$$p(y_i | \mu_i) = \frac{1}{\mu_i} \exp\left(-y_i/\mu_i\right) \quad \mu_i > 0$$

$$E[y_i | \mu_i] = \mu_i$$

$$\theta_i = \frac{1}{\mu_i}$$

a) $\theta_i \sim \text{Gamma}(a_i, b_i)$

$$\frac{a_i}{b_i} = e^{-x_i'\beta} \qquad a_i = 3$$

$$\text{var}(\theta_i) = \tau \exp(x_i'\beta)$$

$$E[y_i] = E[E[y_i | \mu_i]]$$

$$= E[\mu_i]$$

$$= E[1/\theta_i]$$

$$= \int_0^\infty \frac{1}{\theta_i} \frac{1}{\cancel{\theta_i}\Gamma(a_i) b_i^{a_i}} \exp\left(-\theta_i/b_i\right) \theta_i^{a_i - 1} \, d\theta_i$$

$$= \int_0^\infty \theta_i^{a_i-1-1} \frac{1}{\Gamma(a_i) b_i^{a_i}} \exp(-\theta_i/b_i) \, d\theta_i$$

$$= \frac{\Gamma(a_i-1) b_i^{a_i-1}}{\Gamma(a_i) b_i^{a_i}} \underbrace{\int_0^\infty \frac{1}{\Gamma(a_i-1) b_i^{a_i-1}} \theta_i^{(a_i-1)-1} \exp(-\theta_i/b_i) \, d\theta_i}_{1}$$

$$= \frac{\Gamma(a_i-1) b_i^{-1}}{\Gamma(a_i-1)\Gamma(a_i-1)} = \frac{1}{\cancel{\theta}(a_i-1) b_i} = \left[\frac{1}{a_i b_i - a_i}\right] = \frac{1}{2 b_i}$$

with $\frac{a_i}{b_i} = e^{-x_i^T\beta}$ and $a_i = 3$ $\qquad 3e^{x_i^T\beta} = b_i$

$$\Rightarrow b_i = \frac{3}{e^{-x_i^T\beta}}$$

$$\Rightarrow \cancel{\left[E[y_i] = \frac{1}{\frac{9}{e^{-x_i^T\beta}} - \cancel{0}\,3}\right] = \frac{1}{9e^{x_i^T\beta} - 3}}$$

$$\left[E[y_i] = \frac{1}{6e^{x_i^T\beta}}\right] \checkmark$$

a) cont'd

$$var(y_i) = E[var(y_i \mid \mu_i)] + var(E[y_i \mid \mu_i])$$
$$= E[\mu_i^2] + var(\mu_i)$$
$$= E[\mu_i^2] + E[\mu_i^2] - (E[\mu_i])^2$$
$$= 2E[\mu_i^2] - \frac{1}{36e^{2x_i^T\beta}}$$

$$E[\mu_i^2] = E[1/\theta_i^2]$$
$$= \int_0^\infty \frac{1}{\theta_i^2} \frac{1}{\Gamma(a_i) b_i^{a_i}} \exp(-\theta_i/b_i)\, \theta_i^{a_i-1}\, d\theta_i$$
$$= \int_0^\infty \theta_i^{(a_i-2)-1} \frac{1}{\Gamma(a_i) b_i^{a_i}} \exp(-\theta_i/b_i)\, d\theta_i$$
$$= \frac{\Gamma(a_i-2)\, b_i^{a_i-2}}{\Gamma(a_i)\, b_i^{a_i}}$$
$$= \frac{\Gamma(a_i-2)\, b_i^{-2}}{(a_i-1)\,\Gamma(a_i-1)} = \frac{\cancel{\Gamma(a_i-2)}\, b_i^{-2}}{(a_i-1)(a_i-2)\,\cancel{\Gamma(a_i-2)}}$$

$$a_i = 3 \Rightarrow E[\mu_i^2] = \frac{1}{(2)(1)\, b_i^2} = \frac{1}{2b_i^2}$$
$$= \frac{1}{2[3e^{x_i^T\beta}]^2} = \frac{1}{18e^{2x_i^T\beta}}$$

$$\Rightarrow var(y_i) = \frac{1}{9e^{2x_i^T\beta}} - \frac{1}{36e^{2x_i^T\beta}} = \frac{4-1}{36e^{2x_i^T\beta}}$$
$$= \frac{3}{36e^{2x_i^T\beta}}$$
$$= \boxed{\frac{1}{12e^{2x_i^T\beta}}} \checkmark$$

