2009 Theory I #2

2a.i) $EX = E\left(E[X|\lambda]\right) = E[\lambda] = \theta^{-1}$

$$EY = E\left(E[Y|\lambda]\right) = E[\beta\lambda] = \frac{\beta}{\theta}$$

$$Var[X] = E\left(Var[X|\lambda]\right) + Var\left(E[X|\lambda]\right)$$

$$= E[\lambda] + Var[\lambda] = \frac{1}{\theta} + \frac{1}{\theta^2}$$

$$Var[Y] = E\left(E[Y|\lambda]\right) + Var\left(E[Y|\lambda]\right)$$

$$= E[\beta\lambda] + Var[\beta\lambda] = \frac{\beta}{\theta} + \frac{\beta^2}{\theta^2}$$

$$Cov[X,Y] = E\left[(X-EX)(Y-EY)\right]$$

$$= E[XY] - (EX)(EY)$$

$$= E\left(E[XY|\lambda]\right) - (EX)(EY)$$

$$\overset{ind}{=} E\left(E[X|\lambda]\,E[Y|\lambda]\right) - (EX)(EY)$$

$$= E(\lambda \cdot \beta\lambda) - (EX)(EY)$$

$$= \beta\left(\frac{1}{\theta^2} + \frac{1}{\theta^2}\right) - \frac{1}{\theta}\cdot\frac{\beta}{\theta} = \frac{2\beta}{\theta^2} - \frac{\beta}{\theta^2}$$

$$= \frac{\beta}{\theta^2}$$

$$p_{X,Y}(x,y) = \int p(x,y\mid\lambda)\,p(\lambda)\,d\lambda$$

$$= \int_0^\infty \frac{\lambda^x e^{-\lambda}}{x!}\cdot\frac{(\beta\lambda)^y e^{-\beta\lambda}}{y!}\,\theta e^{-\theta\lambda}\,d\lambda$$

$$= \frac{\beta^y\theta}{x!\,y!}\int_0^\infty \lambda^{x+y} e^{-(\beta+\theta+1)\lambda}\,d\lambda$$

$$= \frac{\beta^y\theta}{x!\,y!}\,\frac{\Gamma(x+y+1)}{(\beta+\theta+1)^{x+y+1}} = \frac{\theta}{\beta+\theta+1}\binom{x+y}{x}\left(\frac{1}{\beta+\theta+1}\right)^x\left(\frac{\beta}{\beta+\theta+1}\right)^y$$

2b)

$$\ell_n(\theta,\beta) = \sum\left\{\ln\theta - (x_i+y_i+1)\log(\theta+\beta+1) + y_i\log\beta + c(x_i,y_i)\right\}$$

$$\begin{cases} \partial_\theta \ell_n(\theta,\beta) = \sum\left\{\dfrac{1}{\theta} - \dfrac{x_i+y_i+1}{\theta+\beta+1}\cancel{+\dfrac{y_i}{\beta}}\right\} = \dfrac{n}{\theta} - \dfrac{n(\bar{x}_n+\bar{y}_n+1)}{\theta+\beta+1}\cancel{+\dfrac{n\bar{y}_n}{\beta}} \overset{set}{=} 0 \\[2ex] \partial_\beta \ell_n(\theta,\beta) = \sum\left\{-\dfrac{x_i+y_i+1}{\theta+\beta+1} + \dfrac{y_i}{\beta}\right\} = -\dfrac{n(\bar{x}_n+\bar{y}_n+1)}{\theta+\beta+1} + \dfrac{n\bar{y}_n}{\beta} \overset{set}{=} 0 \end{cases}$$

$$\Rightarrow \begin{cases} \hat{\theta}_n+\hat{\beta}_n+1 - \hat{\theta}_n(\bar{x}_n+\bar{y}_n+1) = 0 \\ \bar{y}_n(\hat{\theta}_n+\hat{\beta}_n+1) - \hat{\beta}_n(\bar{x}_n+\bar{y}_n+1) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} -(\bar{x}_n+\bar{y}_n)\hat{\theta}_n + \hat{\beta}_n = -1 \\ \bar{y}_n\hat{\theta}_n - (\bar{x}_n+1)\hat{\beta}_n = -\bar{y}_n \end{cases} \Rightarrow \hat{\beta}_n = (\bar{x}_n+\bar{y}_n)\hat{\theta}_n - 1$$

