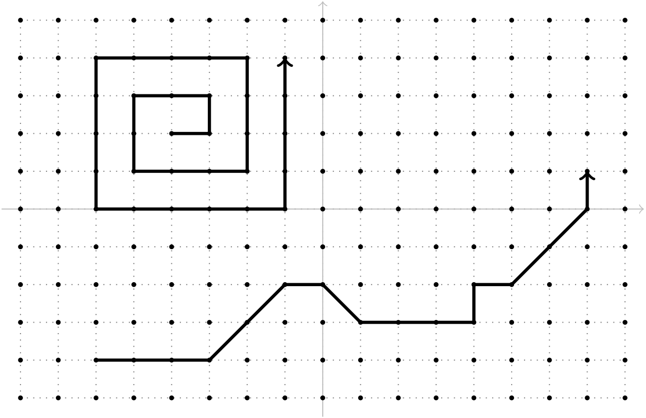
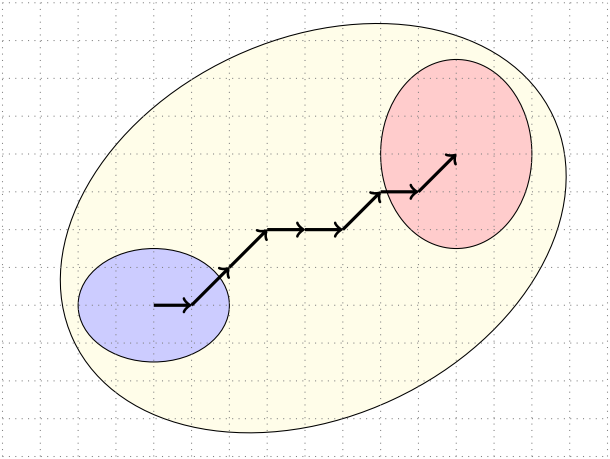
# Lattice paths inside a table

M. Farrokhi D. G.

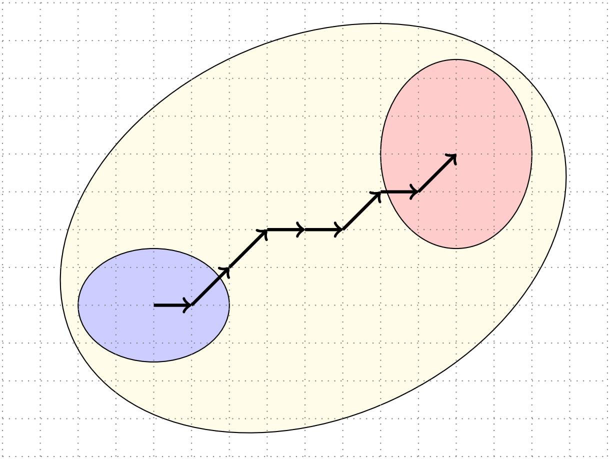
IASBS

Today!





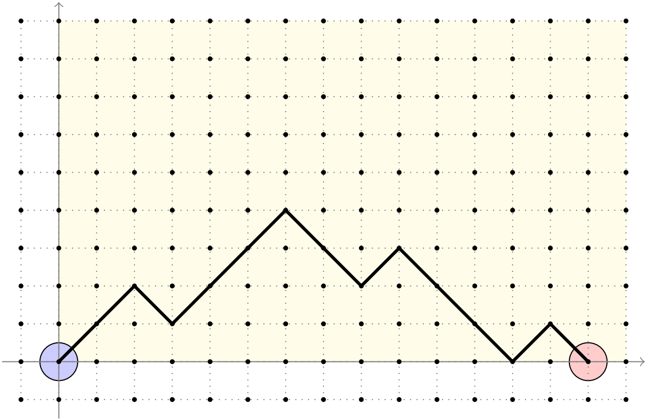
Steps$=\{(1,0)=\longrightarrow, \quad (1,1)=\nearrow, \quad (1,-1)=\searrow\}$



