Cayley numbers

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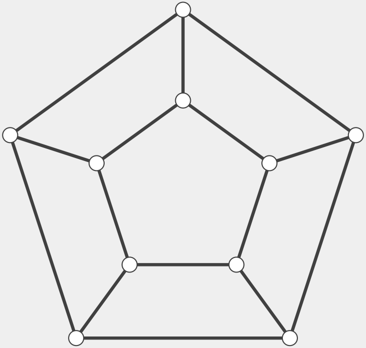
Definitions & examples

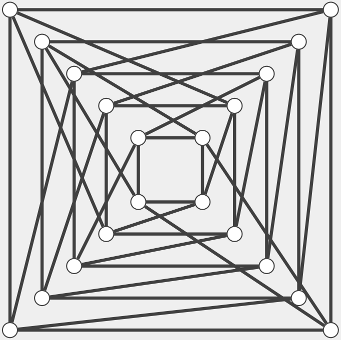
Definition

- A graph Γ is vertex-transitive if its groups of automorphisms acts transitively on its vertex-set.
- A graph Γ is edge-transitive if its groups of automorphisms acts transitively on its edge-set.
- A graph Γ is arc-transitive if its groups of automorphisms acts transitively on its arc-set.
- A graph is symmetric if it is arc-transitive and it is weakly symmetric if it is vertex- and edge-transitive.

Examples

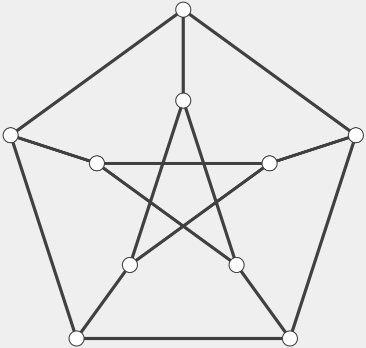


Vertex-transitive but not edge-transitive

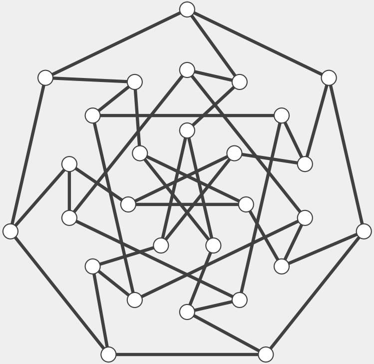


Edge-transitive but not vertex-transitive

Examples



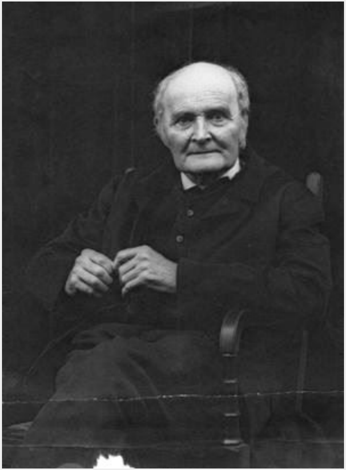
Petersen graph



Coxeter graph

Definition (Cayley 1878[1])

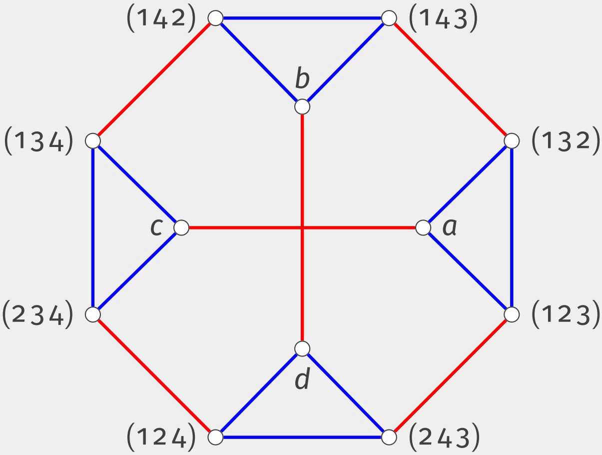
Let G be a group and C be an inversed closed subset of $G \setminus \{1\}$. Then the graph $\mathrm{Cay}(G, C)$ with vertex set G and edges $\{x, y\}$ with $x^{-1}y \in C$ is called the Cayley graph of G with respect to the connection set C.



Arthur Cayley

[1] A. Cayley, Desiderata and Suggestions: No. 2. The Theory of Groups: Graphical Representation, *Amer. J. Math.* **1**(2) (1878), 174–176.