$$\Rightarrow -\bar{y}_n = \bar{y}_n\hat{\theta}_n - (\bar{x}_n+1)\left[(\bar{x}_n+\bar{y}_n)\hat{\theta}_n - 1\right]$$

$$= \cancel{\bar{y}_n\hat{\theta}_n} - \left[\bar{x}_n^2 + \bar{x}_n\bar{y}_n + \bar{x}_n + \cancel{\bar{y}_n}\right]\hat{\theta}_n + \bar{x}_n + 1$$

$$= -\bar{x}_n(\bar{x}_n+\bar{y}_n+1)\hat{\theta}_n + \bar{x}_n + 1 \quad\Rightarrow\quad \hat{\theta}_n = \bar{x}_n^{-1}$$

$$\Rightarrow \quad - \frac{\bar{x}_n + \bar{y}_n}{\bar{x}_n} + \hat{\beta}_n = -1$$

$$\Rightarrow \quad \hat{\beta}_n = \frac{\bar{x}_n + \bar{y}_n}{\bar{x}_n} - 1 = \frac{\bar{y}_n}{\bar{x}_n}$$

$\boxed{2c}$ Under the conditions of Ferguson pg 121,

$$\sqrt{n} \begin{pmatrix} \hat{\theta}_n - \theta \\ \hat{\beta}_n - \beta \end{pmatrix} \xrightarrow{\mathcal{L}} N\left(0, \left[I(\theta,\beta)\right]^{-1}\right)$$

Remains to obtain $\left[I(\theta,\beta)\right]^{-1}$

$$\partial^2_\theta \, \ell(\theta,\beta) = -\frac{1}{\theta^2} + \frac{x+y+1}{(\theta+\beta+1)^2}$$

$$\partial^2_\beta \, \ell(\theta,\beta) = -\frac{y}{\beta^2} + \frac{x+y+1}{(\theta+\beta+1)^2}$$

$$\partial_\theta \partial_\beta \, \ell(\theta,\beta) = \frac{x+y+1}{(\theta+\beta+1)^2}$$

$$E\left[\frac{x+y+1}{(\theta+\beta+1)^2}\right] = \frac{\theta^{-1} + \beta\theta^{-1} + 1}{(\theta+\beta+1)^2} = \frac{\theta^{-1}(1+\beta+\theta)}{(\theta+\beta+1)^2} = \frac{1}{\theta(\theta+\beta+1)}$$

$$I(\theta,\beta) = \begin{bmatrix} \frac{1}{\theta^2} - \frac{1}{\theta(\theta+\beta+1)} & -\frac{1}{\theta(\theta+\beta+1)} \\ -\frac{1}{\theta(\theta+\beta+1)} & \frac{1}{\beta\theta} - \frac{1}{\theta(\theta+\beta+1)} \end{bmatrix}$$

$$= \frac{1}{\beta\theta^2(\theta+\beta+1)} \begin{bmatrix} \beta(\theta+\beta+1) - \beta\theta & -\beta\theta \\ -\beta\theta & \theta(\theta+\beta+1) - \beta\theta \end{bmatrix}$$

$$= \frac{1}{\beta\theta^2(\theta+\beta+1)} \begin{bmatrix} \beta(\beta+1) & -\beta\theta \\ -\beta\theta & \theta(\theta+1) \end{bmatrix}$$

$$[I(\theta,\beta)]^{-1} = \frac{\beta\theta^2(\theta+\beta+1)}{\underbrace{\beta(\beta+1)\theta(\theta+1) - (-\beta\theta)^2}} \begin{bmatrix} \theta(\theta+1) & \beta\theta \\ \beta\theta & \beta(\beta+1) \end{bmatrix}$$

