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Math 516
Homework 4

I collaborated with Damien Huet on this assignment

1. Let $u \in C_c^\infty(\mathbb{R}^n)$. Show that $|u(x_1, \dots, x_n)| \leq \frac{1}{2}\int_{-\infty}^{\infty} |\partial_1 u(x_1, \dots, x_n)| \,\mathrm{d}x_1$.

   Since $u \in C_c^\infty(\mathbb{R}^n)$, exists $R > 0$ s.t. $u(B_0(R)^C) = \frac{\partial u}{\partial x_1}(B_0(R)^C) = \{0\}$. Then by the fundamental theorem of calculus,

$$u(x_1, x_2, \dots, x_n) = \int_{y=-R}^{x_1} \frac{\partial u}{\partial x_1}(y, x_2, x_3, \dots, x_n) \,\mathrm{d}y = -\int_{y=x_1}^{R} \frac{\partial u}{\partial x_1}(y, x_2, x_3, \dots, x_n) \,\mathrm{d}y$$
$$|u(x_1, x_2, \dots, x_n)| \leq \int_{y=-R}^{x_1} \left|\frac{\partial u}{\partial x_1}(y, x_2, x_3, \dots, x_n)\right| \,\mathrm{d}y$$
$$|u(x_1, x_2, \dots, x_n)| \leq \int_{y=x_1}^{R} \left|\frac{\partial u}{\partial x_1}(y, x_2, x_3, \dots, x_n)\right| \,\mathrm{d}y$$

   Since both of the previous are true, we must have

$$\begin{aligned}
2|u(x_1, x_2, \dots, x_n)| \leq& 2\min\left(\int_{y=-R}^{x_1} \left|\frac{\partial u}{\partial x_1}(y, x_2, x_3, \dots, x_n)\right| \,\mathrm{d}y, \int_{y=x_1}^{R} \left|\frac{\partial u}{\partial x_1}(y, x_2, x_3, \dots, x_n)\right| \,\mathrm{d}y\right) \\
\leq& \min\left(\int_{y=-R}^{x_1} \left|\frac{\partial u}{\partial x_1}(y, x_2, x_3, \dots, x_n)\right| \,\mathrm{d}y, \int_{y=x_1}^{R} \left|\frac{\partial u}{\partial x_1}(y, x_2, x_3, \dots, x_n)\right| \,\mathrm{d}y\right) \\
&+ \max\left(\int_{y=-R}^{x_1} \left|\frac{\partial u}{\partial x_1}(y, x_2, x_3, \dots, x_n)\right| \,\mathrm{d}y, \int_{y=x_1}^{R} \left|\frac{\partial u}{\partial x_1}(y, x_2, x_3, \dots, x_n)\right| \,\mathrm{d}y\right) \\
=& \int_{y=-\infty}^{\infty} \left|\frac{\partial u}{\partial x_1}(y, x_2, x_3, \dots, x_n)\right| \,\mathrm{d}y
\end{aligned}$$

   proving the claim.

2. (Gagliardo-Nirenberg interpolation inequality) Let $n \geq 2$, $1 < p < n$, and $1 \leq q < r < \frac{np}{n-p}$. For some $\theta \in (0,1)$ and some constant $C > 0$ we have

$$||u||_{L^r(\mathbb{R}^n)} \leq C||u||_{L^q(\mathbb{R}^n)}^{1-\theta}||Du||_{L^p(\mathbb{R}^n)}^{\theta}; \forall u \in C_c^\infty(\mathbb{R}^n)$$

   (a) Use scaling to find θ.

   (b) Prove the inequality

   Hint: Do an interpretation of L^r in terms of L^q and $L^{\frac{np}{n-p}}$ and then apply Sobolev. (nb. interpretation should be interpolation)

   First we define $u_\lambda(x) = u(\lambda x)$, and note that

$$\int_{\mathbb{R}^n} |u_\lambda|^s \,\mathrm{d}x = \int_{\mathbb{R}^n} |u(\lambda x)|^s \,\mathrm{d}x = \frac{1}{\lambda^n}\int_{\mathbb{R}^n} |u(y)|^s \,\mathrm{d}y$$
$$\int_{\mathbb{R}^n} |Du_\lambda|^s \,\mathrm{d}x = \int_{\mathbb{R}^n} |Du(\lambda x)|^s \,\mathrm{d}x = \frac{\lambda^s}{\lambda^n}\int_{\mathbb{R}^n} |Du(y)|^s \,\mathrm{d}y$$

