2

(a) Under the given condition,

$$\sup_{\theta=\theta_0} P(Y \notin A(\theta_0)) = \sup_{\theta=\theta_0} P(T_{\theta_0}=1) \le \alpha,$$

which is the same as

$$1-\alpha \le \inf_{\theta=\theta_0} P(Y \in A(\theta_0)) = \inf_{\theta=\theta_0} P(\theta_0 \in C(Y)).$$

Since this holds for all θ_0, the result follows from

$$\inf_{P\in\mathcal{P}} P(\theta \in C(Y)) = \inf_{\theta_0\in\Theta} \inf_{\theta=\theta_0} P(\theta_0 \in C(Y)) \ge 1-\alpha.$$

(b)

The likelihood function is given by

$$L(\mu,\gamma) = \left(\sqrt{2\pi\gamma}\,|\mu|\right)^{-n} \exp\left\{-\frac{1}{2\gamma\mu^2}\sum_{i=1}^{n}(X_i-\mu)^2\right\}.$$

After some algebra and differentiation, we can show that the MLE of (μ,γ) is

$$(\hat\mu,\hat\gamma) = (\bar X, \hat\sigma^2/\bar X^2), \text{ where}$$

$$\hat\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\bar X)^2.$$

To see this, note that

$$\log L(\mu, \gamma) = -\frac{n}{2}\log(\gamma) - n\log(\mu) - \frac{1}{2\gamma\mu^2}\sum(x_i-\mu)^2$$

$$\frac{\partial}{\partial\gamma}\log L(\mu,\gamma) = -\frac{n}{2\gamma} + \frac{1}{2\gamma^2\mu^2}\sum(x_i-\mu)^2 = 0$$

$$\Rightarrow \boxed{\hat{\gamma} = \frac{1}{n\mu^2}\sum(x_i-\mu)^2} \qquad (1)$$

Now

$$\frac{\partial}{\partial\mu}\log L(\mu,\gamma) = -\frac{n}{\mu} + \frac{1}{\gamma\mu^3}\sum(x_i-\mu)^2 + \frac{1}{\gamma\mu^2}\sum(x_i-\mu) = 0$$

$$= -\frac{n}{\mu} + \frac{1}{\gamma\mu}\left(\frac{\sum(x_i-\mu)^2}{\mu^2}\right) + \frac{1}{\gamma\mu^2}\sum(x_i-\mu) = 0$$

$$= -\frac{n}{\mu} + \frac{1}{\gamma\mu}(n\gamma) + \frac{1}{\gamma\mu^2}\sum(x_i-\mu) = 0$$

$$= -\frac{n}{\mu} + \frac{n}{\mu} + \frac{1}{\gamma\mu^2}\sum(x_i-\mu) = 0$$

$$\Rightarrow \frac{1}{\gamma\mu^2}\sum(x_i-\mu) = 0$$

$$\Rightarrow \hat{\mu} = \frac{1}{n}\sum x_i = \bar{X}$$

so $\hat{\gamma} = \frac{1}{n\bar{X}^2}\sum_{i=1}^{n}(x_i-\bar{X})^2 = \;= \hat{\sigma}^2/\bar{X}^2$

where $\hat{\sigma}^2 = \frac{1}{n}\sum(x_i-\bar{X})^2$.

Now when $\gamma = \gamma_0$, where γ_0 is a specified scalar, we can obtain the MLE of μ as

$$\hat{\mu}(\gamma_0) = \begin{cases} \mu_+(\gamma_0), & L(\mu_+(\gamma_0), \gamma_0) > L(\mu_-(\gamma_0), \gamma_0) \\ \mu_-(\gamma_0) & L(\mu_+(\gamma_0), \gamma_0) \leq L(\mu_-(\gamma_0), \gamma_0) \end{cases}$$

Where $\mu_\pm(\gamma_0) = \dfrac{-\bar{X} \pm \sqrt{(5\bar{X}^2 + 4\hat{\sigma}^2)/\gamma_0}}{2}$

The likelihood ratio statistic is given by

$$\lambda(\gamma_0) = \frac{e^{n/2}\hat{\sigma}^n}{\gamma_0^{n/2}|\hat{\mu}(\gamma_0)|^n} \exp\left\{-\frac{\left(n\hat{\sigma}^2 + n(\hat{\mu}(\gamma_0) - \bar{x})^2\right)}{2(\hat{\mu}(\gamma_0))^2}\right\}$$

The confidence set is obtained by inverting the acceptance regions of LR tests is

$$\{\gamma : \lambda(\gamma) \geq c(\gamma)\}$$

where $c(\gamma)$ satisfies $P(\lambda(\gamma) < c(\gamma)) = \alpha$.

c

We wish to test

$$H_0: \gamma = \gamma_0 \quad \text{vs.} \quad H_1: \gamma \geq \gamma_0.$$

Take μ fixed for a moment and consider

$$H_0: \gamma = \gamma_0 \quad \text{vs.} \quad H_1: \gamma = \gamma_1, \quad \gamma_1 > \gamma_0.$$

Thus, by the NP lemma, we reject H_0 if

$\frac{f_1}{f_0} > k$, and this test is most powerful.

