2017 Qual, Section 2

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Note: See Practice 5b soln. from 762.

1.

1. (25 points) Consider the linear model

$$Y = X\beta + Z\gamma + \epsilon,$$

where $E(\epsilon) = 0$ and $\text{Cov}(\epsilon) = V$, V is assumed known and positive definite, and (β, γ) are unknown. Further, let $A = X(X'V^{-1}X)^{-}X'V^{-1}$, where $-$ denotes generalized inverse, X is $n \times p$, Z is $n \times q$, and both X and Z may be less than full rank. Let $C(H)$ denote the usual label for the column space of an arbitrary matrix H.

(a) (2 points) Show that $(I - A)'V^{-1}(I - A) = (I - A)'V^{-1} = V^{-1}(I - A)$.

(b) (3 points) Show that A is the projection operator onto $C(X)$ along $C(V^{-1}X)^{\perp}$.

(c) (4 points) Let B denote the projection operator onto $C(X, Z)$ along $C(V^{-1}(X, Z))^{\perp}$. Assume that all matrix inverses exist. Show that

$$B = A + (I - A)Z\left[Z'(I - A)'V^{-1}(I - A)Z\right]^{-1} Z'(I - A)'V^{-1}.$$

(d) (5 points) Show that $(\hat{\gamma}, \hat{\beta})$ are generalized BLUE's for the linear model, where $(\hat{\gamma}, \hat{\beta})$ satisfy

$$\hat{\gamma} = \left[Z'(I - A)'V^{-1}(I - A)Z\right]^{-1} Z'(I - A)'V^{-1}(I - A)Y,$$

and

$$X\hat{\beta} = A(Y - Z\hat{\gamma}).$$

(e) (5 points) Suppose that $\epsilon \sim N_n(0, V)$ and V is known. Further, suppose that (β, γ) are both estimable. From first principles, derive the likelihood ratio test for the hypothesis $H_0 : \gamma = 0$, where (β, γ) are both unknown, and state the exact distribution of the test statistic under the null and alternative hypotheses.

(f) (6 points) Suppose that $\epsilon \sim N_n(0, \sigma^2 R)$, where R is known and positive definite, and $(\beta, \gamma, \sigma^2)$ are all unknown. Further, assume that (β, γ) are both estimable. Derive an exact joint 95% confidence region for $(\beta, \gamma, \sigma^2)$.

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1. a) Show that $(I-A)'V^{-1}(I-A) = (I-A)'V^{-1} = V^{-1}(I-A)$.

i) <u>First, show $(I-A)'V^{-1}(I-A) = (I-A)'V^{-1}$</u>.

$$\underbrace{(I-A)'V^{-1}}(I-A) = (I-A)'V^{-1} - \underbrace{(I-A)'V^{-1}A}_{\text{need to show } = 0}$$

Take $(I-A)'V^{-1}A = V^{-1}A - A'V^{-1}A = \underbrace{V^{-1}A}_{\text{leave as is}} - \underbrace{[X(X'V^{-1}X)^{-}X'V^{-1}]'}_{A}V^{-1}\underbrace{X(X'V^{-1}X)^{-}X'V^{-1}}_{A}$

$= V^{-1}A - \underbrace{V'^{-1}}_{V^{-1}}X\underbrace{(X'V^{-1}X)^{-}}_{G}\underbrace{X'V^{-1}X}_{G}\underbrace{(X'V^{-1}X)^{-}}_{G}X'V^{-1} = V^{-1}A - V^{-1}X\underbrace{G^{-}GG^{-}}_{= G^{-} \text{ by properties of gen-inverse}}X'V^{-1}$

$= V^{-1}A - V^{-1}XG^{-}X'V^{-1} = V^{-1}A - V^{-1}\underbrace{X(X'V^{-1}X)^{-}X'V^{-1}}_{A} = V^{-1}A - V^{-1}A = 0$ ✓

Thus, $(I-A)'V^{-1}(I-A) = (I-A)'V^{-1}$.

ii) <u>Second, show $(I-A)'V^{-1}(I-A) = V^{-1}(I-A)$</u>.

