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Math 516
Homework 3

I collaborated with Damien Huet on this assignment

1. Derive a solution formula for the two-dimensional wave equation with source

$$\begin{cases} u_{tt} = \Delta u + f(x,t) & x \in R^2 \\ u(x,0) = 0, u_t(x,0) = 0 \end{cases}$$

We apply Duhamel's principle. Define $v(x,t;s)$ s.t. it satisfies the following

$$\begin{cases} v_{tt}(x,t;s) - \Delta v(x,t;s) = 0 \\ v(x,s;s) = 0 \\ v_t(x,s;s) = f(x,s) \end{cases}$$

Applying Poisson's formula to compute a solution gives

$$v(x,t;s) = \frac{1}{2} \fint_{B_t(x)} \frac{t^2 f(y;s)}{\sqrt{t^2 - |y-x|^2}} \, \mathrm{d}y$$

Then by Duhamel's principle

$$u(x,t) = \int_0^t v(x,t;s) \, \mathrm{d}s = \frac{1}{2} \int_0^t \fint_{B_t(x)} \frac{t^2 f(y;s)}{\sqrt{t^2 - |y-x|^2}} \, \mathrm{d}y \, \mathrm{d}s$$

Simplifying beyond this doesn't seem possible...

2. Derive a solution formula for the three-dimensional wave equation with radial source

$$\begin{cases} u_{tt} = \Delta u + f(r,t) & t > 0, r > 0 \\ u(r,0) = 0, u_t(r,0) = 0 \end{cases}$$

Again we start with Duhamel's Principle, and define v s.t. it satisfies the following system:

$$\begin{cases} v_{tt}(x,t;s) = \Delta v(x,t;s) & t > 0 \\ v(x,0;s) = 0, v_t(x,0;s) = f(|x|,s) \end{cases}$$

Then since v is three-dimensional, it is solved with Kirchoff's Formula:

$$v(x,t;s) = \fint_{\partial B_t(x)} t f(|y|;s) \, \mathrm{d}S(y)$$

Next, we consider the intersection of $\partial B_t(x)$ with $\partial B_r(0)$ for some $r \in [\max(|x|-t, t-|x|), |x|+t]$. This intersection is a circle. Since f is radially symmetric, it's constant over this circle. So we can rewrite the previous integral by computing the derivative of the area of the spherical cap with respect to the distance r of its boundary circle to the origin. Recall that the area of the spherical cap is given by $\pi(a^2 + h^2)$, where a is the (minimum) distance from the Ox_0 axis to the boundary circle, and h is the distance from the plane containing the boundary circle to the axis' intersection with the spherical cap. On the intersection of these two spheres, we have two cases: $t < |x_0|$ and $t > |x_0|$. For the former, we compute

$$\begin{aligned}
t^2 =& (t-h)^2 + a^2 \to a^2 = t^2 - (t-h)^2 \\
r^2 =& (h + ||x_0| - t|)^2 + a^2 = (h + ||x_0| - t|)^2 + t^2 - (t-h)^2 \\
=& ||x_0| - t|^2 + 2h||x_0| - t| + 2th \\
h =& \frac{r^2 - ||x_0| - t|^2}{2||x_0| - t| + 2t} = \frac{r^2 - ||x_0| - t|^2}{|x_0|} \\
A =& \pi(t^2 - (t-h)^2 + h^2) = 2\pi h t = 2\pi t \frac{r^2 - ||x_0| - t|^2}{2|x_0|}
\end{aligned}$$

For the latter, we compute

$$\begin{aligned}
a^2 =& t^2 - (t-h)^2 \\
r^2 =& (t - |x_0| - h)^2 + a^2 = (t - |x_0| - h)^2 + 2th - h^2 \\
=& 2|x_0|h + t^2 + |x_0|^2 - 2t|x_0| \\
h =& \frac{r^2 - |x_0|^2 + 2t|x_0| - t^2}{2|x_0|} \\
A =& 2\pi h t = 2\pi t \frac{r^2 - |x_0|^2 + 2t|x_0| - t^2}{2|x_0|}
\end{aligned}$$

Then

$$dS(y) = \frac{tr}{|x_0|} \, \mathrm{d}\theta \, \mathrm{d}r$$

Since f is constant w.r.t. θ, the surface integral simplifies to

$$v(x,t;s) = \frac{1}{\alpha_3 t^2} \int\limits_{r=||x|-t|}^{|x|+t} \frac{\alpha_2 tr}{|x|} f(r;s) \, \mathrm{d}r = \frac{1}{2t|x|} \int\limits_{r=||x|-t|}^{|x|+t} r f(r;s) \, \mathrm{d}r$$

Then, by Duhmamel's Principle, we have

$$u(x,t) = \int\limits_{s=0}^{t} v(x,t;s) \, \mathrm{d}s = \frac{1}{2t|x|} \int\limits_{s=0}^{t} \int\limits_{r=||x|-t|}^{|x|+t} r f(r;s) \, \mathrm{d}r \, \mathrm{d}s$$

