>>> **Feature Extraction**
>>> **GRSS Summer School**

Name: Mathieu Fauvel (UMR Dynafor)
Date: *[2017-04-26 Wed 10:30]–[2017-04-26 Wed 12:00]*

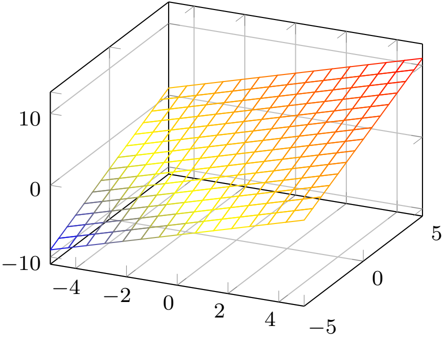
- **Curse of dimensionality**: it is not possible to get enough data to cover all the observation space.

High dimensional saces are mostly empty !

- ⋆ Curse of dimensionality: it is not possible to get enough data to cover all the observation space.

High dimensional saces are mostly empty !

- ⋆ Multivariate data live in a lower dimensional space



- Feature extraction is important in remote sensing because:
  - It reduces the size of the data,
  - It limits the spatial and spectral redundancy,
  - It permits visualization of the data,
  - It mitigates the *curse of dimensionality*.
- Extraction techniques:
  - Spectral
    - Physically based method,
    - Statistical methods.
  - Spatial:
    - Linear filters,
    - Non linear techniques (Mathematical Morphology)

- Spectral indices are a linear/non-linear combination of two (or more) spectral bands.
- They provides information as a *single number* about:
  - Plant structure,
  - Biochemistry,
  - Humidity,
  - Stress.
- Four main types [TLH11]:

Name	Formulae
Difference vegetation index	$R_{\lambda_1} - R_{\lambda_2}$
Ratio vegetation index	$\dfrac{R_{\lambda_1}}{R_{\lambda_2}}$
Normalized difference vegetation index	$\dfrac{R_{\lambda_1} - R_{\lambda_2}}{R_{\lambda_1} + R_{\lambda_2}}$
Soil-adjusted vegetation index	$(1+L) \times \dfrac{R_{\lambda_1} - R_{\lambda_2}}{R_{\lambda_1} - R_{\lambda_2} + L}$

- *The three last indexes are invariant to a multiplicative factor*

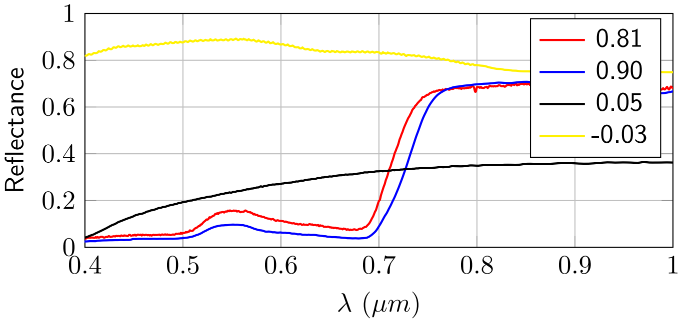
Index database : http://www.indexdatabase.de/

Name	Formulae (λ nm)
Normalized Difference Vnegetation index	$\frac{R_{\lambda_{800}} - R_{\lambda_{670}}}{R_{\lambda_{800}} + R_{\lambda_{670}}}$
Modified Soil-Adjusted Vegetation Index	$\frac{1}{2}\left[2R_{\lambda_{800}} + 1 - \sqrt{(2R_{\lambda_{800}} + 1)^2 - 8(R_{\lambda_{800}} - R_{\lambda_{670}})}\right]$
Modified Chlorophyll Absorption Ratio Index	$[(R_{\lambda_{700}} - R_{\lambda_{670}}) - 0.2(R_{\lambda_{700}} - R_{\lambda_{550}})] \times \frac{R_{\lambda_{700}}}{R_{\lambda_{670}}}$
Normalized Difference Water Index	$\frac{R_{\lambda_{858}} - R_{\lambda_{1240}}}{R_{\lambda_{858}} + R_{\lambda_{1240}}}$
Datt Reflectance Index	$\frac{R_{\lambda_{816}} - R_{\lambda_{2218}}}{R_{\lambda_{816}} + R_{\lambda_{2218}}}$
Normalized Difference Redness Index	$\frac{R_{\lambda_{540}} - R_{\lambda_{700}}}{R_{\lambda_{540}} + R_{\lambda_{700}}}$
Soil Brightness Index	$0.406R_{\lambda 550} + 0.600R_{\lambda 650} + 0.645R_{\lambda 750} + 0.243R_{\lambda 950}$

>>> Normalized difference vegetation index

$$\text{NDVI} = \frac{R_{\lambda_{800}} - R_{\lambda_{670}}}{R_{\lambda_{800}} + R_{\lambda_{670}}}$$

- ⋆ $-1 \leq$ NVDI ≤ 1
- ⋆ NDVI < 0: surfaces other thatn plant cover
- ⋆ NDVI ≈ 0: bare soil
- ⋆ NDVI ≥ 0.1: vegetation cover (higher values correspond to more dense covers)

