Artificial Intelligence/Machine Learning/Deep Learning: 'Bridging the Skills Gap'

Optional: Normal Equation

There is an analytical/closed form solution to a linear regression problem with Mean Squared Error as cost function → no gradient descent needed!

So why don't we use this all the time?

- Because it is heavy taxation on the memory → requires d^2 memory → avoid when dimension d of feature vector is large
- X^TX requires lots of memory if dimension of X is large
- Inverting X^TX is also complex
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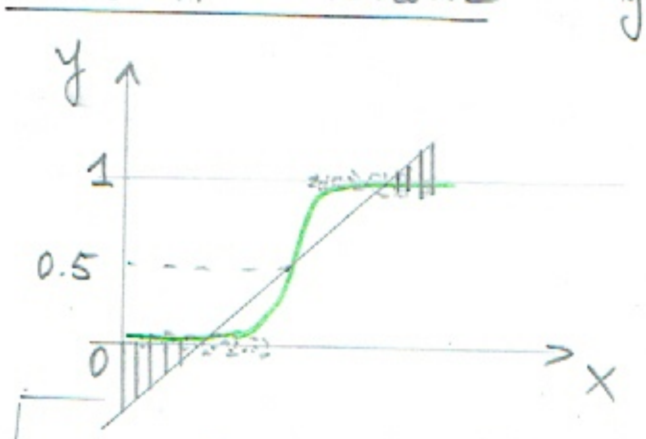
Logistic Regression → classifier < BINARY / Multi-class

Problem statement: y_i is a categorical variable

→ $y=0$ 'no cancer'
$y=1$ 'cancer'



↳ linear Regression $y_p = w_0 + w_1x_1 + w_2x_2 + \dots$

values < 0
> 1 → NOT GOOD!

values between 0 and 1 → NOT Good!

Definition: Odds Ratio : $\frac{\text{Successes}}{\text{Failures}}$

6 coin tosses
↓
4 Hs, 2 Ts

→ $\frac{4}{2}$ → 2 to 1

BERNOULLI
$B \sim p$ ← Probability of success vs Probability of failure → $\frac{4/6}{2/6}$ → $\frac{4}{2}$ → 2

$$\xrightarrow{\ln} \underbrace{\ln\left(\frac{p}{1-p}\right)}_{\text{log odds}} = \underbrace{w_0 + w_1x_1 + \dots + w_dx_d}_{w^Tx} \to \text{linear function of features } x_i$$

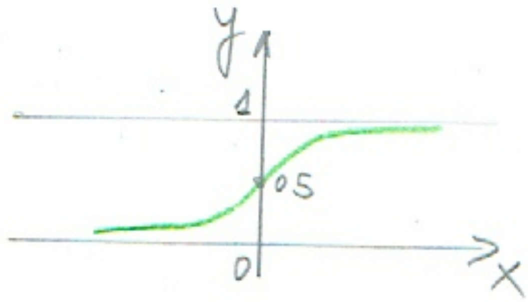
$$\to \frac{p}{1-p} = e^{+w^Tx}$$

$$\to p = \left(e^{+w^Tx}\right)(1-p)$$
$$= e^{+w^Tx} - pe^{+w^Tx}$$

$$\to e^{+w^Tx} = p\left(1+e^{+w^Tx}\right)$$

$$\to p = \frac{e^{+w^Tx}}{1+e^{+w^Tx}} \cdot \frac{e^{-w^Tx}}{e^{-w^Tx}} \to \boxed{p = \frac{1}{1+e^{-w^Tx}}}$$

probability of success



$P(y|x) > 0.5 \rightarrow y=1$ → CANCER

$P(y|x) \leqslant 0.5 \rightarrow y=0$ → NO CANCER

Naive Bayes $P(y|x) \sim \underbrace{P(x|y) \cdot P(y)}$

you get $P(y|x)$ by estimating $P(x|y)$ and $P(y)$

$P(x|y)$ → likelihoods

$P(y)$ → PRIORS

logistics Regression $\boxed{P(y|x) = \frac{1}{1+e^{-W^T x}}} = \sigma(W^T x_i)$ ← sigmoid

Gradient $D = \{(x_1, y_1), \ldots, (x_d, y_d)\}$ m samples!

