Practice Theory 1 Exam 2014

$X_1, \dots, X_n$ iid Unif$(0, \alpha)$
$Y_1, \dots, Y_n$ iid Unif$(0, \beta)$
$\underset{\sim}{X}, \underset{\sim}{Y}$ indep.
Interested in inference on $\theta = \beta/\alpha$

(a) Derive the UMVUEs for α & β & calculate their respective variances
— Either find a complete sufficient statistic whose expectation is α & β (unbiased) or find an unbiased estimator & calculate $E[\hat{\theta} \mid T(x)]$

$$f(\underset{\sim}{x}) = \prod_{i=1}^{n} f(x_i) = \alpha^{-n} I(0 < x_1, \dots, x_n < \alpha)$$

$$= \alpha^{-n} I(x_{(n)} < \alpha)$$

Likewise,

$$f(\underset{\sim}{y}) = \beta^{-n} I(y_{(n)} < \beta)$$

sufficient statistics $= X_{(n)}, Y_{(n)}$

$$F_{X(n)}(z) = P(X_{(n)} < z) = P(X_1, \dots, X_n < z)$$
$$= (P(X_i < z))^n$$

since indep & identically distributed.

$$= \left(\frac{z}{\alpha}\right)^n$$

$$f_{X(n)}(z) = \frac{n z^{n-1}}{\alpha^n}$$

$$E[X_{(n)}] = \int_0^{\alpha} \frac{z \, n z^{n-1}}{\alpha^n} dz$$

$$= \frac{n}{\alpha^n} \int_0^{\alpha} z^n dz = \frac{n}{\alpha^n} \frac{1}{n+1} z^{n+1} \Big|_0^{\alpha}$$

$$= \left(\frac{n}{n+1}\right) \frac{\alpha^{n+1}}{\alpha^n} = \frac{\alpha n}{n+1} \rightarrow$$

$$\Rightarrow E\left[\frac{(n+1)X_{(n)}}{n}\right] = \alpha \checkmark$$

$$\Rightarrow \text{UMVUE of } \alpha = \left(\frac{n+1}{n}\right)X_{(n)}$$

- unbiased ✓
- function of sufficient statistic ✓
  ↳ completeness after part ⓘ

Likewise,

$$\text{UMVUE of } \beta = \left(\frac{n+1}{n}\right)Y_{(n)}$$

ⓘ Calculate their variances

$$Var\left(\frac{n+1}{n}X_{(n)}\right) = \left(\frac{n+1}{n}\right)^2 Var(X_{(n)})$$

$$= \left(\frac{n+1}{n}\right)^2 \left(E(X_{(n)}^2) - (E(X_{(n)}))^2\right)$$

$$E(X_{(n)}^2) = \int_0^\alpha \frac{z^2\, n z^{n-1}}{\alpha^n}\,dz$$

$$= \left(\frac{n}{\alpha^n}\right)\frac{1}{n+2} z^{n+2}\Big|_0^\alpha = \left(\frac{n}{n+2}\right)\alpha^2$$

$$= \left(\frac{n+1}{\cancel{n}}\right)^2\left[\left(\frac{\cancel{n}}{n+2}\right)\alpha^2 - \left(\frac{\cancel{n}}{n+1}\right)^2\alpha^2\right]$$

$$= \alpha^2\left(\frac{(n+1)^2}{n+2} - 1\right)$$

Likewise, $Var\left(\frac{n+1}{n}Y_{(n)}\right) = \beta^2\left(\frac{(n+1)^2}{n+2} - 1\right)$

(ii) Note: Need $X_{(n)}$ & $Y_{(n)}$ to be complete sufficient statistics

- if exponential family, then have this quality
- need to prove for uniform dist.

For completeness, need $E_\theta[g(X_{(n)})] = 0$ for any θ implies $g(\cdot) = 0$

$$E_\theta[g(X_{(n)})] = \int_0^\theta g(t) f_t(t)\, dt$$

where $t = X_{(n)}$ & $f(t)$ = pdf of $X_{(n)}$

$$= \int_0^\theta g(t)\, n\theta^{-n} t^{n-1}\, dt \overset{set}{=} 0 \quad \forall \theta$$

Since integrating over t, can cancel θ^{-n} & n (constants)

$$\Rightarrow \int_0^\theta g(t)\, t^{n-1}\, dt = 0$$

$$\Rightarrow \frac{\partial}{\partial\theta}\int_0^\theta g(t)\, t^{n-1}\, dt \overset{set}{=} \frac{\partial}{\partial\theta} 0 = 0$$

$$\Rightarrow g(\theta)\,\theta^{n-1} = 0$$

Note: $\int_0^\theta f(t)\, dt = F(\theta) - F(0)$

where F = antiderivative of f

$$\frac{\partial}{\partial\theta}\int_0^\theta f(t)\, dt = \frac{\partial}{\partial\theta}F(\theta) - \frac{\partial}{\partial\theta}F(0)$$

$$= f(\theta) + 0$$
$$= f(\theta)$$

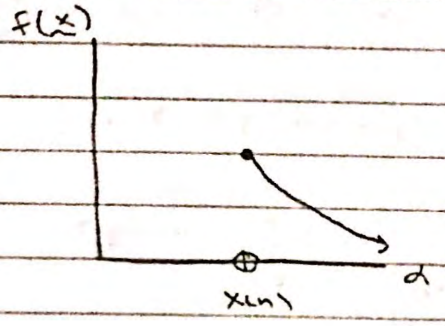
$$g(\theta)\,\theta^{n-1} = 0 \Rightarrow g(\theta) = 0 \Rightarrow g(\cdot) = 0 \checkmark$$

$\Rightarrow X_{(n)}$, & likewise $Y_{(n)}$, is complete (let $\theta = \alpha$ & β, respectively)

(b) Calculate the MLEs for α + β ($\hat{\alpha}$ + $\hat{\beta}$)

- Derive the asymptotic dist of $\hat{\alpha}$ + $\hat{\beta}$ after normalization.