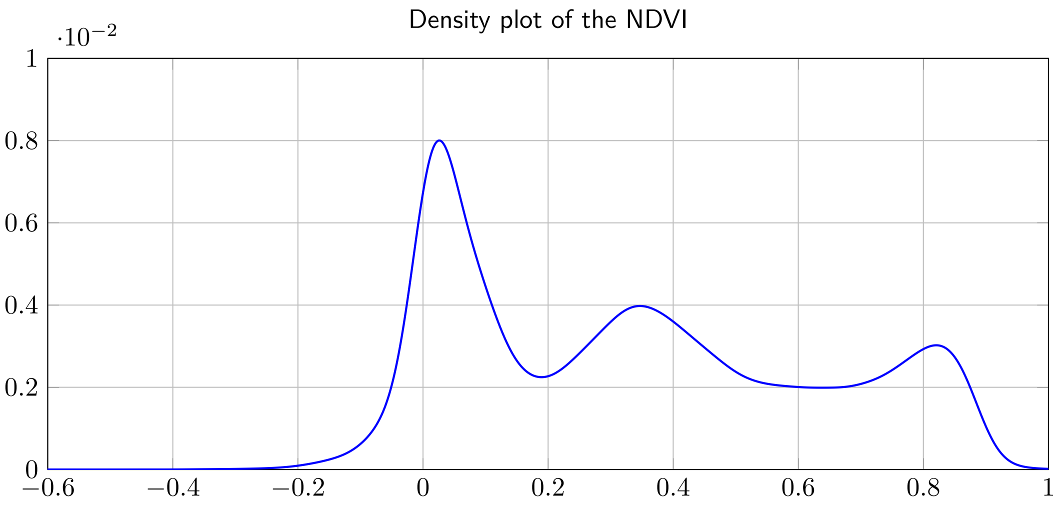


- Peri-urban area
- Rosis-3 sensor
- 103 Spectral bands (400nm-900nm)
- 1.5 meter per pixel spatial resolution
- 610 $\times$ 340 pixels

- OTB is a C++ library for remote sensing images processing.
- It is free, open-source and available for most OS (window, apple, linux)
- OTB-Applications are set of tools appropriated for big/large images
- They are avalaible from QGIS, Python and Bash
- To compute the NDVI

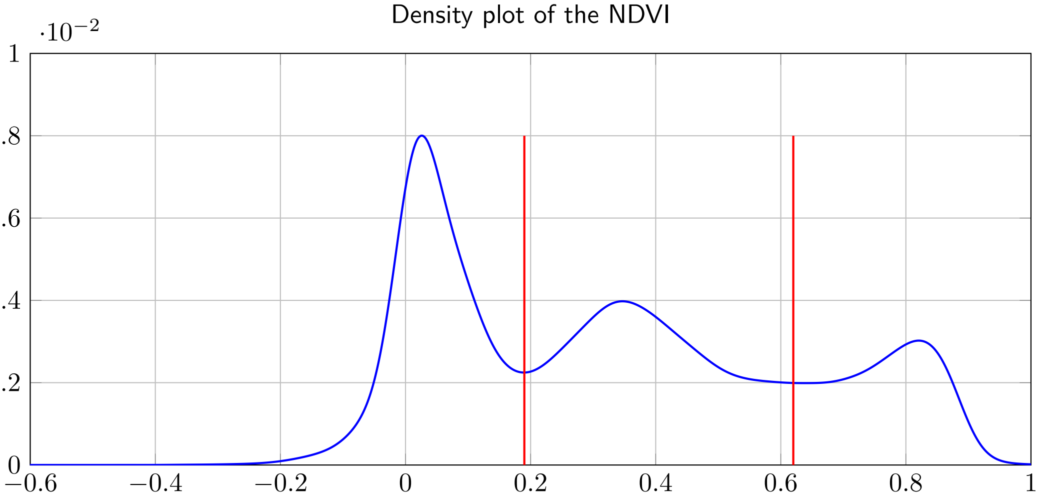
```
# Computation of the NDVI
otbcli_BandMath -il ../Data/university.tif -out ../Data/university_ndvi.tif \
-exp "(im1b83-im1b56)/(im1b83+im1b56)"

# Computation of the SBI
otbcli_BandMath -il ../Data/university.tif -out ../Data/university_sbi.tif \
-exp "0.406*im1b31 + 0.6*im1b52 + 0.645*im1b73"
```

\>>> Where is the vegetation 1/2 ?




```
# Segmentation of the NDVI in three classes
otbcli_BandMath -il ../Data/university_ndvi.tif -out ../Data/university_ndvi_segmented.tif \
-exp "(im1b1<0.19?1:(im1b1<0.62?2:3))"
```

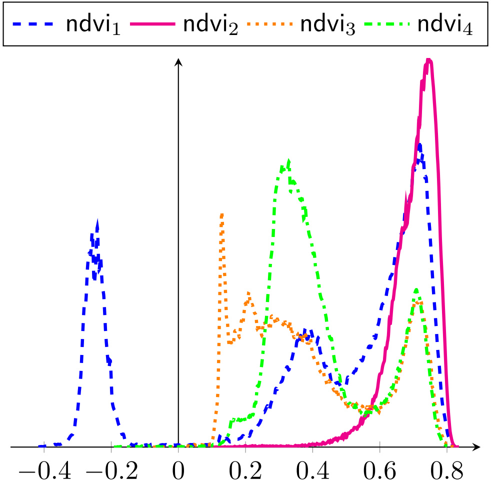
>>> Where is the vegetation 1/2 ?




```
# Segmentation of the NDVI in three classes
otbcli_BandMath -il ../Data/university_ndvi.tif -out ../Data/university_ndvi_segmented.tif \
-exp "(im1b1<0.19?1:(im1b1<0.62?2:3))"
```

>>> Where is the vegetation 2/2 ?

