Unit 1 - Sem 3 - 22MAT220

Mathematics for Computing

Table of Contents

To be added

Nuclear norm(Schatten-0) (WIP)

Weighted L_1 and L_2 norm

Schatten-p norm (p=2 is Frobenius norm)

L_1 regularization or Lasso Regression,

(Least Absolute Shrinkage and Selection Operator)

L_2 regularization or Ridge Regression or Tikhonov regularization, (named for Andrey Tikhonov)

Basis pursuit (solution to system of linear equation with minimal ℓ_1 norm)

Basis Pursuit Denoising (ℓ_1 norm penalized least squares)

Total Variation

Consensus

SVM

Derivation of Pseudoinverse of A

SALSA (Split Augmented Lagrangian Shrinkage Algorithm)

Syllabus

Unit 1
Direct methods for convex functions, sparsity inducing penalty functions. Constrained Convex Optimization problems, Krylov subspace, Conjugate gradient method, formulating problems as LP and QP, support vector machines, solving by packages (CVXOPT), Lagrangian multiplier method, KKT conditions. Introduction to alternating direction method of multipliers (ADMM) - the algorithm. Applications in signal processing, pattern recognition and classification.

Unit 2
Introduction to PDEs. Formulation and numerical solution methods (Finite difference and Fourier) for PDEs in Physics and Engineering.

Unit 3
Inequalities of statistics, Multivariate Gaussian and weighted least squares, Markov chains, Markov decision process.

Component		**Weightage %**
Internal (70)	Weekly Tests	30
	Lab Experiments	10
	Mid Project Review	10
	Final Project Review	20
External (30)	End Semester Exam-Written	20
	End Semester - Coding	10

Notation

$x \neq \mathbf{x}$, $x \in R$, $\mathbf{x} \in R^n$

x is a variable that can take a real value.

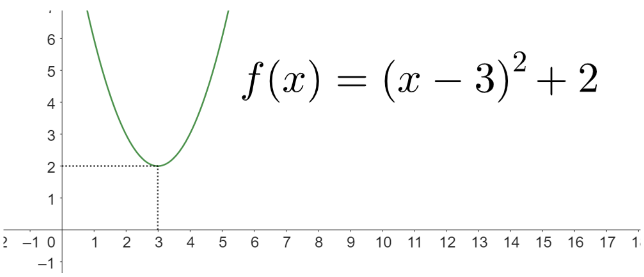
$\mathbf{x}$ is a vector of n variables / unknowns, $x_1, x_2, \cdots x_n$. Think of it as an ordered array of n variables that can take real values.

$\underset{\mathbf{x}}{\operatorname{argmin}} \; f_1(\mathbf{x})$ **Where** is the minimum of the function

$\min\limits_{\mathbf{x}} \; f_1(\mathbf{x})$ **What** is the minimum value of the function

$\underset{\mathbf{x}}{\operatorname{argmax}} \; f_2(\mathbf{x})$ **Where** is the maximum of the function

$\max\limits_{\mathbf{x}} \; f_2(\mathbf{x})$ **What** is the maximum value of the function



Examples : Function of 1 variable , x

$$2 = \underset{x}{\operatorname{argmin}} \; (f(x) = (x-2)^2 + 7)$$

$$7 = \min_{x} \; (f(x) = (x-2)^2 + 7)$$

$$3 = \underset{x}{\operatorname{argmax}} \; (f(x) = 20 - (x-3)^2)$$

$$20 = \max_{x} \; (f(x) = 20 - (x-3)^2)$$

Examples : Function of more than 1 variable, $\mathbf{x}$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \underset{\mathbf{x}}{\operatorname{argmin}} \; f(\mathbf{x}) = \mathbf{x}^T Q \mathbf{x} + 2, \; Q = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$f(x, y) = x^2 + 3y^2 = [x \quad y] \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + 2$$

$$2 = \min_{\mathbf{x}} \; f(\mathbf{x}) = \mathbf{x}^T Q \mathbf{x} + 2$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \underset{\mathbf{x}}{\operatorname{argmax}} \ f(\mathbf{x}) = \mathbf{x}^T Q \mathbf{x} - 4, \ Q = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$$

$$f(x, y) = -x^2 - 3y^2 - 4 = [x \quad y] \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - 4$$

$$-4 = \max_{\mathbf{x}} \ f(\mathbf{x}) = \mathbf{x}^T Q \mathbf{x} - 4$$

Constrained Convex Optimization problem

The general form of a constrained convex optimization problem is

$\min_{\mathbf{x}} \ f(\mathbf{x})$ subject to the constraint $\mathbf{x} \in \mathbf{C}$

where f is a convex function and $\mathbf{C}$ is a convex set.

L_0 norm

(not strictly a norm function)

Understanding the L_0 Norm: Counting the Non-Zero

The L_0 norm of a vector, denoted as $\|x\|_0$ is defined as the number of non-zero elements in that vector. For instance, for the vector $\mathbf{x} = \begin{bmatrix} 0 \\ 5 \\ 0 \\ -2 \\ 3 \end{bmatrix}$, the L_0 norm $\|x\|_0$ is 3, as there are three non-zero elements $5, -2$ and 3.

Mathematical Definition:

For a vector $\mathbf{x} \in R^n$, the L_0 norm is given by:

$$\|x\|_0 = \sum_{i=1}^{n} |x_i|^0$$

where $|x_i|^0 = 1$ if $x_i \neq 0$ and 0 if $x_i = 0$.

L_1 norm

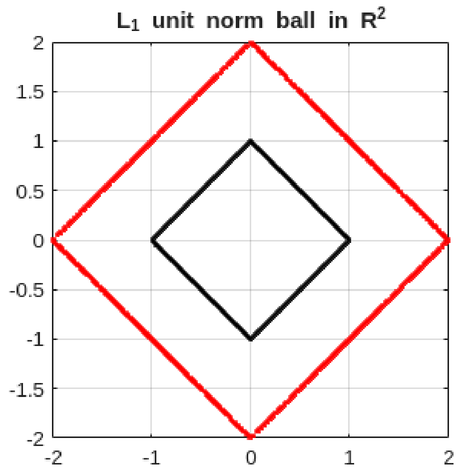
is the sum of the absolute values of each elements of $\mathbf{x}$.

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|$$

Unit norm ball in 2 dimension

$\mathbf{x} \in R^2$

```
clearvars
x = rand(2,2500)-0.5;
x = x./sum(abs(x));
x2 = 2*x;
plot(x(1,:),x(2,:),'k.');hold on
plot(x2(1,:),x2(2,:),'r.');hold off
axis equal
title("L_1 unit norm ball in R^2")
grid on
xlim([-2 2])
ylim([-2 2])
```

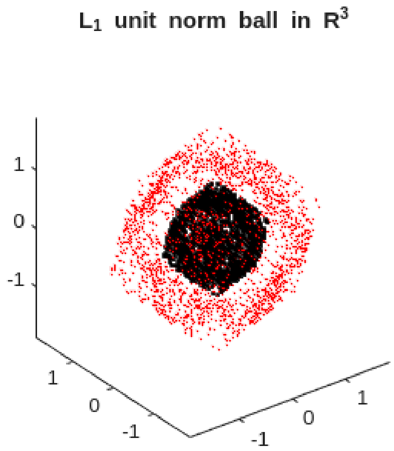


Unit norm ball in 3 dimension

$\mathbf{x} \in R^3$

```
clearvars
x = rand(3,2500)-0.5;
x = x./sum(abs(x));
plot3(x(1,:),x(2,:),x(3,:),'k.');hold on
axis equal
title("L_1 unit norm ball in R^3")

x2 = 2*x;
plot3(x2(1,:),x2(2,:),x2(3,:),'r.',MarkerSize=0.5);
hold off
```



$y = \|\mathbf{x}\|_1$

```
x1 = single(-5:0.2:5);
y = abs(x1);
plot(x1,y)
clearvars
x1 = single(-5:0.2:5);
x2 = x1;
[X1,X2] = meshgrid(x1,x2);
Z1 = abs(X1)+abs(X2);
surf(X1,X2,Z1);hold on
contour(X1,X2,Z1,1:4); hold off
title("|x| in 2d")
```

L_1 norm ball in 3d (function of 3 variables)

```
x = single(rand(5e3,3)-0.5);
x = x./sum(abs(x),2);
size(x)
plot3(x(:,1),x(:,2),x(:,3),'r.')
axis equal
title('L_1 unit norm ball in R^3 - rotate and see')
```

L_2 norm

is the square root of the sum of the squares of each values of $\mathbf{x}$.

