2010 Theory I #2

2a) $p(x;\theta) = \prod_{i=1}^{n} I(\theta < x_i < \theta+1) = I(\theta < x_{(1)} \le x_{(n)} < \theta+1)$

$= I(x_{(n)} - 1 < \theta < x_{(1)})$ so $\hat{\theta} \in \{\theta : x_{(n)} - 1 < \theta < x_{(1)}\}$ is an MLE

2b) The Bayes estimator for absolute error loss is the posterior median. First we find the posterior distribution. We have

$$p(\theta|x) = \frac{p(x|\theta)\,p(\theta)}{\underbrace{p(x)}_{\text{derive this separately since we use later}}} = \frac{\frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left\{-\frac{1}{2\sigma_0^2}(\theta-\mu_0)^2\right\} I(x_{(n)}-1<\theta<x_{(1)})}{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left\{-\frac{1}{2\sigma_0^2}(\theta-\mu_0)^2\right\} I(x_{(n)}-1<\theta<x_{(1)})\,d\theta}$$

$$= \frac{\frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left\{-\frac{1}{2\sigma_0^2}(\theta-\mu_0)^2\right\} I(x_{(n)}-1<\theta<x_{(1)})}{\int_{x_{(n)}-1}^{x_{(1)}} \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left\{-\frac{1}{2\sigma_0^2}(\theta-\mu_0)^2\right\} d\theta}$$

$$= \frac{\frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left\{-\frac{1}{2\sigma_0^2}(\theta-\mu_0)^2\right\} I(x_{(n)}-1<\theta<x_{(1)})}{\Phi\left(\frac{x_{(1)}-\mu_0}{\sigma_0}\right) - \Phi\left(\frac{x_{(n)}-1-\mu_0}{\sigma_0}\right)}$$

Now we wish to find the median. We ~~must~~ solve for m:

$$0.5 \overset{\text{set}}{=} \int_{-\infty}^{m} p(\theta|x)\,d\theta = \int_{x_{(n)}-1}^{\min(m,\,x_{(1)})} \frac{\frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left\{-\frac{1}{2\sigma_0^2}(\theta-\mu_0)^2\right\}}{\Phi\left(\frac{x_{(1)}-\mu_0}{\sigma_0}\right) - \Phi\left(\frac{x_{(n)}-1-\mu_0}{\sigma_0}\right)}\,d\theta$$

$$\Rightarrow \quad \frac{1}{2} p(x) = \Phi\left(\frac{m-\mu_0}{\sigma_0}\right) - \Phi\left(\frac{x_{(n)}-1-\mu_0}{\sigma_0}\right)$$

$$\Rightarrow \quad \Phi\left(\frac{m-\mu_0}{\sigma_0}\right) = \frac{1}{2} p(x) + \Phi\left(\frac{x_{(n)}-1-\mu_0}{\sigma_0}\right)$$

$$\Rightarrow \quad m = \mu_0 + \sigma_0 \Phi^{-1}\left\{\frac{1}{2} p(x) + \Phi\left(\frac{x_{(n)}-1-\mu_0}{\sigma_0}\right)\right\}$$

$$= \mu_0 + \sigma_0 \Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{x_{(n)}-1-\mu_0}{\sigma_0}\right) + \frac{1}{2}\Phi\left(\frac{x_{(1)}-\mu_0}{\sigma_0}\right)\right\}$$

2c) $f_{X_{(1)}}(x) = n\left[1 - F(x)\right]^{n-1} f(x) = n\left[1 - (x-\theta)\right]^{n-1}, \quad 0 < x < \theta+1$

$f_{X_{(n)}}(y) = n\left[F(y)\right]^{n-1} f(y) = n(y-\theta)^{n-1}, \quad 0 < y < \theta+1$

$f_{X_{(1)}, X_{(n)}}(x,y) = n(n-1)\left[F(y) - F(x)\right]^{n-2} f(x) f(y)$

$= n(n-1)(y-x)^{n-2}, \quad 0 < x < y < \theta+1$

Now consider $U_1, \ldots, U_n \overset{iid}{\sim} \text{unif}(0,1)$.

