Practice Theory Section 2 2014

1). $Y_{ij} \sim Poi(\mu_{ij})$ indep. across all i & j

$i = 1, \dots, n$; $j = 1, 2$

$\mu_{i1} = \psi \mu_{i2}$ $\quad \psi > 0$

Interested in ψ (other parameters = nuisance $\rightarrow \mu_{i2}$)

(a) Unconditional maximum likelihood estimate of ψ ($\bar{\psi}$) & its observed info when μ_{i2} varies across i. Interpret the results.

$$p(y_{ij}) = \frac{\mu_{ij}^{y_{ij}} e^{-\mu_{ij}}}{y_{ij}!}$$

$$L(\underset{\sim}{\mu}) = \prod_{i=1}^{n} \prod_{j=1}^{2} p(y_{ij})$$

$$= \frac{\left(\prod_{i=1}^{n} \prod_{j=1}^{2} \mu_{ij}^{y_{ij}}\right) \exp\left(-\sum_{i=1}^{n} \sum_{j=1}^{2} \mu_{ij}\right)}{\prod_{i=1}^{n} \prod_{j=1}^{2} y_{ij}!}$$

$$= c(\underset{\sim}{y}) \exp\left(\sum_{i=1}^{n} \sum_{j=1}^{2} y_{ij} \log \mu_{ij} - \sum_{i=1}^{n} \sum_{j=1}^{2} \mu_{ij}\right)$$

$$= c(\underset{\sim}{y}) \exp\Big(\sum_{i=1}^{n} (y_{i1} \log \mu_{i1} + y_{i2} \log \mu_{i2}) + \\ - \sum_{i=1}^{n} (\mu_{i1} + \mu_{i2}) \Big)$$

$\mu_{i1} = \psi \mu_{i2}$

$$= c(\underset{\sim}{y}) \exp\Big(\sum_{i=1}^{n} (y_{i1} \log \psi \mu_{i2} + y_{i2} \log \mu_{i2}) + \\ - \sum_{i=1}^{n} (\psi \mu_{i2} + \mu_{i2}) \Big)$$

$$= c(\underset{\sim}{y}) \exp\Big(\sum_{i=1}^{n} \big[y_{i1} \log \psi + (y_{i1} + y_{i2}) \log \mu_{i2} + \\ - (\psi + 1) \sum_{i=1}^{n} \mu_{i2} \big] \Big)$$

Note: sufficient stat for $\mu_{i2} = y_{i1} + y_{i2}$

sufficient stat for $\psi = y_{i1}$

$$\ell(\psi) = \log L(\psi)$$

$$= \log c(\underset{\sim}{y}) + (\log \psi) \sum_{i=1}^{n} y_{i1} + \sum_{i=1}^{n} (y_{i1} + y_{i2}) \log \mu_{i2} + \\ - (\psi + 1) \sum_{i=1}^{n} \mu_{i2}$$

$$\frac{\partial}{\partial \psi} \ell(\psi) = \frac{\sum_{i=1}^{n} y_{i1}}{\psi} - \sum_{i=1}^{n} \mu_{i2} \overset{\text{set}}{=} 0$$

$$\Rightarrow \frac{\sum_{i=1}^{n} y_{i1}}{\psi} = \sum_{i=1}^{n} \mu_{i2}$$

$$\Rightarrow \hat{\psi} = \frac{\sum_{i=1}^{n} y_{i1}}{\sum_{i=1}^{n} \hat{\mu}_{i2}}$$

- need to plug in MLE of μ_{i2}

$$\frac{\partial}{\partial \mu_{i2}} \ell(\psi) = \left[\frac{y_{i1} + y_{i2}}{\mu_{i2}} \right] - (\psi + 1) \overset{\text{set}}{=} 0$$

$$\Rightarrow \frac{y_{i1} + y_{i2}}{\mu_{i2}} = \psi + 1$$

$$\Rightarrow \mu_{i2} = \frac{y_{i1} + y_{i2}}{\psi + 1}$$

plug into previous formula

$$\psi = \frac{\sum_{i=1}^{n} y_{i1}}{\left(\sum_{i=1}^{n} \frac{y_{i1} + y_{i2}}{\psi + 1} \right)} = (\psi + 1) \frac{\sum_{i=1}^{n} y_{i1}}{\sum_{i=1}^{n} (y_{i1} + y_{i2})}$$