$$f(\underset{\sim}{x}) = \alpha^{-n} I(x_{(n)} < \alpha)$$



$f(\underset{\sim}{x})$ is maximized at $x_{(n)} = \alpha$

$\Rightarrow \hat{\alpha} = x_{(n)}$

Likewise, $\hat{\beta} = Y_{(n)}$

(ii) Asymptotic Dist:

$$n(\alpha - x_{(n)}) \xrightarrow{d} ?$$

$$F_{n(\alpha - x_{(n)})}(z) = P(n(\alpha - x_{(n)}) < z)$$
$$= P(\alpha - x_{(n)} < z/n)$$
$$= P(x_{(n)} - \alpha > z/n)$$
$$= P(x_{(n)} > z/n + \alpha)$$
$$= P(x_1, \ldots, x_n > z/n + \alpha)$$
$$= 1 - P(x_1, \ldots, x_n < z/n + \alpha)$$
$$= 1 - [P(x_i < z/n + \alpha)]^n \text{ since iid}$$
$$= 1 - \left[\frac{z/n + \alpha}{\alpha}\right]^n$$
$$= 1 - [1 + (z/\alpha)/n]^n$$

$$\lim_{n \to \infty} 1 - \left[1 + \frac{z/\alpha}{n}\right]^n = 1 - \exp(z/\alpha) \sim Exp(\alpha)$$

$n(\mu - \hat{u})$
$\uparrow$ $\frac{Y_{(n)}}{X_{(n)}}$

Consequently,

$n(\alpha - X_{(n)}) \xrightarrow{d} Exp(\alpha)$

and

$n(\beta - Y_{(n)}) \xrightarrow{d} Exp(\beta)$

ⓒ MLE for θ is $\hat{\theta} = \frac{\hat{\beta}}{\hat{\alpha}} = \frac{Y_{(n)}}{X_{(n)}}$

Derive the asymptotic dist of $\hat{\theta}$ after normalization

Options: First principles

· Delta method

By work in ⓓ, we know dist of $U = Y_{(n)}/X_{(n)}$
is $\frac{1}{2n}\left(\frac{\alpha}{\beta}\right)^n u^{n-1}$ = pdf

$F_{n(\mu - U)}(z)$ where $\mu = \beta/\alpha$

$= P(n(\mu - U) < z)$

$= P(\mu - U < z/n) = P(U - \mu > -z/n)$

$= P(U > -z/n + \underline{\beta/\alpha})$

$= 1 - P(U < -z/n + \beta/\alpha)$

$P(U < k) = \int_0^k f(u)\,du$

$= \int_0^k \frac{n}{2}\left(\frac{\alpha}{\beta}\right)^n u^{n-1}\,du$

$\frac{1}{2}\left(\frac{\alpha}{\beta}\right) u^n \Big|_0^k = \frac{1}{2}\left(\frac{\alpha}{\beta}\right)^n k^n$

$P(U < -z/n + \beta/\alpha) = \frac{1}{2}\left(\frac{\alpha}{\beta}\right)^n \left(\frac{-z}{n} + \frac{\beta}{\alpha}\right)^n$

$= \frac{1}{2}\left(-\left(\frac{z\alpha}{\beta}\right)\frac{1}{n} + 1\right)^n$

$Y \sim N(\mu, \sigma^2) \equiv \mu + \sigma(N(0,1))$

$$\lim_{n\to\infty} \frac{1}{2}\left(1 - \left(\frac{z\alpha}{B}\right)\frac{1}{n}\right)^n = \frac{1}{2}\exp(-z\alpha/B)$$

$$n\,(Y(n)/X(n) - B/\alpha) \xrightarrow[z]{d} \frac{1}{2}\exp\left(\frac{-z\alpha}{B}\right)$$

where $z = Y(n)/X(n)$

Note: $= \frac{1}{2} - \frac{1}{2} + \frac{1}{2}\exp\left(\frac{-z\alpha}{B}\right)$ $\qquad \frac{1}{\lambda}\exp(-x/\lambda)$ $\quad E(x) = \lambda$

$$= +\frac{1}{2} - \frac{1}{2}\left(1 - 1\exp(-z\alpha/B)\right) \qquad \chi^2(1)$$

$$= \frac{1}{2} - \frac{1}{2}\underline{\mathrm{Exp}(B/\alpha)} =$$

- alternative way to write above result

(ii) 95% CI for θ $\qquad \longrightarrow \sigma^2\chi^2(1)$

alpha level $\qquad \frac{1}{\sigma^2} \to \chi^2(1)$

$$\frac{\gamma}{2} = \int_0^{k_1} \frac{1}{2}\exp\left(\frac{-z\alpha}{B}\right)dz$$

$$= -\frac{1}{2}\left(\frac{B}{\alpha}\right)\exp\left(\frac{-z\alpha}{B}\right)\Big|_0^{k_1} \qquad -\frac{1}{2}(z - \tfrac{1}{2}) \to \mathrm{Exp}(B/\alpha)$$

$$= -\frac{1}{2}\left(\frac{B}{\alpha}\right)\left[\exp\left(\frac{k_1\alpha}{B}\right) - 1\right]$$

$$\Rightarrow \gamma = -\frac{B}{\alpha}\left[\exp\left(\frac{k_1\alpha}{B}\right) - 1\right]$$

$$\Rightarrow -\frac{\alpha\gamma}{B} = \exp\left(\frac{-k_1\alpha}{B}\right) - 1$$

$$\Rightarrow 1 - \frac{\alpha\gamma}{B} = \exp\left(\frac{-k_1\alpha}{B}\right)$$

$$\Rightarrow \log\left(1 - \frac{\alpha\gamma}{B}\right) = \frac{-k_1\alpha}{B}$$

$$\Rightarrow k_1 = -\frac{B}{\alpha}\log\left(1 - \frac{\alpha\gamma}{B}\right)$$

Since α & β are parameters, replace w/ their MLEs (consistent estimates)

$$\Rightarrow K_1 = -\frac{Y_{(n)}}{X_{(n)}} \log\left(1 - \frac{X_{(n)}}{Y_{(n)}}\gamma\right)$$

$$1 - \frac{\gamma}{2} = \int_{K_2}^{\infty} \frac{1}{2} \exp\left(\frac{-z\alpha}{\beta}\right) dz$$

$$1 - \frac{\gamma}{2} = \frac{1}{2} - \int_0^{K_2} \frac{1}{2} \exp\left(-\frac{z\alpha}{\beta}\right) dz$$

$$= \frac{1}{2} - \frac{1}{2}\left(\frac{\beta}{\alpha}\right)\left(1 - \exp\left(-\frac{K_2\alpha}{\beta}\right)\right)$$

$$\Rightarrow 2 - \gamma = 1 - \left(\frac{\beta}{\alpha}\right)\left(1 - \exp\left(-\frac{K_2\alpha}{\beta}\right)\right)$$

$$\Rightarrow 1 - \gamma = -\frac{\beta}{\alpha}\left(1 - \exp\left(-\frac{K_2\alpha}{\beta}\right)\right)$$

$$\Rightarrow \frac{\alpha}{\beta}(\gamma - 1) = \left(1 - \exp(-K_2\alpha/\beta)\right)$$

$$\Rightarrow 1 + \frac{\alpha}{\beta}(1-\gamma) = \exp(-K_2\alpha/\beta)$$

$$\Rightarrow \log\left(1 + \frac{\alpha}{\beta}(1-\gamma)\right) = -K_2\alpha/\beta$$

$$\Rightarrow K_2 = -\frac{\beta}{\alpha}\log\left(1 + \frac{\alpha}{\beta}(1-\gamma)\right)$$

replace β & α w/ consistent estimates (MLEs)

$$K_2 = -\frac{Y_{(n)}}{X_{(n)}} \log\left(1 + \frac{X_{(n)}}{Y_{(n)}}(1-\gamma)\right)$$

