2015 D1

1) c) $P(Y_n > y, Z_n > z) = \left[\left(1-\frac{z}{n}\right)^{\alpha} - \frac{y^{\alpha}}{n}\right]^n$

(we know: $\left(1-\frac{z}{n}\right)^{\alpha n} \longrightarrow e^{-\alpha z}$

$1-\frac{y^{\alpha}}{n} \longrightarrow e^{-y^{\alpha}}$)

$$\left(1-\frac{z}{n}\right)^{\alpha n} = \left[\left(1-\frac{z}{n}\right)^n\right]^{\alpha} \xrightarrow[n\to\infty]{} \left(e^{-z}\right)^{\alpha} = e^{-z\alpha}$$

$$= \left[\left[\left(1-\frac{z}{n}\right)^n\right]^{\frac{\alpha}{n}} - \frac{y^{\alpha}}{n}\right]^n$$

$$\approx \left[\left(e^{-z}\right)^{\frac{\alpha}{n}} - \frac{y^{\alpha}}{n}\right]^n$$

$$= \left[\left(e^{-z}\right)^{\frac{\alpha}{n}} - \frac{y^{\alpha}}{n} e^{\frac{\alpha z}{n}} \cdot e^{-\frac{\alpha z}{n}}\right]^n$$

$$= \left[e^{-\frac{z\alpha}{n}} - \frac{y^{\alpha}}{n} e^{\frac{\alpha z}{n}} e^{-\frac{\alpha z}{n}}\right]^n$$

$$= e^{-z\alpha}\left[1 - \frac{y^{\alpha}}{n} e^{\frac{\alpha z}{n}}\right]^n$$

$$\approx e^{-z\alpha}\left[1 - \frac{y^{\alpha}}{n}\right]^n \quad \text{since } \frac{\alpha z}{n} \to 0$$

$$\lim_{n\to\infty} e^{-z\alpha}\left[1 - \frac{y^{\alpha}}{n}\right]^n = e^{-z\alpha} e^{-y^{\alpha}}$$

$$= e^{-z\alpha - y^{\alpha}}$$

$Y_n = n^{1/\alpha}(\tilde{\mu}_n - \mu) = n^{1/\alpha}(X_{(1)} - \mu)$.

$n \geq 1 \Rightarrow n^{1/\alpha} \geq 1$ and $X_{(1)} \geq \mu$ by the support of X.

So $Y_n \geq 0$ a.s. for $n \geq 1$.

$Z_n = n(\mu - \hat{\mu}_n) = n(\mu + 1 - X_{(n)})$. $X_{(n)} \leq \mu + 1$ by the support of X.

So $Z_n \geq 0$ a.s. for $n \geq 1$.

2015 D1

1) d) $0 \le \frac{1}{\tilde{\alpha}_n} - \frac{1}{\hat{\alpha}_n} \le C_r |\hat{\mu}_n - \mu|^r \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu)^{-r}$

$r = \min\{s, 1\}$

$$\frac{\hat{\alpha}_n - \tilde{\alpha}_n}{\tilde{\alpha}_n \hat{\alpha}_n} \le C_r |\hat{\mu}_n - \mu|^r \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu)^{-r}$$

$$\hat{\alpha}_n - \tilde{\alpha}_n \le (\tilde{\alpha}_n \hat{\alpha}_n) C_r \underbrace{|\hat{\mu}_n - \mu|^r}_{\star} \left[O_p(1) \right]^{*}$$

by LLN by bound in a)

★ WTS $\hat{\mu}_n \to_p \mu$

Let $\varepsilon > 0$

$$\begin{aligned} P(|\hat{\mu}_n - \mu| > \varepsilon) &= P(|X_{(n)} - 1 - \mu| > \varepsilon) \\ &= P(|X_{(n)} - (\mu+1)| > \varepsilon) \\ &= P(X_{(n)} - (\mu+1) > \varepsilon) + P(X_{(n)} - (\mu+1) < -\varepsilon) \\ &= P(X_{(n)} > \varepsilon + (\mu+1)) + P(X_{(n)} < (\mu+1) - \varepsilon) \\ &= 0 + \int^{\mu+1-\varepsilon} n\alpha (x-\mu)^{n\alpha - 1} dx \\ &= (x-\mu)^{n\alpha} \Big|_{\mu}^{\mu+1-\varepsilon} = (\underbrace{1-\varepsilon}_{<1})^{n\alpha} \xrightarrow[n\to\infty]{} 0 \end{aligned}$$

