Geometric Group Theory

Amenable Groups

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Institute for Advanced Studies
in Basic Sciences

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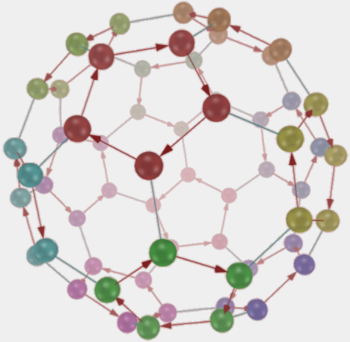

Free Groups

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A group is a set equipped with a binary operation satisfying the associativity law, containing a neutral element, and the inverses of its elements.

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$$\begin{gathered} 1, \\ a, a^{-1}, b, b^{-1}, \\ a^2, ab, ab^{-1}, a^{-2}, a^{-1}b, a^{-1}b^{-1}, b^2, ba, ba^{-1}, b^{-2}, b^{-1}a, b^{-1}a^{-1}, \\ \vdots \end{gathered}$$

  form a the group $\langle a, b \rangle$ generated by a, b.

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Presentation of groups

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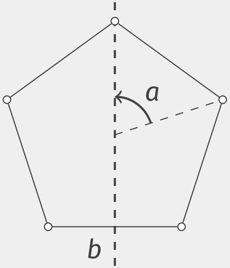

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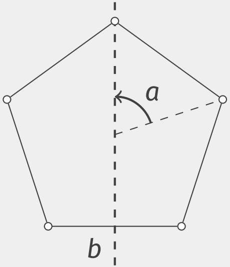

- $S_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i^2 = (\sigma_i \sigma_{i+1})^3 = [\sigma_i, \sigma_j] = 1, |i - j| > 1 \right\rangle$

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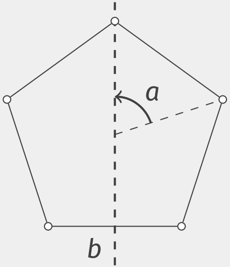

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- $F(X) := \langle X \mid \varnothing \rangle$ is a free group.

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- generating set X, and
- relation set R

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Theorem

If $G = \langle X \mid R\rangle$*, then*

$$G = F/N,$$

where $F := F(X)$ *and*

$$N = \left\langle frf^{-1} \mid r \in R,\ f \in F\right\rangle$$

is the normal closure of R in F.

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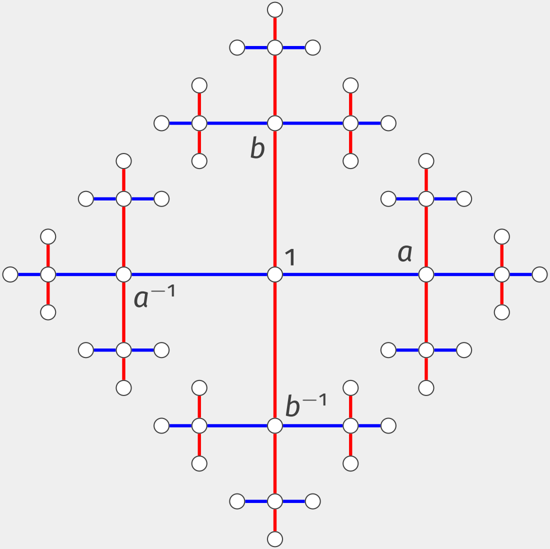
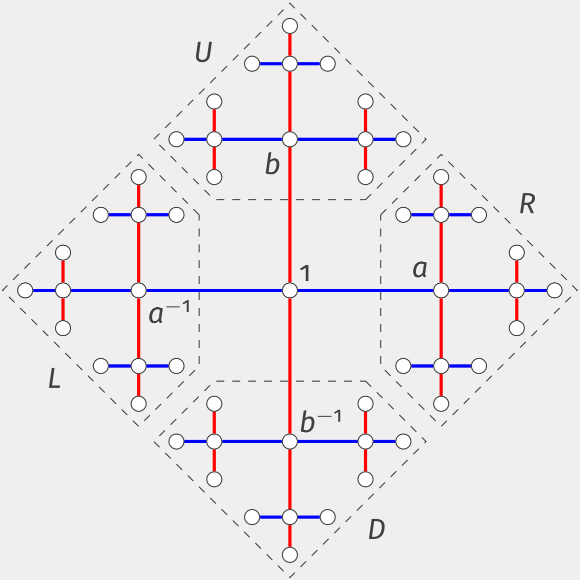
- Rotations, say $\langle \mathrm{Rot}_x(\alpha), \mathrm{Rot}_y(\beta)\rangle$, where
  - $\mathrm{Rot}_x(\alpha)$ is a rotation around *x*-axis by angle α
  - $\mathrm{Rot}_y(\beta)$ is a rotation around *y*-axis by angle β

  in $\mathbb{R}^n$ ($n \geq 3$) and α, β are chosen suitably.