$\|\mathbf{x}\|_2^2 = \sum_{i=1}^{n} x_i^2$ square of the L_2 norm

L_2 norm $= \sqrt{\sum_{i=1}^{n} x_i^2}$

```
x = rand(3,500)-0.5;
```

```
x = x./(sqrt(sum(x.^2)));

plot3(x(1,:),x(2,:),x(3,:),'k.');hold on
axis equal
title("L_2 unit norm ball")
x = 2*x./(sqrt(sum(x.^2)));
plot3(x(1,:),x(2,:),x(3,:),'ro');hold off
```

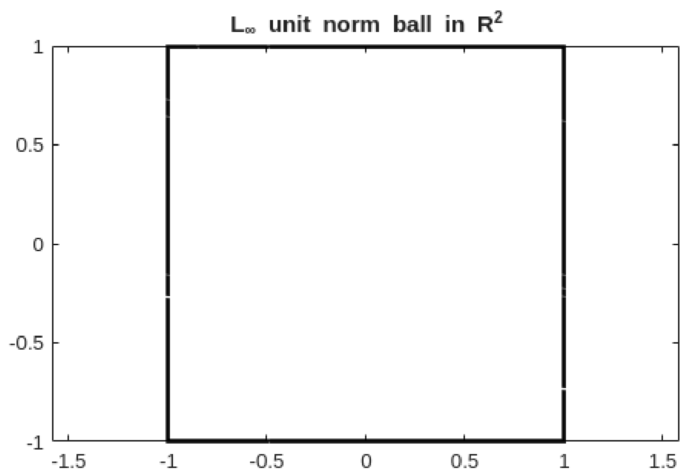
L_∞ norm

is the maximum of the absolute values of $\mathbf{x}$.

$\|\mathbf{x}\|_\infty = \max\{|x_i|\}$

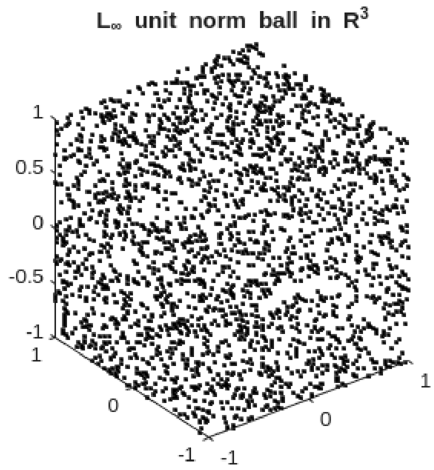
$\mathbf{x} \in R^2$

```
x = rand(2,2500)-0.5;
x = x./(max(abs(x)));
plot(x(1,:),x(2,:),'k.')
axis equal
title("L_\infty unit norm ball in R^2")
```



$\mathbf{x} \in R^3$

```
x = rand(3,2500)-0.5;
x = x./(max(abs(x)));
plot3(x(1,:),x(2,:),x(3,:),'k.')
axis equal
title("L_\infty unit norm ball in R^3")
```



L_p norm

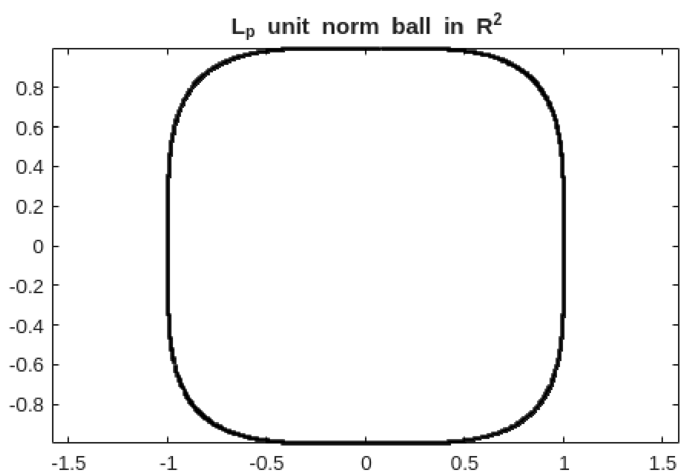
is defined as

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

with $1 \le p < \infty$

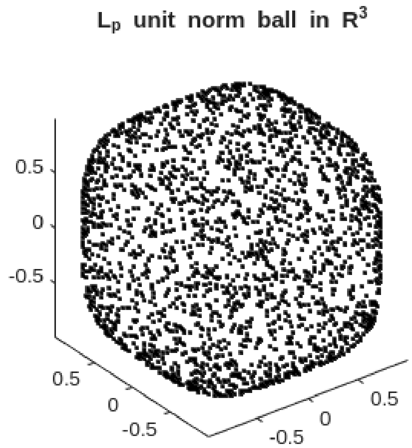
$\mathbf{x} \in R^2$

```
x = rand(2,2500)-0.5;
p = 4;
x = x./power(sum(abs(x).^p),1/p);
plot(x(1,:),x(2,:),'k.')
axis equal
title("L_p unit norm ball in R^2")
```



$\mathbf{x} \in R^3$

```
x = rand(3,2500)-0.5;
p = 4;
x = x./power(sum(abs(x).^p),1/p);
plot3(x(1,:),x(2,:),x(3,:),'k.')
axis equal
title("L_p unit norm ball in R^3")
```



Interesting facts

The 2-norm is the Euclidean length function; its unit ball is spherical. The 1-norm is used by airlines to define the maximal allowable size of a suitcase. The Sergel plaza in Stockholm, Sweden has the shape of the unit ball in the 4-norm; the Danish poet Piet Hein popularized this "superellipse" as a pleasing shape for objects such as conference tables.

Right inverse

If all the rows of a matrix $A \in R^{m\times n}$ are independent, then rank of AA^T is m and hence AA^T is invertible.

So $AA^T(AA^T)^{-1} = I_r$.

$A^+ = A^T(AA^T)^{-1}$ is the right inverse of A.

Right inverse is not unique.

Consider a matrix $A_{m\times n}$ where $m < n$, and A has full row rank. If B is a right inverse ($AB = I_m$), then for any matrix C whose columns are in the RNS of A (i.e., $AC = 0$), the matrix $(B + C)$ will also be a right inverse: $A(B + C) = AB + AC = I_m + 0 = I_m$.

Similar to the left inverse, among the infinitely many right inverses, there is a unique one called the Moore-Penrose pseudoinverse ($A^{\dagger}$). For a matrix A with full row rank, the pseudoinverse is given by $A^{\dagger} = A^T(AA^T)^{-1}$. This is the unique right inverse that minimizes the Frobenius norm of B-- Check this property !.

$A\mathbf{x} = \mathbf{b}$ if all the rows of A are independent, then $\mathbf{x} = A^{+}\mathbf{b}$ is the least norm solution , where A^{+} is the right inverse of A

Left inverse

If all the columns of a matrix $A \in R^{m\times n}$ are independent, then the rank of A^TA is n and hence A^TA is invertible.

So $(A^TA)^{-1}A^TA = I_r$

$A^{+} = (A^TA)^{-1}A^T$ is the left inverse of A.

Left inverse is not unique.

Consider a matrix $A_{m\times n}$ where $m > n$ and A has full column rank. If B is a left inverse ($BA = I_n$), then for any matrix C whose rows are in the LNS of A (i.e., $CA = 0$), the matrix ($B + C$) will also be a left inverse: $(B + C)A = BA + CA = I_n + 0 = I_n$.

Among the infinitely many left inverses, there is a unique one called the Moore-Penrose pseudoinverse (often denoted $A^{\dagger}$). For a matrix A with full column rank, the pseudoinverse is given by $A\dagger = (A^TA)^{-1}A^T$. This particular left inverse has special properties, such as minimizing the Frobenius norm of B. -- Check this property !

$A\mathbf{x} = \mathbf{b}$ if all the columns of A are independent, then $\mathbf{x} = A^{+}\mathbf{b}$ is the only solution possible, where A^{+} is the left inverse of A

Derivation of Pseudoinverse of A using projection

Starting from the CR decomposition of A

$A\mathbf{x} \approx \mathbf{y}$ Project $\mathbf{y}$ onto the column space of A.

