2019, Sect 1, Qual

AMW

1. Let X denote a RV from $N(0,1)$ and let Y be an outcome variable.

The joint distn. of (X,Y) has a finite 2nd moment & $E[X^2Y^2] < \infty$.

Assume that we observe n iid copies of (X,Y) denoted by $(X_1,Y_1),\dots,(X_n,Y_n)$.

The goal is to obtain the best prediction of $Y|X$ for a future subject.

a) One simple prediction is to consider a linear function $\alpha+\beta X$ to minimize the following squared loss:

$$E\left[\{Y-(\alpha+\beta X)\}^2\right]$$

where the expectation is w.r.t. the joint distn. of (Y,X). Show that the optimal soln. for (α,β), denoted by (α^*,β^*) is given by

$$\begin{pmatrix}\alpha^*\\ \beta^*\end{pmatrix} = \begin{pmatrix}E[Y]\\ E[XY]\end{pmatrix}$$

[Let $h(x,y) = E\left[\{Y-(\alpha+\beta X)\}\{Y-(\alpha+\beta X)\}\right]$

$$= E\left[Y^2 - 2(\alpha+\beta X)Y + (\alpha+\beta X)^2\right]$$

$$= E[Y^2] - 2\alpha E[Y] - 2\beta E[XY] + \alpha^2 + 2\alpha\beta E[X] + \beta^2 \underbrace{E[X^2]}_{(Var(X)+(E[X])^2)} \qquad (*)$$

Note that $X \sim N(0,1)$ which doesn't have Y in it

$\Rightarrow E_{(x,y)}[X] = E_x[X] = 0$ and $Var_{(x,y)}(X) = Var_x(X) = 1$

Subbing into $(*)$, get

$$h(x,y) = E[Y^2] - 2\alpha E[Y] - 2\beta E[XY] + \alpha^2 + \cancel{2\alpha\beta(0)}^{\,0} + \beta^2(1)$$

Then, $\frac{\partial h}{\partial \alpha} = -2E[Y] + 2\alpha \overset{set}{=} 0 \Rightarrow \alpha^* = E[Y]$

$\frac{\partial h}{\partial \beta} = -2E[XY] + 2\beta \overset{set}{=} 0 \Rightarrow \beta^* = E[XY]$

$$\Rightarrow \boxed{\begin{pmatrix}\alpha^*\\ \beta^*\end{pmatrix} = \begin{pmatrix}E[Y]\\ E[XY]\end{pmatrix}}$$]

1 b) From (1), we estimate (α^*, β^*) as

$$\hat{\alpha} = \frac{1}{n}\sum_i Y_i \quad , \quad \hat{\beta} = \frac{1}{n}\sum_i X_i Y_i$$

Give the asymptotic distn. of the obtained estimator under proper normalization.

Given finite 2nd moment.

Thus, by multivariate CLT,

$$\sqrt{n}\left(\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\end{pmatrix} - \begin{pmatrix}\alpha^*\\ \beta^*\end{pmatrix}\right) \xrightarrow{d} N\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix} Var(Y_i) & Cov(Y_i, X_iY_i) \\ '' \quad '' & Var(X_iY_i)\end{pmatrix}\right)$$

$$\equiv N\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix} \overset{E[Y_i^2]-(E[Y_i])^2}{\overset{\parallel}{Var(Y_i)}} & E[X_iY_i^2] - E[Y_i]E[X_iY_i] \\ '' \quad '' & \underset{\underset{E[X_i^2Y_i^2] - (\underbrace{E[X_iY_i]}_{\beta^*})^2}{\parallel}}{Var(X_iY_i)}\end{pmatrix}\right)$$

$$\equiv \boxed{N\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix} E[Y_i^2] - \alpha^{*2} & E[X_iY_i^2] - \alpha^*\beta^* \\ '' \quad '' & E[X_i^2Y_i^2] - \beta^{*2}\end{pmatrix}\right)}$$

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1. Now, suppose that we know the distribution of $Y|X$ is from lognormal family, i.e., $\log(Y) = \gamma X + N(0, \sigma^2)$.

c) Obtain the MLEs for α^* and β^* given in (1) and derive their asymptotic distn.