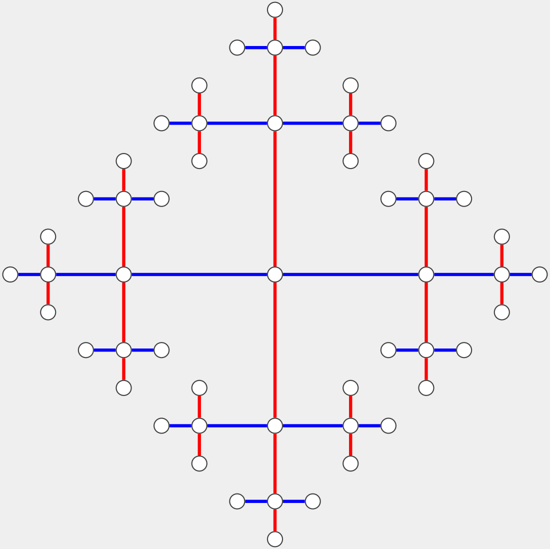
Example: $\mathrm{Cay}(A_4, \{\alpha^{\pm 1}, \beta\})$



$\alpha = (1\,2\,3)$ and $\beta = (1\,2)(3\,4)$

$a = ()$, $b = (1\,4)(2\,3)$, $c = (1\,2)(3\,4)$, and $d = (1\,3)(2\,4)$

Example: $\mathrm{Cay}(F(x,y), \{x^{\pm 1}, y^{\pm 1}\})$



Theorem

Cayley graphs are vertex-transitive.

[2] G. Sabidussi, Vertex-transitive graphs, *Monatsh. Math.* **68** (1964), 426–438.

Theorem

Cayley graphs are vertex-transitive.

Theorem (Sabidussi 1964[2])

A vertex-transitive graph is a Cayley graph if and only if its group of automorphism has a regular subgroup.

[2] G. Sabidussi, Vertex-transitive graphs, *Monatsh. Math.* **68** (1964), 426–438.

Definition

A subgraph Δ of a graph Γ is a retract of Γ if there exists an homomorphism $\theta : \Gamma \longrightarrow \Delta$ such that $\theta|_\Delta$ is the identity map.

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Theorem (Sabidussi 1964[2])

Every connected vertex-transitive graph is a retract of a Cayley graph.

Definition

Let G be a group, H be a subgroup of G, and $C \subseteq G \setminus H$ be an inversed-closed union of double-cosets of G. Then the graph $\mathrm{Cos}(G, H, C)$ with vertex set G/H and edges $\{xH, yH\}$ with $x^{-1}y \in C$ is called a coset graph of G.

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Theorem

Every coset graph is a vertex-transitive graph.

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Theorem

Every coset graph is a vertex-transitive graph.

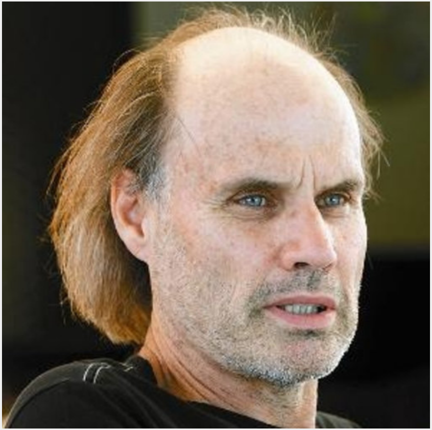
Theorem (Sabidussi 1964[2])

Every vertex-transitive graph is a coset graph.

Cayley numbers

Question (Marušič 1983[3])

For which numbers n there exists a vertex-transitive graph of order n that is not a Cayley graph?



Dragan Marušič

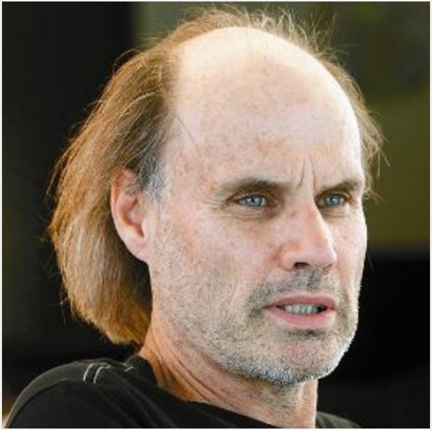
[3] D. Marušič, Cayley properties of vertex symmetric graphs, *Ars Combin.* **16** (1983), 297–302.

