

# A toolbox of smooth effect

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Material available at:

[https://github.com/mfasiolo/workshop\\_BOZEN19](https://github.com/mfasiolo/workshop_BOZEN19)

# Types of smooths

Recall the GAM model structure:

$$y|\mathbf{x} \sim \text{Distr}\{y|\mu(\mathbf{x}), \theta\}$$

where  $\mu(\mathbf{x}) = \mathbb{E}(y|\mathbf{x}) = g^{-1}\left\{\sum_{j=1}^m f_j(\mathbf{x})\right\}$ .

The  $f_j$ 's can be

- parametric e.g.  $f_j(\mathbf{x}) = \beta_1 x_j + \beta_2 x_j^2$
- random effects
- spline-based smooths such as

$$f_j(x_j) = \sum_{i=1}^r \beta_{ji} b_{ji}(x_j)$$

where  $\beta_{ji}$  are coefficients and  $b_{ji}(x_j)$  are known spline basis functions.

NB: we call  $\sum_{j=1}^m f_j(\mathbf{x})$  **linear predictor** because it is linear in  $\beta$ .

# Types of smooths

`mgcv` offers a wide variety of smooths (see `?smooth.terms`).

Univariate types:

- `s(x) = s(x, bs = "tp")` thin-plate-splines
- `s(x, bs = "cr")` cubic regression spline
- `s(x, bs = "ad")` adaptive smooth

Multivariate type:

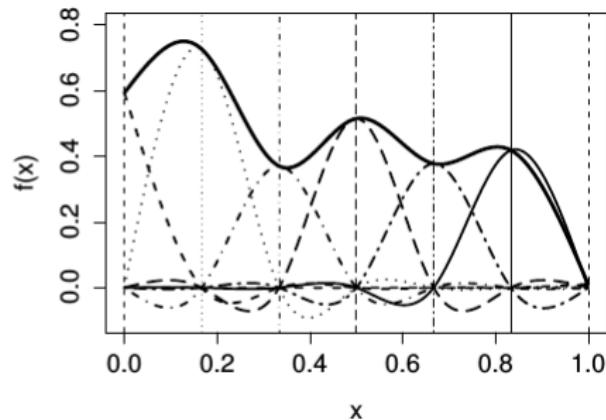
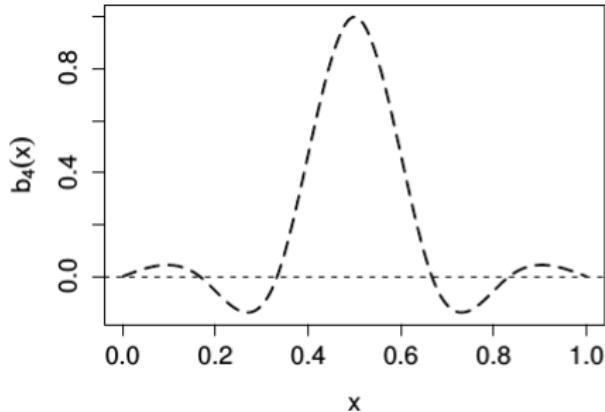
- `s(x1, x2) = s(x1, x2, bs = "tp")` thin-plate-splines (isotropic)
- `te(x1, x2)` tensor-product-smooth (anisotropic)
- `s(x, y, bs = "sos")` smooth on sphere

They can depends on factors:

- `s(x, by = Subject)`
- `s(x, Subject, bs = "fs")`

# Types of smooths

`s(x, bs = "cr", k = 20)`

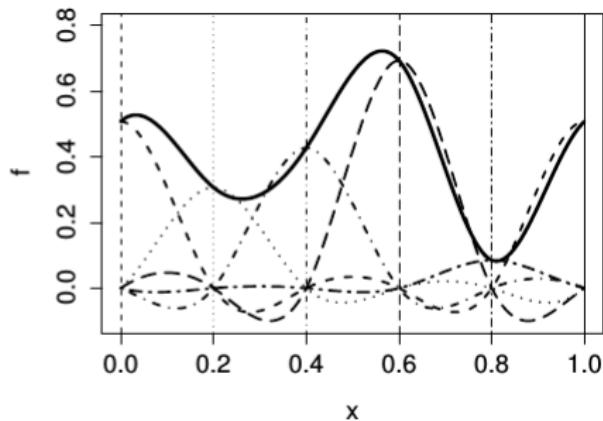
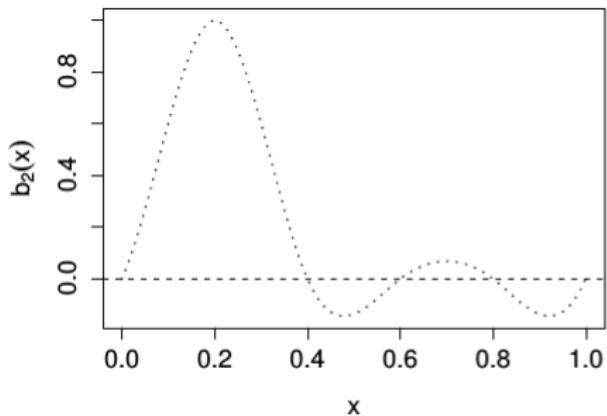


Cubic regression splines are related to the optimal solution to

$$\sum_{i=1}^n \{y_i - f(x_i)\}^2 + \gamma \int f''(x)^2 dx.$$

# Types of smooths

`s(x, bs = "cc")`

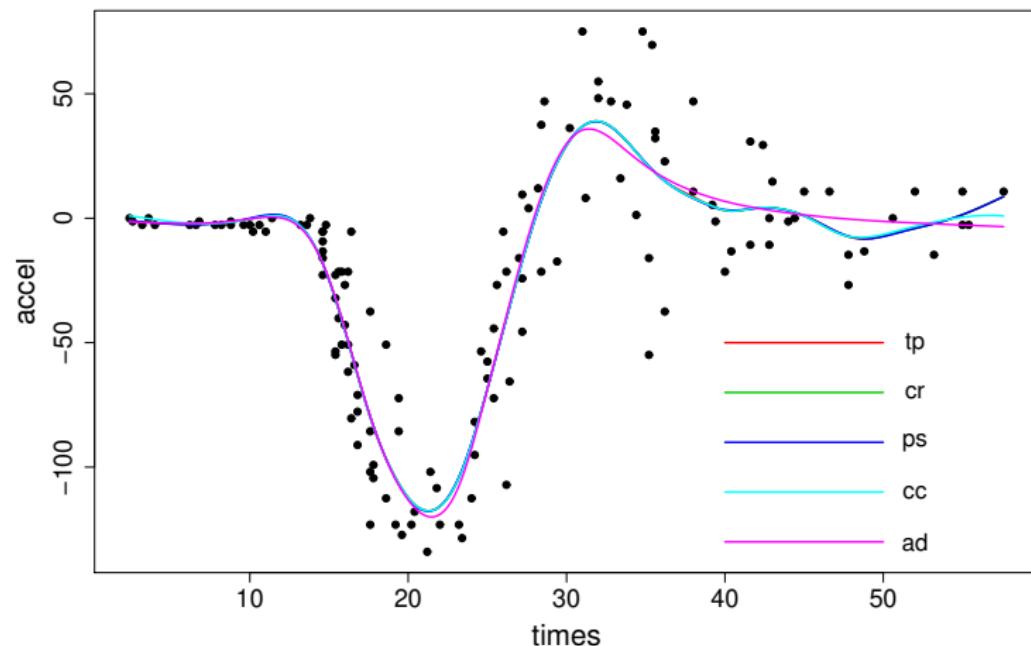


Cyclic cubic regression splines make so that

- $f(x_{min}) = f(x_{max})$
- $f'(x_{min}) = f'(x_{max})$

# Types of smooths

`s(x, bs = "ad")`



The wiggliness or smoothness of  $f(x)$  depends on  $x$ .