① First, find the distn. of $Y|X$

Given $\log(Y) = \gamma X + N(0,\sigma^2) = N(\gamma X, \sigma^2)$

Then, by convolution, let $Z = \log(Y) \Rightarrow f_Z(z) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\log(y)-\gamma x)^2}{2\sigma^2}} \cdot \left|\frac{1}{y}\right|, \ y>0$

$= \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{y} e^{\frac{-(\log(y)-\gamma x)^2}{2\sigma^2}}, \ y > 0. \quad \Rightarrow Y|X \sim \text{lognorm}(\gamma X, \sigma^2)$

② Find α^* in terms of γ and σ^2

Know that $Z = \gamma X + N(0,\sigma^2) = N(\gamma X, \sigma^2)$ has MGF of $M_Z(t) = \exp\{\gamma X t + \sigma^2 t^2/2\}$

Then, for $Z = \log(Y) \Rightarrow \underset{\substack{\shortparallel \\ \exp\{\gamma X t + \sigma^2 t^2/2\}}}{M_Z(t)} = M_{\log(Y)}(t) = E\left[e^{\log(Y)t}\right] = E[Y^t]$

$\Rightarrow E\left[Y^t | X\right] = \exp\{\gamma X t + \sigma^2 t^2/2\}$. For $t=1 \Rightarrow E[Y|X] = \exp\{\gamma X + \sigma^2/2\}$

Then, $E[Y] = E_X[E[Y|X]] = E_X\left[\exp\{\gamma X + \sigma^2/2\}\right] = e^{\sigma^2/2} E\left[_X e^{\gamma X}\right]$ ← $N(0,1)$ — MGF of $N(0,1)$ w/ γ in place of t.

$= e^{\sigma^2/2} \cdot e^{\{\gamma \cdot 0 + 1^2 \gamma^2/2\}}$

$= e^{\sigma^2/2} \cdot e^{\gamma^2/2} = e^{(\gamma^2+\sigma^2)/2}$

$\Rightarrow \alpha^* = E[Y] = \exp\{\tfrac{1}{2}(\gamma^2 + \sigma^2)\}$

③ Find β^* in terms of γ and σ^2

$\beta^* = E[XY] = E\left[E[XY|X]\right] = E\left[X E[Y|X]\right] = E\left[X \cdot \exp\{\gamma X + \sigma^2/2\}\right] = e^{\sigma^2/2} E[Xe^{\gamma X}]$

$= \frac{e^{\sigma^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{\gamma x} e^{-x^2/2} dx = \frac{e^{\sigma^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{(x^2-2\gamma x)}{2}} dx = e^{\frac{\sigma^2+\gamma^2}{2}} \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{(x^2-2\gamma x+\gamma^2)}{2}} dx$

$= e^{\frac{\sigma^2+\gamma^2}{2}} \underbrace{\int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{\frac{-(x-\gamma)^2}{2}} dx}_{\substack{= E[X] \text{ for } X \sim N(\gamma,1) \\ = \gamma}} = \gamma \exp\left\{\frac{\sigma^2+\gamma^2}{2}\right\} = \beta^*$

cont'd →

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1 c) cont'd

(4) Find $\hat{\sigma}^2$ and $\hat{\gamma}^2$ to get MLEs of α^* and β^* by invariance property.