For fixed μ, the most powerful test by the NP lemma rejects H_0 if

$$\frac{f_1}{f_0} > k \iff \frac{(2\pi \gamma_1 \mu^2)^{-n/2} \, e^{-\frac{1}{2\gamma_1 \mu^2} \sum (X_i - \mu)^2}}{(2\pi \gamma_0 \mu^2)^{-n/2} \, e^{-\frac{1}{2\gamma_0 \mu^2} \sum (X_i - \mu)^2}} > k$$

$$\iff \boxed{\sum (X_i - \mu)^2 > k^*}$$

Thus, the rejection region always depends on μ, regardless of the hypothesis concerning γ or μ. Since μ is unknown, this demonstrates that a UMP test cannot exist. (If μ were known, then a UMP test does exist).

d) The log-likelihood is given by

$$\ell(\theta) = -\frac{1}{2\phi}\sum (X_i-\mu)^2 - \frac{n}{2}\log\phi - \frac{n}{2}\log(2\pi)$$

$$\frac{\partial \ell(\theta)}{\partial\theta} = \left(\frac{n(\bar{X}-\mu)}{\phi},\ \frac{1}{2\phi^2}\sum (X_i-\mu)^2 - \frac{n}{2\phi}\right)$$

and the Fisher information is

$$I_n(\theta) = n\begin{pmatrix}\frac{1}{\phi} & 0\\ 0 & \frac{1}{2\phi^2}\end{pmatrix}.$$

The MLE of θ is $\hat{\theta} = (\bar{X}, \hat{\phi})$, where

$$\hat{\phi} = \frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2$$

Consider testing $H_0: \mu=\mu_0$ vs. $H_1: \mu\neq\mu_0$.

For Wald's test, $R(\theta)=\mu-\mu_0$ with

$C = \frac{\partial R}{\partial\theta} = (1, 0)$. Hence Wald's statistic is

$$[R(\hat{\theta})]^2\left\{C^T I_n(\hat{\theta})^{-1} C\right\}^{-1} = \frac{n(\bar{X}-\mu_0)^2}{\hat{\phi}}.$$

Let z_α be the $(1-\alpha)^{th}$ quantil of $N(0,1)$. The $1-\alpha$ asymptotically correct confidence set obtained by inverting the acceptance region of Wald's test is

$$\left\{ \mu : \frac{n(\bar{x}-\mu)^2}{\hat{\phi}} \le z_{\alpha/2}^2 \right\},$$

Which is the interval

$$\left[\bar{X} - z_{\alpha/2}\sqrt{\hat{\phi}/n}\, , \ \bar{X} + z_{\alpha/2}\sqrt{\frac{\hat{\phi}}{n}} \right].$$

Under H_0, the MLE of ϕ is $\hat{\phi}_0 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu_0)^2$

$= \hat{\phi} + (\bar{X}-\mu_0)^2$, When $\hat{\phi} = \frac{1}{n}\sum(X_i-\bar{X})^2$.

Then the likelihood ratio is

$$\lambda = \left(\frac{\hat{\phi}}{\hat{\phi} + (\bar{X}-\mu_0)^2} \right)^{n/2}.$$

The asymptotic LR test rejects H_0 When

$$\lambda < e^{-\frac{z_{\alpha/2}^2}{2}}, \quad \text{i.e.,}$$

$$(\bar{X}-\mu_0)^2 > \left(e^{\frac{z_{\alpha/2}^2}{n}}-1\right)\hat{\phi}.$$

Hence, a $1-\alpha$ asymptotically correct confidence set obtained by inverting the acceptance region of the LR test is given by

$$\left\{\mu : (\bar{X}-\mu_0)^2 \le \left(e^{\frac{z_{\alpha/2}^2}{n}}-1\right)\hat{\phi}\right\},$$

which is the interval

$$\boxed{\left[\bar{X}-\sqrt{\left(e^{\frac{z_{\alpha/2}^2}{n}}-1\right)\hat{\phi}},\ \bar{X}+\left(e^{\frac{z_{\alpha/2}^2}{n}}-1\right)\hat{\phi}\right]}.$$

Let $\hat{\theta} = (\mu_0, \hat{\phi}+(\bar{X}-\mu_0)^2)$ be the MLE of θ under H_0. Then Rao's score test statistic is

$$R_n^2 = S(\hat{\theta})^T\left[I_n(\hat{\theta})\right]^{-1}S(\hat{\theta})$$

$S(\theta) = \frac{\partial\, l(\theta)}{\partial \theta}$, $l(\theta)$ = log-likelihood.

