AMW

2014, Section 2, Qual

2. a) Show that $\begin{pmatrix} \beta_1 + \beta_2 \\ \beta_1 - 2\beta_2 \end{pmatrix}$ is estimable.

To show estimable, either:

1) Show X full rank

OR

2) Show $\Lambda' = P'X$

1) If we row reduce X, get $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \text{rank}(X) = \#\text{ pivots} = 2$

Know X full rank if $\text{rank}(X) = \min(\#\text{rows}, \#\text{cols})$.

Since $\text{rank}(X) = 2 = \min(\underset{4}{\#\text{rows}}, \underset{2}{\#\text{cols}}) = 2$, then X full rank $\Rightarrow$

any $\Lambda'\beta$ is estimable $\Rightarrow \begin{pmatrix} \beta_1 + \beta_2 \\ \beta_1 - 2\beta_2 \end{pmatrix}$ estimable.

2) However, this approach is the way to go b/c we will need a projection matrix for part b).

$\Lambda = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$ and $X = \begin{pmatrix} 1 & 3 \\ 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$. Then $\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$

$\Rightarrow \quad 1 = p_1 + p_2 + p_3 + p_4 \qquad 1 = 3p_1 + p_2 + p_3 + 2p_4$

$1 = p_5 + p_6 + p_7 + p_8 \qquad -2 = 3p_5 + p_6 + p_7 + 2p_8 \Rightarrow$

$$\begin{array}{rl} 1 = & p_5 + p_6 + p_7 + p_8 \\ - \; -2 = & 3p_5 + p_6 + p_7 + 2p_8 \\ \hline 3 = & -2p_5 - p_8 \end{array}$$

Let $p_5 = -4 \Rightarrow p_8 = 5$

Let $p_1 = p_2 = p_4 = 0 \Rightarrow p_3 = 1$

Let $p_5 = -4 \Rightarrow p_8 = 5$ and let $p_6 = p_7 = 0$

Thus, $\Lambda' = P'X$ for $P = \begin{pmatrix} 0 & -4 \\ 0 & 0 \\ 1 & 0 \\ 0 & 5 \end{pmatrix}$. $\Rightarrow \Lambda'\beta$ estimable

Check $\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} 0 & 0 & 1 & 0 \\ -4 & 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$ ✓

2014, Section 2, qual

AMW

2 b) Find the UMVUE of $\begin{pmatrix} \beta_1 + \beta_2 \\ \beta_1 - 2\beta_2 \end{pmatrix}$

By Gauss Markov, we know the BLUE (UMVUE) of $\Lambda'\beta$ is $P'MY$

From part a), know one projection matrix is $P = \begin{pmatrix} 0 & -4 \\ 0 & 0 \\ 1 & 0 \\ 0 & 5 \end{pmatrix}$.

Now, need to find M.

$$(X'X)^{-1} = \left[\begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}\right]^{-1} = \begin{bmatrix} 4 & 7 \\ 7 & 15 \end{bmatrix}^{-1} = \frac{1}{11}\begin{bmatrix} 15 & -7 \\ -7 & 4 \end{bmatrix}$$

Then, $M = X(X'X)^{-1}X' = \frac{1}{11}\begin{pmatrix} 1 & 3 \\ 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 15 & -7 \\ -7 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 2 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} -6 & 5 \\ 8 & -3 \\ 8 & -3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 2 \end{pmatrix}$

$= \frac{1}{11}\begin{pmatrix} 9 & -1 & -1 & 4 \\ -1 & 5 & 5 & 2 \\ -1 & 5 & 5 & 2 \\ 4 & 2 & 2 & 3 \end{pmatrix}$. Then, $\underbrace{\Lambda'\hat{\beta}}_{\text{UMVUE}} = P'MY = \frac{1}{11}\begin{pmatrix} 0 & 0 & 1 & 0 \\ -4 & 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} 9 & -1 & -1 & 4 \\ -1 & 5 & 5 & 2 \\ -1 & 5 & 5 & 2 \\ 4 & 2 & 2 & 3 \end{pmatrix} Y$