Image



NDVI Histogram

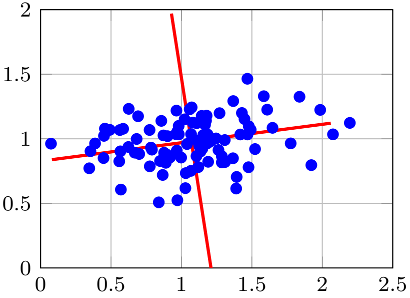
From the histogram, which one does correspond to the NDVI of the image ?

>>> Outline

- Linear transformation used to reduce the dimensionality of the data [Jol02].

$$z_i = \langle \mathbf{v}_i, \mathbf{x} \rangle$$

- Find features $\mathbf{z}$ that account for most of the variability of the data:
  - $z_1,\ z_2,\ z_3, \ldots$ are mutually uncorrelated,
  - $\text{var}(z_i)$ is as large as possible,
  - $\text{var}(z_1) > \text{var}(z_2) > \text{var}(z_3) > \ldots$



>>> Maximization of the variance 1/2

- Search $\mathbf{v}_1$ such as $\max \mathsf{var}(z_1)$:

$$\begin{aligned}\mathsf{var}(z_1) &= \mathsf{var}(\langle \mathbf{v}_1, \mathbf{x} \rangle) \\ &= \mathbf{v}_1^\top \boldsymbol{\Sigma} \mathbf{v}_1\end{aligned}$$

- Search $\mathbf{v}_1$ such as $\max \mathsf{var}(z_1)$:

$$\begin{aligned}\mathsf{var}(z_1) &= \mathsf{var}(\langle \mathbf{v}_1, \mathbf{x} \rangle) \\ &= \mathbf{v}_1^\top \mathbf{\Sigma} \mathbf{v}_1\end{aligned}$$

- Indetermined: if $\hat{\mathbf{v}}_1$ maximizes the variance, $\alpha \hat{\mathbf{v}}_1$ too! Add a constraint: $\langle \mathbf{v}_1, \mathbf{v}_1 \rangle = 1$

>>> Maximization of the variance 1/2

- Search $\mathbf{v}_1$ such as $\max \text{var}(z_1)$:

$$\begin{aligned} \text{var}(z_1) &= \text{var}(\langle \mathbf{v}_1, \mathbf{x} \rangle) \\ &= \mathbf{v}_1^\top \mathbf{\Sigma} \mathbf{v}_1 \end{aligned}$$

- Indetermined: if $\hat{\mathbf{v}}_1$ maximizes the variance, $\alpha \hat{\mathbf{v}}_1$ too! Add a constraint: $\langle \mathbf{v}_1, \mathbf{v}_1 \rangle = 1$
- Lagrangian:

$$\mathcal{L}(\mathbf{v}_1, \lambda_1) = \mathbf{v}_1^\top \mathbf{\Sigma} \mathbf{v}_1 + \lambda_1 (1 - \mathbf{v}_1^\top \mathbf{v}_1)$$

- Search $\mathbf{v}_1$ such as $\max \mathsf{var}(z_1)$:

$$\begin{aligned}\mathsf{var}(z_1) &= \mathsf{var}(\langle \mathbf{v}_1, \mathbf{x} \rangle) \\ &= \mathbf{v}_1^\top \mathbf{\Sigma} \mathbf{v}_1\end{aligned}$$

- Indetermined: if $\hat{\mathbf{v}}_1$ maximizes the variance, $\alpha\hat{\mathbf{v}}_1$ too! Add a constraint: $\langle \mathbf{v}_1, \mathbf{v}_1 \rangle = 1$
- Lagrangian:

$$\mathcal{L}(\mathbf{v}_1, \lambda_1) = \mathbf{v}_1^\top \mathbf{\Sigma} \mathbf{v}_1 + \lambda_1 (1 - \mathbf{v}_1^\top \mathbf{v}_1)$$

- Compute the derivative w.r.t $\mathbf{v}_1$:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{v}_1} = 2\mathbf{\Sigma}\mathbf{v}_1 - 2\lambda_1 \mathbf{v}_1$$

- Search $\mathbf{v}_1$ such as $\max \text{var}(z_1)$:

$$\begin{aligned}\text{var}(z_1) &= \text{var}(\langle \mathbf{v}_1, \mathbf{x}\rangle) \\ &= \mathbf{v}_1^\top \boldsymbol{\Sigma} \mathbf{v}_1\end{aligned}$$