Then 95% CI: $(\gamma = 0.05)$

$$k_1 < n(\theta - \hat{\theta}) < k_2$$

$$\Rightarrow \quad \hat{\theta} + \frac{k_1}{n} < \theta < \hat{\theta} + \frac{k_2}{n}$$

$$\Rightarrow \left(\frac{Y_{(n)}}{X_{(n)}}\right) + \frac{1}{n}\left(-\frac{Y_{(n)}}{X_{(n)}} \log\left(1 - \frac{X_{(n)}}{Y_{(n)}}(0.05)\right)\right) < \theta <$$

$$< \left(\frac{Y_{(n)}}{X_{(n)}}\right) + \frac{1}{n}\left(-\frac{Y_{(n)}}{X_{(n)}} \log\left(1 + \frac{X_{(n)}}{Y_{(n)}}(0.95)\right)\right)$$

ⓓ $H_0: \alpha = \beta$

$H_A: \alpha \neq \beta$

Likelihood ratio statistic?

- Dist of statistic?

$$LRT = \frac{\sup_{\theta \in \Theta_0} L(\alpha, \beta)}{\sup_{\theta \in \Theta} L(\alpha, \beta)}$$

General case:

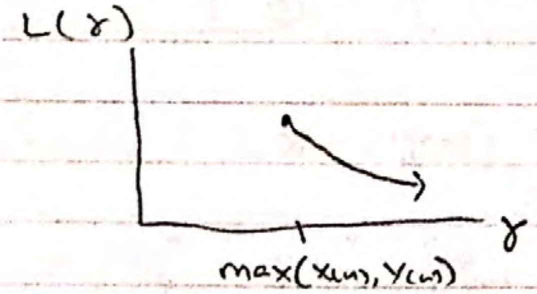
$L(\alpha, \beta)$ maximized at MLE's $X_{(n)} = \hat{\alpha}$, $Y_{(n)} = \hat{\beta}$

$$L(\alpha, \beta) = f(\underset{\sim}{x}) g(\underset{\sim}{y}) = \alpha^{-n} I(X_{(n)} < \alpha)\, \beta^{-n} I(Y_{(n)} < \beta)$$

$$L(\hat{\alpha}, \hat{\beta}) = \hat{\alpha}^{-n} I(X_{(n)} < \hat{\alpha})\, \hat{\beta}^{-n} I(Y_{(n)} < \hat{\beta})$$
$$= (X_{(n)})^{-n} (Y_{(n)})^{-n}$$

Null Hypothesis: $\alpha = \beta$

$$L(\alpha, \beta) = \alpha^{-n} \beta^{-n} I(X_{(n)} < \alpha) I(Y_{(n)} < \alpha)$$
$$\alpha = \beta = \gamma$$
$$\Rightarrow L(\gamma) = \gamma^{-2n} I(X_{(n)} < \gamma) I(Y_{(n)} < \gamma)$$
$$= \gamma^{-2n} I(X_{(n)}, Y_{(n)} < \gamma)$$
$$= \gamma^{-2n} I(\max(X_{(n)}, Y_{(n)}) < \gamma)$$



$$\hat{\gamma} = \max(X_{(n)}, Y_{(n)})$$

Case 1: $\max(X_{(n)}, Y_{(n)}) = X_{(n)}$

$$LRT = \frac{(X_{(n)})^{-2n}}{(X_{(n)})^{-n}(Y_{(n)})^{-n}}$$

$$= \left(\frac{Y_{(n)}}{X_{(n)}}\right)^{n} < k$$

$$\Rightarrow \frac{Y_{(n)}}{X_{(n)}} < k_1$$

$$\Rightarrow \frac{X_{(n)}}{Y_{(n)}} > k_2$$

The likelihood ratio statistic is $\hat{\theta} = \frac{\hat{\alpha}}{\hat{\beta}}$

<u>Exact</u> Distribution:

$U = X_{(n)}/Y_{(n)} \Rightarrow UV = X_{(n)}$

$V = Y_{(n)} \qquad\qquad V = Y_{(n)}$

$$f_{U,V}(u,v) = f_{X_{(n)},Y_{(n)}}(X_{(n)} = UV, Y_{(n)} = V)\,|J|$$

$$|J| = \begin{vmatrix} \partial X_{(n)}/\partial u & \partial X_{(n)}/\partial v \\ \partial Y_{(n)}/\partial u & \partial Y_{(n)}/\partial v \end{vmatrix}$$

$$= \begin{vmatrix} V & U \\ 0 & 1 \end{vmatrix} = V$$

$$f_{X_{(n)},Y_{(n)}}(x_{(n)}, y_{(n)}) = f_{X_{(n)}}(x_{(n)})\, g_{Y_{(n)}}(y_{(n)})$$

Since X & Y indep

$$= \frac{n^2 (UV)^{n-1}(V)^{n-1}(V)}{\alpha^n \beta^n}$$

$\longrightarrow$

$$= \frac{n^2 u^{n-1} v^{2n-1}}{\alpha^n \beta^n}$$

$$f_U(u) = \int_v f_{U,V}(u,v)\, dv$$

$$= \int_0^{\beta} \frac{n^2}{\alpha^n \beta^n} u^{n-1} v^{2n-1}\, dv$$

$$= \frac{n^2}{\alpha^n \beta^n} u^{n-1} \left(\frac{1}{2n}\right) v^{2n} \Big|_0^{\beta}$$

$$= \frac{n}{2} \frac{1}{\alpha^n} \beta^n u^{n-1}$$

$$= \frac{n}{2} \left(\frac{\beta}{\alpha}\right)^n u^{n-1} \qquad (0 < u < \infty)$$

To perform the test, find a k^* such that

$$0.05 = \int_{k^*}^{\infty} \frac{n}{2} \left(\frac{\beta}{\alpha}\right)^n u^{n-1}\, du$$

Or some other α level

where $u = \frac{X_{(n)}}{Y_{(n)}}$

(Is there a named dist for u?)