$CR\mathbf{x} = CC^{+}\mathbf{y}$ Pre multiplying by C^{+}

$C^{+}CR\mathbf{x} = C^{+}CC^{+}\mathbf{y}$, But $C^{+}C = I_r$ where r is the rank of A and C

$\therefore R\mathbf{x} = C^{+}\mathbf{y}$ Pre multiplying by R^{+}

$R^{+}R\mathbf{x} = R^{+}C^{+}\mathbf{y}$, $R^{+}R$ is the projection operator for the row space of A and hence $R^{+}R\mathbf{x}$ is the row space solution vector $\mathbf{x}_r$.

$\therefore \mathbf{x}_r = R^{+}C^{+}\mathbf{y}$

ie., $A^{+} = R^{+}C^{+}$

PseudoInverse

$A = CR$, where $A \in \mathbf{R}^{m \times n}$, $C \in \mathbf{R}^{m \times r}$, $R \in \mathbf{R}^{r \times n}$

rank of C = rank of $R = r$

$A^+ = R^+ C^+ = R^T (RR^T)^{-1} \times (C^T C)^{-1} C^T$

$= R^T (C^T CRR^T)^{-1} C^T$

$= R^T (C^T AR^T)^{-1} C^T$

Derivation of ADMM

Consider the optimization problem

$\underset{\mathbf{x}}{\operatorname{argmin}} f(\mathbf{x}),\ \mathbf{x} \in R^n$

We decompose the multivariate function into 2 functions f and g.

$\underset{\mathbf{x}}{\operatorname{argmin}} f(\mathbf{x}) + g(\mathbf{x})$

This unconstrained optimization problem is converted to constrained by introducing the surrogate variable $\mathbf{z} \in R^n$ with the constraints $\mathbf{x} - \mathbf{z} = \mathbf{0}$.

$\underset{\mathbf{x}}{\operatorname{argmin}} f(\mathbf{x}) + g(\mathbf{z})$ subject to the constraints $\mathbf{x} - \mathbf{z} = \mathbf{0}$

The Lagrangian function for this problem is

$L(\mathbf{x}, \mathbf{z}, \boldsymbol{\lambda}) = f(\mathbf{x}) + g(\mathbf{z}) + \boldsymbol{\lambda}^T(\mathbf{x} - \mathbf{z}),\ \boldsymbol{\lambda} \in R^n$

To show that the first order optimality conditions are preserved, find the gradient of L with respect to $\mathbf{x}$ and $\mathbf{z}$, equate them to zero and add the two expressions.

$\frac{\partial L}{\partial \mathbf{x}} = \nabla_{\mathbf{x}} f(\mathbf{x}) + \boldsymbol{\lambda} = \mathbf{0}$

$\frac{\partial L}{\partial \mathbf{z}} = \nabla_{\mathbf{z}} g(\mathbf{z}) - \boldsymbol{\lambda} = \mathbf{0}$

Adding these two we get $\nabla f(\mathbf{x}) + \nabla g(\mathbf{z}) = \mathbf{0}$

The augmented Lagrangian is

$L_\rho(\mathbf{x}, \mathbf{z}, \boldsymbol{\lambda}) = f(\mathbf{x}) + g(\mathbf{z}) + \boldsymbol{\lambda}^T(\mathbf{x} - \mathbf{z}) + \frac{\rho}{2}\|\mathbf{x} - \mathbf{z}\|_2^2,\ \mathbf{x}, \mathbf{z}, \lambda \in \mathbf{R}^n$ and $\rho \in \mathbf{R}$ and $\rho > 0$

To show that the first order optimality conditions are preserved, find the gradient of L_ρ with respect to $\mathbf{x}$ and $\mathbf{z}$, equate them to zero and add the two expressions.

$$\frac{\partial L}{\partial \mathbf{x}} = \nabla f(\mathbf{x}) + \boldsymbol{\lambda} + \rho(\mathbf{x}-\mathbf{z}) = \mathbf{0}$$

$$\frac{\partial L}{\partial \mathbf{z}} = \nabla g(\mathbf{z}) - \boldsymbol{\lambda} - \rho(\mathbf{x}-\mathbf{z}) = \mathbf{0}$$

Adding these two we get $\nabla f(\mathbf{x}) + \nabla g(\mathbf{z}) = \mathbf{0}$

The minimum location at which the augmented Lagrangian is minimum is computed numerically by updating the variables $\mathbf{x}, \mathbf{z}$ and λ in each iteration. The iteration is continued until the desired precision is achieved.

Assume $\mathbf{z}^{(k)}$ and $\lambda^{(k)}$ are known from the previous iteration.

$$\mathbf{x}^{(k+1)} = \underset{\mathbf{x}}{\operatorname{argmin}} \;\; L_\rho(\mathbf{x}, \mathbf{z}^{(k)}, \boldsymbol{\lambda}^{(k)})$$

$$= \underset{\mathbf{x}}{\operatorname{argmin}} \;\; f(\mathbf{x}) + g(\mathbf{z}^{(k)}) + \boldsymbol{\lambda}^{(k)T}(\mathbf{x} - \mathbf{z}^{(k)}) + \frac{\rho}{2}\|\mathbf{x} - \mathbf{z}^{(k)}\|_2^2$$

$$= \underset{\mathbf{x}}{\operatorname{argmin}} \;\; f(\mathbf{x}) + g(\mathbf{z}^{(k)}) + \boldsymbol{\lambda}^{(k)T}\mathbf{x} - \boldsymbol{\lambda}^{(k)T}\mathbf{z}^{(k)} + \frac{\rho}{2}\|\mathbf{x} - \mathbf{z}^{(k)}\|_2^2$$

$$= \underset{\mathbf{x}}{\operatorname{argmin}} \;\; f(\mathbf{x}) + \boldsymbol{\lambda}^{(k)T}\mathbf{x} + \frac{\rho}{2}\|\mathbf{x} - \mathbf{z}^{(k)}\|_2^2 + \underbrace{g(\mathbf{z}^{(k)}) - \boldsymbol{\lambda}^{(k)T}\mathbf{z}^{(k)}}_{constant}$$

$$= \underset{\mathbf{x}}{\operatorname{argmin}} \;\; f(\mathbf{x}) + \boldsymbol{\lambda}^{(k)T}\mathbf{x} + \frac{\rho}{2}\|\mathbf{x} - \mathbf{z}^{(k)}\|_2^2$$

$$= \underset{\mathbf{x}}{\operatorname{argmin}} \;\; f(\mathbf{x}) + \boldsymbol{\lambda}^{(k)T}\mathbf{x} + \frac{\rho}{2}(\mathbf{x} - \mathbf{z}^{(k)})^T(\mathbf{x} - \mathbf{z}^{(k)})$$

$$= \underset{\mathbf{x}}{\operatorname{argmin}} \;\; f(\mathbf{x}) + \boldsymbol{\lambda}^{(k)T}\mathbf{x} + \frac{\rho}{2}(\mathbf{x}^T\mathbf{x} - 2\mathbf{z}^{(k)T}\mathbf{x} + \mathbf{z}^{(k)T}\mathbf{z}^{(k)})$$

$$= \underset{\mathbf{x}}{\operatorname{argmin}} \;\; f(\mathbf{x}) + \frac{\rho}{2}\left(\mathbf{x}^T\mathbf{x} - 2\mathbf{z}^{(k)T}\mathbf{x} + \mathbf{z}^{(k)T}\mathbf{z}^{(k)} + \frac{2}{\rho}\boldsymbol{\lambda}^{(k)T}\mathbf{x}\right)$$

$$= \underset{\mathbf{x}}{\operatorname{argmin}} \;\; f(\mathbf{x}) + \frac{\rho}{2}\left(\mathbf{x}^T\mathbf{x} - 2\mathbf{z}^{(k)T}\mathbf{x} + \mathbf{z}^{(k)T}\mathbf{z}^{(k)} + \frac{2}{\rho}\boldsymbol{\lambda}^{(k)T}\mathbf{x}\right)$$

$$= \underset{\mathbf{x}}{\operatorname{argmin}} \;\; f(\mathbf{x}) \; + \frac{\rho}{2}\left(\mathbf{x}^T\mathbf{x} - 2\mathbf{z}^{(k)T}\mathbf{x} + \mathbf{z}^{(k)T}\mathbf{z}^{(k)} + \frac{2}{\rho}\boldsymbol{\lambda}^{(k)T}\mathbf{x} + \frac{\boldsymbol{\lambda}^{(k)T}\boldsymbol{\lambda}^{(k)}}{\rho^2} - \frac{2}{\rho}\boldsymbol{\lambda}^{(k)T}\mathbf{z}^{(k)}\right) \underbrace{-\frac{\boldsymbol{\lambda}^{(k)T}\boldsymbol{\lambda}^{(k)}}{2\rho} + \boldsymbol{\lambda}^{(k)T}\mathbf{z}^{(k)}}_{constant}$$

$$= \underset{\mathbf{x}}{\operatorname{argmin}} \;\; f(\mathbf{x}) + \frac{\rho}{2}\left\|\mathbf{x} - \mathbf{z}^{(k)} + \frac{\boldsymbol{\lambda}^{(k)}}{\rho}\right\|_2^2$$