- Indetermined: if $\hat{\mathbf{v}}_1$ maximizes the variance, $\alpha\hat{\mathbf{v}}_1$ too! Add a constraint: $\langle \mathbf{v}_1, \mathbf{v}_1\rangle = 1$
- Lagrangian:

$$\mathcal{L}(\mathbf{v}_1, \lambda_1) = \mathbf{v}_1^\top \boldsymbol{\Sigma} \mathbf{v}_1 + \lambda_1(1 - \mathbf{v}_1^\top \mathbf{v}_1)$$

- Compute the derivative w.r.t $\mathbf{v}_1$:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{v}_1} = 2\boldsymbol{\Sigma}\mathbf{v}_1 - 2\lambda_1 \mathbf{v}_1$$

- $\mathbf{v}_1$ is an eigenvector of the covariance matrix of $\mathbf{x}$:

$$\boldsymbol{\Sigma}\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$$

- ⋆ Search $\mathbf{v}_1$ such as $\max \text{var}(z_1)$:

$$\begin{aligned} \text{var}(z_1) &= \text{var}(\langle \mathbf{v}_1, \mathbf{x} \rangle) \\ &= \mathbf{v}_1^\top \mathbf{\Sigma} \mathbf{v}_1 \end{aligned}$$

- ⋆ Indetermined: if $\hat{\mathbf{v}}_1$ maximizes the variance, $\alpha \hat{\mathbf{v}}_1$ too! Add a constraint: $\langle \mathbf{v}_1, \mathbf{v}_1 \rangle = 1$
- ⋆ Lagrangian:

$$\mathcal{L}(\mathbf{v}_1, \lambda_1) = \mathbf{v}_1^\top \mathbf{\Sigma} \mathbf{v}_1 + \lambda_1 (1 - \mathbf{v}_1^\top \mathbf{v}_1)$$

- ⋆ Compute the derivative w.r.t $\mathbf{v}_1$:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{v}_1} = 2\mathbf{\Sigma} \mathbf{v}_1 - 2\lambda_1 \mathbf{v}_1$$

- ⋆ $\mathbf{v}_1$ is an eigenvector of the covariance matrix of $\mathbf{x}$:

$$\mathbf{\Sigma} \mathbf{v}_1 = \lambda_1 \mathbf{v}_1$$

- ⋆ $\mathbf{v}_1$ is the eigenvector corresponding to the largest eigenvalues !

$$\text{var}(z_1) = \mathbf{v}_1^\top \mathbf{\Sigma} \mathbf{v}_1 = \lambda_1 \mathbf{v}_1^\top \mathbf{v}_1 = \lambda_1$$

- Search $\mathbf{v}_2$ such as $\max \text{var}(z_2)$ and $\langle \mathbf{v}_2, \mathbf{v}_2 \rangle = 1$ and $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$

>>> Maximization of the variance 2/2

- Search $\mathbf{v}_2$ such as $\max \mathrm{var}(z_2)$ and $\langle \mathbf{v}_2, \mathbf{v}_2 \rangle = 1$ and $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$
- Lagrangian:

$$\mathcal{L}(\mathbf{v}_2, \lambda_2, \beta_1) = \mathbf{v}_2^\top \mathbf{\Sigma} \mathbf{v}_2 + \lambda_1(1 - \mathbf{v}_2^\top \mathbf{v}_2) + \beta_1(0 - \mathbf{v}_2^\top \mathbf{v}_1)$$

>>> Maximization of the variance 2/2

- Search $\mathbf{v}_2$ such as $\max \text{var}(z_2)$ and $\langle \mathbf{v}_2, \mathbf{v}_2 \rangle = 1$ and $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$
- Lagrangian:

$$\mathcal{L}(\mathbf{v}_2, \lambda_2, \beta_1) = \mathbf{v}_2^\top \mathbf{\Sigma} \mathbf{v}_2 + \lambda_1 (1 - \mathbf{v}_2^\top \mathbf{v}_2) + \beta_1 (0 - \mathbf{v}_2^\top \mathbf{v}_1)$$

- Compute the derivative w.r.t $\mathbf{v}_2$:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \mathbf{v}_2} &= 2\mathbf{\Sigma}\mathbf{v}_2 - 2\lambda_1 \mathbf{v}_2 - \beta_1 \mathbf{v}_1 \\ \mathbf{\Sigma}\mathbf{v}_2 &= \lambda_1 \mathbf{v}_2 + 2\beta_1 \mathbf{v}_1 \end{aligned}$$

- Search $\mathbf{v}_2$ such as $\max \text{var}(z_2)$ and $\langle \mathbf{v}_2, \mathbf{v}_2 \rangle = 1$ and $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$
- Lagrangian:

$$\mathcal{L}(\mathbf{v}_2, \lambda_2, \beta_1) = \mathbf{v}_2^\top \mathbf{\Sigma} \mathbf{v}_2 + \lambda_1 (1 - \mathbf{v}_2^\top \mathbf{v}_2) + \beta_1 (0 - \mathbf{v}_2^\top \mathbf{v}_1)$$

- Compute the derivative w.r.t $\mathbf{v}_2$:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \mathbf{v}_2} &= 2\mathbf{\Sigma}\mathbf{v}_2 - 2\lambda_1 \mathbf{v}_2 - \beta_1 \mathbf{v}_1 \\ \mathbf{\Sigma}\mathbf{v}_2 &= \lambda_1 \mathbf{v}_2 + 2\beta_1 \mathbf{v}_1 \end{aligned}$$

