2012 Theory I # 3

3a.i) $E_\theta\left[(T_0+c-\theta)^2\right] = E_\theta\left[(T_0-\theta+c)^2\right]$

$= Var_\theta[T_0] + c^2 > Var_\theta[T_0] = E_\theta\left[(T_0-\theta)^2\right]$

Since this is true for all θ, we conclude that

$$\sup_\theta E_\theta\left[(T_0-\theta)^2\right] < \sup_\theta E_\theta\left[(T_0+c-\theta)^2\right]$$

so that T_0+c is not minimax under squared-error loss*

only minimax ~~in case given for (3a.ii)~~

3a.ii) Choose $\theta \neq 0$. Then

$E_\theta\left[(cT_0-\theta)^2\right] = E_\theta\left[(cT_0-T_0+T_0-\theta)^2\right]$

$= E_\theta\left[((c-1)T_0+T_0-\theta)^2\right] = (c-1)^2 ET_0^2 + Var[T_0]$

$> Var[T_0] = E_\theta\left[(T_0-\theta)^2\right]$

where the inequality follows from $(c-1)^2>0$ and $ET_0^2 = Var[T_0] + (ET_0)^2 = Var[T_0]+\theta^2 > 0$ since we assumed $\theta \neq 0$. Thus

$$\sup_\theta E_\theta\left[(T_0-\theta)^2\right] < \sup_\theta E_\theta\left[(cT_0-\theta)^2\right]$$

except in the trivial case described in the problem

maybe not assume this, instead argue that $Var[T_0] > 0$ for a minimax estimator?

(b.i) Let $\delta(a,x)$ be a ~~joint density function~~ decision rule? for the r.v. (A,X) with $A|(X=0) \overset{d}{=} U$ and $A|(X=1) \overset{d}{=} V$. Then

$$E\left[L(p,\delta(A,X))\right] = E\left(E\left[L(p,\delta(A,X)) \mid X\right]\right)$$

$$= \sum_{x=0}^{1} E\left[L(p,\delta(A,X)) \mid X=x\right] P(X=x)$$

$$= (1-p)\, E\left[L(p,\delta(A,X)) \mid X=0\right] + p\, E\left[L(p,\delta(A,X)) \mid X=1\right]$$

$$= (1-p)\, E\left[I(|p-U|>0.25\right] + p\, E\left[I(|p-V|>0.25\right]$$

$$= (1-p)\, P(|p-U|>0.25) + p\, P(|p-V|>0.25)$$

$$= (1-p)\min(1,\, 2|p-0.25|) + p\min(1,\, 2|p-0.75|)$$

(c.i) $\bar{X}_n \sim N\left(\theta, \frac{\sigma^2}{n}\right)$. Suppose now that $\theta \sim N(\mu_0, \tau_0^2)$.

Then

$$p(\theta \mid \bar{x}_n) \propto p(\bar{x}_n \mid \theta)\, p(\theta) \propto \exp\left\{-\frac{n}{2\sigma^2}(\bar{x}_n - \theta)^2 - \frac{1}{2\tau_0^2}(\theta - \mu_0)^2\right\}$$

$$\propto \exp\left\{\frac{n\bar{x}_n}{\sigma^2}\theta - \frac{n}{2\sigma^2}\theta^2 + \frac{\mu_0}{\tau_0^2}\theta - \frac{1}{2\tau_0^2}\theta^2\right\}$$

$$= \exp\left\{-\frac{1}{2}\left(\frac{n}{\sigma^2} + \frac{1}{\tau_0^2}\right)\theta^2 + \left(\frac{n\bar{x}_n}{\sigma^2} + \frac{\mu_0}{\tau_0^2}\right)\theta\right\}$$

$$= \exp\left\{-\frac{1}{2}\left(\frac{n}{\sigma^2} + \frac{1}{\tau_0^2}\right)\left[\theta^2 - 2\left(\frac{n\bar{x}_n}{\sigma^2} + \frac{\mu_0}{\tau_0^2}\right)\left(\frac{n}{\sigma^2} + \frac{1}{\tau_0^2}\right)^{-1}\theta\right]\right\}$$

$$\sim N\left(\left(\frac{n\bar{x}_n}{\sigma^2} + \frac{\mu_0}{\tau_0^2}\right)\left(\frac{n}{\sigma^2} + \frac{1}{\tau_0^2}\right)^{-1}, \left(\frac{n}{\sigma^2} + \frac{1}{\tau_0^2}\right)^{-1}\right)$$

?

Next, to find the Bayes rule we want to minimize

$$E_{\Lambda|X}\left[\frac{1}{\sigma^2}(\theta - a)^2 \mid \bar{X}_n\right]$$

wrt a. We have

$$\frac{d}{da} E_{\Lambda|X}\left[\frac{1}{\sigma^2}(\theta - a)^2 \mid \bar{X}_n\right] = \frac{1}{\sigma^2} E_{\Lambda|X}\left[-2(\theta - a) \mid \bar{X}_n\right]$$

$$= \frac{-2}{\sigma^2} E_{\Lambda|X}\left[\theta \mid \bar{X}_n\right] + \frac{2}{\sigma^2} a \overset{\text{set}}{=} 0 \Rightarrow a = E_{\Lambda|X}\left[\theta \mid \bar{X}_n\right]$$

Let Λ_k be the distribution of a $N(\mu_0, \tau_k^2 = k)$.

then

$$\mathcal{R}(\Lambda_k, \delta_{\Lambda_k}) = E_{X,\Lambda_k}\left[\frac{1}{\sigma^2}(\theta - \delta_{\Lambda_k})^2\right] = \frac{1}{\sigma^2} E_X E_{\Lambda|X}\left[(\theta - E[\theta|X]) \mid X\right]$$

$$= \frac{1}{\sigma^2} E_X\left(Var_{\Lambda|X}[\theta|X]\right) = \frac{1}{\sigma^2} E_X\left[\frac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau_k^2}}\right] \to \frac{1}{\sigma^2} \cdot \frac{1}{\frac{n}{\sigma^2}} = \frac{1}{n}$$

But the frequentist risk is given by (for $\delta(X) = \bar{X}_n$)

$$E_\theta\left[\frac{1}{\sigma^2}(\theta - \delta(X))^2\right] = \frac{1}{\sigma^2} Var_\theta[\bar{X}_n] = \frac{1}{\sigma^2} \cdot \frac{\sigma^2}{n} = \frac{1}{n}$$

Thus $\bar{X}_n$ is minimax.

3c.ii) We wish to estimate $\theta = (\mu, \sigma^2) \in \overbrace{\mathbb{R} \times (0, K)}^{\Theta}$. Let $\Theta_0 = \mathbb{R} \times \{K\}$.

From (3c.i) we obtain that $\bar{X}_n$ is a minimax estimator of $\theta \in \Theta_0$.

Now

$$\sup_{\theta \in \Theta_0} E_\theta\left[\frac{1}{\sigma^2}(\theta - \bar{X}_n)^2\right] = \frac{1}{n} = \sup_{\theta \in \Theta} E_\theta\left[\frac{1}{\sigma^2}(\theta - \bar{X}_n)^2\right]$$

so by Lemma 1.3 (slide 111) we obtain that $\bar{X}_n$ is minimax for (μ, σ^2).