1. $X_1, \dots, X_n \overset{iid}{\sim} f_X(x|\theta, p)$

$$f_X(x_i|\theta,p) = \begin{cases} 0 & p \\ \text{Uniform}[0,\theta] & 1-p \end{cases}$$

Let $Y_i \sim Bern(p)$

$$\Rightarrow X_i = (1-Y_i)U_i \qquad U_i \sim \text{Uniform}(0,\theta) \perp Y_i$$

$$X_i | Y_i \overset{d}{=} \begin{cases} 0 & \text{if } Y_i = 1 \\ U_i & \text{if } Y_i = 0 \end{cases}$$

Suppose p is known, $0 < p < 1$, $\theta > 0$

$$\begin{aligned}
Var\left(\tfrac{2}{1-p} X_1\right) &= \frac{4}{(1-p)^2} Var(X_1) \\
&= \frac{4}{(1-p)^2}\Big[E_Y\big[Var(X|Y)\big] + Var_Y\big[E(X_1|Y_1)\big]\Big] \\
&= \frac{4}{(1-p)^2}\Big[E_Y\big[I(Y_1=0)Var(U_1) + I(Y_1=1)\cdot 0\big] \\
&\qquad + Var_Y\big[I(Y_1=0)E(X_1|Y_1=0) + I(Y_1=1)\cdot 0\big]\Big] \\
&= \frac{4}{(1-p)^2}\Big[Var(U_1)E_Y(I(Y_1=0)) + E(X_1|Y_1=0)\,Var(I(Y_1=0))\Big] \\
&= \frac{4}{(1-p)^2}\Big[\frac{\theta^2}{12}(1-p) + \frac{\theta}{2}(1-p)p\Big]
\end{aligned}$$

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1 a) Method 2: of unbiased estimator Let $T(X_1)$ be unbiased for θ

$$\theta = E[T(X_1)] = E_{Y_1}\left[E(T(X_1)|Y_1)\right] = E_{Y_1}\left[E(T(X_1)|Y_1=0)I(Y_1=0) + E(T(X_1)|Y_1=1)I(Y_1=1)\right]$$

$$= E[T(X_1)|Y_1=0]P(Y_1=0) + \cancel{E[T(X_1)|Y_1=1]P(Y_1=1)} \to 0$$

$$= (1-p)\int_0^\theta \frac{T(x)}{\theta}\,dx = \frac{1-p}{\theta}\int_0^\theta T(x)\,dx$$

$$\Rightarrow \frac{\theta^2}{1-p} = \int_0^\theta T(x)\,dx$$

take deriv of both sides 760 ch4 Slide 27

$$\Rightarrow \frac{d}{d\theta}\left(\frac{\theta^2}{1-p}\right) = \frac{d}{d\theta}\int_0^\theta T(x)\,dx$$

$$\Rightarrow \frac{2\theta}{1-p} = T(\theta)$$

Thus $T(X_1) = \frac{2X_1}{1-p}$

1b)

Let $X_{(n)} = \max\{X_1, \ldots, X_n\}$

Joint X_1, Y_1

$$f_{X_1,Y_1}(x_1, y_1) = f_{X_1|Y_1}(x_1|y_1) f_{Y_1}(y_1)$$

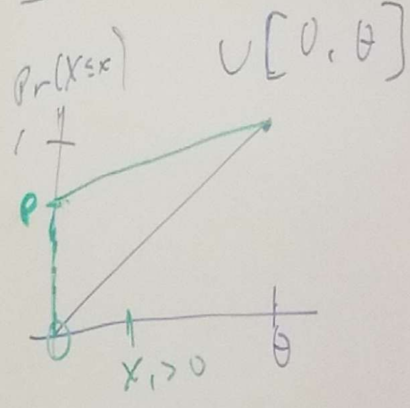
$$= I(Y_1 = 0)\left[f_{X_1|Y_1}(x_1|0) f_{Y_1}(0)\right]$$

$$+ I(Y_1 = 1)\left[f_{X_1|Y_1}(x_1|1) f_{Y_1}(1)\right]$$

$$= I(Y_1 = 0)\left[\frac{1}{\theta} I(0 \le x_1 \le \theta)(1-p)\right]$$

$$+ I(Y_1 = 1)\left[f_{X_1|Y_1}(x_1|1) f_{Y_1}(1)\right]$$

$$\left(\begin{matrix} f_{X_1|Y_1}(0|1) = 1 \\ f_{X_1|Y_1}(x_1|1) = 0 \quad x_1 \neq 0 \end{matrix}\right)$$



