BIOS 779 Homework 3

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12th February 2023

1 Problem 1

Consider the usual linear model given by $Y = X\beta + \epsilon, \epsilon \sim N_n(0, \sigma^2 I)$, and X is $n \times p$ of full rank. Consider the full power prior for β, τ given by equation (2.3) on page 207 of the notes.

(i) Derive marginal posterior distribution of β and τ.

$$
\begin{aligned}
P(Y|\beta,\tau) &\propto \tau^{\frac{n}{2}} \exp\Big[-\frac{\tau}{2}(Y - X\beta)^T(Y - X\beta)\Big] \\
\pi(\beta,\tau) &\propto \tau^{\frac{n_0 a_0 + \delta_0}{2} - 1} \exp\Big[-\frac{a_0 \tau}{2}\Big((Y_0 - X_0\beta)^T(Y_0 - X_0\beta) + \frac{\gamma_0}{a_0}\Big)\Big] \\
P(\beta,\gamma|Y) &\propto \tau^{\frac{n}{2}} \tau^{\frac{n_0 a_0 + \delta_0}{2} - 1} \\
&\quad \exp\Big\{-\frac{\tau}{2}\Big[(Y - X\beta)^T(Y - X\beta) + a_0\Big((Y_0 - X_0\beta)^T(Y_0 - X_0\beta) + \frac{\gamma_0}{a_0}\Big)\Big]\Big\} \\
&= \tau^{\frac{n + n_0 a_0 + \delta_0}{2} - 1} \exp\Big\{-\frac{\tau}{2}\Big[Y^T(I - M)Y + (\beta - \hat{\beta})^T(X^T X)(\beta - \hat{\beta}) \\
&\quad + a_0\Big(Y_0^T(I - M_0)Y_0 + (\beta - \hat{\beta})^T(X_0^T X_0)(\beta - \hat{\beta}) + \frac{\gamma_0}{a_0}\Big)\Big]\Big\}
\end{aligned}
$$

Let $Y^T(I - M)Y = (n - p)S^2, Y_0^T(I - M_0)Y_0 = (n - p)S_0^2$,

$$
\begin{aligned}
P(\beta,\tau|Y) \propto \tau^{\frac{n+n_0a_0+\delta_0}{2}-1} \exp\Big\{-\frac{\tau}{2}\Big[(n-p)S^2+(\beta-\hat{\beta})^T(X^TX)(\beta-\hat{\beta}) \\
+\Big(a_0(n-p)S_0^2+a_0(\beta-\hat{\beta})^T(X_0^TX_0)(\beta-\hat{\beta})+\gamma_0\Big)\Big]\Big\} \\
\propto \tau^{\frac{n+n_0a_0+\delta_0}{2}-1} \exp\Big\{-\frac{\tau}{2}(n-p)S^2-\frac{\tau}{2}a_0(n-p)S_0^2-\frac{\tau}{2}\gamma_0\Big\} \\
\exp\Big\{-\frac{\tau}{2}\Big[(\beta-\hat{\beta})^T(X^TX)(\beta-\hat{\beta})+a_0(\beta-\hat{\beta})^T(X_0^TX_0)(\beta-\hat{\beta})\Big]\Big\} \\
\propto \tau^{\frac{n+n_0a_0+\delta_0}{2}-1} \exp\Big\{-\frac{\tau}{2}\Big[(n-p)S^2+a_0(n-p)S_0^2+\gamma_0\Big]\Big\} \\
\exp\Big\{-\frac{\tau}{2}\Big[(\beta-\tilde{\beta})^T\Big(X^TX+a_0X_0^TX_0\Big)(\beta-\tilde{\beta}) \\
-\tilde{\beta}'(X'X+a_0X_0'X_0)\tilde{\beta}+\hat{\beta}'X'X\hat{\beta}+\hat{\beta}_0^T(a_0X_0^TX_0)\hat{\beta}_0\Big]\Big\} \\
\Lambda = \frac{a_0X_0^TX_0}{X^TX+a_0X_0^TX_0} \\
I-\Lambda = \frac{X^TX}{X^TX+a_0X_0^TX_0} \\
\hat{\beta} = (X'X)^{-1}X^TY \\
\hat{\beta}_0 = (X_0'X_0)^{-1}X_0^TY_0
\end{aligned}
$$

By calculation,

$$
-\tilde{\beta}'(X'X+a_0X_0'X_0)\tilde{\beta}+\hat{\beta}'X'X\hat{\beta}+\hat{\beta}_0^T(a_0X_0^TX_0)\hat{\beta}_0 = (\hat{\beta}-\hat{\beta}_0)^T(\Lambda^TX'X)(\hat{\beta}-\hat{\beta}_0)
$$

