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Math 516
Homework 2

I collaborated with Damien Huet on this assignment

1. (a) Let $\xi \in \partial\Omega$ and $w(x)$ be a barrier on $\Omega_1 \subset\subset \Omega$ with w superharmonic in Ω_1, $w > 0$ in $\overline{\Omega}_1 \setminus \{\xi\}$, $w(\xi) = 0$. Show that w can be extended to a barrier in Ω.

(note w is a local barrier in Ω_1, and $\zeta \in \Omega_1$. So essentially this is straight from the notes for Perron's Method, though with more detail about why the harmonic function of any ball is at most the superharmonic function in the intersection of the ball and a small ball around ξ)

Choose $r > 0$ s.t. $B = B_r(\xi) \subset\subset \Omega_1$, and define $m = \inf_{\Omega_1 \setminus B}$. Then the function

$$W(x) = \begin{cases} w_1(x) = \min(m, w(x)) & x \in B \cap \overline{\Omega} \\ w_2(x) = m & \text{otherwise} \end{cases}$$

This is a barrier at ξ in Ω. First we show that W is continuous in Ω. Since the minimum of two continuous functions is necessarily continuous, w_1 is continuous in $B \cap \overline{\Omega}$. W is also continuous in $B^c \cap \overline{\Omega}$, as it's just constant there. Finally, we note $\lim_{y \to z \in \partial B \cap \overline{\Omega}} W(y) = m$, since $\sup_{x \in B \cap \overline{\Omega}} w_1(x) = m$. If not, then $\exists x \in \partial B$ s.t. $w_1(x) < m$. But this contradicts the definition of m, so this is impossible, and thus, W is continuous on Ω.

Next we need to show that W is superharmonic in Ω. First, we note that w_1 and w_2 are superharmonic in their domains; w_1 by definition, and w_2 because it's just a constant. Thus, to show that W is superharmonic in Ω, we only need to consider every ball B' which intersects the boundary of B, and show that any harmonic function h in B' with $h \leq w$ on $\partial B'$ will have $h \leq w$ in B' as well.

For $x \in B' \cap B^c \cap \Omega$, $W(x) = m = \sup_{y \in \Omega} W(y)$. By the maximum priniple, since on B' $h \leq m$, we have $h \leq m$. Thus, $h \leq W(x)$ for $x \in B' \cap B^c \cap \Omega$.

For $x \in B' \cap B \cap \Omega$, since h is harmonic, $h \leq W(x)$ on $\partial B' \cap B \cap \Omega$, and w_1 is superharmonic in $B' \cap B \cap \Omega$, we have $h \leq W(x)$ in $B' \cap B \cap \Omega$. On $\partial B' \cap B$, we have $h \leq W$, as $h \leq W$ on $\partial B'$ and $h \leq m = W(y)$ for y in $B^c \cap \Omega$. Then if we lift (or I guess lower?) W to a harmonic function g on B, $h \leq g \leq W$ on $B' \cap B \cap \Omega$. This follows from Lemma 2.2.3 from the notes on Perron's method.

Thus, W is superharmonic in Ω. Since $W(\xi) = w_1(\xi) = 0 \leq m$, $W(y) = m > 0$ on $\partial\Omega \setminus B$, and $W(y) = w_1(y) > 0$ on $\partial\Omega \cap B$, W fulfills all of the requirements to be a barrier at ξ relative to Ω.

(b) Let $\Omega = \{x^2 + y^2 < 1\} \setminus \{-1 \leq x \leq 0, y = 0\}$. Show that the function $w := -Re\left(\frac{1}{\ln(z)}\right) = -\frac{\log(r)}{(\log(r))^2 + \theta^2}$ is a local barrier at $\xi = 0$.

First we need to show that there is a neighborhood N s.t. for all $B \subset\subset N$ and all harmonic h in B with $h(x) \leq w(x)$ for $x \in \partial B$, we have $w \geq h$ for $x \in B$.

This can be shown with complex analysis. First note that since Ω is simply connected and excludes 0, the logarithm exists on it, and is holomorphic. Next define $W(z) = \frac{1}{\ln(z)}$. W is a composition of holomorphic functions on Ω, and thus, its real and imaginary parts are harmonic. Note that $w(x, y) = Re(W(x+iy))$ shows that w is harmonic on Ω. From the notes on superharmonic functions, we know that harmonic functions are always superharmonic functions. Therefore, we can take any neighborhood $N \subset\subset \Omega$ we want and w will be superharmonic on it.

Next we need to show that in the closure of such a neighborhood, $w(x, y) > 0$ when $x^2 + y^2 \neq 0$.

Since $0 < r^2 = x^2 + y^2 < 1$, $\log(r) < 0$. Then since $(\log(r))^2 + \theta^2 > 0$, $-\frac{\log(r)}{(log(r))^2 + \theta^2} > 0$. Since this is true for Ω, it's true for N as well.

Finally we note that $0 \leq \lim\limits_{r \to 0+} \left| \frac{\ln(r)}{(\ln(r))^2 + \theta^2} \right| \leq \lim\limits_{r \to 0+} \frac{1}{|\ln(r)|} = 0$, so we must have $w(0) = 0$ for this to be continuous.

Thus, w is a local barrier for $\xi = 0$ in Ω.

2. (a) Show that the problem of minimizing energy

$$I[u] = \int_J x^2 |u'(x)|^2 \, \mathrm{d}x$$

for $u \in C(\overline{J})$ with piecewise continuous derivatives in $J := (-1, 1)$, satisfying the boundary conditions $u(-1) = 0$, $u(1) = 1$ is not attained.

