1. (25 points) Consider the linear model

$$Y = X_1\beta_1 + X_2\beta_2 + \epsilon, \tag{1}$$

where Y is $n \times 1$, X_1 is $n \times p_1$ of rank p_1, β_1 is $p_1 \times 1$, X_2 is $n \times p_2$ of rank p_2, (X_1, X_2) is full rank, β_2 is $p_2 \times 1$, and ϵ is $n \times 1$, with $E(\epsilon) = 0$ and $\text{Cov}(\epsilon) = \sigma^2 I$ where σ^2 is unknown. Suppose in reality that $\beta_2 = 0$, and thus the model used by the data analyst is an overfitted model given by (1) but the true model is

$$Y = X_1\beta_1 + \epsilon. \tag{2}$$

Let $\hat{\sigma}^2_{\text{overfit}}$ denote the usual estimator of σ^2 based on the overfitted model, where (β_1, β_2) are unknown.

(a) (3 points) Derive the expectation of $\hat{\sigma}^2_{\text{overfit}}$ assuming the model in (2) is true. Express your answer in the simplest possible form.

(b) (5 points) Assuming all relevant inverses exist, Derive an explicit expression for the least squares estimate of β_1 based on the model in (1), and derive the expectation of the least squares estimate of β_1 from the model in (1) assuming the model in (2) is true. Express your answers in the simplest possible form.

(c) (6 points) Assume normality of ϵ and the model in (2) is true. Derive an exact $(1-\alpha) \times 100\%$ confidence interval for σ^2 based on the model in (1) and give explicit conditions as to when the expected length of the confidence interval for σ^2 based on the model in (2) is smaller than that of the model in (1).

(d) Now consider the special case in which $\gamma_1, \gamma_2, \gamma_3$ are interior angles of a triangle so that $\gamma_1 + \gamma_2 + \gamma_3 = 180$ degrees. Suppose we have available measurements Y_1, Y_2, Y_3 of $\gamma_1, \gamma_2, \gamma_3$, respectively. We assume that $Y_i \sim N(\gamma_i, \sigma^2)$, $i = 1, 2, 3$, σ^2 is unknown and the Y_i's are independent.

   (i) (7 points) Derive the F test for testing the null hypothesis that the triangle is equilateral, and state the distribution of the F statistic under the null and alternative hypotheses. Express your test statistic in the simplest possible form.

   (ii) (4 points) Derive a 95% joint confidence region for (γ_1, γ_2).

1. Consider the linear model

$$Y = X_1\beta_1 + X_2\beta_2 + \varepsilon \quad (1)$$

$Y_{n\times 1}$ $\quad X_{1\, n\times p_1}$ $\quad r(X_1) = p_1$ $\quad X_{2\, n\times p_2}$ $\quad r(X_2) = p_2$ $\quad r(X_1, X_2) = p$ $\quad \beta_{1\, p_1\times 1}$ $\quad \beta_{2\, p_2\times 1}$ $\quad \varepsilon_{n\times 1}$

$E(\varepsilon) = 0$ $\quad \mathrm{cov}(\varepsilon) = \sigma^2 I$ $\quad \sigma^2$ unknown

Suppose in reality $\beta_2 = 0$ & thus model (1) is overfit & the true model is

$$Y = X_1\beta_1 + \varepsilon \quad (2)$$

Let $\hat{\sigma}^2_{overfit} \equiv \hat{\sigma}^2_0$ denote the usual estimator of σ^2 based on model (2) where (β_1, β_2) are unknown

↳ UMVUE

(a) Derive the expectation of $\hat{\sigma}^2_0$ assuming the model in (2) is true

$\hat{\sigma}^2_0 = \dfrac{Y'(I-M)Y}{n-p}$ $\quad$ where $p = p_1 + p_2$ $\quad X = (X_1, X_2)$

$M = X(X'X)^{-1}X'$

$$E[\hat{\sigma}^2_0] = E\left[\frac{Y'(I-M)Y}{n-p}\right] = \frac{1}{n-p}E[Y'(I-M)Y]$$

$$= \frac{1}{n-p}\left(E[Y'](I-M)E[Y] + tr(\sigma^2(I-M))\right)$$

$$= \frac{1}{n-p}\left[(X_1\beta_1)'(I-M)X_1\beta_1 + tr(\sigma^2(I-M))\right]$$

$$= \frac{1}{n-p}\left[0 + \sigma^2 tr(I-M)\right]$$ since $I-M$ is an o.p.o onto $C(X_1, X_2)^\perp$ $\quad R(I-M) = tr(I-M) = n-p$

$$= \frac{1}{n-p}\sigma^2(n-p) = \boxed{\sigma^2}$$

1.(b) Assuming all relevant inverses exist, Derive an explicit expression for the LSE of β_1 based on (1) & derive the expectation of the LSE of β_1 assuming (2) is true. Express in simplest form. 2018 Theory 2

We must orthogonalize X_1 & X_2 to get an independent estimate for β_1.

$Y = X_1\beta_1 + X_2\beta_2 + \varepsilon$ let $M_2 = X_2(X_2'X_2)^{-1}X_2'$

$= (I-M_2)X_1\beta_1 + M_2X_1\beta_1 + X_2\beta_2 + \varepsilon$

$= (I-M_2)X_1\beta_1 + X_2(X_2'X_2)^{-1}X_2'X_1\beta_1 + X_2\beta_2 + \varepsilon$

$= (I-M_2)X_1\beta_1 + X_2\left[(X_2'X_2)^{-1}X_2'X_1\beta_1 + \beta_2\right] + \varepsilon$

$= (I-M_2)X_1\beta_1 + X_2\beta_2^* + \varepsilon$ let $\beta_2^* = (X_2'X_2)^{-1}X_2'X_1\beta_1 + \beta_2$

$C((I-M_2)X_1) \perp C(X_2)$ as $((I-M_2)X_1)'X_2 = X_1'(I-M_2)X_2 = 0$

$\therefore$ we can use $(I-M_2)X_1$ to find the LSE of β_1.