3. Find the first order and second order weak derivatives for the following function $u : \mathbb{R} \to \mathbb{R}$, if exists:

(a)

$$u(x) = \begin{cases} 1 - |x| & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$

The first order weak derivative is

$$Du(x) = \begin{cases} 1 & -1 < x < 0 \\ -1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

The second order weak derivative doesn't exist, as it's not continuous (see problem 10a). We can prove this by working backwards and using the uniqueness of the weak derivative to prove our claim. We need to show that $\int_{\mathbb{R}} Du\phi \, \mathrm{d}x = -\int_{\mathbb{R}} u\phi' \, \mathrm{d}x$ for any $\phi \in C_C^\infty(\mathbb{R})$. To do so, we simply note that $\frac{\mathrm{d}}{\mathrm{d}x}(1+x) = 1$ and $\frac{\mathrm{d}}{\mathrm{d}x}(1-x) = -1$, which is nice since we need these to be 0 at the -1 and 1, respectively, while having the same value at 0. Then integrating by parts results in:

$$\begin{aligned}
\int Du(x)\phi(x) = &\int_{-1}^{0} \phi(x) \, \mathrm{d}x - \int_{0}^{1} \phi(x) \, \mathrm{d}x \\
=&\phi(x)(1+x)\Big|_{x=-1}^{0} - \int_{-1}^{0} (1+x)\phi'(x) + \phi(x)(1-x)\Big|_{x=0}^{1} - \int_{0}^{1} (1-x)\phi'(x) \, \mathrm{d}x \\
=&\phi(0) - \int_{-1}^{0} (1+x)\phi'(x) - \phi(0) - \int_{0}^{1} (1-x)\phi'(x) \, \mathrm{d}x \\
=&-\int_{-1}^{0} (1+x)\phi'(x) - \int_{0}^{1} (1-x)\phi'(x) \, \mathrm{d}x = -\int_{-1}^{0} (1-|x|)\phi'(x) - \int_{0}^{1} (1-|x|)\phi'(x) \, \mathrm{d}x \\
=&-\int_{-1}^{1} (1-|x|)\phi'(x)
\end{aligned}$$

Thus, Du is the weak derivative of u. The second order weak derivative does not exist, since $Du(x)$ is discontinuous at $x = 0$ and $x = \pm 1$; see problem 10a.

(b)

$$u(x) = |\sin(x)|$$

First we rewrite this in cases:

$$u(x) = \begin{cases} \sin(x) & 2k\pi < x < (2k+1)\pi \\ -\sin(x) & (2k+1)\pi < x < (2k+2)\pi \end{cases}$$

Then the first order weak derivative is

$$Du(x) = \begin{cases} \cos(x) & 2k\pi < x < (2k+1)\pi \\ -\cos(x) & (2k+1)\pi < x < (2k+2)\pi \end{cases}$$

We check by integrating by parts over all of the subintervals:

$$
\begin{aligned}
\int_{\mathbb{R}} |\sin(x)|\phi'(x)\,\mathrm{d}x &= \sum_{k=-\infty}^{\infty} \int_{2k\pi}^{(2k+1)\pi} \sin(x)\phi'(x)\,\mathrm{d}x + \int_{(2k+1)\pi}^{(2k+2)\pi} -\sin(x)\phi'(x)\,\mathrm{d}x \\
&= \sum_{k=-\infty}^{\infty} \sin((2k+1)\pi)\phi((2k+1)\pi) - \sin(2k\pi)\phi(2k\pi) \\
&\quad - \int_{2k\pi}^{(2k+1)\pi} \cos(x)\phi(x)\,\mathrm{d}x \\
&\quad - \sin((2k+2)\pi)\phi((2k+2)\pi) + \sin((2k+1)\pi)\phi((2k+1)\pi) \\
&\quad + \int_{(2k+1)\pi}^{(2k+2)\pi} \cos(x)\phi(x)\,\mathrm{d}x \\
&= \sum_{k=-\infty}^{\infty} - \int_{2k\pi}^{(2k+1)\pi} \cos(x)\phi(x)\,\mathrm{d}x + \int_{(2k+1)\pi}^{(2k+2)\pi} \cos(x)\phi(x)\,\mathrm{d}x \\
&= \sum_{k=-\infty}^{\infty} - \int_{2k\pi}^{(2k+1)\pi} \cos(x)\phi(x)\,\mathrm{d}x - \int_{(2k+1)\pi}^{(2k+2)\pi} -\cos(x)\phi(x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}} Du\phi(x)\,\mathrm{d}x
\end{aligned}
$$

proving our claim.

$Du(x)$ is discontinuous at every $x = 2k\pi$, so the second order weak derivative does not exist.