$= \underset{\mathbf{x}}{\operatorname{argmin}} \;\; f(\mathbf{x}) + \frac{\rho}{2}||\mathbf{x} - \mathbf{z}^{(k)} + \mathbf{u}^{(k)}||_2^2$, where $\mathbf{u} = \frac{\boldsymbol{\lambda}}{\rho}$, the scaled version of λ

Assume $\mathbf{x}^{(k+1)}$ and $\lambda^{(k)}$ are known from the previous iteration.

$$\mathbf{z}^{(k+1)} = \underset{\mathbf{z}}{\operatorname{argmin}} \;\; L_\rho(\mathbf{x}^{(k+1)}, \mathbf{z}, \boldsymbol{\lambda}^{(k)})$$

$$= \underset{\mathbf{z}}{\operatorname{argmin}} \;\; f(\mathbf{x}^{(k+1)}) + g(\mathbf{z}) + \boldsymbol{\lambda}^{(k)T}(\mathbf{x}^{(k+1)} - \mathbf{z}) + \frac{\rho}{2}\|\mathbf{x}^{(k+1)} - \mathbf{z}\|_2^2$$

$$= \underset{\mathbf{z}}{\operatorname{argmin}} \;\; f(\mathbf{x}^{(k+1)}) + g(\mathbf{z}) + \boldsymbol{\lambda}^{(k)T}\mathbf{x}^{(k+1)} - \boldsymbol{\lambda}^{(k)T}\mathbf{z} + \frac{\rho}{2}\|\mathbf{x}^{(k+1)} - \mathbf{z}\|_2^2$$

$$= \underset{\mathbf{z}}{\operatorname{argmin}} \;\; g(\mathbf{z}) - \boldsymbol{\lambda}^{(k)T}\mathbf{z} + \frac{\rho}{2}\|\mathbf{x}^{(k+1)} - \mathbf{z}\|_2^2 + \underbrace{f(\mathbf{x}^{(k+1)}) + \boldsymbol{\lambda}^{(k)T}\mathbf{x}^{(k+1)}}_{constant}$$

$$= \underset{\mathbf{z}}{\operatorname{argmin}} \;\; g(\mathbf{z}) - \boldsymbol{\lambda}^{(k)T}\mathbf{z} + \frac{\rho}{2}\|\mathbf{x}^{(k+1)} - \mathbf{z}\|_2^2$$

$$= \underset{\mathbf{z}}{\operatorname{argmin}} \;\; g(\mathbf{z}) - \boldsymbol{\lambda}^{(k)T}\mathbf{z} + \frac{\rho}{2}(\mathbf{x}^{(k+1)} - \mathbf{z})^T(\mathbf{x}^{(k+1)} - \mathbf{z})$$

$$= \underset{\mathbf{z}}{\operatorname{argmin}} \;\; g(\mathbf{z}) - \boldsymbol{\lambda}^{(k)T}\mathbf{z} + \frac{\rho}{2}(\mathbf{x}^{(k+1)T}\mathbf{x}^{(k+1)} - 2\mathbf{z}^T\mathbf{x}^{(k+1)} + \mathbf{z}^T\mathbf{z})$$

$$= \underset{\mathbf{z}}{\operatorname{argmin}} \;\; g(\mathbf{z}) + \frac{\rho}{2}\left(\mathbf{x}^{(k+1)T}\mathbf{x}^{(k+1)} - 2\mathbf{z}^T\mathbf{x}^{(k+1)} + \mathbf{z}^T\mathbf{z} - \frac{2\boldsymbol{\lambda}^{(k)T}\mathbf{z}}{\rho}\right)$$

$$= \underset{\mathbf{z}}{\operatorname{argmin}} \;\; g(\mathbf{z}) + \frac{\rho}{2}\left(\mathbf{x}^{(k+1)T}\mathbf{x}^{(k+1)} - 2\mathbf{z}^T\mathbf{x}^{(k+1)} + \mathbf{z}^T\mathbf{z} - \frac{2\boldsymbol{\lambda}^{(k)T}\mathbf{z}}{\rho} + \frac{\boldsymbol{\lambda}^{(k)T}\boldsymbol{\lambda}^{(k)}}{\rho^2} + \frac{2\mathbf{x}^{(k+1)T}\boldsymbol{\lambda}^{(k)}}{\rho}\right) \underbrace{-\frac{\boldsymbol{\lambda}^{(k)T}\boldsymbol{\lambda}^{(k)}}{2\rho} - \boldsymbol{\lambda}^{(k)T}\mathbf{x}^{(k+1)}}_{constant}$$

$$= \underset{\mathbf{z}}{\operatorname{argmin}} \;\; g(\mathbf{z}) + \frac{\rho}{2}\left\|\mathbf{x}^{(k+1)} - \mathbf{z} + \frac{\boldsymbol{\lambda}^{(k)}}{\rho}\right\|_2^2$$

$= \underset{\mathbf{z}}{\operatorname{argmin}} \;\; g(\mathbf{z}) + \frac{\rho}{2}\|\mathbf{x}^{(k+1)} - \mathbf{z} + \mathbf{u}^{(k)}\|_2^2$, where $\mathbf{u} = \dfrac{\lambda}{\rho}$, the scaled version of λ

Assume $\mathbf{x}^{(k+1)}$ and $\mathbf{z}^{(k+1)}$ are known from the previous iteration.

$$L_\rho = f(\mathbf{x}^{(k+1)}) + g(\mathbf{z}^{(k+1)}) + \lambda^T(\mathbf{x}^{(k+1)} - \mathbf{z}^{(k+1)}) + \frac{\rho}{2}\|\mathbf{x}^{(k+1)} - \mathbf{z}^{(k+1)}\|_2^2$$

$$\frac{\partial L_\rho}{\partial \lambda} = \mathbf{x}^{(k+1)} - \mathbf{z}^{(k+1)}$$

$\lambda^{(k+1)}$, $\mathbf{u}^{(k+1)}$ update

$\boldsymbol{\lambda}^{(k+1)} = \boldsymbol{\lambda}^{(k)} + \rho(\mathbf{x}^{(k+1)} - \mathbf{z}^{(k+1)})$

$\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)} + (\mathbf{x}^{(k+1)} - \mathbf{z}^{(k+1)})$

Least L_2 norm solution of $A\mathbf{x} = \mathbf{b}$

$$\underset{\mathbf{x}}{\operatorname{argmin}} \ \frac{1}{2}\mathbf{x}^T\mathbf{x} = \frac{1}{2}\mathbf{x}^T I\mathbf{x} = \frac{1}{2}\|\mathbf{x}\|_2^2$$

subject to the constraints $A\mathbf{x} = \mathbf{b}$, $A \in R^{m\times n}$

The Lagrangian of the problem is

$$L(\mathbf{x}, \lambda) = \frac{1}{2}\mathbf{x}^T\mathbf{x} - \boldsymbol{\lambda}^T(A\mathbf{x} - \mathbf{b}) \ ; \ \boldsymbol{\lambda} \in R^m$$

$\frac{\partial L}{\partial \mathbf{x}} = \mathbf{x} - A^T\boldsymbol{\lambda} = \mathbf{0} \Rightarrow \mathbf{x}^* = A^T\boldsymbol{\lambda}$ (Solution in the row space of A)

$\frac{\partial L}{\partial \boldsymbol{\lambda}} = A\mathbf{x} - \mathbf{b} = \mathbf{0} \Rightarrow AA^T\boldsymbol{\lambda} = \mathbf{b} \Rightarrow \boldsymbol{\lambda} = (AA^T)^{-1}\mathbf{b}$ (Assuming rows of A are independent.)