By Gauss-Markov we know the LSE = BLUE $\Rightarrow \hat{\beta} = (X'X)^{-1}X'Y$

$\therefore \hat{\beta}_1 = \left[X_1'(I-M_2)(I-M_2)X_1\right]^{-1}(I-M_2)X_1'Y$

$= \left[X_1'(I-M_2)X_1\right]^{-1}(I-M_2)X_1'Y$

$E[\hat{\beta}_1] = E\left[(X_1'(I-M_2)X_1)^{-1}(I-M_2)X_1'Y\right] = (X_1'(I-M_2)X_1)^{-1}(I-M_2)X_1'E[Y]$

$= (X_1'(I-M_2)X_1)^{-1}(I-M_2)X_1'(X_1\beta_1) = \underbrace{(X_1'(I-M_2)X_1)^{-1}(X_1'(I-M_2)X_1)}_{I \text{ by inverse}}\beta_1 = \boxed{\beta_1}$

1.(c) Assume normality of ε & the model in (2) is true. | 2018 Theory 2

Derive an exact $(1-\alpha)100\%$ CI for σ^2 based on the model in (1) & give explicit conditions as to when the expected length of the CI for σ^2 based on (2) is smaller than that of the model (1).

Model (1)

$Y \sim N(\beta_1 X_1 + \beta_2 X_2, \sigma^2 I)$

$$\frac{Y'(I-M)Y}{\sigma^2} \sim \chi^2(r(I-M)) \equiv \chi^2(n-p)$$

$$\Rightarrow CI = \left\{\sigma^2 : \chi^2(n-p, \alpha/2) \le \frac{Y'(I-M)Y}{\sigma^2} \le \chi^2(n-p, 1-\alpha/2)\right\}$$

where $\chi^2(a,b)$ is the b^{th} percentile of the $\chi^2(df=a)$

$$= \left\{\sigma^2 : \frac{Y'(I-M)Y}{\chi^2(n-p, 1-\alpha/2)} \le \sigma^2 \le \frac{Y'(I-M)Y}{\chi^2(n-p, \alpha/2)}\right\}$$

Model (2) $\quad Y \sim N(\beta_1 X_1, \sigma^2 I) \Rightarrow \dfrac{Y'(I-M_1)Y}{\sigma^2} \sim \chi^2(r(I-M_1)) \equiv \chi^2(n-p_1)$

where $M_1 = X_1(X_1'X_1)^{-1}X_1'$ $\quad r(M_1) = p_1$

$$CI = \left\{\sigma^2 : \frac{Y'(I-M_1)Y}{\chi^2(n-p_1, 1-\alpha/2)} \le \sigma^2 \le \frac{Y'(I-M_1)Y}{\chi^2(n-p_1, \alpha/2)}\right\}$$

(1)

$$E\left[\frac{Y'(I-M)Y}{\chi^2(n-p, \alpha/2)} - \frac{Y'(I-M)Y}{\chi^2(n-p, 1-\alpha/2)}\right] = E[Y'(I-M)Y]\left(\frac{1}{\chi^2(n-p,\alpha/2)} - \frac{1}{\chi^2(n-p, 1-\alpha/2)}\right)$$

$$= \sigma^2(n-p)\left(\frac{1}{\chi^2(n-p,\alpha/2)} - \frac{1}{\chi^2(n-p,1-\alpha/2)}\right) \equiv E[\text{length (1)}]$$

(2)

$$E\left[\frac{Y'(I-M_1)Y}{\chi^2(n-p_1, \alpha/2)} - \frac{Y'(I-M_1)Y}{\chi^2(n-p_1, 1-\alpha/2)}\right] = E[Y'(I-M)Y]\left(\frac{1}{\chi^2(n-p_1,\alpha/2)} - \frac{1}{\chi^2(n-p_1,1-\alpha/2)}\right)$$

$$= \sigma^2(n-p_1)\left(\frac{1}{\chi^2(n-p_1,\alpha/2)} - \frac{1}{\chi^2(n-p_1,1-\alpha/2)}\right) \equiv E[\text{length (2)}]$$

$\therefore$ The expected length of the CI from model (2) is smaller than the expected length of the CI from model (1) when

$$E[\text{length (2)}] < E[\text{length (1)}]$$

$$(n-p_1)\left(\frac{1}{\chi^2(n-p_1,\alpha/2)} - \frac{1}{\chi^2(n-p_1,1-\alpha/2)}\right) < (n-p)\left(\frac{1}{\chi^2(n-p,\alpha/2)} - \frac{1}{\chi^2(n-p,1-\alpha/2)}\right)$$

1(d). Consider the special case in which $\gamma_1, \gamma_2, \gamma_3$ are interior angles of a triangle s.t. $\gamma_1+\gamma_2+\gamma_3=180$ $\boxed{\text{2018 Theory 2}}$

Suppose we have measurements Y_1, Y_2, Y_3 respectively

We assume $Y_i \sim N(\gamma_i, \sigma^2)$ Y_1, Y_2, Y_3 independent, σ^2 unknown

(i) Derive the F-test for testing: $\boxed{H_0: \gamma_1=\gamma_2=\gamma_3=60}$ State the distribution under the null & alternative hypotheses.