From part i), know $(I-A)'V^{-1}(I-A) = (I-A)'V^{-1}$. $(*)$

Notice that $(I-A)'V^{-1}(I-A)$ is symmetric since $[(I-A)'V^{-1}(I-A)]' = (I-A)'V^{-1}(I-A)$.

Thus, transposing both sides of the equality in $(*)$, we get

$[(I-A)'V^{-1}(I-A)]' = [(I-A)'V^{-1}] = (I-A)'V^{-1}(I-A) = \underbrace{V'^{-1}}_{V^{-1}}(I-A)$. ✓

Thus, $(I-A)'V^{-1}(I-A) = V^{-1}(I-A)$.

In conclusion, equating i) & ii), get: <u>$(I-A)'V^{-1}(I-A) = (I-A)'V^{-1} = V^{-1}(I-A)$</u>.

207 Qual, Section 2 NOT O.P.O. Have to show this part since not along $C(X)^\perp$. AMW

1 b) Show that A is the P.O. onto $C(X)$ along $C(V^{-1}X)^\perp$

i) To show A a P.O. onto $C(X)$

Let $V = QQ'$. Since V is PD $\Rightarrow$ Q singular.

Let P be an OPO onto $C(Q^{-1}X)$.

$$\text{Then, } P = (Q^{-1}X)\left[(Q^{-1}X)'(Q^{-1}X)\right]^{-}(Q^{-1}X)'$$

$$= Q^{-1}X[\,X'\underbrace{Q'^{-1}Q^{-1}}_{(QQ')^{-1}}X\,]^{-}X'Q'^{-1}$$

$$= Q^{-1}X[X'V^{-1}X]^{-}X'Q'^{-1}$$

Since P is an OPO onto $C(Q^{-1}X)$, then $P(Q^{-1}X) = Q^{-1}X$

$$\Leftrightarrow Q^{-1}X[X'V^{-1}X]^{-}X'\underbrace{Q'^{-1}Q^{-1}}_{(QQ')^{-1}}X = Q^{-1}X$$

$$\Leftrightarrow Q^{-1}\underbrace{X[X'V^{-1}X]^{-}X'V^{-1}}_{A}X = Q^{-1}X \quad \Leftrightarrow \quad Q^{-1}AX = Q^{-1}X \quad \Leftrightarrow \quad QQ^{-1}AX = QQ^{-1}X$$

$\Leftrightarrow$ $AX = X$ $\Leftrightarrow$ A is a P.O. onto $C(X)$.

ii) To show A projects along $C(V^{-1}X)^\perp$

Take $w \in (V^{-1}X)^\perp \Rightarrow (V^{-1}X)'w = 0 \Rightarrow X'V^{-1}w = 0$

Then, $Aw = X(X'V^{-1}X)^{-}\underbrace{X'V^{-1}w}_{=0} = 0 \Rightarrow A$ projects along $C(V^{-1}X)^\perp$

AMW

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1c) Let B denote the projection operator onto $C(X,Z)$ along $C(V^{-1}(X,Z))^{\perp}$.

Assume that all matrix inverses exist. Show that

$$B = A + (I-A)Z[Z'(I-A)'V^{-1}(I-A)Z]^{-1}Z'(I-A)'V^{-1}$$

Need to show:

i) $B^2 = B \Rightarrow$ B a projection operator (P.O.)

ii) $B(X,Z) = (BX, BZ) = (X,Z) \Rightarrow$ B projects onto $C(X,Z)$

iii) For $w \in C(V^{-1}(X,Z)^{\perp})$, then $Bw = 0 \Rightarrow$ B projects along $C(V^{-1}(X,Z))^{\perp}$.

i) $$B^2 = [A + (I-A)Z[Z'(I-A)'V^{-1}(I-A)Z]^{-1}Z'(I-A)'V^{-1}] \cdot [A + (I-A)Z[Z'(I-A)'V^{-1}(I-A)Z]^{-1}Z'(I-A)'V^{-1}]$$

$$= A^2 + A(I-A)Z[Z'(I-A)'V^{-1}(I-A)Z]^{-1}Z'(I-A)'V^{-1}$$
$$+ (I-A)Z[Z'(I-A)'V^{-1}(I-A)Z]^{-1}Z'(I-A)'V^{-1}A$$
$$+ (I-A)Z[Z'(I-A)'V^{-1}(I-A)Z]^{-1}Z'(I-A)'V^{-1}(I-A)Z[Z'(I-A)'V^{-1}(I-A)Z]^{-1}Z'(I-A)'V^{-1}$$