Steps$=\{(1,0)=\longrightarrow, \quad (1,1)=\nearrow, \quad (1,-1)=\searrow\}$

**Problem: Count the number of all possible legal paths!**

# Dyck paths (Walther von Dyck, 1856–1934)



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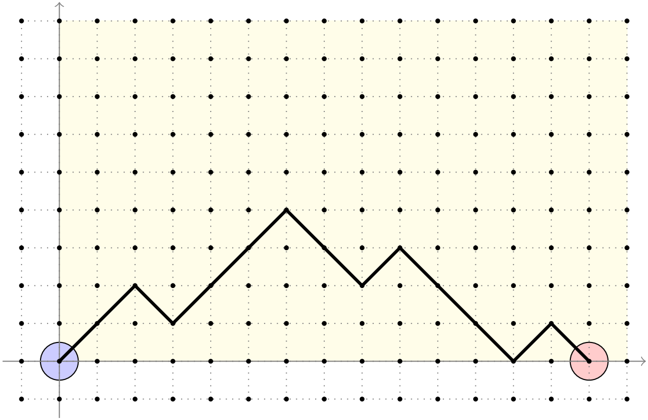
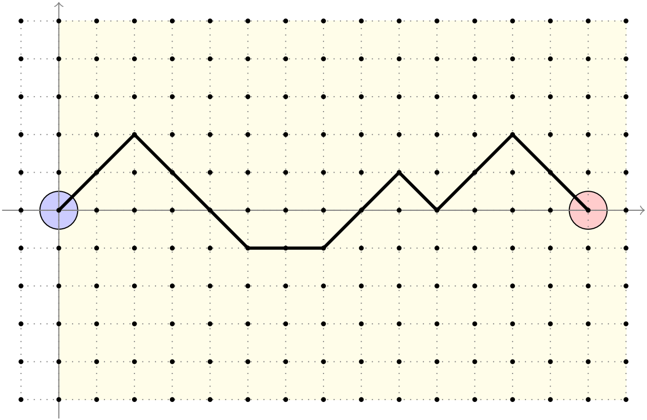


$$S = \{(1,1) = \nearrow, \quad (1,-1) = \searrow\}$$

$l((0,0),(2n,0),S) = \frac{1}{n+1}\binom{2n}{n}$ the Catalan number $C_n$

# Delannoy paths (Henri Delannoy, 1833–1915)



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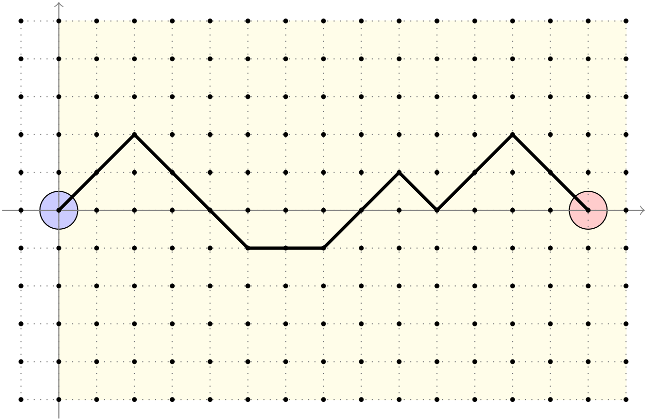
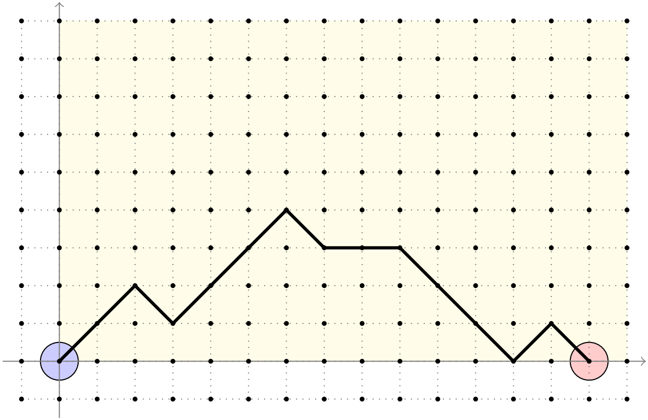


$$S = \{(1,1) = \nearrow, \quad (2,0) = \longrightarrow, \quad (1,-1) = \searrow\}$$

$$D(n,n) := l((0,0),(2n,0),S) = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}$$