- At optimality, $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$. Left-multiplying by $\mathbf{v}_1^\top$ the above equation:

$$\begin{aligned} \mathbf{v}_1^\top \mathbf{\Sigma} \mathbf{v}_2 &= 2\beta_1 \\ \lambda_1 \mathbf{v}_1^\top \mathbf{v}_2 &= 2\beta_1 \\ 0 &= 2\beta_1 \end{aligned}$$

>>> Maximization of the variance 2/2

- Search $\mathbf{v}_2$ such as $\max \text{var}(z_2)$ and $\langle \mathbf{v}_2, \mathbf{v}_2 \rangle = 1$ and $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$
- Lagrangian:

$$\mathcal{L}(\mathbf{v}_2, \lambda_2, \beta_1) = \mathbf{v}_2^\top \mathbf{\Sigma} \mathbf{v}_2 + \lambda_1 (1 - \mathbf{v}_2^\top \mathbf{v}_2) + \beta_1 (0 - \mathbf{v}_2^\top \mathbf{v}_1)$$

- Compute the derivative w.r.t $\mathbf{v}_2$:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \mathbf{v}_2} &= 2\mathbf{\Sigma}\mathbf{v}_2 - 2\lambda_1 \mathbf{v}_2 - \beta_1 \mathbf{v}_1 \\ \mathbf{\Sigma}\mathbf{v}_2 &= \lambda_1 \mathbf{v}_2 + 2\beta_1 \mathbf{v}_1 \end{aligned}$$

- At optimality, $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$. Left-multiplying by $\mathbf{v}_1^\top$ the above equation:

$$\begin{aligned} \mathbf{v}_1^\top \mathbf{\Sigma} \mathbf{v}_2 &= 2\beta_1 \\ \lambda_1 \mathbf{v}_1^\top \mathbf{v}_2 &= 2\beta_1 \\ 0 &= 2\beta_1 \end{aligned}$$

- Hence, we have

$$\mathbf{\Sigma}\mathbf{v}_2 = \lambda_2 \mathbf{v}_2$$

>>> Maximization of the variance 2/2

- Search $\mathbf{v}_2$ such as $\max \text{var}(z_2)$ and $\langle \mathbf{v}_2, \mathbf{v}_2 \rangle = 1$ and $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$
- Lagrangian:

$$\mathcal{L}(\mathbf{v}_2, \lambda_2, \beta_1) = \mathbf{v}_2^\top \mathbf{\Sigma} \mathbf{v}_2 + \lambda_1 (1 - \mathbf{v}_2^\top \mathbf{v}_2) + \beta_1 (0 - \mathbf{v}_2^\top \mathbf{v}_1)$$

- Compute the derivative w.r.t $\mathbf{v}_2$:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \mathbf{v}_2} &= 2\mathbf{\Sigma}\mathbf{v}_2 - 2\lambda_1 \mathbf{v}_2 - \beta_1 \mathbf{v}_1 \\ \mathbf{\Sigma}\mathbf{v}_2 &= \lambda_1 \mathbf{v}_2 + 2\beta_1 \mathbf{v}_1 \end{aligned}$$

- At optimality, $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$. Left-multiplying by $\mathbf{v}_1^\top$ the above equation:

$$\begin{aligned} \mathbf{v}_1^\top \mathbf{\Sigma}\mathbf{v}_2 &= 2\beta_1 \\ \lambda_1 \mathbf{v}_1^\top \mathbf{v}_2 &= 2\beta_1 \\ 0 &= 2\beta_1 \end{aligned}$$

- Hence, we have

$$\mathbf{\Sigma}\mathbf{v}_2 = \lambda_2 \mathbf{v}_2$$

- $\mathbf{v}_2$ is the eigenvector corresponding the *second largest* eigenvalues

- Search $\mathbf{v}_2$ such as $\max \text{var}(z_2)$ and $\langle \mathbf{v}_2, \mathbf{v}_2 \rangle = 1$ and $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$
- Lagrangian:

$$\mathcal{L}(\mathbf{v}_2, \lambda_2, \beta_1) = \mathbf{v}_2^\top \mathbf{\Sigma} \mathbf{v}_2 + \lambda_1 (1 - \mathbf{v}_2^\top \mathbf{v}_2) + \beta_1 (0 - \mathbf{v}_2^\top \mathbf{v}_1)$$

- Compute the derivative w.r.t $\mathbf{v}_2$:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \mathbf{v}_2} &= 2\mathbf{\Sigma}\mathbf{v}_2 - 2\lambda_1 \mathbf{v}_2 - \beta_1 \mathbf{v}_1 \\ \mathbf{\Sigma}\mathbf{v}_2 &= \lambda_1 \mathbf{v}_2 + 2\beta_1 \mathbf{v}_1 \end{aligned}$$

- At optimality, $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$. Left-multiplying by $\mathbf{v}_1^\top$ the above equation:

$$\begin{aligned} \mathbf{v}_1^\top \mathbf{\Sigma} \mathbf{v}_2 &= 2\beta_1 \\ \lambda_1 \mathbf{v}_1^\top \mathbf{v}_2 &= 2\beta_1 \\ 0 &= 2\beta_1 \end{aligned}$$