Then, if we assume that the $||u||_{L^r(\mathbb{R}^n)} \leq C||u||^{1-\theta}_{L^q(\mathbb{R}^n)}||Du||^{\theta}_{L^p(\mathbb{R}^n)}$ holds for any $u \in C_c^\infty(\mathbb{R}^n)$, we have

$$\begin{aligned}\frac{1}{\lambda^{n/r}}||u||_{L^r(\mathbb{R}^n)} \leq & C\frac{1}{\lambda^{(1-\theta)n/q}}||u||^{1-\theta}_{L^q(\mathbb{R}^n)}\frac{\lambda^\theta}{\lambda^{\theta n/p}}||Du||^{\theta}_{L^p(\mathbb{R}^n)} \\ = & C\lambda^{\theta-(1-\theta)n/q-\theta n/p}||u||^{1-\theta}_{L^q(\mathbb{R}^n)}||Du||^{\theta}_{L^p(\mathbb{R}^n)}\end{aligned}$$

Then if $n/r+\theta-(1-\theta)n/q-\theta n/p > 0$, since C is independent of u, any sequence with $\lambda \to 0$ will result in

$$C\lambda^{n/r+\theta-(1-\theta)n/q-\theta n/p}||u||^{1-\theta}_{L^q(\mathbb{R}^n)}||Du||^{\theta}_{L^p(\mathbb{R}^n)} \to 0$$

which contradicts the requirement that

$$||u||_{L^r(\mathbb{R}^n)} \leq C\lambda^{n/r+\theta-(1-\theta)n/q-\theta n/p}||u||^{1-\theta}_{L^q(\mathbb{R}^n)}||Du||^{\theta}_{L^p(\mathbb{R}^n)}$$

for any non-zero $u \in L^r(\mathbb{R}^n)$. Considering $n/r+\theta-(1-\theta)n/q-\theta n/p \leq 0$, any sequence with $\lambda \to \infty$ will result in

$$C\lambda^{n/r+\theta-(1-\theta)n/q-\theta n/p}||u||^{1-\theta}_{L^q(\mathbb{R}^n)}||Du||^{\theta}_{L^p(\mathbb{R}^n)} \to 0$$

causing the same contradiction.

Thus, we must have $n/r+\theta-(1-\theta)n/q-\theta n/p = 0$; or

$$\begin{aligned}\theta = & \frac{\frac{n}{q}-\frac{n}{r}}{1+\frac{n}{q}-\frac{n}{p}} \\ = & \frac{pn}{r}\frac{r-q}{pq+np-nq}\end{aligned}$$

Next, we show that we can choose a C s.t. the original inequality holds when θ satisfies the previous. By the interpolation inequality, if $\frac{1}{r} = \frac{1-\theta}{q} + \frac{\theta}{np/(n-p)}$, there exists a constant C s.t.

$$||u||_{L^r(\mathbb{R}^n)} \leq C||u||^{1-\theta}_{L^q(\mathbb{R}^n)}||u||^{\theta}_{L^{\frac{np}{n-p}}(\mathbb{R}^n)}$$

Happily, the relation $\frac{1}{r} = \frac{1-\theta}{q} + \frac{\theta}{np/(n-p)}$ holds with our choice of θ above.

Then, by the Gagliardo-Nirenberg-Sobolev inequality, since $1 \leq p < n$ and $1 \leq \frac{np}{n-p} < \infty$, for some other C we have

$$\begin{aligned}||u||_{L^{\frac{np}{n-p}}(\mathbb{R}^n)} \leq & C||Du||_{L^p(\mathbb{R}^n)} \\ ||u||^{\theta}_{L^{\frac{np}{n-p}}(\mathbb{R}^n)} \leq & C||Du||^{\theta}_{L^p(\mathbb{R}^n)}\end{aligned}$$

Thus,

$$||u||_{L^r(\mathbb{R}^n)} \leq C||u||^{1-\theta}_{L^q(\mathbb{R}^n)}||Du||^{\theta}_{L^p(\mathbb{R}^n)}$$

completing our proof.

