2012 Theory II #3

3a) $$p(y_{i1}) = \sum_{y_{i2}=0}^{1} p(y_{i1}, y_{i2})$$

$$= [\Delta(\theta,\lambda)]^{-1} \exp\{y_{i1}\theta_1 - C(y_{i1},0)\} + [\Delta(\theta,\lambda)]^{-1} \exp\{y_{i1}\theta_1 + \theta_2 + y_{i1}\lambda - C(y_{i1},1)\}$$

$$= [\Delta(\theta,\lambda)]^{-1} e^{y_{i1}\theta_1}\left[e^{-C(y_{i1},0)} + \exp\{\theta_2 + y_{i1}\lambda - C(y_{i1},1)\}\right]$$

$$p(y_{i2}) = \sum_{y_{i1}=0}^{1} p(y_{i1}, y_{i2})$$

$$= [\Delta(\theta,\lambda)]^{-1} e^{y_{i2}\theta_2}\left[e^{-C(0,y_{i2})} + \exp\{\theta_1 + y_{i2}\lambda - C(1,y_{i2})\}\right]$$

$$p(y_{i2} \mid y_{i1}) = \frac{p(y_{i1}, y_{i2})}{p(y_{i1})}$$

$$= \frac{[\Delta(\theta,\lambda)]^{-1} e^{y_{i1}\theta_1} \exp\{y_{i2}\theta_2 + y_{i1}y_{i2}\lambda - C(y_{i1}, y_{i2})\}}{[\Delta(\theta,\lambda)]^{-1} e^{y_{i1}\theta_1}\left[e^{-C(y_{i1},0)} + \exp\{\theta_2 + y_{i1}\lambda - C(y_{i1},1)\}\right]}$$

$$= \frac{\exp\{y_{i2}\theta_2 + y_{i1}y_{i2}\lambda - C(y_{i1},y_{i2})\}}{e^{-C(y_{i1},0)} + \exp\{\theta_2 + y_{i1}\lambda - C(y_{i1},1)\}}$$

A n.s. condition that $Y_{i1} \perp\!\!\!\perp Y_{i2}$ is that $p(y_{i1},y_{i2}) = g(y_{i1})h(y_{i2})$ for two functions g,h. Thus

$$Y_{i1} \perp\!\!\!\perp Y_{i2} \iff y_{i1}y_{i2}\lambda - C(y_{i1},y_{i2}) = k_1(y_{i1}) + k_2(y_{i2})$$

for two functions k_1, k_2.

3b) $$EY_{i1} = P(Y_{i1}=1) = [\Delta(\theta,\lambda)]^{-1} e^{\theta_1}\left[e^{-C(1,0)} + e^{\theta_2+\lambda-C(1,1)}\right]$$

$$EY_{i2} = P(Y_{i2}=1) = [\Delta(\theta,\lambda)]^{-1} e^{\theta_2}\left[e^{-C(0,1)} + e^{\theta_1+\lambda-C(1,1)}\right]$$

$$E[Y_{i1}Y_{i2}] = P(Y_{i1}=1, Y_{i2}=1)$$
$$= [\Delta(\theta,\lambda)]^{-1} \exp\{\theta_1+\theta_2+\lambda - C(1,1)\}$$

$$\mathrm{Cov}[Y_{i1}, Y_{i2}] = E[Y_{i1}Y_{i2}] - (EY_{i1})(EY_{i2})$$
$$= [\Delta(\theta,\lambda)]^{-1} e^{\theta_1+\theta_2}\Big\{e^{\lambda - C(1,1)} - [\Delta(\theta,\lambda)]^{-1}\left[e^{-C(1,0)} + e^{\theta_2+\lambda-C(1,1)}\right]$$
$$\times\left[e^{-C(0,1)} + e^{\theta_1+\lambda-C(1,1)}\right]\Big\}$$

Under conditions which will be discussed later,

3c) $$V \equiv \frac{\partial(\theta_1,\theta_2,\lambda)}{\partial(\mu_1,\mu_2,\eta_{12})} = \left[\frac{\partial(\mu_1,\mu_2,\eta_{12})}{\partial(\theta_1,\theta_2,\lambda)}\right]^{-1}$$