$Y_1 = \gamma_1 + \varepsilon$

$Y_2 = \gamma_2 + \varepsilon$

$Y_3 = \gamma_3 + \varepsilon = 180 - \gamma_1 - \gamma_2 + \varepsilon$

$\Longleftrightarrow$

$Y_1 - 60 = \gamma_1 - 60 + \varepsilon$

$Y_2 - 60 = \gamma_2 - 60 + \varepsilon$

$Y_3 - 60 = \gamma_3 - 60 + \varepsilon = 120 - \gamma_1 - \gamma_2 + \varepsilon$

$= -(60-\gamma_1) - (60-\gamma_2) + \varepsilon = (\gamma_1 - 60) + (\gamma_2 - 60) + \varepsilon$

let $Y^* = \begin{pmatrix} Y_1-60 \\ Y_2-60 \\ Y_3-60 \end{pmatrix}$ $\beta^* = \begin{pmatrix} \gamma_1-60 \\ \gamma_2-60 \end{pmatrix}$ $X^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}$ $\varepsilon = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix}$

$$F = \frac{Y'(M-M_0)Y \,/\, r(M-M_0)}{Y'(I-M)Y \,/\, r(I-M)}$$

$M = X(X'X)^{-1}X'$

$X'X = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $(X'X)^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

$X(X'X)^{-1}X' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} = M$ $r(M)=2$

$I - M = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ $r(I-M)=1$

$M_0 = 0$ $r(M_0)=0$ $\Rightarrow$ $F = \dfrac{Y'(M)Y \,/\, r(M)}{Y'(I-M)Y \,/\, r(I-M)} \sim F(2,1)$ under H_0

$Y'MY = (MY)'(MY)$ $MY = \begin{pmatrix} y_1-\bar{y} \\ y_2-\bar{y} \\ y_3-\bar{y} \end{pmatrix}$ $\Rightarrow Y'MY = (y_1-\bar{y})^2 + (y_2-\bar{y})^2 + (y_3-\bar{y})^2$

$Y'(I-M)Y = ((I-M)Y)'(I-M)Y$ $(I-M)Y = \begin{pmatrix} \bar{y}-60 \\ \bar{y}-60 \\ \bar{y}-60 \end{pmatrix} \Rightarrow Y'(I-M)Y = 3(\bar{y}-60)^2$

$$\therefore F = \frac{\sum_{i=1}^{3}(y_i-\bar{y})^2/2}{3(\bar{y}-60)^2} = \frac{\sum_{i=1}^{3}(y_i-\bar{y})^2}{6(\bar{y}-60)^2} \sim F(2,1,\delta)$$

under H_0: $\delta = 0$

under H_1: $\delta = \dfrac{(X\beta)'MX\beta}{2\sigma^2}$

$= \dfrac{(\gamma_1-60)^2 + (\gamma_2-60)^2 + (\gamma_3-60)^2}{2\sigma^2}$

1(d). (ii) Derive a 95% joint CI for (γ_1, γ_2)

We know $\beta = \begin{pmatrix} \gamma_1 - 60 \\ \gamma_2 - 60 \end{pmatrix} \Rightarrow \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \beta + \begin{pmatrix} 60 \\ 60 \end{pmatrix}$ $\therefore$ a sufficient CI for (γ_1, γ_2) would be a CI for β

$$\frac{(Y-X\beta)'M(Y-X\beta)/r(M)}{(Y-X\beta)'(I-M)(Y-X\beta)/r(I-M)} \sim F(2,1)$$ because $M(I-M)=0$ the 2 χ^2 distributions are $\perp\!\!\!\perp \Rightarrow \frac{\chi^2}{\chi^2} = F$

numerator

$$(Y-X\beta)'M(Y-X\beta) = (MY-MX\beta)'(MY-MX\beta) = (MX\beta - MY)'(MX\beta - MY)$$

$$= (X\beta - X(X'X)^{-1}X'Y)'(X\beta - X(X'X)^{-1}X'Y) = (\beta - (X'X)^{-1}X'Y)'X'X(\beta - (X'X)^{-1}X'Y)$$

$$(X'X)^{-1}X'Y = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} Y_1 - 60 \\ Y_2 - 60 \\ Y_3 - 60 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \end{pmatrix}\begin{pmatrix} Y_1 - 60 \\ Y_2 - 60 \\ Y_3 - 60 \end{pmatrix} = \begin{pmatrix} y_1 - \bar{y} \\ y_2 - \bar{y} \end{pmatrix} = \hat{\beta}$$

$$(\beta - \hat{\beta})'\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}(\beta - \hat{\beta})$$

denominator

$$(Y-X\beta)'(I-M)(Y-X\beta) = Y'(I-M)Y = 3(\bar{y}-60)^2$$

$\therefore$ a 95% CI for β is

$$\left\{\beta : \frac{(\beta - \hat{\beta})'\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}(\beta - \hat{\beta})}{6(\bar{y}-60)^2} < F(2,1,0.95)\right\} = CI(\beta)$$

and $CI(\gamma) = CI(\beta) + \begin{pmatrix} 60 \\ 60 \end{pmatrix}$

2. (25 points) Suppose that the pair of random variables (X, Y) has the form $X = U + T_1$ and $Y = U + T_2$, where U, T_1 and T_2 have independent Poisson distributions with means $\mathrm{E}[U] = \psi, \mathrm{E}[T_1] = \lambda_1, \mathrm{E}[T_2] = \lambda_2$. Note that the marginal distributions of X and Y are Poisson and that X and Y are independent if and only if $\psi = 0$. The objective of this problem is to develop large-sample tests of independence of X and Y.

   (a) (4 points) Show that the distribution of (X, Y) is given by

$$P(X = x, Y = y) = e^{-(\psi+\lambda_1+\lambda_2)}\lambda_1^x\lambda_2^y \sum_{u=0}^{\min(x,y)} \left(\frac{\psi}{\lambda_1\lambda_2}\right)^u \frac{1}{u!(x-u)!(y-u)!},$$

   for nonnegative integers x and y.