Don't need to consider $f(x,y)$ since $f(x) \perp\!\!\!\perp \sigma^2$ and γ^2. Only consider $f(y|x)$ for likelihood.

$$\mathcal{L}(\gamma,\sigma^2 | \underset{\sim}{x}, \underset{\sim}{y}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{y_i} e^{-\frac{(\log(y_i)-\gamma x_i)^2}{2\sigma^2}}$$

$$\propto (\sigma^2)^{-n/2} \cdot \left(\frac{1}{y_i}\right)^n e^{-\frac{\sum_i (\log(y_i)-\gamma x_i)^2}{2\sigma^2}}$$

$$\Rightarrow \ell(\gamma,\sigma^2 | \underset{\sim}{x}, \underset{\sim}{y}) \propto -\frac{n}{2}\log(\sigma^2) - \frac{\sum_i (\log(y_i)-\gamma x_i)^2}{2\sigma^2}$$

$$\Rightarrow \frac{\partial \ell}{\partial \gamma} = -\cancel{2}\sum_i \left[\frac{(\log(y_i)-\gamma x_i)}{\cancel{2}\sigma^2} \cdot (-x_i) \right] \overset{\text{set}}{=} 0 \Rightarrow \sum_i \log(y_i)x_i - \sum_i \gamma x_i^2 = 0$$

$$\Rightarrow \hat{\gamma} = \frac{\sum_i \log(y_i)x_i}{\sum_i x_i^2}$$

$$\text{\&} \quad \frac{\partial \ell}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{\sum_i (\log(y_i)-\gamma x_i)^2}{2\sigma^4} \overset{\text{set}}{=} 0 \Rightarrow \hat{\sigma}^2 = \frac{1}{n}\sum_i (\log(y_i) - \hat{\gamma} x_i)^2$$

Thus, $\boxed{\hat{\beta}^* = \hat{\gamma}\exp\left\{\frac{\hat{\sigma}^2+\hat{\gamma}^2}{2}\right\} \quad \text{\&} \quad \hat{\alpha}^* = \exp\left\{\frac{1}{2}(\hat{\gamma}^2+\hat{\sigma}^2)\right\}}$

w/ $\hat{\gamma}$ and $\hat{\sigma}^2$ as define above.

cont'd next pg. →

1c) cont'd

(5) Now, want to find asymptotic distn. of $\begin{pmatrix} \hat{\alpha}^* \\ \hat{\beta}^* \end{pmatrix}$.

We were told that the joint distn. of (X, Y) has finite 2nd moment & thus exists.

Then, by properties of MLE, $\sqrt{n}\left(\begin{pmatrix} \hat{\gamma} \\ \hat{\sigma}^2 \end{pmatrix} - \begin{pmatrix} \gamma \\ \sigma^2 \end{pmatrix}\right) \xrightarrow{d} N\Big(0, \underbrace{I_1(\gamma, \sigma^2)^{-1}}_{\text{information matrix for ONLY one obs.}}\Big)$

From previous pg.,

$$\frac{\partial l}{\partial \gamma} = \frac{(\log(Y_i) - \gamma X_i)(X_i)}{\sigma^2} \Rightarrow \frac{\partial^2 l}{\partial \gamma^2} = \frac{-X_i^2}{\sigma^2} \Rightarrow -E\left[\frac{-X_i^2}{\sigma^2}\right] = \frac{E[X_i^2]}{\sigma^2} = \frac{\overset{1}{\cancel{Var[X_i]}} + (\overset{0}{\cancel{E[X_i]}})^2}{\sigma^2} = \frac{1}{\sigma^2}$$

$$\frac{\partial l}{\partial \sigma^2} = \frac{-1}{2\sigma^2} + \frac{(\log(Y_i) - \gamma X_i)^2}{2\sigma^4} \Rightarrow \frac{\partial^2 l}{(\partial \sigma^2)^2} = \frac{1}{2\sigma^4} - \frac{(\log(Y_i) - \gamma X_i)^2}{\sigma^6}$$

$$\Rightarrow -E\left[\frac{1}{2\sigma^4} - \frac{(\log(Y_i) - \gamma X_i)^2}{\sigma^6}\right] = -\frac{1}{2\sigma^4} + \frac{E[(\log(Y_i) - \gamma X_i)^2]}{\sigma^6}$$