Question (Marušič 1983[3])

For which numbers n there exists a vertex-transitive graph of order n that is not a Cayley graph?

Definition

A number n is a Cayley number if all vertex-transitive graphs of order n are Cayley graphs. The set of non-Cayley number is denoted by $\mathcal{NC}$.



Dragan Marušič

[3] D. Marušič, Cayley properties of vertex symmetric graphs, *Ars Combin.* **16** (1983), 297–302.

Theorem (Marušič 1985[4])

For every prime p, the numbers p, p^2, and p^3 are Cayley numbers but p^4 is a non-Cayley number.

[4] D. Marušič, Vertex-transitive graphs and digraphs of order p^k, *Ann. Discrete Math.* **27** (1985), 115–128.

Theorem (Marušič 1985[4])

For every prime p, the numbers p, p^2, and p^3 are Cayley numbers but p^4 is a non-Cayley number.

Theorem

Let p be a prime.

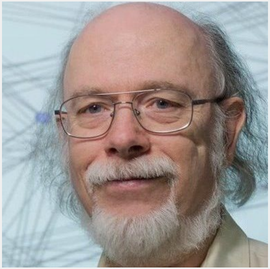
1. *The only p-group of order p is $\mathbb{Z}_p$;*
2. *The p-groups of order p^2 are $\mathbb{Z}_{p^2}$ and $\mathbb{Z}_p \times \mathbb{Z}_p$;*
3. *The p-groups of order p^3 are $\mathbb{Z}_{p^3}$, $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$, $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, $\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$, and $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$.*

[4] D. Marušič, Vertex-transitive graphs and digraphs of order p^k, *Ann. Discrete Math.* **27** (1985), 115–128.

Theorem (McKay and Praeger 1994[5], 1996[6])

Let n be a positive integer that is divisible by the square of a prime p. Then $n \in \mathcal{NC}$ unless $n = p^2$ or $n = p^3$ or $n = 12$.

Cheryl E. Praeger



Brendan McKay

[5] B. D. McKay and C. E. Praeger, Vertex-transitive graphs which are not Cayley graphs, I, *J. Austral. Math. Soc. Ser. A* **56**(1) (1994), 53–63.

[6] B. D. McKay and C. E. Praeger, Vertex-transitive graphs that are not Cayley graphs, II, *J. Graph Theory* **22**(4) (1996), 321–334.

Theorem (Frucht, Graver, and Watkins 1971[7], Marušič and Scapellato 1992[8], Praeger and Xu 1993[9])

Let $p < q$ be primes. Then $pq \in \mathcal{NC}$ if and only if one of the following holds:

(1) p^2 *divides* $q - 1$;

(2) $q = 2p - 1 > 3$ *or* $q = (p^2 + 1)/2$;

(3) $q = 2^t + 1$ *and either* p *divides* $2^t - 1$ *or* $p = 2^{t-1} - 1$;

(4) $q = 2^t - 1$ *and* $p = 2^{t-1} + 1$;

(5) $(p, q) = (7, 11)$.

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[7] R. Frucht, J. Graver, and M. E. Watkins, The groups of the generalized Petersen graphs, *Proc. Cam. Phil. Soc.* **70** (1971), 211–218.

[8] D. Marušič and R. Scapellato, Characterising vertex-transitive *pq*-graphs with an imprimitive automorphism group, *J. Graph Theory* **16** (1992), 375–387.

[9] C. E. Praeger and M. Y. Xu, Vertex-primitive graphs of order a product of two distinct primes, *J. Combin. Theory B* **59** (1993), 245–266.

Theorem (Miller and Praeger 1994[10], Seress 1998[11], Hassani, Iranmanesh, and Praeger 1998[12], Gamble and Praeger 2000[13], Iranmanesh and Praeger 2001[14])

Let $p < q < r$ be primes. Then $pqr \in \mathcal{NC}$ if and only if one of the following holds:

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[10] A. A. Miller and C. E. Praeger, Non-Cayley vertex-transitive graphs of order twice the product of two odd primes, *J. Algebraic Combin.* **3** (1994), 77–111.

[11] Á. Seress, On vertex-transitive, non-Cayley graphs of order *pqr*, *Discrete Math.* **182** (1998), 279–292.

