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a) $E[\{Y-(\alpha+\beta X)\}^2] = E[Y^2 - 2Y(\alpha+\beta X) + (\alpha+\beta X)^2]$

$= E[Y^2] - 2\alpha E[Y] - 2\beta E[XY] + \alpha^2 + 2\alpha\beta E[X] + \beta^2 E[X^2]$

$$\frac{\partial E[\{Y-(\alpha+\beta X)\}^2]}{\partial \alpha} = -2E[Y] + 2\alpha + 2\beta \underbrace{E[X]}_{0} \overset{=}{\text{set}} 0$$

$$\Rightarrow \boxed{\alpha^* = E[Y]}$$

$$\frac{\partial E[\{Y-(\alpha+\beta X)\}^2]}{\partial \beta} = -2E[XY] + 2\alpha \underbrace{E[X]}_{0} + 2\beta \underbrace{E[X^2]}$$

$\text{var}X = E[X^2] - \{E[X]\}^2$

$\Rightarrow E[X^2] = \text{var}X + \{E[X]\}^2$

$= 1 + 0 = 1$

$$= -2E[XY] + 2\beta \overset{=}{\text{set}} 0$$

$$\Rightarrow \boxed{\beta^* = E[XY].}$$

b) $\hat{\alpha} = \frac{1}{n}\sum_{i=1}^{n} Y_i$

$\hat{\beta} = \frac{1}{n}\sum_{i=1}^{n} X_i Y_i$

By Multivariate CLT, since $E[X^2Y^2] < \infty$

$$\sqrt{n}\begin{pmatrix} \hat{\alpha} - \alpha^* \\ \hat{\beta} - \beta^* \end{pmatrix} \to_d N_2\left(0, \begin{pmatrix} \text{var}(Y_i) & \text{cov}(X_iY_i, Y_i) \\ \text{cov}(X_iY_i, Y_i) & \text{var}(X_iY_i) \end{pmatrix}\right)$$

$$\text{cov}(X_iY_i, Y_i) = E[X_iY_i^2] - \underbrace{E[X_iY_i]}_{\beta^*}\underbrace{E[Y_i]}_{\alpha^*}$$

c) Let $Z = \log Y$

$$Z|X \sim N(\gamma X, \; + \sigma^2) \qquad \tfrac{b}{c} \text{ conditional}$$

$$f_{Z|X}(z|x) = \frac{1}{\sqrt{2\pi(+ \sigma^2)}} \exp\left\{ -\frac{1}{2} \frac{(z-\gamma x)^2}{ + \sigma^2} \right\}$$

$$Y = \exp(Z) \quad \Rightarrow \quad Z = \log Y$$

$$\frac{\partial z}{\partial y} = \frac{1}{y}$$

$$f_{Y|X}(y|x) = f_{Z|X}(\log y|x) = \frac{1}{\sqrt{2\pi(+ \sigma^2)}} \exp\left\{ -\frac{1}{2} \frac{(\log y - \gamma x)^2}{ + \sigma^2} \right\} \frac{1}{y}$$

$$\Rightarrow f_{XY}(x,y) = f_{Y|X}(y|x)\, f_X(x)$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2} x^2 \right\} \frac{1}{\sqrt{2\pi(+ \sigma^2)}} \exp\left\{ -\frac{1}{2} \frac{(\log y - \gamma x)^2}{ + \sigma^2} \right\} \frac{1}{y}$$

NOW

$$\alpha^* = E[Y] = E_X[E_{Y|X}[Y|X]]$$

$Z = \log Y$

$Z|X \sim N(\gamma X, \sigma^2)$

$$E[e^{Zt}] = \exp(\gamma X t + \tfrac{1}{2}\sigma^2 t^2) \qquad \text{all conditional on } X$$