Case 2. $\max(X_{(n)}, Y_{(n)}) = Y_{(n)}$

$$\Rightarrow LRT = \left(\frac{X_{(n)}}{Y_{(n)}}\right)^n < k$$

$$\Rightarrow \frac{X_{(n)}}{Y_{(n)}} < k$$

$$= \frac{Y_{(n)}}{X_{(n)}} > k^*$$

(e) $E[X_k] = \alpha/2$ and $E[Y_k] = \beta/2$

Possible estimator: $\tilde{\Theta} = \dfrac{\bar{Y}_n}{\bar{X}_n}$

Derive the asymptotic dist of this estimator after normalization

By the multivariate CLT,

$$\sqrt{n}\left(\begin{bmatrix}\bar{Y}_n \\ \bar{X}_n\end{bmatrix} - \underbrace{\begin{bmatrix}\beta/2 \\ \alpha/2\end{bmatrix}}_{\begin{bmatrix}E(Y_k) \\ E[X_k]\end{bmatrix}}\right) \xrightarrow{d} N\left(\begin{bmatrix}0 \\ 0\end{bmatrix}, \begin{bmatrix}Var(Y_k) & Cov(Y_k, X_k) \\ Cov(Y_k, X_k) & Var(X_k)\end{bmatrix}\right)$$

$Var(Y_k) = E[Y_k^2] - [E(Y_k)]^2$

$$E[Y_k^2] = \int_0^\beta \frac{y^2}{\beta}\,dy = \frac{y^3}{3\beta}\Big|_0^\beta = \frac{\beta^2}{3}$$

$$Var(Y_k) = \frac{\beta^2}{3} - \frac{\beta^2}{4} = \beta^2\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{\beta^2}{12}$$

Likewise, $Var(X_k) = \dfrac{\alpha^2}{12}$

$Cov(X_k, Y_k) = 0$ since X indep of Y

$$\Rightarrow \sqrt{n}\left(\begin{bmatrix}\bar{Y}_n \\ \bar{X}_n\end{bmatrix} - \begin{bmatrix}\beta/2 \\ \alpha/2\end{bmatrix}\right) \xrightarrow{d} N\left(\begin{bmatrix}0 \\ 0\end{bmatrix}, \begin{bmatrix}\beta^2/12 & 0 \\ 0 & \alpha^2/12\end{bmatrix}\right)$$

$\longrightarrow$

By the ~~continuous mapping thm,~~ Delta Method

$$\sqrt{n}\,(g(z_n) - g(\mu)) \xrightarrow{d} N(0,\ \nabla g(\mu)\, \Sigma\, \nabla g(\mu))$$

$$g(y,x) = y/x$$

$$\nabla g(y,x) = \left[\partial g(y,x)/\partial y \quad \partial g(y,x)/\partial x \right]$$

$$= \left[1/x \quad -y/x^2 \right]$$

$$\nabla g(\mu) = \nabla g(y,x) \Big|_{E(y),\, E(x)}$$

$$= \left[2/\alpha,\ -2B/\alpha^2 \right]$$

$$\nabla g(\mu)\, \Sigma\, \nabla g(\mu)^T$$

$$= 2^2 \left[1/\alpha \quad -B/\alpha^2 \right] \begin{bmatrix} B^2/12 & 0 \\ 0 & \alpha^2/12 \end{bmatrix} \begin{bmatrix} 1/\alpha \\ -B/\alpha^2 \end{bmatrix}$$

$$= \frac{2^2}{12} \left[B^2/\alpha \quad -B \right] \begin{bmatrix} 1/\alpha \\ -B/\alpha^2 \end{bmatrix}$$

$$= \frac{1}{3} \left(B^2/\alpha^2 + B^2/\alpha^2 \right)$$

$$= \frac{2}{3} \frac{B^2}{\alpha^2}$$

$$\Rightarrow \sqrt{n} \left(\frac{\bar{Y}_n}{\bar{X}_n} - \frac{B}{\alpha} \right) \xrightarrow{d} N\left(0,\ \frac{2}{3}\frac{B^2}{\alpha^2}\right)$$

(ii) Asymptotic relative efficiency wrt $\hat{\theta}$, $\frac{2\bar{Y}_n}{\hat{\alpha}}$, + $\frac{\hat{\beta}}{2\bar{X}_n}$

$$a_n\left(\frac{2\bar{Y}_n}{\hat{\alpha}} - \frac{2(1/2\,\beta)}{\alpha}\right) = a_n\left(\frac{2\bar{Y}_n}{\hat{\alpha}} - \frac{\beta}{\alpha}\right)$$

↑ should match θ

– to compute asymptotic relative efficiencies, need to compare estimates of same parameter θ

$$= \frac{a_n}{\hat{\alpha}}\left(2\bar{Y}_n - \frac{\beta(\hat{\alpha})}{\alpha}\right)$$

we know $\sqrt{n}(\bar{Y}_n - \beta/2) = O_p(1)$
$\Rightarrow \sqrt{n}(2\bar{Y}_n - \beta) = O_p(1)$

$$= \frac{a_n}{\hat{\alpha}}\left(2\bar{Y}_n - \beta + \beta - \frac{\beta(\hat{\alpha})}{\alpha}\right)$$

let $a_n = \sqrt{n}$

$$= \frac{2\sqrt{n}}{\hat{\alpha}}(\bar{Y}_n - \beta/2) + \frac{\sqrt{n}(\beta)}{\hat{\alpha}}\left(1 - \frac{\hat{\alpha}}{\alpha}\right)\left(\frac{\alpha}{\alpha}\right)$$

$$= \frac{2\sqrt{n}}{\hat{\alpha}}(\bar{Y}_n - \beta/2) + \frac{\sqrt{n}(\beta)}{\hat{\alpha}(\alpha)}(\alpha - \hat{\alpha})$$

Since $\sqrt{n}(\bar{Y}_n - \beta/2) \xrightarrow{d} N(0, \beta^2/12)$

By Slutsky's thm

$$\Rightarrow \frac{2\sqrt{n}}{\hat{\alpha}}(\bar{Y}_n - \beta/2) \xrightarrow{d} \left(\frac{2}{\alpha}\right) N(0, \beta^2/12) \equiv N\left(0, \frac{\beta^2}{3\alpha^2}\right)$$

Since $n(\alpha - \hat{\alpha}) \xrightarrow{d} Exp(\alpha)$

$\Rightarrow$ By Slutsky's thm,

$$\frac{1}{\sqrt{n}}\left(\frac{\beta}{\hat{\alpha}(\alpha)}\right) n(\alpha - \hat{\alpha}) \xrightarrow{d} 0\, Exp(\alpha) = 0$$

By Slutsky's thm

$$\Rightarrow \frac{2\sqrt{n}}{\hat{\alpha}}(\bar{Y}_n - \beta/2) + \frac{\sqrt{n}\,\beta}{\hat{\alpha}\alpha}(\alpha - \hat{\alpha}) \xrightarrow{d} N\left(0, \frac{1}{3}\theta^2\right)$$

* Similar steps for $\hat{\beta}/2\bar{X}_n$

2). $\Theta = \{\theta_1, \dots, \theta_\ell\}$ → finite # parameters.