b) $p(y_i) = \int p(y_i|\theta_i)\, p(\theta_i)\, d\theta_i$

$$p(y_i|\mu_i) = \frac{1}{\mu_i} \exp(-y_i/\mu_i)$$

$$\theta_i = \frac{1}{\mu_i}$$

$$\Rightarrow p(y_i|\theta_i) = \theta_i \exp(-y_i\theta_i)$$

$$\Rightarrow p(y_i) = \int_0^\infty \theta_i \exp(-y_i\theta_i) \frac{1}{\Gamma(a_i) b_i^{a_i}} \exp(-\theta_i/b_i)\, \theta_i^{a_i - 1}\, d\theta_i$$

$$= \frac{1}{\Gamma(a_i)\, b_i^{a_i}} \int_0^\infty \theta_i^{(a_i+1)-1} \exp\left(-\theta_i\left[y_i + \frac{1}{b_i}\right]\right) d\theta_i$$

$$a^* = a_i + 1$$

$$b^* = \left(y_i + \frac{1}{b_i}\right)^{-1}$$

$$= \frac{\Gamma(a_i+1)\left(\left(y_i + \frac{1}{b_i}\right)^{-1}\right)^{a_i+1}}{\Gamma(a_i)\quad b_i^{a_i}}$$

$$= \frac{a_i \cancel{\Gamma(a_i)}}{\cancel{\Gamma(a_i)}} \frac{\left(\frac{y_i b_i + 1}{b_i}\right)^{-(a_i+1)}}{b_i^{a_i}}$$

$$= \frac{a_i \quad (y_i b_i + 1)^{-(a_i+1)}}{b_i^{-(a_i+1)} \quad b_i^{a_i}} = \frac{a_i\, (y_i b_i + 1)^{-(a_i+1)}}{b_i^{-1}}$$

$$a_i = 3$$

$$\Rightarrow$$

$$p(y_i) = \frac{3\, b_i}{(y_i b_i + 1)^4}$$

Recall $b_i = 3\, e^{x_i^T \beta}$

$$\Rightarrow \boxed{p(y_i) = \frac{9 \exp(x_i^T \beta)}{(3 y_i e^{x_i^T \beta} + 1)^4}}$$

c) $H_0: \tau = 0$

$var(\theta_i) = \tau e^{x_i^T\beta}$

$$\frac{a_i}{b_i} = e^{-x_i^T\beta}$$

$$\Rightarrow a_i e^{x_i^T\beta} = b_i$$

$\boxed{\text{overdispersion score test}}$

$$E[\theta_i] = a_ib_i = 3(3\exp(x_i^T\beta)) = 9\exp(x_i^T\beta)$$

$$\mu_i = \frac{1}{\theta_i} = \dot{b}(\theta_i) = \theta_i^{-1}$$

$$var(\theta_i) = \tau\exp(x_i^T\beta)$$

$$9e(2x_i^T\beta)$$

$$\Rightarrow f_i = \exp(x_i^T\beta)$$

$$\Rightarrow \ddot{b}(\theta_i) = -\theta_i^{-2} = \frac{-1}{\theta_i^2}$$

$$\Rightarrow \dddot{b}(\theta_i) = 2\theta_i^{-3} = \frac{2}{\theta_i^3}$$

$$\Rightarrow b^{(4)}(\theta_i) = \frac{-6}{\theta_i^4}$$

$$\tau = 9\exp(x_i^T\beta)$$

$$\partial_\tau \ell n(\alpha) = \sum_{i=1}^{n} \frac{1}{2}(\exp(x_i^T\beta))\left\{(y_i-\mu_i)^2 + \frac{1}{\theta_i}\right\}$$

$$w_1 = -\frac{1}{\theta_i^2}$$

$$w_2 = 0.5\exp(x_i^T\beta)\frac{2}{\theta_i^3}$$

$$\mu_i = \frac{1}{\theta_i}$$

$$w_{1i} = \ddot{b}(\theta_i) = \frac{-1}{\theta_i^2} = \frac{-1}{(1/\mu_i)^2} = -\mu_i^2$$

$$\Rightarrow \frac{1}{\theta_i} = \frac{1}{6}\exp(-x_i^T\beta)$$

$$\Rightarrow \frac{1}{\frac{1}{6}\exp(x_i^T\beta)} = \theta_i$$

$$\Rightarrow 6\exp(x_i^T\beta) = \theta_i$$

Now $E[y_i] = \dot{b}(\theta_i)$

$$E[y_i] = \frac{1}{6}e^{-x_i^T\beta} = \mu_i$$

$$\Rightarrow \boxed{w_{1i} = -\left(\frac{1}{6}e^{-x_i^T\beta}\right)^2 = -\frac{1}{36}e^{-2x_i^T\beta}}$$