$$E[e^{Zt}] = E[e^{\log Y t}] = E[e^{\log(Y^t)}] = E[Y^t]$$

$$\Rightarrow E[Y^t|X] = \exp\{\gamma X t + \tfrac{1}{2}\sigma^2 t^2\}$$

$$\Rightarrow E[Y|X] = \exp(\gamma X + \tfrac{1}{2}\sigma^2)$$

$$\Rightarrow \alpha^* = E_X[\exp(\gamma X + \tfrac{1}{2}\sigma^2)]$$

c) con't

$$E\left[\exp(\gamma X + \tfrac{1}{2}\sigma^2)\right] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) \exp\left(\gamma x + \tfrac{1}{2}\sigma^2\right) dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x-\gamma)^2\right) \exp\left(\tfrac{1}{2}(\sigma^2+\gamma^2)\right) dx$$

$$= -\frac{1}{2}x^2 + \frac{2}{2}x\gamma - \frac{1}{2}\gamma^2$$

$$= \exp\left(\tfrac{1}{2}(\sigma^2+\gamma^2)\right) \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x-\gamma)^2\right) dx}_{\text{pdf of } N(\gamma,1)}$$

$$\Rightarrow \alpha^* = \exp\left(\tfrac{1}{2}(\sigma^2+\gamma^2)\right)$$

$$\beta^* = E[XY] = E_X[X\, E_{Y|X} E[Y]]$$

$$= E_X[X \exp(\gamma X + \tfrac{1}{2}\sigma^2)]$$

$$E[X\exp(\gamma X + \tfrac{1}{2}\sigma^2)] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) x \exp\left(\gamma x + \tfrac{1}{2}\sigma^2\right) dx$$

$$= \exp\left(\tfrac{1}{2}(\sigma^2+\gamma^2)\right) \underbrace{\int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x-\gamma)^2\right) dx}_{E[N(\gamma,1)]}$$

$$= \gamma \exp\left(\tfrac{1}{2}(\sigma^2+\gamma^2)\right).$$

$$\Rightarrow \beta^* = \gamma \exp\left(\tfrac{1}{2}(\sigma^2+\gamma^2)\right)$$

NOW

$$L(\gamma, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}x_i^2\right\} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\frac{(\log y_i - \gamma x_i)^2}{\sigma^2}\right\} \frac{1}{y_i}$$

$$= (2\pi)^{-\frac{n}{2}} (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2}\left[\textstyle\sum_{i=1}^n x_i^2\right] - \frac{1}{2\sigma^2}\sum_{i=1}^n (\log y_i - \gamma x_i)^2\right\} \prod_{i=1}^n \frac{1}{y_i}$$

$$\ell(\gamma,\sigma^2) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2}\sum_{i=1}^n x_i^2 - \frac{1}{2\sigma^2}\sum_{i=1}^n (\log y_i)^2 + \frac{1}{\sigma^2}\sum_{i=1}^n \log y_i\, \gamma x_i$$

$$- \frac{1}{2\sigma^2}\sum_{i=1}^n \gamma^2 x_i^2 + \log\left(\prod_{i=1}^n \frac{1}{y_i}\right)$$

$$\frac{\partial \ell(\gamma,\sigma^2)}{\partial \gamma} = \frac{1}{\sigma^2}\sum_{i=1}^n \log(y_i)x_i - \frac{\gamma}{\sigma^2}\sum_{i=1}^n x_i^2 \underset{\text{set}}{=} 0$$

$$\Rightarrow \boxed{\frac{\sum_{i=1}^n \log(y_i)x_i}{\sum_{i=1}^n x_i^2} = \hat{\gamma}}$$

c) cont'd

η^2

$$\frac{\partial \ell(\gamma,\sigma^2)}{\partial \sigma^2} = -\frac{n}{2}\frac{\cancel{2\pi}}{\cancel{2\pi}\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^n (\log y_i)^2 - \frac{\gamma}{\sigma^4}\sum_{i=1}^n \log y_i x_i + \frac{\gamma^2}{2\sigma^4}\sum_{i=1}^n x_i^2 \overset{set}{=}$$

$$\Rightarrow \frac{1}{2}\sum_{i=1}^n (\log y_i)^2 + \frac{\gamma^2}{2}\sum_{i=1}^n x_i^2 = \frac{n\sigma^2}{2} + \gamma\sum_{i=1}^n \log(y_i)x_i$$

$$\sum_{i=1}^n (\log y_i)^2 + \hat{\gamma}^2 \sum_{i=1}^n x_i^2 - 2\hat{\gamma}\sum_{i=1}^n \log(y_i)x_i = n\hat{\sigma}^2$$