(a) Show that a Bayes rule d_B wrt prior Λ on Θ having positive probabilities $\lambda_1, \dots, \lambda_\ell > 0$ is admissible

$R = E_\theta[\text{Risk}] = E_\theta[E_{x|\theta}[\text{Loss}]]$

$R(\theta, d_B) = \sum_{i=1}^{\ell} \lambda_i R(\theta_i, d_B)$

Admissible: if not inadmissible

Inadmissible (d_B):

$\exists\, d^*$ such that

$R(\theta, d^*) \leq R(\theta, d_B)$ for all θ and

$R(\theta, d^*) < R(\theta, d_B)$ for some θ.

Bayes rule: minimizes Bayes risk.

$\Rightarrow R(\theta, d_B) \leq R(\theta, d)$ for any other decision rule d.

Suppose d_B is not admissible (is inadmissible) under these conditions

$\Rightarrow$ $\exists$ some d^* such that for at least one θ_j,

$R(\theta_j, d^*) < R(\theta_j, d_B)$

and $R(\theta_k, d^*) \leq R(\theta_k, d_B)$ for all θ_k $k = 1, \dots, \ell$

Suppose for some d^*, the following holds:

Suppose for $\theta_A = \{\theta_1, \dots, \theta_k\}$ $R(\theta_A, d^*) < R(\theta_A, d_B)$

and for $\theta_B = \{\theta_{k+1}, \dots, \theta_\ell\}$, $R(\theta_A, d^*) = R(\theta_A, d_B)$

$R(\theta, d^*) =$

$\Rightarrow \sum_{i=1}^{l} \lambda_i R(\theta_i, d^*) =$

$= \sum_{i\in A} \hat{\lambda_i} R(\theta_i, d^*) + \sum_{i\in B} \hat{\lambda_i} R(\theta_i, d^*)$

$= \sum_{i\in A} \hat{\lambda_i} R(\theta_i, d^*) + \sum_{i\in B} \hat{\lambda_i} R(\theta_i, d_B)$

$< \sum_{i\in A} \hat{\lambda_i} R(\theta_i, d_B) + \sum_{i\in B} \hat{\lambda_i} R(\theta_i, d_B)$

$= \sum_{i=1}^{l} \hat{\lambda_i} R(\theta_i, d_B)$

$= R(\theta, d_B)$

$\Rightarrow R(\theta, d^*) < R(\theta, d_B)$

This is a contradiction because if d_B is a Bayes rule, it should minimize the Bayes risk, which it doesn't under these conditions

(b) Suppose $\lambda_i = 0$ for some i in $1, \dots, l$.

Now show d_B (Bayes rule may not be admissible.

Suppose we consider the same set-up as before

For some d^*, there exists a set Θ_A such that for θ_i in set Θ_A (Θ_A arbitrarily $= \{\theta_1, \dots, \theta_k\}$)

$R(\theta_i, d^*) < R(\theta_i, d_B)$

And for set Θ_B, the complement of Θ_A, for all θ_i in set Θ_B ($\Theta_B = \{\theta_{k+1}, \dots, \theta_l\}$)

$R(\theta_i, d^*) = R(\theta_i, d_B)$

Under these conditions, show d_B inadmissible.

Suppose $\lambda_1 = \dots = \lambda_k = 0$

and $\lambda_{k+1}, \dots, \lambda_l > 0$

$\longrightarrow$ Continued.

$R(\theta, d^*) = \sum_{i=1}^{\ell} \lambda_i R(\theta_i, d^*)$

$= \sum_{i \in A} \lambda_i R(\theta_i, d^*) - \sum_{i \in B} \lambda_i R(\theta_i, d^*)$

$= 0 + \lambda_{k+1} R(\theta_{k+1}, d^*) - \sum_{i=k+2}^{\ell} \lambda_i R(\theta_i, d^*)$

$= 0 + \sum_{i=k+2}^{\ell} \lambda_i R(\theta_i, d_B)$ since

$R(\theta_i, d_B) = R(\theta_i, d^*)$ in these cases

$R(\theta, d_B) = \sum_{i=1}^{\ell} \lambda_i R(\theta_i; d_B)$

$= \sum_{i \in A} \lambda_i R(\theta_i, d_B) + \sum_{i \in B} \lambda_i R(\theta_i, d_B)$

for $i \in A$, $\lambda_i = 0$, and $i \in B \Rightarrow i = k+1, \dots, \ell$

$= 0 + \sum_{i=k+1}^{\ell} \lambda_i R(\theta_i, d_B)$

$= 0 + \sum_{i=k+2}^{\ell} \lambda_i R(\theta_i, d_B)$ since $\lambda_{k+1} = 0$

$R(\theta, d^*) = \sum_{i=1}^{\ell} \lambda_i R(\theta_i, d^*)$

$= \sum_{i \in A} \lambda_i R(\theta_i, d^*) + \sum_{i \in B} \lambda_i R(\theta_i, d^*)$

since for all $i \in A$, $\lambda_i = 0$, then

$= 0 + \sum_{i \in B} \lambda_i R(\theta_i, d^*)$

$= \sum_{i=k+1}^{\ell} \lambda_i R(\theta_i, d^*)$

$R(\theta, d_B) = \sum_{i=1}^{\ell} \lambda_i R(\theta_i, d_B)$

$= \sum_{i \in A} \lambda_i R(\theta_i, d_B) + \sum_{i \in B} \lambda_i R(\theta_i, d_B)$

$= 0 + \sum_{i=k+1}^{\ell} \lambda_i R(\theta_i, d_B)$

In this case, $R(\theta, d_B) = R(\theta, d^*)$

Since we are given that d_B is a Bayes Rule, we know it minimizes the Bayes Risk

We have found a situation where d_B is inadmissible, but the "improved" rule d^* does not result in a smaller Bayes Risk thereby producing a Bayes rule that is not admissible.