$\therefore \mathbf{x}^* = A^T\boldsymbol{\lambda} = A^T(AA^T)^{-1}\mathbf{b} = A^+b$ (since rows are A are assumed to be independent, A has right inverse)

$\underset{\mathbf{x}}{\operatorname{argmin}} \ \frac{1}{2}\mathbf{x}^T B\mathbf{x}$ where $B^T = B$, B is positive definite , subject to the constraints $A\mathbf{x} = \mathbf{b}$ $\rightarrow (1)$

$$L(\mathbf{x}, \lambda) = \frac{1}{2}\mathbf{x}^T B\mathbf{x} - \boldsymbol{\lambda}^T(A\mathbf{x} - \mathbf{b}) \qquad \rightarrow (2)$$

$$\frac{\partial L}{\partial \mathbf{x}} = B\mathbf{x} - A^T\boldsymbol{\lambda} = 0 \Rightarrow \mathbf{x}^* = B^{-1}A^T\boldsymbol{\lambda} \qquad \rightarrow (3)$$

$$\frac{\partial L}{\partial \boldsymbol{\lambda}} = A\mathbf{x} - \mathbf{b} = 0 \Rightarrow AB^{-1}A^T\boldsymbol{\lambda} = \mathbf{b} \Rightarrow \boldsymbol{\lambda} = (AB^{-1}A^T)^{-1}\mathbf{b}$$

$\rightarrow (4)$

Is the square matrix $AB^{-1}A^T$(size $m \times m$) ($A \in R^{m\times n}$, $B \in R^{n\times n}$) invertible ?

Explain in terms of independent rows or columns of the square matrix

$$\begin{bmatrix} B & -A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix}$$

$$\begin{bmatrix}\mathbf{x}\\ \lambda\end{bmatrix} = \begin{bmatrix}B & -A^T\\ A & 0\end{bmatrix}^{-1}\begin{bmatrix}0\\ \mathbf{b}\end{bmatrix}$$

$$\therefore \mathbf{x} = B^{-1}A^T(AB^{-1}A^T)^{-1}\mathbf{b} = B^{-1}A^T(A^T)^{+}BA^{+}\mathbf{b} = B^{-1}P_rBA^{+}\mathbf{b} \qquad \rightarrow (5)$$

Lets find the difference between the least L_2 norm sol and this case

$$d\mathbf{x} = B^{-1}P_rBA^{+}\mathbf{b} - A^{+}\mathbf{b}$$

$$= (B^{-1}P_rB - I)A^{+}\mathbf{b} \qquad \rightarrow (6)$$

Since the least L_2 norm sol is in the row space A, the sol of the current problem will not be completely in the row space A, the difference $d\mathbf{x}$ is in right null space and hence orthogonal to the row space sol.

Or equivalently,

The gradient of L with respect to $\mathbf{x}$ and λ equated to zero can be solved simultaneously to find $\mathbf{x}^*$.

```
A = [1 2 3; 3 4 5]
b = [1 2]'
x1 = pinv(A)*b   % least norm sol using pinv
x2 = A'*inv(A*A')*b % least norm sol using formula
```

Projection

$\underset{\mathbf{x}}{\operatorname{argmin}} \frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T(\mathbf{x}-\boldsymbol{\mu}) = \frac{1}{2}\|\mathbf{x}-\boldsymbol{\mu}\|_2^2$ subject to the constraints $A\mathbf{x} = \mathbf{b}$

$$L = \frac{1}{2}\|\mathbf{x}-\boldsymbol{\mu}\|_2^2 - \boldsymbol{\lambda}^T(A\mathbf{x}-\mathbf{b})$$

$\frac{\partial L}{\partial \mathbf{x}} = \mathbf{x} - \boldsymbol{\mu} - A^T\boldsymbol{\lambda} = \mathbf{0} \Rightarrow \mathbf{x}^* = \boldsymbol{\mu} + A^T\boldsymbol{\lambda}$ (part of the solution in the row space of A, see figure below)

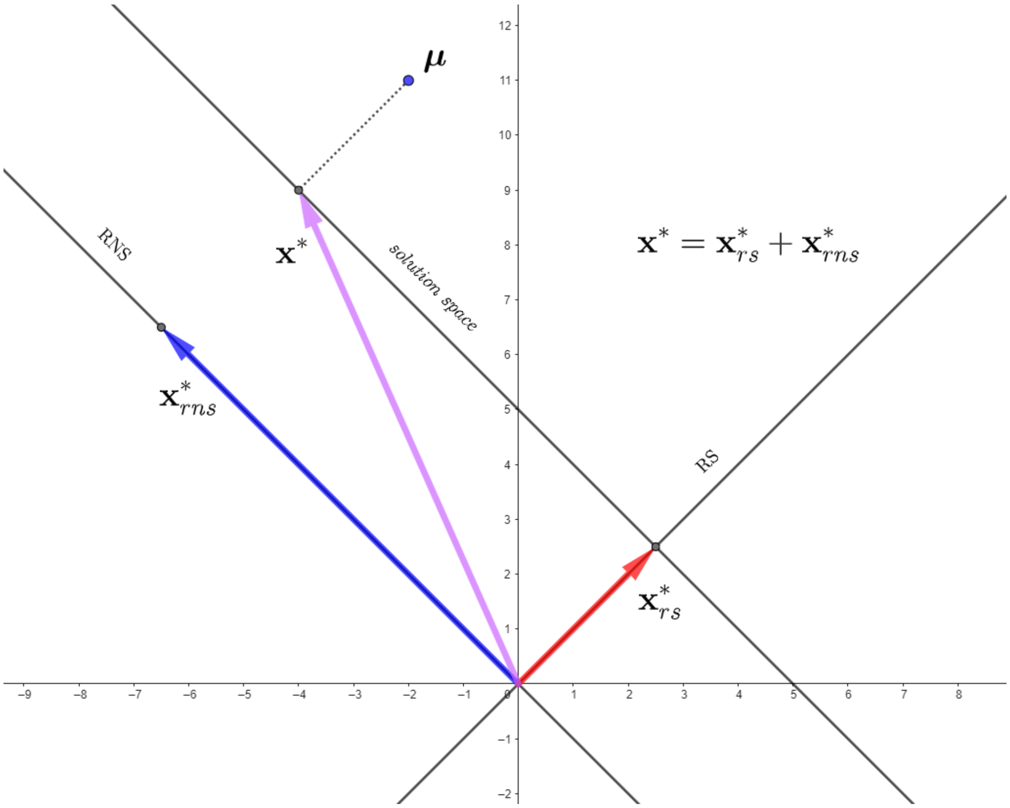
$$\frac{\partial L}{\partial \lambda} = A\mathbf{x} - \mathbf{b} = \mathbf{0} \Rightarrow A(\boldsymbol{\mu} + A^T\boldsymbol{\lambda}) - \mathbf{b} = \mathbf{0} \Rightarrow \boldsymbol{\lambda} = (AA^T)^{-1}(\mathbf{b} - A\boldsymbol{\mu})$$

$$\therefore \mathbf{x}^* = \boldsymbol{\mu} + A^T(AA^T)^{-1}(\mathbf{b} - A\boldsymbol{\mu})$$

Rearranging,

$$\mathbf{x}^* = \boldsymbol{\mu} + A^T(AA^T)^{-1}\mathbf{b} - A^T(AA^T)^{-1}A\boldsymbol{\mu}$$

$$= \underbrace{(I - A^T(AA^T)^{-1}A)\boldsymbol{\mu}}_{\mathbf{x}^*_{rns}} + \underbrace{A^T(AA^T)^{-1}\mathbf{b}}_{\mathbf{x}^*_{rs}}$$



$\boldsymbol{\mu} = \mathbf{0}$ in the first example.