4. Suppose $u : (a, b) \to R$ and the weak derivative exists and satisfies

$$Du = 0, \text{ a.e. in } (a, b)$$

Prove that u is constant a.e. in (a, b)

First we note that since $\int_a^b Du\phi\,\mathrm{d}x = -\int_a^b u\phi'\,\mathrm{d}x$ for any $\phi \in C_c^\infty((a, b))$, we have $\int_a^b u\phi'\,\mathrm{d}x = 0$. Next, we assume that u is not constant. Then since u is continuous and almost everywhere differentiable (as it has a weak derivative), there must be two disconnected neighborhoods N_1 and N_2 where $|u(N_1)| \cap |u(N_2)| = \emptyset$. WLOG, let $\inf_{N_1} |u| \leq \sup_{N_1} |u| < \inf_{N_2} |u| \leq \sup_{N_2} |u|$. Next, consider ϕ_n s.t.

$$
\phi_n'(x) = \begin{cases} \sqrt{n}e^{-\frac{1}{1-n|x-x_1|^2}} & |x - x_1| < \frac{1}{\sqrt{n}} \\ \sqrt{n}e^{-\frac{1}{1-n|x-x_2|^2}} & |x - x_2| < \frac{1}{\sqrt{n}} \\ 0 & \text{otherwise} \end{cases}
$$

with $x_1 \in N_1, x_2 \in N_2$ (this is continuous, and thus integrable, so ϕ_n exists and $\phi_n \in C_C^\infty$). Then we can choose N s.t. for all $n \geq N$, $\left(x_1 - \frac{1}{n}, x_1 + \frac{1}{n}\right) \subset N_1$ and $\left(x_2 - \frac{1}{n}, x_2 + \frac{1}{n}\right) \subset N_2$; we assume we are using such an n for the rest of the proof. We also note that $\int_{N_1} \frac{\mathrm{d}^k}{\mathrm{d}x^k}\phi_n(x)\,\mathrm{d}x = \int_{N_2} \frac{\mathrm{d}^k}{\mathrm{d}x^k}\phi_m(x)\,\mathrm{d}x$ for distinct n and m; this can be shown by a simple substitution.

Then

$$\begin{aligned} 0 = \int_a^b u\phi_n' \,\mathrm{d}x &= \int_{x_1-1/n}^{x_1+1/n} u\phi_n' \,\mathrm{d}x + \int_{x_2-1/n}^{x_2+1/n} u\phi_n' \,\mathrm{d}x \\ -\int_{x_1-1/n}^{x_1+1/n} u\phi_n' \,\mathrm{d}x &= \int_{x_2-1/n}^{x_2+1/n} u\phi_n' \,\mathrm{d}x \end{aligned}$$

But

$$\begin{aligned} -\int_{x_1-1/n}^{x_1+1/n} u\phi_n' \,\mathrm{d}x &= \int_{x_2-1/n}^{x_2+1/n} u\phi_n' \,\mathrm{d}x \\ C = \int_{x_1-1/n}^{x_1+1/n} \phi_m' \,\mathrm{d}x &= \int_{x_2-1/n}^{x_2+1/n} \phi_m' \,\mathrm{d}x \\ -C \sup_{N_1} u \leq -\int_{x_1-1/n}^{x_1+1/n} u\phi_n' \,\mathrm{d}x &\leq -C \inf_{N_1} u \\ C \inf_{N_2} u \leq \int_{x_2-1/n}^{x_2+1/n} u\phi_n' \,\mathrm{d}x &\leq C \sup_{N_2} u \end{aligned}$$

Since we have $(-C \sup_{N_1} u, -C \inf_{N_1} u) \cap (C \inf_{N_1} u, C \sup_{N_2} u) = \emptyset$, this is a with a contradiction, and thus, u must be constant.

5. Let $1 < p < \infty$. Show that if $u, v \in W^{1,p}(\Omega)$, then $\max(u,v), \min(u,v) \in W^{1,p}(\Omega)$. Show that this is not true for $W^{2,p}(\Omega)$.

$u, v \in W^{1,p}(\Omega)$. First, assuming $D^{x_i} \max(u,v)$ exists, we have $\max(u,v) \in W^{1,p}(\Omega)$, since

$$\begin{aligned} \int_\Omega |\max(u,v)|^p + |D^{x_i} \max(u,v)|^p \,\mathrm{d}x &\leq \int_\Omega |\max(u,v)|^p + \max(|D^{x_i}u|, |D^{x_i}v|)^p \,\mathrm{d}x \\ &\leq \int_\Omega |u|^p + |v|^p + |D^{x_i}u|^p + |D^{x_i}v|^p \,\mathrm{d}x \end{aligned}$$

is a finite upper bound to these quantities. By the same argument, we have $\max(u,v) \in W^{1,p}(\Omega)$.