(c) Suppose that the frequentist risk of d_B in part (b) is finite + constant on those θ_i's having $\lambda_i > 0$.
Show that this decision rule is minimax (minimizes the maximum risk) on those θ_i's w/ $\lambda_i > 0$

Use same set-up as (a) + (B), where
$i \in A \Rightarrow \lambda_i = 0$,
$\boxed{i \in B \Rightarrow \lambda_i > 0}$ restrict ourselves to discussion of just $i \in B$

For all $i \neq j \in B$, $R(\theta_i, d_B) = R(\theta_j, d_B) = r$
– in given assumption, finite + constant risk.

$R(\theta, d_B) = \sum_{i=1}^{\ell} \lambda_i R(\theta_i, d_B)$
$= 0 + \sum_{i \in B} \lambda_i R(\theta_i, d_B)$
$= \sum_{i \in B} \lambda_i (r)$
$= r \sum_{i \in B} \lambda_i$
since Λ is a prior where $\sum_i \lambda_i = 1$, then
$= r < \infty$

~~if can't assume $\sum_i \lambda_i = 1 \Rightarrow r \sum_{i \in B} \lambda_i$~~

d_M is minimax if

$$\inf_{d \in D} \{ \sup_{\theta \in \Theta} R(\theta, d) \} = \sup_{\theta \in \Theta} R(\theta, d_M)$$

restrict ourselves to Θ_B

We know since d_B is a Bayes rule that it minimizes Bayes risk
$\Rightarrow R(\theta, d_B)$ minimizes possible $R(\theta, d)$
$\Rightarrow r < R(\theta, d)$ for all other $d \in D$
~~(or $r \sum_{i \in B} \lambda_i < R(\theta, d)$)~~

Suppose d_B is not minimax & there exists a d_M s.t.

$$\Rightarrow \inf_{d \in D} \left\{ \sup_{\theta_i \in \Theta_B} R(\theta_i, d) \right\} = \sup_{\theta_i \in \Theta_B} R(\theta_i, d_M) < \sup_{\theta_i \in \Theta} R(\theta_i, d_B)$$

$$\Rightarrow \sup_{\theta_i \in \Theta} R(\theta_i, d_M) < r \quad \text{for some } d_M \in D$$

because $R(\theta_i, d_B) = \text{constant } r \ \forall i \in B$

This implies that for all $i \in B$,

$$R(\theta_i, d_M) < r = R(\theta_i, d_B)$$

$$\Rightarrow \sum_{i \in B} \lambda_i R(\theta_i, d_M) < \sum_{i \in B} \lambda_i r$$

$$\Rightarrow \sum_{i \in B} \lambda_i R(\theta_i, d_M) < \mathcal{R}(\Theta, d_B)$$

$$\Rightarrow \mathcal{R}(\Theta, d_M) < \mathcal{R}(\Theta, d_B)$$

This contradicts the fact that as a Bayes Rule, d_B minimizes the Bayes risk

$\Rightarrow$ d_B must be minimax on those θ_i's w/ $\lambda_i > 0$

(d) Can anything be said about whether or not d_B in part (c) is minimax on Θ_i, $i = 1, \ldots, \ell$?

$\boxed{\text{No}}$

Consider same set-up as before.

It is possible that for $i \in A$ for some $d_M \in \mathcal{D}$

$$\sup_{\theta_i \in \Theta_A} R(\theta_i, d_M) < \sup_{\theta_i \in \Theta} R(\theta_i, d_B)$$

This would result in d_B not being minimax

However, d_B could still be Bayes rule because for $i \in A$, $\lambda_i = 0$ and the calculation for Bayes risk would not incorporate the smaller risk values in $i \in A$.

First draft → see additional steps in Second Draft

Classification problem:

X obs from

$$p(x|\theta) = \theta^{-1} I(0 < x < \theta)$$

$$\Theta = \{1, 2, 3\}$$

Want to classify x as arising from $p(x|1)$, $p(x|2)$ or $p(x|3)$

Loss : 0-1 loss

(e) Find the form of the Bayes rule for this problem.

Under 0-1 Loss, by a thm, the rule that minimizes the Bayes risk is the rule that maximizes the mode of the posterior dist

Prior $\Lambda = \{\lambda_1, \lambda_2, \lambda_3\}$ where $\sum_{i=1}^{3} \lambda_i = 1$

Posterior $p(\theta_i|x) \propto p(x|\theta_i)\lambda_i$

Maximize $p(\theta_i|x)$ by maximizing $p(x|\theta_i)\lambda_i$

pick θ_1 over θ_2 if

$$\lambda_1 p(x|\theta_1) > \lambda_2 p(x|\theta_2)$$

$$\Rightarrow \frac{\lambda_1}{\theta_1} I(0 < x < \theta_1) > \frac{\lambda_2}{\theta_2} I(0 < x < \theta_2)$$

Likewise, pick θ_j over θ_k if

$$\frac{\lambda_j}{\theta_j} I(0 < x < \theta_j) > \frac{\lambda_k}{\theta_k} I(0 < x < \theta_k)$$

Note: $\theta_1 = 1$, $\theta_2 = 2$, $\theta_3 = 3$

Case 1: $0 < x < 1$

all indicators $I(0<x<1)$, $I(0<x<2)$, & $I(0<x<3)$ are $= 1$

In this region, choose θ_j if

$\frac{\lambda_j}{\theta_j} >$ both $\frac{\lambda_k}{\theta_k}$ & $\frac{\lambda_\ell}{\theta_\ell}$

If $\frac{\lambda_j}{\theta_j} = \frac{\lambda_k}{\theta_k}$ in this region, choose

θ_j w/ prob γ & θ_k w/ prob $1-\gamma$

If $\frac{\lambda_1}{\theta_1} = \frac{\lambda_2}{\theta_2} = \frac{\lambda_3}{\theta_3}$, then choose

θ_1 w/ prob α_1, θ_2 w/ prob α_2, & θ_3 w/ prob α_3

where $\alpha_3 = 1 - \alpha_1 - \alpha_2$ $\left(\sum_{i=1}^{3} \alpha_i = 1\right)$

Case 2: $1 < x < 2$

$I(0<x<1) = 0$

$I(0<x<2) = I(0<x<3) = 1$

Choose $\theta_1 = 1$ w/ prob 0 since

$\lambda_1 p(x|\theta_1) = 0$ in this region

If $\frac{\lambda_j}{\theta_j} > \frac{\lambda_k}{\theta_k}$ for $\{j,k\} = \{2,3\}$, then

pick θ_j w/ prob 1

If $\frac{\lambda_2}{\theta_2} = \frac{\lambda_3}{\theta_3}$ then pick θ_2 w/ prob γ &

θ_3 w/ prob $1-\gamma$

- Case 3 $\quad 2 < x < 3$

$I(0<x<1) = I(0<x<2) = 0$

$\Rightarrow$ choose $\theta_3 = 3$ w/ prob 1 in this region
since $\lambda_1 p(x|\theta_1) = \lambda_2 p(x|\theta_2) = 0$ in this region.