$$w_{2i} = 0.5 f_i \dddot{b}(\theta_i)$$

$$= 0.5\exp(x_i^T\beta)\frac{2}{\theta_i^3} = 0.5\exp(x_i^T\beta)\frac{2}{(1/\mu_i)^3} = \exp(x_i^T\beta)\mu_i^3$$

$$= \exp(x_i^T\beta)\left(\frac{1}{6}e^{-x_i^T\beta}\right)^3 = \frac{1}{216}\exp(x_i^T\beta - 3x_i^T\beta)$$

$$\boxed{w_{2i} = \frac{1}{216}\exp(-2x_i^T\beta)}$$

$$\boxed{\partial_\tau \ell n(\alpha) = \sum_{i=1}^{n}\frac{1}{2}\exp(x_i^T\beta)\left\{\left(y_i - \frac{1}{6}\exp(-x_i^T\beta)\right)^2 - \frac{1}{36}\exp(-2x_i^T\beta)\right\}}$$

$$\frac{\partial\theta}{\partial\beta} = 6\exp(x_i^T\beta)x_i$$

$$\Rightarrow \boxed{D\theta(\beta)^T = \begin{pmatrix} 6\exp(x_i^T\beta)x_{11} & \cdots & 6\exp(x_i^T\beta)x_{n1} \\ \vdots & \ddots & \vdots \\ 6\exp(x_i^T\beta)x_{1p} & \cdots & 6\exp(x_i^T\beta)x_{np} \end{pmatrix}}$$

c) continued

$$\boxed{W_1 = \mathrm{diag}(w_{1i})}$$

$$\boxed{W_2 = \mathrm{diag}(w_{2i})}$$

$$I_{\beta\tau} = D\theta(\beta)^T W_2 1_n$$

$$I_{\beta\beta} = D\theta(\beta)^T W_1 D\theta(\beta)$$

$$I_{\tau\tau} = \sum_{i=1}^{n} \frac{1}{4} f_i^2 \{ 2(\ddot{b}(\theta_i))^2 + b^{(4)}(\theta_i) \}$$

$$= \sum_{i=1}^{n} \frac{1}{4} \exp(2x_i^T\beta) \{ 2(- \frac{1}{36} \exp(-2x_i^T\beta))^2 + (-6/\theta_i^4) \}$$

$$= \sum_{i=1}^{n} \frac{1}{4} \exp(2x_i^T\beta) \{ \frac{2}{36^2} \exp(-4x_i^T\beta) - 6\mu_i^4 \}$$

$\uparrow$

$\frac{1}{6}\exp(-x_i^T\beta)$.

$$\sigma_\tau^2 = I_{\tau\tau} - I_{\tau\beta} I_{\beta\beta}^{-1} I_{\beta\tau}$$

$$S_\tau = \frac{\partial_\tau \ell n(\hat{\alpha})}{\sigma_\tau^2} I(\partial_\tau \ell n(\hat{\alpha}) > 0) \Big|_{\tilde{\alpha}} \xrightarrow{D} 0.5\chi_0^2 + 0.5\chi_1^2$$

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d) $var(y_i) = \sigma^2(v_i - \mu_i)$

(i) $\mu_i = \exp(x_i^T\beta)$

Now
$$S_n(\beta) = \sum_{i=1}^{n} \frac{\partial \mu_i}{\partial \beta} \frac{y_i - \mu_i}{Var(y_i)}$$

$\mu_i = \exp(x_i^T\beta)$

$\Rightarrow \frac{\partial \mu_i}{\partial \beta} = \exp(x_i^T\beta)\, x_i$

$var(y_i) = \sigma^2(v_i + \mu_i)$
$= \sigma^2(\mu_i^2 + \mu_i)$

$$\Rightarrow \boxed{S_n(\beta) = \sum_{i=1}^{n} \frac{(y_i - \mu_i)}{\sigma^2 \cancel{\mu_i}(\mu_i + 1)} \cancel{\mu_i}\, x_i = \sum_{i=1}^{n} \frac{(y_i - \mu_i)}{\sigma^2(\mu_i + 1)} x_i}$$

