2018 Qual, Section 2

AMW

1 a) $Y = X_1\beta_1 + X_2\beta_2 + \epsilon$. Let $X = (X_1, X_2)$ and $\beta = \begin{pmatrix}\beta_1 \\ \beta_2\end{pmatrix}$

$\Rightarrow$ $Y = X\beta + \epsilon$. $\quad M = X(X'X)^{-1}X$ is an opo onto $X = (X_1, X_2)$

Then, $\hat{\sigma}^2_{overfit} = \dfrac{Y'(I-M)Y}{n-p}$ where $p = rank(W) = rank(X_1, X_2)$.

Then, $E[\hat{\sigma}^2_{overfit}] = \dfrac{1}{n-p} E[Y'(I-M)Y] = \dfrac{1}{n-p}\{E[Y]'(I-M)E[Y] + tr(\sigma^2(I-M))\}$

$$= \frac{1}{n-p}\{(X_1\beta_1)'\overset{0}{\cancel{(I-M)}}(X_1\beta_1) + \sigma^2(n-p)\}$$

$$= \boxed{\sigma^2}$$

since $(I-M)X_1 = X_1 - \underbrace{MX_1}_{X_1} = 0$.

b) Take $Y = X_1\beta_1 + X_2\beta_2 + \epsilon = X_1\beta_1 + M_2X_1\beta_1 - M_2X_1\beta_1 + X_2\beta_2 + \epsilon$

$$= X_1\beta_1 + X_2(X_2'X_2)^{-1}X_2'X_1\beta_1 - X_2(X_2'X_2)^{-1}X_2'X_1\beta_1 + X_2\beta_2 + \epsilon$$

$$= (I + M_2)X_1\beta_1 + X_2[\beta_2 - (X_2'X_2)^{-1}X_2'X_1\beta_1] + \epsilon$$

Then, $\boxed{\hat{\beta}_1 = \{[(I+M_2)X_1]'[(I+M_2)X_1]\}^{-1}[(I+M_2)X_1]'Y}$

under true model (2)

Then, $E[\hat{\beta}_1] = \{X_1'\underbrace{(I+M_2)'(I+M_2)}_{(I+M_2)}X_1\}^{-1}X_1'(I+M_2)E[Y]$

$$= \underbrace{\{X_1'(I+M_2)X_1\}^{-1}X_1'(I+M_2)X_1}_{I}\beta_1 = \boxed{\beta_1}$$

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1.c) Know that $\frac{Y'(I-M)Y}{n-p}$ is the UMVUE of σ^2. If $Y \sim N(X\beta, \sigma^2 I)$,

and $(I-M)$ is an OPO of rank $n-p$, it follows by a thm given in the slides on "distribution of quadratic forms" that $\frac{1}{\sigma^2} Y'(I-M)Y \sim \chi^2(n-p, \gamma)$

where $\gamma = \frac{(X\beta)'(I-M)(X\beta)}{2\sigma^2} = \frac{\beta' X'(I-M)X\beta}{2\sigma^2} = 0$ since $(I-M)X = 0$.

Thus, $\frac{1}{\sigma^2} Y'(I-M)Y \sim \chi^2(n-p)$

So, for model (1): $\frac{1}{\sigma^2} Y'(I-M)Y \sim \chi^2(r(I-M)) \equiv \chi^2(n-p)$

$\Rightarrow$ $(1-\alpha) \times 100\%$ CI (σ^2) based on (1) is:

$$\boxed{C_1 = \left\{ \chi^2_{(n-p)}(\alpha/2) \le \frac{Y'(I-M)Y}{\sigma^2} \le \chi^2_{(n-p)}(1-\alpha/2) \right\} = \left\{ \frac{Y'(I-M)Y}{\chi^2_{(n-p)}(1-\alpha/2)} \le \sigma^2 \le \frac{Y'(I-M)Y}{\chi^2_{(n-p)}(\alpha/2)} \right\}}$$

For model (2): $\frac{1}{\sigma^2} Y'(I-M_1)Y \sim \chi^2(r(I-M_1)) \equiv \chi^2(n-p_1)$

$\Rightarrow$ $(1-\alpha) \times 100\%$ CI (σ^2) based on (2) is:

$$C_2 = \left\{ \chi^2_{(n-p_1)}(\alpha/2) \le \frac{Y'(I-M_1)Y}{\sigma^2} \le \chi^2_{(n-p_1)}(1-\alpha/2) \right\} = \left\{ \frac{Y'(I-M_1)Y}{\chi^2_{(n-p_1)}(1-\alpha/2)} \le \sigma^2 \le \frac{Y'(I-M_1)Y}{\chi^2_{(n-p_1)}(\alpha/2)} \right\}$$