$$\Rightarrow \psi \left(1 - \frac{\sum_{i=1}^{n} y_{i1}}{\sum_{i=1}^{n} (y_{i1} + y_{i2})} \right) = \frac{\sum_{i=1}^{n} y_{i1}}{\sum_{i=1}^{n} (y_{i1} + y_{i2})}$$

$$\Rightarrow \boxed{\hat{\psi} = \frac{\left(\frac{\sum_{i=1}^{n} y_{i1}}{\sum_{i=1}^{n} (y_{i1} + y_{i2})} \right)}{\left(1 - \frac{\sum_{i=1}^{n} y_{i1}}{\sum_{i=1}^{n} (y_{i1} + y_{i2})} \right)}}$$

$$\frac{\sum (y_{i1} + y_{i2}) - \sum_{i=1}^{n} y_{i1}}{\sum (y_{i1} + y_{i2})} = \frac{\sum_{i=1}^{n} y_{i2}}{\sum_{i=1}^{n} (y_{i1} + y_{i2})}$$

(b) Derive the conditional likelihood for conducting inference on ψ.

- Derive the conditional maximum likelihood estimate denoted $\hat{\psi}_c$ in closed form
- Also find the conditional Fisher Information.

$$P(\psi \mid \text{suff stat of } \mu_{i2}) = \frac{P(\text{joint dist})}{P(\text{suff stat of } \mu_{i2})}$$

suff stat of $\mu_{i2} = y_{i1} + y_{i2}$

$$y_{i1} + y_{i2} \sim \text{Poi}(\mu_{i1} + \mu_{i2})$$
$$\equiv \text{Poi}(\mu_{i2}(\psi + 1))$$

$$P(y_{i1} + y_{i2}) = \frac{(\mu_{i2}(\psi+1))^{y_{i1}+y_{i2}} \exp(-\mu_{i2}(\psi+1))}{(y_{i1}+y_{i2})!}$$

We know

$$Y_{i1} \mid Y_{i1} + Y_{i2} = m \sim \text{Bin}\left(m_i, \frac{\mu_{i1}}{\mu_{i1}+\mu_{i2}}\right)$$
$$\equiv \text{Bin}\left(m_i, \frac{\psi\mu_{i2}}{\psi\mu_{i2}+\mu_{i2}}\right)$$
$$\equiv \text{Bin}\left(m_i, \frac{\psi}{\psi+1}\right)$$

$$P(Y_{i1} \mid Y_{i1} + Y_{i2} = m_i) = \binom{m_i}{y_{i1}} \left(\frac{\psi}{\psi+1}\right)^{y_{i1}} \left(\frac{1}{\psi+1}\right)^{m-y_{i1}}$$

Since all Y_{ij} indep

$\Rightarrow$ Conditional Likelihood is

$$\prod_{i=1}^{n} \binom{m_i}{y_{i1}} \left(\frac{\psi}{\psi+1}\right)^{y_{i1}} \left(\frac{1}{\psi+1}\right)^{m-y_{i1}}$$

$\longrightarrow$

$$= c(\underset{\sim}{y}) \left(\frac{\psi}{\psi+1}\right)^{\sum_{i=1}^n y_{i1}} \left(\frac{1}{\psi+1}\right)^{nm - \sum_{i=1}^n y_{i1}} = L_c(\psi)$$

where $c(\underset{\sim}{y}) = \prod_{i=1}^n \binom{m_i}{y_{i1}}$ ✓

(ii) $\hat{\psi}_c$:

$$l_c(\psi) = \log L_c(\psi)$$
$$= \log c(\underset{\sim}{y}) + \sum_{i=1}^n y_{i1} \left(\log\psi - \log(\psi+1)\right) + -\left(nm - \sum_{i=1}^n y_{i1}\right)\left(\log(\psi+1)\right)$$
$$= \log c(\underset{\sim}{y}) + \sum_{i=1}^n y_{i1} (\log \psi) - nm \log(\psi+1).$$

$$\frac{\partial}{\partial\psi} l_c(\psi) = \frac{\sum_{i=1}^n y_{i1}}{\psi} - \frac{nm}{\psi+1} \overset{\text{set}}{=} 0$$

$$\Rightarrow \frac{(\psi+1)\sum_{i=1}^n y_{i1} - \psi nm}{\psi(\psi+1)} = 0$$

$$\Rightarrow \psi\left(\sum_{i=1}^n y_{i1} - nm\right) + \sum_{i=1}^n y_{i1} = 0$$

$$\Rightarrow \boxed{\hat{\psi}_c = \frac{\sum_{i=1}^n y_{i1}}{\sum_{i=1}^n m_i - \sum_{i=1}^n y_{i1}}}$$

where $m_i = \cancel{\sum_{i=1}^n} (y_{i1} + y_{i2})$ (obs. value)