$$\boxed{\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (\log y_i)^2 + \frac{\hat{\gamma}^2}{n}\sum_{i=1}^n x_i^2 - \frac{2\hat{\gamma}}{n}\sum_{i=1}^n \log(y_i)x_i}$$

$$\frac{\partial^2 \ell(\gamma,\sigma^2)}{\partial \gamma^2} = -\frac{1}{\sigma^2}\sum_{i=1}^n x_i^2$$

$$E\left[-\frac{\partial^2 \ell(\gamma,\sigma^2)}{\partial \gamma^2}\right] = \frac{1}{\sigma^2}\sum_{i=1}^n E[x_i^2] = \frac{1}{\sigma^2}\sum_{i=1}^n \{1+0^2\} = \frac{n}{\sigma^2}$$

$$\frac{\partial^2 \ell(\gamma,\sigma^2)}{\partial \gamma\,\partial \sigma^2} = -\frac{1}{\sigma^4}\sum_{i=1}^n \log(y_i)x_i + \frac{\gamma}{\sigma^4}\sum_{i=1}^n x_i^2$$

$$E\left[-\frac{\partial^2 \ell(\gamma,\sigma^2)}{\partial \gamma\,\partial \sigma^2}\right] = \frac{1}{\sigma^4}\sum_{i=1}^n E[\log(y_i)x_i] - \frac{\gamma}{\sigma^4}(n)$$

$$E[\log(y_i)x_i] = E[z_i x_i] = E_x[x_i E[z_i | x_i]] = E_x[\gamma x_i^2] = \gamma E[x_i^2] = \gamma$$

$$\Rightarrow E\left[-\frac{\partial^2 \ell(\gamma,\sigma^2)}{\partial \gamma\,\partial \sigma^2}\right] = \frac{\gamma}{\sigma^4}n - \frac{\gamma}{\sigma^4}n = 0.$$

$$\frac{\partial^2 \ell(\gamma,\sigma^2)}{\partial (\sigma^2)^2} = \frac{n}{2\sigma^4} - \frac{1}{\sigma^6}\sum_{i=1}^n (\log y_i)^2 + \frac{2\gamma}{\sigma^6}\sum_{i=1}^n \log y_i x_i - \frac{\gamma^2}{\sigma^6}\sum_{i=1}^n x_i^2$$

$$E\left[-\frac{\partial^2 \ell(\gamma,\sigma^2)}{\partial (\sigma^2)^2}\right] = \frac{-n}{2\sigma^4} + \frac{1}{\sigma^6}\sum_{i=1}^n E[z_i^2] - \frac{2\gamma}{\sigma^6}n\gamma + \frac{n\gamma^2}{\sigma^6}$$

$$E[z_i^2] = E_x[E[z_i^2 | x_i]] = E_x[\sigma^2 + \gamma^2 x_i^2] = \sigma^2 + \gamma^2 E[x_i^2] = \sigma^2 + \gamma^2$$

c) con't'd

$$E\left[-\frac{\partial^2 \ell(\gamma,\sigma^2)}{\partial(\sigma^2)^2}\right] = \frac{-n}{2\sigma^4} + \frac{n}{\sigma^6}(\sigma^2+\gamma^2) - \frac{n\gamma^2}{\sigma^6}$$

$$= \frac{-n}{2\sigma^4} + \frac{n}{\sigma^4} + \cancel{\frac{n\gamma^2}{\sigma^6}} - \cancel{\frac{n\gamma^2}{\sigma^6}}$$

$$= \frac{2n-n}{2\sigma^4} = \frac{n}{2\sigma^4}$$

$$\Rightarrow \sqrt{n}\begin{pmatrix}\hat{\gamma} - \gamma \\ \hat{\sigma}^2 - \sigma^2\end{pmatrix} \to N\left(\begin{pmatrix}0\\0\end{pmatrix}, \underbrace{\begin{bmatrix}\sigma^2 & 0 \\ 0 & 2\sigma^4\end{bmatrix}}_{\Sigma}\right)$$

Still want asymptotic distribution of α^*, β^* MLEs.