Consider

$$u_k(x) = \begin{cases} 0 & -1 \le x \le -\frac{1}{k} \\ \frac{k^2}{2}x^2 + kx + \frac{1}{2} & -\frac{1}{k} \le x \le 0 \\ -\frac{k^2}{2}x^2 + kx + \frac{1}{2} & 0 \le x \le \frac{1}{k} \\ 1 & \frac{1}{k} \le x \le 1 \end{cases}$$

u_k has piecewise continuous derivatives, is 0 at -1 and 1 at 1. Furthermore, $I[u_k] \le \frac{C}{k}$ for some positive C indepenedent of k, and thus, $\lim\limits_{k\to\infty} I[u_k] = 0$ is our minimal energy. However, $\lim\limits_{k\to\infty} u_k$ does not have piecewise continuous derivatives though, so it's not in the admissible set. Thus, $\inf_u I[u] = 0$, and now we must show it's not actually attainable.

By the intermediate value theorem, since $u(-1) = 0$ and $u(1) = 1$, there is some $y \in (-1, 1)$ s.t. $u'(y) = \frac{1}{2}$. Then since $u'(x)$ is piecewise continuous, we can choose some $\delta > 0$ s.t. $0 < \epsilon < u'(x)$ whenever $|x - y| < \delta$. Then $\int\limits_{-1}^{1} x^2 u'(x)^2 \, \mathrm{d}x \ge \int\limits_{y-\delta}^{y+\delta} x^2 \epsilon^2 \, \mathrm{d}x > 0$. Thus, since the optimal energy is 0, and any element in the admissible set has non-zero energy, the minimizer is not attained.

(b) Consider the problem of minimizing the energy

$$I[u] = \int_0^1 (1 + |u'(x)|^2)^{\frac{1}{4}} \, \mathrm{d}x$$

for all $u \in C^1((0, 1)) \cap C([0, 1])$ satisfying $u(0) = 0, u(1) = 1$. Show that the minimum is 1 and is not attained.

Consider

$$u_k(x) = \begin{cases} -k^2x^2 + 2kx & 0 \le x \le \frac{1}{k} \\ 1 & \frac{1}{k} \le x \le 1 \end{cases}$$

u_k has piecewise continuous derivatives, is 0 at 0 and 1 at 1. Furthermore, $I[u_k] \le \frac{k-1}{k} + \frac{(1+4k^2)^{1/4}}{k}$, and thus, $\lim\limits_{k\to\infty} I[u_k] = 1$ is our minimal energy. However, $\lim\limits_{k\to\infty} u_k$ does not have piecewise continuous derivatives though, so it's not in the admissible set. Thus, $\inf_u I[u] = 1$, and now we must show it's not actually attainable; this is largely the same process as in 2a.

By the intermediate value theorem, since $u(0) = 0$ and $u(1) = 1$, there is some $y \in (0, 1)$ s.t. $u'(y) = 1$. Then since $u'(x)$ is piecewise continuous, we can choose some $\delta > 0$ s.t. $0 < \epsilon < u'(x)$ whenever $|x - y| < \delta$. Then

$$\int\limits_0^1 (1 + u'(x)^2)^{1/4} \, \mathrm{d}x \ge \int\limits_{y-\delta}^{y+\delta} (1 + \epsilon^2)^{1/4} \, \mathrm{d}x + 1 - 2\delta = 1 - 2\delta + 2\delta(1 + \epsilon^2)^{1/4} > 1$$

Thus, since the optimal energy is 1, and any element in the admissible set has energy greater than 1, the minimizer is not attained.

3. Discuss the Dirichlet Principle for
$$\begin{cases} \Delta^2 u(x) = f(x) & x \in \Omega \\ u(x) = \frac{\partial u}{\partial \nu}(x) = 0 & x \in \partial\Omega \end{cases}$$

Dirichlet's principle for $\Delta u = f(x)$ says that a function u that satisifes
$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$
will minimize
$$I[w] = \int_\Omega \frac{1}{2}|Dw|^2 - wf dx$$
for the admissible set $A = \{w \in C^1(\Omega) \cap C^0(\overline{\Omega}, w = g \text{ on } \partial\Omega\}$

We guess that the energy functional
$$E[w] = \int_\Omega 0 \, \mathrm{d}x = \int_\Omega \frac{1}{2}|\Delta w|^2 - \Delta w f \, \mathrm{d}x$$
is minimized by a solution to our problem with the admissible set
$$A = \{w \in C^1(\Omega) \cap C^0(\overline{\Omega}), w = g \text{ on } \partial\Omega\}$$
Next, assume u satisifies the problem statement. Then we consider the integral below:
$$0 = \int_\Omega (-\Delta^2 u - f)(\Delta u - \Delta w) \, \mathrm{d}x$$
Integrating by parts gives and noting that $\frac{\partial \Delta u}{\partial \nu} = \Delta \frac{\partial u}{\partial \nu} = 0$ gives
$$\begin{aligned} 0 &= \int_\Omega D(\Delta u) \cdot D(\Delta u - \Delta w) - (\Delta u - \Delta w) f \, \mathrm{d}x \\ &= -\int_\Omega D(\Delta u) \cdot D(\Delta u - \Delta w) - (\Delta u - \Delta w) f \, \mathrm{d}x + \int_{\partial\Omega} (\Delta u - \Delta w) \frac{\partial \Delta u}{\partial \nu} \, \mathrm{d}S \\ &= -\int_\Omega D(\Delta u) \cdot D(\Delta u - \Delta w) - (\Delta u - \Delta w) f \, \mathrm{d}x \end{aligned}$$
Then, rearranging terms and applying Cauchy Schwarz gives
$$\begin{aligned} \int_\Omega (D\Delta u) \cdot (D\Delta w) - \Delta w f \, \mathrm{d}x &= \int_\Omega |D(\Delta u)|^2 - \Delta u f \, \mathrm{d}x \\ &\leq \int_\Omega \frac{1}{2}(|D(\Delta u)|^2 + |D(\Delta w)|^2) - \Delta w f \, \mathrm{d}x \end{aligned}$$
Thus, $E[u] \leq E[w]$ when $w \in A$.