3. Fix $\alpha > 0$, $1 < p < \infty$, and let $U = B_1(0)$. Show that there exists a constant C, depending on n, p, α s.t.

$$\int_U u^p \, \mathrm{d}x \le C \int_U |Du|^p \, \mathrm{d}x$$

provided

$$u \in W^{1,p}(U), |\{x \in U | u(x) = 0\}| \ge \alpha$$

First we consider the case where $p < n$. Then $p \in \left[1, \frac{np}{n-p}\right]$, so by Theorem 5.6.3 in Evans, there exists a constant C s.t. $||u||_{L^p(U)} \le C||Du||_{L^p(U)}$. This leaves us with the cases with $p \ge n$.

Assume that the claim is false; ie for any $k \in \mathbb{N}$, we can find a u_k with $||u_k||_{L^p(U)} > k\,||Du_k||_{L^p(U)}$. Define $v_k = \frac{u_k}{||u_k||_{L^p(U)}}$; then $||v_k||_{L^p(U)} = 1 > k\,||Dv_k||_{L^p(U)}$. Then $||v_k||_{W^{p,1}(U)} \le 2$, so by the compactness of $W^{k,p}(U)$ in $L^p(U)$, there must be a subsequence $\{v_{k_j}\}$ and $v \in L^p(U)$ with $\left|\left|v_{k_j} - v\right|\right|_{L^{\to}(U)} 0$. Then by the dominated convergence theorem

$$\begin{aligned}\int_U v\phi_{x_i} \, \mathrm{d}x &= \int_U \lim_{j\to\infty} v_{k_j}\phi_{x_i} \, \mathrm{d}x \\ &= \lim_{j\to\infty} \int_U v_{k_j}\phi_{x_i} \, \mathrm{d}x = \lim_{j\to\infty} -\int_U v_{k_j,x_i}\phi \, \mathrm{d}x = 0\end{aligned}$$

Thus, $Dv = 0$ almost everywhere, implying v is constant. Since $|\{x|v_k(x) = 0\}| \ge \alpha$, we must also have $|\{x|v(x) = 0\}| \ge \alpha > 0$ Then $v = 0$, contradicting $||v||_{L^p(U)} = 1$, and thus, that the sequence $\{v_k|\, ||v_k||_{L^p(U)} > k\,||Dv_k||_{L^p(U)}\}$ exists.

4. (a) Show that $W^{1,2}(\mathbb{R}^N) \subset L^2(\mathbb{R}^N)$ is not compact.

First define $U = \mathbb{R}^n$.

Then given a nonzero $u \in W^{2,2}(U)$, define $u_k(x_1, x_2, \ldots, x_n) = u(x_1 - k, x_2, \ldots, x_n)$. $\{u_k\}$ does not have a subsequence $\{u_{k_j}\}$ with $\lim_{j\to\infty} \left|\left|u_{k_j} - v\right|\right|_{L^2(U)} = 0$ for any v, while $||u_k||_{L^2(U)} = ||u||_{L^2(U)} < \infty$.

Given an $0 < 2\epsilon < ||u||_{W^{2,2}(U)}$, consider any v s.t. a given u_k, $||u_k - v||_{L^2(U)} < \epsilon$. Since $u_k \in W^{1,2}(U)$, it has compact support, so there is an $R > 0$ with $u_k(B_0(R)^C) = \{0\}$. Then

$$\epsilon > ||u_k - v||^2_{L^2(U)} = \int_{B_0(R)} |u_k - v|^2 \, \mathrm{d}x + \int_{B_0(R)^C} |v|^2 \, \mathrm{d}x \ge \int_{B_0(R)^C} |v|^2 \, \mathrm{d}x$$