$= \frac{1}{11}\begin{pmatrix} -1 & 5 & 5 & 2 \\ -16 & 14 & 14 & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$

$= \boxed{\frac{1}{11}\begin{pmatrix} -y_1 + 5y_2 + 5y_3 + 2y_4 \\ -16y_1 + 14y_2 + 14y_3 - y_4 \end{pmatrix}}$

AMW

2014 Section 2 Qual

2c) Find the distn. of the UMVUE in b)

Know $\underbrace{\Lambda'\hat{\beta} = P'MY}_{\text{UMVUE}} \sim N(E[P'MY], Cov(P'MY))$

$$\equiv N(\Lambda'\beta,\ P'M\,Cov(Y)(P'M)')$$

$$\equiv N(\Lambda'\beta,\ P'M\sigma^2 I\,MP)$$

$$\equiv N(\Lambda'\beta,\ \sigma^2 P'MP)$$

Know from b) $P'M = \frac{1}{11}\begin{pmatrix} -1 & 5 & 5 & 2 \\ -16 & 14 & 14 & -1 \end{pmatrix}$

Since $P'MP = (P'M)(P'M)' = \frac{1}{121}\begin{pmatrix} -1 & 5 & 5 & 2 \\ -16 & 14 & 14 & -1 \end{pmatrix}\begin{pmatrix} -1 & -16 \\ 5 & 14 \\ 5 & 14 \\ 2 & -1 \end{pmatrix}$

$$= \frac{1}{11}\begin{pmatrix} 5 & 14 \\ 14 & 59 \end{pmatrix}$$

$$\Rightarrow \boxed{P'MY \sim N\left(\begin{pmatrix} \beta_1 + \beta_2 \\ \beta_1 - 2\beta_1 \end{pmatrix}, \frac{\sigma^2}{11}\begin{pmatrix} 5 & 14 \\ 14 & 59 \end{pmatrix}\right)}$$

Note Could have also recognized that:

$$\Lambda'\hat{\beta} = P'MY = N(\Lambda'\beta, \sigma^2 P'MP)$$

Since $\Lambda' = P'X \Rightarrow P'MP = \underbrace{P'X}_{\Lambda'}(X'X)^{-1}\underbrace{X'P}_{\Lambda} = \Lambda'(X'X)^{-1}\Lambda$

$$\Rightarrow P'MY = N(\Lambda'\beta, \sigma^2 \Lambda'(X'X)^{-1}\Lambda)$$

Know from b) that $(X'X)^{-1} = \frac{1}{11}\begin{bmatrix} 15 & -7 \\ -7 & 4 \end{bmatrix}$ and $\Lambda = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$

$$\Rightarrow \Lambda'(X'X)^{-1}\Lambda = \frac{1}{11}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 15 & -7 \\ -7 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 8 & -3 \\ 29 & -15 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$$

29
30
59

$$= \frac{1}{11}\begin{pmatrix} 5 & 14 \\ 14 & 59 \end{pmatrix}$$

AMW

2014, Section 2, Quad

2.d) Now $E(\epsilon)=0$ and $Cov(\epsilon) = \Sigma = \sigma^2 V$ w/ V P.D.

I dentify a model, call it model (2), such that model (2) is in the form of an ordinary linear model (i.e., for the error term, the mean is 0 and variance is $\sigma^2 I$) and it contains the same parameters as in model (1)

Have Model (1) : $Y = X\beta + \epsilon$ where $\epsilon \sim N(0, \sigma^2 V)$

Let $V = QQ'$ where Q invertible since V P.D.

