A toolbox of smooth effects

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Material available at:

https://github.com/mfasiolo/workshop_BOZEN19

Types of smooths

Recall the GAM model structure:

$$y|\mathbf{x} \sim \text{Distr}\{y|\mu(\mathbf{x}), \boldsymbol{\theta}\}$$

where $\mu(\mathbf{x}) = \mathbb{E}(y|\mathbf{x}) = g^{-1}\big\{ \sum_{j=1}^m f_j(\mathbf{x})\big\}$.

The f_j's can be

- parametric e.g. $f_j(\mathbf{x}) = \beta_1 x_j + \beta_2 x_j^2$
- random effects
- spline-based smooths such as

$$f_j(x_j) = \sum_{i=1}^{r} \beta_{ji} b_{ji}(x_j)$$

  where β_{ji} are coefficients and $b_{ji}(x_j)$ are known spline basis functions.

NB: we call $\sum_{j=1}^m f_j(\mathbf{x})$ **linear predictor** because it is linear in $\boldsymbol{\beta}$.

Types of smooths

`mgcv` offers a wide variety of smooths (see `?smooth.terms`).

Univariate types:

- `s(x) = s(x, bs = "tp")` thin-plate-splines
- `s(x, bs = "cr")` cubic regression spline
- `s(x, bs = "ad")` adaptive smooth

Multivariate type:

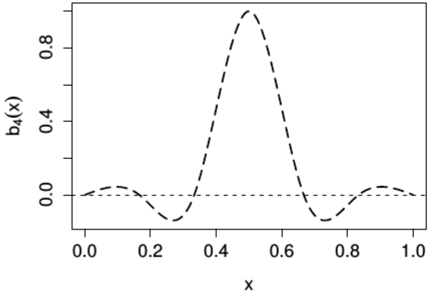
- `s(x1, x2) = s(x1, x2, bs = "tp")` thin-plate-splines (isotropic)
- `te(x1, x2)` tensor-product-smooth (anisotropic)
- `s(x, y, bs = "sos")` smooth on sphere

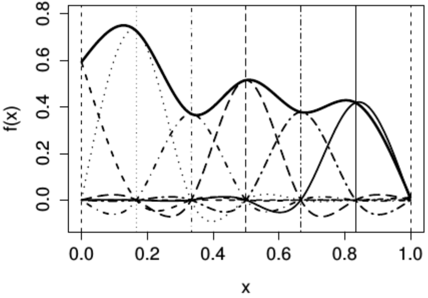
They can depends on factors:

- `s(x, by = Subject)`
- `s(x, Subject, bs = "fs")`

Types of smooths

`s(x, bs = "cr", k = 20)`



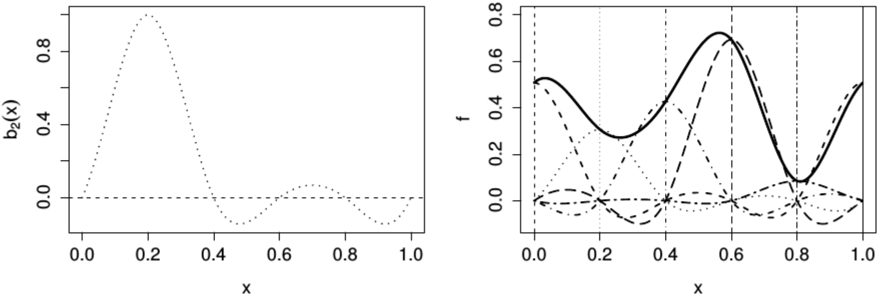


Cubic regression splines are related to the optimal solution to

$$\sum_{i=1}^{n}\{y_i - f(x_i)\}^2 + \gamma \int f''(x)^2 \, dx.$$

Types of smooths

`s(x, bs = "cc")`

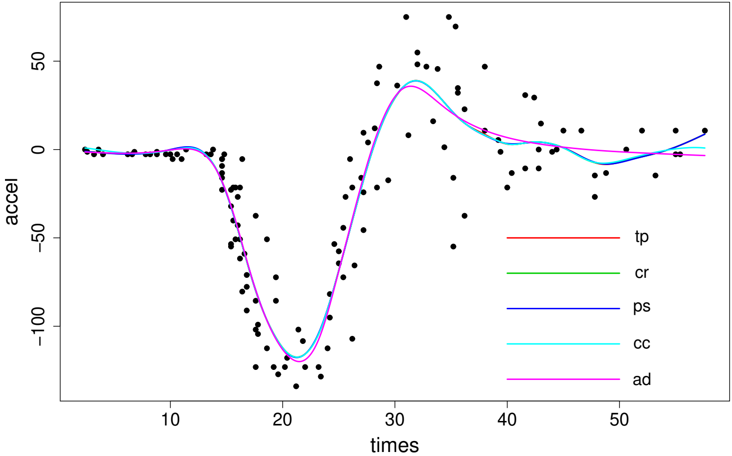


Cyclic cubic regression splines make so that

- $f(x_{min}) = f(x_{max})$
- $f'(x_{min}) = f'(x_{max})$

Types of smooths

```
s(x, bs = "ad")
```



The wiggliness or smoothness of $f(x)$ depends on x.

Types of smooths

`s(x1, x2)`, `s(x1, x2, x3)`, ...

Based on thin plate regression splines basis.

Related to optimal solution to:

$$\sum_i \{y_i - f(x_i, z_i)\}^2 + \gamma \int f_{xx}^2 + 2f_{xz}^2 + f_{zz}^2 dxdz$$

A single smoothing parameter γ.

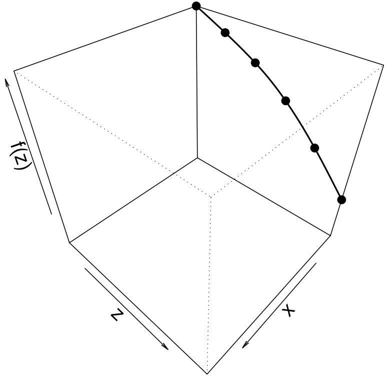
Isotropic: same smoothness along x_1, x_2, ...

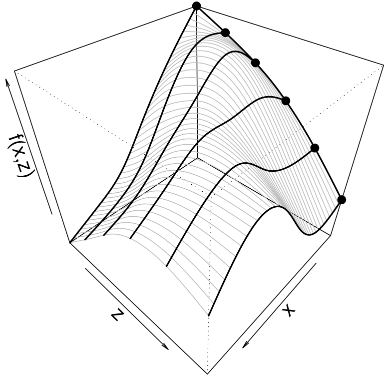
Types of smooths

Isotropic effect of x_1, x_2 are in same unit (e.g. Km).

If different units better use tensor product smooths `te(x1, x2)`.

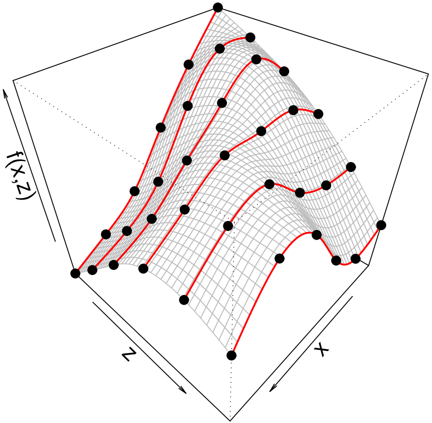
Construction: make a spline $f_z(z)$ a function of x by letting its coefficients vary smoothly with x

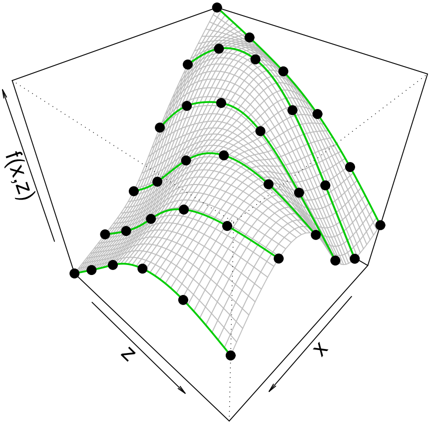




Types of smooths

- x-penalty: average wiggliness of red curves
- z-penalty: average wiggliness of green curves





Types of smooths

Can use (almost) any kind of marginal:

- `te(x1, x2, x3)` product of 3 cubic regression splines bases
- `te(x1, x2, bs = c("cc", "cr"), k = c(10, 6))`
- `te(LO, LA, t, d=c(2,1), k=c(20,10), bs=c("tp","cc"))`

Basis of `te` contains functions of the form $f(x_1)$ and $f(x_2)$.

To fit $f(x_1) + f(x_2) + f(x_1, x_2)$ separately use:

```
y ~ ti(x1) + ti(x2) + ti(x1, x2)
```

Types of smooths

By-factor smooths

Approach (1) is `s(x, by = subject)`, which means

- $\mu(x) = f_1(x) + \ldots \qquad \text{if subject} = 1$
- $\mu(x) = f_2(x) + \ldots \qquad \text{if subject} = 2$
- ...