The $E[\text{length}(C_1)] = E\left[\frac{Y'(I-M)Y}{\chi^2_{(n-p)}(\alpha/2)} - \frac{Y'(I-M)Y}{\chi^2_{(n-p)}(1-\alpha/2)}\right] = \underbrace{\left[\frac{1}{\chi^2_{(n-p)}(\alpha/2)} - \frac{1}{\chi^2_{(n-p)}(1-\alpha/2)}\right]}_{a_1} E\left[Y'(I-M)Y + Y'(I-M)Y\right]$

$= a_1 \{E[Y'(I-M)Y] + E[Y'(I-M)Y]\}$

$= a_1 \{2\cancel{(X\beta)'(I-M)(X\beta)}^{0} + 2tr(\sigma^2(I-M))\} = 2a_1(n-p)$

$a_2 = \frac{1}{\chi^2_{(n-p_1)}(\alpha/2)} - \frac{1}{\chi^2_{(n-p_1)}(1-\alpha/2)}$

Similarly, $E[\text{length}(C_2)] = 2a_2(n-p_1)$ $\Rightarrow$ $E[\text{length}(C_1)] > E[\text{length}(C_2)]$

when $\cancel{2}a_1(n-p) > \cancel{2}a_2(n-p_1)$ $\Rightarrow$ $\boxed{(n-p)\left[\frac{1}{\chi^2_{(n-p)}(\alpha/2)} - \frac{1}{\chi^2_{(n-p)}(1-\alpha/2)}\right] > (n-p_1)\left[\frac{1}{\chi^2_{(n-p_1)}(\alpha/2)} - \frac{1}{\chi^2_{(n-p_1)}(1-\alpha/2)}\right]}$

AMW

1 d) (i) Constraint $\gamma_1+\gamma_2+\gamma_3=180$

where $y_1=\gamma_1+\epsilon_1$

$y_2=\gamma_2+\epsilon_2$

$y_3=\gamma_3+\epsilon_3=180-\gamma_1-\gamma_2+\epsilon_3$

$\iff$

$y_1-60=\gamma_1-60+\epsilon_1$

$y_2-60=\gamma_2-60+\epsilon_2$

$y_3-60=120-\gamma_1-\gamma_2+\epsilon_3$

Let $Y=\begin{pmatrix} y_1-60\\ y_2-60\\ y_3-60\end{pmatrix}$, $\gamma=\begin{pmatrix}\gamma_1-60\\ \gamma_2-60\end{pmatrix}$, $\epsilon=\begin{pmatrix}\epsilon_1\\ \epsilon_2\\ \epsilon_3\end{pmatrix}$, $X=\begin{pmatrix}1&0\\0&1\\-1&-1\end{pmatrix}$

Then, $Y=X\beta+\epsilon$

$H_0: \gamma=0$ vs. $H_1: \gamma\neq 0$

$$F=\frac{Y'(M-M_0)Y/r(M-M_0)}{MSE}\sim F((M-M_0),(I-M),\gamma)$$

where $\gamma=\dfrac{(X\gamma)'(M-M_0)(X\gamma)}{2\sigma^2}$, $M_0=0$, and $M=X(X'X)^{-1}X'=\begin{pmatrix}1&0\\0&1\\-1&-1\end{pmatrix}\left[\begin{pmatrix}1&0&-1\\0&1&-1\end{pmatrix}\begin{pmatrix}1&0\\0&1\\-1&-1\end{pmatrix}\right]^{-1}\begin{pmatrix}1&0&-1\\0&1&-1\end{pmatrix}$

$\overset{\text{in R}}{=}\frac{1}{3}\begin{pmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{pmatrix}$

Numerator: Know $\underbrace{\frac{1}{3}\begin{pmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{pmatrix}}_{M}\underbrace{\begin{pmatrix}y_1-60\\y_2-60\\y_3-60\end{pmatrix}}_{Y}=\frac{1}{3}\begin{pmatrix}2(y_1-60)-(y_2-60)-(y_3-60)\\-(y_1-60)+2(y_2-60)-(y_3-60)\\-(y_1-60)-(y_2-60)+2(y_3-60)\end{pmatrix}=\frac{1}{3}\begin{pmatrix}2y_1-y_2-y_3\\-y_1+2y_2-y_3\\-y_1-y_2+2y_3\end{pmatrix}$

$=\begin{pmatrix}y_1-\bar{y}\\y_2-\bar{y}\\y_3-\bar{y}\end{pmatrix}$

Know $Y'(M-\overset{=0}{M_0})Y=Y'MY=(MY)'(MY)$

$=(y_1-\bar{y})^2+(y_2-\bar{y})^2+(y_3-\bar{y})^2$ where $r(M-\overset{=0}{M_0})=r(M)$

