2016 Qual, Section 1

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3. (25 points) Suppose $S \sim \text{Binomial}(n, p)$ and conditional on $S = s$, let $X_1, \dots, X_{s+1}$ be iid from a $N(\mu, 1)$ distribution. The value of n is known whereas (p, μ) are both unknown, $0 < p < 1$, and $-\infty < \mu < \infty$. We observe $(S, X_1, \dots, X_{S+1})$ and we wish to test $H_0 : \mu \leq 0$ versus $H_1 : \mu > 0$ at level α.

   (a) (5 points) Write out the joint density of $(S, X_1, \dots, X_{S+1})$ and show that it belongs to a full-rank exponential family, and find the two dimensional complete sufficient statistic. Do the same thing for the special case that $\mu = 0$.

   (b) (3 points) Derive the joint MLE's of (p, μ), denoted by $(\hat{p}, \hat{\mu})$.

   (c) (5 points) Assuming that standard MLE theory applies, derive the joint asymptotic distribution of $(\hat{p}, \hat{\mu})$, properly normalized.

   (d) (6 points) Let $\phi(S, X_1, \dots, X_{S+1})$ be *any* unbiased level α test of H_0 versus H_1. Write out what unbiasedness means for the power function $\beta(p, \mu)$ of such a test, and explain in detail why unbiasedness implies that $\beta(p, 0) = \alpha$ for all p.

   (e) (6 points) Find the complete form of the UMPU test of H_0 versus H_1, including specification of the rejection region in terms of the sample mean of the X_i's and the $1 - \alpha$ quantile of a well known distribution.

3. Given $S \sim Binom(n,p)$ & $X_i | S \overset{iid}{\sim} N(\mu, 1)$ for $i = 1, \ldots, S+1$.

Here ; n: Known
(p,μ) both unknown
$0 < p < 1$
$-\infty < \mu < \infty$

Observe $(S, X_1, \ldots, X_{S+1})$

Want to test $H_0: \mu \le 0$ vs. $H_1: \mu > 0$ at level α.

a) Write the joint density of $(S, X_1, \ldots, X_{S+1})$ and show that it belongs to a full rank exp family and find the 2D CSS.

Do the same for the special case that $\mu = 0$

[Given $S \sim Binom(n,p)$
$X_i | S \overset{iid}{\sim} N(\mu, 1)$

By Bayes Thm, $f(\underset{\sim}{x}, s) = f(\underset{\sim}{x} | s) \cdot f(s) = \prod_{i=1}^{s+1} \left[\frac{1}{\sqrt{2\pi}} e^{-\frac{(x_i - \mu)^2}{2}} \right] \cdot \binom{n}{s} p^s (1-p)^{n-s}$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^{s+1} e^{-\sum_{i=1}^{s+1} (x_i - \mu)^2 / 2} \cdot \binom{n}{s} \cdot p^s \cdot (1-p)^{n-s}$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^{s+1} \cdot \binom{n}{s} \exp\left\{ -\frac{1}{2} \sum_{i=1}^{s+1} (x_i - \mu)^2 + s \log(p) + (n-s)\log(1-p) \right\}$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^{s+1} \cdot \binom{n}{s} \cdot \exp\left\{ -\frac{1}{2} \sum_{i=1}^{s+1} x_i^2 + \sum_{i=1}^{s+1} x_i \mu - \underbrace{\overbrace{\frac{(s+1)}{2}\mu^2}^{= -\frac{s}{2}\mu^2 - \frac{1}{2}\mu^2} + s\,\text{logit}(p) + n \log(1-p)} \right\} \quad s(\text{logit}(p) - \tfrac{1}{2}\mu^2)$$

$$= \underbrace{\left(\frac{1}{\sqrt{2\pi}}\right)^{s+1} \cdot \binom{n}{s} \cdot \exp\left\{ -\frac{1}{2} \sum_{i=1}^{s+1} x_i^2 \right\}}_{h(s, \bar{x})} \underbrace{\exp\left\{ -\frac{1}{2}\mu^2 + n\log(1-p) \right\}}_{c(\theta) = c(\mu, p)} \underbrace{\exp\left\{ \mu (s+1) \bar{x} - \frac{s}{2}\mu^2 + s\,\text{logit}(p) \right\}}_{\exp\{\theta u - \sum_{i=1}^{k} \xi_i T_i\}}$$

