2) a. Clearly, $\begin{pmatrix}1\\1\end{pmatrix} \in C(X^T)$. So $\begin{pmatrix}1\\-2\end{pmatrix} \in C(X^T)$

Also, $\begin{pmatrix}1\\-2\end{pmatrix} = 4\cdot\begin{pmatrix}1\\1\end{pmatrix} - 3\cdot\begin{pmatrix}1\\2\end{pmatrix}$.

So $\lambda \in C(X^T)$ and $\Lambda^T\beta$ is estimable.

{alternatively If we can find a P st $P^TX = \Lambda^T$ then $\Lambda^T\beta$ is est}

b) UMVUE for $\Lambda^T\beta$ is P^TMY

$$P^TMY = \frac{1}{11}\begin{pmatrix}-y_1 + 5y_2 + 5y_3 + 2y_4\\ -16y_1 + 14y_2 + 14y_3 - y_4\end{pmatrix}$$

$$M = \begin{pmatrix}9 & -1 & -1 & 4\\ -1 & 5 & 5 & 2\\ -1 & 5 & 5 & 2\\ 4 & 2 & 2 & 3\end{pmatrix}$$

c) $Y \sim N_4(X\beta, \sigma^2 I_4)$ $P'MY$ lin trans of Y

$$E(P'MY) = P'ME(Y) = P'MX\beta = P'X\beta = \Lambda^T\beta$$

$$Cov(P'MY) = (P'M)Cov(Y)(M'P) = \sigma^2(P'M)(M'P)$$

$$= \frac{\sigma^2}{121}\begin{pmatrix}55 & 154\\ 154 & 649\end{pmatrix} = \Sigma$$

$$\Rightarrow P'MY \sim N(\Lambda^T\beta, \Sigma)$$

d) M2 : $V^{-1/2}Y = (V^{-1/2}X)\beta + V^{-1/2}\varepsilon$

$\Leftrightarrow Y^* = X^*\beta + \varepsilon^*$

e) suppose $\lambda^T\beta$ is estimable in M1

$\Leftrightarrow \exists c$ st $\lambda' = c'X$

$\Leftrightarrow \lambda' = c'\underbrace{(V^{1/2}V^{-1/2})}_{I}X$

$\Leftrightarrow \lambda' = (V^{1/2}c)^T(V^{-1/2}X) = c_2^T X^*$

Thus $\lambda^T\beta$ is estimable in M(2)

$\equiv N_4(\underset{\sim}{0}, I)$. $\leftarrow \Sigma^{-1/2}Y$ is a 4-dim. vector of indep. std normal RVs.

all χ^2's here are central. → We know that the square of a std. normal RV has χ^2_1 distr'n, and that the sum of n indep. χ^2_1 RVs is χ^2_n.

Established above, just defining notation here. → Let $\Sigma^{-1/2}Y = (y_1^*, y_2^*, y_3^*, y_4^*)^T$ where $y_i^* \sim N(0,1)$ $\forall i = 1, \dots, 4$.

Then,

$$Y^T \Sigma^{-1} Y = (\Sigma^{-1/2}Y)^T(\Sigma^{-1/2}Y) = (y_1^*, y_2^*, y_3^*, y_4^*)\begin{pmatrix} y_1^* \\ y_2^* \\ y_3^* \\ y_4^* \end{pmatrix}$$

$$= y_1^{*2} + y_2^{*2} + y_3^{*2} + y_4^{*2}$$