- Hence, we have

$$\mathbf{\Sigma}\mathbf{v}_2 = \lambda_2 \mathbf{v}_2$$

- $\mathbf{v}_2$ is the eigenvector corresponding the *second largest* eigenvalues
- $\mathbf{v}_k$ is the eigenvector corresponding the k^{th} *largest* eigenvalues

1. Empirical estimation the mean value:

$$\boldsymbol{\mu} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i$$

2. Empirical estimation the covariance matrix:

$$\boldsymbol{\Sigma} = \frac{1}{n-1}\sum_{i=1}^{n}(\mathbf{x}_i - \boldsymbol{\mu})(\mathbf{x}_i - \boldsymbol{\mu})^\top$$

3. Compute p first eigenvalues/eigenvectors... How to choose p ? Explained variance:

$$\frac{\sum_{i=1}^{p}\lambda_i}{\sum_{i=1}^{d}\lambda_i}$$

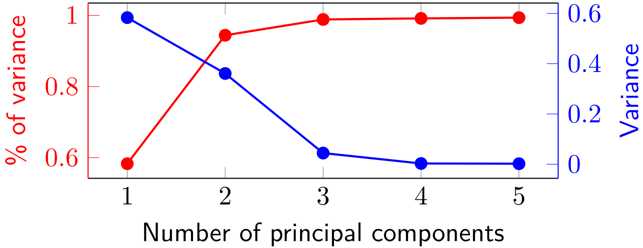
4. Tips for high dimensional data set: if $n < d$ see [MLC16] page 420

>>> PCA case study 1/3

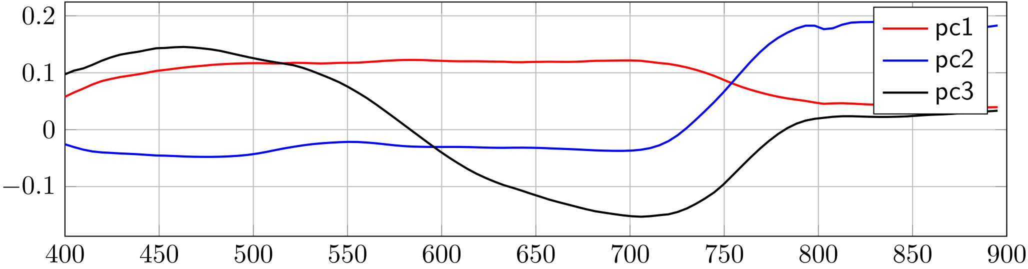
```
import rasterTools as rt
import scipy as sp
from sklearn.decomposition import PCA
import matplotlib.pyplot as plt

# Load data set
im,GeoT,Proj = rt.open_data('../Data/university.tif')
[h,w,b]=im.shape
im.shape=(h*w,b)
wave = sp.loadtxt('../Data/waves.csv',delimiter=',')

# Do PCA
pca = PCA()
pca.fit(im)

# Plot explained variance
l = pca.explained_variance_ratio_
print l[:5]
print (l.cumsum()/l.sum())[:5]

# Save Eigenvectors
D = sp.concatenate((wave[:,sp.newaxis],pca.components_[:3,:].T),axis=1)
sp.savetxt('../FeatureExtraction/figures/pca_pcs.csv',D,delimiter=',')
```

\>>> PCA case study 2/3

- ⋆ Explained variance



- ⋆ Principal components



```
# Projection of the first PCs
imp = sp.dot(im,pca.components_[:3,:].T)
imp.shape = (h,w,3)

# Save image
rt.write_data('../Data/pca_university.tif',imp,GeoT,Proj)
```

>>> Kernel PCA

- $\star$ PCA is limited to second order information
- $\star$ To capture higher-order statistics, it is possible to map the data onto another space $\mathcal{H}$

$$\begin{array}{rcl} \phi : \mathbb{R}^d & \rightarrow & \mathcal{H} \\ \mathbf{x} & \mapsto & \phi(\mathbf{x}). \end{array}$$

- $\star$ In $\mathcal{H}$, conventional PCA can be applied.
- $\star$ Using the *kernel trick* it is possible to directly work on the *kernel matrix* in $\mathbb{R}^d$

$$\mathbf{K} = \left(\begin{array}{cccc} k(\mathbf{x}_1, \mathbf{x}_1) & k(\mathbf{x}_1, \mathbf{x}_2) & \ldots & k(\mathbf{x}_1, \mathbf{x}_n) \\ k(\mathbf{x}_2, \mathbf{x}_1) & k(\mathbf{x}_2, \mathbf{x}_2) & \ldots & k(\mathbf{x}_2, \mathbf{x}_n) \\ \vdots & \vdots & \ddots & \vdots \\ k(\mathbf{x}_n, \mathbf{x}_1) & k(\mathbf{x}_n, \mathbf{x}_2) & \ldots & k(\mathbf{x}_n, \mathbf{x}_n) \end{array} \right).$$

