Induced cycles in circulant graphs

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- G =group, $1 \notin S = S^{-1} \subseteq G$
  $\Gamma := \mathrm{Cay}(G, S)$ is a graph with

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- A graph is Γ-free if it does not have any induced subgraph isomorphic to Γ.

Babai, 1976[1]

There is no minimal Cayley graph containing $K_4 \setminus e$ or $K_{3,5}$ as a subgraph.

[1] L. Babai, Chromatic number and subgraphs of Cayley graphs, *Theory and applications of graphs (Proc. Internat. Conf., Western Mich. Univ., Kalamazoo, Mich., 1976)*, pp. 10–22.

[2] J. Spencer, What's not inside a Cayley graph, *Combinatorica* **3**(2) (1983), 239–241.

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There exists a class of graphs of bounded degree and arbitrary girth which cannot be embedded into minimal Cayley graphs as induced subgraphs.

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Definition

A graph is perfect if the chromatic and clique numbers of its induced subgraphs are equal.

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Chudnovsky, Robertson, Seymour, and Thomas, 2006[1]

A graph Γ is perfect iff neither Γ nor Γ^c has an induced odd cycle of length ≥ 5.

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Let G be a nontrivial finite group. Then all minimal Cayley graphs of G are bipartite if and only if G is a 2-group.

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Let G be a nontrivial finite group. Then all minimal Cayley graphs of G are perfect if and only if

- G is a 2-group; or
- G is isomorphic to C_3, C_6, S_3, $C_3 \times C_3$, A_4, or E, where

$$E = \langle a, b : a^3 = b^3 = [b, a, b] = [b, a, a] = 1 \rangle$$

is a group of order 27.

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The size of induced cycles inside $\mathrm{Cay}(C_m, S)$ is bounded above by $m/2+1$.

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(3) $\left|\sum_{i=1}^{j}(-1)^{l_0}x_i - \sum_{i=1}^{k}(-1)^{l_0}x_i\right|_n \notin S$ for $|j-k|_n \geq 2$;

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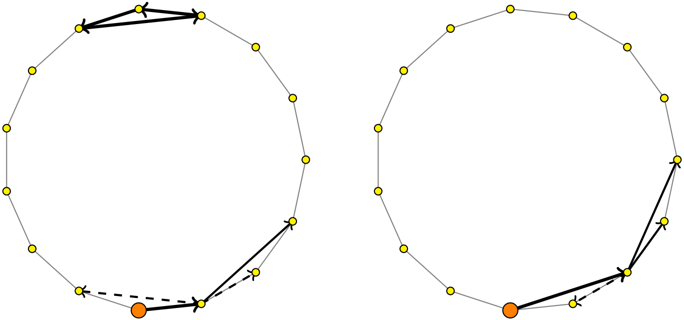
$\mathfrak{F}(n) :=$ maximum non-negative integer m such that all circulant graphs on C_m are C_n-free.

First lemma

Assume $m = 6k + i$ with $i \in \{0, 1, 2, 3, 4, 5\}$. Then the length of induced cycles in $\Gamma = \mathrm{Cay}(C_m, \{\pm 1, \pm 2\})$ are exactly

$$3, 3k + \left\lceil \frac{i}{2} \right\rceil, \ldots, 4k + \left\lceil \frac{i}{2} \right\rceil - \delta_{1i}.$$

Proof of the first lemma



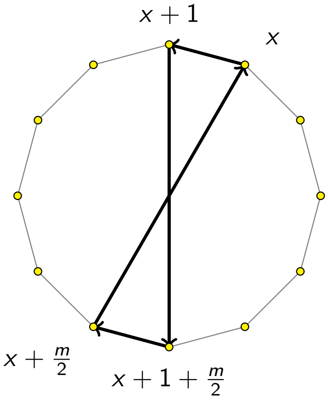
There is a one-to-one correspondence between induced cycles of length $k > 3$ and circular sequences of 1's and 2's of length k with no two consecutive 1 whose sum equal m.

Second lemma

Assume $m \neq 2$ is even. Then the length of induced cycles in $\Gamma = \mathrm{Cay}(C_m, \{\pm 1, \frac{m}{2}\})$ are exactly

$$4, 1 + \frac{m}{2}, 3 + \frac{m}{2}, \ldots, 2\left\lfloor \frac{m+4}{8} \right\rfloor - 1 + \frac{m}{2}.$$

Proof of the second lemma

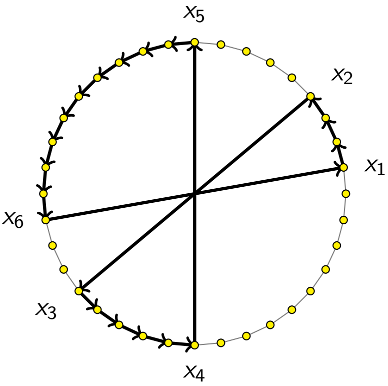


Γ has squares with vertex sets of the form

$$\left\{x, x+1, x+1+\frac{m}{2}, x+\frac{m}{2}, x\right\}$$

for some $x = 0, \ldots, m-1$.