(iii) Fisher Info:

$$I_c(\psi) = E\left[-\partial^2/\partial\psi^2\, l_c(\psi)\right] \quad \sum_{i=1}^n (y_{i1} + y_{i2}) = \sum_{i=1}^n m_i$$

$$= E\left[-\left(\frac{\partial}{\partial\psi} \frac{\sum_{i=1}^n y_{i1}}{\psi} - \frac{\sum_{i=1}^n m_i}{\psi+1}\right) \middle| \sum_{i=1}^n m_i\right]$$

$$= E\left[-\left(\frac{-\sum_{i=1}^n y_{i1}}{\psi^2} + \frac{\sum_{i=1}^n m_i \; \cancel{nm}}{(\psi+1)^2}\right) \middle| \sum_{i=1}^n m_i\right]$$

$$= \frac{E\left[\sum_{i=1}^n y_{i1} \,\middle|\, \sum m_i\right]}{\psi^2} - \frac{nm}{(\psi+1)^2}$$

$$= \boxed{\frac{\sum_{i=1}^n m_i \left(\frac{\psi}{\psi+1}\right)}{\psi^2} - \frac{\sum_{i=1}^n m_i \; \cancel{nm}}{(\psi+1)^2}}$$

$Y_{i1} \mid \overbrace{Y_{i1} + Y_{i2}}^{m_i} \sim \mathrm{Bin}(m_i, \psi/\psi+1)$

$E\left[\sum_{i=1}^n Y_{i1} \mid \sum_{i=1}^n m_i\right]$

$= \sum_{i=1}^n E[Y_{i1} \mid m_i]$ (since it's indep)

$= \sum_{i=1}^n m_i \left(\frac{\psi}{\psi+1}\right)$

(d) Compare the observed info of $\hat{\varphi}$ to the Fisher info of $\hat{\varphi}_c$ & derive a $100(1-\alpha)\%$ CI interval for φ_c w/o resorting to large sample theory.

(i) Observed Info of $\hat{\varphi}$ vs. Fisher info of $\hat{\varphi}_c$:

$$-\frac{\partial^2}{\partial \varphi^2} L(\varphi) = -\frac{\partial}{\partial \varphi}\left(\frac{\sum_{i=1}^n y_{i1}}{\varphi} - \sum_{i=1}^n \mu_{i2}\right)$$

$$= \frac{\sum_{i=1}^n y_{i1}}{\varphi^2} \Rightarrow \frac{\sum_{i=1}^n y_{i1}}{\hat{\varphi}^2} = \frac{(\sum_{i=1}^n y_{i2})^2}{\sum_{i=1}^n y_{i1}}$$

$=$ observed info of $\hat{\varphi}$

After simplification, Fisher info of $\hat{\varphi}_c$ is

$$\frac{\sum_{i=1}^n m_i}{\varphi(1+\varphi)^2}$$

obs. info of $\hat{\varphi}$

$$\frac{\sum_{i=1}^n y_{i1}}{\hat{\varphi}_c^2} - \frac{\sum_{i=1}^n m_i}{(\hat{\varphi}_c+1)^2} \quad (?)$$

$$\hat{\varphi}_c = \hat{\varphi}$$

$\Rightarrow$ obs. info of $\hat{\varphi}_c$ smaller than obs info of the regular MLE $\hat{\varphi}$

$\Rightarrow$ asymptotic var of $\hat{\varphi}$ larger than asymptotic var of $\hat{\varphi}_c$ since $\sqrt{n}(\hat{\varphi}_{(c)} - \varphi) \xrightarrow{d} N(0, I(\hat{\varphi}_{(c)})^{-1})$

approximate of $I(\varphi)^{-1}$

(e) Repeat (a) when we set $\mu_{i1} = \exp(x_i^T B)\,\mu_{i2}$

$\underset{\sim}{x}_i$ = 2x1 vector of covariates.