Next, we can choose given any subsequence $\{u_{k_j}\}$, we can choose a J s.t. $\forall j \ge J$, $k_j > 2R$. These u_{k_j} have compact support in a region excluding $B_0(R)$. We compute

$$\begin{aligned}\left|\int_{B_0(R)} |v|^2 \, \mathrm{d}x - \int_{B_0(R)} |u_k|^2 \, \mathrm{d}x\right| &\le \int_{B_0(R)} |u_k - v|^2 \, \mathrm{d}x < \epsilon \\ -\epsilon + \int_{B_0(R)} |u_k|^2 \, \mathrm{d}x &< \int_{B_0(R)} |v|^2 \, \mathrm{d}x \\ \epsilon &< \int_{B_0(R)} |v|^2 \, \mathrm{d}x \\ \left|\left|u_{k_j} - v\right|\right|^2_{L^2(U)} &= \int_{B_0(R)} |v|^2 \, \mathrm{d}x + \int_{B_0(R)^C} |u_{k_j} - v|^2 \, \mathrm{d}x \\ &> \epsilon + \int_{B_0(R)^C} |u_{k_j} - v|^2 \, \mathrm{d}x > \epsilon\end{aligned}$$

showing that we can never choose a J with $||u_{k_j} - v||^2_{L^2(U)} < \epsilon$ for all $j \geq J$
Thus, $W^{1,2}(U)$ is not compactly contained in $L^2(U)$.

(b) Let $n > 4$ Show that the embedding $W^{2,2}(U) \to L^{\frac{2n}{n-4}}(U)$ is not compact.
Given a non-negative $u \in W^{2,2}(U)$ with finite $||u||_{W^{2,2}(U)}$ and $||u||_{L^2(U)}$, define $u_\lambda = \lambda^\theta u(\frac{x}{\lambda})$. Then

$$||u_\lambda||^2_{W^{2,2}(U)} = \lambda^{2\theta-n} \int_U u^2 \, \mathrm{d}x + \lambda^{2\theta-n-2} \int_U |Du|^2 \, \mathrm{d}x + \lambda^{2\theta-n-4} \int_U |D^2u|^2 \, \mathrm{d}x$$

Then if we choose θ such that $2\theta - n \leq 0$, this remains bounded as $\lambda \to \infty$. However, the $L^{\frac{2n}{n-4}}(U)$ norm does not necessarily remain bounded:

$$\begin{aligned} ||u_\lambda||^2_{L^{\frac{2n}{n-4}}(U)} &= \lambda^{\frac{2n}{n-4}\theta - n} \int_U u^2 \, \mathrm{d}x \\ 0 &\geq \frac{2n}{n-4}\theta - n = \frac{(2\theta - n + 4)n}{n-4} \\ 0 &\geq 2\theta - n + 4 \end{aligned}$$

Choosing $n - 4 < 2\theta < n$ gives a sequence which is unbounded in the $L^{\frac{2n}{n-4}}(U)$ space, but bounded and convergent to 0 in the $W^{2,2}(U)$ space. This violates $||u_\lambda||_{L^{\frac{2n}{n-4}}(U)} \leq C \, ||u_\lambda||_{W^{2,2}(U)}$, since for any C, we can increase λ s.t. the previous no longer holds, proving that there is not a compact embedding.
Also, for any $v \in L^{\frac{2n}{n-4}}(U)$, we have

$$\begin{aligned} \infty = \lim_{\lambda \to \infty} \left| ||u_\lambda||_{L^{\frac{2n}{n-4}}(U)} - ||v||_{L^{\frac{2n}{n-4}}(U)} \right| &\leq \lim_{\lambda \to \infty} |||u_\lambda| - |v|||_{L^{\frac{2n}{n-4}}(U)} \\ \leq \lim_{\lambda \to \infty} ||u_\lambda - v||_{L^{\frac{2n}{n-4}}(U)} \end{aligned}$$

Since this is true for any sequence with $\lambda \to \infty$, $W^{2,2}(U)$ cannot be compactly contained in $L^{\frac{2n}{n-4}}(U)$.

(c) Describe the embedding of $W^{3,p}(U)$ in different dimensions. State if the embedding is continuous or compact.
By the Rellich-Kondrachov Compactness theorem, whenever $1 \leq p < n$, $W^{3,p}(U)$ is compactly contained in $L^q(U)$ for $1 \leq q < \frac{np}{n-p}$. Then by the General Sobolev inequality, if $3 < \frac{n}{p}$ and $\frac{1}{q} = \frac{1}{p} - \frac{3}{n}$, the embedding is continuous.
7 cases, 2 borderline cases, $n = 3$, compact embedding, rest are cases from the Sobolev inequality and Morrey's inequality.