$\mu_i = \exp(x_i^T\beta)$

Denote
$$\boxed{\begin{aligned} D^T &= \left(\frac{\partial \mu}{\partial \beta}\right)^T = \left(\exp(x_i^T\beta)x_1, \dots, \exp(x_i^T\beta)x_n\right) \\ V &= diag\{\mu_i^2 + \mu_i\} \\ \Rightarrow S_n(\beta) &= \frac{1}{\sigma^2} D^T V^{-1}(Y - \mu) \end{aligned}}$$

Moment Estimator

$$E\left[\sum_{i=1}^{n} \frac{(y_i - \mu_i)^2}{V(\mu_i)}\right] = n\sigma^2$$

$$\Rightarrow \cancel{\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} \frac{(y_i - \hat{\mu}_i)^2}{\hat{\mu}_i(\hat{\mu}_i + 1)}}$$

$$\boxed{\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} \frac{(y_i - \hat{\mu}_i)^2}{\hat{\mu}_i(\hat{\mu}_i + 1)}}$$

$\boxed{n-p}$

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d) continued

(ii) $$0 = S_n(\hat{\beta}_P) \overset{TS}{\approx} S_n(\beta) + \partial_\beta S_n(\beta)(\hat{\beta}_P - \beta)$$

$$\Rightarrow -\frac{\partial S_n(\beta)}{\partial \beta}(\hat{\beta}_P - \beta) \approx S_n(\beta)$$

$$\Rightarrow \hat{\beta}_P - \beta \approx \left[-\frac{\partial S_n(\beta)}{\partial \beta}\right]^{-1} S_n(\beta)$$

$$\Rightarrow \sqrt{n}(\hat{\beta}_P - \beta) \approx \left[-\frac{1}{n}\frac{\partial S_n(\beta)}{\partial \beta}\right]^{-1} \frac{1}{\sqrt{n}} S_n(\beta)$$

$$\Rightarrow \operatorname{cov}(\sqrt{n}\hat{\beta}) \approx \left[-\frac{1}{n}\partial_\beta S_n(\beta)\right]^{-1} \operatorname{cov}\left(\frac{1}{\sqrt{n}} S_n(\beta)\right)\left[-\frac{1}{n}\partial_\beta S_n(\beta)\right]^{-1}$$

Now
$$-\frac{1}{n}\partial_\beta S_n(\beta) = -\frac{1}{n}\partial_\beta\left[\sum_{i=1}^n \partial_\beta \mu_i(\beta) V_i(\beta)^{-1} e_i(\beta)\right]$$

$$= -\frac{1}{n}\sum_{i=1}^n \partial_\beta\left[\partial_\beta \mu_i(\beta) V_i(\beta)^{-1}\right] e_i(\beta) + \frac{1}{n}\sum_{i=1}^n \partial_\beta \mu_i(\beta) V_i(\beta)^{-1} \partial_\beta \mu_i(\beta)^T$$

$$\approx \frac{1}{n}\sum_{i=1}^n \partial_\beta \mu_i(\beta) V_i(\beta)^{-1} \partial_\beta \mu_i(\beta)^T$$

$$\approx \frac{1}{n} D^T V^{-1} D$$

and
$$\operatorname{cov}\left(\frac{1}{\sqrt{n}} S_n(\beta)\right) = \operatorname{cov}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n \partial_\beta \mu_i(\beta) V_i(\beta)^{-1} e_i(\beta)\right)$$

$$= \frac{\sigma^2}{n}\sum_{i=1}^n \partial_\beta \mu_i(\beta) V_i(\beta)^{-1} \partial_\beta \mu_i(\beta)^T$$

$$= \frac{\sigma^2 D^T V^{-1} D}{n}$$

$$\Rightarrow \operatorname{cov}(\sqrt{n}\hat{\beta}) \approx \left[\frac{1}{n} D^T V^{-1} D\right]^{-1} \frac{\sigma^2 D^T V^{-1} D}{n}\left[\frac{1}{n} D^T V^{-1} D\right]^{-1}$$

$$= \sigma^2 n (D^T V^{-1} D)^{-1} \cancel{(D^T V^{-1} D)} \cancel{(D^T V^{-1} D)^{-1}}$$

$$\Rightarrow \boxed{\begin{aligned} \operatorname{cov}(\hat{\beta}) &\approx \frac{1}{n}\operatorname{cov}(\sqrt{n}\hat{\beta}) \\ &\approx \sigma^2 (D^T V^{-1} D)^{-1}. \end{aligned}}$$