First term: $A^2 = A$ since A a P.O. (recall: A is NOT an O.P.O – see notes)

Second term: $\underbrace{A(I-A)}_{\substack{A-A^2 \\ =A-A=0}} Z[Z'(I-A)'V^{-1}(I-A)Z]^{-1}Z'(I-A)'V^{-1} = 0$

Third Term: $(I-A)Z[Z'(I-A)'V^{-1}(I-A)Z]^{-1}Z'\underbrace{(I-A)'V^{-1}}_{=(I-A)'V^{-1}(I-A) \text{ from a)}}A$

$$= (I-A)Z[Z'(I-A)'V^{-1}(I-A)Z]^{-1}Z'(I-A)'V^{-1}\underbrace{(I-A)A}_{=A-A^2=A-A=0} = 0$$

Fourth Term: $(I-A)Z[\underbrace{Z'(I-A)'V^{-1}(I-A)Z}_{G}]^{-1}\underbrace{Z'(I-A)'V^{-1}(I-A)Z}_{G}[\underbrace{Z'(I-A)'V^{-1}(I-A)Z}_{G}]^{-1}Z'(I-A)'V^{-1}$

$$= (I-A)Z\underbrace{G^{-1}GG^{-1}}_{G^{-1}}Z'(I-A)'V^{-1} = (I-A)ZG^{-1}Z'(I-A)'V^{-1}$$

$$\Rightarrow B^2 = A + 0 + 0 + (I-A)ZG^{-1}Z'(I-A)'V^{-1} = A + (I-A)Z[Z'(I-A)'V^{-1}(I-A)Z]^{-1}Z'(I-A)'V^{-1}$$
$$= B$$

Thus, $B^2 = B$, as we wanted to show for i) $\Rightarrow$ B a P.O.

Cont'd next pg. →

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1c) cont'd.

ii) Show $B(X,Z) = (BX, BZ) = (X,Z)$.

$$B(X,Z) = \{A + (I-A)Z[Z'(I-A)'V^{-1}(I-A)Z]^{-1}Z'(I-A)'V^{-1}\}(X,Z)$$

$$= AX + (I-A)Z[Z'(I-A)'V^{-1}(I-A)Z]^{-1}Z'(I-A)'V^{-1}X$$

$$+ AZ + (I-A)Z[Z'(I-A)'V^{-1}(I-A)Z]^{-1}Z'(I-A)'V^{-1}Z$$

Since A is a P.O. onto $C(X)$ along $C(V^{-1}X)^{\perp}$ $\Rightarrow$ $AX = X$ and $(I-A)'V^{-1}X = 0$,

have $\underbrace{AX}_{X} + (I-A)Z[\ ]^{-1}Z'\underbrace{(I-A)'V^{-1}X}_{0} = X + 0 = X$

Also, $AZ + (I-A)Z[\ ]^{-1}Z'\underbrace{(I-A)'V^{-1}}_{(I-A)'V^{-1}(I-A) \text{ from a)}}Z = AZ + (I-A)Z[\underbrace{Z'(I-A)'V^{-1}(I-A)Z}_{G}]^{-1}\underbrace{Z'(I-A)'V^{-1}(I-A)Z}_{G}$

$$= AZ + (I-A)Z\underbrace{G^{-1}G}_{I} = AZ + (I-A)Z = \cancel{AZ} + Z - \cancel{AZ} = Z$$

Thus, $B(X,Z) = (X,Z)$ $\Rightarrow$ B projects onto $C(X,Z)$.

iii) Show for $w \in C(V^{-1}(X,Z)^{\perp})$, then $Bw = 0$

Take $w \in C(V^{-1}(X,Z)^{\perp})$ $\Rightarrow$ $[V^{-1}(X,Z)]'w = (0,0) \Rightarrow (X',Z')V^{-1}w = (0,0)$