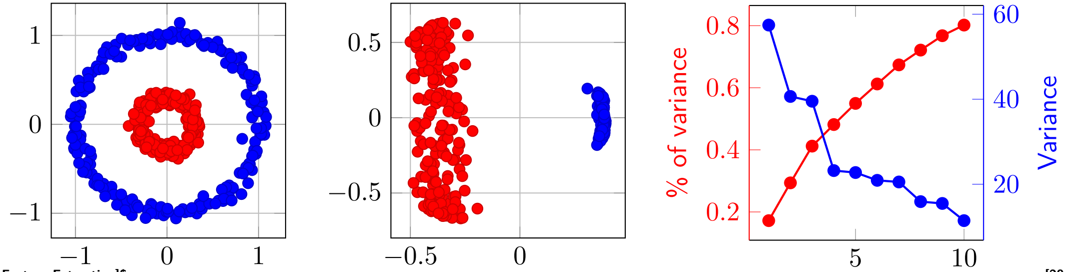
- ⋆ PCA is limited to second order information
- ⋆ To capture higher-order statistics, it is possible to map the data onto another space $\mathcal{H}$

$$
\begin{aligned}
\phi : \mathbb{R}^d &\rightarrow \mathcal{H} \\
\mathbf{x} &\mapsto \phi(\mathbf{x}).
\end{aligned}
$$

- ⋆ In $\mathcal{H}$, conventional PCA can be applied.
- ⋆ Using the *kernel trick* it is possible to directly work on the *kernel matrix* in $\mathbb{R}^d$

$$
\mathbf{K} = \begin{pmatrix}
k(\mathbf{x}_1, \mathbf{x}_1) & k(\mathbf{x}_1, \mathbf{x}_2) & \dots & k(\mathbf{x}_1, \mathbf{x}_n) \\
k(\mathbf{x}_2, \mathbf{x}_1) & k(\mathbf{x}_2, \mathbf{x}_2) & \dots & k(\mathbf{x}_2, \mathbf{x}_n) \\
\vdots & \vdots & \ddots & \vdots \\
k(\mathbf{x}_n, \mathbf{x}_1) & k(\mathbf{x}_n, \mathbf{x}_2) & \dots & k(\mathbf{x}_n, \mathbf{x}_n)
\end{pmatrix}.
$$

- ⋆ KPCA versus PCA:



- Choose the kernel and its parameters
- Compute the kernel matrix $\mathbf{K}$ for all the pixels (or a subset)
- Center the matrix

$$\mathbf{K}_c = \mathbf{K} - \mathbf{1}_n\mathbf{K} - \mathbf{K}\mathbf{1}_n + \mathbf{1}_n\mathbf{K}\mathbf{1}_n$$

- Solve the eigenproblems

$$\lambda\boldsymbol{\alpha} = \mathbf{K}\boldsymbol{\alpha} \text{ subject to } \|\boldsymbol{\alpha}\|_2 = \frac{1}{\lambda}$$

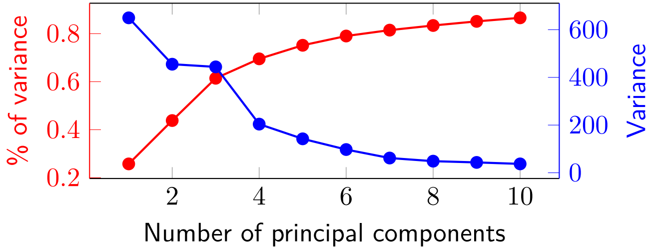
- Project on the p first *kernel principal components*: $\phi^{kpc}(\mathbf{x}) = \begin{bmatrix}\phi_1^{kpc}(\mathbf{x}) & \dots & \phi_p^{kpc}(\mathbf{x})\end{bmatrix}^t$

$$\phi_j^{kpc}(\mathbf{x}) = \sum_{i=1}^{n} \alpha_{ki} k(\mathbf{x}_i, \mathbf{x})$$

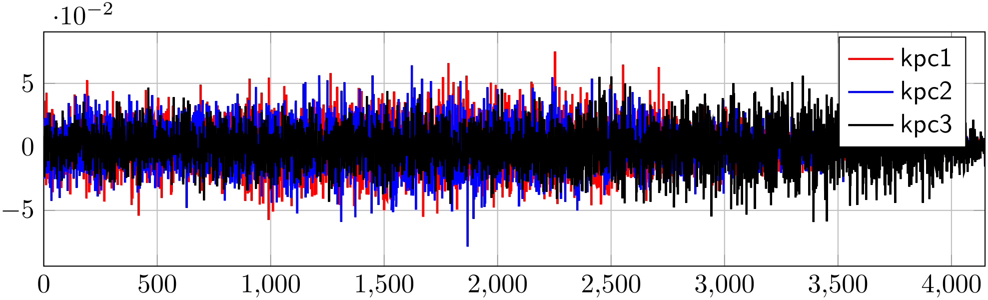
From [FCB09].

```
import rasterTools as rt
import scipy as sp
from sklearn.decomposition import KernelPCA
import matplotlib.pyplot as plt
from sklearn.preprocessing import StandardScaler

# Load data set
im,GeoT,Proj = rt.open_data('../Data/university.tif')
[h,w,b]=im.shape
im.shape=(h*w,b)
wave = sp.loadtxt('../Data/waves.csv',delimiter=',')

# Scale data
sc = StandardScaler()
im = sc.fit_transform(im)

# Do KPCA
kpca = KernelPCA(kernel='rbf',gamma=1.0/b,n_jobs=-1)
kpca.fit(im[::50,:]) # Use a subset of the total number of pixels
```