Note that

$$S(\tilde{\theta}) = \left(\frac{n(\bar{X}-\mu_0)}{\hat{\phi} + (\bar{X}-\mu_0)^2}, 0 \right).$$

Hence

$$R_n^2 = \frac{n(\bar{X}-\mu_0)^2}{\hat{\phi} + (\bar{X}-\mu_0)^2},$$

and the $1-\alpha$ asymptotically correct confidence set obtained by inverting the acceptance region of Rao's score test is

$$\left\{ \mu : (n - z_{\alpha/2}^2)(\bar{X}-\mu)^2 < z_{\alpha/2}^2 \hat{\phi} \right\},$$

which is the interval

$$\boxed{\left[\bar{X} - z_{\alpha/2}\sqrt{\frac{\hat{\phi}}{(n - z_{\alpha/2}^2)}}, \bar{X} + z_{\alpha/2}\sqrt{\frac{\hat{\phi}}{n - z_{\alpha/2}^2}} \right]}$$

(ii) Yes, these sets are always intervals as just shown.

(e) Let $\bar{X}$ denote the sample mean and $\hat{\sigma}^2 = \frac{1}{n}\Sigma(X_i-\bar{X})^2$. It follows from the CLT and Slutsky's theorem that

$$\sqrt{n}\left[\begin{pmatrix}\bar{X}\\ \hat{\sigma}^2\end{pmatrix} - \begin{pmatrix}\mu\\ \sigma^2\end{pmatrix}\right] \xrightarrow{d} N_2\left(0, \begin{pmatrix}\sigma^2 & \gamma\\ \gamma & \kappa\end{pmatrix}\right)$$

$\gamma = E(X_1-\mu)^3$ and $\kappa = E(X_1-\mu)^4 - \sigma^4$.

This result is based on the fact that if we let

$$Y_i = (X_i-\mu, (X_i-\mu)^2), \quad i=1,\ldots,n,$$

then $Y_1,\ldots,Y_n$ are iid random 2-vectors with $E(Y_i) = (0,\sigma^2)$ and covariance matrix

$$\Sigma = \begin{pmatrix} \sigma^2 & \gamma \\ \gamma & \kappa \end{pmatrix}.$$

Note that $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i = (\bar{X}-\mu, \tilde{S}^2)$,

where $\tilde{S}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2$.

Applying the CLT to the Y_i's, we obtain that

$$\sqrt{n}\left(\bar{X}-\mu, \tilde{S}^2-\sigma^2\right) \xrightarrow{d} N_2(0,\Sigma).$$

Since

$$S^2 = \frac{n}{n-1}\left[\tilde{S}^2 - (\bar{X}-\mu)^2\right]$$

and $\bar{X} \xrightarrow{a.s.} \mu$, an application of Slutsky's theorem leads to

$$\sqrt{n}\left(\bar{X}-\mu, S^2-\sigma^2\right) \xrightarrow{d} N_2(0,\Sigma).$$

Now let $g(x,y) = \frac{x}{\sqrt{y}}$. Then

$\frac{\partial g}{\partial x} = \frac{1}{\sqrt{y}}$ and $\frac{\partial g}{\partial y} = \frac{-x}{2y^{3/2}}$. By the delta method,

$$\sqrt{n}\left(\frac{\bar{X}}{\hat{\sigma}} - \frac{\mu}{\sigma}\right) \xrightarrow{d} N\left(0,\ 1 + \frac{\mu^2 K}{4\sigma^6} - \frac{\mu\gamma}{\sigma^4}\right).$$

Let
$$\hat{\gamma} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^3$$

and
$$\hat{K} = \frac{1}{n}\sum(X_i - \bar{X})^4 - \hat{\sigma}^4.$$

By the SLLN, $\hat{\gamma} \xrightarrow{p} \gamma$ and $\hat{K} \xrightarrow{p} K$. Let

$W = 1 + \frac{\bar{X}^2 \hat{K}}{4\hat{\sigma}^6} - \frac{\bar{X}\hat{\gamma}}{\hat{\sigma}^4}$. By Slutsky's theorem,

$$\frac{\sqrt{n}}{\sqrt{W}}\left(\frac{\bar{X}}{\hat{\sigma}} - \theta\right) \xrightarrow{d} N(0,1)$$

and, hence a $1-\alpha$ asymptotically correct confidence interval for θ is

$$\left[\frac{\bar{X}}{\hat{\sigma}} - z_{\alpha/2}\frac{\sqrt{W}}{\sqrt{n}},\ \frac{\bar{X}}{\hat{\sigma}} + z_{\alpha/2}\frac{\sqrt{W}}{\sqrt{n}}\right],$$

where z_α is the $(1-\alpha)$th quantile of $N(0,1)$.