[12] A. Hassani, M. A. Iranmanesh, and C. E. Praeger, On vertex-imprimitive graphs of order a product of three distinct odd primes, *J. Combin. Math. Combin. Comput.* **28** (1998), 187–213.

[13] G. Gamble and C. E. Praeger, Vertex-primitive groups and graphs of order twice the product of two distinct odd primes, *J. Group Theory* **3** (2000), 247–269.

[14] M. A. Iranmanesh and C. E. Praeger, On non-Cayley vertex-transitive graphs of order a product of three primes, J. *Combin. Theory Ser. B* **81**(1) (2001), 1–19.

Theorem (Continued)

(1) $q \overset{p^2}{\equiv} 1$, or $r \overset{p^2}{\equiv} 1$, or $r \overset{q^2}{\equiv} 1$;

(2) $2s-1$ and $(s^2+1)/2$ belongs to $\{p,q,r\}$ for some odd $s \in \{p,q,r\}$;

(3) $\{p,q,r\}$ contains 2^t+1 and also contains either $2^{t-1}-1$ or a divisor of 2^t-1 for some t;

(4) $\{p,q,r\}$ contains 2^t-1 and $2^{t-1}+1$ for some t;

(5) $7, 11 \in \{p,q,r\}$;

(6) $pqr = (2^{2^t}+1)(2^{2^{t+1}}+1)$ for some t, or $pqr = (2^{s\pm1}+1)(2^s-1)$ for some prime s;

Theorem (Continued)

(7) $\{p, q, r\} = \{p', q', r'\}$ *with* $p'q'$ *being equal to*

  (a) $2r' \pm 1$ *with* $p > 2$;

  (b) $(r' + 1)/2$;

  (c) $(r'^2 + 1)/2$ *with* $p > 2$;

  (d) $(r'^2 - 1)/24x$ *with* $x \in \{1, 2, 5\}$ *and* $p > 2$, *or*

  (e) $2^t + 1$ *with* r *dividing* $2^t - 1$ *for some* t *and* $p > 2$;

(8) $p^p \| q - 1$ *and* $q^q \| r - 1$;

(9) $q = (3p + 1)/2$ *and* $r = 3p + 2$ *with* $p > 2$;

(10) $q = 6p - 1$ *and* $r = 6p + 1$ *with* $p > 2$;

(11) $q = (r - 1)/2$ *with* $p \mid r + 1$ *and* $p > 2$;

(12) $p = (r - 1)/2$ *and* $q = (p + 1)/2$ *with* $q \mid r + 1$ *and* $p > 2$;

Theorem (Continued)

(13) $p = (k^{d/2}+1)/(k+1)$, $q = (k^{d/2}-1)/(k-1)$, *and* $r = (k^{d-1}-1)/(k-1)$ *with* k, $d-1$, *and* $d/2$ *all prime;*

(14) $p = (k^{(d-1)/2}+1)/(k+1)$, $q = (k^{(k-1)/2}-1)/(k-1)$, *and* $r = (k^{d}-1)/(k-1)$ *with* k, d, *and* $(d-1)/2$ *all prime;*

(15) $p = k^2-k+1$, $q = (k^5-1)/(k-1)$, *and* $r = (k^7-1)/(k-1)$ *with* k *prime;*

(16) $p = 3$, $q = (2^d+1)/3$, *and* $r = 2^d-1$ *with* d *prime;*

(17) $p = (2^d+1)/3$, $q = 2^d-1$, *and* $r = 2^{2d\pm2}+1$ *with* $d = 2^t \mp 1$ *prime, or*

(18) $(p,q,r) = (2,7,19), (5,11,19)$, *or* $(7,73,257)$.

Question (McKay and Praeger 1996[6])

Is there a number $k > 0$ such that every product of k distinct primes is in $\mathcal{NC}$?

[15] T. Dobson and P. Spiga, Pablo, Cayley numbers with arbitrarily many distinct prime factors, *J. Combin. Theory Ser. B* **122** (2017), 301–310.

Question (McKay and Praeger 1996[6])

Is there a number $k > 0$ such that every product of k distinct primes is in $\mathcal{NC}$?

Theorem (Dobson and Spiga 2017[15])

There exists an infinite set of primes such that every finite product of its distinct elements is a Cayley number.

[15]T. Dobson and P. Spiga, Pablo, Cayley numbers with arbitrarily many distinct prime factors, *J. Combin. Theory Ser. B* **122** (2017), 301–310.

