Positive matching decomposition of graphs

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Preliminaries

Orthogonal representations

- $[n] = \{1, \ldots, n\}$

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- A set of vectors X in $\mathbb{R}^d$ is in general position if

any d-subset of X is linearly independent.

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- $L^{\mathbb{K}}_{\bar{\Gamma}}(d) = \left(x_{i,1}x_{j,1} + \cdots + x_{i,d}x_{j,d} : \quad \{i, j\} \in E(\bar{\Gamma})\right)$
- $V(L^{\mathbb{K}}_{\bar{\Gamma}}(d)) = \{\text{orthogonal representations of } \Gamma\}$

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Positive matching decompositions (PMD)

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$$M \text{ is positive if } M = \{e \in E \colon \omega(e) > 0\}$$

for some weight function $\omega \colon V \to \mathbb{R}$

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Remark

$\mathbb{R}$ can be replaced with $\mathbb{Z}$

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$(E_1, \ldots, E_p)$ is a pmd of Γ if

E_i is a positive matching in $\Gamma \setminus E_1 \cup \cdots \cup E_{i-1}$,

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W is alternathing w.r.t. M if

the edges of W alternate between M and M^c

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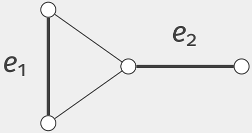

PMD of Graphs

Simple cases

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- $\mathrm{pmd}(\Gamma) = 1$ iff $\Gamma = sK_2 \cup tK_1$
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Question

Any characterization of graphs Γ with $\mathrm{pmd}(\Gamma) = 3$?

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Every graph Γ with maximum valency at least three has a ***subdivision*** *Γ' satisfying $\mathrm{pmd}(\Gamma') = \Delta(\Gamma')$.*

Antler graphs

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$$\mathrm{pmd}(\Gamma') = \max\{\mathrm{pmd}(\Gamma), \Delta(\Gamma')\}.$$

Definition (Cactus graph)

A connected graph with any two cycles having at most one vertex in common.

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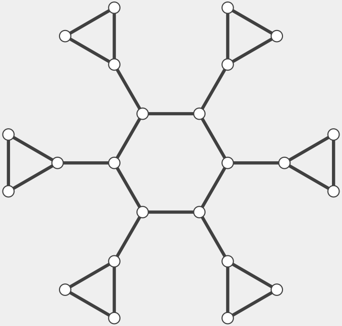
$\mathrm{pmd}(\Gamma) = \Delta(\Gamma)$ *if* Γ *is triangle-free and non-cycle*

Problem

Any characterization of cacti with given pmd?



A cactus with $\mathrm{pmd} = 4$

Complete multipartite graphs

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Conjecture

$\mathrm{pmd}(\Gamma) \leq 2\Delta(\Gamma) - 1$ for any graph Γ.

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$$\mathrm{pmd}(\Gamma) \leq \mathrm{pmd}(\Gamma - M) + \Delta(\Gamma) - 1.$$

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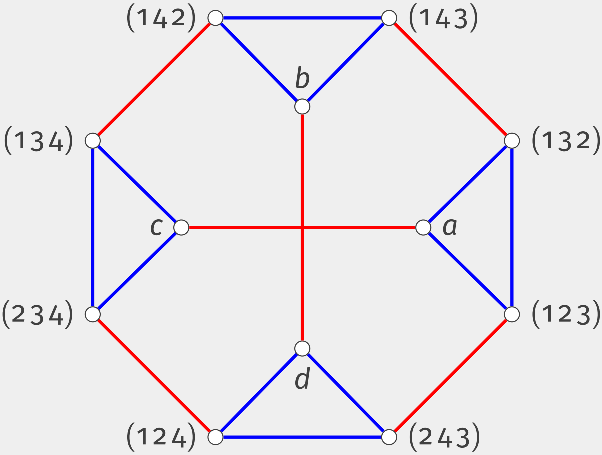
$$\mathrm{Cay}(G, C) = \left(G, \left\{\text{edges } \{g, gc\}\right\}\right)$$



$\alpha = (1\,2\,3)$ and $\beta = (1\,2)(3\,4)$

$a = ()$, $b = (1\,4)(2\,3)$, $c = (1\,2)(3\,4)$, and $d = (1\,3)(2\,4)$

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(3) *if $I(C) = \mathit{Involutions}(C)$ and $H \in \{H_c\}_{c \in C}$ has min. order, then*

$$\mathrm{pmd}(\Gamma) \leq \left(|C| - \frac{1}{2}|I(C)|\right)[G : H].$$

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Conjecture

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Thanks!