Demonstrate that if $\boldsymbol{\mu}$ is in the row space, the right null space component of the solution (the point projected on to the solution space is zero) !?!?

```
A = [1 2 3; 3 4 5];
b = [1 2]';
mu = [-2 11 2]';
x1 = (eye(3)-pinv(A)*A)*mu  + pinv(A)*b
x2 = 2*(eye(3)-pinv(A)*A)*mu  + pinv(A)*b
A*x1
A*x2
norm(x1)
norm(x2)
```

Shrinkage function

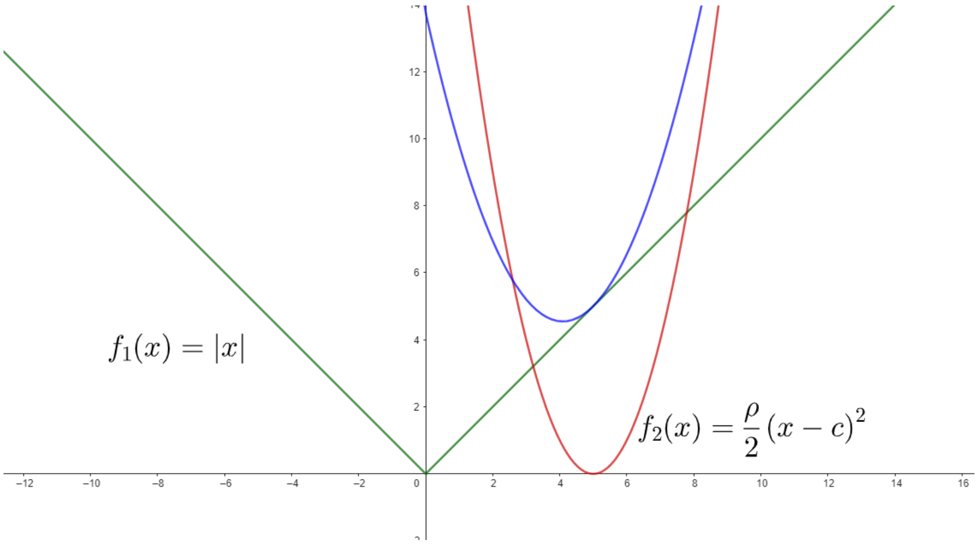
Consider the optimization problem in one variable

$$\underset{\mathbf{x}}{\operatorname{argmin}} \quad \|x\|_1 + \frac{\rho}{2}\|x - c\|_2^2 \qquad\qquad \rightarrow (1)$$

This is equivalent to finding the point where the minimum of the sum of the abs function and the quadratic function is.

$f_1(x) = |x|$

$f_2(x) = \frac{\rho}{2}(x - c)^2$



In more than one variable the equivalent form of (1) is

$$\underset{\mathbf{x}}{\operatorname{argmin}} \ \|\mathbf{x}\|_1 + \frac{\rho}{2}\|\mathbf{x} - \mathbf{c}\|_2^2$$

$$= \underset{\mathbf{x}}{\operatorname{argmin}} \ \sum_1^n |x_i| + \frac{\rho}{2}\|\mathbf{x} - \mathbf{c}\|_2^2$$

Taking the derivative at $\mathbf{x} \neq 0$

$$= \operatorname{sgn}(\mathbf{x}) + \rho(\mathbf{x} - \mathbf{c}) = 0$$

$\therefore \ \mathbf{x}^* = \mathbf{c} - \frac{1}{\rho}\operatorname{sgn}(\mathbf{x}^*)$ with shrink towards zero if $\operatorname{sgn}(\mathbf{x}) \neq \operatorname{sgn}(\mathbf{c})$

This can be rewritten as

$\mathbf{x}^* = \mathbf{c} - \frac{1}{\rho}\operatorname{sgn}(\mathbf{c})$ with shrink towards zero if $\operatorname{sgn}(\mathbf{x}) \neq \operatorname{sgn}(\mathbf{c})$

since the sign of the solution is same as that of $\mathbf{c}$.

In the case of one variable, it can be easily seen that the solution $\mathbf{x}^*$ is between 0 and c and has the same sign as c.

This result is used in the X update equation in the L_1 norm optimization using ADMM.

Examples for one variable case

Plot the graph for the following cases

Suppose $c = 3$ and $\rho = 1$

$$x^* = c - \frac{\text{sgn}(c)}{\rho} = 3 - \frac{1}{1} = 2$$

Suppose $c = 3$ and $\rho = 0.5$

$$x^* = c - \frac{\text{sgn}(c)}{\rho} = 3 - \frac{1}{0.1} = -7 \to 0$$

Suppose $c = -4$ and $\rho = 0.5$

$$x^* = c - \frac{\text{sgn}(c)}{\rho} = -4 - \frac{-1}{0.5} = -2$$

Suppose $c = -3$ and $\rho = 0.25$

$$x^* = c - \frac{\text{sgn}(c)}{\rho} = -3 - \frac{-1}{0.25} = 1 \to 0$$

Examples for multivariate case

Suppose $\mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 2 \end{bmatrix}$ and $\rho = 1$, $\mathbf{x}^* = \mathbf{c} - \frac{sgn(\mathbf{c})}{\rho} = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -4 \\ 1 \end{bmatrix}$

Suppose $\mathbf{c} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 2 \end{bmatrix}$ and $\rho = 0.5$, $\mathbf{x}^* = \mathbf{c} - \frac{sgn(\mathbf{c})}{\rho} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Suppose $\mathbf{c} = \begin{bmatrix} 4 \\ 0.5 \\ -2 \\ 5 \end{bmatrix}$ and $\rho = 0.5$, $\mathbf{x}^* = \mathbf{c} - \frac{sgn(\mathbf{c})}{\rho} = \begin{bmatrix} 4 \\ 0.5 \\ -2 \\ 5 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1.5 \\ 0 \\ 3 \end{bmatrix} \to \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}$

Basis Pursuit

(shrinkage, projection, gradient)

$\underset{\mathbf{x}}{\text{argmin}} \ \|\mathbf{x}\|_1$ subject to the constraints $A\mathbf{x} = \mathbf{b}$

In pursuit of a sparse basis

Finding the least ℓ_1 norm solution to $A\mathbf{x} = \mathbf{b}$ is called basis pursuit because the method seeks to find a small subset of columns from the matrix A (the "basis") that can be used to represent the vector b. The term "pursuit" refers to the optimization process that "pursues" a sparse solution.

ℓ_1-Norm and Sparsity

The ℓ_1-norm of a vector is the sum of the absolute values of its elements. When you minimize the ℓ_1-norm of a vector $\mathbf{x}$ subject to the constraint $A\mathbf{x} = \mathbf{b}$, you are encouraging sparse solution. A sparse solution is one where most of the elements of x are zero.

Basis Selection

The equation $A\mathbf{x} = \mathbf{b}$ can be seen as a linear combination of the columns of A, where the coefficients of the combination are the elements of $\mathbf{x}$. If $\mathbf{x}$ is sparse, it means that only a few of its elements are non-zero. These non-zero elements correspond to the columns of A that are actively used to form $\mathbf{b}$. In essence, the optimization process is selecting a small "basis" of vectors from the columns of A.

"Pursuit"

The "pursuit" part of the name refers to the algorithmic process of searching for this sparse basis among all the possible combinations of columns in A.

L_1 norm optimization problem using ADMM

Least L_1 norm solution to $A\mathbf{x} = \mathbf{b}$

$\underset{\mathbf{x}}{\operatorname{argmin}} \ \|\mathbf{x}\|_1$ **subject to the constraints** $A\mathbf{x} = \mathbf{b} \qquad \rightarrow (1)$

To convert this problem to ADMM format

Assume $f(\mathbf{x}) = \|\mathbf{x}\|_1$ and $g(\mathbf{z}) = 0$ if $A\mathbf{z} = \mathbf{b}$; ∞ if $A\mathbf{z} \neq \mathbf{b} \qquad \rightarrow (2)$ Indicator function

Write the simplified augmented lagrangian function can be written

$$L_\rho(\mathbf{x}, \mathbf{z}, \mathbf{u}) = \|\mathbf{x}\|_1 + g(\mathbf{z}) + \frac{\rho}{2}\|\mathbf{x} - \mathbf{z} + \mathbf{u}\|_2^2 \qquad \rightarrow (3)$$

Write the update equations for the variables $\mathbf{x}, \mathbf{z}$ and $\mathbf{u}$.

$$\mathbf{x}^{(k+1)} = \underset{\mathbf{x}}{\operatorname{argmin}} L_\rho(\mathbf{x}, \mathbf{z}^{(k)}, \mathbf{u}^{(k)}) = \underset{\mathbf{x}}{\operatorname{argmin}} \ \|\mathbf{x}\|_1 + \frac{\rho}{2}\|\mathbf{x} - \mathbf{z}^{(k)} + \mathbf{u}^{(k)}\|_2^2 \qquad \rightarrow (4)$$

Using shrinkage function, update $\mathbf{x}$.

