#6 Let $Y = X\beta + \varepsilon$ where $\varepsilon \sim N_n(0, \sigma^2 I)$ and let $H = X(X'X)^{-1}X'$. In addition, let $d_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$. ← i^{th} spot has one in it

a) Let $B_i = \dfrac{(n-p)\hat{\sigma}^2}{\sigma^2} - \dfrac{\hat{\varepsilon}_i^2/(1-h_{ii})}{\sigma^2}$.

Note that $(n-p)\hat{\sigma}^2 = \sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2 = Y'(I-H)Y$.

In addition, $1 - h_{ii} = d_i'(I-H)d_i$ and
$\hat{\varepsilon}_i = d_i'\hat{\varepsilon} = d_i'(I-H)Y$.

Hence, $\hat{\varepsilon}_i/(1-h_{ii}) = Y'(I-H)\dfrac{d_i d_i'}{1-h_{ii}}(I-H)Y$.

Thus, $B_i = \left(\dfrac{Y}{\sigma}\right)'\left[(I-H) - (I-H)\dfrac{d_i d_i'}{1-h_{ii}}(I-H)\right]\left(\dfrac{Y}{\sigma}\right)$

Note that $Y/\sigma \sim N_n(0, I)$.

Moreover, $\left[(I-H) - (I-H)\dfrac{d_i d_i'}{1-h_{ii}}(I-H)\right]\left[(I-H) - (I-H)\dfrac{d_i d_i'}{1-h_{ii}}(I-H)\right]$

$= (I-H) - (I-H)\dfrac{d_i d_i'}{1-h_{ii}}(I-H)$

since $(I-H)^2 = I-H$ and $d_i'(I-H)d_i = 1-h_{ii}$.

Also, $(I-H) - (I-H)\dfrac{d_i d_i'}{1-h_{ii}}(I-H)$ is symmetric and so we've got a projection matrix.

Therefore, $B_i \sim \chi^2_r$ where $r = \text{rank}\left[(I-H) - (I-H)\dfrac{d_i d_i'}{1-h_{ii}}(I-H)\right]$.

Now, $\text{rank}\left[(I-H) - (I-H)\frac{d_i d_i'}{1-h_{ii}}(I-H)\right]$

$$= tr\left[(I-H) - (I-H)\frac{d_i d_i'}{1-h_{ii}}(I-H)\right]$$

$$= n-p - \frac{1}{1-h_{ii}} tr\left[(I-H)\, d_i d_i' (I-H)\right]$$

$$= n-p - \frac{1}{1-h_{ii}} tr\left[d_i' (I-H) d_i\right]$$

$$= n-p - \frac{1-h_{ii}}{1-h_{ii}}$$

$$= n-p-1.$$

And so $B_i \sim \chi^2_{n-p-1}$.

b) Note that $A_i = \frac{1}{\sigma^2} Y'(I-H)\frac{d_i d_i'}{1-h_{ii}}(I-H)Y$.

We look at "A" & "B" (the matrices between Y' & Y)

$$\Rightarrow BA = \frac{1}{\sigma^2}\left[(I-H) - (I-H)\frac{d_i d_i'}{1-h_{ii}}(I-H)\right] \cdot \frac{1}{\sigma^2}\left[(I-H)\frac{d_i d_i'}{1-h_{ii}}(I-H)\right]$$

$$= \frac{1}{(\sigma^2)^2}\left[(I-H)\frac{d_i d_i'}{1-h_{ii}}(I-H) - (I-H)\frac{d_i d_i'}{1-h_{ii}}(I-H)\frac{d_i d_i'}{1-h_{ii}}(I-H)\right]$$

$$= \frac{1}{(\sigma^2)^2}\left[(I-H)\frac{d_i d_i'}{1-h_{ii}}(I-H) - (I-H)\frac{d_i d_i'}{1-h_{ii}}(I-H)\right]$$

since $d_i'(I-H)d_i = 1-h_{ii}$

$$= 0.$$

Thus, $A_i \perp\!\!\!\perp B_i$.

c) For $r_i = \dfrac{\hat{\varepsilon}_i}{\hat{\sigma}\sqrt{1-h_{ii}}}$, note that

$$A_i = \frac{\hat{\sigma}^2}{\sigma^2} r_i^2 .$$

In addition, $A_i = \left(\frac{Y}{\sigma}\right)' \underbrace{(I-H) \frac{d_i d_i'}{1-h_{ii}} (I-H)}_{\text{symmetric, idempotent, rank}=1.} \left(\frac{Y}{\sigma}\right)$

$\sim \chi^2_1$ (similar to part a) process).

And so since $A_i \perp\!\!\!\perp B_i$: $\dfrac{A_i}{A_i+B_i} \sim \text{Beta}\left(\frac{1}{2}, \frac{n-p-1}{2}\right)$.

Thus, $$\frac{A_i}{A_i+B_i} = \frac{\frac{\hat{\sigma}^2}{\sigma^2} r_i^2}{\frac{\hat{\sigma}^2}{\sigma^2}(n-p)} = \frac{r_i^2}{n-p} \sim \text{Beta}\left(\frac{1}{2}, \frac{n-p-1}{2}\right). ✓$$

d) The notes used a "trick" where we consider $(I-H)d_i\phi$ instead (to "orthogonalize" things).