$\Rightarrow (X'V^{-1}w, Z'V^{-1}w) = (0,0)$

Then, $Bw = \{A + (I-A)Z[Z'(I-A)'V^{-1}(I-A)Z]^{-1}Z'(I-A)'V^{-1}\}w$

$$= Aw + (I-A)Z[\ ]^{-1}Z'(I-A)'V^{-1}w$$

$$= X(X'V^{-1}X)^{-}\underbrace{X'V^{-1}w}_{0} + (I-A)Z[\ ]^{-1}Z'\underbrace{(I-A)'V^{-1}}_{V^{-1}(I-A)}w$$

$$= 0 + (I-A)Z[\ ]^{-1}Z^{-1}V^{-1}(I-A)w$$

$$= 0 + (I-A)Z[\ ]^{-1}(\underbrace{Z^{-1}V^{-1}w}_{=0} - Z^{-1}V^{-1}\underbrace{Aw}_{})$$

Since $w \in C(V^{-1}(X,Z))^{\perp} \Rightarrow (X'V^{-1}w, Z'V^{-1}w) = (0,0)$, then $Aw = X(X'V^{-1}X)^{-}\underbrace{X'V^{-1}w}_{0} = 0$

$$= 0 + 0 = 0$$

$\Rightarrow Bw = 0$, so B projects along $C(V^{-1}(X,Z)^{\perp})$.

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1 d) Show that $(\hat{\gamma}, \hat{\beta})$ are generalized BLUEs for the LM, where $(\hat{\gamma}, \hat{\beta})$ satisfy $\hat{\gamma} = [Z'(I-A)'V^{-1}(I-A)Z]^{-1} Z'(I-A)'V^{-1}(I-A)Y$

and $X\hat{\beta} = A(Y - Z\hat{\gamma})$.

$\ulcorner$ Model : $Y = X\beta + Z\gamma + \epsilon$ where $E(\epsilon) = 0$ and $Cov(\epsilon) = V$.

Let $V = QQ'$. Since V P.D., then Q invertible.

Take $\underbrace{Q^{-1}Y}_{Y^*} = \underbrace{Q^{-1}X}_{X^*}\beta + \underbrace{Q^{-1}Z}_{Z^*}\gamma + \underbrace{Q^{-1}\epsilon}_{\epsilon^*} \Rightarrow Y^* = X^*\beta + Z^*\gamma + \epsilon^*$ (*)

where $Cov(Q^{-1}\epsilon) = Q^{-1}Cov(\epsilon)Q'^{-1} = Q^{-1}VQ'^{-1} = \underbrace{Q^{-1}Q}_{I}\underbrace{Q'Q'^{-1}}_{I} = I$

and $E(Q^{-1}\epsilon) = Q^{-1}E(\epsilon) = Q^{-1}.0 = 0$.

By Gauss-Markov, the LSE of (*) are the BLUEs of (β, γ).

Know by the normal eqns, $\begin{pmatrix} X^* \\ Z^* \end{pmatrix}'(X^* Z^*)\begin{pmatrix}\hat{\beta} \\ \hat{\gamma}\end{pmatrix} = \begin{pmatrix} X^* \\ Z^* \end{pmatrix}' Y^*$

(Normal Eqns. Given a matrix $Ax = b$, the normal eqn. is $A^TAx = A^Tb$, where $b - Ax$ is normal to the range of A) — Wolfram Alpha

Then, need to solve $\begin{pmatrix} X^{*\prime}X^* & X^{*\prime}Z^* \\ Z^{*\prime}X^* & Z^{*\prime}Z^* \end{pmatrix}\begin{pmatrix}\hat{\beta} \\ \hat{\gamma}\end{pmatrix} = \begin{pmatrix} X^{*\prime} \\ Z^{*\prime}\end{pmatrix} Y^*$

First Eqn : $(X^{*\prime}X^*)\hat{\beta} + (X^{*\prime}Z^*)\hat{\gamma} = X^{*\prime}Y^*$

2nd Eqn : $(Z^{*\prime}X^*)\hat{\beta} + (Z^{*\prime}Z^*)\hat{\gamma} = Z^{*\prime}Y^*$

Have, from above, $X^* = Q^{-1}X$, $Z^* = Q^{-1}Z$, and $Y^* = Q^{-1}Y$

First Eqn : $(X'Q'^{-1}Q^{-1}X)\hat{\beta} + (X'Q'^{-1}Q^{-1}Z)\hat{\gamma} = (X'Q'^{-1}Q^{-1}Y)$

