Theory Exam Section I 2013

1). $(X_1, Y_1), \ldots, (X_n, Y_n)$ iid

$f(x, y; \alpha) = \alpha(\alpha+1)(1+x+y)^{-(\alpha+2)} \; [x, y > 0, \; \alpha > 0]$

- joint density of x & y

(a) Show MLE of α ($\hat{\alpha}$) has the following properties

(i) $\hat{\alpha}$ exists, is unique, & has form $g^{-1}(\hat{\mu}_n)$

$$\hat{\mu}_n = \frac{1}{n} \sum_{i=1}^{n} \log(1 + x_i + y_i)$$

g^{-1} = inverse of some function g.

- give form of g & show g^{-1} exists.

$$l(\alpha) = \log \alpha + \log(\alpha+1) - (\alpha+2)\log(1+x+y)$$

$$l_n(\alpha) = \sum_{i=1}^{n} [\log \alpha + \log(\alpha+1) - (\alpha+2)\log(1+x_i+y_i)] = \sum_{i=1}^{n} l(x_i, y_i; \alpha)$$

$$\frac{\partial}{\partial \alpha} l_n(\alpha) = \frac{n}{\alpha} + \frac{n}{\alpha+1} - \sum_{i=1}^{n} \log(1+x_i+y_i) \overset{set}{=} 0$$

$$\Rightarrow n\left(\frac{\overbrace{(\alpha+1)+\alpha}^{2\alpha+1}}{\alpha(\alpha+1)}\right) = \sum_{i=1}^{n} \log(1+x_i+y_i)$$

$$\Rightarrow \frac{2\alpha+1}{\alpha(\alpha+1)} = \frac{1}{n}\sum_{i=1}^{n} \log(1+x_i+y_i)$$

$$\Rightarrow g(\alpha) = \frac{1}{n}\sum_{i=1}^{n} \log(1+x_i+y_i)$$

$$\Rightarrow \hat{\alpha} = g^{-1}\left(\tfrac{1}{n}\textstyle\sum_{i=1}^{n} \log(1+x_i+y_i)\right) = b$$

$$g(\alpha) = \frac{2\alpha+1}{\alpha(\alpha+1)} = b$$

$$\Rightarrow \alpha = \frac{2b+1}{b(b+1)}$$ rearrange & solve for b to get inverse eqn.

$\Rightarrow b(b+1)\alpha = 2b+1$

$\Rightarrow b^2\alpha + b\alpha = 2b+1$

$\Rightarrow \alpha b^2 + (\alpha-2)b - 1 = 0$

$$\Rightarrow b = \frac{(\alpha-2) \pm \sqrt{(\alpha-2)^2 - 4\alpha(-1)}}{2\alpha}$$

$$= \frac{(\alpha-2) \pm \sqrt{\alpha^2 - \cancel{4\alpha} + 4 + \cancel{4\alpha}}}{2\alpha}$$

$$= \frac{(\alpha-2) \pm \sqrt{\alpha^2+4}}{2\alpha} \Rightarrow \boxed{\hat{\alpha} = \frac{(b-2) \pm \sqrt{b^2+4}}{2b}}$$

$g(g^{-1}(\alpha)) \overset{OR}{=} g^{-1}(g(\alpha)) = \alpha$ Check:

$$g^{-1}(g(\alpha)) = \frac{g(\alpha) - 2 \pm \sqrt{g(\alpha)^2 + 4}}{2\alpha}$$

$$= \frac{\left(\frac{2\alpha+1}{\alpha(\alpha+1)}\right) - 2 \pm \sqrt{\frac{(2\alpha+1)^2 + 4}{\alpha^2(\alpha+1)^2}}}{2\left(\frac{2\alpha+1}{\alpha(\alpha+1)}\right)}$$

$$= \frac{2\alpha + 1 - 2 \pm \sqrt{(4\alpha^2 + 2\alpha + 1 + 4(\alpha^2(\alpha+1)^2)}}{4\alpha + 2.}$$

...

Uniqueness:

$$\frac{\partial^2}{\partial\alpha^2} \ln(\alpha) = \frac{-n}{\alpha^2} - \frac{n}{(\alpha+1)^2} < 0 \text{ for all } \alpha \text{ + } n = 1, \dots, \infty$$

$\Rightarrow g^{-1}(\hat{\alpha}) = \hat{\mu}_n$ is the unique maximum.