## Types of smooths

$s(x_1, x_2)$ ,  $s(x_1, x_2, x_3)$ , ...

Based on thin plate regression splines basis.

Related to optimal solution to:

$$\sum_i \{y_i - f(x_i, z_i)\}^2 + \gamma \int f_{xx}^2 + 2f_{xz}^2 + f_{zz}^2 dx dz$$

A single smoothing parameter  $\gamma$ .

Isotropic: same smoothness along  $x_1, x_2, \dots$

# Types of smooths

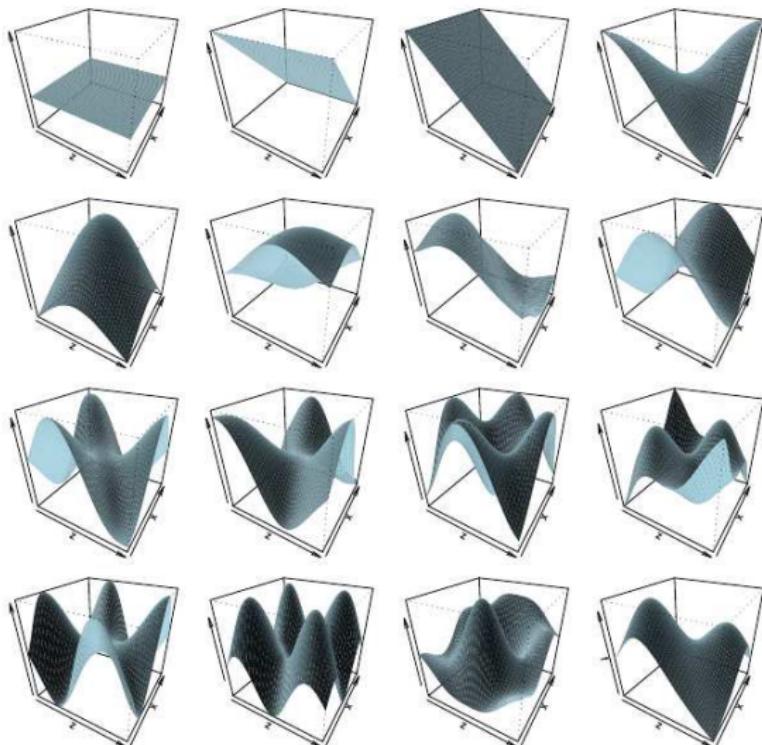


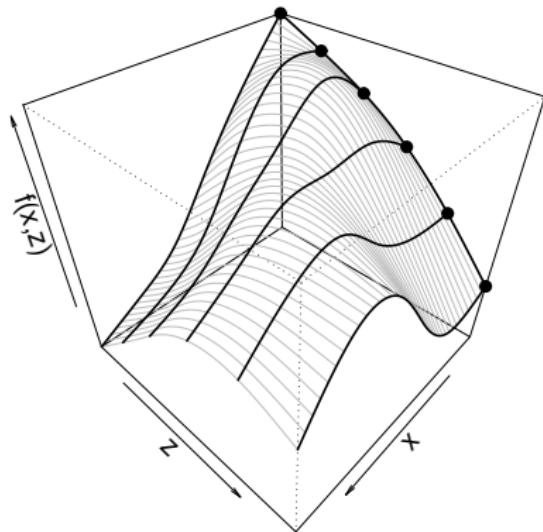
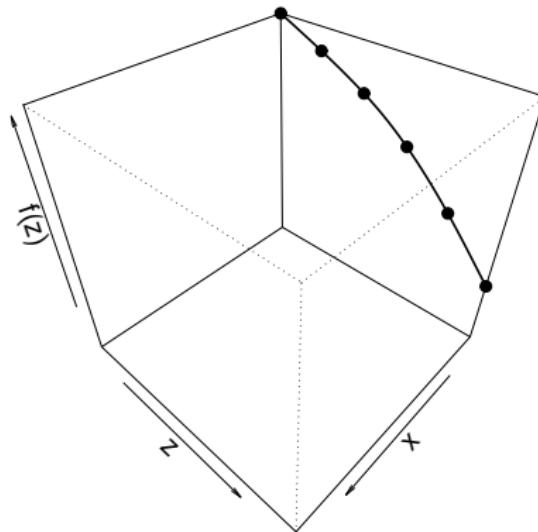
Figure: Rank 17 2D TPRS basis. Courtesy of Simon Wood.