Next we need to show that these have weak derivatives. Consider $\int_\Omega \max(u,v) D^{x_i}\phi \,\mathrm{d}x$ for any $k \leq p$. Because u and v have weak derivatives up to order p, we know that they are continuous and differentiable almost everywhere, and thus, so is $u-v$. Then we can partition Ω into countably many disjoint neighborhoods $\{N_i^u\}$ in which $u - v > 0$ and $\{N_j^v\}$ in which $u - v < 0$, with $\overline{\Omega} = \overline{(\cup_i N_i^u \cup_j N_j^v)}$. Then we can rewrite our integral

$$\begin{aligned} \int_\Omega \max(u,v) D^{x_k}\phi \,\mathrm{d}x &= \sum_i \int_{N_i^u} \max(u,v) D^{x_k}\phi \,\mathrm{d}x + \sum_j \int_{N_i^v} \max(u,v) D^{x_k}\phi \,\mathrm{d}x \\ &= \sum_i \int_{N_i^u} u D^{x_k}\phi \,\mathrm{d}x + \sum_j \int_{N_i^v} v D^{x_k}\phi \,\mathrm{d}x \end{aligned}$$

and noting that since u and v have weak derivatives in Ω, and therefore in any neighborhood contained in Ω, we have

$$\begin{aligned}\int_\Omega \max(u,v) D^{x_k}\phi \, \mathrm{d}x &= -\left(\sum_i \int_{N_i^u} D^{x_k} u\phi \, \mathrm{d}x + \sum_j \int_{N_j^v} D^{x_k} v\phi \, \mathrm{d}x\right) \\ &= -\left(\sum_i \int_{N_i^u} D^{x_k} \max(u,v)\phi \, \mathrm{d}x + \sum_j \int_{N_j^v} D^{x_k} \max(u,v)\phi \, \mathrm{d}x\right) \\ &= -\int_\Omega D^{x_k} \max(u,v)\phi \, \mathrm{d}x\end{aligned}$$

thus proving our claim that $\max(u,v)$ has weak derivatives up to order k.

Next we show that for $W^{2,p}(\Omega)$, the previous does not necessarily hold. Consider $\Omega = (-1,1)$ with

$$\begin{aligned}u &= x \\ v &= -x\end{aligned}$$

Clearly, $\max(u,v) = |x|$, which isn't in $W^{2,p}(\Omega)$.

6. Consider the following function

$$u(x) = \frac{1}{|x|^\gamma}$$

in $\Omega = B_1(0)$.

Show that if $\gamma + 1 < n$, the weak derivatives are given by

$$\partial_j u = -\gamma \frac{x_j}{|x|^{\gamma+2}}$$

i.e., you need to show rigorously that

$$\int u \partial_j \phi = \int \phi \gamma \frac{x_j}{|x|^{\gamma+2}}$$

Find the condition on γ s.t. $u \in W^{1,p}$ and $u \in W^{2,p}$

For $u \in W^{1,p}$, we just follow Evans' example 3. This comes out to $\gamma < \frac{n-p}{p}$.

First we note that u is smooth away from 0, with $u_{x_i}(x) = \frac{-\gamma x_i}{|x|^{\gamma+2}}$, so $|Du(x)| = \frac{\gamma}{|x|^{\gamma+1}}$ for any distance at least $\epsilon > 0$ from the origin.

Then, for any $\phi \in C_C^\infty(\Omega)$, the weak derivative v satisfies

$$\begin{aligned}\int_\Omega u\phi_{x_i} \, \mathrm{d}x &= -\int_\Omega v\phi \, \mathrm{d}x \\ \int_{\epsilon<|x|<1} u\phi_{x_i} \, \mathrm{d}x &= -\int_{\epsilon<|x|<1} v\phi \, \mathrm{d}x\end{aligned}$$

Integrating by parts gives

$$\int_{\epsilon<|x|<1} u\phi_{x_i} \, \mathrm{d}x = \int_{\partial B_\epsilon(0)} u\phi\nu^i \, \mathrm{d}S - \int_{\epsilon<|x|<1} u_{x_i}\phi \, \mathrm{d}x$$

where ν is the inward normal of $\partial B_\epsilon(0)$.

Since u and ϕ are bounded on the boundary, we have

$$\begin{aligned}\left|\int_{\partial B_\epsilon(0)} u\phi\nu^i \,\mathrm{d}S\right| &\leq \int_{\partial B_\epsilon(0)} \epsilon^{-\gamma}\,\mathrm{d}S \leq C\epsilon^{n-1-\alpha} \\ &\leq ||\phi||_{L^\infty} \int_{\partial B_\epsilon} \epsilon^{-\gamma}\,\mathrm{d}S \leq C\epsilon^{n-1-\alpha}\end{aligned}$$

This goes to 0 as $\epsilon \to 0$ if $n - 1 - \alpha > 0$.

Then

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{\epsilon<|x|<1} u\phi_{x_i}\,\mathrm{d}x &= \lim_{\epsilon\to 0} -\int_{\epsilon<|x|<1} u_{x_i}\phi\,\mathrm{d}x \\ \int_{B_1(0)} u\phi_{x_i}\,\mathrm{d}x &= \int_{|x|<\epsilon} u\phi_{x_i}\,\mathrm{d}x + \int_{\epsilon<|x|<1} u\phi_{x_i}\,\mathrm{d}x \\ &= \lim_{\epsilon\to 0}\int_{|x|<\epsilon} u\phi_{x_i}\,\mathrm{d}x - \int_{\epsilon<|x|<1} u_{x_i}\phi\,\mathrm{d}x \\ &= \lim_{\epsilon\to 0} -\int_{\epsilon<|x|<1} u_{x_i}\phi\,\mathrm{d}x\end{aligned}$$

proving that this weak derivative works over the entire domain excluding 0 when $\gamma + 1 < n$.