$\Rightarrow$ θ = parameter of interest = μ ✓

U = CSS for parameter of interest = $(s+1)\bar{x}$ ✓

ξ = nuisance parameter = $\text{logit}(p) - \frac{1}{2}\mu^2$ ✓

T = CSS for nuisance parameter = s ✓

Since $\mu \in (-\infty, \infty)$ and $\xi = \underbrace{\text{logit}(p)}_{\in(-\infty,\infty)} - \underbrace{\frac{1}{2}\mu^2}_{\in(-\infty, 0)} \in (-\infty, \infty) \Rightarrow \mu \times \xi \overset{\text{obviously}}{=} (-\infty,\infty) \times (-\infty, \infty)$
$= \mathbb{R}^2$ contains an open set in $\mathbb{R}^2$

$\Rightarrow f(\underset{\sim}{x}, s)$ is a member of the full rank (rank = 2) multiparameter exponential family with 2D CSS = $((s+1)\bar{x}, s)$

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3 a) cont'd.

For the special case that $\mu = 0$, have to sub $\mu = 0$ into the previous pdf to get

$$f(\underset{\sim}{x}, s) = \underbrace{\left(\frac{1}{\sqrt{2\pi}}\right)^{s+1} \binom{n}{s} \exp\left\{-\frac{1}{2}\sum_{i=1}^{s+1} x_i^2\right\}}_{h(s,\bar{x})} \underbrace{\exp\left\{n\log(1-p)\right\}}_{c(\theta)=c(p)} \underbrace{\exp\left\{s\,\text{logit}(p)\right\}}_{\exp\left\{-\sum_{i=1}^{k} \xi_i T_i\right\}}$$

$\Rightarrow$ θ = parameter of interest $= 0$

ξ = nuisance parameter = logit(p)

T = CSS for nuisance parameter $= s$

$\rightarrow$ CSS is 1D in this case and is given by $T = s$ $\rfloor$

3 b) Derive the joint MLEs of (p, μ), denoted by $(\hat{p}, \hat{\mu})$. AMW

From a), know

$$f(\underset{\sim}{x}, s) = \left(\frac{1}{\sqrt{2\pi}}\right)^{s+1} \binom{n}{s} \exp\left\{-\frac{1}{2}\sum_{i=1}^{s+1} x_i^2\right\} \exp\left\{-\frac{1}{2}\mu^2 + n\log(1-p)\right\} \exp\left\{\mu(s+1)\bar{x} + s(\underbrace{\text{logit}(p)}_{\log(p) - \log(1-p)} - \tfrac{1}{2}\mu^2)\right\}$$

$$\Rightarrow \mathcal{L}(p, \mu \mid \underset{\sim}{x}, s) \propto \exp\left\{-\tfrac{1}{2}\mu^2 + n\log(1-p)\right\} \exp\left\{\mu(s+1)\bar{x} + s(\text{logit}(p) - \tfrac{1}{2}\mu^2)\right\}$$

$$\Rightarrow \ell(p, \mu \mid \underset{\sim}{x}, s) = -\tfrac{1}{2}\mu^2 + n\log(1-p) + \mu(s+1)\bar{x} + s(\log(p) - \log(1-p) - \tfrac{1}{2}\mu^2)$$

$$\Rightarrow \frac{\partial \ell}{\partial p} = \frac{-n}{1-p} + s\left(\frac{1}{p} + \frac{1}{1-p}\right) = \frac{-n}{1-p} + \underbrace{s\left(\frac{1-\cancel{p}+\cancel{p}}{p(1-p)}\right)}_{s(p-p^2)^{-1}} \Rightarrow \hat{p} = \frac{s}{n}$$

Since
$$\left.\frac{\partial^2 \ell}{\partial p^2}\right|_{p=\hat{p}} = \frac{-n}{(1-p)^2} - s(p-p^2)^{-2}(1-2p) = \left.\frac{-n}{(1-p)^2} - \frac{s(1-2p)}{(p-p^2)^2}\right|_{p=\hat{p}}$$

$$= \frac{-n}{(1-\hat{p})^2} - \frac{s(1-2\hat{p})}{(\hat{p}-\hat{p}^2)^2} = \frac{-n}{(1-\frac{s}{n})^2} - \frac{s(1-2(\frac{s}{n}))}{(\frac{s}{n}-\frac{s^2}{n^2})^2} = \underbrace{\frac{-n}{(1-\frac{s}{n})^2}}_{<0} - \underbrace{\frac{(\overset{>0}{s} + \overset{>0}{2s^2/n})}{(\frac{s}{n}-\frac{s^2}{n^2})^2}}_{<0} \Rightarrow \left.\frac{\partial^2 \ell}{\partial p^2}\right|_{p=\hat{p}} < 0$$

$\Rightarrow \hat{p}$ occurs @ a max.