By invariance of MLE,

$$\hat{\alpha}^* = \exp\left(\tfrac{1}{2}(\hat{\sigma}^2 + \hat{\gamma}^2)\right)$$

$$\hat{\beta}^* = \hat{\gamma}\exp\left(\tfrac{1}{2}(\hat{\sigma}^2 + \hat{\gamma}^2)\right)$$

so by Delta Method

$$g(x,y) = \begin{bmatrix}\exp(\tfrac{1}{2}(y+x^2)) \\ x\exp(\tfrac{1}{2}(y+x^2))\end{bmatrix}$$

$$\nabla g(x,y) = \begin{bmatrix} x\exp(\tfrac{1}{2}(y+x^2)) & \tfrac{1}{2}\exp(\tfrac{1}{2}(y+x^2)) \\ x^2\exp(\tfrac{1}{2}(y+x^2)) + \exp(\tfrac{1}{2}(y+x^2)) & \tfrac{1}{2}x\exp(\tfrac{1}{2}(y+x^2)) \end{bmatrix}$$

c) cont'd

$$\sqrt{n}\begin{pmatrix} \hat{\alpha}^* - \alpha^* \\ \hat{\beta}^* - \beta^* \end{pmatrix} \to_d N\left(0, \nabla g(\gamma,\sigma^2)\Sigma\nabla g^T(\gamma,\sigma^2)\right)$$

$$\nabla g(\gamma,\sigma^2)\Sigma = \begin{bmatrix} \gamma\exp(\frac{1}{2}(\sigma^2+\gamma^2)) & \frac{1}{2}\exp(\frac{1}{2}(\sigma^2+\gamma^2)) \\ (\gamma^2+1)\exp(\frac{1}{2}(\sigma^2+\gamma^2)) & \frac{1}{2}\gamma\exp(\frac{1}{2}(\sigma^2+\gamma^2)) \end{bmatrix}\begin{bmatrix} \sigma^2 & 0 \\ 0 & 2\sigma^4 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma^2\gamma\exp(\frac{1}{2}(\sigma^2+\gamma^2)) & \sigma^4\exp(\frac{1}{2}(\sigma^2+\gamma^2)) \\ \sigma^2(\gamma^2+1)\exp(\frac{1}{2}(\sigma^2+\gamma^2)) & \gamma\sigma^4\exp(\frac{1}{2}(\sigma^2+\gamma^2)) \end{bmatrix}$$

$$\nabla g(\gamma,\sigma^2)\Sigma(\nabla g(\gamma,\sigma^2))^T = \exp(\tfrac{2}{2}(\sigma^2+\gamma^2))\begin{bmatrix} \sigma^2\gamma & \sigma^4 \\ \sigma^2(\gamma^2+1) & \gamma\sigma^4 \end{bmatrix}\begin{bmatrix} \gamma & \gamma^2+1 \\ \frac{1}{2} & \frac{1}{2}\gamma \end{bmatrix}$$

$$= \exp(\sigma^2+\gamma^2)\begin{bmatrix} \sigma^2\gamma^2+\frac{1}{2}\sigma^4 & \sigma^2\gamma^3+\sigma^2\gamma+\frac{1}{2}\gamma\sigma^4 \\ \gamma\sigma^2(\gamma^2+1)+\frac{1}{2}\gamma\sigma^4 & \sigma^2(\gamma^2+1)^2+\frac{1}{2}\gamma^2\sigma^4 \end{bmatrix}$$

$$\Rightarrow \sqrt{n}\begin{pmatrix} \hat{\alpha}^* - \alpha^* \\ \hat{\beta}^* - \beta^* \end{pmatrix} \to_d N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Sigma^*\right)$$

$$\Sigma^* = \exp(\sigma^2+\gamma^2)\begin{bmatrix} \sigma^2\gamma^2+\frac{1}{2}\sigma^4 & \gamma\sigma^2(\gamma^2+1)+\frac{1}{2}\gamma\sigma^4 \\ \gamma\sigma^2(\gamma^2+1)+\frac{1}{2}\gamma\sigma^4 & \sigma^2(\gamma^2+1)^2+\frac{1}{2}\gamma^2\sigma^4 \end{bmatrix}.$$