Since $\log(Y_i) = \gamma X_i + N(0, \sigma^2)$

$\Rightarrow \log(Y_i) - \gamma X_i = N(0, \sigma^2)$

$\Rightarrow E[(\log(Y_i) - \gamma X_i)^2] = Var[(\log(Y_i) - \gamma X_i)^2] - (E[\overset{0}{\cancel{(\log(Y_i) - \gamma X_i)}}])^2 = \sigma^2$

$$= -\frac{1}{2\sigma^4} + \frac{\sigma^2}{\sigma^6}$$

$$= -\frac{1}{2\sigma^4} + \frac{1}{\sigma^4} \cdot \frac{2}{2}$$

$$= \frac{-1+2}{2\sigma^4} = \frac{1}{2\sigma^4}$$

$$\frac{\partial^2 l}{\partial \gamma \sigma^2} = \frac{\partial}{\partial \sigma^2}\left[\frac{(\log(Y_i) - \gamma X_i)(X_i)}{\sigma^2}\right] = -\frac{(\log(Y_i) - \gamma X_i)(X_i)}{\sigma^4} \Rightarrow -E\left[-\frac{(\log(Y_i) - \gamma X_i)(X_i)}{\sigma^4}\right]$$

$$= \frac{1}{\sigma^4}\Big[\underbrace{E[E[\log(Y_i)X_i | X_i]]}_{} - \gamma \overset{1}{\cancel{E[X_i^2]}}\Big] = \frac{1}{\sigma^4}[\gamma - \gamma] = 0$$

$$= E[X_i \underbrace{E[\log(Y_i)|X_i]}_{\gamma X_i}] = \gamma \overset{1}{\cancel{E[X_i^2]}}$$

$$\Rightarrow I_1(\gamma, \sigma^2)^{-1} = \begin{bmatrix} \sigma^2 & 0 \\ 0 & 2\sigma^4 \end{bmatrix}$$

$$\Rightarrow \sqrt{n}\left(\begin{pmatrix} \hat{\gamma} \\ \hat{\sigma}^2 \end{pmatrix} - \begin{pmatrix} \gamma \\ \sigma^2 \end{pmatrix}\right) \xrightarrow{d} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2 & 0 \\ 0 & 2\sigma^4 \end{pmatrix}\right)$$

cont'd. $\longrightarrow$

1 c) cont'd.

⑥ Then, we can now use delta method to get the resulting distn.

$$\sqrt{n}\left(g(\hat{\gamma},\hat{\sigma}^2) - g(\gamma,\sigma^2)\right) \xrightarrow{d} N\left(\begin{pmatrix}0\\0\end{pmatrix}, \nabla g' I_1^{-1} \nabla g\right)$$

where $g(a,b) = \begin{pmatrix} \exp\{\frac{1}{2}(a^2+b)\} \\ a\exp\{\frac{1}{2}(a^2+b)\} \end{pmatrix} \Rightarrow \nabla g(a,b) = \begin{pmatrix} \frac{\partial}{\partial a}[\exp\{\frac{1}{2}(a^2+b)\}] & \frac{\partial}{\partial b}[\exp\{\frac{1}{2}(a^2+b)\}] \\ \frac{\partial}{\partial a}\underbrace{[a\exp\{\frac{1}{2}(a^2+b)\}]} & \frac{\partial}{\partial b}[a\exp\{\frac{1}{2}(a^2+b)\}] \end{pmatrix}$

$$a e^{\frac{1}{2}(a^2+b)}$$

$\downarrow$ derivative via chain rule

$$= e^{\frac{1}{2}(a^2+b)} + a^2 e^{\frac{1}{2}(a^2+b)}$$

$$= (1+a^2)e^{\frac{1}{2}(a^2+b)}$$

$$= \begin{pmatrix} a\exp\{\frac{1}{2}(a^2+b)\} & \frac{1}{2}\exp\{\frac{1}{2}(a^2+b)\} \\ (1+a^2)\exp\{\frac{1}{2}(a^2+b)\} & \frac{a}{2}\exp\{\frac{1}{2}(a^2+b)\} \end{pmatrix}$$