(ii) $\hat{\alpha}_n \xrightarrow{a.s.} \alpha_0$ where α_0 = true value of α

If we can show $g(\hat{\alpha}) = \frac{1}{n}\sum_{i=1}^{n} \log(1+x_i+y_i) \xrightarrow{a.s.} g(\alpha_0)$

Then by the continuous mapping thm
$\hat{\alpha} \xrightarrow{a.s.} \alpha_0$

By the strong law of large #s,

$$\frac{1}{n}\sum_{i=1}^{n} \log(1+x_i+y_i) \xrightarrow{a.s.} E[\log(1+x_i+y_i)]$$

$$E[\log(1+x_i+y_i)] = \int_0^\infty \int_0^\infty \alpha(\alpha+1)\log(1+x+y)(1+x+y)^{-(\alpha+2)}\,dx\,dy$$

$$u = \log(1+x+y)^{\alpha+1}$$

$$du = \frac{1}{(1+x+y)^{\alpha+1}}(\alpha+1)(1+x+y)^{\alpha}$$

...

$$\begin{cases} z = \log(1+x+y) \\ w = x \end{cases}$$

$$\Rightarrow \begin{cases} x = w \\ y = e^z - 1 - w \end{cases}$$

$$f_{(z,w)}(z,w) = f_{x,y}(x=w,\ y=e^z-1-w)\,|J|$$

$$J = \begin{vmatrix} \partial x/\partial z & \partial x/\partial w \\ \partial y/\partial z & \partial y/\partial w \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ e^z & -1 \end{vmatrix} = -e^z$$

$$|J| = |-e^z| = e^z$$

$$f_{Z,W}(z,w) = \alpha(\alpha+1)(\cancel{1}+\cancel{w}+e^z-\cancel{1}-\cancel{w})^{-(\alpha+2)}(e^z)$$
$$= \alpha(\alpha+1)\, e^{z(-(\alpha+2)+1)} \qquad -\alpha-2+1 = -\alpha-1 = -(\alpha+1)$$
$$= \alpha(\alpha+1)\exp(-z(\alpha+1))$$

$$f_Z(z) = \int_w f_{Z,W}(z,w)\,dw$$
$$= \int_w \alpha(\alpha+1)\exp(-z(\alpha+1))\,dw$$

$w = x$

~~$y = e^z - 1 - w \Rightarrow w = e^z - 1 - y$~~

$\log(1+x+y) = z \Rightarrow e^z = 1+x+y$

$\Rightarrow x+y = e^z - 1$

$\Rightarrow \max(x) = x+y$ (if $y=0$) $= e^z - 1$

$\Rightarrow$ range of w: 0 to $e^z - 1$

$$= \int_0^{e^z-1} \alpha(\alpha+1)\exp(-z(\alpha+1))\,dw$$
$$= \alpha(\alpha+1)\exp(-\overbrace{z(\alpha+1)}^{-z\alpha-z})(e^z-1)$$
$$= \alpha(\alpha+1)\left[\exp(-z\alpha) - \exp(-z(\alpha+1))\right]$$

$$E[z] = (E[\log(1+x+y)])$$
$$= \int_0^\infty z\,\alpha(\alpha+1)\left[\exp(-z\alpha) - \exp(-z(\alpha+1))\right]dz$$

Note: $z = \log(1+x+y)$

if $x = y = 0 \Rightarrow z = \log(1) = 0$

if $x+y \to \infty \Rightarrow z = \log(\infty) \to \infty$

$\Leftrightarrow \quad u = z\alpha \quad ; \quad v = (\alpha+1)z \quad \Leftrightarrow \quad z = v/(\alpha+1)$

$dz = du/\alpha \quad du = \alpha\, dz \qquad dv = (\alpha+1)\,dz \qquad dz = dv/(1+\alpha)$

$$\int_0^\infty u e^{-u}\,du = 1!$$

$$= \int_0^\infty u \frac{(\alpha+1)}{\alpha} \exp(-u)\,du - \int_0^\infty v \frac{\alpha}{1+\alpha} \exp(-v)\,dv$$

$$= \left(\frac{\alpha+1}{\alpha}\right) 1! - \left(\frac{\alpha}{1+\alpha}\right) 1!$$

$$= \frac{(\alpha+1)(\alpha+1) - \alpha^2}{\alpha(1+\alpha)}$$

$$= \frac{\cancel{\alpha^2} + 2\alpha + 2 - \cancel{\alpha^2}}{\alpha(\alpha+1)}$$

$$= \frac{2\alpha + 2}{\alpha(\alpha+1)}$$

$$= g(\alpha) \checkmark$$

$$\Rightarrow \frac{1}{n} \sum_{i=1}^n \log(1 + x_i + y_i) \xrightarrow{a.s.} g(\alpha_0)$$