$f_{U_{(1)}}(x) = n\left[1 - F(x)\right]^{n-1} f(x) = n(1-x)^{n-1}, \quad 0 < x < 1$

$f_{U_{(n)}}(y) = n\left[F(x)\right]^{n-1} f(y) = n y^{n-1}, \quad 0 < y < 1$

$f_{U_{(1)}, U_{(n)}}(x,y) = n(n-1)\left[F(y) - F(x)\right]^{n-2} f(x) f(y)$

$= n(n-1)(y-x)^{n-2}, \quad 0 < x < y < 1$

Let $V = U_{(1)} + \theta$, $W = U_{(n)} + \theta$. Then $\theta < V < W < \theta + 1$, $U_{(1)} = V - \theta$ and $U_{(n)} = W - \theta$ so that

$$f_{V,W}(x,y) = f_{U_{(1)},U_{(n)}}(x-\theta, y-\theta) \begin{vmatrix} \frac{\partial(x-\theta)}{\partial x} & \frac{\partial(x-\theta)}{\partial y} \\ \frac{\partial(y-\theta)}{\partial x} & \frac{\partial(y-\theta)}{\partial y} \end{vmatrix}$$

$$= n(n-1)\left[y-\theta-(x-\theta)\right]^{n-2} = n(n-1)(y-x)^{n-2}, \quad \theta < x < y < \theta+1$$

Thus $(U_{(1)}+\theta, U_{(n)}+\theta) \stackrel{d}{=} (X_{(1)}, X_{(n)})$ and since the joint distribution completely characterizes the marginal distributions we also obtain $U_{(1)}+\theta = X_{(1)}$, $U_{(n)}+\theta = X_{(n)}$. Now

$$E[U_{(1)}] = \int_0^1 x\, n(1-x)^{n-1}\,dx = n\int_0^1 (1-x)x^{n-1}\,dx = n\left[\frac{x^n}{n} - \frac{x^{n+1}}{n+1}\right]_{x=0}^{x=1} = \frac{1}{n+1}$$

$$E[U_{(1)}^2] = \int_0^1 x^2 n(1-x)^{n-1}\,dx = n\int_0^1 (1-x)^2 x^{n-1}\,dx = n\int_0^1 (1-2x+x^2)x^{n-1}\,dx$$

$$= n\left[\frac{x^n}{n} - \frac{2x^{n+1}}{n+1} + \frac{x^{n+2}}{n+2}\right]_{x=0}^{x=1} = \frac{n\left[(n+1)(n+2) - 2n(n+2) + n(n+1)\right]}{n(n+1)(n+2)}$$

$$= \frac{n\left[n^2+3n+2-2n^2-4 + n^2+n\right]}{n(n+1)(n+2)} = \frac{2}{(n+1)(n+2)}$$

$$E[U_{(n)}] = \int_0^1 y n y^{n-1}\,dy = \frac{ny^{n+1}}{n+1}\Big|_0^1 = \frac{n}{n+1}$$

$$E[U_{(n)}^2] = \int_0^1 y^2 n y^{n-1}\,dy = \frac{ny^{n+2}}{n+2}\Big|_0^1 = \frac{n}{n+2}$$

$$E[U_{(1)}U_{(n)}] = \int_0^1 \int_x^1 xy\, n(n-1)(y-x)^{n-2}\, dy\, dx$$

$$= n(n-1)\int_0^1 x \int_x^1 y(y-x)^{n-2}\, dy dx$$

let $z = y - x$
when $y = x$ then $z = 0$
$y = 1$ then $z = 1 - x$.

$$= n(n-1)\int_0^1 x \int_0^{1-x} (z+x)z^{n-2}\, dz\, dx$$

$$= n(n-1)\int_0^1 x \left[\frac{z^n}{n} + \frac{xz^{n-1}}{n-1}\right]_{z=0}^{z=1-x} dx$$

$$= n(n-1)\int_0^1 \frac{x(1-x)^{n-1}(n-1+x)}{n(n-1)}\, dx$$

$$= (n-1)\int_0^1 x(1-x)^{n-1}\, dx + \int_0^1 x^2(1-x)^{n-1}\, dx$$

$$= (n-1)\int_0^1 (1-x)x^{n-1}\, dx + \int_0^1 (1-x)^2 x^{n-1}\, dx$$

$$= (n-1)\int_0^1 (1-x)x^{n-1}\, dx + \int_0^1 (1-2x+x^2)\, x^{n-1}\, dx$$

$$= (n-1)\left[\frac{x^n}{n} - \frac{x^{n+1}}{n+1}\right]_{x=0}^{x=1} + \left[\frac{x^n}{n} - \frac{2x^{n+1}}{n+1} + \frac{x^{n+2}}{n+2}\right]_{x=0}^{1}$$