$$L(\underset{\sim}{\mu}) = c(\underset{\sim}{y}) \exp\left(\sum_{i=1}^n \left(y_{i1} \log \mu_{i1} + y_{i2} \log \mu_{i2} - (\mu_{i1} + \mu_{i2})\right)\right)$$

like before in (a)

$$\mu_{i1} = \exp(x_i^T B)\,\mu_{i2}$$

$$= c(\underset{\sim}{y}) \exp\left(\sum_{i=1}^n \left(y_{i1}(x_i^T B) + y_{i1} \log \mu_{i2} - \exp(x_i^T B)\mu_{i2} - \mu_{i2}\right)\right)$$

$$\ell(B) = \log L(\underset{\sim}{\mu})$$

$$= \log c(y) + \sum_{i=1}^n y_{i1} x_i^T B + \sum_{i=1}^n y_{i1} \log \mu_{i2} + $$
$$- \sum_{i=1}^n \exp(x_i^T B)\mu_{i2} - \sum_{i=1}^n \mu_{i2}$$

$$\frac{\partial}{\partial B} \ell(B) = \sum_{i=1}^n y_{i1} \underset{\sim}{x}_i - \sum_{i=1}^n \exp(x_i^T B)\mu_{i2} \underset{\sim}{x}_i \overset{set}{=} 0$$

$$\frac{\partial}{\partial \mu_{i2}} \ell(B) = \frac{y_{i1}}{\mu_{i2}} - \exp(x_i^T B) - 1 \overset{set}{=} 0$$

$$\Rightarrow \frac{y_{i1}}{\mu_{i2}} = \exp(x_i^T B) + 1$$

$$\Rightarrow \mu_{i2} = \frac{y_{i1}}{\exp(x_i^T B) + 1}$$

$$\frac{\partial \ell(B)}{\partial B} = \sum_{i=1}^n y_{i1} \underset{\sim}{x}_i - \sum_{i=1}^n \exp(x_i^T B)\left(\frac{y_{i1}}{\exp(x_i^T B) + 1}\right) \underset{\sim}{x}_i \overset{set}{=} 0$$

Can solve for $\hat{B}$ using Newton Raphson ...

$$I(B, \underset{\sim}{\mu}) = \begin{bmatrix} -\partial^2/\partial B_1^2 & -\partial^2/\partial B_1 \partial B_2 & \cdots \\ -\partial^2/\partial B_1 \partial B_2 & \partial^2/\partial B_2^2 & \\ -\partial^2/\partial B_1 \partial \mu_{12} & & \\ \vdots & & \\ \partial^2/\partial B_1 \partial \mu_{n2} & \cdots & \end{bmatrix}_{(n+2)\times(n+2)}$$

2).

$Y = XB + \varepsilon$

$Y = (y_1, y_2, y_3, y_4)^T$

$\varepsilon \sim N(0, \sigma^2 I)$

$B = [B_1, B_2]^T$

$$X = \begin{bmatrix} 1 & 3 \\ 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = [J_4, x_1] \qquad X_1 = [x_{11}, x_{12}, x_{13}, x_{14}]^T$$

(a) Show $\begin{bmatrix} B_1 + B_2 \\ B_1 - 2B_2 \end{bmatrix}$ is estimable

Estimable if $\lambda' B = \rho' E[Y] = \rho' X B$

$\Rightarrow \lambda' = \rho' X$

$\Rightarrow \lambda = X' \rho$

$\Rightarrow \lambda \in C(X') \equiv$ rowspace of X

$$X = \begin{bmatrix} 1 & 3 \\ 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C(X') = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$\lambda = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} B_1 + B_2 \\ B_1 - 2B_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \Rightarrow \Lambda' = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

$c_1 = c_2 = 1$ $\qquad$ $c_1 = 1, \; c_2 = -2.$

$\lambda_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in C(X') \checkmark \qquad \lambda_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \in C(X') \checkmark$

(b) Find the UMVUE of $\Lambda' B = \begin{bmatrix} B_1 + B_2 \\ B_1 - 2B_2 \end{bmatrix}$

