BASIC PHD WRITTEN EXAMINATION IN BIOSTATISTICS

THEORY, SECTION 2

(9:00 AM- 1:00 PM
Wednesday, July 30, 2014)

INSTRUCTIONS:

a) This is a **CLOSED-BOOK** examination.

b) The time limit for this Examination is four hours.

c) Answer any TWO (2) (BUT ONLY TWO) of the THREE (3) questions that follow.

d) Put the answers to different questions on separate sets of paper.

e) Put your code letter, **NOT YOUR NAME**, on each page. The same code will be used for Section 1 and Section 2 of the PhD Theory Exam. Please keep the code confidential and do not share this information with any students or faculty. Sharing your code with either students or faculty is viewed as a violation of the UNC honor code.

f) Return the examination with a signed statement of the UNC honor pledge, separately from your answers. The pledge statement is given on the last page of the exam handout.

g) In the questions to follow, you are required to answer only what is asked, and not to tell all you know about the topics involved.

1. Suppose that $Y_{ij} \sim Poisson(\mu_{ij})$ and independent for $i = 1, \cdots, n$ and $j = 1, 2$. Moreover, it is assumed that $\mu_{i1} = \psi\mu_{i2}$ for all i with $\psi > 0$ and $\lim_{n\to\infty} \sum_{i=1}^{n} \mu_{i2} = \infty$. Recall the important fact that if $Y_1 \sim Poission(\lambda_1)$ and $Y_2 \sim Possion(\lambda_2)$ are independent, then Y_1 given $Y_1 + Y_2 = m$ is a binomial distribution (m, π), where π is a function of (λ_1, λ_2). Our primary interest is to make inference on the ratio ψ with unknown μ_{i2}'s.

   (a) Derive the unconditional maximum likelihood estimate of ψ, denoted by $\hat{\psi}$, and its observed information when μ_{i2} varies across i. Please interpret the results.

   (b) Derive the conditional likelihood for conducting inference on ψ. Derive the conditional maximum likelihood estimate, denoted by $\hat{\psi}_c$, in closed form and its Fisher information.

   (c) Derive the conditional likelihood for conducting inference on ψ. Derive the conditional maximum likelihood estimate, denoted by $\hat{\psi}_c$, in closed form and its Fisher information.

   (d) Compare the observed information of $\hat{\psi}$ to the Fisher information of $\hat{\psi}_c$ and derive a $100(1-\alpha)\%$ confidence interval for ψ_c without resorting large sample theory.

   (e) Repeat (a) when we set $\mu_{i1} = \exp(\mathbf{x}_i^T\beta)\mu_{i2}$, where $\mathbf{x}_i$ is a 2×1 vector of covariates.

2. Consider the linear model

$$Y = X\beta + \epsilon, \quad (1)$$

where $Y = (y_1, y_2, y_3, y_4)'$, $\epsilon \sim N_4(0, \sigma^2 I_4)$, I_4 denotes the 4×4 identity matrix, $\beta = (\beta_1, \beta_2)'$, and $X = \begin{pmatrix} 1 & 3 \\ 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} = (J_4, X_1)$ where $J_n = (1, \cdots, 1)'_{n\times 1}$ and $X_1 = (3, 1, 1, 2)' = (x_{1,1}, x_{1,2}, x_{1,3}, x_{1,4})'$.

(a) Show that $\begin{pmatrix} \beta_1 + \beta_2 \\ \beta_1 - 2\beta_2 \end{pmatrix}$ is estimable.

(b) Find the UMVUE of $\begin{pmatrix} \beta_1 + \beta_2 \\ \beta_1 - 2\beta_2 \end{pmatrix}$.

(c) Find the distribution of the UMVUE in (b).

Now suppose the error term in the above linear model changes to

$$E(\epsilon) = 0 \text{ and } \mathrm{Cov}(\epsilon) = \Sigma = \sigma^2 V$$

where V is a known positive definite matrix.