(*) See extended f after Second Draft

(f) Find the minimax rule & the corresponding least favorable prior dist.

By a thm, If the Bayes rule results in a constant risk w/ finite Bayes risk (aka proper prior), then the Bayes rule is minimax.

In Region $0 < x < 1$, want
$R(1, d_M) = R(2, d_M) = R(3, d_M)$
- constant risk

And in region $1 < x < 2$, want
$R(2, d_M) = R(3, d_M)$

Can only get these equal risk situations if
$\frac{\lambda_1}{1} = \frac{\lambda_2}{2} = \frac{\lambda_3}{3} \rightarrow \lambda_2 = 2\lambda_1, \ \lambda_3 = 3\lambda_1$

$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 1$

$\Rightarrow \lambda_1 + 2\lambda_1 + 3\lambda_1 = 1$

$\Rightarrow 6\lambda_1 = 1$

$\Rightarrow$ $\boxed{\begin{array}{l} \lambda_1 = 1/6 \\ \lambda_2 = 1/3 \\ \lambda_3 = 1/2 \end{array}}$

(*) Continued after second draft of part (e)

Second draft / Edit of first draft

Classification problem (2014 Theory I)

(e) Find the form of the Bayes rule for this problem

(i) First, as done in the first draft, split into cases based on range of x ✓

(ii) Write out explicitly the decision rules $\phi_1(x)$, $\phi_2(x)$, + $\phi_3(x)$ for each range.

Case 3: $2 < x < 3$

$$\phi_3(x) = 1, \quad \phi_1(x) = \phi_2(x) = 0$$

– Decision rules for picking θ_i

Case 2: $1 < x < 2$

If $\frac{\lambda_3}{3} > \frac{\lambda_2}{2} \Rightarrow \phi_3(x) = 1$

If $\frac{\lambda_2}{2} > \frac{\lambda_3}{3} \Rightarrow \phi_2(x) = 1$

If $\frac{\lambda_2}{2} = \frac{\lambda_3}{3} \Rightarrow \phi_3(x) = \gamma_0, \quad \phi_2(x) = 1 - \gamma_0$

Thus:

$$\phi_3(x) = I(\lambda_1 > \lambda_2/2) + \gamma_0 I(\lambda_1 = \lambda_2/2)$$

$$\phi_2(x) = I(\lambda_1 < \lambda_2/2) + (1-\gamma_0) I(\lambda_1 = \lambda_2/2)$$

$$\phi_1(x) = 0$$

$\longrightarrow$

Case 1: $0 < x < 1$

List out all cases & their results.

Note: probabilities here must be distinguished from the γ_0 used in Case 2

$\lambda_1 > \frac{\lambda_2}{2} > \frac{\lambda_3}{3} \Rightarrow \phi_1(x) = 1$ others $= 0$

$\lambda_1 > \frac{\lambda_3}{3} > \frac{\lambda_2}{3}$ same as above

$\lambda_1 > \frac{\lambda_2}{2} = \frac{\lambda_3}{3}$ same

$\lambda_1 = \frac{\lambda_2}{2} > \frac{\lambda_3}{3}$ $\phi_1(x) = \gamma_1$, $\phi_2(x) = 1 - \gamma_1$,
$\phi_3(x) = 0$

$\frac{\lambda_2}{2} > \lambda_1 > \frac{\lambda_3}{3}$ $\phi_2(x) = 1$, others $= 0$

$\frac{\lambda_2}{2} > \frac{\lambda_3}{3} > \lambda_1$ same

$\frac{\lambda_2}{2} = \frac{\lambda_3}{3} > \lambda_1$ $\phi_2(x) = \gamma_2$, $\phi_3(x) = 1 - \gamma_2$,
$\phi_1(x) = 0$

$\frac{\lambda_3}{3} > \frac{\lambda_2}{2} > \lambda_1$ $\phi_3(x) = 1$, others $= 0$

$\frac{\lambda_3}{3} > \lambda_1 > \frac{\lambda_2}{2}$ same

$\frac{\lambda_3}{3} = \lambda_1 > \frac{\lambda_2}{2}$ $\phi_3(x) = \gamma_3$, $\phi_1(x) = 1 - \gamma_3$,
$\phi_2(x) = 0$

$\lambda_1 = \frac{\lambda_2}{2} = \frac{\lambda_3}{3}$ $\phi_1(x) = \alpha_1$, $\phi_2(x) = \alpha_2$,
$\phi_3(x) = \alpha_3 = 1 - \alpha_1 - \alpha_2$

$\phi_1(x) = I(\lambda_1 > \lambda_2/2)\, I(\lambda_1 > \lambda_3/3) +$
$\gamma_1 I(\lambda_1 = \lambda_2/2)\, I(\lambda_1 > \lambda_3/3) +$
$(1-\gamma_3)\, I(\lambda_1 = \lambda_3/3)\, I(\lambda_1 > \lambda_2/2) +$
$+ \alpha_1 I(\lambda_1 = \lambda_2/2 = \lambda_3/3)$

$\phi_2(x) = (1-\gamma_1)\, I(\lambda_1 = \lambda_2/2)\, I(\lambda_2/2 > \lambda_3/3) +$
$+ I(\lambda_2/2 > \lambda_1)\, I(\lambda_2/2 > \lambda_3/3) +$
$+ \gamma_2 I(\lambda_2/2 = \lambda_3/3)\, I(\lambda_2/2 > \lambda_1) +$
$+ \alpha_2 I(\lambda_1 = \lambda_2/2 = \lambda_3/3)$

$\phi_3(x) = (1-\gamma_2)\, I(\lambda_3/3 = \lambda_2/2)\, I(\lambda_3/3 > \lambda_1) +$
$+ I(\lambda_3/3 > \lambda_2/2)\, I(\lambda_3/3 > \lambda_1) +$
$+ \gamma_3 I(\lambda_3/3 = \lambda_1)\, I(\lambda_3/3 > \lambda_2/2) +$
$+ \alpha_3 I(\lambda_1 = \lambda_2/2 = \lambda_3/3)$

(f) Minimax rule
As explained in the first draft,
want to find a Beyes rule w/ constant risk

$$R(\theta_i, \phi) = \sum_{j=1}^{3} L(\theta_i, a_j)\, E_{\theta}[\phi_j(x)]$$

since loss is 0-1, then

$$= \sum_{j \neq i} E_{\theta_i}[\phi_j(x)]$$

$$= 1 - E_{\theta_i}[\phi_i(x)]$$

reasoning: For each θ_i, $\sum_{j=1}^{3} \phi_j(x) = 1$

For $R(1,\phi) = R(2,\phi) = R(3,\phi)$
$\Rightarrow E_{\theta_1}[\phi_1(x)] = E_{\theta_2}[\phi_2(x)] = E_{\theta_3}[\phi_3(x)]$

In homework soln, "it is easy to verify that $E_{\theta_i}[\phi_i(x)]$ is only constant under $\lambda_1 = \frac{\lambda_2}{2} = \frac{\lambda_3}{3}$ case"

Least favorable prior found in first draft is therefore correct.