Next, we note that the weak derivative is the same form as the original; just multiplied by a constant. Then applying the same process above, except starting with $\gamma + 1$, for the second weak derivative we end up with $\gamma + 2 < n$, with

$$|D^2u(x)| = \frac{|\gamma(\gamma+1)|}{|x|^{\gamma+2}}$$

Finally, we need to determine when the L^p norms exist for all of these. Again, for any $0 < \epsilon < 1$, with $n - p(\gamma + 2) > 0$, we have

$$\begin{aligned}||u||_{L^p} &= \int_{B_1(0)} |x|^{-p\gamma}\,\mathrm{d}x = \int_{r<\epsilon} Cr^{n-1-p\gamma}\,\mathrm{d}r + \int_{\epsilon<r<1} Cr^{n-1-p\gamma}\,\mathrm{d}r \\ &= \lim_{\epsilon\to 0}\int_{\epsilon<r<1} Cr^{n-1-p\gamma}\,\mathrm{d}r = \lim_{\epsilon\to 0} Cr^{n-p\gamma}\big|_{r=\epsilon}^1 = C\end{aligned}$$

$$\begin{aligned}||Du||_{L^p} &= \int_{B_1(0)} C|x|^{-p(\gamma+1)}\,\mathrm{d}x = \int_{r<\epsilon} Cr^{n-1-p(\gamma+1)}\,\mathrm{d}r + \int_{\epsilon<r<1} Cr^{n-1-p(\gamma+1)}\,\mathrm{d}r \\ &= \lim_{\epsilon\to 0}\int_{\epsilon<r<1} Cr^{n-1-p(\gamma+1)}\,\mathrm{d}r = \lim_{\epsilon\to 0} Cr^{n-p(\gamma+1)}\big|_{r=\epsilon}^1 = C\end{aligned}$$

$$\begin{aligned}||D^2u||_{L^p} &= \int_{B_1(0)} C|x|^{-p(\gamma+2)}\,\mathrm{d}x = \int_{r<\epsilon} Cr^{n-1-p(\gamma+2)}\,\mathrm{d}r + \int_{\epsilon<r<1} Cr^{n-1-p(\gamma+2)}\,\mathrm{d}r \\ &= \lim_{\epsilon\to 0}\int_{\epsilon<r<1} Cr^{n-1-p(\gamma+2)}\,\mathrm{d}r = \lim_{\epsilon\to 0} Cr^{n-p(\gamma+2)}\big|_{r=\epsilon}^1 = C\end{aligned}$$

Thus, this function is in $W^{2,p}(\Omega)$ whenever $n - p(\gamma + 2) > 0$, since that is the most restrictive bound on it (as $p > 0$).

7. Let $\eta \in C^p(\Omega)$ with $\eta(t) = 1$ for $t \leq 0$ and $\eta(t) = 0$ for $t > 1$. Let $f \in W^{k,p}(R^n)$ and $f_m = f(x)\eta(|x| - m)$. Show that $||f_m - f||_{W^{k,p}} \to 0$ as $m \to \infty$. As a consequence show that $W^{k,p}(R^n) = W_0^{k,p}(R^n)$.

   We want to show that

$$\begin{aligned}
0 &= \lim_{k\to\infty} \sum_{|\alpha|\leq p} \int_{x\in R^n} |D^\alpha(f(x) - f(x)\eta(|x| - k))|^p \, \mathrm{d}x \\
&= \sum_{|\alpha|\leq p} \lim_{k\to\infty} \int_{x\in B_k^C} |D^\alpha f(x)|^p \, \mathrm{d}x
\end{aligned}$$

   For $k = 0$, we can just apply the monotone convergence theorem, since $|f - f_m|^p$ converges monotonically to 0, showing that $\lim_{m\to\infty} \int_{R^n} |f - f_m|^p \, \mathrm{d}x = \int_{R^n} \lim_{m\to\infty} |f - f_m|^p \, \mathrm{d}x = 0$. For $p \geq 1$, we need to consider $\int_{R^n} |D^\alpha(f - f_m)|^p \, \mathrm{d}x$. Breaking this into three integrals, we have

$$\begin{aligned}
||f - f_m||_{W^{k,p}} = \sum_{|\alpha|=k} &\int_{|x|<m} [D^\alpha 0]^p \, \mathrm{d}x \\
&+ \int_{m<|x|<m+1} [D^\alpha[f(x)(1 - \eta(|x| - m))]]^p \, \mathrm{d}x \\
&+ \int_{m+1<|x|} [D^\alpha f(x)]^p \, \mathrm{d}x \\
= \sum_{|\alpha|=k} &\int_{m<|x|<m+1} [D^\alpha[f(x)(1 - \eta(|x| - m))]]^p \, \mathrm{d}x \\
&+ \int_{R^n} [D^\alpha f(x)]^p \, \mathrm{d}x - \int_{|x|<m+1} [D^\alpha f(x)]^p \, \mathrm{d}x
\end{aligned}$$

