q}) \cdots U_{a_j-i}(\bar{p},\bar{q}) \right) \times \\
&S_{m-j}(n-k-i(m-j)); \bar{p}, \bar{q}; 1)
\end{aligned}
$$

and

Sums of products: Example 2

Theorem (Continued)

$$\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} a_i \\ b_i \end{pmatrix} = \begin{pmatrix} U_{m+1} & U_m \\ -qU_m & -qU_{m-1} \end{pmatrix} \begin{pmatrix} a_{i-1} \\ b_{i-1} \end{pmatrix} + \begin{pmatrix} 0 \\ \left\{ {m \atop i} \right\} \end{pmatrix},$$

for $i = 1, \ldots, m.$

Sums of products: Example 3

Definition

Let $m \geq 1$. Then for any $n \geq 1$,

$$T_m(n; p, q) := \sum_{i=1}^{n-1} U_{n-i} U_i^m.$$

Definition

Let $m \geq 1$. Then for any $n \geq 1$,

$$T_m(n; p, q) := \sum_{i=1}^{n-1} U_{n-i} U_i^m.$$

Examples

We have

$$T_2(n; 1, -1) = \binom{F_{n+1}}{2} - \binom{F_n}{2},$$
$$T_3(n; 1, -1) = \frac{1}{2} F_{n-1} F_n F_{n+1} - \binom{F_n + 1}{3}.$$

Theorem (MFDG, 2009[34])

$$\sum_{x=1}^{n} U_{ax+b} = \frac{q^a U_{na+b} - U_{(n+1)a+b} - q^a U_{b-a} + U_b}{1 + q^a - V_a} - U_b.$$

[34] M. Farrokhi D. G., Generalization of an identity involving the generalized Fibonacci numbers and its applications, *Integers* **9** (2009), Article 39, 497–513.

Theorem (MFDG, 2009[35])

$$
\begin{aligned}
&\sum_{x+y=n} U_{ax+b}U_{cy+d} \\
&= \big[(q^c V_a - q^a V_c)U_d U_{(n+1)a+b} - q^{2c}U_{d-c}U_{(n+1)a+b} \\
&\quad + q^{2a}U_d U_{na+b} - q^{2a+c}U_{d-c}U_{(n-1)a+b} + q^{a+2c}U_{d-2c}U_{na+b} \\
&\quad + (q^a V_c - q^c V_a)U_b U_{(n+1)c+d} - q^{2a}U_{b-a}U_{(n+1)c+d} \\
&\quad + q^{2c}U_b U_{nc+d} - q^{a+2c}U_{b-a}U_{(n-1)c+d} + q^{2a+c}U_{b-2a}U_{nc+d}\big] \\
&\quad / \big[(q^a + q^c)^2 + q^a(V_{2c} - V_{a+c} - q^a V_{c-a}) + q^c(V_{2a} - V_{a+c} - q^c V_{a-c})\big] \\
&\quad - U_b U_{nc+d} - U_{na+b}U_d.
\end{aligned}
$$

[35] M. Farrokhi D. G., Generalization of an identity involving the generalized Fibonacci numbers and its applications, *Integers* **9** (2009), Article 39, 497–513.

Theorem (MFDG, 2009[36])

$$T_m(n) = \frac{qa_{m-1}\beta_{m,n+m-1} + b_{m-1}\beta_{m,n+m} - a_{m-1}b_{m-1}U_n^m}{b_{m-1}^2 + pa_{m-1}b_{m-1} + qa_{m-1}^2},$$

where

$$\beta_{m,n} = \alpha_{m,n} - \sum_{i=1}^{m-1} a_i U_{n-i-1}^m,$$

$$\alpha_{m,n} = \sum_{i=1}^{n-1} U_{n-i} U_{mi-\binom{m+1}{2}} - \sum_{j=1}^{m} \left\{ {m \atop j} \right\} \sum_{i=1}^{j-1} U_{n-i} U_{i-j}^m$$

and

$$\begin{pmatrix} a_{-1} \\ b_{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} a_i \\ b_i \end{pmatrix} = \begin{pmatrix} p & 1 \\ -q & 0 \end{pmatrix} \begin{pmatrix} a_{i-1} \\ b_{i-1} \end{pmatrix} + \begin{pmatrix} 0 \\ \left\{ {m \atop i+1} \right\} \end{pmatrix},$$

for $i = 0, 1, \ldots, m$.

[36] M. Farrokhi D. G., Generalization of an identity involving the generalized Fibonacci numbers and its applications, *Integers* **9** (2009), Article 39, 497–513.

Theorem (MFDG, 2009[37])

Let $\{U_n\}$ be a generalized Fibonacci sequence and let m be a natural number. Then for any $n \geq 1$,

$$\sum_{i=1}^{n} U_i^m = \frac{1}{\sum_{i=1}^{m} \left\{ {m \atop i} \right\}} \left(\sum_{i=1}^{n} U_{im-\binom{m+1}{2}} + \sum_{i=1}^{m} \left\{ {m \atop i} \right\} \sum_{j=1}^{i} (U_{n-i+j}^m - U_{j-i}^m) \right).$$

[37] M. Farrokhi D. G., Generalization of an identity involving the generalized Fibonacci numbers and its applications, *Integers* **9** (2009), Article 39, 497–513.