$\Rightarrow (X'V^{-1}X)\hat{\beta} + (X'V^{-1}Z)\hat{\gamma} = (X'V^{-1}Y)$ (1)

2nd Eqn : $(Z'Q'^{-1}Q^{-1}X)\hat{\beta} + (Z'Q'^{-1}Q^{-1}Z)\hat{\gamma} = Z'Q'^{-1}Q^{-1}Y$

$\Rightarrow (Z'V^{-1}X)\hat{\beta} + (Z'V^{-1}Z)\hat{\gamma} = (Z'V^{-1}Y)$ (2)

Solve Eqn. 1 $(X'V^{-1}X)\hat{\beta} + (X'V^{-1}Z)\hat{\gamma} = (X'V^{-1}Y)$

$\Rightarrow (X'V^{-1})X\hat{\beta} = (X'V^{-1}Y) - (X'V^{-1}Z)\hat{\gamma}$

$= X'V^{-1}(Y - Z\hat{\gamma})$

$\Rightarrow \underbrace{X(X'V^{-1}X)^{-}X'V^{-1}}_{A} X\hat{\beta} = \underbrace{X(X'V^{-1}X)^{-}X'V^{-1}}_{A}(Y - Z\hat{\gamma}) \Rightarrow \overbrace{AX}^{=X}\hat{\beta} = A(Y - Z\hat{\gamma})$

$\Rightarrow \underline{X\hat{\beta} = A(Y - Z\hat{\gamma})}$ ✓

Cont'd next pg. →

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1 d) cont'd.

Solve Eqn. 2 Sub $X\hat{\beta} = A(Y - Z\hat{\gamma})$ from previous pg. into (2) to get:

$$Z'V^{-1}A(Y - Z\hat{\gamma}) + (Z'V^{-1}Z)\hat{\gamma} = Z'V^{-1}Y$$

$$\Rightarrow Z'V^{-1}AY - \underline{Z'V^{-1}AZ\hat{\gamma}} + \underline{Z'V^{-1}Z\hat{\gamma}} = Z'V^{-1}Y$$

$$\Rightarrow (Z'V^{-1}Z - Z'V^{-1}AZ)\hat{\gamma} = Z'V^{-1}Y - Z'V^{-1}AY$$

$$\Rightarrow [Z'V^{-1}(I-A)Z]\hat{\gamma} = Z'V^{-1}(I-A)Y$$

$$\Rightarrow \hat{\gamma} = [Z'\underbrace{V^{-1}(I-A)}_{(I-A)'V^{-1}(I-A) \text{ from a)}}Z]^{-1}Z'\overbrace{V^{-1}(I-A)}^{(I-A)'V^{-1}(I-A) \text{ from a)}}Y$$

$$\Rightarrow \hat{\gamma} = [Z'(I-A)'V^{-1}(I-A)Z]^{-1}Z'(I-A)'V^{-1}(I-A)Y \quad \rfloor$$

1.e) Suppose that $\epsilon \sim N_n(0,V)$ and V is known. Further, suppose that (β,γ) are both estimable. From first principles, derive the LRT for the hypothesis $H_0: \gamma=0$, where (β,γ) are both unknown, and state the exact distribution of the test statistic under the null & alternative hypothesis.

$\top$ Aim: Test $H_0: \gamma = 0$.

How: Use $\Lambda(y) = \dfrac{\sup_{\theta\in\Theta_0} L(\theta|y)}{\sup_{\theta\in\Theta} L(\theta|y)}$ to get the rejection region $R=\{y : \lambda(y) \le c\}$, $0 \le c \le 1$.

Model under Θ_0: $Y^* = X^*\beta + \epsilon^*$ $\Rightarrow$ Likelihood under Θ_0: $L(\beta,\gamma_0|y)$

Model under Θ: $Y^* = X^*\beta + Z^*\gamma + \epsilon^*$ $\Rightarrow$ Likelihood under Θ: $L(\beta,\gamma|y)$

where had $y = X\beta + Z\gamma + \epsilon$ $\Rightarrow$ $\underbrace{Q^{-1}y}_{Y^*} = \underbrace{Q^{-1}X}_{X^*}\beta + \underbrace{Q^{-1}Z}_{Z^*}\gamma + \underbrace{Q^{-1}\epsilon}_{\epsilon^*}$ and $W^* = (X^*, Z^*)$ w/ $\hat\delta = \begin{pmatrix}\hat\beta\\ \hat\gamma\end{pmatrix}$ (note here $Var(\epsilon^*) = I$)

Then, for both β and γ unknown, let $M_0^* = X^*(X^{*\prime}X^*)^- X^{*\prime}$ and $M^* = W^*(W^{*\prime}W^*)^- W^{*\prime}$.