# Schröder paths (Ernst Schröder, 1841–1902)



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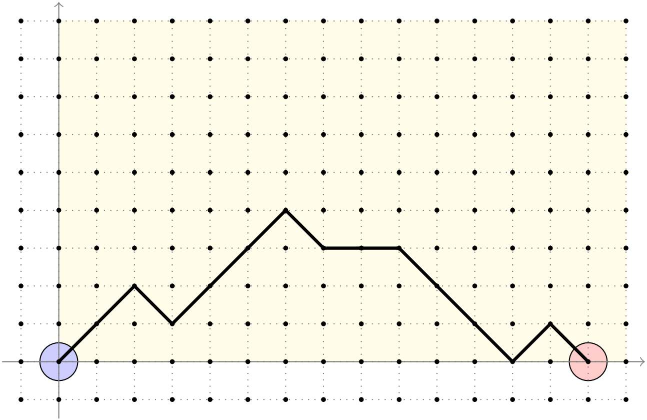
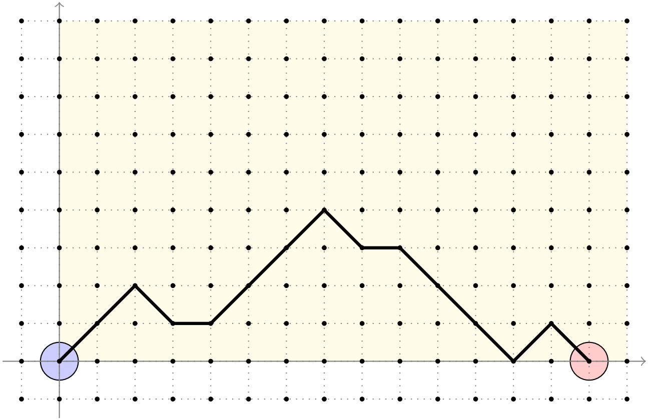


$$S = \{(1,1) = \nearrow, \quad (2,0) = \longrightarrow, \quad (1,-1) = \searrow\}$$

$$S(n) := I((0,0),(2n,0),S) = \sum_{k=0}^{n} C_{n-k}\binom{2n-k}{k}$$

# Motzin paths (Theodore Motzkin, 1908–1970)



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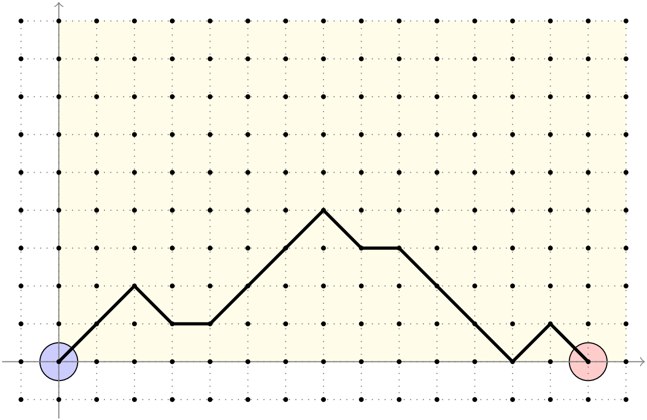
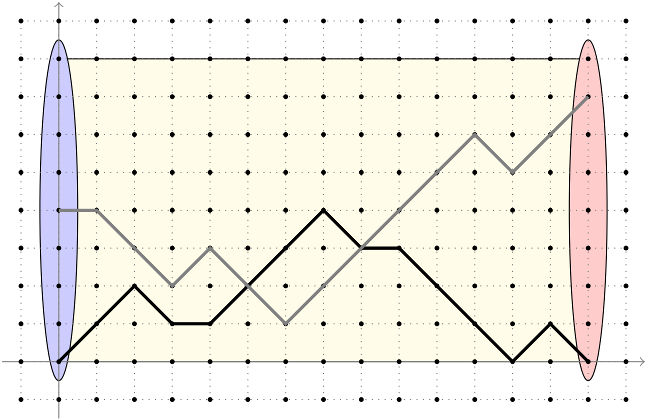


$$S = \{(1,1) = \nearrow, \quad (1,0) = \longrightarrow, \quad (1,-1) = \searrow\}$$

$$M_n := I((0,0),(n,0),S) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k$$