Theorem (MFDG, 2009[38])

Let r be an odd prime and let $\left(\frac{a}{b}\right)$ denote the Legendre's symbol. Then for any $m \geq 0$,

[38] M. Farrokhi D. G., Generalization of an identity involving the generalized Fibonacci numbers and its applications, *Integers* **9** (2009), Article 39, 497–513.

Theorem (MFDG, 2009[38])

Let r be an odd prime and let $\left(\frac{a}{b}\right)$ denote the Legendre's symbol. Then for any $m \geq 0$,

(1) $U_n(p,q) = \sum_{i=0}^{m} p^i(-q)^{m-i}\binom{m}{i}U_{n-m-i}(p,q)$; *and if p, q are integers, then*

[38] M. Farrokhi D. G., Generalization of an identity involving the generalized Fibonacci numbers and its applications, *Integers* **9** (2009), Article 39, 497–513.

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[38] M. Farrokhi D. G., Generalization of an identity involving the generalized Fibonacci numbers and its applications, *Integers* **9** (2009), Article 39, 497–513.

Divisibility criterion: Example 5

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(3) $U_{rn}(p,q) \overset{r}{\equiv} U_r(p,q)U_n(p,q)$;

(4) $U_{r+1}(p,q) \overset{r}{\equiv} \frac{1}{2}p\left(\left(\frac{\Delta}{r}\right)+1\right)$, $U_r(p,q) \overset{r}{\equiv} \left(\frac{\Delta}{r}\right)$ *and if $r \nmid q$, then* $U_{r-1}(p,q) \overset{r}{\equiv} \frac{p}{2q}\left(\left(\frac{\Delta}{r}\right)-1\right)$.

[38] M. Farrokhi D. G., Generalization of an identity involving the generalized Fibonacci numbers and its applications, *Integers* **9** (2009), Article 39, 497–513.

Theorem (MFDG, 2009[39])

Let p and q be integers and r be an odd prime such that $r \nmid pq(p^2 - 4q)$.

[39] M. Farrokhi D. G., Generalization of an identity involving the generalized Fibonacci numbers and its applications, *Integers* **9** (2009), Article 39, 497–513.

Divisibility criterion: Example 5

Theorem (MFDG, 2009[39])

Let p and q be integers and r be an odd prime such that $r \nmid pq(p^2 - 4q)$.

(1) *If $q = 1$, or $q = -1$ and $4 | r - 1$, then $r | U_i$ for some $3 \leq i \leq r - 2$.*

[39] M. Farrokhi D. G., Generalization of an identity involving the generalized Fibonacci numbers and its applications, *Integers* **9** (2009), Article 39, 497–513.

Divisibility criterion: Example 5

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(1) *If $q = 1$, or $q = -1$ and $4|r-1$, then $r|U_i$ for some $3 \leq i \leq r-2$.*

(2) *If $r|U_{r-1}$, then $r|U_{\frac{r-1}{2}}$.*

[39] M. Farrokhi D. G., Generalization of an identity involving the generalized Fibonacci numbers and its applications, *Integers* **9** (2009), Article 39, 497–513.

Open Problems

Conjecture (R. S. Melham, 2011[40])

If
$$\Delta_n(x_1,\ldots,x_k) := F_{n+x_1+\cdots+x_k}\prod_{i=1}^{k}F_{n-x_i} - F_{n-x_1-\cdots-x_k}\prod_{i=1}^{k}F_{n+x_i},$$
then
$$\Delta_n(x_1,\ldots,x_{2k+2}) = (-1)^{n+x_1+\cdots+x_{2k+2}}L_n\left(c_{2k}F_n^{2k}+\sum_{i=1}^{2k-1}c_{2k-i}F_{n-1}^{2k-i}F_{n+1}^{i}\right)$$
and
$$\Delta_n(x_1,\ldots,x_{2k+3}) = (-1)^{n+x_1+\cdots+x_{2k+3}}F_{2n}\left(c_{2k}F_n^{2k}+\sum_{i=1}^{2k-1}c_{2k-i}F_{n-1}^{2k-i}F_{n+1}^{i}\right),$$
where $c_{2k-i}=c_i$ for all $i=1,\ldots,k-1$.

[40] R. S. Melham, On product difference Fibonacci identities, *Integers* **11** (2011), A10, 8 pp.

Problem

Find all s-tuples $(a_1, \ldots, a_s)$ and $(b_1, \ldots, b_s)$ and r-tuple $(c_1, \ldots, c_r)$ $(r < s)$ such that

$$\prod_{i=1}^{s} F_{n+a_i} - \prod_{i=1}^{s} F_{n+b_i} = (-1)^{\theta(n)} \lambda \prod_{i=1}^{r} F_{n+c_i}$$

for some constant λ and function θ.