Some new results

Some new results

Some algebraic properties of Sierpiński-Type graphs

Joint work with E. Ghorbani, H. R. Maimani, and F. Rahimi Mahid

Definition (Klavžar and Milutinović 1997[16])

The Sierpiński graph $S(n,k)$ $(n,k \geq 1)$ is a graph with vertex set $\{1,\ldots,k\}^n$ such that two distinct vertices $(u_1,\ldots,u_n)$ and $(v_1,\ldots,v_n)$ are adjacent if there exists $t \in \{1,\ldots,n\}$ such that

- $u_i = v_i$ for $i = 1,\ldots,t-1$,
- $u_t \neq v_t$,
- $u_j = v_t$ and $v_j = u_t$ for $j = t+1,\ldots,n$.

[16] S. Klavžar and U. Milutinović, Graphs $S(n,k)$ and a variant of the tower of Hanoi problem, *Czechoslovak Math. J.* **47**(122) (1997), 95–104.

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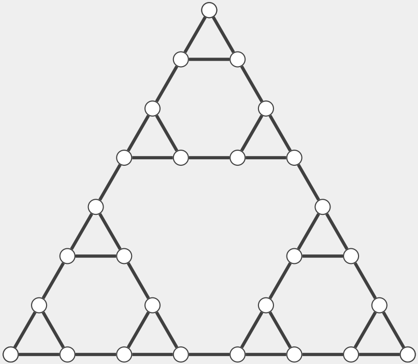
- $u_i = v_i$ for $i = 1, \dots, t-1$,
- $u_t \neq v_t$,
- $u_j = v_t$ and $v_j = u_t$ for $j = t+1, \dots, n$.

Remark

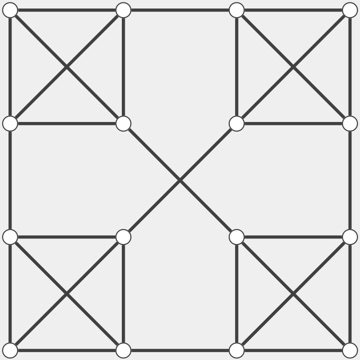
The graph $S(n, k)$ has $k^n - k$ vertex of degree k and k extreme vertices of degree $k-1$, namely $(i, \dots, i)$ for $1 \leq i \leq k$.

[16] S. Klavžar and U. Milutinović, Graphs $S(n, k)$ and a variant of the tower of Hanoi problem, *Czechoslovak Math. J.* **47**(122) (1997), 95–104.

New results



$S(3,3)$



$S(2,4)$

Definition (Klavžar and Mohar 2005[17])

The Sierpiński-type graphs $S^{++}(n,k)$ are defined as follows:

- For $n = 1$, $S^{++}(n,k)$ is the complete graph K_{k+1};
- For $n \geq 2$, $S^{++}(n,k)$ is the graph obtained from a disjoint union of $k+1$ copies of $S(n-1,k)$ in which the extreme vertices in distinct copies of $S(n-1,k)$ are connected as the complete graph K_{k+1}.

[17] S. Klavžar and B. Mohar, Crossing numbers of Sierpiński-like graphs, *J. Graph Theory* **50** (2005), 186–198.

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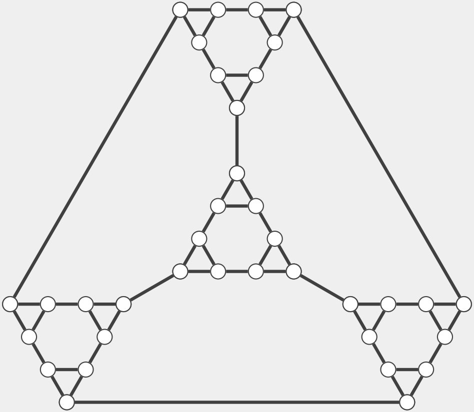
Remark

- $S^{++}(n, 1) \cong K_2$;
- $S^{++}(n, 2) \cong C_{3 \cdot 2^{n-1}}$;
- $S^{++}(1, k) \cong K_{k+1}$.

[17] S. Klavžar and B. Mohar, Crossing numbers of Sierpiński-like graphs, *J. Graph Theory* **50** (2005), 186–198.



$S^{++}(3,3)$

Theorem

The graph $S^{++}(n,k)$ is vertex-transitive if and only if either $n \leq 2$ or $k \leq 2$.