Diagram of the free group $F(a, b)$




U
b
R
1
a
a^{-1}
L
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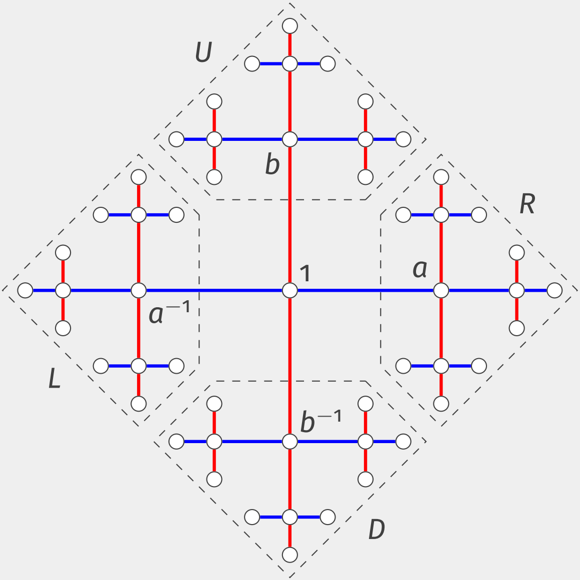
D



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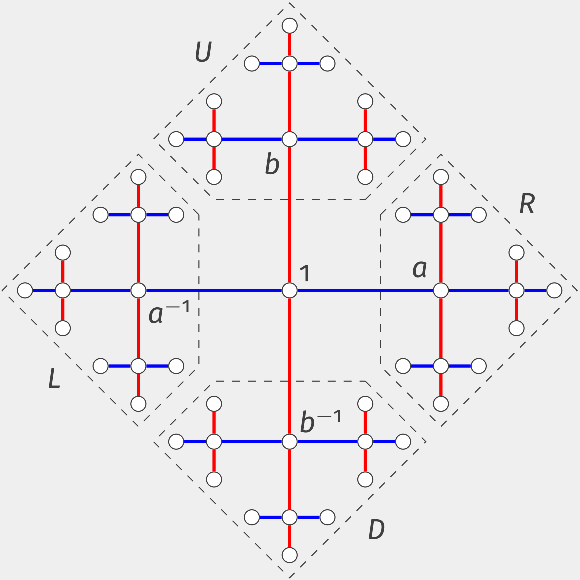

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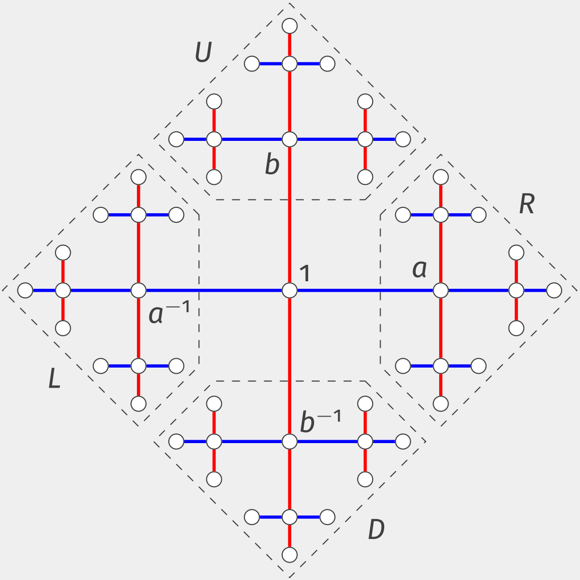