4. Let
$$\Phi(x - y, t) = (4\pi t)^{-n/2} e^{-\frac{|x-y|^2}{4t}}$$

(a) Show that there exists a generic constant C_n s.t.

$$\Phi(x-y,t) \leq C_n|x-y|^{-n}$$

(Hint: Maximize the function in t)
Following the hint, we compute

$$0 = \frac{\mathrm{d}\Phi(r,t)}{\mathrm{d}t} = -\frac{n}{2}(4\pi t)^{-\frac{n}{2}-1}4\pi e^{-\frac{r^2}{4t}} + (4\pi t)^{-n/2}e^{-\frac{r^2}{4t}}\left(\frac{r^2}{4t^2}\right)$$

Then, after some simplification we get

$$\frac{n}{2}(4\pi t)^{-\frac{n}{2}-1}4\pi = (4\pi t)^{-n/2}\left(\frac{r^2}{4t^2}\right)$$

which leaves us with just

$$t_{crit} = \frac{r^2}{2n}$$

Then for this to be a local maximum, we need $\frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2}(r,t_{crit}) < 0$. After some computation, we arrive at

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2}(r,t_{crit}) = -\frac{2n^3\left(\frac{2\pi r}{n}\right)^{-n/2}e^{-n/2}}{r^4} < 0$$

Thus, this is a local maximum. Then

$$\Phi(r,t) \leq \Phi(r,t_{crit}) = \left(r\sqrt{\frac{2\pi}{n}}\right)^{-n} e^{-n/2} \leq Cr^{-n}$$

for some C independent of R.

(b) Let $n=1$ and $f(x)$ be a bounded measurable function s.t. $f(x_0-)$ and $f(x_0+)$ exists. Show that

$$\lim_{t\to 0}\int_R \Phi(x_0-y,t)f(y)\,\mathrm{d}y = \frac{1}{2}(f(x_0-)+f(x_0+))$$

f is bounded and measurable. This is essentially a generalization of the proof for Theorem 2.3.1 in the book.
First define $u(x,t) = \int_{\mathbb{R}}\Phi(x_0-y,t)f(y)\,\mathrm{d}y$. Then consider $u(x,t) - \frac{1}{2}f(x_0+) - \frac{1}{2}f(x_0-)$ Since $f(x_0-)$ and $f(x_0+)$ exists, given any $\epsilon > 0$, we can choose $\delta > 0$ s.t. for $|y-x_0| < \delta$, we have

$$\begin{cases} |f(y) - f(x_0+)| < \epsilon & y < x_0 \\ |f(y) - f(x_0-)| < \epsilon & y > x_0 \end{cases}$$

Now we write

$$\begin{aligned}
&u(x_0,t) - \frac{1}{2}f(x_0+) - \frac{1}{2}f(x_0-)\\
&\quad= \int_{\mathbb{R}} \Phi(x_0-y,t)\left[f(y) - \frac{1}{2}f(x_0+) - \frac{1}{2}f(x_0-)\right]\,\mathrm{d}y\\
&\quad= \int_{B_\delta(x_0)} \Phi(x_0-y,t)\left[f(y) - \frac{1}{2}f(x_0+) - \frac{1}{2}f(x_0-)\right]\,\mathrm{d}y\\
&\qquad+ \int_{B_\delta(x_0)^c} \Phi(x_0-y,t)\left[f(y) - \frac{1}{2}f(x_0+) - \frac{1}{2}f(x_0-)\right]\,\mathrm{d}y\\
&\quad= I_t + J_t
\end{aligned}$$

$\lim\limits_{t\to 0} J_t = 0$ as shown in the book, as $f(y) - \frac{1}{2}f(x_0+) - \frac{1}{2}f(x_0+) < C$ since f is bounded, letting us bound $|J_t| < \frac{C}{\sqrt{t}}\int_{B_\delta(x_0)^c} e^{-\frac{|x_0-y|^2}{4t}}\,\mathrm{d}y \leq C\int_{B_{(\delta/\sqrt{t})}(x_0)^c} e^{-\frac{|\tilde{x}_0-\tilde{y}|^2}{16}}\,\mathrm{d}\tilde{y}$
Thus we only need to concern ourselves with $\lim\limits_{t\to 0} I_t$.

$$\begin{aligned}
I_t &= \int_{B_\delta(x_0)} \Phi(x_0-y,t)\left[f(y) - \frac{1}{2}f(x_0+) - \frac{1}{2}f(x_0-)\right]\,\mathrm{d}y\\
&= \int_{x_0-\delta}^{x_0} \Phi(x_0-y,t)\left[f(y) - \frac{1}{2}f(x_0+) - \frac{1}{2}f(x_0-)\right]\,\mathrm{d}y\\
&\quad+ \int_{x_0}^{x_0+\delta} \Phi(x_0-y,t)\left[f(y) - \frac{1}{2}f(x_0+) - \frac{1}{2}f(x_0-)\right]\,\mathrm{d}y\\
&= -\int_0^\delta \Phi(x_0-(x_0-z),t)\left[f(x_0-z) - \frac{1}{2}f(x_0+) - \frac{1}{2}f(x_0-)\right]\,\mathrm{d}z\\
&\quad+ \int_0^\delta \Phi(x_0-(x_0+z),t)\left[f(x_0+z) - \frac{1}{2}f(x_0+) - \frac{1}{2}f(x_0-)\right]\,\mathrm{d}z\\
&= -\int_0^\delta \Phi(z,t)\left[f(x_0-z) - \frac{1}{2}f(x_0+) - \frac{1}{2}f(x_0-)\right]\,\mathrm{d}z\\
&\quad+ \int_0^\delta \Phi(-z,t)\left[f(x_0+z) - \frac{1}{2}f(x_0+) - \frac{1}{2}f(x_0-)\right]\,\mathrm{d}z\\
&= \int_0^\delta \Phi(z,t)\left[f(x_0+z) - \frac{1}{2}f(x_0+) - \frac{1}{2}f(x_0-) - f(x_0-z) + \frac{1}{2}f(x_0+) + \frac{1}{2}f(x_0-)\right]\,\mathrm{d}z
\end{aligned}$$

Then we can bound $|I_t|$ by noting that $|f(x_0+z)-f(x_0-z)| < |f(x_0+z)-f(x_0)|+|f(x_0)-f(x_0-z)| \le 2\epsilon$.

$$\begin{aligned}|I_t| \le& \int_0^\delta |\Phi(z,t)\left[f(x_0+z)-f(x_0-z)\right]| \, \mathrm{d}z \\ \le& \int_0^\delta \Phi(z,t)\,|f(x_0+z)-f(x_0-z)| \, \mathrm{d}z \\ \le& \int_0^\delta \Phi(z,t) 2\epsilon \, \mathrm{d}z \\ =& 2\epsilon\end{aligned}$$

Thus, $\lim_{t\to 0}\left|u(x_0,t) - \frac{f(x_0+)+f(x_0-)}{2}\right| \le 3\epsilon$, proving our claim.