# Types of smooths

Isotropic effect of  $x_1, x_2$  are in same unit (e.g. Km).

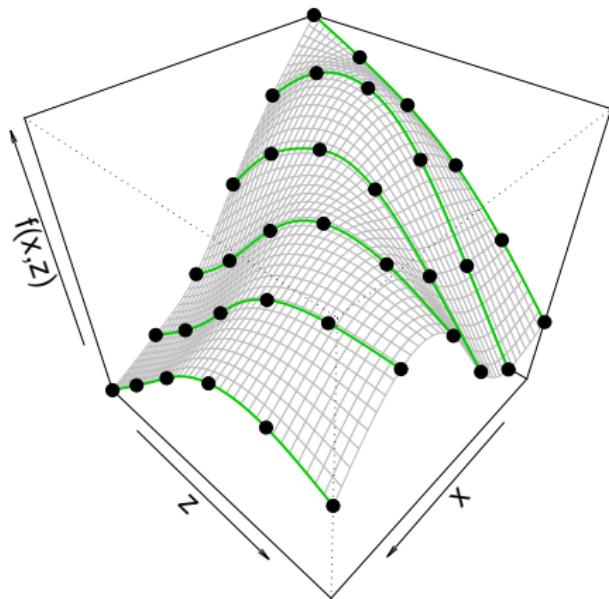
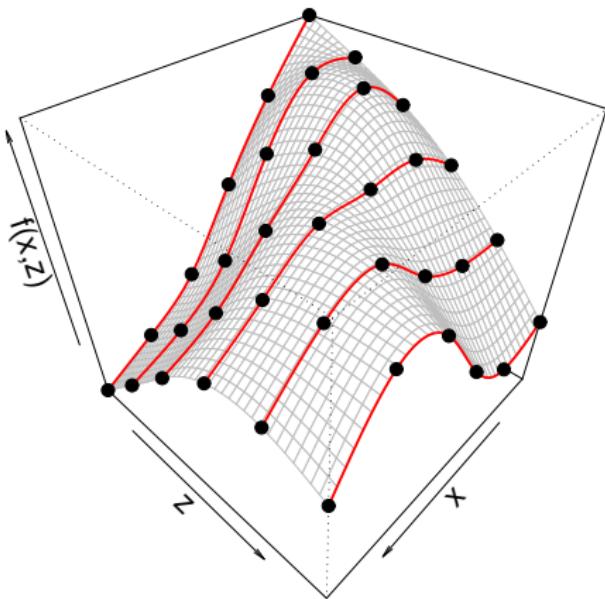
If different units better use tensor product smooths  $\text{te}(x_1, x_2)$ .

Construction: make a spline  $f_z(z)$  a function of  $x$  by letting its coefficients vary smoothly with  $x$



# Types of smooths

- x-penalty: average wiggliness of red curves
- z-penalty: average wiggliness of green curves



## Types of smooths

Can use (almost) any kind of marginal:

- `te(x1, x2, x3)` product of 3 cubic regression splines bases
- `te(x1, x2, bs = c("cc", "cr"), k = c(10, 6))`
- `te(L0, LA, t, d=c(2,1), k=c(20,10), bs=c("tp","cc"))`

Basis of `te` contains functions of the form  $f(x_1)$  and  $f(x_2)$ .