- ⋆ Explained variance



- ⋆ Principal components



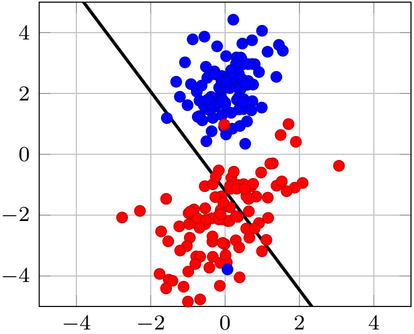
>>> KPCA case study 3/3

```
imp = kpca.transform(im)[:,:3]
imp.shape = (h,w,3)

# Save image
rt.write_data('../Data/kpca_university.tif',imp,GeoT,Proj)
```



>>> Outline

- We observe some $\{\mathbf{x}_i, y_i\}_{i=1}^n$
- Use the label information to find the linear features that highlight differences among classes



- FDA: find $\mathbf{a}$ such as the ratio between the *between projected variance* and the *sample projected variance* is maximal

- Between-class covariance matrix:

$$\mathbf{B} = \frac{1}{n}\sum_{c=1}^{C} n_c(\boldsymbol{\mu}_c - \boldsymbol{\mu})(\boldsymbol{\mu}_c - \boldsymbol{\mu})^\top$$

- Sample covariance matrix

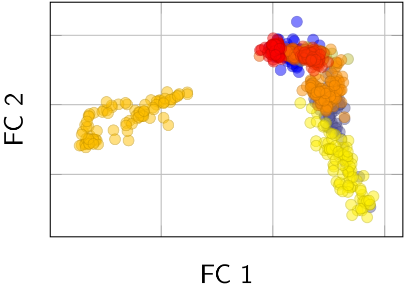
$$\boldsymbol{\Sigma} = \frac{1}{n-1}\sum_{i=1}^{n}(\mathbf{x}_i - \boldsymbol{\mu})(\mathbf{x}_i - \boldsymbol{\mu})^\top$$

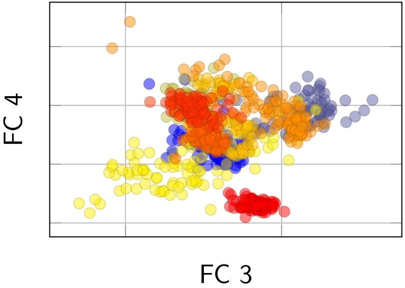
- The Fisher discriminant subspace is given by the eigenvectors of $\boldsymbol{\Sigma}^{(-1)}\mathbf{B}$
- Remark: there are at most $C-1$ eigenvectors because Rank$(\mathbf{B}) \leq C-1$.

>>> FDA case study 1/3

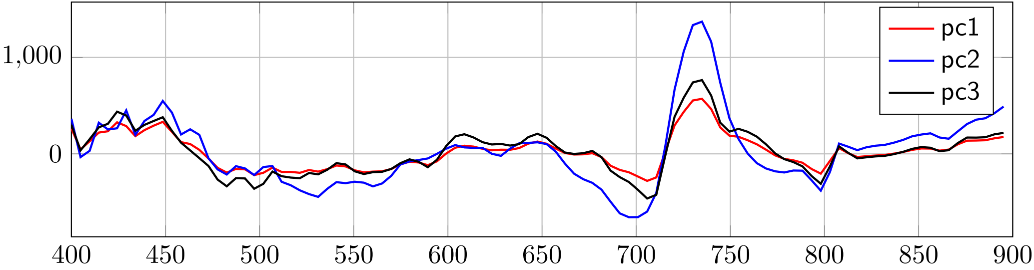
```
import rasterTools as rt
import scipy as sp
from sklearn.discriminant_analysis import LinearDiscriminantAnalysis

# Load data set
X,y=rt.get_samples_from_roi('../Data/university.tif','../Data/university_gt.tif')
wave = sp.loadtxt('../Data/waves.csv',delimiter=',')

# Select the same number of samples
nt = 900
xt,yt=[],[]
for i in sp.unique(y):
    t = sp.where(y==i)[0]
    nc = t.size
    rp =  sp.random.permutation(nc)
    xt.extend(X[t[rp[0:nt]],:])
    yt.extend(y[t[rp[0:nt]]])

xt = sp.asarray(xt)
yt = sp.asarray(yt)

# Do LDA
lda = LinearDiscriminantAnalysis(solver='eigen',shrinkage='auto')
lda.fit(xt,yt.ravel())
```

- Projection on Fisher components





- Fisher components



```
im,GeoT,Proj = rt.open_data('../Data/university.tif')
[h,w,b]=im.shape
im.shape=(h*w,b)
imp = lda.transform(im)[:,:3]
imp.shape = (h,w,3)
# Save image
rt.write_data('../Data/lda_university.tif',imp,GeoT,Proj)
```