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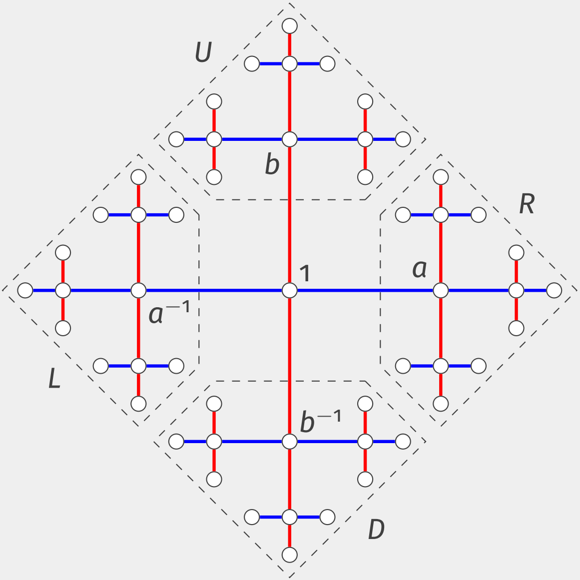

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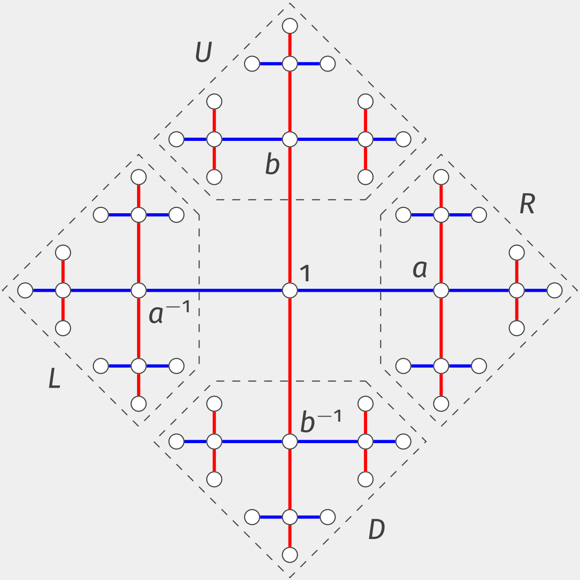

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Definition (John von Neumann, 1929)

A finitely presented discrete group G is amenable if it admits a probability measure μ on 2^G satisfying

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- Non-abelian free groups are not amenable.

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Conjecture (John von Neumann, 1929)

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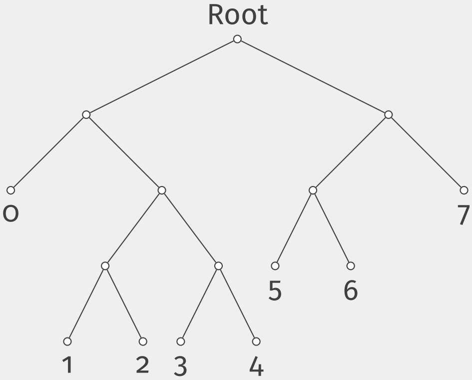
Richard Thompson's Group F

First construction: rooted binary trees



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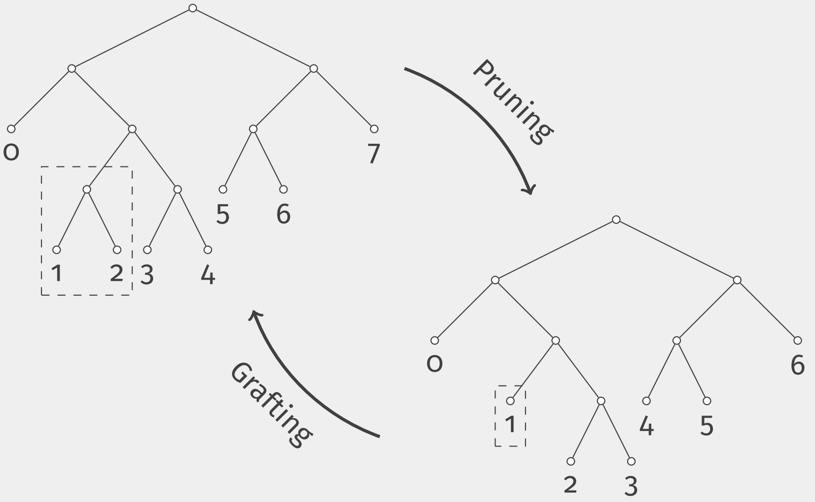
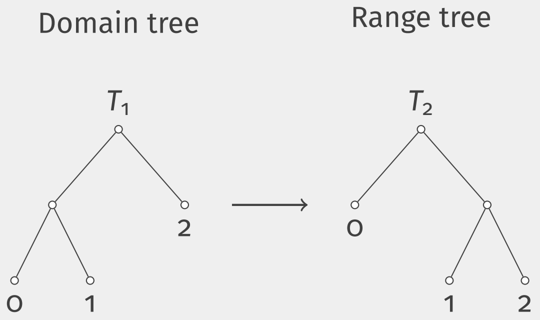
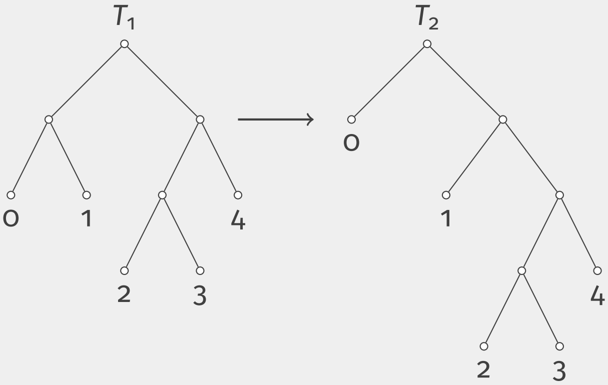