$\hat{\mu}_n - \mu \to_p 0$

$\Rightarrow |\hat{\mu}_n - \mu|^r \to_p 0$

$\hat{\mu}_n - \mu = o_p(1)$

$\Rightarrow |\hat{\mu}_n - \mu| \operatorname{sign}(\hat{\mu}_n - \mu) = o_p(1)$

$\Rightarrow |\hat{\mu}_n - \mu| = o_p(\operatorname{sign}(\hat{\mu}_n - \mu) \cdot 1) = o_p(1)$

* by LLN

$\frac{1}{n} \sum (X_i - \mu)^{-r} \to_p E[(X_i - \alpha)^{-r}] < M$ (by (a)

so $\underbrace{\frac{1}{n} \sum (X_i - \mu)^{-r} - E[(X_i - \mu)^{-r}]}_{= o_p(1)} + \underbrace{E[(X_i - \mu)^{-r}]}_{O(1)}$

$o_p(1) + O(1) = O_p(1)$

$\sqrt{n}(\tilde{\alpha}_n - \alpha) \to_d N(0, \alpha^2)$

$\sqrt{n}(\tilde{\alpha}_n - \alpha) = O_p(1)$

$\tilde{\alpha}_n - \alpha = O_p(1/\sqrt{n})$

$\tilde{\alpha}_n = O_p(\frac{1}{\sqrt{n}}) + \alpha$

(i) $\tilde{\alpha}_n^{-1} - \hat{\alpha}_n^{-1} = O_p(n^{-(s\wedge 1)}) \Rightarrow$ (ii) $\tilde{\alpha}_n - \hat{\alpha}_n = O_p(n^{-(s\wedge 1)})$

$$\hat{\alpha}^{-1} = \tilde{\alpha}_n^{-1} - O_p(n^{-r}) \Rightarrow \hat{\alpha}_n = \frac{1}{\tilde{\alpha}_n^{-1} - O_p(n^{-r})}$$

$\tilde{\alpha}_n = \alpha + O_p(1)$ since MLE

$$\Rightarrow \tilde{\alpha}_n^{-1} = \frac{1}{\alpha + O_p(1)} \overset{?}{=} \frac{1}{\alpha} + O_p(1)$$

$$\hat{\alpha}_n = \frac{1}{\frac{1}{\alpha} + O_p(1) - O_p(n^{-r})} = \frac{1}{\alpha^{-1} + O_p(n^{-r})}$$

$$= \alpha \frac{1}{\alpha^{-1} + O_p(n^{-r})} + O_p(n^{-r})$$

$$\tilde{\alpha}_n - \hat{\alpha}_n = O_p(n^{-r})$$

$C_r = O(1)$

$*$ $\frac{1}{n}\sum_{i=1}(x_i - \mu)^{-r} = O_p(1)$

wts $|\hat{\mu}_n - \mu|^r = O_p(n^{-r})$

$\forall \varepsilon > 0$ $\iff |\hat{\mu}_n - \mu| = O_p(\frac{1}{n})$

$\iff n|\hat{\mu}_n - \mu| = O_p(1)$

$$P(n|\hat{\mu}_n - \mu| > \varepsilon) = P(|\hat{\mu}_n - \mu| > \tfrac{\varepsilon}{n}) \le \frac{E_1(\hat{\mu}_n - \mu)^2}{(\frac{\varepsilon}{n})^2} \text{ by } = \frac{n^2 E(\hat{\mu}_n - \mu)^2}{\varepsilon^2}$$