   (b) (7 points) The hypothesis $H_0 : \psi = 0$ is of interest. Obtain an explicit expression for the log-likelihood, $l(\psi, \lambda_1, \lambda_2; x, y)$, based on a single pair (x, y). Then show that the score vector evaluated at $\psi = 0$ is

$$\begin{pmatrix} \frac{xy}{\lambda_1\lambda_2} - 1 \\ \frac{x}{\lambda_1} - 1 \\ \frac{y}{\lambda_2} - 1 \end{pmatrix}.$$

   Further, obtain the expected information matrix (3×3), also evaluated at $\psi = 0$. Important: The general form of the score vector and expected information matrix may be complicated. Only the form evaluated at $\psi = 0$ is required in this problem.

   (c) (5 points) Now consider testing the hypothesis $H_0 : \psi = 0$ using independent pairs $(X_1, Y_1), \cdots, (X_n, Y_n)$ from the same distribution as (X, Y) above. Derive an explicit expression for the score test statistic, and identify its asymptotic (as $n \to \infty$) distribution under H_0.

   (d) Since $\mathrm{cov}(X, Y) = \psi$ it is natural to develop a test statistic based on the "sample covariance". The sample covariance is S/n where $S := \sum_{i=1}^n (X_i - \bar{X})(Y_i - \bar{Y})$, $\bar{X} = \sum_{i=1}^n X_i/n$ and $\bar{Y} = \sum_{i=1}^n Y_i/n$. Since $(\bar{X}, \bar{Y})$ is a minimal sufficient statistic under H_0, a test can be based on the conditional distribution of S given $(\bar{X}, \bar{Y})$ under H_0.

      (i) (7points) Compute the ***exact*** conditional variance of S given $\bar{X} = \bar{x}$ and $\bar{Y} = \bar{y}$ under H_0.

      (ii) (2 points) By standardizing S, obtain a test statistic that is asymptotically standard normal under H_0.

2. Suppose the pair of RVs (X,Y) has the form 2018 Theory 2

$X = U + T_1$ and $Y = U + T_2$ where U, T_1, T_2 have independent Poisson

$E[U] = \psi$ $E[T_1] = \lambda_1$ $E[T_2] = \lambda_2$ $X \sim$ Poisson $Y \sim$ Poisson $X \perp\!\!\!\perp Y \Leftrightarrow \psi = 0$

(a) Show the distribution of (X,Y) is given by

$$P(X=x, Y=y) = e^{-(\psi + \lambda_1 + \lambda_2)} \lambda_1^x \lambda_2^y \sum_{u=0}^{\min(x,y)} \left(\frac{\psi}{\lambda_1 \lambda_2}\right)^u \frac{1}{u!(x-u)!(y-u)!}$$

for non-negative integers x and y.

we start with the joint dist. of (U, T_1, T_2) independent Poissons.

$$P(U=u, T_1=t_1, T_2=t_2) = \frac{e^{-\psi}\psi^u}{u!} \frac{e^{-\lambda_1}\lambda_1^{t_1}}{t_1!} \frac{e^{-\lambda_2}\lambda_2^{t_2}}{t_2!}$$

let $X = U + T_1$, $Y = U + T_2$, $U = U \Rightarrow T_1 = X - U$, $T_2 = Y - U$

$0 < u < \min(x,y)$

$u < x < \infty$

$u < y < \infty$

$$|J| = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 1 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$f_{X,Y,U}(x,y,u) = \frac{e^{-\psi}\psi^u}{u!} \frac{e^{-\lambda_1}\lambda_1^{x-u}}{(x-u)!} \frac{e^{-\lambda_2}\lambda_2^{y-u}}{(y-u)!} = e^{-(\psi+\lambda_1+\lambda_2)} \lambda_1^x \lambda_2^y \left(\frac{\psi}{\lambda_1\lambda_2}\right)^u \frac{1}{u!(x-u)!(y-u)!}$$

$$f_{X,Y}(x,y) = \sum_u f_{X,Y,U}(x,y,u)$$

$$= \sum_{u=0}^{\min(x,y)} e^{-(\psi+\lambda_1+\lambda_2)} \lambda_1^x \lambda_2^y \left(\frac{\psi}{\lambda_1\lambda_2}\right)^u \frac{1}{u!(x-u)!(y-u)!}$$

$$= e^{-(\psi+\lambda_1+\lambda_2)} \lambda_1^x \lambda_2^y \sum_{u=0}^{\min(x,y)} \left(\frac{\psi}{\lambda_1\lambda_2}\right)^u \frac{1}{u!(x-u)!(y-u)!}$$

$x = 0, 1, 2, \dots$

$y = 0, 1, 2, \dots$

2(b) The hypothesis $H_0: \Psi = 0$ is of interest. Obtain an explicit expression for $\ell(\Psi, \lambda_1, \lambda_2)$ based on a single pair (x,y). Then show the score value evaluated at $\Psi=0 = \cdots$ & find the expected info, also evaluated at $\Psi=0$

$$\ell_{x,y}(x,y) = -(\Psi+\lambda_1+\lambda_2) + x\log\lambda_1 + y\log\lambda_2 + \log\left[\sum_u \left(\frac{\Psi}{\lambda_1\lambda_2}\right)^u \frac{1}{u!(x-u)!(y-u)!}\right]$$

let $A = \sum_u \left(\frac{\Psi}{\lambda_1\lambda_2}\right)^u \frac{1}{u!(x-u)!(y-u)!}$

$$\therefore \ell_{x,y}(x,y) = -(\Psi+\lambda_1+\lambda_2) + x\log\lambda_1 + y\log\lambda_2 + \log A$$

$$\frac{\partial \ell_{x,y}}{\partial\Psi} = -1 + \frac{1}{A}\frac{\partial A}{\partial\Psi} \qquad \frac{\partial \ell_{x,y}}{\partial\lambda_1} = -1 + \frac{x}{\lambda_1} + \frac{1}{A}\frac{\partial A}{\partial\lambda_1} \qquad \frac{\partial \ell_{x,y}}{\partial\lambda_2} = -1 + \frac{y}{\lambda_2} + \frac{1}{A}\frac{\partial A}{\partial\lambda_2}$$