# Our paths



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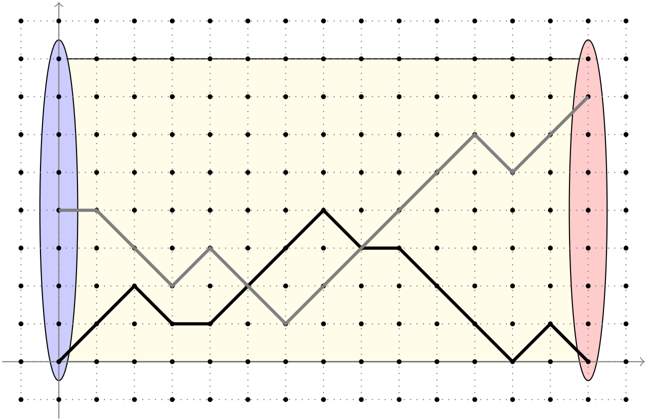
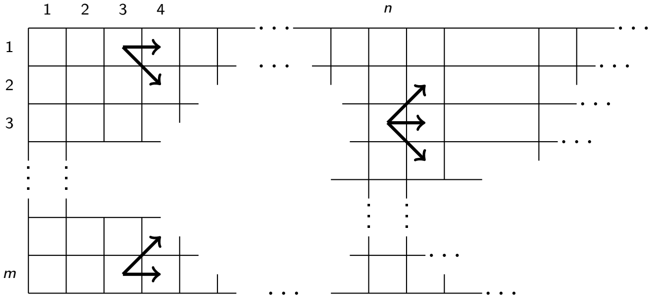


$$S = \{(1,1) = \nearrow, \quad (1,0) = \longrightarrow, \quad (1,-1) = \searrow\}$$

$$I(\text{blue area}, \text{red area}, S) = ?$$

# An instant table




The $m \times \infty$ table

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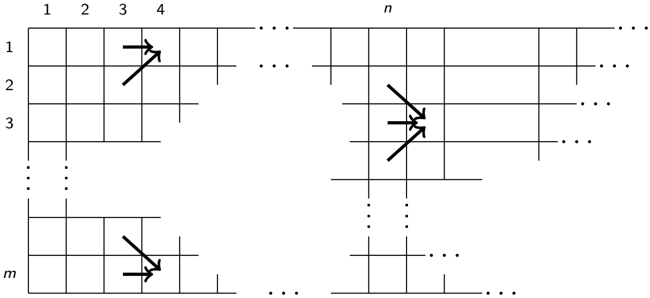


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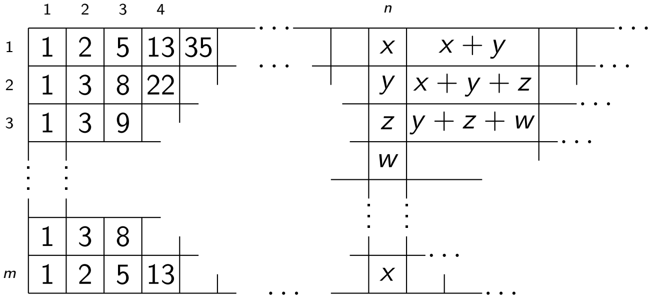
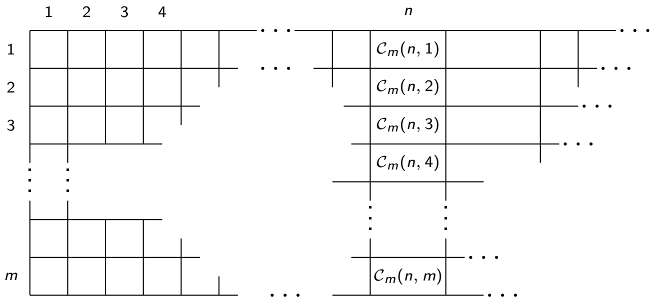



The $m \times \infty$ table

# Notations




The $m \times \infty$ table

$$\mathcal{I}_m(n) := \mathcal{C}_m(n,1) + \cdots + \mathcal{C}_m(n,m).$$

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(3) $\mathcal{I}_3(n) = Q_{n+1}$, a Pell-Lucas number defined as $Q_1 = 1$, $Q_2 = 3$, and $Q_{n+2} = 2Q_n + Q_{n+1}$ for $n \geqslant 0$.