(c) Let u satisfy

$$\begin{cases} u_t = \Delta u & x \in \mathbb{R}^n, t > 0 \\ u(x,0) = f(x) \end{cases}$$

Suppose that f is continuous and has compact support. Show that $\lim_{t\to+\infty} u(x,t) = 0$ for all x

First, note that since f has compact support, $\exists R > 0$ s.t. $f(B_R^c) = \{0\}$. Next, consider some positive $T >> t$ and an associated $R' >> R$. On $B_{R'} \times [0,T]$, the heat equation has a unique solution if we assume that it's between $-\epsilon$ and ϵ on the boundary of $B_{R'}(0)$ for $0 < t < T$. (n.b. This makes sense from a physical point of view; I haven't come up with a mathematical justification yet. A potentially just as bad alternative is to assume that our solution is bounded by an exponential growth estimate, but that also appears to rely on a set value of T in addition to bringing in the assumption on growth rates.)

Next, since the fundamental solution of the heat equation gives us a solution on the domain, it is the solution. That is, we can write $u(x,t) = \int_{B_{R'}(0)} \Phi(x-y,t)f(y)\, \mathrm{d}y$. Since f has compact support, we can reduce this to $u(x,t) = \int_{B_R} \Phi(x-y,t)f(y)\, \mathrm{d}y$. We can also interchange the integral and a limit (if we ignore that we chose R' based on T so the boundary conditions hold...), so

$$\lim_{t\to\infty} u(x,t) = \int_{B_R(0)} \lim_{t\to\infty} \Phi(x-y,t)f(y)\, \mathrm{d}y = \int_{B_R(0)} 0 \, \mathrm{d}y = 0$$

5. Derive a solution formula for

$$\begin{cases} u_t = \Delta u + cu + f(x,t) & t > 0 \\ u(x,0) = g(x) \end{cases}$$

We apply the Fourier transform to everything, (assuming that the transform exists for u, f, and g), with $\hat{u} = F[u], \hat{f} = F[f], \hat{g} = F[g]$. Then we note that $F[u_t] = \frac{\mathrm{d}}{\mathrm{d}t}F[u]$ and $F[\Delta u] = -|\omega|^2\hat{u}$. This gives us the following conditions:

$$\begin{cases} \hat{u}_t + |\omega|^2\hat{u} - c\hat{u} + \hat{f}(x,t) & t > 0 \\ \hat{u}(x,0) = \hat{g}(x) \end{cases}$$

which is an ODE.

Using the integrating factor $\hat{I}(\omega, t) = \exp((|\omega|^2 - c)t)$, we compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[\hat{I}(\omega.t)\hat{u}(\omega,t)\right] =& \hat{I}(\omega,t)\hat{f}(\omega,t)\\ \hat{I}(\omega,t)\hat{u}(\omega,t) =& \hat{g}(\omega) + \int_0^t \hat{I}(\omega,\tau)\hat{f}(\omega,\tau)\,\mathrm{d}\tau\end{aligned}$$

or

$$\hat{u} = \hat{g}(\omega)e^{(c-|\omega|^2)t} + e^{(c-|\omega|^2)t}\int_0^t e^{(|\omega|^2-c)\tau}\hat{f}(\omega,\tau)\,\mathrm{d}\tau$$

Then

$$\begin{aligned}u(x,t) =& \frac{1}{(2\pi)^{n/2}}\left[\int_{\mathbb{R}^n}\hat{g}(\omega)e^{(c-|\omega|^2)t}e^{i\omega\cdot x}\,\mathrm{d}\omega + \int_{\mathbb{R}^n} e^{i\omega\cdot x}e^{(c-|\omega|^2)t}\left(\int_0^t e^{(|\omega|^2-c)\tau}\hat{f}(\omega,\tau)\,\mathrm{d}\tau\right)\,\mathrm{d}\omega\right]\\ =& \frac{1}{(2\pi)^{n/2}}\left[G+F\right]\end{aligned}$$

We handle the terms separately, substituting in the definition of the Fourier transform and making use of Fubini's theorem to interchange integrals, since the terms are all integrable over $\mathbb{R}^n$.

$$\begin{aligned}G =& \int_{\omega\in\mathbb{R}^n}\int_{y\in\mathbb{R}^n} e^{(c-|\omega|^2)t+i\omega\cdot x-i\omega\cdot y}g(y)\,\mathrm{d}y\,\mathrm{d}\omega\\ =& \int_{y\in\mathbb{R}^n}\int_{\omega\in\mathbb{R}^n} e^{(c-|\omega|^2)t+i\omega\cdot(x-y)}g(y)\,\mathrm{d}\omega\,\mathrm{d}y\\ =& e^{ct}\int_{y\in\mathbb{R}^n}\int_{\omega\in\mathbb{R}^n} e^{-|\omega|^2t+i\omega\cdot(x-y)}g(y)\,\mathrm{d}\omega\,\mathrm{d}y\\ =& e^{ct}\int_{y\in\mathbb{R}^n}\int_{\omega\in\mathbb{R}^n} e^{-\left|\omega-\frac{i(x-y)}{2t}\right|^2t-\frac{|x-y|^2}{4t}}g(y)\,\mathrm{d}\omega\,\mathrm{d}y\end{aligned}$$

