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3) a)

$$f(s, x_1, \dots, x_{s+1}) = f(s)\prod_{i=1}^{s+1} f(x_i|s)$$

$$= \exp\Big\{ s\log\frac{p}{1-p} + \mu\sum_{i=1}^{s+1} x_i - \frac{s}{2}\mu^2 - \frac{\mu^2}{2} + n\log(1-p) \underbrace{- \frac{1}{2}\sum_{i=1}^{s+1} x_i^2 + \log\binom{n}{s} - \frac{n}{2}\log(2\pi)}_{-c(s,x)}\Big\}$$

$$= \exp\{ T(s,x)^T Q(p,\mu) - b(p,\mu) - c(s,x)\}$$

where

$$T(s,x)^T = \Big(s, \sum^{s+1} x_i\Big) \longleftarrow \text{CSS}$$

$$Q(p,\mu)^T = \Big(\log\frac{p}{1-p} - \frac{\mu^2}{2},\ \mu\Big) = (\theta_1, \theta_2)$$

$$b(p,\mu) = \frac{\mu^2}{2} - n\log(1-p)$$

$\ddot{b}(\theta)$ is invertible $\Rightarrow$ exp fam is full rank

$\underline{\mu = 0}$

$$f(s, x_1, \dots, x_{s+1}|\mu=0) = \exp\Big\{ s\log\frac{p}{1-p} + n\log(1-p) - c(s,x)\Big\}$$

$$= \exp\{ s\,\theta_1 - b_2(\theta_1) - c(s,x)\}$$

b) $\hat{p} = s/n \qquad \hat{\mu} = \frac{1}{s+1}\sum_{i=1}^{s+1} = \bar{X}$

c) $$I_n(p,\mu) = \begin{pmatrix} \frac{n}{p(1-p)} & 0 \\ 0 & np+1 \end{pmatrix}$$

$$I(p,\mu) = \lim_{n\to\infty}\frac{1}{n}\begin{pmatrix} \frac{n}{p(1-p)} & 0 \\ 0 & np+1 \end{pmatrix} = \lim_{n\to\infty}\begin{pmatrix} \frac{1}{p(1-p)} & 0 \\ 0 & p+\frac{1}{n} \end{pmatrix} = \begin{pmatrix} \frac{1}{p(1-p)} & 0 \\ 0 & p \end{pmatrix}$$

$$\Rightarrow \sqrt{n}\left(\begin{pmatrix}\hat{p}\\ \hat{\mu}\end{pmatrix} - \begin{pmatrix}p\\ \mu\end{pmatrix}\right) \to_d N\left(\begin{pmatrix}0\\0\end{pmatrix}, \underbrace{\begin{pmatrix} p(1-p) & 0 \\ 0 & 1/p \end{pmatrix}}_{I(p,\mu)^{-1}}\right)$$

from b) $$\frac{d}{dp}\ell = \frac{s}{p} - \frac{(n-s)}{1-p}$$

$$\frac{d}{d\mu}\ell = \sum^{s+1} x_i - (s+1)\mu$$

$$\Rightarrow \frac{d^2}{dp\,d\mu}\ell = 0$$

$$\frac{d^2}{dp^2}\ell = \frac{-s}{p^2} - \frac{(n-s)}{(1-p)^2} \Rightarrow -E\Big(\frac{d^2}{dp^2}\ell\Big) = \frac{E(S)}{p^2} + \frac{n-E(S)}{(1-p)^2} = \frac{np}{p^2} + \frac{n(1-p)}{(1-p)^2}$$

$$= n\Big(\frac{1}{p} + \frac{1}{1-p}\Big) = \frac{n}{p(1-p)}(1-\cancel{p}+\cancel{p}) = \frac{n}{p(1-p)}$$

$$\frac{d^2}{d\mu^2}\ell = -(s+1) \Rightarrow -E\Big(\frac{d^2}{d\mu^2}\ell\Big) = E(S)+1 = np+1$$

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Since $\phi$ is unbiased, $\beta_\phi(p,\mu) \le \alpha$ $\forall \theta \in \Theta_0$
& $\beta_\phi(p,\mu) \le \alpha$ $\forall \theta \in \Theta_1$

since we have exp fam, $\beta_\phi(p,\mu)$ is continuous in $\mu$

Fix $p$

Since $\phi$ is level

$\Rightarrow \sup_{\mu \le 0} \beta_\phi(p, \mu) = \alpha$

$\Rightarrow \beta_\phi(p, \mu) \le \alpha \quad \forall \mu \le 0$

$\beta_\phi$ continuous in $\mu$

$\Rightarrow \lim_{\mu \to a} \beta_\phi(p, \mu) = \beta_\phi(p, a)$ for $a \in \mathbb{R}$

$\Rightarrow \lim_{\mu \to 0^-} \beta_\phi(p, \mu) \le \alpha$

$\Rightarrow \beta_\phi(p, 0) \le \alpha$

Unbiased $\Rightarrow \beta_\phi(p, \mu) \ge \alpha \quad \forall \mu > 0$

$\Rightarrow \lim_{\mu \to 0^+} \beta_\phi(p, \mu) \ge \alpha$

$\Rightarrow \beta_\phi(p, 0) \ge \alpha$

$\Rightarrow \beta_\phi(p, 0) = \alpha$

b/, gen. of $p$, holds $\forall\ 0 < p < 1$

3d) let $\phi(S, X_1, \dots, X_{S+1})$ be any unbiased level $\alpha$ test of $H_0: \mu \leq 0$ v $H_1: \mu > 0$. Write what unbiasedness means for the power function $\beta(p, \mu)$ & explain in detail why it implies $\beta(p, 0) = \alpha$ $\forall p$

let $\theta = \{p, \mu\}$

a test $\phi$ is unbiased if $\beta_\phi(p, \mu) \leq \alpha$ $\forall \theta \in \Theta_0$

& $\beta_\phi(p, \mu) \geq \alpha$ $\forall \theta \in \Theta_1$

Thus since $\phi$ is unbiased, $\beta(p, 0) \leq \alpha$ $\forall p \in (0,1)$

However, since $\phi$ is level $\alpha$ & we have a simple null

$\alpha = \sup_{\theta \in \Theta_0} E_\theta[\phi(X)] = \sup_{\mu \leq 0} \beta_\phi(p, \mu)$

since $\beta_\phi(p, \mu)$ is increasing in $\mu$, we have

$\alpha = \sup_{\mu \leq 0} \beta_\phi(p, \mu) = \beta_\phi(p, 0)$ $\forall p$

(note: to prove this, suppose $\exists p^*$ s.t. $\alpha \neq \beta_\phi(p^*, 0)$

$\Rightarrow$ $\phi$ is not a size $\alpha$ test CONTRADICTION!

Thus $\alpha = \beta_\phi(p, 0)$ $\forall p$)