$$\frac{\partial A}{\partial\Psi} = \sum_u u\left(\frac{1}{\lambda_1\lambda_2}\right)^u \Psi^{u-1}\frac{1}{u!(x-u)!(y-u)!} = \sum_{u=1}^{\min(x,y)} u\left(\frac{1}{\lambda_1\lambda_2}\right)^u \Psi^{u-1}\frac{1}{u!(x-u)!(y-u)!}$$

assuming $0^0=1$

$$\left.\frac{\partial A}{\partial\Psi}\right|_{\Psi=0} = \sum_{u=1}^{\min(x,y)} u\left(\frac{1}{\lambda_1\lambda_2}\right)^u 0^{u-1}\frac{1}{u!(x-u)!(y-u)!} = \frac{1}{\lambda_1\lambda_2}\frac{1}{(x-1)!(y-1)!}$$

$$A\big|_{\Psi=0} = \sum_{u=0}^{\min(x,y)} \left(\frac{0}{\lambda_1\lambda_2}\right)^u \frac{1}{u!(x-u)!(y-u)!} = \frac{1}{x!y!}$$

$$\frac{\partial A}{\partial\lambda_1} = \sum_u \left(\frac{\Psi}{\lambda_2}\right)^u (-u)\lambda_1^{-u-1}\frac{1}{u!(x-u)!(y-u)!} \qquad \left.\frac{\partial A}{\partial\lambda_1}\right|_{\Psi=0} = \sum_{u=1}^{\min(x,y)} \left(\frac{0}{\lambda_2}\right)^u (-u)\lambda_1^{-u-1}\frac{1}{u!(x-u)!(y-u)!} = 0$$

$$\frac{\partial A}{\partial\lambda_2} = \sum_u \left(\frac{\Psi}{\lambda_1}\right)^u (-u)\lambda_2 \frac{1}{u!(x-u)!(y-u)!} \qquad \left.\frac{\partial A}{\partial\lambda_2}\right|_{\Psi=0} = 0$$

$$\left.\frac{\partial \ell}{\partial\Psi}\right|_{\Psi=0} = -1 + x!y!\left(\frac{1}{\lambda_1\lambda_2}\frac{1}{(x-1)!(y-1)!}\right) = \frac{xy}{\lambda_1\lambda_2} - 1$$

$$\left.\frac{\partial \ell}{\partial\lambda_1}\right|_{\Psi=0} = -1 + \frac{x}{\lambda_1} + x!y!\cdot 0 = \frac{x}{\lambda_1} - 1 \qquad \left.\frac{\partial \ell}{\partial\lambda_2}\right|_{\Psi=0} = -1 + \frac{y}{\lambda_2} + x!y!\,0 = \frac{y}{\lambda_2} - 1$$

$$\therefore \partial\ell_{x,y}(x,y)\big|_{\Psi=0} = \begin{pmatrix} \frac{xy}{\lambda_1\lambda_2} - 1 \\ \frac{x}{\lambda_1} - 1 \\ \frac{y}{\lambda_2} - 1 \end{pmatrix}$$

2.(b) con't

$$\frac{\partial^2 l}{\partial\psi^2} = -\frac{1}{A^2}\left(\frac{\partial A}{\partial\psi}\right)^2 + \frac{1}{A}\frac{\partial^2 A}{\partial\psi^2} \qquad \frac{\partial^2 l}{\partial\lambda_1^2} = -\frac{x}{\lambda_1^2} - \frac{1}{A^2}\left(\frac{\partial A}{\partial\lambda_1}\right)^2 + \frac{1}{A}\frac{\partial^2 A}{\partial\lambda_1^2}$$

$$\frac{\partial^2 l}{\partial\lambda_2^2} = -\frac{y}{\lambda_2^2} - \frac{1}{A^2}\left(\frac{\partial A}{\partial\lambda_2}\right)^2 + \frac{1}{A}\frac{\partial^2 A}{\partial\lambda_2^2} \qquad \frac{\partial^2 l}{\partial\psi\partial\lambda_1} = -\frac{1}{A^2}\frac{\partial A}{\partial\psi}\frac{\partial A}{\partial\lambda_1} + \frac{1}{A}\frac{\partial^2 A}{\partial\psi\partial\lambda_1}$$

$$\frac{\partial^2 l}{\partial\psi\partial\lambda_2} = -\frac{1}{A^2}\frac{\partial A}{\partial\psi}\frac{\partial A}{\partial\lambda_2} + \frac{1}{A}\frac{\partial^2 A}{\partial\psi\partial\lambda_2} \qquad \frac{\partial^2 l}{\partial\lambda_1\partial\lambda_2} = -\frac{1}{A^2}\frac{\partial A}{\partial\lambda_1}\frac{\partial A}{\partial\lambda_2} + \frac{1}{A}\frac{\partial A^2}{\partial\lambda_1\partial\lambda_2}$$

$$\frac{\partial^2 A}{\partial\psi^2} = \frac{\partial}{\partial\psi}\left(\sum_u u\left(\frac{1}{\lambda_1\lambda_2}\right)^u \psi^{u-1}\frac{1}{u!(y-u)!(x-u)!}\right) = \sum_{u=2}^{\min(x,y)} u(u-1)\left(\frac{1}{\lambda_1\lambda_2}\right)^u \psi^{u-2}\frac{1}{u!(y-u)!(x-u)!}$$

$$\left.\frac{\partial^2 A}{\partial\psi^2}\right|_{\psi=0}^{(u=2)} = 2\left(\frac{1}{\lambda_1\lambda_2}\right)^2\frac{1}{2!(y-2)!(x-2)!} = \left(\frac{1}{\lambda_1\lambda_2}\right)^2\frac{1}{(y-2)!(x-2)!}$$