Now

$$1 = \sum_{y_{i1}=0}^{1}\sum_{y_{i2}=0}^{1} p(y_{i1},y_{i2}) = [\Delta(\theta,\lambda)]^{-1}\exp\left\{y_{i1}\theta_1 + y_{i2}\theta_2 + y_{i1}y_{i2}\lambda - C(y_{i1},y_{i2})\right\}$$

$$\Rightarrow \Delta(\theta,\lambda) = \underbrace{e^{-C(0,0)}}_{A_{00}} + \underbrace{e^{\theta_2 - C(0,1)}}_{A_{01}} + \underbrace{e^{\theta_1 - C(1,0)}}_{A_{10}} + \underbrace{e^{\theta_1+\theta_2+\lambda - C(1,1)}}_{A_{11}}$$

Then

$$\mu_1 = \frac{A_{10} + A_{11}}{\sum_i \sum_j A_{ij}}, \quad \mu_2 = \frac{A_{01} + A_{11}}{\sum_i \sum_j A_{ij}}, \quad \eta_{12} = \frac{A_{11}}{\sum_i \sum_j A_{ij}}$$

so that

$$V^{-1} = \frac{\partial(\mu_1, \mu_2, \eta_{12})}{\partial(\theta_1, \theta_2, \lambda)} = \frac{\partial(\mu_1, \mu_2, \eta_{12})}{\partial(A_{00}, A_{01}, A_{10}, A_{11})} \cdot \frac{\partial(A_{00}, A_{01}, A_{10}, A_{11})}{\partial(\theta_1, \theta_2, \lambda)}$$

$$= \frac{1}{[\Delta(\theta,\lambda)]^2} \begin{bmatrix} -(A_{10}+A_{11}) & -(A_{10}+A_{11}) & \sum\sum A_{ij} - (A_{10}+A_{11}) & \sum A_{ij} - (A_{10}+A_{11}) \\ -(A_{01}+A_{11}) & \sum\sum A_{ij} - (A_{01}+A_{11}) & -(A_{01}+A_{11}) & \sum\sum A_{ij} - (A_{01}+A_{11}) \\ -A_{11} & -A_{11} & -A_{11} & \sum\sum A_{ij} - A_{11} \end{bmatrix}$$

$$\times \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{01} & 0 \\ A_{10} & 0 & 0 \\ A_{11} & A_{11} & A_{11} \end{bmatrix}$$

$$= \frac{1}{[\Delta(\theta,\lambda)]^2} \begin{bmatrix} -(A_{10}+A_{11}) & -(A_{10}+A_{11}) & A_{00}+A_{01} & A_{00}+A_{01} \\ -(A_{01}+A_{11}) & A_{00}+A_{10} & -(A_{01}+A_{11}) & A_{00}+A_{10} \\ -A_{11} & -A_{11} & -A_{11} & A_{00}+A_{01}+A_{10} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{01} & 0 \\ A_{10} & 0 & 0 \\ A_{11} & A_{11} & A_{11} \end{bmatrix}$$

$$= \frac{1}{[\Delta(\theta,\lambda)]^2} \begin{bmatrix} (A_{00}+A_{01})(A_{10}+A_{11}) & A_{11}(A_{00}+A_{01}) - A_{01}(A_{10}+A_{11}) & A_{11}(A_{00}+A_{01}) \\ A_{11}(A_{00}+A_{10}) - A_{10}(A_{01}+A_{11}) & (A_{00}+A_{10})(A_{01}+A_{11}) & A_{11}(A_{00}+A_{10}) \\ A_{11}(A_{00}+A_{01}+A_{10}) - A_{10}A_{11} & A_{11}(A_{00}+A_{01}+A_{10}) - A_{01}A_{11} & A_{11}(A_{00}+A_{01}+A_{10}) \end{bmatrix}$$

$$= \frac{1}{[\Delta(\theta,\lambda)]^2}\begin{bmatrix} (A_{00}+A_{01})(A_{10}+A_{11}) & A_{00}A_{11} - A_{01}A_{10} & A_{11}(A_{00}+A_{01}) \\ A_{00}A_{11} - A_{01}A_{10} & (A_{00}+A_{10})(A_{01}+A_{11}) & A_{11}(A_{00}+A_{10}) \\ A_{11}(A_{00}+A_{01}) & A_{11}(A_{00}+A_{10}) & A_{11}(A_{00}+A_{01}+A_{10}) \end{bmatrix}$$

$$= \begin{bmatrix} \mu_1(1-\mu_1) & & \eta_{12}(1-\mu_1) \\ (1-\mu_1-\mu_2+\eta_{12})\eta_{12} - (\mu_1-\eta_{12})(\mu_2-\eta_{12}) & (1-\mu_2)\mu_2 & \eta_{12}(1-\mu_2) \\ \eta_{12}(1-\mu_1) & \eta_{12}(1-\mu_2) & \eta_{12}(1-\eta_{12}) \end{bmatrix}$$

$$= \begin{bmatrix} \mu_1(1-\mu_1) & \eta_{12} - \mu_1\mu_2 & \eta_{12}(1-\mu_1) \\ & \mu_2(1-\mu_2) & \eta_{12}(1-\mu_2) \\ & & \eta_{12}(1-\eta_{12}) \end{bmatrix}$$