$\swarrow$ sub $a=\gamma$ and $b=\sigma^2$

$$= \begin{pmatrix} \gamma\exp\{\frac{1}{2}(\gamma^2+\sigma^2)\} & \frac{1}{2}\exp\{\frac{1}{2}(\gamma^2+\sigma^2)\} \\ (1+\gamma^2)\exp\{\frac{1}{2}(\gamma^2+\sigma^2)\} & \frac{\gamma}{2}\exp\{\frac{1}{2}(\gamma^2+\sigma^2)\} \end{pmatrix} = \exp\{\tfrac{1}{2}(\gamma^2+\sigma^2)\}\begin{pmatrix} \gamma & 1/2 \\ (1+\gamma^2) & \gamma/2 \end{pmatrix}$$

Then,

$$\nabla g' I_1(\gamma,\sigma^2)^{-1}\nabla g = \exp\{\gamma^2+\sigma^2\}\begin{pmatrix} \gamma & (1+\gamma^2) \\ 1/2 & \gamma/2 \end{pmatrix}\begin{pmatrix} \sigma^2 & 0 \\ 0 & 2\sigma^4 \end{pmatrix}\begin{pmatrix} \gamma & 1/2 \\ (1+\gamma^2) & \gamma/2 \end{pmatrix}$$

$$= \exp\{\gamma^2+\sigma^2\}\begin{pmatrix} \gamma\sigma^2 & 2\sigma^4(1+\gamma^2) \\ \frac{1}{2}\sigma^2 & \gamma\sigma^4 \end{pmatrix}\begin{pmatrix} \gamma & 1/2 \\ (1+\gamma^2) & \gamma/2 \end{pmatrix}$$

$$= \underbrace{\exp\{\gamma^2+\sigma^2\}\begin{pmatrix} [\gamma^2\sigma^2 + 2\sigma^4(1+\gamma^2)^2] & [\frac{1}{2}\gamma\sigma^2 + \gamma\sigma^4(1+\gamma^2)] \\ [\frac{1}{2}\gamma\sigma^2 \quad \gamma(1+\gamma^2)\sigma^4] & [\frac{1}{4}\sigma^2 + \frac{1}{2}\gamma^2\sigma^4] \end{pmatrix}}$$

Thus, $\sqrt{n}\left(\begin{pmatrix}\hat{\alpha}^*\\ \hat{\beta}^*\end{pmatrix} - \begin{pmatrix}\alpha^*\\ \beta^*\end{pmatrix}\right) \xrightarrow{d} N\left(\begin{pmatrix}0\\0\end{pmatrix}, \Sigma\right)$

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1.d) Calculate the asymptotic relative efficiency between the MLE for β^* and $\hat{\beta}$ given in b).

$$\gamma \exp\left\{\frac{\sigma^2+\gamma^2}{2}\right\} \text{ from c)}$$

In b), $\hat{\beta} = \frac{1}{n}\sum_i X_i Y_i$ where $Var(\hat{\beta}) = \frac{1}{\cancel{n^2}} \cancel{n} \cdot \left[E[X_i^2 Y_i^2] - \left(E[X_i Y_i]\right)^2\right]$ (*)

In c), $\hat{\beta}^* = \hat{\gamma} \exp\{\frac{1}{2}(\hat{\sigma}^2+\hat{\gamma}^2)\}$ where $Var(\hat{\beta}^*) = \frac{1}{n}\exp\{\gamma^2+\sigma^2\}\left[\frac{1}{4}\sigma^2 + \frac{1}{2}\gamma^2\sigma^4\right]$

$$ARE(\hat{\beta}^*, \hat{\beta}) = \frac{Var(\hat{\beta})}{Var(\hat{\beta}^*)}$$

In (*), need to find $E[X_i^2 Y_i^2] = E[E[X_i^2 Y_i^2 | X_i]] = E[X_i^2 E[Y_i^2 | X_i]]$