$$\frac{\partial^2 A}{\partial\psi\partial\lambda_1} = \frac{\partial}{\partial\lambda_1}\left(\sum_u u\left(\frac{1}{\lambda_1\lambda_2}\right)^u\psi^{u-1}\frac{1}{u!(y-u)!(x-u)!}\right) = \sum_u u(-u)\left(\frac{1}{\lambda_2}\right)^u\lambda_1^{-u-1}\psi^{u-1}\frac{1}{u!(y-u)!(x-u)!}$$

$$\left.\frac{\partial^2 A}{\partial\psi\partial\lambda_1}\right|_{\psi=0}^{(u=1)} = -\frac{1}{\lambda_2\lambda_1^2}\frac{1}{(x-1)!(y-1)!} \qquad \text{similarly, } \left.\frac{\partial^2 A}{\partial\psi\partial\lambda_2}\right|_{\psi=0} = -\frac{1}{\lambda_1\lambda_2^2}\frac{1}{(x-1)!(y-1)!}$$

$$\frac{\partial^2 A}{\partial\lambda_1^2} = \frac{\partial}{\partial\lambda_1}\left(\sum_u\left(\frac{\psi}{\lambda_2}\right)^u(-u)\lambda_1^{-u-1}\frac{1}{u!(y-u)!(x-u)!}\right) = \sum_u\left(\frac{\psi}{\lambda_2}\right)^u(-u)(-u-1)\lambda_1^{-u-2}\frac{1}{u!(y-u)!(x-u)!}$$

$$\left.\frac{\partial A}{\partial\lambda_1^2}\right|_{\psi=0} = 0 \qquad \text{similarly } \left.\frac{\partial A}{\partial\lambda_2^2}\right|_{\psi=0} = 0 \text{ and } \left.\frac{\partial A}{\partial\lambda_1\partial\lambda_2}\right|_{\psi=0} = 0$$

$$\therefore \left.\frac{\partial^2 l}{\partial\psi^2}\right|_{\psi=0} = -(x!y!)^2\left(\frac{1}{\lambda_1\lambda_2}\frac{1}{(x-1)!(y-1)!}\right)^2 + x!y!\left(\frac{1}{\lambda_1\lambda_2}\right)^2\frac{1}{(y-2)!(x-2)!}$$

$$= -\frac{x^2y^2}{\lambda_1^2\lambda_2^2} + \frac{x(x-1)y(y-1)}{\lambda_1^2\lambda_2^2} = -\frac{xy(x+y-1)}{\lambda_1^2\lambda_2^2}$$

$$\left.\frac{\partial^2 l}{\partial\lambda_1^2}\right|_{\psi=0} = -\frac{x}{\lambda_1^2} - (x!y!)^2\cdot 0^2 + (x!y!)0 = -\frac{x}{\lambda_1^2} \qquad \frac{\partial^2 l}{\partial\lambda_2^2} = -\frac{y}{\lambda_2^2} \qquad \left.\frac{\partial^2 l}{\partial\lambda_1\partial\lambda_2}\right|_{\psi=0} = 0$$

$$\left.\frac{\partial^2 l}{\partial\psi\partial\lambda_1}\right|_{\psi=0} = \frac{-x!y!}{\lambda_2\lambda_1^2(x-1)!(y-1)!} = -\frac{xy}{\lambda_2\lambda_1^2} \qquad \left.\frac{\partial^2 l}{\partial\psi\partial\lambda_2}\right|_{\psi=0} = -\frac{xy}{\lambda_2^2\lambda_1}$$

$$\left.\partial^2 l\right|_{\psi=0} = \begin{pmatrix} -\frac{xy(x+y-1)}{\lambda_1^2\lambda_2^2} & -\frac{xy}{\lambda_1^2\lambda_2} & -\frac{xy}{\lambda_1\lambda_2^2} \\ & -\frac{x}{\lambda_1^2} & 0 \\ & & -\frac{y}{\lambda_2^2} \end{pmatrix}$$

2.(b) con't   when $\psi=0$, $X \perp\!\!\!\perp Y$, $X=T_1$, $Y=T_2$

$$E[XY(X+Y-1)] = E[X^2Y] + E[XY^2] - E[XY]$$
$$= E[T_1^2]E[T_2] + E[T_1]E[T_2^2] - E[T_1]E[T_2]$$
$$= (var(T_1) + E[T_1]^2)E[T_2] + E[T_1](var(T_2) + E(T_2)^2) - E[T_1]E[T_2]$$
$$= (\lambda_1 + \lambda_1^2)\lambda_2 + \lambda_1(\lambda_2 + \lambda_2^2) - \lambda_1\lambda_2$$
$$= \lambda_1\lambda_2 + \lambda_1^2\lambda_2 + \lambda_1\lambda_2 + \lambda_1\lambda_2^2 - \lambda_1\lambda_2 = \lambda_1\lambda_2 + \lambda_1^2\lambda_2 + \lambda_1\lambda_2^2$$

$$E[XY] = \lambda_1\lambda_2$$

$$E[-\partial^2 \ell|_{\psi=0}] = \begin{pmatrix} \frac{\lambda_1\lambda_2 + \lambda_1^2\lambda_2 + \lambda_1\lambda_2^2}{\lambda_1^2\lambda_2^2} & \frac{\lambda_1\lambda_2}{\lambda_1^2\lambda_2} & \frac{\lambda_1\lambda_2}{\lambda_1\lambda_2^2} \\ & \frac{\lambda_1}{\lambda_1^2} & 0 \\ & & \frac{\lambda_2}{\lambda_2^2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1+\lambda_1+\lambda_2}{\lambda_1\lambda_2} & \frac{1}{\lambda_1} & \frac{1}{\lambda_2} \\ & \frac{1}{\lambda_1} & 0 \\ & & \frac{1}{\lambda_2} \end{pmatrix} \equiv \text{expected information}$$