(T_1, T_2)



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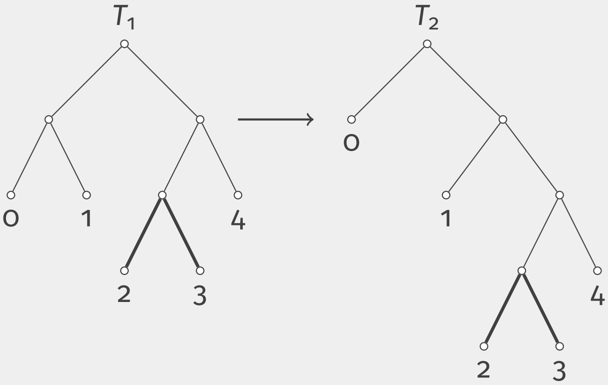

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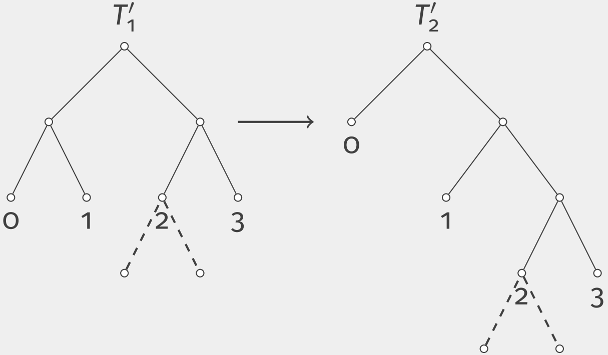
First construction: reductions



(T_1', T_2')

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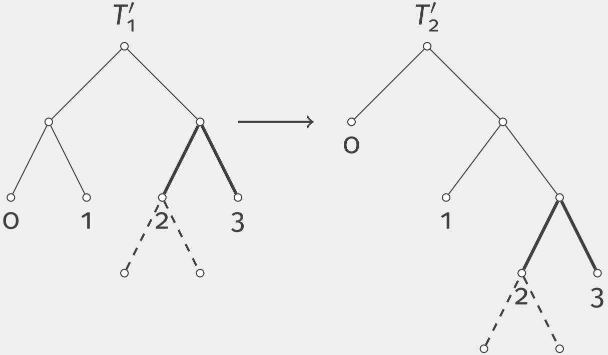