$$= (n-1)\,\frac{n+1-n}{n(n+1)} + \frac{(n+1)(n+2) - 2n(n+2) + n(n+1)}{n(n+1)(n+2)}$$

$$= \frac{(n-1)(n+2)+2}{n(n+1)(n+2)} = \frac{1}{n+2}$$

Then

$$E[X_{(1)}] = E[U_{(1)} + \theta] = \frac{1}{n+1} + \theta$$

$$E[X_{(1)}^2] = E[(U_{(1)} + \theta)^2] = E[U_{(1)}^2] + 2\theta E[U_{(1)}] + \theta^2$$

$$= \frac{2}{(n+1)(n+2)} + \frac{2\theta}{n+1} + \theta^2$$

$$E[X_{(n)}] = E[U_{(n)} + \theta] = \frac{n}{n+1} + \theta$$

$$E[X_{(n)}^2] = E[(U_{(n)} + \theta)^2] = E[U_{(n)}^2] + 2\theta E[U_{(n)}] + \theta^2$$

$$= \frac{n}{n+2} + \frac{2n\theta}{n+1} + \theta^2$$

$$E[X_{(1)} X_{(n)}] = E[(U_{(1)} + \theta)(U_{(n)} + \theta)]$$

$$= E[U_{(1)} U_{(n)}] + \theta E[U_{(1)}] + \theta E[U_{(n)}] + \theta^2$$

$$= \frac{1}{n+2} + \frac{\theta}{n+1} + \frac{n\theta}{n+1} + \theta^2$$

Pitman's estimator is given by

$$\frac{\int_{\mathbb{R}} \theta\, p(\theta|x)\, d\theta}{\int_{\mathbb{R}} p(\theta|x)\, d\theta} = \frac{\int_{X_{(n)}-1}^{X_{(1)}} \theta\, d\theta}{\int_{X_{(n)}-1}^{X_{(1)}} d\theta} = \frac{\left.\frac{\theta^2}{2}\right|_{X_{(n)}-1}^{X_{(1)}}}{\left.\theta\right|_{X_{(n)}-1}^{X_{(1)}}}$$

$$= \frac{X_{(1)}^2 - (X_{(n)}-1)^2}{2} \cdot \frac{1}{X_{(1)} - (X_{(n)}-1)} = \frac{X_{(1)} + X_{(n)} - 1}{2}$$

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Pitman's estimator is admissible if there exists a location invariant estimator T_0 s.t. $E|T_0(\underline{X})|^3 < \infty$. Let $T_0(X) = X_{(1)}$. Then by a similar argument as before $T_0(X_1+c, \dots X_n+c) \stackrel{d}{=} T(X_1, \dots, X_n) + c$ so that T_0 is location invariant. Now

$$E|T_0(\underline{X})|^3 = X_{(1)}^3 = (U_{(1)} + \theta)^3 = U_{(1)}^3 + 3\theta U_{(1)}^2 + 3\theta^2 U_{(1)} + \theta^3$$

so that $E|T_0(\underline{X})|^3 < \infty \iff U_{(1)}^3 < \infty$. We have

$$E\, U_{(1)}^3 = \int_0^1 x^3 n(1-x)^{n-1} dx < \int_0^1 1^3 n\, 1^{n-1} dx = n < \infty$$

so that the Pitman estimator is admissible.

2d) Since $T(X) \equiv \frac{1}{2}(X_{(1)} + X_{(n)} - 1)$ is admissible it must be a Bayes rule (see class text for theorem). Thus if we can show that $T(X)$ has constant frequentist risk wrt θ then T will be minimax.

Now

$$Var[X_{(1)}] = \frac{2}{(n+1)(n+2)} + \frac{2\theta}{n+1} + \theta^2 - \left(\frac{1}{n+1} + \theta\right)$$

$$= \frac{2}{(n+1)(n+2)} + \frac{2\theta}{n+1} + \theta^2 - \frac{1}{(n+1)^2} - \frac{2\theta}{n+1} - \theta^2$$

$$= \frac{2(n+1)-(n+2)}{(n+1)^2(n+2)} = \frac{n}{(n+1)^2(n+2)}$$

$\boxed{\text{write as } E_\theta \text{ so we don't have to carry } |\theta\text{'s}}$

Now,

$$E_{X|\theta}\left[(T(X)-\theta)^2 \mid \theta\right] = E_{X|\theta}\left[\left(T(X) - E_{X|\theta}[T(X)|\theta] + E_{X|\theta}[T(X)|\theta] - \theta\right)^2 \mid \theta\right]$$

$$= Var_{X|\theta}[T(X)|\theta] + \left(E_{X|\theta}[T(X)|\theta] - \theta\right)^2$$