The vertex of the parabola in (4) is at $\mathbf{c} = \mathbf{z}^{(k)} - \mathbf{u}^{(k)}$

The updated $\mathbf{x}$ is obtained using the shrinkage function discussed in the previous section.

$\mathbf{x}^{(k+1)} = \mathbf{c} - \frac{1}{\rho} sgn(\mathbf{c})$ and shrink towards zero

$$\mathbf{z}^{(k+1)} = \underset{\mathbf{z}}{\operatorname{argmin}} \;\; L_\rho(\mathbf{x}^{(k+1)}, \mathbf{z}, \mathbf{u}^{(k)}) = \underset{\mathbf{z}}{\operatorname{argmin}} \;\; g(\mathbf{z}) \; + \frac{\rho}{2} \|\mathbf{x}^{(k+1)} - \mathbf{z} + \mathbf{u}^{(k)}\|_2^2 \qquad \rightarrow (5)$$

Using projection of $\mathbf{x}^{(k+1)} + \mathbf{u}^{(k)}$ on to $A\mathbf{z} = \mathbf{b}$, update $\mathbf{z}$.

The updated $\mathbf{z}$ is obtained by projecting $\mathbf{x}^{(k+1)} + \mathbf{u}^{(k)}$ on $A\mathbf{z} = \mathbf{b}$.

We will use the result from the section "Projection"

$$\boldsymbol{\mu} = \mathbf{x}^{(k+1)} + \mathbf{u}^{(k)}$$

$$\mathbf{z}^{(k+1)} = (I - A^T(AA^T)^{-1}A)\boldsymbol{\mu} \;\; + A^T(AA^T)^{-1}\mathbf{b}$$

$$\mathbf{z}^{(k+1)} = (I - A^\dagger A)\boldsymbol{\mu} \;\; + A^\dagger \mathbf{b}$$

$$\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)} + (\mathbf{x}^{(k+1)} - \mathbf{z}^{(k+1)}) \qquad \rightarrow (6)$$

Using gradient, update $\mathbf{u}$.

Repeat the iteration until the desired precision is obtained.

L_1 norm optimized solution for Linear Programming

$\underset{\mathbf{x}}{\operatorname{argmin}} \;\; \|\mathbf{x}\|_1$ subject to the constraints $A\mathbf{x} = \mathbf{b} \qquad \rightarrow (1)$

Let $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2 \qquad \rightarrow (2)$

such that $\|\mathbf{x}\|_1 = \|\mathbf{x}_1\|_1 + \|\mathbf{x}_2\|_1 \qquad \rightarrow (3)$

and $\mathbf{x}_{1,2} \geq \mathbf{0} \qquad \rightarrow (4)$

$$A\mathbf{x} = A(\mathbf{x}_1 - \mathbf{x}_2) \;=\; A\mathbf{x}_1 - A\mathbf{x}_2 = [A \quad -A] \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = B\mathbf{z} = \mathbf{b} \qquad \rightarrow (5)$$

where $B = [A \quad -A] \qquad \rightarrow (6)$

and

$$\mathbf{z} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \qquad \rightarrow (7)$$

Now

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} \mathbf{x}_{1i} + \mathbf{x}_{2i} = \mathbf{e}^T\mathbf{z} \qquad \rightarrow (8)$$

where $\mathbf{e} = [1 \quad \cdots \quad 1]^T$ $\rightarrow (9)$

is a constant vector of all ones.

Now the optimization problem in (1) can be recast as a linear programming problem as

$\underset{\mathbf{z}}{\operatorname{argmin}} \quad \mathbf{e}^T\mathbf{z}$ subject to the constraints $B\mathbf{z} = \mathbf{b}$ and $\mathbf{z} \geq 0$

Linear Programming with equality constraints using ADMM

$\underset{\mathbf{x}}{\operatorname{argmin}} \quad \mathbf{c}^T\mathbf{x}$ subject to the constraints $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq 0$

Define $f(\mathbf{x}) = \mathbf{c}^T\mathbf{x}$ with $A\mathbf{x} = \mathbf{b}$

and $g(\mathbf{z}) = 0$ if $\mathbf{z} \geq 0$ and ∞ if $\mathbf{z} < 0$ (Indicator function)

The Lagrangian is

$$L(\mathbf{x}, \mathbf{z}, \boldsymbol{\lambda}) = f(\mathbf{x}) + g(\mathbf{z}) + \boldsymbol{\lambda}^{\mathbf{T}}(\mathbf{x} - \mathbf{z})$$

The augmented Lagrangian is

$$L(\mathbf{x}, \mathbf{z}, \boldsymbol{\lambda}) = f(\mathbf{x}) + g(\mathbf{z}) + \boldsymbol{\lambda}^{\mathbf{T}}(\mathbf{x} - \mathbf{z}) + \frac{\rho}{2}\|\mathbf{x} - \mathbf{z}\|_2^2$$

The simplified augmented Lagrangian function is

$L_\rho(\mathbf{x}, \mathbf{z}, \mathbf{u}) = f(\mathbf{x}) + g(\mathbf{z}) + \frac{\rho}{2}\|\mathbf{x} - \mathbf{z} + \mathbf{u}\|_2^2$ (after assuming initial values for z and u vector)

We find the minimum of this simplified augmented Lagrangian function numerically.

Write the update equations for the variables $\mathbf{x}, \mathbf{z}$ and $\mathbf{u}$.

$$\mathbf{x}^{(k+1)} = \underset{\mathbf{x}}{\operatorname{argmin}} L_\rho(\mathbf{x}, \mathbf{z}^{(k)}, \mathbf{u}^{(k)}) = \underset{\mathbf{x}}{\operatorname{argmin}} \quad f(\mathbf{x}) + \frac{\rho}{2}\|\mathbf{x} - \mathbf{z}^{(k)} + \mathbf{u}^{(k)}\|_2^2 \qquad \rightarrow (4)$$

$$L_\rho(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{c}^T\mathbf{x} + \boldsymbol{\eta}^T(A\mathbf{x} - \mathbf{b}) + \frac{\rho}{2}\|\mathbf{x} - \mathbf{z}^{(k)} + \mathbf{u}^{(k)}\|_2^2 \qquad \rightarrow (5)$$

$$\frac{\partial L_\rho}{\partial \mathbf{x}} = \mathbf{c} + A^T\boldsymbol{\eta} + \rho(\mathbf{x} - \mathbf{z}^{(k)} + \mathbf{u}^k) = \mathbf{0} \qquad \rightarrow (6)$$

$$\frac{\partial L_\rho}{\partial \boldsymbol{\lambda}} = A\mathbf{x} - \mathbf{b} = \mathbf{0} \qquad \rightarrow (7)$$

Eq:s (5) and (6) can be combined as below

$$\begin{bmatrix} \rho I_{n\times n} & A^T \\ A & 0_{m\times m} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\eta} \end{bmatrix} = \begin{bmatrix} \rho(\mathbf{z}^{(k)} - \mathbf{u}^{(k)}) - \mathbf{c} \\ \mathbf{b} \end{bmatrix} \qquad \rightarrow (8)$$

The solution to this can be found by inverting the square matrix (of size $(m+n)\times(m+n)$) on the left side of eq (8).

This gives $\mathbf{x}^{(k+1)}$. Before iterating, we can compute the inverse of this matrix and use it repeatedly as this matrix is not constant.

$$\mathbf{z}^{(k+1)} = \underset{\mathbf{x}}{\operatorname{argmin}} L_\rho(\mathbf{x}^{(k+1)}, \mathbf{z}, \mathbf{u}^{(k)}) = \underset{\mathbf{x}}{\operatorname{argmin}} \quad g(\mathbf{z}) \ + \frac{\rho}{2} \|\mathbf{x}^{(k+1)} - \mathbf{z} + \mathbf{u}^{(k)}\|_2^2 \qquad \rightarrow (9)$$

Using projection of $\mathbf{x}^{(k+1)} + \mathbf{u}^{(k)}$ on to R_+, update $\mathbf{z}$.

$$\mathbf{z}^{(k+1)} = |\mathbf{x}^{(k+1)} + \mathbf{u}^{(k)}|_{R_+}$$

$$\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)} + (\mathbf{x}^{(k+1)} - \mathbf{z}^{(k+1)}) \qquad \rightarrow (10)$$

Using gradient, update $\mathbf{u}$.