$y_i \sim$ Bernoulli

MLE: $\underset{W}{\operatorname{argmax}}\, P(D|W)$ → Find Ws that maximizes the likelihood of seeing the DATA D

$$P(D|W) = \prod_{i=1}^{m} P(y^{(i)} | x^{(i)}, w)$$

SET $\alpha^{(i)} = \sigma(W^T x^{(i)})$

$$= \prod_{i=1}^{m} \alpha^{(i)\, y^{(i)}} \left(1 - \alpha^{(i)}\right)^{1-y^{(i)}} \quad y_i \in \{0, 1\}$$

↑ pmf Bernoulli

↳ We will not be able to solve for W because non-linearity of α

→ Newton's method $\left\langle \begin{matrix} \nabla \\ H \end{matrix} \right.$ → 2nd order optimization ③

- no α
- very fast convergence

$$\boxed{w_1 = w_0 - \frac{C'(w_0)}{C''(w_0)}}$$

$\hookrightarrow H^{-1}$ → can be challenge to calculate

$\xrightarrow{\text{newton}}$ if $d < 1000$

$$\mathcal{L}(w) \overset{\text{SET}}{=} \underbrace{-\log p(D|w)}_{-\log\text{-likelihood}} \rightarrow \text{we want to minimize!}$$

$$= -\sum_{i=1}^{m} y_i \log \alpha_i + (1-y_i)\log(1-\alpha_i)$$

INTERMEZZO: $\frac{\partial}{\partial w_j} \log \alpha_i = + \frac{x_j e^{-w^T x}}{1+e^{-w^T x}} = \boxed{x_j(1-\alpha)}$

$$\log \alpha = \log \sigma(w^T x) = \log \frac{1}{1+e^{-w^T x}} = 0 - \log(1+e^{-w^T x})$$

$$\log(1-\alpha) = -w^T x - \log(1+e^{-w^T x})$$

$$\frac{\partial}{\partial w_j} \log(1-\alpha) = -x_j + x_j(1-\alpha) = \boxed{-\alpha x_j}$$

$$\rightarrow \mathcal{L}(w) = -\sum_{i=1}^{m} y_i \log \alpha_i + (1-y_i)\log(1-\alpha_i)$$

$$\frac{\partial \mathcal{L}(w)}{\partial w_j} = -\sum_{i=1}^{m} y_i x_{ij}(1-\alpha_i) - (1-y_i)x_{ij}\alpha_i$$

$$= -\sum_{i=1}^{m} y_i x_{ij} - \cancel{y_i x_{ij}\alpha_i} - x_{ij}\alpha_i + \cancel{y_i x_{ij}\alpha_i}$$

DOT product of column j of X WITH $(\alpha - y_i)$ $\rightarrow$

$$= \sum_{i=1}^{m} (\alpha_i - y_i) x_{ij} \rightarrow (\text{predict } \hat{y} - \text{real } y_i)$$

DOT

1st ROW $(x_1)^T$ $\rightarrow$ design MATRIX

$$X = \begin{pmatrix} x_{11} & x_{i1} & x_{1d} \\ \vdots & \vdots & \vdots \\ x_{m1} & x_{im} & x_{md} \end{pmatrix}$$

d-features $\rightarrow x_{i1}, \dots, x_{id}$ $\downarrow$ i-th sample!