   We note that $\lim_{m\to\infty} \int_{R^n} [D^\alpha f(x)]^p \, \mathrm{d}x - \int_{|x|<m+1} [D^\alpha f(x)]^p \, \mathrm{d}x = 0$, since $f(x) \in W^{k,p}(R^n)$. Then we're left with

$$\lim_{m\to\infty} ||f - f_m||_{W^{k,p}} = \sum_{|\alpha|=k} \lim_{m\to\infty} \int_{m<|x|<m+1} [D^\alpha[f(x)(1 - \eta(|x| - m))]]^p \, \mathrm{d}x$$

   We just need to show that this goes to zero. Let $S_\alpha = \{(\beta, \gamma) | \alpha_i = \beta_i + \gamma_i, \beta_i, \gamma_i \geq 0\}$ Then we can rewrite the weak derivative above with the product rule:

$$D^\alpha[f(x)(1 - \eta(|x| - m))] = \sum_{(\beta,\gamma)\in S_\alpha} [D^\beta f(x)] D^\gamma (1 - \eta(|x| - m)) \, \mathrm{d}x \leq C \sum_{(\beta,\gamma)\in S_\alpha} D^\beta f(x)$$

   Then

$$\begin{aligned}
\lim_{m\to\infty} ||f - f_m||_{W^{k,p}} &\leq \lim_{m\to\infty} \sum_{|\alpha|=k} \int_{m<|x|<m+1} [D^\alpha[f(x)(1 - \eta(|x| - m))]]^p \, \mathrm{d}x \\
&\leq \lim_{m\to\infty} \sum_{|\alpha|=k} \int_{m<|x|} C D^\alpha f(x) \, \mathrm{d}x = 0
\end{aligned}$$

completing our proof.

Next, we note that $f_m \in W_0^{k,p}(R^n)$, so $\forall\epsilon > 0, \exists g_i \in C_c^\infty(R^n)$ s.t. $|f_m - g_i| < \frac{\epsilon}{2}$, and also we can choose $M \in N$ s.t. $\forall m \geq M$, $|f_m - f| < \frac{\epsilon}{2}$. Through the triangle inequality we then have $g_i \to f$. Thus, $f \in W_0^{k,p}(R^n)$. Since this is true for any $f \in W^{k,p}(R^n)$, and since $W_0^{k,p}(R^n) \subset W^{k,p}(R^n)$, we have $W^{k,p}(R^n) = W_0^{k,p}(R^n)$.

8. Let $u \in C^\infty(\overline{R}_+^n)$. Extend u to Eu on R^n such that

$$\begin{aligned} Eu =& u, x \in R_+^n \\ Eu \in& C^{3,1}(R^n) \cap W^{4,p}(R^n) \\ ||Eu||_{W^{4,p}} \leq& C||u||_{W^{4,p}} \end{aligned}$$

Here $R_+^n = \{(x', x_n); x_n > 0\}$ and $C^{3,1} = \{u \in C^3, D^\alpha u \text{ is Lipschitz, } |\alpha| = 3\}$

First we assume the form

$$Eu(x', x_n) = \begin{cases} u(x', x_n) & x_n \geq 0 \\ c_0 u(x', -x_n) + c_1 u(x', -x_n/2) + c_2 u(x', -x_n/4) + c_3 u(x', -x_n/8) + c_4 u(x', -x_n/16) & x_n \leq 0 \end{cases}$$

while requiring $c_0 + c_1 + c_2 + c_3 + c_4 = 1$. This clearly satisfies $Eu(x', x_n) = u(x', x_n)$ when $x_n \geq 0$, and is continuous since $\lim_{x_n \to 0} Eu(x', x_n) = u(x', 0)$. We also have $Eu \in C^\infty(R_-^n)$, and just need to show that $\lim_{x_n \to 0} D^{x_n^k} Eu$ exists for $k \leq 3$.