To fit  $f(x_1) + f(x_2) + f(x_1, x_2)$  separately use:

```
y ~ ti(x1) + ti(x2) + ti(x1, x2)
```

# Types of smooths

## By-factor smooths

Approach (1) is  $s(x, \text{by} = \text{subject})$ , which means

- $\mu(x) = f_1(x) + \dots$  if subject = 1
- $\mu(x) = f_2(x) + \dots$  if subject = 2
- ...

Approach (2) is  $s(x, \text{subject}, \text{bs} = \text{"fs"})$ , which means

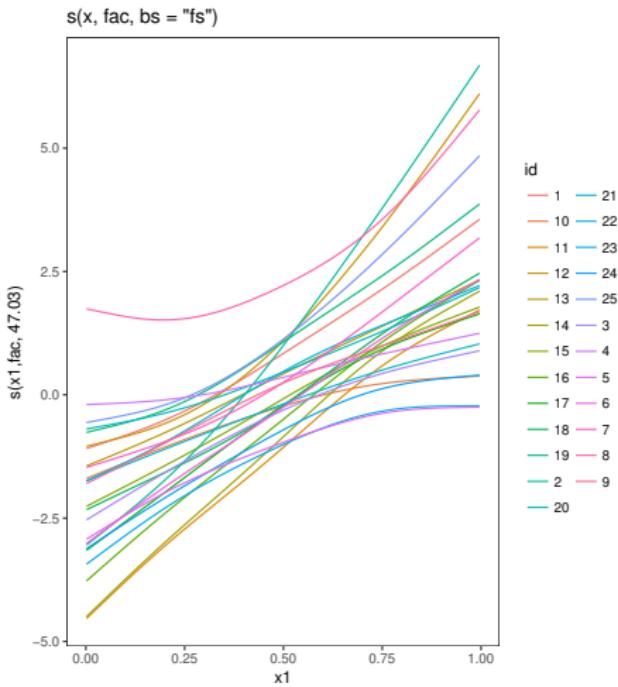
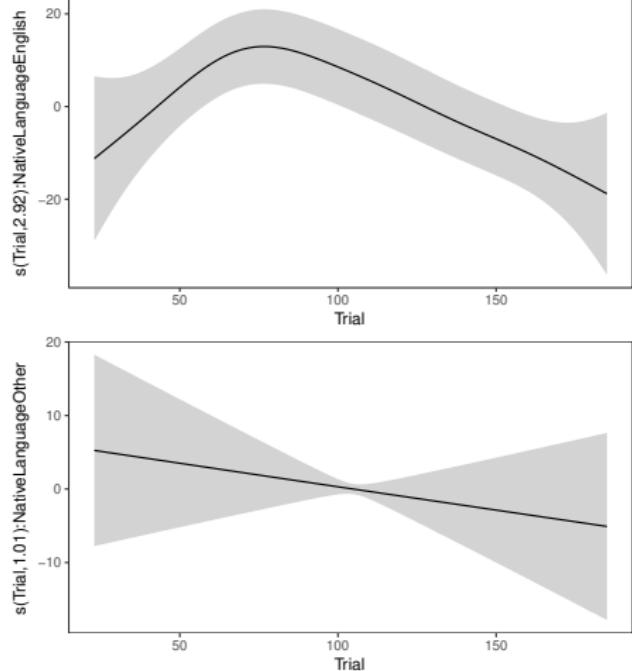
- $\mu(x) = b_1 + f_1(x) + \dots$  if subject = 1
- $\mu(x) = b_2 + f_2(x) + \dots$  if subject = 2
- ...

where  $b_1, b_2, \dots \sim N(0, \gamma_b \mathbf{I})$  are random effects.

In (1) each  $f_j$  has its own smoothing parameter.

In (2) all  $f_j$ 's have the same smoothing parameter.