From there $\hat{\phi} = \left[d_i'(I-H)d_i\right]^{-1} d_i'(I-H)Y$

$$= \frac{\hat{\varepsilon}_i}{1-h_{ii}} \quad \text{(as seen earlier, above).}$$

So to test $\phi = 0$:

consider $\dfrac{A_i}{\sqrt{B_i}} \sim t(n-p-1)$. Then

$$t^* = \frac{A_i}{\sqrt{B_i}} = \frac{\hat{\phi} - 0}{\left[(n-p)\hat{\sigma}^2 - \hat{\phi}\hat{\varepsilon}_i\right]^{1/2}} \sim t(n-p-1); \text{ and compare to } t\left(\tfrac{2-\alpha}{2}, n-p-1\right).$$

Let $P_I = X_I(X'X)^{-1}X_I'$.

e) For a single i, $\hat{\beta}_{(i)} = \hat{\beta} - \dfrac{(X'X)^{-1}x_i'\hat{\varepsilon}_i}{1-h_{ii}}$

$$= \hat{\beta} - (X'X)^{-1}(X'd_i)\left[d_i'(I-H)d_i\right]^{-1}d_i'\hat{\varepsilon}$$

So, in general, $\hat{\beta}_I = \hat{\beta} - (X'X)^{-1}(X'd_I)\left[d_I'(I-H)d_I\right]^{-1}d_I'\hat{\varepsilon}$

$$= \hat{\beta} - (X'X)^{-1}X_I\left[d_I'd_I - d_I'X(X'X)^{-1}X'd_I\right]\hat{\varepsilon}_I$$

$$= \hat{\beta} - (X'X)^{-1}X_I\left[I_m - X_I(X'X)^{-1}X_I'\right]\hat{\varepsilon}_I$$

$$= \hat{\beta} - (X'X)^{-1}X_I\left[I - P_I\right]^{-1}\hat{\varepsilon}_I.$$

We need to show that equivalently

$$(\hat{\beta}-\hat{\beta}_I)'(X'X)(\hat{\beta}-\hat{\beta}_I) = \sum_{i=1}^{m}\frac{(\gamma_i'\hat{\varepsilon}_I)^2}{1-\lambda_i}\left(\frac{\lambda_i}{1-\lambda_i}\right).$$

Note that $(\hat{\beta}-\hat{\beta}_I)'(X'X)(\hat{\beta}-\hat{\beta}_I)$

$$= \hat{\varepsilon}_I'(I-P_I)^{-1}X_I'(X'X)^{-1}(X'X)(X'X)^{-1}X_I(I-P_I)^{-1}\hat{\varepsilon}_I$$

Now, $I - P_I = QQ' - Q\Lambda Q'$
$= Q[I-\Lambda]Q'$

And $(I-P)^{-1} = Q[I-\Lambda]^{-1}Q'$

$= Q\,\mathrm{diag}\left(\frac{1}{1-\lambda_i}\right)Q'.$

Note: $\left[Q[I-\Lambda]Q'Q[I-\Lambda]^{-1}Q' = Q[I-\Lambda][I-\Lambda]^{-1}Q' = QQ' = I \right]$

So $(\hat\beta - \hat\beta_I)'(X'X)(\hat\beta-\hat\beta_I)$

$= \hat\varepsilon_I'\left(Q\,\mathrm{diag}\left(\frac{1}{1-\lambda_i}\right)Q'\right)\left(Q\Lambda Q'\right)\left(Q\,\mathrm{diag}\left(\frac{1}{1-\lambda_i}\right)Q'\right)\hat\varepsilon_I$

$= \hat\varepsilon_I' Q\,\mathrm{diag}\left(\frac{\lambda_i}{(1-\lambda_i)^2}\right)Q'\hat\varepsilon_I$

$= \begin{bmatrix} \gamma_1'\hat\varepsilon_I \\ \vdots \\ \gamma_m'\hat\varepsilon_I \end{bmatrix}' \mathrm{diag}\left(\frac{\lambda_i}{(1-\lambda_i)^2}\right)\begin{bmatrix} \gamma_1'\hat\varepsilon_I \\ \vdots \\ \gamma_m'\hat\varepsilon_I \end{bmatrix}$

$= \sum_{i=1}^{m} \frac{\lambda_i}{(1-\lambda_i)^2}\left(\gamma_i'\hat\varepsilon_I\right)^2$

$= \sum_{i=1}^{m} \frac{(\gamma_i'\hat\varepsilon_I)^2}{1-\lambda_i}\left(\frac{\lambda_i}{1-\lambda_i}\right)$, as needed.

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By equivalence,

$$\frac{(\hat\beta-\hat\beta_I)(X'X)(\hat\beta-\hat\beta_I)}{p\hat\sigma^2} = \frac{1}{p}\sum_{i=1}^{m} h_i^2 \frac{\lambda_i}{1-\lambda_i}$$

where $h_i^2 = \dfrac{(\gamma_i'\hat\varepsilon_I)^2}{\hat\sigma^2(1-\lambda_i)}$