LRT: $\Lambda(y) = \dfrac{\sup_{\Theta_0} L(\beta,\gamma_0|y)}{\sup_{\Theta} L(\beta,\gamma|y)} = \dfrac{\exp\{-\frac{1}{2}(Y^*-X^*\hat\beta)'(Y^*-X^*\hat\beta)\}}{\exp\{-\frac{1}{2}(Y^*-W^*\hat\delta)'(Y^*-W^*\hat\delta)\}} = \dfrac{\exp\{-\frac{1}{2}(Y^*-M_0^*Y^*)'(Y^*-M_0^*Y^*)\}}{\exp\{-\frac{1}{2}(Y^*-M^*Y^*)'(Y^*-M^*Y^*)\}}$

since $X^*\hat\beta = M_0^*Y^*$

since $W^*\hat\delta = M^*Y^*$

$$= \exp\Big\{-\tfrac{1}{2}\Big[\overbrace{(Y^*-M_0^*\hat\beta)'(Y^*-M_0^*\hat\beta)}^{Y^{*\prime}(I-M_0^*)Y^*} - \overbrace{(Y^*-M^*\hat\delta)'(Y^*-M^*\hat\delta)}^{Y^{*\prime}(I-M^*)Y^*}\Big]\Big\}$$

$$= \exp\Big\{-\tfrac{1}{2} Y^{*\prime}(M^*-M_0^*)Y^*\Big\}$$

Rejection Region:

$\Rightarrow R = \{y : \exp\{-\frac{1}{2} Y^{*\prime}(M^*-M_0^*)Y^*\} \le c_1\} = \{y : Y^{*\prime}(M^*-M_0^*)Y^* \ge c_2\}$.

Since $\boxed{Y^{*\prime}(M^*-M_0^*)Y^* \overset{H_0}{\sim} \chi^2(r(M^*-M_0^*))}$ (central) $\Rightarrow c_2 = \chi^2(1-\alpha, r(M^*-M_0^*))$.

(the $(1-\alpha)\times 100\%$th percentile of a $\chi^2(r(M^*-M_0^*))$ distribution)

Thus, $\boxed{R = \{y : Y^{*\prime}(M^*-M_0^*)Y^* \ge \chi^2(1-\alpha, r(M^*-M_0^*))\}}$

Also, $\boxed{Y^{*\prime}(M^*-M_0^*)Y^* \overset{H_1}{\sim} \chi^2(\theta, r(M^*-M_0^*)) = \chi^2(r(M^*-M_0^*))}$

where $\theta = \dfrac{(W^*\delta)'(I-M_0^*)(W^*\delta)}{2}$ (expected value of numerator)

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1f) Suppose that $\epsilon \sim N_n(0, \sigma^2 R)$, where R is known and P.D., and $(\beta, \gamma, \sigma^2)$ are all unknown.

Further, assume that (β, γ) are both estimable

Derive an exact joint 95% CR for $(\beta, \gamma, \sigma^2)$.

Damn this was one lonnng problem. God help us all...

Given $\epsilon \sim N_n(0, \sigma^2 R)$ where R P.D. Do cholesky decomposition on $R = QQ'$ $\ni$ Q invertible since R P.D.