Similarly,
$$\frac{\partial \ell}{\partial \mu} = -\mu + (s+1)\bar{x} - s\mu \overset{\text{set}}{=} 0 \Rightarrow -\mu + s\bar{x} + \bar{x} - s\mu = 0$$
$$\Rightarrow \mu(1+s) = (1+s)\bar{x}$$
$$\Rightarrow \mu = \bar{x}$$

Since $\left.\frac{\partial^2 \ell}{\partial \mu^2}\right|_{\mu=\hat{\mu}} = -1-s < 0 \Rightarrow \hat{\mu} = \bar{x}$ occurs @ a max.

Then, $\boxed{(\hat{p}, \hat{\mu}) = \left(\frac{s}{n}, \bar{x}\right)}$

3c) Assuming that standard MLE theory applies, derive the joint distn. of $(\hat{p}, \hat{\mu})$ properly normalized.

⌈ Told to assume standard MLE theory applies (i.e., $\log f(p,\mu)$ must be twice continuously differentiable)

Know $\sqrt{n\, I_1(p,\mu)}\left(\begin{pmatrix}\hat{p}\\ \hat{\mu}\end{pmatrix} - \begin{pmatrix}p\\ \mu\end{pmatrix}\right) \xrightarrow{d} N\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)$

$$\Rightarrow \sqrt{n}\left(\begin{pmatrix}\hat{p}\\ \hat{\mu}\end{pmatrix} - \begin{pmatrix}p\\ \mu\end{pmatrix}\right) \xrightarrow{d} N\left(\begin{pmatrix}0\\0\end{pmatrix}, \underbrace{I_1^{-1}(p,\mu)}_{\text{inv fisher info for a single obs.}}\right)$$

From b), $\frac{\partial l}{\partial p} = \frac{-n}{1-p} + \frac{s}{p(1-p)}$ $\qquad \frac{\partial l}{\partial \mu} = -(s+1)\mu + (s+1)\bar{x} = (s+1)(\bar{x}-\mu)$

$\frac{\partial^2 l}{\partial p^2} = \frac{-n}{(1-p)^2} - \frac{s(1-2p)}{p^2(1-p)^2}$ $\qquad \frac{\partial^2 l}{\partial \mu^2} = -1-s$

$$\frac{\partial l}{\partial p\, \partial \mu} = \frac{\partial}{\partial p}\left((s+1)(\bar{x}-\mu)\right) = 0$$

$$\Rightarrow I_n^{-1}(p,\mu) = \begin{bmatrix} \frac{n}{(1-p)^2} + \frac{E(S)(1-2p)}{p^2(1-p)^2} & 0 \\ 0 & 1+E(S)\end{bmatrix}^{-1} = \begin{bmatrix} \frac{n}{(1-p)^2} + \frac{n\cancel{p}(1-2p)}{p^{\cancel{2}}(1-p)^2} & 0 \\ 0 & 1+np\end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \frac{np+n-2np}{p(1-p)^2} & 0 \\ 0 & 1+np\end{bmatrix}^{-1} = \begin{bmatrix} \frac{n\cancel{(1-p)}}{p(1-p)^{\cancel{2}}} & 0 \\ 0 & 1+np\end{bmatrix}^{-1} = \underbrace{\begin{bmatrix} \frac{n}{p(1-p)} & 0 \\ 0 & 1+np\end{bmatrix}}_{\text{diagonal matrix}}^{-1}$$

$$= \begin{bmatrix} \frac{p(1-p)}{n} & 0 \\ 0 & (1+np)^{-1}\end{bmatrix}$$

Recall that $I_n(\theta) = n I_1(\theta) \Rightarrow I_n^{-1}(\theta) = \frac{1}{n} I_1^{-1}(\theta) \Rightarrow I_1^{-1}(\theta) = n I_n^{-1}(\theta)$

$$\Rightarrow I_1^{-1}(p,\mu) = n\begin{bmatrix} \frac{p(1-p)}{n} & 0 \\ 0 & \frac{1}{(1+np)}\end{bmatrix} = \begin{bmatrix} p(1-p) & 0 \\ 0 & \frac{1}{1/n + p}\end{bmatrix} \longrightarrow \underbrace{\begin{bmatrix} p(1-p) & 0 \\ 0 & \frac{1}{p}\end{bmatrix}}_{\text{when we move to RHS, cannot depend on } n}$$