2.(c) Now consider testing the hypothesis $H_0: \psi = 0$ using independent pairs $(X_1, Y_1) \cdots (X_n, Y_n)$. Derive an expression of the score test statistic & identify its asymptotic distribution under H_0 as $n \to \infty$.

$$l_n = \sum_{i=1}^n l(x_i, y_i) \Rightarrow \partial l_n \Big|_{\psi=0} = \sum_{i=1}^n \partial l(x_i, y_i) \Big|_{\psi=0} = \begin{pmatrix} \sum \frac{x_i y_i}{\lambda_1 \lambda_2} - n \\ \sum x_i / \lambda_1 - n \\ \sum y_i / \lambda_2 - n \end{pmatrix} \overset{\text{set}}{=} 0$$

$$\frac{\sum x_i}{\lambda_1} - n \overset{\text{set}}{=} 0 \rightarrow \frac{\sum x_i}{n} = \tilde{\lambda}_1 \qquad \frac{\sum y_i}{\lambda_2} - n \overset{\text{set}}{=} 0 \Rightarrow \frac{\sum y_i}{n} = \tilde{\lambda}_2$$

$$l_n(\tilde{\xi}) = \begin{pmatrix} \frac{\sum x_i y_i}{\tilde{\lambda}_1 \tilde{\lambda}_2} - n \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{n^2 \sum x_i y_i}{\sum x_i \sum y_i} - n \\ 0 \\ 0 \end{pmatrix}$$

$$I_n(\xi) = \sum_{i=1}^n E\left[-\partial^2 l(\xi)\right] = nI = n \begin{pmatrix} \frac{1 + \lambda_1 + \lambda_2}{\lambda_1 \lambda_2} & \frac{1}{\lambda_1} & \frac{1}{\lambda_2} \\ \frac{1}{\lambda_1} & \frac{1}{\lambda_1} & 0 \\ \frac{1}{\lambda_2} & 0 & \frac{1}{\lambda_2} \end{pmatrix}$$

$$SC_n = l_n(\tilde{\xi})' I_n(\tilde{\xi})^{-1} l_n(\tilde{\xi})$$

$$= \left(\frac{n^2 \sum x_i y_i}{\sum x_i y_i} - n\right) I_n^{11}(\tilde{\xi}) \left(\frac{n^2 \sum x_i y_i}{\sum x_i y_i} - n\right)$$

$$I_n^{11}(\tilde{\xi}) = \frac{1}{\det(I_n(\tilde{\xi}))} \det(I_{n,22}(\tilde{\xi})) = \frac{\lambda_1^2 \lambda_2^2}{n^3} \frac{n^2}{\lambda_1 \lambda^2} = \frac{\lambda_1 \lambda_2}{n}$$

$$\det(I_n(\tilde{\xi})) = n^3\left(\frac{1 + \lambda_1 + \lambda_2}{\lambda_1 \lambda_2}\left(\frac{1}{\lambda_1 \lambda_2}\right) - \frac{1}{\lambda_1}\left(\frac{1}{\lambda_1 \lambda_2}\right) - \frac{1}{\lambda_2}\left(\frac{1}{\lambda_1 \lambda_2}\right)\right) = \frac{n^3}{\lambda_1^2 \lambda_2^2}$$

$$\det(I_{n,22}(\tilde{\xi})) = \frac{n^2}{\lambda_1 \lambda_2}$$

$$SC_n = \left(\frac{n^2 \sum x_i y_i}{\sum x_i y_i} - n\right)^2 \frac{\lambda_1 \lambda^2}{n} = n \lambda_1 \lambda_2 \left(\frac{n \sum x_i y_i}{\sum x_i \sum y_i} - 1\right)^2 \xrightarrow{d} \chi_1^2 \text{ under } H_0.$$

2.(d) cont.

Let $\alpha = \begin{pmatrix} X_1 - \bar{X} \\ \vdots \\ X_n - \bar{X} \end{pmatrix}$ and $\beta = \begin{pmatrix} Y_1 - \bar{Y} \\ \vdots \\ Y_n - \bar{Y} \end{pmatrix}$ we know $\alpha \perp\!\!\!\perp \beta$ & $E(\alpha) = E(\beta) = 0$

Since conditionally $X_1 \cdots X_n \mid \Sigma X_i \sim$ Multi,

$X_i \mid \sum_{i=1}^{n} X_i = N_1 \sim \text{Binomial}(n\bar{X}, 1/n)$

$\Rightarrow \underline{var(X_i)} = n\bar{X}(\frac{1}{n})(1 - 1/n) = n\bar{X}(\frac{1}{n})(\frac{n-1}{n}) = \bar{X}\frac{n-1}{n}$

$\underline{Cov(X_i, X_j)}$ $i \neq j$ note $(X_1, \dots, X_n) = \sum_{k=1}^{n\bar{X}} (Z_{k1}, \dots, Z_{kn})$

where $Z_{k1}, \dots, Z_{kn}$ are iid Multinomial $(1, (1/n, \dots, 1/n))$

$$\Rightarrow Cov(X_i, X_j) = n\bar{X} \cdot Cov(Z_{ki}, Z_{kj})$$
$$= n\bar{X}\left[E[Z_{ki}Z_{kj}] - E[Z_{ki}]E[Z_{kj}]\right]$$
$$= n\bar{X}\left[0 - (\tfrac{1}{n})(\tfrac{1}{n})\right] = -\frac{\bar{X}}{n}$$

$$\Rightarrow cov(\alpha \mid \bar{X}) = \bar{X}\begin{pmatrix} \frac{n-1}{n} & -1/n & \cdots & -1/n \\ -1/n & \frac{n-1}{n} & & \vdots \\ \vdots & & \ddots & \vdots \\ -1/n & \cdots & & \frac{n-1}{n} \end{pmatrix} = \bar{X}(I - 1/n\, J_n J_n') \quad \text{where } J_n = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}_{n \times 1}$$