Definition

Let Γ be a graph and Δ be a subgraph of Γ. Then Γ is strongly Δ-partitioned if

(1) the vertex set of Γ is partitioned by the vertex sets of copies $\Delta_0, \dots, \Delta_k$ of Δ;

(2) besides $\Delta_0, \dots, \Delta_k$, the graph Γ contains no further copies of Δ.

Definition

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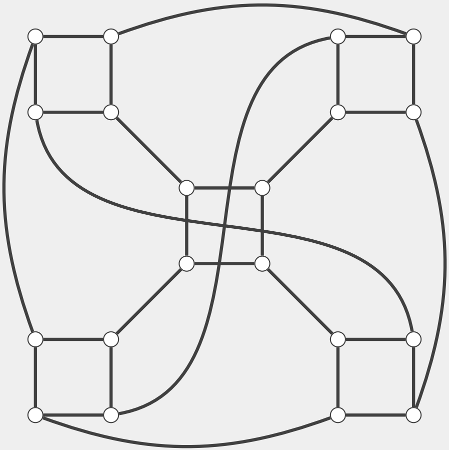
(2) besides $\Delta_0, \ldots, \Delta_k$, the graph Γ contains no further copies of Δ.

Definition

Let Γ be a strongly Δ-partitioned graph. Then Γ has connection constant c if there are exactly c edges between any two copies of Δ in Γ. The set of all strongly Δ-partitioned graphs with connection constant c is denoted by $\mathcal{SP}_c(\Delta)$.

New results



The only vertex-transitive graph
among seven strongly C_4-partitioned graphs.

Theorem

The graph $S^{++}(n,k)$ is strongly $S(n-1,k)$-partitioned when $n \geq 2$ and $k \geq 3$.

Theorem

Let Γ and Δ be regular graphs with $\Gamma \in \mathcal{SP}_1(\Delta)$. If Γ is a Cayley graph $\mathrm{Cay}(G, C)$, then

(1) $|\Delta| + 1 = p^m$ *is prime power,*

(2) $G = N \rtimes H$ *is a Frobenius group with minimal normal Frobenius kernel* $N \cong \mathbb{Z}_p^m$ *and Frobenius complement* H,

(3) $C = C' \cup \{c\}$ *with* $\Delta \cong \mathrm{Cay}(H, C')$ *and* $c^2 = 1$, *and either*

(i) $c \in N$ *and* $H = \langle C' \rangle$, *or*

(ii) $c = h^n$ *for some* $h \in H \setminus \{1\}$ *and* $n \in N \setminus \{1\}$, *and* $H = \langle C', h \rangle$.

Theorem

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  (ii) $c = h^n$ *for some* $h \in H \setminus \{1\}$ *and* $n \in N \setminus \{1\}$, *and* $H = \langle C', h \rangle$.

Conversely, if Δ satisfies the above conditions, then $\mathrm{Cay}(G, C) \in \mathcal{SP}_1(\Delta)$.

Corollary

The graph $S^{++}(n,k)$ is a Cayley graph if and only if either

(1) $n = 1$;

(2) $k \leq 2$, *or*

(3) $n = 2$ *and* $k + 1 = p^m$ *is a prime power.*

Corollary

The graph $S^{++}(n,k)$ is a Cayley graph if and only if either

(1) $n = 1$;

(2) $k \leq 2$, *or*

(3) $n = 2$ *and* $k + 1 = p^m$ *is a prime power.*

Furthermore, in case (3), we have

$$S^{++}(n,k) \cong \mathrm{Cay}(G, (H \setminus \{1\}) \cup \{c\}),$$

for every Frobenius group G with complement H, elementary abelian minimal normal Frobenius kernel of order p^m, and involution $c \in G \setminus H$.

Theorem

Let k be any positive integer such that $k(k+1)$ is square-free and $k+1$ is not a prime. Then $k(k+1) \in \mathcal{NC}$.

Theorem

Let k be any positive integer such that $k(k+1)$ is square-free and $k+1$ is not a prime. Then $k(k+1) \in \mathcal{NC}$.

Theorem

The density of the set

$$\{k : k(k+1) \text{ is square-free and } k+1 \text{ is not a prime}\}$$

is equal to $2C_{\text{Feller-Tornier}} - 1 \approx 0.3226$, where the Feller-Tornier constant

$$\frac{1}{2} + \frac{1}{2}\prod_p \left(1 - \frac{2}{p^2}\right) = \frac{1}{2} + \frac{3}{\pi^2}\prod_p \left(1 - \frac{1}{p^2 - 1}\right) \approx 0.6613$$

is the density of integers having an even number of non-prime prime powers factors.