Note that after integration over a single term, $e^{-|\omega-\frac{i(x-y)}{2t}|^2}$ results in a constant independent of the other terms. So, substituting $z = \omega - \frac{i(x-y)}{2t}$ with $\mathrm{d}z = \mathrm{d}\omega$ gives

$$\begin{aligned}G =& e^{ct}\int_{y\in\mathbb{R}^n}\int_{z\in\mathbb{R}^n} e^{-|z|^2t}\,\mathrm{d}z\,e^{-\frac{|x-y|^2}{4t}}g(y)\,\mathrm{d}y\\ =& e^{ct}(2t)^{-n/2}\int_{y\in\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}g(y)\,\mathrm{d}y\\ (2\pi)^{-n/2}G =& e^{ct}(4\pi t)^{-n/2}\int_{y\in\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}g(y)\,\mathrm{d}y\end{aligned}$$

Next we compute the second term, F using similar tricks

$$\begin{aligned}
F &= \int_{\mathbb{R}^n} e^{i\omega\cdot x}\left(\int_0^t e^{(|\omega|^2-c)(\tau-t)}\hat{f}(\omega,\tau)\ \mathrm{d}\tau\right)\ \mathrm{d}\omega \\
&=(2\pi)^{-n/2}\int_{\omega\in\mathbb{R}^n}\int_{\tau=0}^t\int_{y\in\mathbb{R}^n} e^{i\omega\cdot x+(|\omega|^2-c)(\tau-t)-i\omega\cdot y}f(y,\tau)\ \mathrm{d}y\ \mathrm{d}\tau\ \mathrm{d}\omega \\
&=(2\pi)^{-n/2}\int_{\tau=0}^t e^{c(t-\tau)}\int_{\omega\in\mathbb{R}^n}\int_{y\in\mathbb{R}^n} e^{i\omega\cdot(x-y)-|\omega|^2(t-\tau)}f(y,\tau)\ \mathrm{d}y\ \mathrm{d}\omega\ \mathrm{d}\tau \\
&=(2\pi)^{-n/2}\int_{\tau=0}^t e^{c(t-\tau)}\int_{\omega\in\mathbb{R}^n}\int_{y\in\mathbb{R}^n} e^{-\frac{1}{t-\tau}\left|\omega-\frac{i}{2}(x-y)\right|^2}\cdot e^{\frac{|x-y|^2}{4(t-\tau)}}f(y,\tau)\ \mathrm{d}y\ \mathrm{d}\omega\ \mathrm{d}\tau \\
&=(2\pi)^{-n/2}\int_{\tau=0}^t e^{c(t-\tau)}\int_{y\in\mathbb{R}^n} e^{\frac{|x-y|^2}{4(t-\tau)}}f(y,\tau)\int_{\omega\in\mathbb{R}^n} e^{-\frac{1}{t-\tau}\left|\omega-\frac{i}{2}\frac{x-y}{t-\tau}\right|^2}\ \mathrm{d}\omega\ \mathrm{d}y\ \mathrm{d}\tau
\end{aligned}$$

Substituting $z=\omega-\frac{i(x-y)}{2(t-\tau)}$, with $\mathrm{d}z=\mathrm{d}\omega$ gives

$$\begin{aligned}
F =&(2\pi)^{-n/2}\int_{\tau=0}^t e^{c(t-\tau)}\int_{y\in\mathbb{R}^n} e^{\frac{|x-y|^2}{4(t-\tau)}}f(y,\tau)\int_{\omega\in\mathbb{R}^n} e^{-\frac{|z|^2}{t-\tau}}\ \mathrm{d}z\ \mathrm{d}y\ \mathrm{d}\tau \\
=&\int_{\tau=0}^t (2(t-\tau))^{-n/2}e^{c(t-\tau)}\int_{y\in\mathbb{R}^n} e^{\frac{|x-y|^2}{4(t-\tau)}}f(y,\tau)\ \mathrm{d}y\ \mathrm{d}\tau \\
\frac{F}{(2\pi)^{n/2}} =&\int_{\tau=0}^t (4\pi(t-\tau))^{-n/2}e^{c(t-\tau)}\int_{y\in\mathbb{R}^n} e^{\frac{|x-y|^2}{4(t-\tau)}}f(y,\tau)\ \mathrm{d}y\ \mathrm{d}\tau
\end{aligned}$$

We define the fundamental solution Φ:

$$\Phi(x,t)=(4\pi t)^{-n/2}e^{ct}e^{\frac{|x|^2}{4t}}$$

and conclude

$$u(x,t)=(\Phi*g)(x,t)+\int_0^t(\Phi*f)(x,\tau)\ \mathrm{d}\tau$$

6. Consider the following general parabolic equation

$$L[u]=a(x,t)u_{xx}+b(x,t)u_x+c(x,t)u-u_t$$

where

$$0<C_1<a(x,t)<C_2,|b(x,t)|\le C_3,c(x,t)\le C_4$$

(a) Show that if $L[u]\ge 0$, then

$$\max_{\overline{\Omega}_T}u\le e^{C_4T}\max_{\partial'\Omega_T}u^+$$

Here $\Omega_T = (0,L)\times(0,T), \partial'\Omega_T = \partial\Omega_T \setminus ((0,L)\times\{T\})$, and $u^+ = \max(u,0)$.
(Hint: Consider the function $v := ue^{-C_4 t}$)
Incomplete...
First, we follow the hint and compute:

$$L[v] = (au_{xx} + bu_x + (c+C_4)u - u_t)e^{-C_4 t} = L[u]e^{-C_4 t} + C_4 ue^{-C_4 t} \geq C_4 ue^{-C_4 t}$$

