1. Why |X|?

Suppose you have a test $\phi(X)$. A sign transformation invariant test must give you the same result for X and -X. Say, if you are given either X=2 or X=-2, $\phi(2) = \phi(-2)$. This should be true for all X.

So a natural statistic we can come up with is T(X)=|X|.
(Note absolute value loses information only as much as the problem wants. A test based on nothing (eg. T(X)=0, T(X)=1.) is also sign invariant test, but it loses too much information. Of course, we can also think about $T(X) = X^2$ as an alternative to T(X)=|X|)

2. Need to find $\bar{g}(\theta)$ that corresponds to $g(X) = -X$.

-X	-2	-1	0	1	2
P_θ(-X) $= P_\theta(g(X))$	$\theta_1\theta_2$	$\left(\frac{1}{2} - \alpha\right)\left(\frac{1-\theta_1}{1-\alpha}\right)$	$\alpha\left(\frac{1-\theta_1}{1-\alpha}\right)$	$\left(\frac{1}{2} - \alpha\right)\left(\frac{1-\theta_1}{1-\alpha}\right)$	$\theta_1(1-\theta_2)$

To have $\theta_1\theta_2 = \bar{\theta}_1(1-\bar{\theta}_2)$ (and same for others), $\bar{g}(\theta) = (\theta_1, 1-\theta_2)$

3. Then check Hypothesis invariance, test invariance,…
After this, it becomes a problem of UMP test using |X| and new parameters $(\bar{\theta}_1, \bar{\theta}_2)$. And forget about the original data (X) and original parameters.

4. Based on |X|, draw a distribution table.

|X|	0	1	2
P(|X|)	$\alpha\left(\frac{1-\bar{\theta}_1}{1-\alpha}\right)$	$2\left(\frac{1}{2} - \alpha\right)\left(\frac{1-\bar{\theta}_1}{1-\alpha}\right)$	$\bar{\theta}_1$

The likelihood ratio comparing $\bar{\theta}_1 < \bar{\theta}_1'$:

|X|	0	1	2
$\frac{P_{\bar{\theta}_1}(\lvert X\rvert)}{P_{\bar{\theta}_1'}(\lvert X\rvert)}$	$\left(\frac{1-\bar{\theta}_1}{1-\bar{\theta}_1'}\right)$	$\left(\frac{1-\bar{\theta}_1}{1-\bar{\theta}_1'}\right)$	$\frac{\bar{\theta}_1}{\bar{\theta}_1'}$
	> 1	> 1	< 1

Note the ratio is nondecreasing in $|X|$

Since it's flat between 0 and 1, we can also say that it is nondecreasing in $I(|X| = 2)$.

I(|X|=2)	0	1
$\frac{P_{\bar{\theta}_1}(\mathrm{I}(\lvert X\rvert = 2))}{P_{\bar{\theta}_1'}(\mathrm{I}(\lvert X\rvert = 2))}$	$\left(\frac{1-\bar{\theta}_1}{1-\bar{\theta}_1'}\right)$	$\frac{\bar{\theta}_1}{\bar{\theta}_1'}$

5. If we use NP lemma based on $I(|X| = 2)$, we end up getting a randomized rule

$$\phi(X) = \begin{cases} 1 & I(|X| = 2) < 0 \\ \gamma & I(|X| = 2) = 0 \\ 0 & I(|X| = 2) > 0 \end{cases} \quad with\, \gamma = \frac{\alpha}{1-\alpha}$$

If we use NP lemma based on $|X|$, we will get a nonrandomized rule

$$\phi(X) = \begin{cases} 1 & |X| < 2 \\ \gamma & |X| = 2 \\ 0 & |X| > 2 \end{cases} \quad with\, \gamma = 0$$

So for 2(c), the answer depends on the statistic we choose.