But

$$Var_{X|\theta}[T(X)|\theta] = Var_{X|\theta}\left[\tfrac{1}{2}(X_{(1)} + X_{(n)} - 1)\right]$$

$$= \frac{1}{4}\left\{Var[X_{(1)}|\theta] + Var[X_{(n)}|\theta] + Cov[X_{(1)}, X_{(n)}|\theta]\right\}$$

$$= \frac{1}{4}\left\{Var[U_{(1)}+\theta|\theta] + Var[U_{(n)}+\theta|\theta] + Cov[U_{(1)}+\theta, U_{(n)}+\theta|\theta]\right\}$$

$$= \frac{1}{4}\left\{ Var[U_{(1)}|\theta] + Var[U_{(n)}|\theta] + Cov[U_{(1)}, U_{(n)}|\theta] \right\}$$

$$= \frac{1}{4}\left\{ \frac{n}{(n+1)^2(n+2)} + \frac{n}{(n+1)^2(n+2)} + \frac{1}{(n+1)^2(n+2)} \right\}$$

$$= \frac{2n+1}{4(n+1)^2(n+2)}$$

and

$$\left(E[T(x)|\theta] - \theta\right)^2 = \left(E\left[\frac{X_{(1)} + X_{(n)} - 1}{2} - \theta \,\middle|\, \theta\right]\right)^2$$

$$= \left(E\left[\frac{U_{(1)} + \theta + U_{(n)} + \theta - 1}{2} - \theta \,\middle|\, \theta\right]\right)^2$$

$$= \frac{1}{4}\left\{E[U_{(1)}] + E[U_{(n)}] - 1\right\}^2$$

$$= \frac{1}{4}\left\{\frac{1}{n+1} + \frac{n}{n+1} - 1\right\}^2 = 0$$

so that T is minimax

2e) $R_n = X_{(n)} - X_{(1)} \overset{d}{=} U_{(n)} + \theta - (U_{(1)} + \theta) \overset{d}{=} U_{(n)} - U_{(1)}$

Let $W = U_{(n)}$, then $0 < R_n < W < 1$. $U_{(1)} = W - R_n$, $U_{(n)} = W$

and $$J(r,w) = \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} = -1$$

so that

$$f_{R_n,W}(r,w) = f_{U_{(1)},U_{(n)}}(w-r, w)\,|-1| = n(n-1)(w-(w-r))^{n-2}$$

$$= n(n-1)r^{n-2}, \quad 0<r<w<1$$

and
$$f_R(r) = \int_r^1 n(n-1)r^{n-2}\,dw = n(n-1)r^{n-2}w\Big|_{w=r}^{w=1}$$

$$= n(n-1)r^{n-2}(1-r), \quad 0<r<1 \quad \sim \text{beta}(n-1, 2)$$

Consider the limiting distribution of $n(1-R_n)$. We have

$$P(n(1-R_n) < t) = P\left(1-R_n < \frac{t}{n}\right) = P\left(R_n > 1 - \frac{t}{n}\right)$$

$$= 1 - P\left(R_n \le 1 - \frac{t}{n}\right) = 1 - \int_0^{1-t/n} n(n-1)r^{n-2}(1-r)\,dr$$

$$= 1 - \int_0^{1-t/n} n(n-1)(r^{n-2} - r^{n-1})\,dr = 1 - n(n-1)\left[\frac{r^{n-1}}{n-1} - \frac{r^n}{n}\right]_{r=0}^{r=1-t/n}$$

$$= 1 - n(n-1)\frac{r^{n-1}[n-(n-1)r]}{n(n-1)}\Bigg|_{r=0}^{r=1-t/n}$$

$$= 1 - \left(1 - \frac{t}{n}\right)^{n-1} \left[n - (n-1) + (n-1)\frac{t}{n} \right]$$

$$= 1 - \left(1 - \frac{t}{n}\right)^{n-1} - \left(t - \frac{t}{n}\right)\left(1 - \frac{t}{n}\right)^{n-1}$$

$$\to 1 - e^{-t} - te^{-t}$$

Then

$$\frac{\partial}{\partial t}\left(1 - e^{-t} - te^{-t}\right) = e^{-t} - e^{-t} + te^{-t}$$

$$= te^{-t} \sim \text{gamma}(2, 1)$$