By a thm, UMVUE of $\Lambda' B$ = least squares estimate of $\Lambda' B$

least squares estimate = $\rho' M Y$

$\Lambda' = \rho' X$

- find a possible ρ that works

$\rho' = \begin{bmatrix} \rho_1' \\ \rho_2' \end{bmatrix}$

$[1 \;\; 1] = \rho_1' \begin{bmatrix} 1 & 3 \\ 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} =$

$= [\rho_{11} + \rho_{12} + \rho_{13} + \rho_{14} \, , \; 3\rho_{11} + \rho_{12} + \rho_{13} + 2\rho_{14}]$

$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 2 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 1 & 0 \end{array}\right]$

Result: solve $[X' | b]$

$\rho_{11} + \rho_{12} + \rho_{13} + \rho_{14} = 1$

$2\rho_{11} + \rho_{14} = 0$

let $\rho_{11} = \rho_{14} = 0$

$\rho_{12} = \rho_{13} = 1/2$

$\rho_1' = [0 \;\; 1/2 \;\; 1/2 \;\; 0]$ works ✓

$\longrightarrow$

$$\begin{array}{cccc} p_{21} & p_{22} & p_{23} & p_{24} \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 2 & -2 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 1 & -3 \end{array}\right]$$

$p_{11} + p_{12} + p_{13} + p_{14} = 1$

$2p_{11} + p_{14} = -3$

let $p_{11} = p_{14} = -1$

then $p_{12} = 2$, $p_{13} = 1$

$p_2' = \begin{bmatrix} -1 & 2 & 1 & -1 \end{bmatrix}$ works ✓

$M = X(X'X)^{-1}X'$

$$X'X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 7 & 15 \end{bmatrix}$$

$$(X'X)^{-1} = \frac{1}{\underbrace{4(15)}_{60} - \underbrace{7(7)}_{49}} \begin{bmatrix} 15 & -7 \\ -7 & 4 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 15 & -7 \\ -7 & 4 \end{bmatrix}$$

$60 - 49 = 11$

$$X(X'X)^{-1} = \frac{1}{11} \begin{bmatrix} 1 & 3 \\ 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 15 & -7 \\ -7 & 4 \end{bmatrix}$$

$$= \frac{1}{11} \begin{bmatrix} -6 & 5 \\ 8 & -3 \\ 8 & -3 \\ 1 & 1 \end{bmatrix}$$

$X(X'X)^{-1}X'$

$$= \frac{1}{11}\begin{bmatrix} -6 & 5 \\ 8 & -3 \\ 8 & -3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 2 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} 9 & -1 & -1 & 4 \\ -1 & 5 & 5 & 2 \\ -1 & 5 & 5 & 2 \\ 4 & 2 & 2 & 3 \end{bmatrix}$$ symmetric ✓

$$P'M = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 \\ -1 & 2 & 1 & -1 \end{bmatrix}\left(\frac{1}{11}\right)\begin{bmatrix} 9 & -1 & -1 & 4 \\ -1 & 5 & 5 & 2 \\ -1 & 5 & 5 & 2 \\ 4 & 2 & 2 & 3 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} -1 & 5 & 5 & 2 \\ -16 & 14 & 14 & -1 \end{bmatrix}$$

$$P'MY = \frac{1}{11}\begin{bmatrix} -1 & 5 & 5 & 2 \\ -16 & 14 & 14 & -1 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} -y_1 + 5y_2 + 5y_3 + 2y_4 \\ -16y_1 + 14y_2 + 14y_3 - y_4 \end{bmatrix}$$

= UMVUE of $\Lambda'B$ given.

(c) Find the distribution of the UMVUE in (b)

Since $\varepsilon \sim N(0, \sigma^2 I)$

$\Rightarrow Y \sim N(XB, \sigma^2 I)$

$\Rightarrow P'MY \sim N(P'MXB, P'M(\sigma^2 I)MP)$

$\equiv N(P'MB, \sigma^2 P'MP)$

Note: $MX = X$ since M = orthog proj operator onto $C(X)$

Also, M symmetric & idempotent

$M' = M, \quad M^2 = M$ ✓

(d) Suppose $E(\varepsilon) = 0$, $Cov(\varepsilon) = \Sigma = \sigma^2 V$

V = known positive definite

identify a model (model (2)) such that it is in the form of an ordinary linear model