The list of the numbers whose membership in $\mathcal{NC}$ are not yet determined begins with

$$9982,\ 12958,\ 18998,\ 19646,\ 20398,\ 21574,\ 24662,\ 25438,\ 25606,\ \ldots.$$

Among the numbers $\leq 10^8$, there are 2763 square-free integers of the form $k(k+1)$ with $k+1$ not a prime of which the following eight integers are new non-Cayley numbers:

$$1386506,\ 2668322,\ 15503906,\ 23985506,\ 38359442,\ 74261306,\ 89898842,\ 95912642.$$

Related problems

Smallest valency of non-Cayley vertex-transitive graphs

Question (Feng 2002[18])

What is the smallest valency $d(n)$ of a non-Cayley vertex-transitive graph of order n.

Yan-Quan Feng

[18] Y.-Q. Feng, On vertex-transitive graphs of odd prime-power order, *Discrete Math.* **248**(1-3) (2002), 265–269.

Smallest valency of non-Cayley vertex-transitive graphs

Theorem (Feng 2002[18])

Every vertex-transitive graph of odd prime power order p^k with valency less than $2p+2$ is a Cayley graph.

Smallest valency of non-Cayley vertex-transitive graphs

Theorem (Feng 2002[18])

Every vertex-transitive graph of odd prime power order p^k with valency less than $2p+2$ is a Cayley graph.

Theorem (Marušič 1985[4] and McKay and Praeger 1994[6])

There exists a non-Cayley vertex-transitive graph of odd prime power order p^k and valency $2p+2$.

Smallest valency of non-Cayley vertex-transitive graphs

Theorem (Feng 2002[18])

Every vertex-transitive graph of odd prime power order p^k with valency less than $2p+2$ is a Cayley graph.

Theorem (Marušič 1985[4] and McKay and Praeger 1994[6])

There exists a non-Cayley vertex-transitive graph of odd prime power order p^k and valency $2p+2$.

Corollary

We have $d(p^k) = 2p+2$ for all odd prime powers p^k.

Definition (Marušič, Scapellato, and Zagaglia Salvi 1992[19])

Let G be a group, C be a subset of G, and θ be an automorphism of G satisfying

(1) $\theta^2 = 1$;

(2) $\theta(x^{-1})x \notin C$ for all $x \in G$;

(3) $\theta(x^{-1})y \in C$ implies $\theta(y^{-1})x$ for all $x, y \in G$.

Then the graph $\mathrm{GCay}(G, C, \theta)$ with vertex set G and edges $\{x, y\}$ if $\theta(x^{-1})y \in C$ is called a generalized Cayley graph of G.

[19] D. Marušič, R. Scapellato, and N. Zagaglia Salvi, Generalized Cayley graphs, *Discrete Math.* **102** (1992), 279–285.

Remark (Marušič, Scapellato, and Zagaglia Salvi 1992[19])

Let

- $G = \mathbb{Z}_n \times \mathbb{Z}_n$,
- $C = \{(1, 0), (1, 1), (0, n-1), (n-1, n-1)\} \subseteq G$, and
- $\theta : (x, y) \mapsto (y, x)$ be an automorphism of G.

Then the graph $\mathrm{GCay}(\mathbb{Z}_n \times \mathbb{Z}_n, C, \theta)$ is not vertex-transitive.

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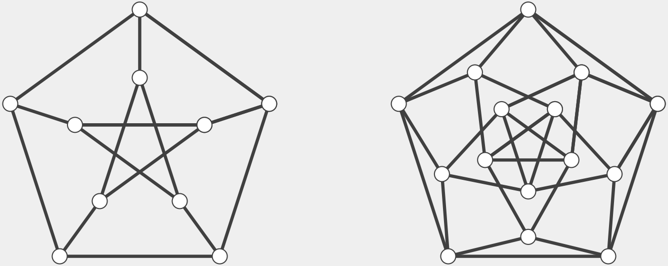
Then the graph $\mathrm{GCay}(\mathbb{Z}_n \times \mathbb{Z}_n, C, \theta)$ is not vertex-transitive.

Question (Marušič, Scapellato, and Zagaglia Salvi 1992[19])

Are there vertex-transitive generalized Cayley graphs which are not Cayley graphs?