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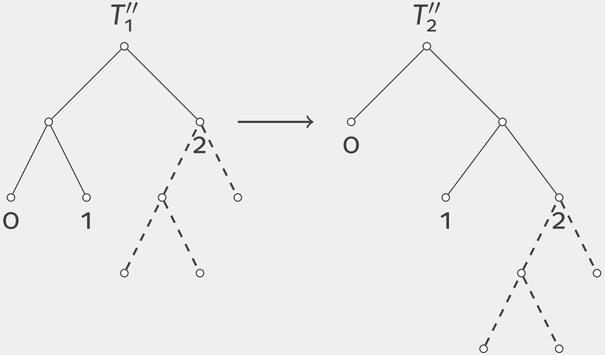
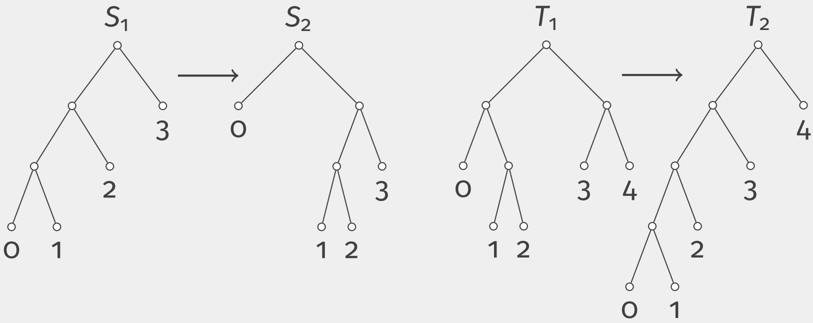


(T_1'', T_2'')

Lemma

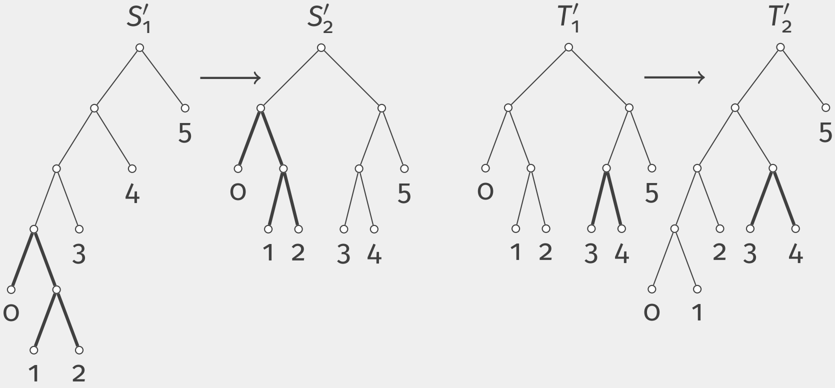
- *Every tree diagram has a unique reduced tree diagrams.*
- *Having same reduced tree diagrams is an equivalence relation on the set of all tree diagrams.*

First construction: combination



$$[(T_1, T_2)] \circ [(S_1, S_2)]$$

First construction: combination (grafting)



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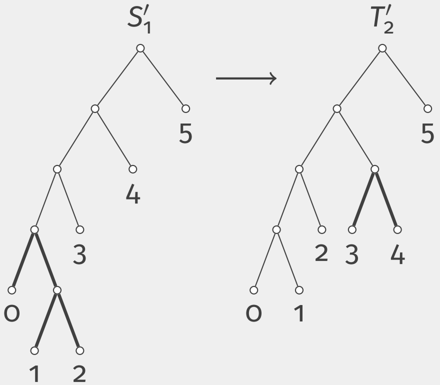

$$[(T_1, T_2)] \circ [(S_1, S_2)] := [(S'_1, T'_2)]$$

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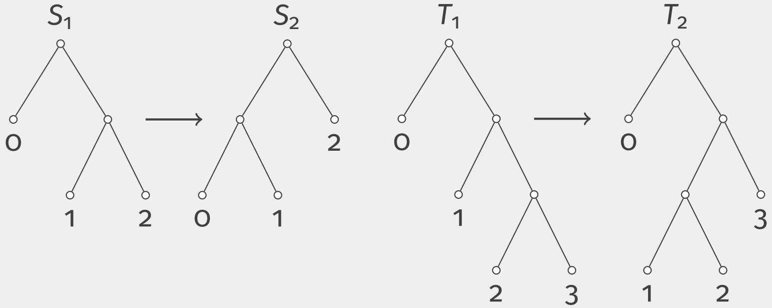
$$[(T_1, T_2)]^{-1} = [(T_2, T_1)]$$



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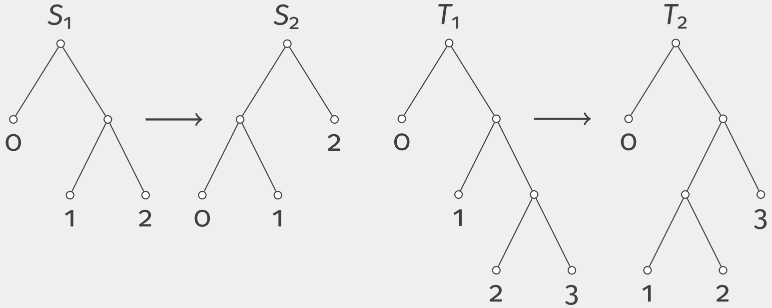