↳ $E(\varepsilon^*) = 0$, $E(\varepsilon^*) = \sigma^2 I$

- contains same parameters

By spectral decomposition, a positive definite matrix V can be decomposed into

$V = P' \Lambda P$

$= P' \Lambda^{1/2} \Lambda^{1/2} P$

$= QQ' \qquad Q = P' \Lambda^{1/2}$

where P = eigenvectors of V & $\Lambda = \text{diag}(\text{eigenvalues})$

$Y = XB + \varepsilon$

$\Rightarrow Q^{-1} Y = Q^{-1} X B + Q^{-1} \varepsilon \qquad Y^* = Q^{-1} Y, \; X^* = Q^{-1} X$

$\Rightarrow Y^* = X^* B + \varepsilon^*$

⟶

$E(\varepsilon^*) = E(Q^{-1}\varepsilon) = Q^{-1}E(\varepsilon) = Q^{-1}(0) = 0$ ✓

$Cov(\varepsilon^*) = Cov(Q^{-1}\varepsilon) = Q^{-1}Cov(\varepsilon)(Q^{-1})' =$

$= Q^{-1}(\sigma^2 V)(Q^{-1})'$

$= \sigma^2 Q^{-1} Q Q'(Q')^{-1}$

Note: $(Q^{-1})' = (Q')^{-1}$

$= \sigma^2 I$ ✓

Still has same parameters β ✓

(e) Show $\lambda'\beta$ is estimable in model (1) iff it is estimable in model (2)

Model (1):

$\lambda'\beta$ estimable if $= \rho'E(Y) = \rho' X\beta$.

$\Rightarrow \lambda' = \rho'X \Leftrightarrow \lambda = X'\rho$

Model (2):

$\lambda'\beta$ estimable if $\rho^{*\prime}E(Y^*) = \rho^{*\prime}X^*\beta$

$\Rightarrow \lambda' = \rho' X^*$

$\Leftrightarrow \lambda = X^{*\prime}\rho$

Want to show if $\lambda = X'\rho \Rightarrow \lambda = X^{*\prime}\rho^*$ & vice versa.

(i) $X'\rho = X'(Q^{-1})'Q'\rho = (Q^{-1}X)'Q'\rho$

$= X^{*\prime}Q'\rho$

Let $Q'\rho = \rho^*$

$= X^{*\prime}\rho^*$.

Note: Some thm about $\underline{C(A) = C(AB)}$ under certain circumstances

If we define $\rho^* = Q'\rho$ then $X'\rho = X^{*\prime}\rho^*$

$\Rightarrow \lambda$ is in both the column space of X' & $X^{*\prime}$

$\Rightarrow \lambda'\beta$ is estimable under both models.

(f) Show $Y^T \Sigma^{-1} Y - \frac{y_1^2}{\sigma_{11}^2} \sim \chi^2(3)$

given that $B = 0$ & $\Sigma = (\sigma_{ij})$

$$Y^T \underbrace{(\Sigma^{-1} + A)}_{B} Y$$

$$\frac{1}{\sigma_{11}^2} = (c^T \Sigma c)^{-1} \quad \text{where } c = [1\ 0\ 0\ 0]$$

$$\frac{y_1^2}{\sigma_{11}^2} = (c^T Y)(c^T \Sigma c)^{-1}(Y^T c)$$

$$Y^T \Sigma^{-1} Y = Y^T \Sigma^{-1/2} \Sigma^{-1/2} Y$$

$$\Sigma^{-1/2} Y \sim N(0, \underbrace{\Sigma^{-1/2} \Sigma \Sigma^{-1/2}}_{I})$$

$$B_K \sim N(0, I_K)$$

$$B_K^T B_K \sim \chi^2(K)$$

(9) Show that the best linear predictor of y_i is $\bar{y} + (x_{1i} - \bar{x}_1)\hat{\beta}_1^*$

where:

$$\hat{\beta}_1^* = (x_1'(I - \tfrac{1}{n}J_n^n)x_1)^{-1} x_1 (I - \tfrac{1}{n}J_n^n) Y$$

with $\bar{x}_1 = \frac{1}{n}\sum_{i=1}^{n} x_{1i}$, $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$

& $J_n^n = J_n J_n'$ ($n \times n$ matrix of all ones)

$n=4$ in this problem.

- Best linear predictors have the smallest variance of other linear predictors