5. (a) Let $u \in W^{1,2}_r(\mathbb{R}^n) \subset H^1_r(\mathbb{R}^n) = \{u \in W^{1,2}(\mathbb{R}^n) | u = u(r)\}$. Show that $|u(r)| \leq C \, ||u||_{W^{1,2}(\mathbb{R}^n)} \, r^{\frac{1-n}{2}}$.
First define $u(x) = f(|x|)$. Note that $\frac{\partial}{\partial x_i} u(x) = f'(|x|) \frac{x_i}{|x|}$. Next, we consider $\frac{\mathrm{d}}{\mathrm{d}r}[r^{2m} f^2(r)]$ and compare it to $[\frac{\mathrm{d}}{\mathrm{d}r}(r^m f(r))]^2 + (r^m f(r))^2$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r}[r^{2m} f^2(r)] &= r^{2m-1} f^2(r) + r^{2m} f(r) f'(r) \\ [(r^m f(r))']^2 + (r^m f(r))^2 &= r^{2m}[(f'(r))^2 + f^2(r)] + m \frac{\mathrm{d}}{\mathrm{d}r}(r^{2m-1} f^2(r)) - m(m-1) r^{2m-2} f^2(r) \\ 0 &\leq \left[\frac{\mathrm{d}}{\mathrm{d}r}(r^m f(r))\right]^2 + (r^m f(r))^2 - \frac{\mathrm{d}}{\mathrm{d}r}[r^{2m} f^2(r)] \\ &= r^{2m}[f(r) - f'(r)]^2 + m r^{2m-2}((m - 2r) f^2(r) + 2r f(r) f'(r)) \\ &= r^{2m-2}[-m f(r) + r f(r) - r f'(r)]^2 \end{aligned}$$

The last equation shows the inequality is true, since $r > 0$ and $[-mf(r) + rf(r) - rf'(r)]^2 > 0$. Thus, $\frac{\mathrm{d}}{\mathrm{d}r}(r^{2m}f^2(r)) \leq [\frac{\mathrm{d}}{\mathrm{d}r}(r^m f(r))]^2 + (r^m f(r))^2$.

Next, we apply compact support of our function with the fundamental theorem of calculus and triangle inequality to put an upper bound on the pointwise value of the function. When $m = \frac{n-1}{2}$, $n \geq 3$, we have

$$\begin{aligned} r^{2m}f(r)^2 &= \left| \int_r^\infty \frac{\mathrm{d}}{\mathrm{d}s}[s^{2m}f(s)] \, \mathrm{d}s \right| \\ &\leq \int_r^\infty \left[\frac{\mathrm{d}}{\mathrm{d}s}(s^m f(s))\right]^2 \, \mathrm{d}s + \int_r^\infty (s^m f(s))^2 \, \mathrm{d}s \\ &\leq \frac{1}{n\alpha_n} \, ||f(s)||^2_{W^{1,2}(\mathbb{R}^n)} \end{aligned}$$

Then $|u(x)| = |f(r)| \leq \frac{C}{r^{\frac{n-1}{2}}} \, ||u(x)||_{W^{1,2}(\mathbb{R}^n)}$, completing our proof.

(b) Show that for $n \geq 2$, the embedding $W_r^{1,2} \subset L^p$ is compact when $2 < p < \frac{2n}{n-2}$.

First, recall that $W^{1,2}(\mathbb{R}^n) = W_C^{1,2}(\mathbb{R}^n)$. Then consider any sequence $\{u_k\} \subset W_r^{1,2}(\mathbb{R}^n)$ with $||u_k||_{W^{1,2}(\mathbb{R}^n)} < C$.

(c) Let $u = D_r^{1,2} = \{\int |\nabla u|^2 \, \mathrm{d}x < \infty | u = u(r)\}$. Show that $D_r^{1,2} \subset L^{\frac{2n}{n-2}}$ and $|u(r)| \leq C \, ||Du||_{L^2()} \, r^{\frac{2-n}{2}}$. However show that $D_r^{1,2} \subset L^{\frac{2n}{n-2}}$ is not compact.