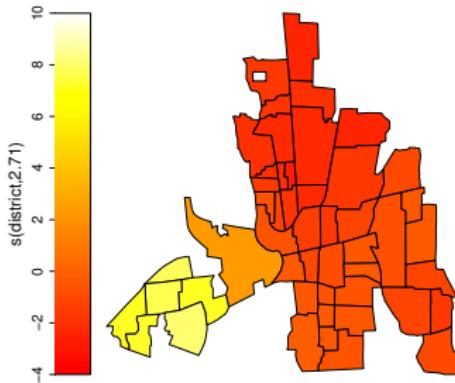
# Types of smooths



# Types of smooths

## Markov random field effects

Sometimes data come allocated to irregular partitions of space.

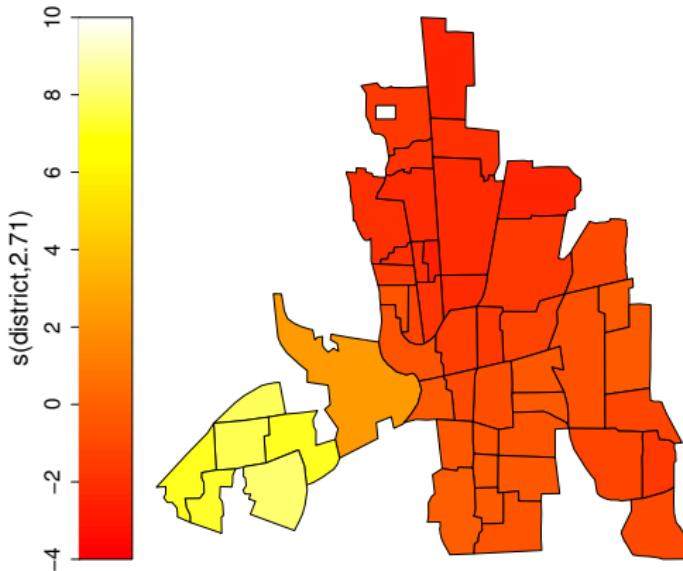


- Markov random fields are a popular way of smoothing such data.
- The smooth has a coefficient,  $\beta_i$ , for each region.
- $N_i$  is the set of indices of the neighbours of region  $i$ , then penalty is

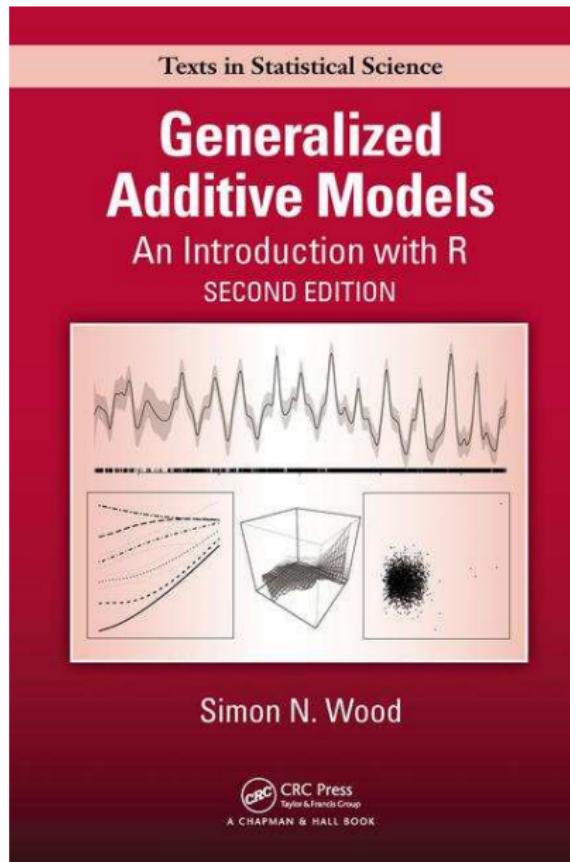
$$\sum_i \left( \sum_{j \in N_i} (\beta_i - \beta_j) \right)^2.$$

# Types of smooths

```
data(columb.polys) ## district shapes list  
xt <- list(polys=columb.polys)  
gam(crime ~ s(district,bs="mrf",xt=xt),data=columb)
```



# Further reading



# References |

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