Similarly $Cov(\beta \mid \bar{Y}) = \bar{Y}(I - 1/n\, J_n J_n')$

Conditional on $\bar{X}, \bar{Y}, J$

$$var(S) = var(\alpha'\beta) = var(E[\alpha'\beta \mid \alpha]) + E[var(\alpha'\beta \mid \alpha)]$$
$$= var(\alpha' E[\beta \mid \alpha]) + E[\alpha' \, var(\beta \mid \alpha)\alpha$$
$$= var(\alpha' E[\beta]) + E[\alpha' \, var(\beta)\alpha]$$
$$= 0 + tr(E[\alpha' var(\beta)\alpha])$$
$$= tr(var(\beta)E[\alpha'\alpha]) = tr(var(\beta) + var(\alpha) - E[\alpha]E[\alpha'])$$
$$= tr(var(\beta)\, var(\alpha)) = tr(\bar{X}(I - 1/n)J_nJ_n' \, \bar{Y}(I - 1/n)J_nJ_n')$$
$$= tr(\bar{X}\bar{Y}(I - 1/n)J_nJ_n') = \bar{X}\bar{Y}\, tr((I - 1/n)J_nJ_n')$$
$$= \bar{X}\bar{Y}\, n\left(\frac{n-1}{n}\right) = (n-1)\bar{X}\bar{Y}$$

2.(d) $\text{cov}(X,Y)=\psi$ so we can develop a test-statistic based on the sample covariance $\frac{S}{n}$ where $S=\sum_{i=1}^{n}(X_i-\bar{X})(Y_i-\bar{Y})$

Since $(\bar{X},\bar{Y})$ is a minimal S.S. under H_0, a test can be derived based on the conditional distribution of $S\,|\,\bar{X},\bar{Y}$ under H_0.

Compute the <u>exact</u> conditional variance of S given $\bar{X}=\bar{x}$ and $\bar{Y}=\bar{y}$ under H_0.

$X=T_1$, $Y=T_2$ under $H_0:\psi=0$

$\therefore$ WTS $(X_1,\dots,X_n)\,|\,\Sigma X_i$ and $(Y_1,\dots,Y_n)\,|\,\Sigma Y_i$ are Multinomial

$X_i=T_i\sim \text{Poisson}(\lambda_1)$

$$P(X_1,X_2,\dots,X_n)=\prod_{i=1}^{n}\frac{e^{-\lambda_1}\lambda_1^{x_i}}{x_i!}=\frac{e^{-n\lambda_1}\lambda_1^{\Sigma x_i}}{\prod_{i=1}^{n}x_i!}$$

let $Z_1=X_1,\dots,Z_{n-1}=X_{n-1},\ Z_n=\sum_{i=1}^{n}X_i \Rightarrow |J|=1$

$$\Rightarrow f(z_1,\dots,z_n)=\frac{e^{-n\lambda_1}\lambda_1^{z_n}}{\prod_{i=1}^{n-1}z_i!\,(z_n-\sum_{i=1}^{n-1}z_i)!}$$ we know $\sum_{i=1}^{n}X_i\sim\text{Poisson}(n\lambda_1)$

$$\Rightarrow f_{Z_n}(z_n)=\frac{e^{-n\lambda_1}(n\lambda_1)^{z_n}}{z_n!}$$

$$\Rightarrow f(z_1,\dots,z_{n-1}|z_n)=\frac{f(z_1,\dots,z_n)}{f(z_n)}=\frac{\cancel{e^{-n\lambda_1}}\,\cancel{\lambda_1^{z_n}}\,z_n!}{z_1!\cdots z_{n-1}!(z_n-z_1-\cdots-z_{n-1})!\,\cancel{e^{-n\lambda_1}}(n\cancel{\lambda_1})^{z_n}}$$

$$=\frac{n^{-z_n}\,z_n!}{z_1!\cdots z_{n-1}!(z_n-z_1-\cdots-z_n)!}$$

$$\Rightarrow f(x_1,\dots,x_n|\Sigma x_i=N_1)=\frac{n^{-\Sigma x_i}\,N_1!}{x_1!\cdots x_{n-1}!(N_1-x_1-\cdots-x_{n-1})!}\Rightarrow (X_1,\dots,X_n\,|\,\sum_{i=1}^{n}X_i=N_1)\sim\text{Multin}(N_1,(\tfrac{1}{n},\dots,\tfrac{1}{n}))$$

Similarly, $(Y_1,\dots,Y_n\,|\,\sum_{i=1}^{n}Y_i=N_2)\sim\text{Multinomial}(N_2,(\tfrac{1}{n},\dots,\tfrac{1}{n}))$

2(d)(ii) By standardizing S, obtain a test statistic that is asymptotically standard normal under H_0.

$$Var(S|\bar{X},\bar{Y}) = (n-1)\bar{X}\bar{Y} \Rightarrow var\left(\frac{S}{\sqrt{(n-1)\bar{X}\bar{Y}}} \,\Big|\, \bar{X},\bar{Y}\right) = 1$$

$$E(S|\bar{X},\bar{Y}) = E[\alpha'\beta|\bar{X},\bar{Y}] = E[E[\alpha'\beta|\alpha,\bar{X},\bar{Y}]]$$

$$= E[\alpha' E[\beta|\bar{X},\bar{Y}]] = 0$$

$$\Rightarrow var(S) = var(E[S|\bar{X},\bar{Y}]) + E[var(S|\bar{X},\bar{Y})]$$

$$= 1$$

Just subtract Expectation & divide by s.e.