6. Let $U = (-1, 1)$. Show that the dual space of $H^1(U)$ is isomorphic to $H^{-1}(U) + E^*$ where E^* is the two dimensional subspace of $H^1(U)$ spanned by the orthogonal vectors $\{e^x, e^{-x}\}$.

Let $u \in H_0^1(U)$, $v \in H^1(U) \cap (H_0^1(U))^\perp$, with $(f, g) = \int_U uv + u'v' \, \mathrm{d}x$ for $f, g \in H^1(U)$. Then $(u, v) = 0 = \int_U uv + u'v' \, \mathrm{d}x$. If $v \in C^2(U)$, integrating by parts gives

$$\begin{aligned} 0 &= \int_U uv - uv'' \, \mathrm{d}x + u(1)v'(1) - u(-1)v'(-1) \\ &= \int_U u(v - v'') \, \mathrm{d}x \end{aligned}$$

Then v is a weak solution to the ODE $-v'' + v = 0$.

Recall that given boundary conditions, the strong form has a unique solution, namely $v = c_1e^x + c_2e^{-x}$. Since the solution to the strong form is a solution to the weak form, and solutions to this weak form are unique, this solution is the only solution to our weak form. Thus, if $v \in H^1(U)$, we must have $v = u + c_1e^x + c_2e^{-x}$ for some $u \in H_0^1(U)$, $c_1, c_2 \in \mathbb{R}$, or equivalently, $H^1 = H_0^1 + E$.

7. (a) Assume that U is connected. A function $u \in W^{1,2}(U)$ is a weak solution of the Neumann problem

$$\begin{cases} -\Delta u(x) = f(x) & x \in U \\ \frac{\partial u}{\partial \nu} = 0 & x \in \partial U \end{cases}$$

if

$$\int_U Du \cdot Dv \, \mathrm{d}x = \int_U fv \, \mathrm{d}x, \forall v \in W^{1,2}(U)$$

Suppose that $f \in L^2(U)$. Show that this only has a solution iff $\int_U f(x) \, \mathrm{d}x = 0$

If U is bounded, then we can just take $v = 1 \in W^{1,2}(U)$, as $Dv = 0 \to 0 = \int_U Du \cdot Dv =< f, v >$.
If U is unbounded, then we note that since $u \in W^{1,2}(U)$, there must be some $R > 0$ with $u(B_R^C \cap U) = \{0\}$. Then $Du(B_R^C \cap U) = \{0\}$. If we then choose $v = 1 \in B_R$ and $v = 0 \in B_{R+1}$ with some smooth transition, we must have $0 = \int_U Du \cdot Dv \text{ d}x = \int_{U \cap B_R} f \text{ d}x + \int_{U \cap B_R^C \cap B_{R+1}} fv \text{ d}x$

(b) Discuss how to define a weak solution of the Poisson equation with Robin boundary conditions

$$\begin{cases} -\Delta u(x) = f(x) & x \in U \\ u + \frac{\partial u}{\partial \nu} = 0 & x \in \partial U \end{cases}$$

Starting with the form of the PDE, we require that $< f, v >= -\int_U \Delta u(x) v(x) \text{ d}x$. Integrating by parts gives

$$\begin{aligned} < f, v >&= \int_U Du \cdot Dv \text{ d}x - \int_{\partial U} \frac{\partial u}{\partial \nu} v \text{ d}S \\ &= \int_U Du \cdot Dv \text{ d}x + \int_{\partial U} uv \text{ d}S \end{aligned}$$

with $q \in \mathbb{R}$. Thus, any solution to the strong problem will be a weak solution as well. Next we need to show that this problem is well defined.
By the infinite differentiability up to the boundary theorem, if $f \in C^\infty(\overline{U})$ and $\partial U \in C^\infty$, then $u \in W^{1,2}(U) \cap C(\overline{U})$. Then by the trace theorem, if $\partial U \in C^1$ and $u \in W^{1,2}(U) \cap C(\overline{U})$, then $Tu = u|_{\partial U}$ and $||Tu||_{L^2(\partial U)} \le C\,||u||_{W^{1,2}(U)}$. So if we define a bilinear operator $B[u, v] = \int_U Du \cdot Dv \text{ d}x + \int_{\partial U} uv \text{ d}S$, then $B[u, u] = ||Du||_{L^2(U)}^2 + ||u||_{L^2(\partial U)}^2 \le C\,||u||_{W^{1,2}(U)}$ whenever these assumptions hold. We can also show that B is coercive in most cases, and then Lax Millgram applies, proving that this has a unique solution.