The expectation under the prior:

$$E_{\theta_i}[\phi_i(x)] = \sum_{j=1}^{3} P(\text{being in x range } j \mid \theta_i \text{ true}) * P(\text{choosing } \theta_i \mid \text{x range + prior})$$

In $\phi_i(x)$ for each range, only look at term in front of $I(\lambda_1 = \lambda_2/2 = \lambda_3/3)$

$$E_{\theta_1}[\phi_1(x)] = P_1(0<x<1)(\alpha_1)$$

$$E_{\theta_2}[\phi_2(x)] = P_2(0<x<1)(\alpha_2) + P_2(1<x<2)(1-\gamma_0)$$

$$E_{\theta_3}[\phi_3(x)] = P_3(0<x<1)(\alpha_3) + P_3(1<x<2)(\gamma_0) + \\ + P_3(2<x<3)(1)$$

$P_1(0<x<1) = 1$

$P_2(0<x<1) = P(0<x<1 \mid \theta_2 \text{ true})$

Since $X \sim \text{Unif}(0,2)$ under θ_2 true,

$= 1/2$

Same way, $P_2(1<x<2) = 1/2$

Likewise, $P_3(0<x<1) = P_3(1<x<2) = P_3(2<x<3) = 1/3$

$E_{\theta_1}[\phi_1(x)] = \alpha_1$

$E_{\theta_2}[\phi_2(x)] = (1/2)\alpha_2 + (1/2)(1-\gamma_0)$

$E_{\theta_3}[\phi_3(x)] = (1/3)\alpha_3 + (1/3)\gamma_0 + (1/3)$

For $\alpha_1, \alpha_2, \gamma_0 \in [0,1]$ and $\alpha_3 = 1-\alpha_1-\alpha_2$,

The minimax rule is:

$\phi_1(x) = I(0<x<1)\alpha_1$

$\phi_2(x) = \alpha_2 I(0<x<1) + (1-\gamma_0) I(1<x<2)$

$\phi_3(x) = \alpha_3 I(0<x<1) + \gamma_0 I(1<x<2) + I(2<x<3)$

coefficients need to satisfy:

$\alpha_1 = (1/2)\alpha_2 + (1/2)(1-\gamma_0) = (1/3)\alpha_3 + (1/3)\gamma_0 + 1/3$

and $\alpha_1 + \alpha_2 + \alpha_3 = 1$

so that $E_{\theta_1}[\phi_1(x)] = E_{\theta_2}[\phi_2(x)] = E_{\theta_3}[\phi_3(x)]$

(g) Find the decision rule that minimizes the maximum risk over $\theta_1 = 1$ & $\theta_2 = 2$ & the corresponding least favorable prior distribution
- is this minimax rule the same as in (f)? Explain

Using same logic as in (f), want
$E_{\theta_1}(\phi_1(x)) = E_{\theta_2}(\phi_2(x))$

Since we no longer are concerned w/ $\theta_3 = 3$, need to recalculate decision rules in regions of x.

Case 1: $0 < x < 1$

$\frac{\lambda_1}{1} > \frac{\lambda_2}{2} \Rightarrow \phi_1(x) = 1, \; \phi_2(x) = 0$

$\frac{\lambda_2}{2} > \lambda_1 \Rightarrow \phi_2(x) = 1, \; \phi_1(x) = 0$

$\lambda_1 = \frac{\lambda_2}{2} \Rightarrow \phi_1(x) = \gamma_0, \; \phi_2(x) = 1 - \gamma_0$

$\phi_1(x) = I(\lambda_1 > \lambda_2/2) + \gamma_0 I(\lambda_1 = \lambda_2/2)$

$\phi_2(x) = I(\lambda_2/2 > \lambda_1) + (1-\gamma_0) I(\lambda_1 = \lambda_2/2)$

Case 2: $1 < x < 2$

In this case, $\lambda_1 p(x|1) = 0$ since $I(0<x<1) = 0$

Therefore, choose θ_2 w/ prob 1

$\phi_2(x) = 1$

$E_{\theta_1}[\phi_1(x)] = E_{\theta_2}[\phi_2(x)]$ only if $\lambda_1 = \frac{\lambda_2}{2}$

$E_{\theta_1}[\phi_1(x)] = P(0<x<1 | \theta_1 \text{ true}) P(\text{choose } \theta_1 \text{ in } 0<x<1 | \lambda_1 = \lambda_2/2)$

$E_{\theta_1}[\phi_1(x)] = E_{\theta_2}[\phi_2(x)]$ only if $\lambda_1 = \frac{\lambda_2}{2}$

$E_{\theta_1}[\phi_1(x)] = P(0<x<1 \mid \theta_1 \text{ true})\, P(\text{choose } \theta_1 \text{ in } 0<x<1 \mid \lambda_1 = \lambda_2/2)$

$= (1)(\gamma_0)$

$E_{\theta_2}[\phi_2(x)] = P(0<x<1 \mid \theta_2 \text{ true})\, P(\text{choose } \theta_2 \text{ in } 0<x<1 \mid \lambda_1 = \lambda_2/2)$

$+ P(1<x<2 \mid \theta_2 \text{ true})\, P(\text{choose } \theta_2 \text{ in } 1<x<2 \mid \lambda_1 = \lambda_2/2)$

$= (1/2)(1-\gamma_0) + (1/2)(1)$

Want $\gamma_0 = 1/2(1-\gamma_0) + 1/2$

$= 1 - 1/2\, \gamma_0$

$\Rightarrow 3/2\, \gamma_0 = 1$

$\Rightarrow \boxed{\gamma_0 = 2/3}$

Least favorable prior:

$\lambda_1 = \frac{\lambda_2}{2}$ and $\lambda_1 + \lambda_2 = 1$

$\Rightarrow \lambda_1 + 2\lambda_1 = 1$

$\Rightarrow \boxed{\lambda_1 = 1/3, \quad \lambda_2 = 2/3}$

- Clearly not equivalent to (f)
- Not equivalent due to focus on 2 θ instead of 3.