Markov $\hookrightarrow P(|X| > \varepsilon) \le \frac{E(g|X|)}{g(\varepsilon)}$

$$E(\hat{\mu}_n - \mu)^2 = E\Big[\underbrace{(X_{(n)} - 1 - \mu)^2}_{(X_{(n)} - (\mu+1))^2}\Big]$$

$$= E\left[X_{(n)}^2 - 2(\mu+1)X_{(n)} + (\mu+1)^2\right] = \frac{2}{(\alpha n+1)(\alpha n+2)}$$

$$E(X_{(n)}) = \int_\mu^{\mu+1} x n\alpha (x-\mu)^{\alpha n-1} dx = \quad \begin{matrix} \text{let } u = x & dv = \alpha n (x-\mu)^{\alpha n-1} dx \\ du = dx & v = (x-\mu)^{\alpha n} \end{matrix}$$

$$= x(x-\mu)^{\alpha n}\Big|_\mu^{\mu+1} - \int_\mu^{\mu+1} (x-\mu)^{\alpha n} dx$$

$$= \mu + 1 - 0 - \left[\frac{(x-\mu)^{\alpha n+1}}{\alpha n+1}\right]_\mu^{\mu+1} = \mu + 1 - \frac{1}{\alpha n+1} = \mu + \frac{\alpha n + 1 - 1}{\alpha n+1} = \mu + \frac{\alpha n}{\alpha n+1}$$

$$E(X_{(n)}^2) = \int_\mu^{\mu+1} x^2 n\alpha (x-\mu)^{\alpha n-1} dx \quad : \text{let } z = x - \mu \Rightarrow dz = dx \quad \Rightarrow x = z + \mu$$

$$= \int (z+\mu)^2 z^{\alpha n-1} dz = \int (z^2 + 2z\mu + \mu^2) z^{\alpha n-1} dz$$

$$= \int\left[z^{\alpha n+1} + 2\mu z^{\alpha n} + \mu^2 z^{\alpha n-1}\right] dz = \left[\frac{1}{\alpha n+2} z^{\alpha n+2} + \frac{2\mu}{\alpha n+1} z^{\alpha n+1} + \frac{\mu^2}{\alpha n} z^{\alpha n}\right]$$

$$= \left[\frac{1}{\alpha n+2}(x-\mu)^{\alpha n+2} + \frac{2\mu}{\alpha n+1}(x-\mu)^{\alpha n+1} + \frac{\mu^2}{\alpha n}(x-\mu)^{\alpha n}\right]_\mu^{\mu+1}$$

$$= \frac{1}{\alpha n+2} + \frac{2\mu}{\alpha n+1} + \frac{\mu^2}{\alpha n} =$$

$$\mu + \frac{n\alpha}{\alpha n+2}$$

$r = \min\{1, s\}$

e) Show for $\frac{1}{2} < \alpha < \infty$

$\sqrt{n}(\hat{\alpha}_n - \alpha) \to_d N(0, \alpha^2)$

from d) $\hat{\alpha}_n - \tilde{\alpha}_n = O_p(n^{-r})$

$\hat{\alpha}_n - \alpha + \alpha - \tilde{\alpha}_n = O_p(n^{-r})$

$\sqrt{n}(\hat{\alpha}_n - \alpha + \alpha - \tilde{\alpha}_n) = O_p(n^{-r+\frac{1}{2}})$

$\sqrt{n}(\hat{\alpha}_n - \alpha) - \sqrt{n}(\tilde{\alpha}_n - \alpha) = O_p(n^{-r+\frac{1}{2}})$

$\sqrt{n}(\hat{\alpha} - \alpha) =_d O_p(n^{-r+\frac{1}{2}}) + \underbrace{\sqrt{n}(\tilde{\alpha}_n - \alpha)}_{\to N(0, \alpha^2)}$

if μ known. Hope its normal if μ unknown

$\hat{\alpha}_n = \left[\frac{1}{n}\sum_{i=1}^{n} \ln(X_i - \mu)\right]^{-1}$

Let $Z_i = \ln(X_i - \mu)$ *Note: Z_i is a RV BUT not a statistic as μ is unknown

X_i iid $\Rightarrow$ Z_i are iid

$\Rightarrow$ Can use CLT to show $\sqrt{n}(\bar{Z} - E(Z_1)) \to_d N(0, \sigma_Z^2)$

$\sigma_Z^2 = Var(Z_1)$

Let $g(x) = x^{-1}$