BASIC PHD WRITTEN EXAMINATION IN BIOSTATISTICS

THEORY, SECTION 1

(9:00 AM- 1:00 PM
Tuesday, August 6, 2013)

INSTRUCTIONS:

a) This is a **CLOSED-BOOK** examination.

b) The time limit for this Examination is four hours.

c) Answer any TWO (2) (BUT ONLY TWO) of the THREE (3) questions that follow.

d) Put the answers to different questions on separate sets of paper.

e) Put your exam code, **NOT YOUR NAME**, on each page. The same code will be used for Section 1 and Section 2 of the PhD Theory Exam. Please keep the code confidential and do not share this information with any students or faculty. Sharing your code with either students or faculty is viewed as a violation of the UNC honor code.

f) Return the examination with a signed statement of the UNC honor pledge, separately from your answers. The pledge statement is given on the last page of the exam handout.

g) In the questions to follow, you are required to answer only what is asked, and not to tell all you know about the topics involved.

1. Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be an i.i.d. sample of n pairs of random variables, each pair having joint density

$$f(x, y; \alpha) = \alpha(\alpha+1)(1+x+y)^{-(\alpha+2)}, \; x, y > 0,$$

for parameter $0 < \alpha < \infty$. Do the following:

(a) Show that the maximum likelihood estimator for α, $\hat{\alpha}_n$, has the following properties:

(i) $\hat{\alpha}_n$ exists and is unique and has the form $g^{-1}(\hat{\mu}_n)$, where $\hat{\mu}_n = n^{-1}\sum_{i=1}^n \log(1 + X_i + Y_i)$ and g^{-1} is the inverse of some function g, i.e., if $b = g(a)$, then $g^{-1}(b) = a$. Give the form of g and show g^{-1} exists.

(ii) $\hat{\alpha}_n \to \alpha_0$, almost surely, where α_0 is the true value of α.

(iii) $\sqrt{n}(\hat{\alpha}_n - \alpha_0)$ is asymptotically normal with mean zero and variance

$$\sigma_1^2 = \frac{\alpha_0^2(\alpha_0+1)^2}{\alpha_0^2 + (\alpha_0+1)^2}.$$

(b) Suppose now that $X_1, \ldots, X_n$ are fixed and known, i.e., $(X_1, \ldots, X_n) = (x_1, \ldots, x_n)$, and we observe the sample of independent observations $Y_1, \ldots, Y_n$, where, for $i = 1, \ldots, n$, Y_i is drawn from the conditional distribution of Y_i given $X_i = x_i$, and where the unconditional joint density of (X_i, Y_i) is given above. Show that for $i = 1, \ldots, n$, the density of Y_i given $X_i = x_i$ is

$$\tilde{f}_i(y; \alpha) = (\alpha+1)(1+x_i)^{-1}\left(1 + \frac{y}{1+x_i}\right)^{-(\alpha+2)}, \; y > 0.$$

(c) In the setting of (b), verify that the maximum likelihood estimator, $\tilde{\alpha}_n$, has the following properties:

(i) $\tilde{\alpha}_n$ exists, is unique, and can be expressed in explicit closed form.

(ii) $\tilde{\alpha}_n \to \alpha_0$, almost surely. Hint: Consider $U_i = Y_i/(1 + x_i)$.

(iii) $\sqrt{n}(\tilde{\alpha}_n - \alpha_0)$ is asymptotically normal with mean zero and variance $\sigma_2^2 = h(\alpha_0)$, and give the form of h.

(d) What is the asymptotic relative efficiency of $\tilde{\alpha}_n$ to $\hat{\alpha}_n$?

2. Consider the following:

   (a) For each $\theta_0 \in \Theta$, let T_{θ_0} be a test of $H_0 : \theta = \theta_0$ (versus some H_1) with significance level α and acceptance region $A(\theta_0)$. For each y in the range of the random variable Y, define

   $$C(y) = \{\theta : y \in A(\theta)\}.$$

   Show that $C(Y)$ is a level $1 - \alpha$ confidence set for θ.

   (b) Suppose $X_1, \ldots, X_n$ is a random sample from $N(\mu, \sigma^2)$, where $\sigma^2 = \gamma\mu^2$ and $-\infty < \mu < \infty$ and $\gamma > 0$ are both unknown scalar parameters, and $\mu \neq 0$. Using part (a), derive a confidence set for γ with confidence coefficient $1 - \alpha$ by inverting the acceptance region of the likelihood ratio test for testing $H_0 : \gamma = \gamma_0$ versus $H_1 : \gamma \neq \gamma_0$.

   (c) Under the set-up of part (b), show that a UMP test does not exist for testing $H_0 : \gamma = \gamma_0$ versus $H_1 : \gamma \geq \gamma_0$

   (d) Suppose $X_1, \ldots, X_n$ is a random sample from $N(\mu, \sigma^2)$, where μ and σ^2 are scalar parameters. Assume that $\theta = (\mu, \phi)$, where $\phi = \sigma^2$, is unknown. Derive $1 - \alpha$ asymptotically correct confidence sets for μ by

      (i) inverting acceptance regions for (1) the likelihood ratio test, (2) the Wald test, and (3) the score test.

      (ii) Are these sets always intervals? Justify your answer.

Hint: Using the notation from part (a), if $\lim_{n\to\infty} P(\theta \in C(Y)) = 1 - \alpha$ for all θ, then $C(Y)$ is a $1 - \alpha$ asymptotically correct confidence set for θ.

   (e) Now, under the set-up in part (d), suppose we do not make any distributional assumptions about $X_i, i = 1, \ldots, n$, but only assume that $X_1, \ldots, X_n$ are i.i.d. with mean μ and variance σ^2 with finite fourth moment. Derive an asymptotically correct confidence interval for $\theta = \mu/\sigma$.

3. Consider an r-sided coin and suppose that on each flip of the coin exactly one of the sides appears: side i with probability $P_i, \sum_{i=1}^{r} P_i = 1$. For given positive integers $n_1, n_2, \ldots, n_r$, let N_i denote the number of flips required until side i has appeared for the n_i time, $i = 1, \ldots, r$, and let $N = \min_{i=1,\ldots,r} N_i$. Thus, N is the number of flips required until some side i has appeared n_i times, for $i = 1, \ldots, r$.

   (a) Derive the marginal distribution of N_i, for $i = 1, \ldots, r$.

   (b) Prove whether or not $N_i, i = 1, \ldots, r$ are independent random variables.

   Now, suppose that the flips are performed at random times generated by a Poisson process with rate $\lambda = 1$. Let T_i denote the time until side i has appeared for the n_i time, $i = 1, \ldots, r$, and let $T = \min_{i=1,\ldots,r} T_i$.

   (c) Derive the marginal distribution of T_i, for $i = 1, \ldots, r$.

   (d) Prove whether or not $T_i, i = 1, \ldots, r$ are independent random variables.

   (e) Derive the density of T.

   (f) Obtain an expression for $E(N)$ as a function of $E(T)$.

2013 PhD Theory Exam, Section 1

Statement of the UNC honor pledge:

"In recognition of and in the spirit of the honor code, I certify that I have neither given nor received aid on this examination and that I will report all Honor Code violations observed by me."

(Signed) ____

NAME

(Printed) ____

NAME