Mapping LP with inequality constraints to LP with equality constraints

$\underset{\mathbf{y}}{\operatorname{argmin}} \quad \mathbf{c}^T\mathbf{y}$ subject to the constraints $M\mathbf{y} \leq \mathbf{b}$ and $\mathbf{y} \geq \mathbf{0}$

where $\mathbf{c}, \mathbf{y} \in R^n$, $\mathbf{b} \in R^m$ and $M \in R^{m\times n}$

To remove the inequality we introduce a slack variable $\mathbf{u} \in R^m$ where $\mathbf{u} \geq \mathbf{0}$.

$$M\mathbf{y} + I\mathbf{u} = \mathbf{b}$$

$$[M \quad I]\begin{bmatrix}\mathbf{y}\\ \mathbf{u}\end{bmatrix} = \mathbf{b}$$

The objective function and the constraints are transformed as below.

The m (equations) constraints in n unknowns (initially $\mathbf{y}$) is changed to m constraints in $n+m$ unknowns $\mathbf{x} = \begin{bmatrix}\mathbf{y}\\ \mathbf{u}\end{bmatrix}$

$$\mathbf{y} \in R^n \ \rightarrow \ \mathbf{x} = \begin{bmatrix}\mathbf{y}\\ \mathbf{u}\end{bmatrix} \in R^{m+n}, \ \mathbf{y}, \mathbf{u} \geq \mathbf{0}$$

The corresponding coefficient matrix M is changed to $A = \begin{bmatrix}M_{m\times n} & I_{m\times m}\end{bmatrix}$

$$M \in R^{m\times n} \ \rightarrow \ A = \begin{bmatrix}M_{m\times n} & I_{m\times m}\end{bmatrix} \in R^{m\times(m+n)}$$

The constant vector $\mathbf{c}$ is changed to $\mathbf{a} = \begin{bmatrix}\mathbf{c}_{n\times 1}\\ \mathbf{0}_{m\times 1}\end{bmatrix}$

The new optimization problem is

$\underset{\mathbf{x}}{\operatorname{argmin}} \ \mathbf{a}^T\mathbf{x}$ subject to the constraints $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$

Least Absolute Deviation (LAD)

$\underset{\mathbf{x},\mathbf{z}}{\operatorname{argmin}} \ \|\mathbf{z}\|_1$ subject to the constraints $A\mathbf{x} - \mathbf{b} = \mathbf{z}$

Note the constraint here is not $\mathbf{x} - \mathbf{z} = \mathbf{0}$, Instead it is $A\mathbf{x} - \mathbf{b} = \mathbf{z}$

The Lagrangian is

$$L = \|\mathbf{z}\|_1 + \boldsymbol{\lambda}^T(A\mathbf{x} - \mathbf{b} - \mathbf{z}^{(k)}) + \frac{\rho}{2}\|A\mathbf{x} - \mathbf{b} - \mathbf{z}^{(k)}\|_2^2$$

The update equation for $\mathbf{x}, \mathbf{z}$ and $\boldsymbol{\lambda}$ are :

$$\mathbf{x}^{(k+1)} = \underset{\mathbf{x}}{\operatorname{argmin}} \ L(\mathbf{x}, \mathbf{z}^{(k)}, \boldsymbol{\lambda}^{(k)})$$

$$\Rightarrow A^T\boldsymbol{\lambda}^{(k)} + \rho A^T(A\mathbf{x} - \mathbf{b} - \mathbf{z}^{(k)}) = \mathbf{0}$$

$$\Rightarrow \rho A^T A\mathbf{x} = \rho A^T(\mathbf{b} + \mathbf{z}^{(k)}) - A^T\boldsymbol{\lambda}^{(k)}$$

$$\therefore \mathbf{x}^{(k+1)} = (A^T A)^{-1}\left(A^T(\mathbf{b} + \mathbf{z}^{(k)}) - \frac{A^T\boldsymbol{\lambda}^{(k)}}{\rho}\right)$$

$$\mathbf{z}^{(k+1)} = \underset{\mathbf{z}}{\operatorname{argmin}} \ L(\mathbf{x}^{(k+1)}, \mathbf{z}, \boldsymbol{\lambda}^{(k)})$$

$$= \underset{\mathbf{z}}{\operatorname{argmin}} \ \|\mathbf{z}\|_1 + \boldsymbol{\lambda}^{(k)T}(A\mathbf{x}^{(k+1)} - \mathbf{b} - \mathbf{z}) + \frac{\rho}{2}\|A\mathbf{x}^{(k+1)} - \mathbf{b} - \mathbf{z}\|_2^2$$

Since this Lagrangian function involves the sum of absolute function and a quadratic function,

the solution can be obtained by using the shrinkage function discussed above. Find the vertex of the quadratic term by simplyfing the lagrangian and the augmented lagrangian terms.

$$\boldsymbol{\lambda}^{(k+1)} = \boldsymbol{\lambda}^{(k)} + (A\mathbf{x}^{(k+1)} - \mathbf{b} - \mathbf{z}^{(k+1)})$$

$\underset{\mathbf{x},\mathbf{z}}{\operatorname{argmin}} \|\mathbf{z}\|_2^2$ subject to the constraints $A\mathbf{x} - \mathbf{b} = \mathbf{z}$, $\mathbf{z}$ is the error vector.

$\underset{\mathbf{x},\mathbf{z}}{\operatorname{argmin}} \|\mathbf{z}\|_1$ subject to the constraints $A\mathbf{x} - \mathbf{b} = \mathbf{z}$, $\mathbf{z}$ is the error vector.

Exercise

$\underset{\mathbf{x}}{\operatorname{argmin}} \ \|\mathbf{x}\|_1$ subject to the constraint $2x_1 + 3x_2 = 4$

$\underset{\mathbf{x}}{\operatorname{argmin}} \ \|\mathbf{x}\|_1$ subject to the constraints $2x_1 + 3x_2 - x_3 = 4$ and $2x_1 - 3x_2 - x_3 = 1$

Coding exercises - 09-July-2025

Write Matlab/Octave function to compute

1. L_0, L_1, L_2, L_∞ norm of given vectors
2. Shrinkage function
3. Projection
4. Least L_1 norm solution to $A\mathbf{x} = \mathbf{b}$ using ADMM
5. Least L_2 norm solution to $A\mathbf{x} = \mathbf{b}$
6. closest point to origin on the plane $A\mathbf{x} = \mathbf{b}$
7. closest point to $\boldsymbol{\mu}$ on the plane $A\mathbf{x} = \mathbf{b}$
8. LP with equality

function name : L1norm

input : a vector

output : L_1 norm of the input vector

Logic : sum of absolute values of the components of the input vector

function name : L2norm

input : a vector

output : L_2 norm of the input vector

function name : L0norm

input : a vector

output : L_0 norm of the vector

```
function result = L1norm(x)
    result = sum(abs(x));
end

function result = L2norm(x)
    result = sqrt(sum(x.^2));
end

function result = L0norm(x)
    result = sum(abs(x)>0);
```

```
end

function result = LInfnorm(x)
    result = max(abs(x));
end
```

```
x=[-3 4 0]
L0norm(x)
L1norm(x)
L2norm(x)
LInfnorm(x)
```

References

https://en.wikipedia.org/wiki/Augmented_Lagrangian_method#Alternating_direction_method_of_multipliers

https://en.wikipedia.org/wiki/Quadratic_programming

https://en.wikipedia.org/wiki/Total_variation_denoising

https://developers.google.com/optimization/introduction

https://developers.google.com/optimization/introduction


```
cd("/media/user/DATA4LINUX/new1/Repos/Mine/MFC4_22MAT230/")
mlxfile = matlab.desktop.editor.getActive().Filename;
outfile = mlxfile + ".pdf"
export(matlab.desktop.editor.getActive().Filename, outfile);
```