Then when $x_n > 0$, we have $D^{x_n^k} Eu(x', x_n) = D^{x_n^k} u(x', x_n)$ For $x_n < 0$, we have

$$\begin{aligned} \lim_{x_n \to 0, x_n < 0} D^{x_n^k} Eu(x', x_n) &= \lim_{x_n \to 0} (-1)^k c_0 D^{x_n^k} u(x', -x_n) + \left(\frac{-1}{2}\right)^k c_1 D^{x_n^k} u(x', -x_n/2) \\ &\quad + \left(\frac{-1}{4}\right)^k c_2 D^{x_n} u(x', -x_n/4) + \left(\frac{-1}{8}\right)^k c_3 D^{x_n} u(x', -x_n/8) \\ &\quad + \left(\frac{-1}{16}\right)^k c_4 D^{x_n} u(x', -x_n/16) \\ &= \left[(-1)^k c_0 + \left(\frac{-1}{2}\right)^k c_1 + \left(\frac{-1}{4}\right)^k c_2 + \left(\frac{-1}{8}\right)^k c_3 + \left(\frac{-1}{16}\right)^k c_4\right] D^{x_n} u(x', 0) \end{aligned}$$

Then for $k \in \{0, 1, 2, 3, 4\}$ we need

$$1 = (-1)^k c_0 + \left(\frac{-1}{2}\right)^k c_1 + \left(\frac{-1}{4}\right)^k c_2 + \left(\frac{-1}{8}\right)^k c_3 + \left(\frac{-1}{16}\right)^k c_4$$

This is a nonsingular linear system; it has a unique solution of $c_0 = \frac{51}{7}$, $c_1 = -\frac{1020}{7}$, $c_2 = -816$, $c_3 = -\frac{10880}{7}$, $c_4 = \frac{6144}{7}$. This extension has 4 derivatives by construction, and thus, has at least 3 continuous derivatives, so it's in $C^{3,1}$.

Next, we show that $||Eu||_{W^{4,p}} \leq C||u||_{W^{4,p}}$.

$$\begin{aligned}
\sum_{|\alpha|\leq 4}\int_{R^n} |D^\alpha Eu|^p \text{ d}x =& \sum_{|\alpha|\leq 4}\int_{R^{n-1}\times(0,\infty)} |D^\alpha u|^p \text{ d}x + \int_{R^{n-1}\times(-\infty,0)} |D^\alpha Eu|^p \text{ d}x \\
=&||u||_{W^{4,p}} + \sum_{|\alpha|\leq 4}\int_{R^{n-1}\times(-\infty,0)} |D^\alpha Eu|^p \text{ d}x \\
=&||u||_{W^{4,p}} + \sum_{|\alpha|\leq 4}\int_{R^{n-1}\times(-\infty,0)} \left|D^\alpha \sum_{i=0}^{4} c_i u(x_1, x_2, \dots, x_{n-1}, -x_n/2^i)\right|^p \text{ d}x \\
\leq&||u||_{W^{4,p}} + \sum_{|\alpha|\leq 4}\int_{R^{n-1}\times(0,\infty)} C\left|\sum_{i=0}^{4} D^\alpha c_i u(x_1, x_2, \dots, x_{n-1}, x_n)\right|^p \text{ d}x \leq C||u||_{W^{4,p}}
\end{aligned}$$

9. Let $\mathbb{R}^+ = \{x \in \mathbb{R}; x > 0\}$ and assume that $u \in W^{2,p}(\mathbb{R}^+)$. Define the symmetric extension of u by setting $Eu(x) = u(|x|)$. Prove that $Eu \in W^{1,p}(\mathbb{R})$ but $Eu \notin W^{2,p}(\mathbb{R})$ in general.

   $Eu(x) = u(|x|)$ Then $||Eu||_{L^p(\mathbb{R})} = \int_0^\infty |u(x)|^p \text{ d}x + \int_0^\infty |u(x)|^p \text{ d}x = 2||u||_{L^p(\mathbb{R})}$. Next, we show that Eu has a weak derivative by some substitions and integration by parts.

$$\begin{aligned}
\int_{-\infty}^{\infty} Eu(x)\phi'(x) \text{ d}x =& \int_{-\infty}^{0} u(-x)\phi'(x) \text{ d}x + \int_0^\infty u(x)\phi'(x) \text{ d}x \\
=& -\int_{\infty}^{0} u(x)\phi'(-x) \text{ d}x + \int_0^\infty u(x)\phi'(x) \text{ d}x \\
=& \int_0^\infty u(x)(\phi'(x) + \phi'(-x)) \text{ d}x = -\int_0^\infty u'(x)(\phi(x) + \phi(-x)) \text{ d}x \\
=& -\int_{-\infty}^{0} u'(-x)\phi(x) \text{ d}x - \int_0^\infty u'(x)\phi(x) \text{ d}x = -\int_{-\infty}^{\infty} [DEu(x)]\phi(x) \text{ d}x
\end{aligned}$$

   Next, we show that $||DEu||_{L^p(\mathbb{R})}$ is finite, thus proving that $Eu \in W^{2,p}$

$$\begin{aligned}
\int_{-\infty}^{\infty} |DEu|^p \text{ d}x =& \int_{-\infty}^{0} |D[u(-x)]|^p \text{ d}x + \int_0^\infty |D[u(x)]|^p \text{ d}x \\
=& \int_{-\infty}^{0} |Du(-x)|^p \text{ d}x + ||Du||_{L^p(R_+)} \\
=& 2||Du||_{L^p(R_+)}
\end{aligned}$$