Let (x^*,t^*) be the maximum of u in $\overline{\Omega}_T$. First assume that $u(x^*,t^*) > 0$, since otherwise we're in a trivial case, and $u(x^*,t^*) \leq 0 = e^{C_4 T}\max_{\partial'\Omega} u^+$.
We first consider the case where $(x^*,t^*) \in \Omega_T$. Since this is a maximum, we have $u_t(x^*,t^*) = u_x(x^*,t^*) = 0$ and $u_{xx}(x^*,t^*) < 0$, $u_{tt}(x^*,t^*) < 0$ (assuming u_{tt} exists). Then

$$L[u](x^*,t^*) = a(x^*,t^*)u_{xx}(x^*,t^*) + c(x^*,t^*)u(x^*,t^*) \geq 0$$

Then since $a > 0$ and $u_{xx} < 0$, we have $\text{sign}(c) = \text{sign}(u)$, so $C_4 \geq c > 0$, and therefore $L[u] \geq L[u]e^{-C_4 t}$.
Now we consider $(x^*,t^*) \in \partial'\Omega_T$. In $(0,L)\times\{0\}$, at (x^*,t^*), we trivially satisfy this maximum principle, since $e^{C_4 t^*} = 1$, and $\max_{\overline{\Omega}} u = \max_{\partial'\Omega_T} u^+$.
For the other cases we just need to show that $C_4 \geq 0$, which we can do by showing that $\exists (y,s)$ s.t. $c(y,s) \geq 0$. In $\{0\}\times(0,T)$, we only require that $u_x \leq 0$, $u_t = 0$, and $u_{tt} < 0$. In $\{L\}\times(0,T)$, we only require that $u_x \geq 0$, $u_t = 0$, and $u_{tt} < 0$ (if it exists). In these cases, we have

$$L[u](x^*,t^*) = a(x^*,t^*)u_{xx}(x^*,t^*) + b(x^*,t^*)u_x(x^*,t^*) + c(x^*,t^*)u(x^*,t^*) \geq 0$$

(b) Prove the uniqueness of the initial value problem

$$\begin{cases} Lu(x,t) = f(x,t) & x \in \Omega_T \\ u(x,0) = \phi(x) & x \in \Omega \\ u(x,t) = g(x,t) & x \in \partial\Omega, t \in (0,T] \end{cases}$$

Assume we have two solutions u and v which solve the problem. Then $L[u-v] = L[u] - L[v] = 0$, and $u - v = 0$ on the parabolic cylinder's boundary. Then since $L[u-v] \geq 0$, we know from part a that $\max_{\overline{\Omega}} u - v \leq e^{C_4 t}\max_{\partial'\Omega}(u-v)^+ = 0$. We also know that $L[v-u] \leq 0$, so $\max_{\overline{\Omega}} v - u \leq e^{C_4 t}\max_{\partial'\Omega}(v-u)^+ = 0$. Thus, $0 \leq u - v \leq 0$, so $u = v$, proving uniqueness.

7. (a) Use d'Alembert's formula to show that the Maximum Principle does not hold for wave equation, i.e., $\exists$ u satisfying

$$\begin{cases} u_{tt} = u_{xx}, & -L < x < L, 0 < t < T \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

s.t.

$$\max_{\overline{U}_T} > \max_{\partial' U_T} u(x,t)$$

(Hint: Let $f = 0$ and $g \in C_0^\infty([-1,1]), g \geq 0$ and choose a small T.
Following the hint, we consider the case

$$\begin{cases} u_{tt} = u_{xx}, & -1 < x < 1, 0 < t < T \\ u(x,0) = 0 \\ 0 \leq u_t(x,0) = g(x) \in C_0^\infty([-1,1]) \end{cases}$$

and choose $u_t(x,0) = 0$ when $|x| > 1/2$
We can extend our problem to this domain by the reflection principle. We define $\tilde{u} : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ in the following manner: If $-1 < x < 1$, take $\tilde{u}(x,t) = u(x,t)$ If $x < -1$, take $\tilde{u}(x,t) = -\tilde{u}(-(x+1)-1,t) = -\tilde{u}(-x-2,t)$ If $x > 1$, take $\tilde{u}(x,t) = -\tilde{u}(-(x-1)+1,t) = -\tilde{u}(-x+2,t)$
We generalize this by defining $I_k = (-1+2k, 1+2k)$ for any integer k; then

$$\tilde{u}(x,t) = \begin{cases} u(x-2k,t) & x \in I_{2k}, 0 < t < T \\ -u(-x+2(k+1),t) & x \in I_{2k+1}, 0 < t < T \end{cases}$$

$$\tilde{g}(x,t) = \begin{cases} g(x-2k,t) & x \in I_{2k}, 0 < t < T \\ -g(-x+2(k+1),t) & x \in I_{2k+1}, 0 < t < T \end{cases}$$

Then

$$\begin{cases} \tilde{u}_{tt} = \tilde{u}_{xx} & x \in \mathbb{R} \times (0,T) \\ \tilde{u}(x,0) = 0 & x \in \mathbb{R} \\ \tilde{u}_t(x,0) = \tilde{g}(x) & x \in \mathbb{R} \end{cases}$$

is solved by d'Alembert's formula:

$$\tilde{u}(x,t) = \frac{1}{2} \int\limits_{x-t}^{t+x} \tilde{g}(y)\ \mathrm{d}y$$

Considering $T < \frac{1}{4}$ ensures us that for $\frac{3}{4} < |x| < 1$

$$\tilde{u}(x,t) = \frac{1}{2} \int\limits_{x-t}^{x+x} \tilde{g}(y)\ \mathrm{d}y = 0$$

as $\frac{1}{2} < x+t < \frac{3}{2}$, so $\tilde{g} = 0$. For $|x| < \frac{1}{4}$, we have $\tilde{g}(y) > 0$ for a set of non-zero measure. Since $\tilde{g}(y) \geq 0$, that means that the integral must be non-zero, so $\sup u(x,t) > 0$. But $\tilde{u}(\pm 1, t) = u(\pm 1, t) = 0$ for $0 < t < T$ and $u(x,0) = 0$ for $-1 < x < 1$. Thus, on the boundaries of $(-1,1) \times (0,T)$, $\sup u(x,t) = 0$. This shows that the maximum principle does not necessarily hold for the wave equation.