Denominator: $\cancel{I-M=\frac{1}{3}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}}$ $(I-M)Y=Y-MY=\begin{pmatrix}y_1-60\\y_2-60\\y_3-60\end{pmatrix}-\begin{pmatrix}y_1-\bar{y}\\y_2-\bar{y}\\y_3-\bar{y}\end{pmatrix}=\begin{pmatrix}\bar{y}-60\\\bar{y}-60\\\bar{y}-60\end{pmatrix}$

Then, $Y'(I-M)Y=[(I-M)Y]'[(I-M)Y]=3(\bar{y}-60)^2$

Conclude: Thus, $F=\dfrac{Y'MY/\overset{=2}{r(M)}}{Y'(I-M)Y/\underset{3-2=1}{r(I-M)}}=\dfrac{(y_1-\bar{y})^2+(y_2-\bar{y})^2+(y_3-\bar{y})^2}{6(\bar{y}-60)^2}\sim F(2,1,\delta)$

Under H_0: $F\sim F(2,1)$

Under H_1: $F\sim F(2,1,\delta)$ where $\delta=\dfrac{(X\gamma)'M(X\gamma)}{2\sigma^2}=\dfrac{[M(X\gamma)]'[M(X\gamma)]}{2\sigma^2}$ since $M_0=0$

Where $M(X\gamma)=\underbrace{\frac{1}{3}\begin{pmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{pmatrix}\begin{pmatrix}1&0\\0&1\\-1&-1\end{pmatrix}}\begin{pmatrix}\gamma_1-60\\\gamma_2-60\end{pmatrix}=\underset{3\times2}{\begin{pmatrix}1&0\\0&1\\-1&-1\end{pmatrix}}\underset{2\times2}{\begin{pmatrix}\gamma_1-60\\\gamma_2-60\end{pmatrix}}=\begin{pmatrix}\gamma_1-60\\\gamma_2-60\\-(\gamma_1+\gamma_2)+120\end{pmatrix}$

$=\begin{pmatrix}1&0\\0&1\\-1&-1\end{pmatrix}$ since $MX=X$

$-(\gamma_1+\gamma_2)+120 = \gamma_3-60$

$\Rightarrow \boxed{\delta=\dfrac{(\gamma_1-60)^2+(\gamma_2-60)^2+(\gamma_3-60)^2}{2\sigma^2}}$

1d ii) $\ulcorner$ Want to derive joint 95% $CR(\gamma_1, \gamma_2)$

Note $\delta = 0$ ✓ b/c subtracted mean of Y, namely $E[Y] = X\gamma$

Know $F = \dfrac{(\overset{X\hat{\gamma}}{\overset{\shortparallel}{Y}} - X\gamma)'(M - \overset{0}{\overset{\shortparallel}{M_0}})(\overset{X\hat{\gamma}}{\overset{\shortparallel}{Y}} - X\gamma)/r(M - \overset{0}{\overset{\shortparallel}{M_0}})}{MSE} \sim F(\underbrace{r(M - \overset{0}{\overset{\shortparallel}{M_0}})}_{\overset{\shortparallel}{r(M)}}, r(I-M))$

From i), know $r(M) = 2$, $r(I-M) = 1$, $MSE = (\bar{Y} - 60)^2/1 = (\bar{Y} - 60)^2$

Need to find $(Y - X\gamma)'M(Y - X\gamma) = (X\hat{\gamma} - X\gamma)'M(X\hat{\gamma} - X\gamma) = (\hat{\gamma} - \gamma)'\underbrace{X'X(X'X)^{-1}X'X}_{(X'X)}(\hat{\gamma} - \gamma)$

$= (\hat{\gamma} - \gamma)'\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}(\hat{\gamma} - \gamma) = (\gamma - \hat{\gamma})'\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}(\gamma - \hat{\gamma})$

Then, $F = \dfrac{(\gamma - \hat{\gamma})'\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}(\gamma - \hat{\gamma})/2}{(\bar{Y} - 60)^2} \sim F(2,1)$

$\Rightarrow (1-\alpha) \times 100\%\ CR(\overset{\gamma}{\overset{\shortparallel}{\gamma_1, \gamma_2}})$

$= \left\{ \gamma : (\gamma - \hat{\gamma})'\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}(\gamma - \hat{\gamma}) \leq 2(\bar{Y} - 60)^2 \underbrace{F(0.95, 2, 1)}_{\text{The 95\% percentile of a central } F(2,1) \text{ distn.}} \right\}$ $\lrcorner$