Model: $Y = X\beta + Z\gamma + \epsilon \Rightarrow \underbrace{Q^{-1}Y}_{Y^*} = \underbrace{Q^{-1}X}_{X^*}\beta + \underbrace{Q^{-1}Z}_{Z^*}\gamma + \underbrace{Q^{-1}\epsilon}_{\epsilon^*}$

where $E(\epsilon^*) = E(Q^{-1}\epsilon) = Q^{-1}E(\epsilon) = 0$

$Var(\epsilon^*) = Var(Q^{-1}\epsilon) = Q^{-1}Var(\epsilon)Q'^{-1} = Q^{-1}\sigma^2 R Q'^{-1} = Q^{-1}\sigma^2 QQ'Q'^{-1} = \sigma^2 \underbrace{Q^{-1}Q}_{I}\underbrace{Q'Q'^{-1}}_{I}$
$= \sigma^2 I$

Then, for $W^* = (X^*, Z^*)$, $\delta = (\beta, \gamma)'$, and $M^* = W^*(W^{*\prime}W^*)^{-}W^{*\prime}$

$$\frac{M^*Y^* - E[M^*Y^*]}{(Var(M^*Y^*))^{1/2}} = \frac{M^*Y^* - W^*\delta}{\sigma^2} \sim N_n(0, M^*) \rightarrow \overbrace{\frac{\|M^*Y^* - W^*\delta\|^2}{\sigma^2} \sim \chi^2(r(M^*))}^{\text{slide 177 (bottom theorem)}}$$

b/c $E[M^*Y^*] = M^*E[Y^*] = M^*E[W^*\delta - \epsilon^*] = M^*[\underbrace{E(W^*\delta)}_{W^*\delta} - \underbrace{E(\epsilon^*)}_{0}] = M^*W^*\delta = W^*\delta$ (since M^* an OPO onto $C(W^*)$)

$Var[M^*Y^*] = M^*Var[Y^*]M^{*\prime} = M^*Var[W^*\delta - \epsilon^*]M^{*\prime}$

$= M^*\{\underbrace{Var[W^*\delta]}_{0} + \underbrace{Var(\epsilon^*)}_{\sigma^2 I} - \underbrace{2Cov(W^*\delta, \epsilon^*)}_{0}\}M^{*\prime}$

$= M^*\sigma^2 I \cdot M^{*\prime} = \sigma^2 M^*M^{*\prime} = \sigma^2 M^*$ (since $M^*M^{*\prime} = M^*$)

Know by thm on slide 176, since $Y^* \sim N_n(0, \sigma^2 I)$, then

$\frac{1}{\sigma^2}(Y'(I-M^*)Y) \sim \chi^2(r(I-M))$ since $I-M^*$ is an O.P.O. of rank $= r(I-M^*)$.

Then, $P\left(\chi^2_a(r(M^*)) \leq \frac{\|M^*Y^* - W^*\delta\|^2}{\sigma^2} \leq \chi^2_{1-a}(r(M^*)), \chi^2_b(r(I-M^*)) \leq \frac{\|(I-M^*)Y\|^2}{\sigma^2} \leq \chi^2_{1-b}(r(I-M^*))\right)$

$= 1 - \alpha = 0.95$

Since $M^*Y^* \perp\!\!\!\perp (I-M^*)Y^*$ (since $(M^*Y^*)'(I-M^*)Y^* = Y^{*\prime}M^{*\prime}Y^* - \underbrace{M^*Y^{*\prime}}_{Y^{*\prime}}M^*Y^* = 0$), then can break probability into,

$P\left(\chi^2_a(r(M^*)) \leq \frac{\|M^*Y^* - W^*\delta\|^2}{\sigma^2} \leq \chi^2_{1-a}(r(M^*))\right) \cdot P\left(\chi^2_b(r(I-M^*)) \leq \frac{\|(I-M^*)Y\|^2}{\sigma^2} \leq \chi^2_{1-b}(r(I-M^*))\right) = 0.95$

Need a & b $\ni$ $(1-a-a)(1-b-b) = (1-2a)(1-2b) = 0.95$

Thus, 95% CR$(\underbrace{\beta, \gamma}_{=\delta}, \sigma^2) =$

$$\left\{(\delta, \sigma^2): \chi^2_a(r(M^*)) \leq \frac{\|M^*Y^* - W^*\delta\|^2}{\sigma^2} \leq \chi^2_{1-a}(r(M^*)), \chi^2_b(r(I-M^*)) \leq \frac{\|(I-M^*)Y\|^2}{\sigma^2} \leq \chi^2_{1-b}(r(I-M^*))\right\}$$

$\ni$ (a, b) satisfy $(1-2a)(1-2b) = 0.95$