   Thus, $||Eu||_{W^{1,p}(\mathbb{R})}$ is finite, completing the proof.a

   Next, we give a counter-example $u \in W^{2,p}(\mathbb{R}^+)$ with $Eu \notin W^{2,p}(\mathbb{R})$. Consider $u(x) = e^{-x}$. Then $Eu(x) = e^{-|x|}$, and

$$DEu(x) = \begin{cases} -e^{-x} & x > 0 \\ e^x & x < 0 \end{cases}$$

Suppose that this has a weak derivative. Then integrating by parts gives

$$\begin{aligned}\int_{-\infty}^{\infty} DEu(x)\phi'(x) = \int_{-\infty}^{\infty} v\phi(x)\,\mathrm{d}x = &\int_{-\infty}^{0} e^{x}\phi'(x)\,\mathrm{d}x - \int_{0}^{\infty} e^{-x}\phi'(x)\,\mathrm{d}x \\ =&\int_{0}^{\infty} e^{-x}\phi'(-x)\,\mathrm{d}x + \phi(0) + \int_{0}^{\infty} e^{-x}\phi(x)\,\mathrm{d}x \\ =&\phi(0) - \int_{0}^{\infty} e^{-x}\phi(-x)\,\mathrm{d}x + \phi(0) + \int_{0}^{\infty} e^{-x}\phi(x)\,\mathrm{d}x \\ =&2\phi(0) + \int_{0}^{\infty} e^{-x}(\phi(x) - \phi(-x))\,\mathrm{d}x\end{aligned}$$

Consider a monotonic bounded sequence ϕ_m with $\phi_m(0) = 1$, $\phi_m(x) = \phi_m(-x)$, and $\phi_m(x) \to 0$ when $x \neq 0$, and we have a contradiction as then by the monotone convergence theorem, $0 = \lim \int_{-\infty}^{\infty} v\phi_m\,\mathrm{d}x \neq 2$. Thus, Eu does not necessarily have 2 weak derivatives.

10. (a) If $n = 1$ and $u \in W^{1,1}(\Omega)$ then show that $u \in L^{\infty}$ and u is continuous.

First, WLOG, assume $\Omega = \mathbb{R}$, since we can extend u to $W^{1,1}(\mathbb{R})$ if it's not. Then by problem 7, we have $u \to 0$ as $x \to \pm\infty$. $u \in W^{1,1}(\Omega)$, so Du exists, $\int_{\Omega} |u|\,\mathrm{d}x$, $\int_{\Omega} |Du|\,\mathrm{d}x$ are finite. If we assume that Ω is $\mathbb{R}$, we have $W^{1,p}(\Omega) = W_0^{1,p}(\Omega)$, and we can assume it u compact support. Then, since we have $|u(x)| \leq \int_{-\infty}^{x} |u'|\,\mathrm{d}x \leq ||Du||_{L^1(\mathbb{R})}$, $u \in L^{\infty}(\mathbb{R})$.

Next, we need to show that u is continuous. Assume not, let x_d be a point of discontinuity, $x_{min} = \inf(\Omega)$, $x_{max} = \sup(\Omega)$, $u^+ = \lim\limits_{x \to x_d, x > x_d} u(x)$, $u^- = \lim\limits_{x \to x_d, x < x_d} u(x)$. Then

$$\begin{aligned}\int_{\Omega} u(x)\phi'(x)\,\mathrm{d}x = &\int_{x_{min}}^{x_0} u(x)\phi'(x)\,\mathrm{d}x + \int_{x_0}^{x_{max}} u(x)\phi'(x)\,\mathrm{d}x \\ =&u^-\phi(x_d) - u(x_{min})\phi(x_{min}) - \int_{x_{min}}^{x_0} Du(x)\phi(x)\,\mathrm{d}x \\ &+ u(x_{max})\phi(x_{max}) - u^+\phi(x_d) - \int_{x_{min}}^{x_0} Du(x)\phi(x)\,\mathrm{d}x \\ =&(u^- - u^+)\phi(x_d) + u(x_{max})\phi(x_{max}) - u(x_{min})\phi(x_{min}) \\ &- \int_{x_{min}}^{x_0} Du(x)\phi(x)\,\mathrm{d}x - \int_{x_{min}}^{x_0} Du(x)\phi(x)\,\mathrm{d}x\end{aligned}$$

As we did in problem 9, let ϕ_m be a monotonic sequence with $\phi_m(x_d) \neq 0$ and $\phi_m(x) \to 0$ for $x \neq x_d$. Then $\int_{\Omega} Du(x)\phi_m(x)\,\mathrm{d}x \to 0 \neq (u^- - u^+)\phi(x_d)$, contradicting the claim that u has a discontinuity.

(b) If $n > 1$, find an example of $u \in W^{1,n}(B_1)$ and $u \notin L^{\infty}$

The previous functions don't work, becuase $\gamma < 0$ puts the function in L^{∞}. Example 4 from Evans doesn't work, since that requires $\gamma \in \emptyset$.