(b) Let u solve the initial value problem for the wave equation in one dimension

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t) & (x,t) \in \mathbb{R} \times (0, +\infty) \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

where f and g have compact support in $\mathbb{R}$. Let $k(t) = \frac{1}{2} \int\limits_{-\infty}^{\infty} u_t(x,t)^2\ \mathrm{d}x$ be the kinetic energy and $p(t) = \frac{1}{2} \int\limits_{-\infty}^{\infty} u_x(x,t)^2\ \mathrm{d}x$ be the potential energy. Show that

i. $k(t) + p(t)$ is constant in t
ii. $k(t) = p(t)$ for all large enough time t

First we show that $k(t) = p(t)$ for large enough time t:

Assuming $f \in C^2(\mathbb{R})$, $g \in C^1(\mathbb{R})$, d'Alembert's formula gives

$$2u(x,t) = f(x+t) + f(x-t) + \int_{x-t}^{x+t} g(y)\, \mathrm{d}y$$

Differentiating and squaring these gives

$$\begin{aligned} 4u_x(x,t)^2 =& [f'(x+t) + f'(x-t) + g(x+t) - g(x-t)]^2 \\ 4u_t(x,t)^2 =& [f'(x+t) - f'(x-t) + g(x+t) + g(x-t)]^2 \end{aligned}$$

Then we have

$$\begin{aligned} 2k(t) &= \int_{-\infty}^{\infty} [f'(x+t) + f'(x-t) + g(x+t) - g(x-t)]^2\, \mathrm{d}x \\ 2p(t) &= \int_{-\infty}^{\infty} [f'(x+t) - f'(x-t) + g(x+t) + g(x-t)]^2\, \mathrm{d}x \end{aligned}$$

Since f and g have compact support, there is a $R > 0$ s.t. $f(B_R(0)^c) = f'(B_R(0)^c) = g(B_R(0)^c) = \{0\}$. Then for a given t we can split the integral limits into an integration over 0 and over a non-zero term.

$$\begin{aligned} 2k(t) =& 0 + \int_{-R-t}^{R+t} [f'(x+t) + f'(x-t) + g(x+t) - g(x-t)]^2\, \mathrm{d}x \\ 2p(t) =& 0 + \int_{-R-t}^{R+t} [f'(x+t) - f'(x-t) + g(x+t) + g(x-t)]^2\, \mathrm{d}x \end{aligned}$$

Subtracting the two and removing terms leaves us with

$$\begin{aligned} 2k(t) - 2p(t) =& \int_{-R-t}^{R+t} [f'(x+t) + f'(x-t) + g(x+t) - g(x-t)]^2 \\ & \qquad - [f'(x+t) - f'(x-t) + g(x+t) + g(x-t)]^2\, \mathrm{d}x \\ =& \int_{-R-t}^{R+t} - 4g(x-t)g(x+t) - 4g(x-t)f'(x+t) \\ & \qquad + 4g(x+t)f'(x-t) + 4f'(x-t)f'(x+t)\, \mathrm{d}x \end{aligned}$$

For this to be non-zero, we need both $x + t < R$ and $x - t > -R$, as otherwise the remaining terms inside the integral all go to zero.

If we consider $t > R$, then $x < 0$ must hold for $x + t \in B_R(0)$. Similarly, $x > 0$ must hold for $x - t \in B_R(0)$. But $\{x > 0\} \cap \{x < 0\} = \emptyset$, so the integral goes to 0 everywhere. Thus, for large enough t, $k(t) - p(t) = 0$.

To show that $k(t) + p(t)$ is constant, we show that $\frac{\mathrm{d}k+p}{\mathrm{d}t} = 0$.

First, $k(t) + p(t) = \int\limits_{-\infty}^{\infty} u_t(x,t)^2 \,\mathrm{d}x + \int\limits_{-\infty}^{\infty} u_x(x,t)^2 \,\mathrm{d}x$

Since both of these integrals converge, we can combine them: $k(t) + p(t) = \int\limits_{-\infty}^{\infty} u_t(x,t)^2 + u_x(x,t)^2 \,\mathrm{d}x$

Then, since u has compact support (shown in the previous part), for a given t we can choose an R s.t. $u(B_R(0)^c) = u_x(B_R(0)^c) = u_t(B_R(0)^c) = \emptyset$ Then our integral becomes

$$k(t) + p(t) = \int\limits_{-R}^{R} u_t(x,t)^2 + u_x(x,t)^2 \,\mathrm{d}x$$

Since u_t and u_x are continuous on $[-R, R]$, they are uniformly continuous, so we can interchange the derivative and integral:

$$\frac{\mathrm{d}k(t) + p(t)}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{-R}^{R} u_t(x,t)^2 + u_x(x,t)^2 \,\mathrm{d}x = \int\limits_{-R}^{R} u_t(x,t)u_{tt}(x,t) + u_x(x,t)u_{xt}(x,t) \,\mathrm{d}x$$

Integrating the second term by parts gives us

$$\begin{aligned}\frac{\mathrm{d}k(t) + p(t)}{\mathrm{d}t} &= u_x(R,t)u_t(R,t) - u_x(-R,t)u_t(-R,t) + \int\limits_{-R}^{R} u_t(x,t)u_{tt}(x,t) - u_{xx}(x,t)u_t(x,t) \,\mathrm{d}x \\ &= \int\limits_{-R}^{R} u_t(x,t)u_{tt}(x,t) - u_{xx}(x,t)u_t(x,t) \,\mathrm{d}x\end{aligned}$$

Then, noting that by our definition $u_{xx} = u_{tt}$ leaves us with an integral containing 0

$$\frac{\mathrm{d}k(t) + p(t)}{\mathrm{d}t} = \int\limits_{-R}^{R} u_t(x,t)u_{tt}(x,t) - u_{tt}(x,t)u_t(x,t) \,\mathrm{d}x = 0$$

Thus, the derivative of the sum of our energies is 0, and the sum must be constant.