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(4) $\mathcal{I}_4(n) = 2F_{n+1}$ twice a Fibonacci number. Moreover,

$$\mathcal{C}_4(n,1) = F_{2n-1} \quad \text{and} \quad \mathcal{C}_4(n,2) = F_{2n}.$$

## Theorem

*For $m \geqslant n$ we have*

$$\mathcal{I}_{m+1}(n) - \mathcal{I}_m(n) = \sum_{i=0}^{n-1} \mathcal{C}_m(i,1)\mathcal{C}_m(n-i,1),$$

*where $\mathcal{C}_m(0,1) := 1$.*

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*where $\mathcal{C}_m(0,1) := 1$.*

### Theorem

*For $m \geqslant 2n-2$ we have*

$$\mathcal{I}_{m+1}(n) - \mathcal{I}_m(n) = 3^{n-1}.$$

## Theorem

*We have*

$$\mathcal{I}_m(a+b-1)=\sum_{i=1}^{m}\mathcal{C}_m(a,i)\mathcal{C}_m(b,i)$$

*for all* $a, b \geqslant 1$.

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## Theorem

*We have*

$$\mathcal{I}_m(n) = m3^{n-1} - 2\sum_{k=1}^{n-1} 3^{n-k-1}\mathcal{C}_m(k,1).$$

*If* $m \geqslant n$ *and* $1 \leqslant k \leqslant n$*, then*

$$\mathcal{C}_m(k,1) = 3^{k-1} - \sum_{i=1}^{k-1} 3^{i-1} M_{k-i-1}.$$

## Theorem

*Inside the $m \times \infty$ table, a nontrivial linear combination of columns entries is a constant if and only if $m \equiv 1 \pmod 4$, and the equation is given by*

$$\alpha_1\mathcal{C}_m(n,1)+\alpha_3\mathcal{C}_m(n,3)+\cdots+\alpha_{m-2}\mathcal{C}_m(n,m-2)+\alpha_m\mathcal{C}_m(n,m)=\lambda$$

*for all $n \geqslant 1$, where $\lambda \neq 0$ is a fixed number and*

$$\alpha_{2i+1}+\alpha_{m-2i}=(-1)^i 2\lambda,$$

*for all $i = 0, \ldots, (m-1)/4$.*

# Multipliers

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$$\mathcal{M}_m(0) = \begin{cases} 2, & m \text{ is odd,} \\ 1, & m \text{ is even,} \end{cases}$$

$\mathcal{M}_m(1) = \Delta - 2 + \mathcal{M}_m(0)$, and

$$\mathcal{M}_m(n+2) = \Delta\mathcal{M}_m(n+1) - \mathcal{M}_m(n)$$

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  for all $n \geqslant 0$. Set $\mathcal{M}_o := \mathcal{M}_1$ and $\mathcal{M}_e := \mathcal{M}_2$.
- $\mathcal{M}' = \mathcal{M}'(\Delta)$: the multiplier function defined as $\mathcal{M}'(0) = 1$, $\mathcal{M}'(1) = \Delta$, and

$$\mathcal{M}'(n+2) = \Delta \mathcal{M}'(n+1) - \mathcal{M}'(n)$$

  for all $n \geqslant 0$.

### Lemma

*Inside the $m \times \infty$ table, we have*

$$\mathcal{M}_m(k-b)\mathcal{C}_m(n,a) = \mathcal{M}_m(k-a)\mathcal{C}_m(n,b)$$

*for all $a, b = 1, \ldots, k$, where $k = \lceil m/2 \rceil$.*

## Corollary

*Inside the $m \times \infty$ table, we have*

$$\mathcal{M}_m(k-a)\mathcal{I}_m(n) = 2(\mathcal{M}_m(k-1) + \cdots + \mathcal{M}_m(1) + 1)\mathcal{C}_m(n, a)$$

*for all $a = 1, \ldots, k$ and $n \geqslant 1$, where $k = \lceil m/2 \rceil$.*

*In particular, for $a = k$, we have*

$$\mathcal{I}_m(n) = \frac{2}{\mathcal{M}_m(0)}(\mathcal{M}_m(k-1) + \cdots + \mathcal{M}_m(1) + 1)\mathcal{C}_m(n, k)$$