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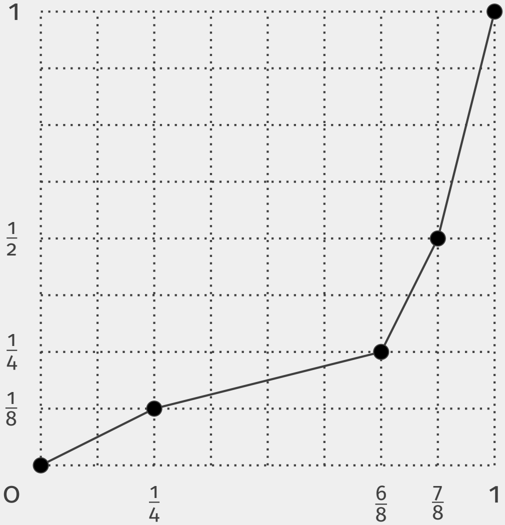
$$F = \langle a, b \mid [ab^{-1}, aba^{-1}] = [ab^{-1}, a^2ba^{-2}] = 1 \rangle$$

Second construction



Thompson's group F is the set of all bijective functions $f : [0,1] \to [0,1]$ that are linear on sub-intervals

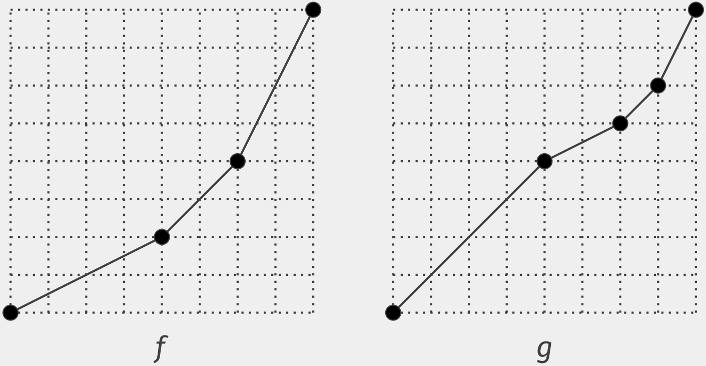
$$\left[\frac{a_0}{2^n}, \frac{a_1}{2^n}\right], \left[\frac{a_1}{2^n}, \frac{a_2}{2^n}\right], \ldots, \left[\frac{a_{m-1}}{2^n}, \frac{a_m}{2^n}\right] \quad (0 = a_0 < \cdots < a_m = 2^n)$$

with slopes

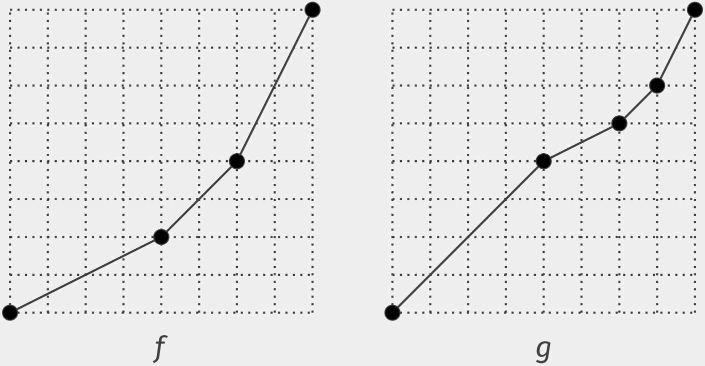
$$\frac{f(a_i) - f(a_{i-1})}{a_i - a_{i-1}} = 2^{k_i} \quad (k_i \in \mathbb{Z})$$

for all $i = 1, \ldots, m$.

Second construction: presentation



Second construction: presentation



$$F = \langle f,\ g \mid [fg^{-1}, gfg^{-1}] = [fg^{-1}, g^2fg^{-2}] = 1 \rangle$$

Definition

The class $\mathcal{EG}$ of elementary amenable groups is the smallest class of groups that

- contains finite groups and abelian groups, and it is
- closed under subgroups, quotients, extensions, and direct limits.

Proposition

$\mathcal{EG} \subseteq \mathcal{AG}$.

Amenability

Thanks!