Then, $\underbrace{Q^{-1}Y}_{Y^*} = \underbrace{Q^{-1}X}_{X^*}\beta + \underbrace{Q^{-1}\epsilon}_{\epsilon^*} \Rightarrow Y^* = X^*\beta + \epsilon^*$ where $E[\epsilon^*] = E[Q^{-1}\epsilon] = Q^{-1}E[\epsilon] = 0$

$$
\begin{aligned}
Cov[\epsilon^*] &= Cov[Q^{-1}\epsilon] = Q^{-1}Cov(\epsilon)Q'^{-1} \\
&= Q^{-1}\sigma^2 V Q'^{-1} \\
&= \sigma^2 Q^{-1}(QQ')Q'^{-1} \\
&= \sigma^2 I \cdot I = \sigma^2 I
\end{aligned}
$$

2 e) Show that $\lambda'\beta$ is estimable in model (1) iff $\lambda'\beta$ is estimable in model (2)

($\Rightarrow$) Assume $\Lambda'\beta$ estimable in model (1) $\Rightarrow \Lambda' = P'X \Rightarrow \Lambda' = P'\underbrace{QQ^{-1}}_{I}X$

$\Rightarrow \Lambda' = \underbrace{P'Q}_{P^{*\prime}}\underbrace{Q^{-1}X}_{X^*} \Rightarrow \Lambda' = P'\underbrace{Q^{-1}X}_{X^*} \Rightarrow \Lambda' = P'X^* \Rightarrow \Lambda'\beta$ estimable in model (2).

($\Leftarrow$) Assume $\Lambda'\beta$ estimable in model (2) $\Rightarrow \Lambda' = P'X^* \Rightarrow \Lambda' = \underbrace{P'Q^{-1}}_{P^{*\prime}}X \Rightarrow \Lambda' = P^{*\prime}X$

$\Rightarrow \Lambda'\beta$ estimable in model (1).

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2014, Section 2, Quad

2f. Show that $Y^T\Sigma^{-1}Y - Y_1^2/\sigma_{11}^2 \sim \chi_3^2$ given that $\beta = 0$ and $\Sigma = (\sigma_{ij})$.

Γ $Y'\Sigma^{-1}Y = \frac{1}{\sigma^2} Y'(QQ')^{-1}Y = \frac{1}{\sigma^2} Y'Q'^{-1}Q^{-1}Y = \frac{1}{\sigma^2}\underbrace{(Q^{-1}Y)'}_{Y^*}\underbrace{(Q^{-1}Y)}_{Y^*}$

Since $\underbrace{Q^{-1}Y}_{Y^*} \sim N(0, \sigma^2 I) \Rightarrow \frac{1}{\sigma} Y^* \sim N(0,1)$

$\Rightarrow \frac{1}{\sigma^2} Y^{*\prime}Y^* \sim \chi_4^2$

Also, Let $\Sigma_{11}^{-1} = \frac{1}{\sigma_{11}^2}\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{bmatrix}$

Then, $\frac{Y_1^2}{\sigma_{11}^2} = Y'\Sigma_{11}Y$. Since $Y^* = Q^{-1}Y \Rightarrow QY^* = Y$

$\Rightarrow Y'\Sigma_{11}^{-1}Y = (QY^*)'\Sigma_{11}^{-1}(QY^*) = Y^{*\prime}Q'\Sigma_{11}^{-1}QY^*$

Then, $\underbrace{Y'\Sigma^{-1}Y}_{\frac{1}{\sigma^2}Y^{*\prime}Y^*} - \underbrace{Y_1^2/\sigma_{11}^2}_{Y^{*\prime}Q'\Sigma_{11}^{-1}QY^*} = \frac{1}{\sigma^2}Y^{*\prime}Y^* - Y^{*\prime}Q'\Sigma_{11}^{-1}QY^*$

$= Y^{*\prime}\left(\frac{1}{\sigma^2}I - Q'\Sigma_{11}^{-1}Q\right)Y^*$

$\sim \chi_3^2$ ↖ rank$\left(\frac{1}{\sigma^2}I - Q'\Sigma_{11}^{-1}Q\right)$ $\lrcorner$