We observe that

$$V^{-1} = \mathrm{Var}\left(\begin{bmatrix} Y_{i1} \\ Y_{i2} \\ Y_{i1}Y_{i2} \end{bmatrix}\right)$$

So a nonsingularity condition on $\mathrm{Var}[\cdot]$ would ensure that V is locally invertible at each $(\mu_1, \mu_2, \eta_{12})$ and hence 1-1.

3d) $$\frac{\partial}{\partial(\alpha,\beta)} \ell_n(\theta,\lambda) = \sum_{i=1}^{n} \frac{\partial}{\partial(\alpha,\beta)} \ell_i(\theta_i,\lambda_i) = \sum_{i=1}^{n} \left[\frac{\partial \ell_i(\theta_i,\lambda_i)}{\partial(\theta_i,\lambda_i)}\right]' \left[\frac{\partial(\theta_i,\lambda_i)}{\partial(\mu_{i1},\mu_{i2},\eta_{12})}\right] \times \left[\frac{\partial(\mu_{i1},\mu_{i2},\eta_{12})}{\partial(\alpha,\beta)}\right]$$

$$\ell_n(\theta,\lambda) = -\sum_{i=1}^{n} \log[\Delta(\theta_i,\lambda_i)] + \sum_{i=1}^{n}\left\{ y_{i1}\theta_{i1} + y_{i2}\theta_{i2} + y_{i1}y_{i2}\lambda_i - C(y_{i1},y_{i2})\right\}$$

Define $y_i^* = (y_{i1}, y_{i2}, y_{i1}y_{i2})$

$\mu_i^* = (EY_{i1}, EY_{i2}, E[Y_{i1}Y_{i2}])$

since $\log[\Delta(\theta_i,\lambda_i)]$ is $b(\theta_i,\lambda_i)$

$$\frac{\partial}{\partial(\theta_i,\lambda_i)} l_i(\theta_i,\lambda_i) = y_i^* - \mu_i^*$$

$$\frac{\partial(\mu_i,\eta_{i12})}{\partial(\alpha,\beta)} = \begin{bmatrix} 0 & \frac{\partial \mu_{i1}}{\partial \beta'} \\ 0 & \frac{\partial \mu_{i2}}{\partial \beta'} \\ \frac{\partial \eta_{i12}}{\partial \alpha} & \frac{\partial \eta_{i12}}{\partial \beta'} \end{bmatrix}$$

$$\frac{\partial(\theta_i,\lambda_i)}{\partial(\mu_i,\lambda_i)} = V_i^{-1}$$

Thus

$$\frac{\partial}{\partial(\alpha,\beta)} l_n(\theta,\lambda) = \sum_{i=1}^{n} (y_i^* - \mu_i^*)' V_i^{-1} \begin{bmatrix} 0 & \frac{\partial \mu_{i1}}{\partial \beta'} \\ 0 & \frac{\partial \mu_{i2}}{\partial \beta'} \\ \frac{\partial \eta_{i12}}{\partial \alpha} & \frac{\partial \eta_{i12}}{\partial \beta'} \end{bmatrix}$$

* Expression does not include $C(y_{i1}, y_{i2})$ for any i

3e) Define $V_n = \left(E[\dot{s}_n(\theta)]\right)^{-1} Var[s_n(\theta)] \left(E[\dot{s}_n(\theta)]\right)^{-1}$. Then under appropriate regularity conditions,

$$V_n^{-1/2}(\hat{\theta}_n - \theta) \xrightarrow{L} N(0, I_k)$$

where

$$s_n(\theta) = \sum_{i=1}^{n} \frac{\partial(\mu_i, \sigma_{i12})}{\partial(\alpha,\beta)'} \cdot \frac{\partial l_i(\theta_i,\lambda_i)}{\partial(\theta_i,\lambda_i)} = \sum_{i=1}^{n} \frac{\partial(\mu_i,\sigma_{i12})}{\partial(\alpha,\beta)'} (y_i^* - \mu_i^*)$$

and ???