*for all $n \geqslant 1$.*

## Lemma

*Inside the $m \times \infty$ table, we have*

$$\mathcal{M}'(b-1)\mathcal{C}_m(n,a) = \mathcal{M}'(a-1)\mathcal{C}_m(n,b)$$

*for all $a, b = 1, \ldots, k$, where $k = \lceil m/2 \rceil$.*

## Corollary

*Inside the $m \times \infty$ table, we have*

$$\mathcal{M}'(a-1)\mathcal{I}_m(n) = 2\left(\frac{\mathcal{M}'(k-1)}{\mathcal{M}_m(0)} + \mathcal{M}'(k-2) + \cdots + \mathcal{M}'(0)\right)\mathcal{C}_m(n,a)$$

*for all $a = 1, \ldots, k$ and $n \geqslant 1$, where $k = \lceil m/2 \rceil$.*

*In particular, for $a = 1$, we have*

$$\mathcal{I}_m(n) = 2\left(\frac{\mathcal{M}'(k-1)}{\mathcal{M}_m(0)} + \mathcal{M}'(k-2) + \cdots + \mathcal{M}'(0)\right)\mathcal{C}_m(n,1)$$

*for all $n \geqslant 1$.*

Let

$$\mathcal{T}_m := \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}$$

be the matrix of the $m \times \infty$ table, and $\mathcal{C}_m(n)$ be the $n^{th}$ column of the $m \times \infty$ table. Then

$$\mathcal{T}_m \mathcal{C}_m(n) = \mathcal{C}_m(n+1) \quad \text{and} \quad \mathcal{I}_m(n) = \mathbf{1}^T \mathcal{T}_m^n \mathbf{1}$$

for all $n \geqslant 1$.

Let

$$\mathcal{O}_1 = [1], \quad \mathcal{O}_2 = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \quad \mathcal{O}_k = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 2 & 1 \end{bmatrix} \quad (k \geqslant 3),$$

and

$$\mathcal{E}_1 = [2], \quad \mathcal{E}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad \mathcal{E}_k = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 2 \end{bmatrix} \quad (k \geqslant 3).$$

Also, let

$$\mathcal{T}_m^* = \begin{cases} \mathcal{O}_{\lceil \frac{m}{2} \rceil}, & m \text{ is odd,} \\ \mathcal{E}_{\lceil \frac{m}{2} \rceil}, & m \text{ is even} \end{cases}$$

be the reduced matrix of the $m \times \infty$ table for all $m \geqslant 1$.

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Assume $C_m^*(n)$ is the reduced $n^{th}$ column in the $m \times \infty$ table including entries in the rows $1, \ldots, \lceil m/2 \rceil$.

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Then

$$\mathcal{T}_m^* C_m^*(n) = C_m^*(n+1)$$

for all $n \geqslant 1$.

## Lemma

*For every $m \geqslant 1$, we have*

$$\det \mathcal{T}_m^* = \begin{cases} (-1)^{\left\lfloor \frac{\lceil \frac{m}{2} \rceil + 1}{3} \right\rfloor} 2^{\chi_{3\mathbb{Z}}(\lceil \frac{m}{2} \rceil)}, & m \text{ is odd,} \\ (-1)^{\left\lfloor \frac{\lceil \frac{m}{2} \rceil}{3} \right\rfloor} 2^{\chi_{3\mathbb{Z}+1}(\lceil \frac{m}{2} \rceil)}, & m \text{ is even,} \end{cases}$$

*where $\chi$ denotes the characteristic function.*

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*where $\chi$ denotes the characteristic function.*

## Corollary

*Let $k = \lceil m/2 \rceil$. Then*

$$\det([C_m^*(n+1) \cdots C_m^*(n+k)]) = \det(\mathcal{T}_m^*)^n$$

*for all $n \geqslant 0$.*

## Theorem

*Let $k = \lceil m/2 \rceil$ and $[\alpha_1 \cdots \alpha_k]^T$ be the solution to the matrix equation*

$$[C^*_m(1) \cdots C^*_m(k)][\alpha_1 \cdots \alpha_k]^T = [C^*_m(k+1)]$$

*inside the $m \times \infty$ table. Then*

$$\mathcal{C}_m(n+k, i) = \alpha_1 \mathcal{C}_m(n, i) + \cdots + \alpha_k \mathcal{C}_m(n+k-1, i) \qquad (1)$$