$$\boxed{\begin{aligned} \beta(\beta+1)\theta(\theta+1) - \beta^2\theta^2 &= \beta^2\theta(\theta+1) + \beta\theta(\theta+1) - \beta^2\theta^2 \\ &= \beta^2\theta^2 + \beta^2\theta + \beta\theta^2 + \beta\theta - \beta^2\theta^2 = \beta^2\theta + \beta\theta^2 + \beta\theta = \beta\theta(\beta+\theta+1) \end{aligned}}$$

$$= \theta \begin{bmatrix} \theta(\theta+1) & \beta\theta \\ \beta\theta & \beta(\beta+1) \end{bmatrix} = \begin{bmatrix} \theta^2(\theta+1) & \beta\theta^2 \\ \beta\theta^2 & \theta\beta(\beta+1) \end{bmatrix}$$

$\boxed{2d.i}$ $$T_2 = \sqrt{n\bar{X}_n/2}\left(\frac{T_n}{\bar{X}_n} - 1\right) = \sqrt{n}\left(\sqrt{\frac{\bar{X}_n}{2}}\left(\frac{T_n}{\bar{X}_n} - 1\right)\right)$$

$$= \sqrt{n}\left((2\hat{\theta}_n)^{-1/2}(\hat{\beta}_n - 1)\right) = \sqrt{n}\left(g(\hat{\theta}_n, \hat{\beta}_n)\right)$$

where $g(a,b) = (2a)^{-1/2}(b-1)$. Now

$$g(\theta,1) = 0, \quad \frac{\partial}{\partial a} g(a,b) = 2^{-1/2}(b-1)\left[-\frac{1}{2}a^{-3/2}\right] = -(2a)^{-3/2}(b-1)$$

$$\frac{\partial}{\partial b} g(a,b) = (2a)^{-1/2}. \quad \text{Let } \Sigma = [I(\theta,1)]^{-1} = \begin{bmatrix} * & * \\ * & 2\theta \end{bmatrix}$$

so that by the delta method

$$T_1 = \sqrt{n}\left(g(\hat{\theta}_n,\hat{\beta}_n) - g(\theta,\beta)\right) \xrightarrow[H_0]{\mathcal{L}} N(0, \dot{g}(\theta,1)\Sigma[\dot{g}(\theta,1)]')$$

$$\stackrel{d}{=} N(0, (2\theta)^{-1}2\theta) \stackrel{d}{=} N(0,1)$$

$\boxed{2.d.ii}$

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{L} N(0, \theta^2(\theta+1))$$

Let $g(x) = \frac{1}{x}$, $\dot{g}(x) = -\frac{1}{x^2}$

$$\Rightarrow \sqrt{n}(\hat{\theta}_n^{-1} - \theta^{-1}) \overset{d}{=} \sqrt{n}(g(\hat{\theta}_n^{-1}) - g(\theta)) \to$$

$$N(0, [\dot{g}(\theta)]^2 \theta^2(\theta+1)) \overset{d}{=} N(0, \theta+1)$$

$\Rightarrow \sqrt{n}\,\hat{\theta}_n^{-1} = O_p(1)$

$$\boxed{T_1 - T_2 = \sqrt{n}\left[\frac{1}{2\hat{\theta}_n}\left(\hat{\beta}_n - 1 + \log\hat{\beta}_n\right)\right]}$$

$$\begin{pmatrix} \bar{X}_n \\ \bar{Y}_n \end{pmatrix} \xrightarrow{P} \begin{pmatrix} \theta^{-1} \\ \beta\theta^{-1} \end{pmatrix}$$ by SLLN so that by CMT

$$\hat{\beta}_n = \frac{\bar{Y}_n}{\bar{X}_n} \xrightarrow{P} \frac{\beta\theta^{-1}}{\theta^{-1}} = \beta$$

Furthermore by the CMT we obtain (under H_0)

$$\hat{\beta}_n - 1 + \log\hat{\beta}_n \xrightarrow{P} 1 - 1 + \log(1) = 0$$

Thus,

$$T_1 - T_2 = O_p(1)\,o_p(1) = o_p(1)$$

$\boxed{2d, iii.}$ Immediate from Slutsky's thm

Still need (e)!