(d) Identify a model, call it model (2), such that model (2) is in the form of an ordinary linear model (i.e, for the error term, the mean is 0 and the variance is $\sigma^2 I$) and it contains the same parameters as in model (1)

(e) Show that $\lambda'\beta$ is estimable in model (1) if and only if $\lambda'\beta$ is estimable in model (2).

(f) Show that $Y^T \Sigma^{-1} Y - y_1^2/\sigma_{11}^2 \sim \chi_3^2$ given that $\beta = 0$ and $\Sigma = (\sigma_{ij})$.

(g) Show that the best linear predictor of y_i is $\bar{y} + (x_{1,i} - \bar{x}_1)'\hat{\beta}_{1*}$ where $\hat{\beta}_{1*} = (X_1'(I - n^{-1}J_n^n)X_1)^{-1}X_1'(I - n^{-1}J_n^n)Y$ with $\bar{x}_1 = \frac{1}{n}\sum_{i=1}^n x_{1,i}$, $\bar{y} = \frac{1}{n}\sum_{i=1}^n y_i$ and $J_n^n = J_n J_n'$ (and $n = 4$ in this question).

3. Consider the following:

(a) Suppose we have k multivariate normal p-dimensional populations $N_p(\mu_i, \Sigma)$, $i = 1, \ldots, k$, $\mu_i = (\mu_{i1}, \ldots \mu_{ip})'$, and Σ is a $p \times p$ positive definite covariance matrix. Assume that samples of size n_i from $N_p(\mu_i, \Sigma)$ are drawn, $i = 1, \ldots, k$. Suppose we wish to test the null hypothesis

$$H_0 : \mu_i - \mu_j = \delta_{ij} J, \quad i = 1, \ldots k, \; j = 1, \ldots, k, \; i \neq j,$$

where δ_{ij} is an unknown and unspecified scalar and J is a $p \times 1$ vector of ones. Derive a UMPU size α test for this hypothesis and determine the exact distribution of the test statistic under H_0.

(b) Suppose that $Y_1, \ldots, Y_n$ is a random sample from a $N_p(\mu, \Sigma)$ population. Derive an exact uniformly most accurate $1 - \alpha$ confidence region for μ and and calculate its expected volume. Note: If $A \sim W_a(b, C)$, then the density of A is given by

$$f(A) = C \, \frac{| A |^{\frac{1}{2}(b-a-1)} \exp\{-\frac{1}{2}\mathrm{tr}(C^{-1}A)\}}{| \Sigma |^{b/2}}$$

where $C = 2^{-\frac{ab}{2}} \, \pi^{-\frac{a(a-1)}{4}} \left[\prod_{i=1}^{a} \Gamma\left(\frac{b+1-i}{2}\right)\right]^{-1}$.

(c) Suppose that $Y_1, \ldots, Y_n$ is a random sample from a $N_p(\mu, \Sigma)$ distribution, where $\mu = (\mu_1, \ldots, \mu_p)'$ and $\Sigma = \sigma^2[(1 - \rho) + \rho J J']$, where J is a $p \times 1$ vector of ones and $(p-1)^{-1} < \rho < 1$. Assume that (μ, σ^2, ρ) are all unknown.

(i) Derive the maximum likelihood estimates of (μ, σ^2, ρ).

(ii) Derive the simplest possible expression for the size α likelihood ratio test for $H_0 : \rho = 0$ versus $H_1 : \rho \neq 0$ and find the exact distribution of the test statistic under H_0 as well as an explicit closed-form expression for the critical value of the test.

(iii) Find an exact $1 - \alpha$ confidence interval for ρ.

2014 PhD Theory Exam, Section 2

Statement of the UNC honor pledge:

"In recognition of and in the spirit of the honor code, I certify that I have neither given nor received aid on this examination and that I will report all Honor Code violations observed by me."

(Signed) ______________________________

NAME

(Printed) ______________________________

NAME