8. Find the explicit formula for the following wave equation

$$\begin{cases} u_{tt} - \Delta u = 0, & x \in \mathbb{R}^3, t > 0 \\ u(x,0) = 0 & x \in \mathbb{R}^3 \\ u_t(x,0) = h(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 1 \end{cases} & x \in \mathbb{R}^3 \end{cases}$$

(Hint: u is radially symmetric)

Just follow what we did in class :), trick to make it one dimensional.

Since $n = 3$, we can follow Kirchoff's formula.

Set

$$\begin{aligned} U(x;r,t) =& \fint_{\partial B_r(x)} u(y,t)\ \mathrm{d}S(y) \\ \tilde{U} =& rU \\ H(x;r) =& \fint_{\partial B_r(x)} \begin{cases} 1 & |y|<1 \\ 0 & |y|>1 \end{cases} \mathrm{d}S(y) \\ \tilde{H} =& rH \end{aligned}$$

Note that $u(x,t) = \lim_{r\to 0} U(x,r,t) = \lim_{r\to 0} \frac{1}{r}\tilde{U}(x,r,t)$, $h(x) = \lim_{r\to 0} H(x,r) = \lim_{r\to 0} \frac{1}{r}\tilde{H}(x,r)$. Also note that $H(x;r)$ is the ratio of the spherical cap area in Figure 1 to the sphere's surface area.

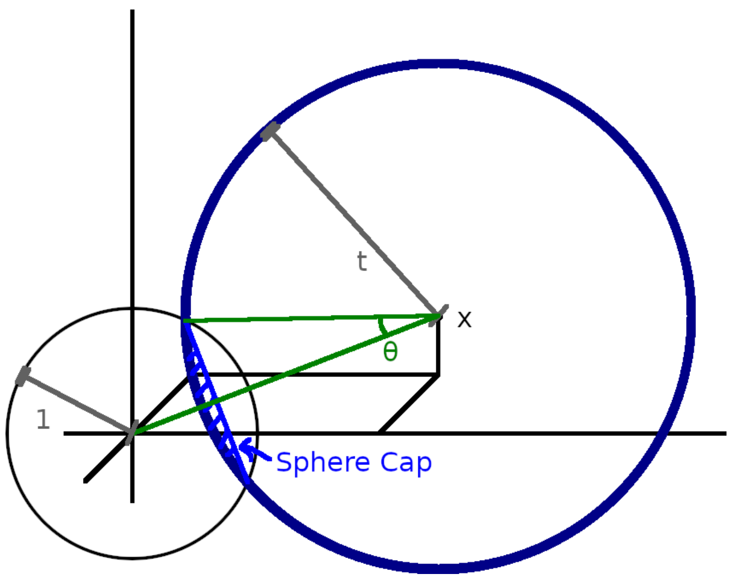

Figure 1: Illustration of the sphere and sphere-cap the integral computes the area of

Then

$$\begin{aligned} \tilde{U}_{tt} - \tilde{U}_{rr} = 0 \text{ in } \mathbb{R}_+ \times (0,\infty) \\ \tilde{U} = \tilde{G}, \tilde{U}_t = \tilde{H} \text{ on } \mathbb{R}_+ \times \{t=0\} \\ \tilde{U} = 0 \text{ on } \{r=0\} \times (0,\infty) \end{aligned}$$

$$u(x,t) = t\fint_{\partial B_t(x)} h(y)\ \mathrm{d}S(y)$$

To compute the area of the spherical cap, we first note

$$1 = t^2 \sin^2(\theta) + (|x| - t\cos(\theta))^2 = t^2 + |x|^2 - 2|x|t\cos(\theta)$$

Then, recall the surface area of a sphere cap for a sphere of radius t and an angle θ:

$$A = 2\pi t^2(1 - \cos(\theta))$$

Some manipulations of the previous two equations gives

$$A = -\frac{\pi t}{|x|}\left((|x| + t)^2 - 1\right)$$

Then, when $-1 < |x| - t < 1$ (since the sphere must actually intersect the unit sphere at the origin), and $t \neq 0$, $u(x.t)$ is as follows:

$$u(x,t) = t\frac{A}{4\pi t^2} = -\frac{(|x| + t)^2 - 1}{4|x|}$$

In full:

$$u(x,t) = \begin{cases} -\frac{(|x|+t)^2-1}{4|x|} & ||x| - t| < 1, t \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

To check our calculations, we can rewrite the original problem in spherical coordinates:

$$\begin{cases} u_{tt} - \Delta u = u_{tt} - \frac{2}{r}u_r - u_{rr} = 0, & r > 0, \theta \in \mathbb{R}^2, |\theta| = 1, t > 0 \\ u(r,\theta,0) = 0 & r > 0, \theta \in \mathbb{R}^2, |\theta| = 1 \\ u_t(r,\theta,0) = \begin{cases} 1 & r < 1 \\ 0 & r > 1 \end{cases} & r > 0, \theta \in \mathbb{R}^2, |\theta| = 1 \end{cases}$$

Our solution is

$$u(r,\theta,t) = \begin{cases} -\frac{(r+t)^2-1}{4r} & |r - t| < 1, t \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

For $|r - t| < 1, t \neq 0$, we compute

$$\begin{aligned} u_t =& \frac{t + r}{2r} \\ u_{tt} =& \frac{1}{2r} \\ u_r =& \frac{1 - t^2 + r^2}{4r^2} \\ u_{rr} =& \frac{t^2 - 1}{2r^3} \\ u_{tt} =& \frac{2}{r}u_r + u_{rr} = \frac{t^2 - 1}{2r^3} - \frac{1}{2r} + \frac{t^2 - 1}{2r^3} = -\frac{1}{2r} \end{aligned}$$

For $r < 1$, $\lim_{t \to 0} u_t = \frac{1}{2}$, suggesting this is off by a factor of 2, so we amend the solution to

$$u(r, \theta, t) = \begin{cases} -\frac{(r+t)^2 - 1}{2r} & |r - t| < 1, t \neq 0 \\ 0 & \text{otherwise} \end{cases}$$