Thus, $\boxed{\sqrt{n}\left(\begin{pmatrix}\hat{p}\\ \hat{\mu}\end{pmatrix} - \begin{pmatrix}p\\ \mu\end{pmatrix}\right) \xrightarrow{d} N\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}p(1-p) & 0\\ 0 & \frac{1}{p}\end{pmatrix}\right)}$ ⌋

3 d) Let $\phi(S, X_1, \ldots, X_{S+1})$ be any unbiased level α test of H_0 vs. H_1.

Write out what unbiasedness means for the power function $\beta(p,\mu)$ of such a test, and explain in detail why unbiasedness implies that $\beta(p,0) = \alpha \; \forall p$.

⌈ Recall (slide 306) that multiparameter exponential families have the property that the power function, $\beta_\phi(\theta)$ for a test ϕ is continuous in θ $\forall \phi$.

Now, suppose that ϕ is unbiased. Then,

- $\beta_\phi(\theta) \leq \alpha$ for $\theta \in \Theta_0$
- $\beta_\phi(\theta) \geq \alpha$ for $\theta \in \Theta_1$

Define $\Theta_B := \{\theta = (p,\mu) : \mu = 0\}$

Fix a value of p, call it p^*.

By defn. of continuity and existence of a limit, it must hold that $\theta \to (p^*, 0)$ from the left and right of the boundary, the power approaches a value β^*. This result, in combination with the defn. of unbiasedness, guarantees that $\beta^* = \alpha$. Since p^* was arbitrary, this holds across the entire boundary.

More specifically, if $\beta^* < \alpha$, the test could not be both continuous and unbiased as required. We see this since the limit on the RHS of the boundary would approach a value greater than or equal to α. However, the value at the boundary would be less than α. Thus, we have a contradiction to the assumption of continuity.

Were $\beta^* > \alpha$, we again would not meet the defn. of unbiasedness by similar argument and thus again arrive at a contradiction.

Thus, $\beta(p,0) = \alpha$. ⌋

3e) Find the complete form of the UMPU test of H_0 vs. H_1, including specification of the rejection region in terms of the sample mean of the X_i's and the $1-\alpha$ quantile of a well known distn.

⌈ From 3a), know

U = CSS for parameter of interest $(s+1)\bar{X}$

T = CSS for nuisance parameter $= S$.

Then, the UMPU level α test is of the form

$$\phi(u) = \begin{cases} 1 & \text{if } U > c(T) \\ 0 & \text{if } U \le c(T) \end{cases} = \begin{cases} 1 & \text{if } (s+1)\bar{X} > c(s) \\ 0 & \text{if } (s+1)\bar{X} < c(s) \end{cases}$$

where $\alpha = E_0[\phi(u) \mid T] = E_0[\phi[(s+1)\bar{x}] \mid S=s]$

$= 1 \cdot P_0((s+1)\bar{x} > \underbrace{c(s)}_{\text{call this } c} \mid S=s) = 1 - P_0((s+1)\bar{x} \le c \mid S=s)$

Since $X_i \mid S=s \sim N(\mu, 1) \Rightarrow \frac{(\bar{X}-\mu)}{1} \mid S=s \sim N(0,1)$.

Since, under H_0, $\mu = 0 \Rightarrow \bar{X} \mid S=s \sim N(0,1) \Rightarrow (s+1)\bar{X} \mid S=s \sim N(0, (s+1)^2)$

Then, $\alpha = 1 - (s+1)\Phi(c) \Rightarrow (s+1)\Phi(c) = 1-\alpha$

$\Rightarrow \Phi(c) = \frac{1-\alpha}{(s+1)} \Rightarrow c = \Phi^{-1}\left(\frac{1-\alpha}{(s+1)}\right)$

Thus, $$\phi((s+1)\bar{x}) = \begin{cases} 1 & \text{if } (s+1)\bar{x} > \Phi^{-1}\left(\frac{1-\alpha}{(s+1)}\right) \\ 0 & \text{if } (s+1)\bar{x} < \Phi^{-1}\left(\frac{1-\alpha}{(s+1)}\right) \end{cases}$$

for $\Phi^{-1}(\cdot)$ denoting the inverse cdf of a $N(0,1)$ RV. ⌋