We can write the marginal posterior distribution $P(\tau|Y)$

$$
\begin{aligned}
P(\beta,\tau|Y) = P(\beta|\tau,Y)P(\tau|Y) \\
P(\tau|Y) = \int P(\beta,\tau|Y)d\beta \\
P(\tau|Y) \propto \tau^{\frac{n_0a_0+\delta_0}{2}-1} \exp\Big\{-\frac{\tau}{2}\Big[(n-p)S^2+a_0(n-p)S_0^2+\gamma_0+(\hat{\beta}-\hat{\beta}_0)^T(\Lambda^TX'X)(\hat{\beta}-\hat{\beta}_0)\Big]\Big\} \\
a = \frac{n_0a_0+\delta_0}{2} \\
b = \Big[(n-p)S^2+a_0(n-p)S_0^2+\gamma_0+(\hat{\beta}-\hat{\beta}_0)^T(\Lambda^TX'X)(\hat{\beta}-\hat{\beta}_0)\Big] \\
P(\tau|Y) \sim \Gamma(a,b)
\end{aligned}
$$

the marginal posterior distribution $P(\beta|Y)$

$$
\begin{aligned}
P(\beta|Y) = \int P(\beta,\tau|Y)d\tau \\
\propto \int \tau^{\frac{n+n_0a_0+\delta_0}{2}-1} \exp\Big\{-\frac{\tau}{2}\Big[(n-p)S^2+a_0(n-p)S_0^2+\gamma_0+(\hat{\beta}-\hat{\beta}_0)^T(\Lambda^TX'X)(\hat{\beta}-\hat{\beta}_0)\Big]\Big\} \\
\exp\Big\{-\frac{\tau}{2}\Big[(\beta-\tilde{\beta})^T\Big(X^TX+a_0X_0^TX_0\Big)(\beta-\tilde{\beta})\Big]\Big\}d\tau \\
\tilde{S}^2 = (n+n_0a_0+\delta_0)^{-1}\Big[(n-p)S^2+a_0(n-p)S_0^2+\gamma_0+(\hat{\beta}-\hat{\beta}_0)^T(\Lambda^TX'X)(\hat{\beta}-\hat{\beta}_0)\Big] \\
P(\beta|Y) \sim S_p\Big(n+n_0a_0+\delta_0,\tilde{\beta},\tilde{S}^2(X^TX+a_0X_0^TX_0\Big)^{-1}\Big)
\end{aligned}
$$

(ii) Let $z_{q\times 1}$ vector of future observations take at X_f , where X_f is $q \times p$. Derive the predictive distribution of z based on the full power prior for β, τ given by equation (2.3) on page 207 of the notes.

Derive the predictive distribution of z

$$\begin{aligned}
P(z|Y) &= \int P(\beta,\tau|Y)P(z|\beta,\tau)d\beta d\tau \\
&= \int \tau^{\frac{n+n_0a_0+\delta_0}{2}-1} \exp\Big\{-\frac{\tau}{2}\Big[(n-p)S^2 + a_0(n-p)S_0^2 + \gamma_0 + (\hat\beta-\hat\beta_0)^T(\Lambda^T X'X)(\hat\beta-\hat\beta_0)\Big]\Big\} \\
&\exp\Big\{-\frac{\tau}{2}\Big[(\beta-\tilde\beta)^T\Big(X^TX + a_0X_0^TX_0\Big)(\beta-\tilde\beta)\tau^{\frac{q}{2}} \exp\Big(-\frac{\tau}{2}(z-X_f\beta)^T(z-X_f\beta)\Big)d\beta d\tau \\
\Sigma^{-1} &= X^TX + a_0X_0^TX_0 \\
\tilde\beta_f &= \Big(X_f'X_f + \Sigma^{-1}\Big)^{-1}\Big[(X_f'X_f)\hat\beta_z + \Sigma^{-1}\tilde\beta\Big] \\
\Lambda_f &= \Big(X_f'X_f + \Sigma^{-1}\Big)^{-1} X_f'X_f \\
P(z|Y) &\sim S_q\Big(n + n_0a_0 + \delta_0 - p, X_f\tilde\beta, \tilde S^2(I + X_f\Sigma X_f')\Big)
\end{aligned}$$

2 Problem 2

Derive the distribution of $(z|X_{(i)}, Y_{(i)})$ given on page 286 of the notes. Use this result to obtain $p(z|X_{(i)}, Y_{(i)})$.

We have

$$\begin{aligned}
Y_{(i)} &= X_{(i)}\beta + \epsilon \\
\pi(\beta,\tau) &= \tau^{-1}
\end{aligned}$$

The posterior distribution of β, τ

$$\begin{aligned}
P(\beta,\tau|Y_{(i)}) &\propto \tau^{\frac{n-1}{2}}\tau^{-1}\exp\Big\{-\frac{\tau}{2}\Big[(Y_{(i)}-X_{(i)}\beta)^T(Y_{(i)}-X_{(i)}\beta)\Big]\Big\} \\
&\propto \tau^{\frac{n-1}{2}-1}\exp\Big\{-\frac{\tau}{2}\Big[Y_{(i)}^T(I-M_{(i)})Y_{(i)} + (\beta-\hat\beta)^TX_{(i)}'X_{(i)}(\beta-\hat\beta)\Big]\Big\} \\
&\propto \tau^{\frac{n-1}{2}-1}\exp\Big\{-\frac{\tau}{2}\Big[(n-p-1)S^2 + (\beta-\hat\beta_{(i)})^TX_{(i)}'X_{(i)}(\beta-\hat\beta_{(i)})\Big]\Big\}
\end{aligned}$$