8. (a) Discuss the definition of weak solutions $u \in H_0^2(U)$ to

$$\begin{cases} \Delta^2 u(x) = f(x) & x \in U \\ u(x) = \frac{\partial u}{\partial \nu}(x) = 0 & x \in \partial U \end{cases}$$

Consider $v \in C^2(U)$. Then $\int_U fv \text{ d}x = \int_U (\Delta^2 u) v \text{ d}x$. Integrating by parts and noting that $u \in H_0^2(U) \to D^\alpha u = 0$ gives

$$\begin{aligned} \int_U fv \text{ d}x &= -\int_U D(\Delta u) \cdot Dv \text{ d}x + \int_{\partial U} \left(\frac{\partial}{\partial \nu} \Delta u \right) v \text{ d}S \\ &= -\int_U D(\Delta u) \cdot Dv \text{ d}x = \int_U (\Delta u)(\Delta v) \text{ d}x - \int_{\partial U} \Delta u \frac{\partial}{\partial \nu} v \text{ d}S \\ &= \int_U (\Delta u)(\Delta v) \text{ d}x \end{aligned}$$

Then any solution to the strong problem will be a solution to the above weak problem. We can then define the bilinear form $B[u, v] = \int_U (\Delta u)(\Delta v) \text{ d}x$, with $B : H^2(U) \times H^2(U) \to \mathbb{R}$. We can use the Sobolev Inequality and a modification of the argument for Poincare's Inequality to prove that this is a bounded coercive form. Thus, Lax-Millgram applies, so we know that there is a unique $u \in H_0^2(U)$ with $B[u, v] =< f, v >$.

(b) Given $f \in L^2(U)$ prove that there exists a unique weak solution to $\Delta^2 u(x) = f(x)$.
First

$$
\begin{aligned}
|B[u,v]| &= \left| \int_U (\Delta u)(\Delta v) \, \mathrm{d}x \right| \\
&\leq \int_U |\Delta u||\Delta v| \, \mathrm{d}x \leq C \, ||\Delta u||_{L^2(U)} \, ||\Delta v||_{L^2(U)} \\
&\leq C \, ||u||_{H^2(U)} \, ||v||_{H^2(U)}
\end{aligned}
$$

Thus, B is a bounded bilinear form.
Next, we need to show that B is coercive, i.e. that there is a constant C s.t. $||u||_{H^2(U)} \leq CB[u,u]$.
If $n \geq 3$, we can apply Evans 5.6.3, so

$$
\begin{aligned}
||u||_{H^2(U)}^2 &= \int_U u^2 \, \mathrm{d}x + \int_U |Du|^2 \, \mathrm{d}x + \int_U |\Delta u|^2 \, \mathrm{d}x \\
&\leq C \int_U |Du|^2 \, \mathrm{d}x + \int_U |\Delta u|^2 \, \mathrm{d}x \\
&\leq C \int_U |\Delta u|^2 \, \mathrm{d}x = CB[u,u]
\end{aligned}
$$

Otherwise, assume that there is a sequence $\{u_k\}$ with $||u_k||_{H^2(U)}^2 > kB[u_k u_k]$. Note that $u_k \neq 0$, as otherwise the above inequality does not hold. Then since B is linear, we can assume that $||u_k||_{H^2(U)} = 1$. Then

$$
\begin{aligned}
0 = \lim_{k \to \infty} \frac{1}{k} &> \lim_{k \to \infty} ||\Delta u_k||_{L^2(U)}^2 \\
0 = \lim_{k \to \infty} \Delta u_k &= \Delta u
\end{aligned}
$$

Then (as shown in a previous homework) Du is constant in U. But $Du = 0$ on ∂U, so $Du = 0$ everywhere. Then by the same argument, $u = 0$ everywhere, contradicting our assumption.
Thus, B is coercive and bounded, so Lax-Millgram applies, proving that there is a unique $u \in H_0^2(U)$ s.t. $B[u,v] =< f, v >$ whenever $v \in C^\infty(U)$.