$$= I(Y_1 = 0)\left[\frac{1}{\theta} I(0 \le X_1 \le \theta)(1-p)\right]$$

$\frac{1}{\theta} I(0 \le x_i \le \theta)$ $$+ I(Y_1 = 1)\left[I(X_1 = 0)p\right]$$

$$= \left[I(X_1 = 0) I(Y_1 = 1) + \frac{1}{\theta} I(0 \le x_1 \le \theta) I(Y_1 = 0)\right] p^{Y_1}(1-p)^{1-Y_1}$$

$$f_{X_1}(x_1) = \sum_{Y_1=0}^{1} f_{X_1,Y_1}(x_1, y_1)$$

$$= \frac{1}{\theta} I(0 < x_1 \le \theta)(1-p) + I(X_1 = 0)p$$

$$= \frac{1}{\theta} I(0 < x_1) I(x_1 \le \theta)(1-p) + I(x_1 = 0)p$$

$$f(\vec{x}) = \prod_{i=1}^{n} \left(\frac{1-p}{\theta}\right)^{I(0 < x_i) I(x_i \le \theta)} p^{I(x_i = 0)} I(0 \le x_i \le \theta)$$

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c) For $\hat{\theta} = X_{(n)}$, CDF is

$$F_{X_{(n)}}(x) = Pr(X_{(n)} \le x)$$

$$= \left[F_{X_1}(x)\right]^n \quad \text{by iid}$$

$$= \begin{cases} \left[Pr(X_1 = 0)\right]^n & x = 0 \\ \left[Pr(X_1 = 0) + Pr(0 < X_1 \le x)\right]^n & x > 0 \end{cases}$$

$$Pr(X_1 = 0) = p$$

$$Pr(0 < X_1 \le x) = \frac{x}{\theta}(1-p)$$

$$F_{X_{(n)}}(x) = \begin{cases} 0 & x < 0 \\ \left[p + (1-p)\frac{x}{\theta}\right]^n & 0 \le x \le \theta \\ 1 & x > \theta \end{cases}$$

c)

1) For $\hat{\theta}$: $X_{(n)}$ CDF is

$$F_{X_{(n)}}(x) \triangleq Pr(X_{(n)} \le x)$$
$$= Pr(X_1, \dots X_n \le x)$$
$$= (F_{X_1}(x))^n \text{ by iid}$$
$$= (Pr(0 \le X_1 \le x))^n$$
$$= \begin{cases} [Pr(X_1 = 0)]^n & \text{for } x = 0 \\ [Pr(X_1 = 0) + Pr(0 < X_1 \le x)]^n & x > 0 \end{cases}$$

$$Pr(X_1 = 0) = Pr(Y_1 = 0, X_1 = 0) + Pr(Y_1 = 1, X_1 = 0)$$
$$= Pr(X_1 = 0 | Y_1 = 0) Pr(Y_1 = 0) + Pr(X_1 = 0 | Y_1 = 1) Pr(Y_1 = 1)$$
$$= 0 + 1 \cdot p \text{ by continuous uniform}$$
$$= p$$

$$Pr(0 < X_1 \le x) = Pr(0 < X_1 \le x, Y_1 = 0) + Pr(0 < X_1 \le x, Y_1 = 1)$$
$$= Pr(0 < X_1 \le x | Y_1 = 0) Pr(Y_1 = 0)$$
$$+ Pr(0 < X_1 \le x | Y_1 = 1) Pr(Y_1 = 1)$$
$$= Pr(0 < X_1 \le x | Y_1 = 0)(1-p)$$
$$= \frac{x}{\theta}(1-p)$$

$$\Rightarrow F_{X_{(n)}}(x) = \begin{cases} p^n & x = 0 \\ \left[p + (1-p)\frac{x}{\theta}\right]^n & x > 0 \end{cases}$$

$$\Rightarrow f_{X_{(n)}}(0) = p^n$$
$$f_{X_{(n)}}(x) = n\left[p + (1-p)\frac{x}{\theta}\right]^{n-1} \cdot \frac{1-p}{\theta}$$

Check: $\int_0^\theta f_{X_{(n)}}(x)\,dx = p^n + \int_0^\theta n\left(\frac{1-p}{\theta}\right)\left[p + \frac{(1-p)}{\theta}x\right]^{n-1} dx$

$= \rho^n + n\left(\frac{1-\rho}{\theta}\right) \int_0^\theta \left(\rho + \frac{1-\rho}{\theta}x\right)^{n-1} dx =$

$\int_0^\theta \left(\rho + \frac{1-\rho}{\theta}x\right)^{n-1} dx = \int_a^b u^{n-1} \cdot \frac{\theta}{1-\rho} du = \frac{\theta}{1-\rho} \int_a^b u^{n-1} du$