The predictive distribution of $z|X_{(i)}, Y_{(i)}$,

$$\begin{aligned}
P(z|X_{(i)}, Y_{(i)}) &= \int\int P(\beta,\tau|Y_{(i)})P(z|\beta,\tau)d\beta d\tau \\
&\propto \int\int \tau^{\frac{n-1}{2}-1}\exp\Big\{-\frac{\tau}{2}(n-p-1)S^2\Big\}\exp\Big\{-\frac{\tau}{2}(\beta-\hat\beta_{(i)})^TX_{(i)}'X_{(i)}(\beta-\hat\beta_{(i)})\Big\} \\
&\tau^{\frac{n}{2}}\exp\Big\{-\frac{\tau}{2}(z-X\beta)^T(z-X\beta)\Big\}d\beta d\tau
\end{aligned}$$

In which, we have

$$
\begin{aligned}
&(z-X\beta)^T(z-X\beta)=z^T(I-M)z+(\beta-\hat\beta)^TX'X(\beta-\hat\beta)\\
P(z|X_{(i)},Y_{(i)})\propto&\int\int\tau^{\frac{n-1}{2}-1}\exp\Big\{-\frac{\tau}{2}(n-p-1)S^2_{(i)}\Big\}\exp\Big\{-\frac{\tau}{2}(\beta-\hat\beta_{(i)})^TX'_{(i)}X_{(i)}(\beta-\hat\beta_{(i)})\Big\}\\
&\tau^{\frac{n}{2}}\exp\Big\{-\frac{\tau}{2}\Big(\beta-\hat\beta)^TX'X(\beta-\hat\beta+z^T(I-M)z\Big)\Big\}d\beta d\tau\\
&\propto\int\int\tau^{\frac{2n-1}{2}-1}\exp\Big\{-\frac{\tau}{2}(n-p-1)S^2_{(i)}\Big\}\exp\Big\{-\frac{\tau}{2}\Big[z^T(I-M)z\\
&+(\beta-\hat\beta_{(i)})^TX'_{(i)}X_{(i)}(\beta-\hat\beta_{(i)})+(\beta-\hat\beta)^TX'X(\beta-\hat\beta)\Big]\Big\}d\beta d\tau
\end{aligned}
$$

In which,

$$
\begin{aligned}
&(\beta-\hat\beta_{(i)})^TX'_{(i)}X_{(i)}(\beta-\hat\beta_{(i)})+(\beta-\hat\beta)^TX'X(\beta-\hat\beta)\\
&=(\beta-\tilde\beta)^T(X'_{(i)}X_{(i)}+X'X)(\beta-\tilde\beta)-\tilde\beta'(X'_{(i)}X_{(i)}+X'X)\tilde\beta+\hat\beta'_{(i)}X'_{(i)}X_{(i)}\hat\beta_{(i)}+\hat\beta'X'X\hat\beta\\
&=(\beta-\tilde\beta)^T\Big(X'_{(i)}X_{(i)}+X'X\Big)(\beta-\tilde\beta)-\tilde\beta'\Big(X'_{(i)}X_{(i)}+X'X\Big)\tilde\beta+\hat\beta'_{(i)}X'_{(i)}X_{(i)}\hat\beta_{(i)}+\hat\beta'X'X\hat\beta\\
&=(\beta-\tilde\beta)^T\Big(X'_{(i)}X_{(i)}+X'X\Big)(\beta-\tilde\beta)+(\hat\beta_{(i)}-\hat\beta)^T(\Lambda'X'X)(\hat\beta_{(i)}-\hat\beta)\\
&\Lambda=\frac{X'X}{X'_{(i)}X_{(i)}+X'X}
\end{aligned}
$$

So we have

$$
\begin{aligned}
P(z|X_{(i)},Y_{(i)})\propto&\int_0^\infty\int_{-\infty}^\infty\int\int\tau^{\frac{2n-1}{2}-1}\exp\Big\{-\frac{\tau}{2}\Big[(n-p-1)S^2_{(i)}+(\beta-\tilde\beta)^T\Big(X'_{(i)}X_{(i)}+X'X\Big)(\beta-\tilde\beta)\Big]\Big\}\\
&\exp\Big\{-\frac{\tau}{2}\Big[z^T(I-M)z+(\hat\beta_{(i)}-\hat\beta)^T(\Lambda'X'X)(\hat\beta_{(i)}-\hat\beta)\Big]\Big\}d\beta d\tau\\
P(z|X_{(i)},Y_{(i)})\sim&\, S_n\Big(n-p-1,X\hat\beta_{(i)},S^2_{(i)}(I+X(X'_{(i)}X_{(i)})^{-1}X')\Big)\\
S^2_{(i)}=&\frac{Y'_{(i)}(I-M_{(i)})Y_{(i)}}{n-p-1}
\end{aligned}
$$

3 Problem 3