Conjecture (MFDG, 2009[41])

Let $\{U_n\}$ be a generalized Fibonacci sequence with $\sum_{i=1}^{m} \left\{ {m \atop i} \right\} = 0$, for some m. Then

$$\sum_{i=1}^{m+n} \left\{ {m+n \atop i} \right\} U_{x_1-n_1 i} \cdots U_{x_k-n_k i} = 0,$$

where $n \geq 0$, $n_1 + \cdots + n_k = n$ and $x_1, \ldots, x_k$ are arbitrary integers.

[41] M. Farrokhi D. G., Generalization of an identity involving the generalized Fibonacci numbers and its applications, *Integers* **9** (2009), Article 39, 497–513.

Let $\mathbb{Q}_F$ be the multiplicative group of Fibonacci numbers.

Theorem (Florian Luca and Štefan Porubský, 2003[42])

For every $\epsilon > 0$,

$$\#\mathbb{Q}_F \cap \mathbb{N} \cap [1, x] \ll \frac{x}{(\log x)^\epsilon}.$$

As a result,

$$\sum_{n \in \mathbb{Q}_F \cap \mathbb{N}} \frac{1}{n} < \infty$$

[42] F. Luca and Š. Porubský, The multiplicative group generated by the Lehmer numbers, *Fibonacci Quart.* **41**(2) (2003) 122–132.

Group of Fibonacci numbers

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Question

Describe the group $\mathbb{Q}_F$?

[42] F. Luca and Š. Porubský, The multiplicative group generated by the Lehmer numbers, *Fibonacci Quart.* **41**(2) (2003) 122–132.

Fibonacci-like sequences of composite numbers

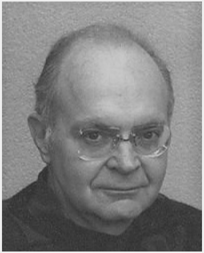
Definition

A sequence $\{U_n\}$ satisfying the recursive relation

$$U_n = U_{n-1} + U_{n-2}$$

is called a *Fibonacci-like sequence*.

Ronald Graham

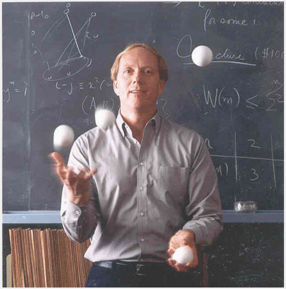


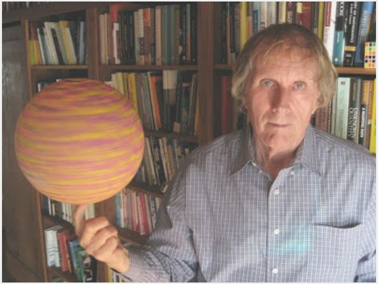
Donald Knuth

John Nicol

Maxim Vsemirnov

Fibonacci-like sequences of composite numbers





Theorem

Let $\{U_n\}$ *be a Fibonacci-like sequence with* U_0 *and* U_1 *given by*

U_0	U_1	Author
1786772701928802632268715130455793	1059683225053915111058165141686995	*R. L. Graham, 1964*[43]
62638280004239857	49463435743205655	*D. E. Knuth, 1990*[44]
407389224418	76343678551	*J. W. Nicol, 1999*[45]
106276436867	35256392432	*M. Vsemirnov, 2004*[46]

Then $\gcd(U_0, U_1) = 1$ *and* $\{U_n\}$ *contains only composite numbers.*

---

[43] R. L. Graham, A Fibonacci-like sequence of composite numbers, *Math. Mag.* **37**(5) (1964), 322–324.

[44] D. E. Knuth, A Fibonacci-like sequence of composite numbers, *Math. Mag.* **63**(1) (1990), 21–25.

[45] J. W. Nicol, A Fibonacci-like sequence of composite numbers, *Electron. J. Combin.* **6** (1999), Research Paper 44, 6 pp.

[46] M. Vsemirnov, A new Fibonacci-like sequence of composite numbers, *J. Integer Seq.* **7**(3) (2004), Aricle 04.3.7, 3 pp.

Problem

Find the smallest pairs (U_0, U_1) of coprime natural numbers such that the resulting Fibonacci-like sequence contains only composite numbers.

Definition

An odd positive integer k is a Sierpiński number if the sequence $\{k2^n + 1\}$ contains only composite numbers.

Wacław Sierpiński

John Selfridge

A problem of the same nature

Theorem (Wacław Sierpiński, 1960[47])

There are infinitely many Sierpińki numbers.

[47] W. Sierpiński, Sur un problème concernant les nombres $k2^n + 1$, *Elem. Math.* **15** (1960), 73–74.

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Conjecture (Wacław Sierpiński and John Selfridge, 1962)

The smallest Sierpińki number is equal to 78557.

[47] W. Sierpiński, Sur un problème concernant les nombres $k2^n + 1$, *Elem. Math.* **15** (1960), 73–74.

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Remark

The smallest Sierpińki number belongs to the set

$$\{21181, 22699, 24737, 55459, 67607, 78557\}.$$

[47] W. Sierpiński, Sur un problème concernant les nombres $k2^n + 1$, *Elem. Math.* **15** (1960), 73–74.

Topics not covered by this talk

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- Fibonacci cubes

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References

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Thanks!