$u = \rho + \frac{1-\rho}{\theta}x$

$du = \frac{1-\rho}{\theta} dx \Rightarrow \frac{\theta}{1-\rho} du = dx$

$= \frac{\theta}{1-\rho} \cdot \frac{1}{n}\left[u^n \Big|_a^b\right]$

$= \frac{\theta}{1-\rho} \cdot \frac{1}{n}\left[\left(\rho + \frac{1-\rho}{\theta}x\right)^n \Big|_0^\theta\right]$

$= \frac{\theta}{1-\rho} \cdot \frac{1}{n}\left[\left(\rho + \frac{1-\rho}{\theta}\theta\right)^n - \left(\rho + \frac{1-\rho}{\theta} \cdot 0\right)^n\right]$

$= \frac{\theta}{1-\rho} \cdot \frac{1}{n}\left[(\rho + 1 - \rho)^n - \rho^n\right]$

$= \frac{\theta}{1-\rho} \cdot \frac{1}{n}\left[1 - \rho^n\right]$

$= \rho^n + n\left(\frac{1-\rho}{\theta}\right) \cdot \frac{\theta}{1-\rho} \cdot \frac{1}{n}\left[1 - \rho^n\right] = \rho^n + 1 - \rho^n = 1$ ✓

then $E(X_{(n)}) =$

$\rho^n \theta + \int_0^\theta x \cdot n\left(\frac{1-\rho}{\theta}\right)\left[\rho + \frac{1-\rho}{\theta}x\right]^{n-1} dx$

$= \int_0^\theta n\left(\frac{1-\rho}{\theta}\right) x \left[\rho + \frac{1-\rho}{\theta}x\right]^{n-1} dx = n\left(\frac{1-\rho}{\theta}\right) \int_0^\theta x\left[\rho + \frac{1-\rho}{\theta}x\right]^{n-1} dx$

$\int_0^\theta x\left(\rho + \frac{1-\rho}{\theta}x\right)^{n-1} dx = \int_a^b \frac{\theta}{1-\rho}(u-\rho)\, u^{n-1} \cdot \frac{\theta}{1-\rho} du = \left(\frac{\theta}{1-\rho}\right)^2 \int_a^b (u-\rho) u^{n-1} du$

$u = \rho + \frac{1-\rho}{\theta}x \qquad \frac{1-\rho}{\theta}x = u - \rho$

$\Rightarrow du = \frac{1-\rho}{\theta} dx, \qquad x = \frac{\theta}{1-\rho}(u-\rho)$

$\Rightarrow \frac{\theta}{1-\rho} du = dx$

$\int_a^b (u-\rho) u^{n-1} du = \int_a^b u^n du - \rho \int_a^b u^{n-1} du$

$= \frac{1}{n+1}\left[u^{n+1} \Big|_a^b\right] - \rho \frac{1}{n}\left[u^n \Big|_a^b\right]$

$= \frac{1}{n+1}\left[\left(\rho + \frac{1-\rho}{\theta}x\right)^{n+1} \Big|_0^\theta\right] - \frac{\rho}{n}\left[\left(\rho + \frac{1-\rho}{\theta}x\right)^n \Big|_0^\theta\right]$

$= \frac{1}{n+1}\left[(\rho + 1 - \rho)^{n+1} - (\rho)^{n+1}\right] - \frac{\rho}{n}\left[(\rho + 1 - \rho)^n - (\rho)^n\right]$

$= \frac{1}{n+1}\left[1 - \rho^{n+1}\right] - \frac{\rho}{n}\left[1 - \rho^n\right]$

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$= \frac{1}{n+1} - \frac{1}{n+1}p^{n+1} - \frac{p}{n} + \frac{1}{n}p^{n+1}$

$\Rightarrow E(X_{(n)}) =$

$\left(\frac{\theta}{1-p}\right)^2 \left[\frac{1}{n+1} - \frac{1}{n+1}p^{n+1} - \frac{1}{n}p + \frac{1}{n}p^{n+1}\right]\left[n\,\frac{1-p}{\theta}\right]$

$= \frac{\theta}{1-p} \cdot \left[\frac{n}{n+1} - \frac{n}{n+1} - p + p^{n+1}\right]$

$\rightarrow \frac{\theta}{1-p} \cdot [1-p] = \theta$ as $n \to \infty$

$\Rightarrow X_n = \hat{\theta}_n$ satisfies $E(\hat{\theta}_n) \to \theta$ in n

for consistency need $Var(\hat{\theta}_n) \to 0$

2) $Var(\hat{\theta}) = Var(X_{(n)}) = E(X_{(n)}^2) - E(X_{(n)})^2$

$E(X_{(n)}^2) =$

$\int_0^\theta x^2 n\left(\frac{1-p}{\theta}\right)\left[p + \frac{1-p}{\theta}x\right]^{n-1} dx$