Proof of the second lemma



An induced cycle with $t = 3$

An induced cycle C of Γ of length > 4 is the union of some spokes and paths

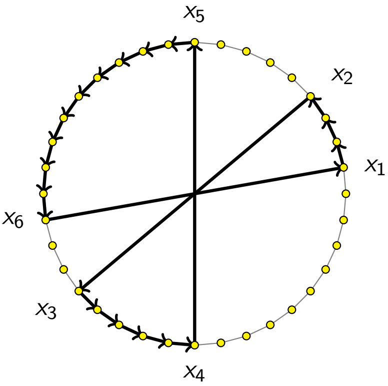
$$P(x_1, x_2), \ldots, P(x_{2t-1}, x_{2t})$$

in which $x_{2s-1} < x_{2s}$ and

$$|x_{2s+1} - x_{2s}| = \frac{m}{2}$$

for all $s = 1, \ldots, t$ (with $x_{2t+1} := x_1$).

Proof of the second lemma



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The length of C equals

$$n := \sum_{s=1}^{t} (x_{2s} - x_{2s-1} + 1)$$

$$= t + \sum_{s=1}^{t} (x_{2s} - x_{2s+1}).$$

It follows that

$$n = t + (t \bmod 2)\frac{m}{2}.$$

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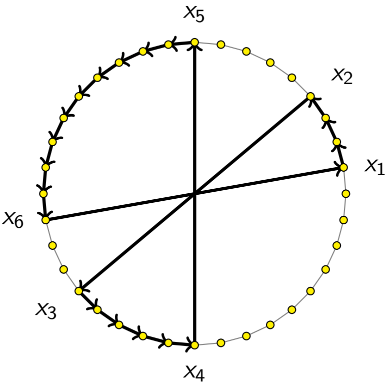

An induced cycle with $t = 3$

Every vertex of $C_m \setminus C$ is a counterpoint to some vertex of $C \setminus \{x_1, \ldots, x_{2t}\}$. Hence

$$m = 2n - 2t.$$

So, t is odd. Also, $t \leq \frac{m}{4}$ for

$$|V(P(x_{2s-1}, x_{2s}))| \geq 3$$

for all $s = 1, \ldots, t$.

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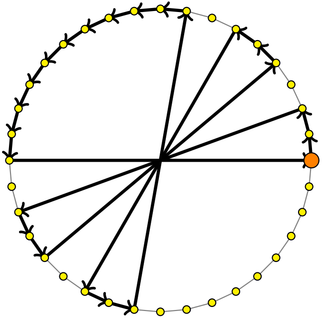

An induced cycle with $t = 5$

For any odd integer $t \leq \frac{m}{4}$

$$0, 1, 2, 2+\frac{m}{2}, 3+\frac{m}{2}, 4+\frac{m}{2},$$
$$\ldots, 2t-5, 2t-4, 2t-3,$$
$$2t-3+\frac{m}{2}, 2t-2+\frac{m}{2},$$
$$2t-1+\frac{m}{2}, 2t-1, \ldots, \frac{m}{2}, 0.$$

is a cycle of length $t + \frac{m}{2}$.

Theorem

For every $n > 2$ we have

$$\mathfrak{F}(n) = 12k + \left\lceil \frac{3i}{2} \right\rceil - \delta_{1,\lfloor \frac{i}{2} \rfloor} - 1,$$

where $n = 8k + i$ with $i \in \{0, \ldots, 7\}$.

Proof of the theorem

Let $m := \mathfrak{F}(n) > n$. We claim that

$$m \geq 12k + \left\lceil \frac{3i}{2} \right\rceil - \delta_{1,\left\lfloor \frac{i}{2} \right\rfloor}.$$

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$$\rho : C_n \hookrightarrow \Gamma := \mathrm{Cay}(C_m, S)$$

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- Every vertex in $\rho(C_n)$ is adjacent to $|S| - 2$ vertices of $C_m \setminus \rho(C_n)$. Thus

$$(m - n)|S| \geq n(|S| - 2)$$

so that

$$m \geq 2n(1 - |S|^{-1}).$$

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- If $|S| = 3$ then m is even and $S = \{\pm 1, \frac{m}{2}\}$.
  - C_n is an induced subgraph of a circulant graph on C_{m+1}. Thus

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  Suppose $m = 12k + \left\lceil \frac{3i}{2} \right\rceil - 1$.

  - $i = 2$, 3, 6, or 7.
  - $n - \frac{m}{2}$ is odd by the second Lemma. Thus $i = 2$ or 3.

Corollary

Let G be a finite group whole all Cayley graphs are C_n-free. Then order of elements of G are bounded above by $\mathfrak{F}(n)$.

Thank You for Your Attention!