Example (Watkins 1990[20])

The line graph of the Petersen graph is a non-Cayley vertex-transitive generalized Cayley graph.



Petersen graph and its line graph

[20] M. E. Watkins, Vertex-transitive graphs that are not Cayley graphs, in: G. Hahn, et al. (Eds.), *Cycles and Rays*, Kluwer, Netherlands, 1990, 243–256.

Theorem (Hujdurović, Kutnar, and Marušič 2015[21])

Every generalized Cayley graph of prime order is a Cayley graph.

[21] A. Hujdurović, K. Kutnar, and D. Marušič, Vertex-transitive generalized Cayley graphs which are not Cayley graphs, *European J. Combin.* **46** (2015), 45–50.

Theorem (Hujdurović, Kutnar, and Marušič 2015[21])

Every generalized Cayley graph of prime order is a Cayley graph.

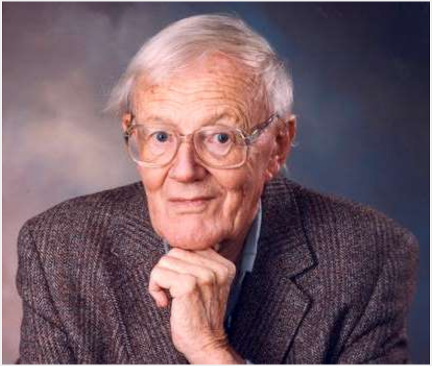
Theorem (Hujdurović, Kutnar, and Marušič 2015[21])

There are vertex-transitive generalized Cayley graphs of orders $4(2n^2+2n+1)$ and $20n$ $(5 \nmid n)$ that are not Cayley graphs.

[21] A. Hujdurović, K. Kutnar, and D. Marušič, Vertex-transitive generalized Cayley graphs which are not Cayley graphs, *European J. Combin.* **46** (2015), 45–50.

Question (Tutte 1966[22])

For which numbers n there exists a weakly symmetric graph of order n that is not a symmetric graph?



William Thomas Tutte

[22] W. T. Tutte, *Connectivity in Graphs*, University of Toronto Press, Toronto, 1966.

Theorem (Tutte 1966[22])

Every weakly symmetric but not symmetric graph has even valency.

[23] I. Z. Bouwer, Vertex and edge transitive, but not 1-transitive, graphs, *Canad. Math. Bull.* **13** (1970), 231–237.

Theorem (Tutte 1966[22])

Every weakly symmetric but not symmetric graph has even valency.

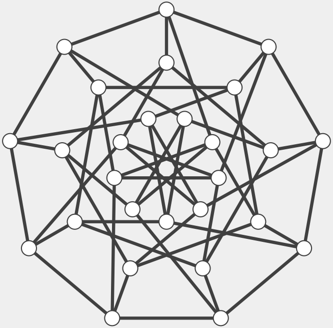
Theorem (Bouwer 1971[23])

For every number n, there exists a connected $2n$-regular weakly symmetric graph of order $6 \cdot 9^{n-1}$ that is not a symmetric graph.

[23] I. Z. Bouwer, Vertex and edge transitive, but not 1-transitive, graphs, *Canad. Math. Bull.* **13** (1970), 231–237.

Theorem (Holt 1981[24])

The following graph of order 27 is a weakly symmetric graph that is not a symmetric graph.



[24] D. F. Holt, A graph which is edge transitive but not arc transitive, *J. Graph Theory*, **5** (1981), 201–204.

Theorem

Let p be a prime.

1. *Every weakly symmetric graph of order p is a symmetric graph (Chao 1971[25]).*
2. *Every weakly symmetric graph of order $2p$ is a symmetric graph (Cheng and Oxley 1987[26])*
3. *Every weakly symmetric graph of order $2p^2$ is a symmetric graph (Zhou and Zhang 2018[27])*

[25] C.-Y. Chao, On the classification of symmetric graphs with a prime number of vertices, *Trans. Amer. Math. Soc.* **158** (1971), 247–256.

[26] Y. Cheng and J. Oxley, On weakly symmetric graphs of order twice a prime, *J. Combin. Theory Ser. B* **42** (1987), 196–211.

[27] J.-X. Zhou and M.-M. Zhang, On weakly symmetric graphs of order twice a prime square, *J. Combin. Theory Ser. A* **155** (2018), 458–475.

Thanks!