m-samples

$$\rightarrow (\alpha - y)^T X \rightarrow \left((\alpha-y)^T X\right)^T = \underbrace{\boxed{X^T(\alpha - y)}}_{\nabla_W \mathcal{L}}$$

next: Hessian

$$H_W \rightarrow \frac{\partial^2}{\partial w_j \partial w_k} \mathcal{L}_W$$

jk entry of H

$$H_W \Rightarrow \sum_{i=1}^{m} x_{ij} \frac{\partial}{\partial w_k} \alpha_i \ (1)$$

$$\partial \log \alpha = \frac{\partial \alpha}{\alpha}$$

$$\partial \alpha = \alpha\, \partial \log \alpha = \alpha x_j (1-\alpha)$$

$$\frac{\partial}{\partial w_k} \alpha_i = x_{ik} \alpha_i (1-\alpha_i) \rightarrow x_i$$

$$(1) = \sum_{i=1}^{m} \underbrace{x_{ij} x_{ik}}_{\text{ROW}^{(i)}} \alpha_i (1-\alpha_i)$$

$$= z_j^T B z_k$$

WITH $z_j = (x_{1j}, \dots, x_{mj})$, $z_k = (x_{1k}, \dots, x_{mk})$

DIAGONAL

WITH $B \begin{pmatrix} \alpha_1(1-\alpha_1) & 0 \\ 0 & \ddots \ \alpha_n(1-\alpha_m) \end{pmatrix}$

$x_j = (x_{j1}, \dots, x_{jd})^T \rightarrow$ ROW (i)

$z_j = x_{ij}, \dots, x_{mj} \rightarrow$ COLUMN (j)

to make it a ROW

$X^T B X$

$$\rightarrow \boxed{\nabla^2_W \ell = X^T B X}$$

positive semi Definite!

ℓ is convex

$\alpha_i = \sigma(w^T x_i)$

$\hookrightarrow$ always > 0 AND < 1

$0 < \alpha_i(1-\alpha_i) < 1$

<u>Newton</u>: $\longrightarrow$ <u>iterative Reweighted least squares!</u>

$\hookrightarrow$ Fast!

$w_{t+1} = w_t - H^{-1} \nabla$ (see page 3)

$$\rightarrow \boxed{w_{t+1} = w_t - \underbrace{(X^T B A)^{-1}}_{\text{ASSUME is INVERTIBLE}} \cdot X^T(\alpha - Y)}$$

$$= w_t = (X^T B X)^{-1} X^T B \underbrace{\left(A w_t - B^{-1}(\alpha - Y)\right)}_{z_t}$$

$$= w_t = (X^T B X)^{-1} X^T B z_t$$

Normal Equation

multi-variate regression $\boxed{y_p = w^T x}$ (1)

Mean squared error : $\|e\|^2 = (y_p - y)^2$ (2)

(1) and (2) $\rightarrow (w^T x - y)^T (w^T x - y)$ $\quad \|\bar{e}\|^2 = \bar{e} \cdot \bar{e}^T = \bar{e}^T \cdot \bar{e}$

$$\rightarrow \left((w^T x)^T - y^T\right)\left(w^T x - y\right)$$

$$= (w^T x)^T (w^T x) - y^T (w^T x) - y (w^T x)^T + \cancel{y^T y}$$

↳ drop as no function of w

$$= x^T \underbrace{w w^T}_{\|w\|^2} x - \underbrace{y^T (w^T x) - (w^T x)^T y}_{-2 (w^T x)^T y}$$

m samples
d features.

$$\rightarrow \nabla_w C = 2 x^T x w - 2 x^T y$$

$$\nabla_w C \overset{\text{SET}}{=} 0$$

$$\rightarrow \cancel{2} x^T x w - \cancel{2} x^T y^T = 0$$

$$\rightarrow x^T x \, w = x^T y$$

$$\rightarrow \boxed{w = \underbrace{(x^T x)^{-1} x^T}_{\text{pseudo-inverse of } X} y}$$

→ sometimes $(x^T x)^{-1}$ is not defined

ex. $(x^T x)$ is singular (features are related)