$= \boxed{n\left(\frac{1-p}{\theta}\right)} \int_0^\theta x^2 \left[p + \frac{1-p}{\theta}x\right]^{n-1} dx$

$\int_0^\theta x^2 \left[p + \frac{1-p}{\theta}x\right]^{n-1} dx = \int_a^b \left(\frac{\theta}{1-p}(u-p)\right)^2 [u]^{n-1} \frac{\theta}{1-p}\, du$

$u = p + \frac{1-p}{\theta}x$

$\frac{\theta}{1-p}\, du = dx$

$x^2 = \left(\frac{\theta}{1-p}(u-p)\right)^2$

from before

$= \boxed{\left(\frac{\theta}{1-p}\right)^3} \int_a^b (u-p)^2 u^{n-1}\, du$

$\int_a^b (u^2 - 2pu + p^2) u^{n-1}\, du$

$= \int_a^b u^{n+1}\, du - 2p \int_a^b u^n\, du + p^2 \int_a^b u^{n-1}\, du$

$= \frac{1}{n+2}\left[u^{n+2}\Big|_a^b\right] - 2p \cdot \frac{1}{n+1}\left[u^{n+1}\Big|_a^b\right] + p^2 \cdot \frac{1}{n}\left[u^n\Big|_a^b\right]$

$= \frac{1}{n+2}\left[1 - p^{n+2}\right] - 2\frac{p}{n+1}\left[1 - p^{n+1}\right] + \frac{p^2}{n}\left[1 - p^n\right]$

$\Rightarrow E(X_{(n)}^2) = \frac{n(1-p)}{\theta}\left(\frac{\theta}{1-p}\right)^3 \left[\frac{1}{n+2}(1-p^{n+2}) - \frac{2p}{n+1}[1-p^{n+1}] + \frac{p^2}{n}[1-p^n]\right]$

$$= \frac{\theta^2}{(1-p)^2}\left[\frac{n}{n+2}(1-p^{n+2}) - \frac{2pn}{n+1}(1-p^{n+1}) + p^2(1-p^n)\right]$$

then $Var(X_n)$ is

$$\frac{\theta^2}{(1-p)^2}\left[\frac{n}{n+1} - \frac{n}{n+1}p^{n+1} - p + p^{n+1}\right]^2 - \frac{\theta^2}{(1-p)^2}\left[\frac{n}{n+2}(1-p^{n+2}) - \frac{2pn}{n+1}(1-p^{n+1}) + p^2(1-p^n)\right]$$

$$\rightarrow \frac{\theta^2}{(1-p)^2}\left[1-p\right]^2 - \frac{\theta^2}{(1-p)^2}\left[1-2p+p^2\right]$$

$$= \frac{\theta^2}{(1-p)^2}(1-p)^2 - \frac{\theta^2}{(1-p)^2}(1-p)^2$$

$$= 0$$

$\Rightarrow$ since $Var(\hat{\theta}) \to 0$ and $E(\hat{\theta}) \to \theta$

$\hat{\theta}$ is consistent

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$$E[g(X_1)] = E_Y[E(g(X_1)|Y_1)]$$

$$= E_Y[E(g(X_1)|Y_1=0)I(Y_1=0) + E(g(X_1)|Y_1=1)I(Y_1=1)]$$

$$= E(g(X_1)|Y_1=0)p + (1-p)E(g(X_1)|Y_1=1)$$

$$[g(0)\cdot 1]p + (1-p)\int_0^\theta \frac{1}{\theta} g(x)dx = \theta$$

$$g(0)\cdot p + \frac{1-p}{\theta}\int_0^\theta g(x)dx = \theta$$

$$\theta g(0)p + (1-p)\int_0^\theta g(x)dx = \theta^2$$

$$\frac{\partial}{\partial\theta} \text{ " } \quad \text{ " } = \frac{\partial}{\partial\theta}\theta^2$$

$$g(0)p + (1-p)g(\theta) = 2\theta \Rightarrow$$ any $g()$ satisfing this is unbiased

$$g(x) = \begin{cases} \frac{2x_i}{1-p} - \frac{g(0)p}{1-p} & x>0 \\ g(0) & x=0 \end{cases}$$

$*\ \theta > 0$

Suppose $g(0)=1$

$$\Rightarrow p + (1-p)g(\theta) = 2\theta$$

$$\Rightarrow g(\theta) = \frac{2\theta - p}{1-p}$$

Suppose $g(0)=0$

$$\Rightarrow g(\theta) = \frac{2\theta}{1-p}$$

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