2013 Theory I #3

3a) We can consider each flip that lands on side i as a success, and as a failure if it lands otherwise. Then the number of flips required until side i has appeared for the n_i^{th} time follows a negative binomial distribution with pmf

$$P(N_i = k) = \binom{k-1}{n_i - 1} p_i^{n_i} (1-p_i)^{k-n_i}, \qquad k = n_i, n_i + 1, \ldots$$

3b) Two r.v.'s X, Y are independent iff for all $C, D \in \mathcal{B}(\mathbb{R})$

$$P(X \in C, Y \in D) = P(X \in C) P(Y \in D)$$

We have

$$P(N_1 = n_1) = p_1^{n_1}$$

$$P(N_2 = n_2) = p_2^{n_2}$$

$$P(N_1 = n_1, N_2 = n_2) = 0$$

have to condition on something

3c) Let $N_i(t)$ represent the number of flips landing on side i by time t. By proposition 2.3.2 in Ross (pg 69), N_i is a Poisson process with rate $\lambda P_i = P_i$. Furthermore, by proposition 2.2.1 in Ross (pg. 64), the arrival time for each flip landing on side i follows an exponential distribution with mean P_i^{-1}. Thus T_i is the sum of n_i iid r.v.'s with mean P_i^{-1} so that $T_i \sim \text{gamma}(n_i, P_i)$

3d) Let $N_{tot}(t)$ denote the number of coin flips by time t. Let $W \sim \text{mult}(k, (P_1, \dots, P_r))$, where $k = \sum_{i=1}^{r} k_i$ for $k_i \in \mathbb{N}_0$, $i = 1, \dots, r$. Since the outcome of a coin flip is independent of the time of flip we have

$$P(N_1(t) = k_1, \dots, N_r(t) = k_r) = P(N_1(t) = k_1, \dots, N_r(t) = k_r \mid N_{tot}(t) = k) P(N_{tot}(t) = k)$$

$$= P(W = (k_1, \dots, k_r)) P(N_{tot}(t) = k) = \frac{k!}{k_1! \cdots k_r!} P_1^{k_1} \cdots P_r^{k_r} \frac{t^k e^{-t}}{k!}$$

$$= \frac{(P_1 t)^{k_1} e^{-P_1 t}}{k_1!} \cdots \frac{(P_r t)^{k_r} e^{-P_r t}}{k_r!}.$$

so we conclude that $N_1(t), \dots, N_r(t)$ are mutually independent Poisson processes with means $P_1 t, \dots, P_r t$, respectively. Thus for $(s_1, \dots, s_r) \in (0, \infty)^r$

$$P(T_1 < s_1, \dots, T_r < s_r) = P(N_1(s_1) \geq n_1, \dots, N_r(s_r) \geq n_r)$$

$$= P(N_1(s_1) \geq n_1) \cdots P(N_r(s_r) \geq n_r) = P(T_1 < s_1) \cdots P(T_r < s_r)$$

Then

$$p_{T_1,\dots,T_r}(s_1,\dots,s_r) = \frac{\partial^r P(T_1 < s_1, \dots, T_r < s_r)}{\partial s_1 \cdots \partial s_r}$$

$$= \frac{\partial^r P(T_1 < s_1) \cdots P(T_r < s_r)}{\partial s_1 \cdots \partial s_r}$$

$$= \frac{\partial P(T_1 < s_1)}{\partial s_1} \cdots \frac{\partial P(T_r < s_r)}{\partial s_r}$$

$$= p_{T_1}(s_1) \cdots p_{T_r}(s_r)$$

3e) $F_T(s) = P(T \le s) = 1 - P(T > s) = 1 - P(T_1 > s, \dots, T_r > s)$

$$= 1 - P(T_1 > s) \cdots P(T_r > s) = 1 - P(N_1(s) \le n_1 - 1) \cdots P(N_r(s) \le n_r - 1)$$

$$= 1 - \prod_{i=1}^{r} \sum_{k=0}^{n_i - 1} \frac{(p_i s)^k e^{-p_i s}}{k!}$$

Then

$$f_T(s) = \frac{\partial}{\partial s} F_T(s)$$

3f) Let X_i denote the length of time between the $(i-1)^{th}$ and i^{th} flip. Notice that $X_1, X_2, \dots \overset{iid}{\sim} \exp(1)$. Then $T = \sum_{i=1}^{N} X_i$ so that

$$ET = E\left[\sum_{i=1}^{N} X_i\right] = E\left(E\left[\sum_{i=1}^{N} X_i \,\middle|\, N\right]\right) = E[N\, EX_1] = EN$$