*for all $1 \leqslant i \leqslant m$ and $n \geqslant 1$. As a result,*

$$\mathcal{I}_m(n+k) = \alpha_1 \mathcal{I}_m(n) + \cdots + \alpha_k \mathcal{I}_m(n+k-1) \qquad (2)$$

*for all $n \geqslant 1$. Moreover, these recurrence relations are of the minimum degree.*

## Corollary

*For every $m \geqslant 1$, the following polynomials are equal:*

(1) $\det(xI - \mathcal{T}_m^*)$;

(2) $\mathcal{M}_m(k)(x-1)$;

(3) $x^k - [1 \cdots x^{k-1}][C_m^*(1) \cdots C_m^*(k)]^{-1}[C_m^*(k+1)]$,

*where $k = \lceil m/2 \rceil$.*

## Corollary

*The recurrence relations* (1) *and* (2) *are given by any of the formulas*

$$\det((\Delta+1)I-\mathcal{T}_m^*)\mathcal{C}_m(n,i)=0 \quad \textit{or} \quad \mathcal{M}_m(k)\mathcal{C}_m(n,i)=0$$

*and*

$$\det((\Delta+1)I-\mathcal{T}_m^*)\mathcal{I}_m(n)=0 \quad \textit{or} \quad \mathcal{M}_m(k)\mathcal{I}_m(n)=0,$$

*respectively, for all* $i=1,\ldots,m$ *and* $n\geqslant 1$*, where* $k=\lceil m/2\rceil$*.*

## Theorem

*For any $m, k \geqslant 1$ and $1 \leqslant a \leqslant k$, we have*

$$\mathcal{M}'(a-1)\mathcal{M}_m(k-1) \equiv \mathcal{M}_m(k-a) \pmod{\mathcal{M}_m(k)}.$$

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## Theorem

*For any $m \geqslant 1$ and $a, b \geqslant 1$, we have*

$$\mathcal{M}_m(a+b) = \mathcal{M}'(a)\mathcal{M}_m(b) - \mathcal{M}'(a-1)\mathcal{M}_m(b-1)$$

*and*

$$\mathcal{M}'(a+b) = \mathcal{M}'(a)\mathcal{M}'(b) - \mathcal{M}'(a-1)\mathcal{M}'(b-1).$$

## Lemma

*Let* $m \geqslant 1$, $k = \lceil m/2 \rceil$, $0 \leqslant a \leqslant k$, *and* $0 \leqslant b \leqslant \min\{a, k-a\}$. *Then*

$$\mathcal{M}_o(b)\mathcal{C}_m(n,a) = \mathcal{C}_m(n,a-b) + \mathcal{C}_m(n,a+b).$$

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## Theorem

*For any* $m \geqslant 1$ *and* $a \geqslant b \geqslant 0$, *we have*

$$\mathcal{M}_o(b)\mathcal{M}_m(a) = \mathcal{M}_m(a+b) + \mathcal{M}_m(a-b)$$

*and*

$$\mathcal{M}_o(b)\mathcal{M}'(a) = \mathcal{M}'(a+b) + \mathcal{M}'(a-b).$$

**Theorem**

*For any $a, b \geqslant 1$ we have*

$$\mathcal{M}_m(ab) = \mathcal{M}'(a-1)\,(\mathcal{M}_o(b))\,\mathcal{M}_m(b) - \mathcal{M}'(a-2)\,(\mathcal{M}_o(b))\,\mathcal{M}_m(0)$$

*and*

$$\mathcal{M}'(ab) = \mathcal{M}'(a-1)\,(\mathcal{M}_o(b))\,\mathcal{M}'(b) - \mathcal{M}'(a-2)\,(\mathcal{M}_o(b))\,\mathcal{M}'(0).$$

**Theorem (Factorization theorem for $\mathcal{M}_o$)**

For all $a, b \geqslant 1$, we have

$$\mathcal{M}_o(ab) = \mathcal{M}_o(a) \circ \mathcal{M}_o(b) = \mathcal{M}_o(b) \circ \mathcal{M}_o(a).$$

As a result, if $n = p_1^{a_1} \dots p_k^{a_k}$ is the canonical factorization of $n$ into distinct primes $p_1, \dots, p_k$, then

$$\mathcal{M}_o(n) = \mathcal{M}_o(p_1)^{a_1} \dots \mathcal{M}_o(p_k)^{a_k},$$

where all the products are the combination of functions.