d) Recall from (b)

$$\sqrt{n}\,(\hat{\beta} - \beta^*) \to_d N(0, var(X_i Y_i))$$

Now we know

$$var(X_i Y_i) = E[(X_i Y_i)^2] - \{E[X_i Y_i]\}^2$$

Now

$$E[(X_i Y_i)^2] = E_X[X_i^2 E[Y_i^2 | X_i]]$$

by (c) $\quad E[Y_i^2 | X_i] = \exp(\gamma X_i (2) + \frac{1}{2}\sigma^2 (2)^2)$

$$= \exp(2\gamma X_i + 2\sigma^2)$$

$$\Rightarrow E[(X_i Y_i)^2] = E_X[X_i^2 \exp(2\gamma X_i + 2\sigma^2)]$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}x^2)\, x^2 \exp(2\gamma x + 2\sigma^2)\, dx$$

$$-\tfrac{1}{2}(x - 2\gamma)^2 = -\tfrac{1}{2}[x^2 - 4x\gamma + 4\gamma^2]$$

$$= -\tfrac{1}{2}x^2 + 2\gamma x - 2\gamma^2$$

$$= \exp(2\sigma^2) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\tfrac{1}{2}(x - 2\gamma)^2\right) x^2 \exp(2\gamma^2)\, dx$$

$$= \exp(2\sigma^2 + 2\gamma^2)\, E[U^2]$$

where $U \sim N(2\gamma, 1)$

$$\Rightarrow E[U^2] = var(U) + \{EU\}^2$$

$$= 1 + 4\gamma^2$$

$$\Rightarrow E[(X_i Y_i)^2] = \exp(2(\sigma^2 + \gamma^2))(1 + 4\gamma^2)$$

$$E[X_i Y_i] = E_X[X_i E[Y_i | X_i]] = \gamma \exp(\tfrac{1}{2}(\sigma^2 + \gamma^2))$$

$$\Rightarrow var(X_i Y_i) = (1 + 4\gamma^2)\exp(2(\sigma^2 + \gamma^2)) - \gamma^2 \exp(\sigma^2 + \gamma^2)$$

$$\Rightarrow \sqrt{n}\,(\hat{\beta} - \beta^*) \to_d N(0, \exp(2(\sigma^2 + \gamma^2)) + 4\gamma^2 \exp(2(\sigma^2 + \gamma^2)) - \gamma^2 \exp(\sigma^2 + \gamma^2))$$

$$\sqrt{n}\,(\hat{\beta}^* - \beta^*) \to_d N(0, \exp(\sigma^2 + \gamma^2)\sigma^2(\gamma^2 + 1)^2 + \tfrac{1}{2}\gamma^2\sigma^4 \exp(\sigma^2 + \gamma^2))$$

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d) cont'd

Asym Rel Eff $\hat{\beta}^*$ to $\hat{\beta}$ = $\dfrac{\exp(\sigma^2+\gamma^2)\,\sigma^2(\gamma^2+1)^2 + \frac{1}{2}\gamma^2\sigma^4\exp(\sigma^2+\gamma^2)}{\exp(2(\sigma^2+\gamma^2)) + 4\gamma^2\exp(2(\sigma^2+\gamma^2)) - \gamma^2\exp(\sigma^2+\gamma^2)}$

e)

$$E\left[\{Y-g(x)\}^2\right] = E\left[Y^2 - 2Y g(x) + g^2(x)\right]$$

$$E\left[\{Y-g(x)\}^2 \mid x\right] = E[Y^2 \mid x] - 2g(x)E[Y \mid x] + g^2(x)$$

$$\frac{\partial E[\{Y-g(x)\}^2 \mid x]}{\partial g(x)} = -2E[Y \mid x] + 2g(x) \overset{\text{set}}{=} 0$$

$$\Rightarrow \hat{g}(x) = E[Y \mid x]$$

$$= \exp(\gamma x)\exp(\tfrac{1}{2}\sigma^2)$$ $\boxed{\text{by part c}}$

→ optimal $g(x)$ in terms of (γ, σ^2) is

$$g(x) = \exp(\gamma x)\exp(\tfrac{1}{2}\sigma^2)$$