Approach (2) is `s(x, subject, bs = "fs")`, which means

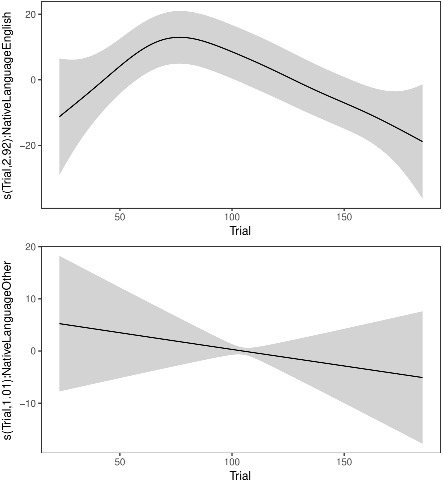
- $\mu(x) = b_1 + f_1(x) + \ldots \qquad \text{if subject} = 1$
- $\mu(x) = b_2 + f_2(x) + \ldots \qquad \text{if subject} = 2$
- ...

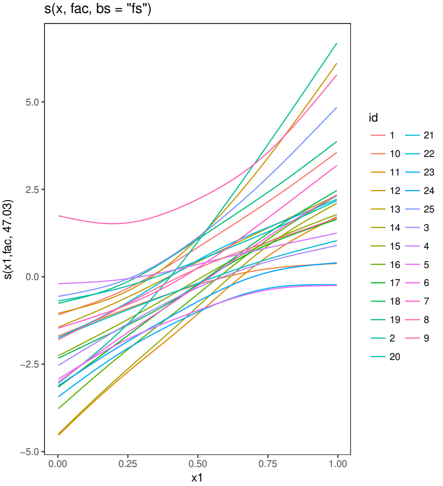
where $b_1, b_2, \cdots \sim N(0, \gamma_{\mathbf{b}} \mathbf{I})$ are random effects.

In (1) each f_j has its own smoothing parameter.

In (2) all f_j's have the same smoothing parameter.

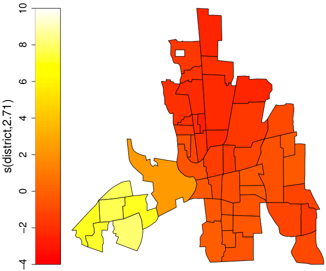
Types of smooths





Markov random field effects

Sometimes data come allocated to irregular partitions of space.

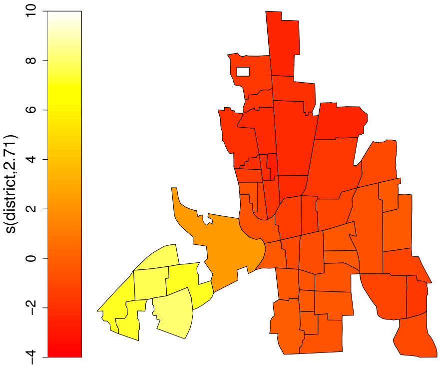


- Markov random fields area a popular way of smoothing such data.
- The smooth has a coefficient, β_i, for each region.
- N_i is the set of indices of the neighbours of region i, then penalty is

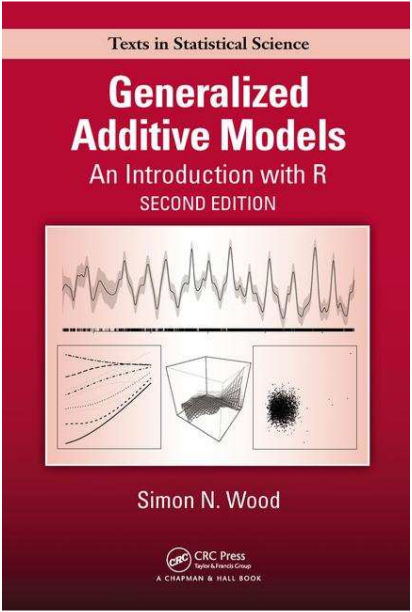
$$\sum_i \Big(\sum_{j \in N_i} (\beta_i - \beta_j) \Big)^2.$$

Types of smooths

```
library(mgcv)
data(columb); data(columb.polys)
xt <- list(polys=columb.polys)
gam(crime ~ s(district, bs="mrf", xt=xt), data=columb)
```



Further reading



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