# Prime functions

- $\mathcal{F}_{m,p}$ is a function defined on the set of multipliers $\{\mathcal{M}_m(n)\}$ as

$$\mathcal{F}_{m,p}(\mathcal{M}_m(n)) = \mathcal{M}'(p-1)(\mathcal{M}_e(n))\mathcal{M}_m(n) - \mathcal{M}'(p-2)(\mathcal{M}_e(n))\mathcal{M}_m(0).$$

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- $\mathcal{F}'_p$ is a function defined on the set of multipliers $\{\mathcal{M}'(n)\}$ as

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## Theorem (Uniform factorization theorem)

Let $n = p_1^{a_1} \dots p_k^{a_k}$ be the canonical factorization of $n$ into distinct primes $p_1, \dots, p_k$. Then

$$\mathcal{M}_m(n) = \mathcal{F}_{m,p_1}^{a_1} \dots \mathcal{F}_{m,p_k}^{a_k} \mathcal{M}_m(1)$$

and

$$\mathcal{M}'(n) = \mathcal{F'}_{p_1}^{a_1} \dots \mathcal{F'}_{p_k}^{a_k} \mathcal{M}'(1),$$

where all the products are the combination of functions.

## Lemma

*For all $n \geqslant 1$, we have*

$$\mathcal{M}_e(n) = \mathcal{M}'(n) - \mathcal{M}'(n-1)$$

*and*

$$\mathcal{M}_o(n) = \mathcal{M}_e(n) + \mathcal{M}_e(n-1) = \mathcal{M}'(n) - \mathcal{M}'(n-2).$$

## Corollary

*For all $n \geqslant 1$, we have*

$$1+\mathcal{M}_m(1)+\cdots+\mathcal{M}_m(n)=\mathcal{M}'(n)+(\mathcal{M}_m(0)-1)\mathcal{M}'(n-1).$$

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*For all $n \geqslant 1$, we have*

$$1 + \mathcal{M}_m(1) + \cdots + \mathcal{M}_m(n) = \mathcal{M}'(n) + (\mathcal{M}_m(0) - 1)\mathcal{M}'(n-1).$$

## Corollary

*Inside the $m \times \infty$ table, we have*

$$\mathcal{M}_m(k-a)\mathcal{I}_m(n) = 2(\mathcal{M}'(k-1) + (\mathcal{M}_m(0)-1)\mathcal{M}'(k-2))\mathcal{C}_m(n,a)$$

*for all $a = 1, \ldots, k$ and $n \geqslant 1$, where $k = \lceil m/2 \rceil$.*

## Theorem

*For all $n \geqslant 1$, we have*

$$\mathcal{M}_o(n) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \left[ \binom{n+1-i}{i} - \binom{n-1-i}{i-2} \right] \Delta^{n-2i},$$

$$\mathcal{M}_e(n) = \sum_{i=0}^{n} (-1)^{\lceil \frac{i}{2} \rceil} \binom{n - \lceil \frac{i}{2} \rceil}{\lfloor \frac{i}{2} \rfloor} \Delta^{n-i},$$

$$\mathcal{M}'(n) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \binom{n-i}{i} \Delta^{n-2i}.$$

# Relations to other polynomials

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- $|\mathcal{M}_e(n)|(x)$ is the Fibonacci polynomial $F_{n+1}(x)$;
- $|\mathcal{M}'(n)|(x)$ is the Lucas polynomial $L_{n+1}(x)$.

# Open problem

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$$k = 1, 2, 3, 5, 9, 11, 14, 23, 26, 29, 35, 39, 41, 53, ?, ?, ?, \ldots.$$

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**Question**

Which $\mathcal{E}_k$ are Singer?

Thank You for Your Attention!