From c), had $E[Y^t | X] = \exp\{\gamma X t + \sigma^2 t^2/2\} \Rightarrow E[Y_i^2 | X_i] = \exp\{\gamma X_i 2 + \sigma^2 2^2/2\}$

$$\Rightarrow E[X_i^2 Y_i^2] = E[X_i^2 \cdot e^{2\gamma X_i + 2\sigma^2}] = e^{2\sigma^2} E[X_i^2 e^{2\gamma X_i}] = e^{2\sigma^2}\int_{-\infty}^{\infty} X_i^2 \frac{1}{\sqrt{2\pi}} e^{-X_i^2/2} \cdot e^{2\gamma X_i} dX_i$$

$$= e^{2\sigma^2}\int_{-\infty}^{\infty} X_i^2 e^{\frac{-X_i^2+4\gamma X_i}{2}} dX_i = e^{2\sigma^2+2\gamma^2}\int_{-\infty}^{\infty} X_i^2 e^{\frac{-(X_i^2-4\gamma X_i+4\gamma^2)}{2}} dX_i$$

$$= e^{2\sigma^2+2\gamma^2}\underbrace{\int_{-\infty}^{\infty} X_i^2 e^{\frac{-(X_i-2\gamma)^2}{2}} dX_i}_{= E[X_i^2] \text{ for } X_i \sim N(2\gamma, 1)}$$

b/c $\frac{(X_i - 2\gamma)^2}{1} \sim \chi^2_1$

Then, $E[X_i^2] = E\left[\underbrace{\frac{(X_i-2\gamma)^2}{1}}_{E[X_i^2 \underbrace{-4\gamma X_i + 4\gamma^2}_{\text{add in}}]}\right] + \underbrace{4\gamma E[X_i] - 4\gamma^2}_{\text{compensate}} = 1 + 4\gamma \cdot 2\gamma - 4\gamma^2 = 1 + 8\gamma^2 - 4\gamma^2 = 1 + 4\gamma^2$

$$= e^{2\sigma^2+2\gamma^2} \cdot (1+4\gamma^2) = (1+4\gamma^2)e^{2(\sigma^2+\gamma^2)} \overset{(*)}{\Rightarrow} Var(\hat{\beta}) = (1+4\gamma^2)e^{2(\sigma^2+\gamma^2)} - \gamma^2 e^{(\sigma^2+\gamma^2)}$$

$$\text{Then, } ARE(\hat{\beta}^*, \hat{\beta}) = \frac{Var(\hat{\beta})}{Var(\hat{\beta}^*)} = \frac{\cancel{n}(1+4\gamma^2)e^{2(\sigma^2+\gamma^2)} - \gamma^2 e^{2(\sigma^2+\gamma^2)}}{\cancel{n}\frac{1}{2}(\frac{1}{2}\sigma^2 + \gamma^2\sigma^4)e^{\gamma^2+\sigma^2}}$$

$$= \frac{(1+3\gamma^2)e^{\cancel{2}(\sigma^2+\gamma^2)}}{\frac{1}{2}(\frac{1}{2}\sigma^2+\gamma^2\sigma^4)\cancel{e^{\gamma^2+\sigma^2}}} = \boxed{\frac{(1+3\gamma^2)}{\frac{1}{2}(\frac{1}{2}\sigma^2+\gamma^2\sigma^4)} e^{\sigma^2+\gamma^2}}$$

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1.e) If we allow the prediction function to be arbitrary, that is, we aim to find the best fcn, $g(X)$, to minimize

$$E[\{Y - g(X)\}^2],$$

What is the optimal $g(X)$ in terms of (γ, σ^2)?

Hint: Consider minimization conditional on X.

Let $h(x, Y) = E[\{Y - g(X)\}^2 | X] = E[Y^2 - 2Yg(X) + g(X)^2 | X]$

$= E[Y^2 | X] - 2g(X) \cdot E[Y|X] + g(X)^2$

$\Rightarrow \dfrac{\partial h}{\partial g(X)} = -2E[Y|X] + 2g(X) \overset{\text{set}}{=} 0 \Rightarrow \hat{g}(X) = E[Y|X]$ ← optimal $g(X)$

$= \exp\{\gamma X + \sigma^2/2\}$