By the continuous mapping thm

$$\Rightarrow g^{-1}\left(\frac{1}{n} \sum_{i=1}^n \log(1 + x_i + y_i)\right) \xrightarrow{a.s.} \alpha_0 \checkmark$$

(iii) $\sqrt{n}(\hat{\alpha}-\alpha_0) \xrightarrow{d} N(0, \sigma^2)$

$$\sigma^2 = \frac{\alpha_0^2(\alpha_0+1)^2}{\alpha_0^2+(\alpha_0+1)^2}$$

By MLE theory,

$\sqrt{n}(\hat{\alpha}-\alpha_0) \xrightarrow{d} N(0, I^{-1}(\alpha))$

$$I(\alpha) = E[-\partial^2/\partial\alpha^2 \, l(\alpha)]$$

$$= E\left[-\left(\frac{\partial}{\partial\alpha}\left(\frac{1}{\alpha}+\frac{1}{\alpha+1}-\log(1+x_i+y_i)\right)\right)\right]$$

$$= E\left[-\left(\frac{-1}{\alpha^2}-\frac{1}{(\alpha+1)^2}\right)\right]$$

$$= E\left[\frac{1}{\alpha^2}+\frac{1}{(\alpha+1)^2}\right]$$

$$= \frac{(\alpha+1)^2+\alpha^2}{\alpha^2(\alpha+1)^2}$$

$$\Rightarrow I^{-1}(\alpha_0) = \frac{\alpha_0^2(\alpha_0+1)^2}{(\alpha_0+1)^2+\alpha_0^2}$$ ✓

(b) $X_1, \dots, X_n$ known + fixed

$Y_1, \dots, Y_n$ indep. sample

Y_i is drawn from conditional dist $Y_i | X_i = x_i$

Unconditional joint density (Y_i, X_i) given in part (a)

Show dist of $Y_i | X_i = x_i$ is:

$$\tilde{f}_i(y_i, \alpha) = (\alpha+1)(1+x_i)^{-1}\left(1 + \frac{y_i}{1+x_i}\right)^{-(\alpha+2)} \quad y > 0$$

$$P(Y | X) = \frac{P(Y, X)}{P(X)}$$

Find $g(x) = \int_0^\infty f(x, y; \alpha)\, dx$

$$\Rightarrow \tilde{f}(y; \alpha) = \frac{f(x, y; \alpha)}{g(x)}$$

$-\alpha - 2 + 1 = -\alpha - 1 = -(\alpha+1)$

$$g(x) = \int_0^\infty \alpha(\alpha+1)(1+x+y)^{-(\alpha+2)}\, dy$$

$$= \alpha \cancel{(\alpha+1)} \frac{(1+x+y)^{-(\alpha+1)}}{-\cancel{(\alpha+1)}} \Bigg|_{y=0}^{y=\infty}$$

$$= -\alpha\left[\cancel{\frac{1}{(\infty)^{\alpha+1}}}^{\to 0} - \frac{1}{(1+x)^{\alpha+1}}\right]$$

$$= \frac{\alpha}{(1+x)^{\alpha+1}}$$

$$\frac{\cancel{\alpha}(\alpha+1)(1+x+y)^{-(\alpha+2)}}{\cancel{\alpha}(1+x)^{-(\alpha+1)}}$$

$$= (\alpha+1)\frac{(1+x+y)^{-(\alpha+2)}}{(1+x)^{-(\alpha+2)}(1+x)}$$

$$= (\alpha+1)(1+x)^{-1}\left(\frac{1+x+y}{1+x}\right)^{-(\alpha+2)}$$

$$= (\alpha+1)(1+x)^{-1}\left(1 + \frac{y}{1+x}\right)^{-(\alpha+2)}$$

✓

ⓒ (Setting of ⓑ)

Verify the MLE $\tilde{\alpha}_n$ has following properties:

ⓘ $\tilde{\alpha}_n$ exists, unique, & can be expressed in closed form

$$l_n(\alpha|x) = \sum_{i=1}^{n}\left[\log(\alpha+1) - \log(1+x_i) - (\alpha+2)\log(1+y_i/1+x_i)\right]$$

$$\frac{\partial}{\partial\alpha} l_n(\alpha|x) = \frac{n}{\alpha+1} - \sum_{i=1}^{n}\log(1+y_i/1+x_i) \overset{\text{set}}{=} 0$$

$$\Rightarrow \frac{n}{\alpha+1} = \sum_{i=1}^{n}\log\left(1+\frac{y_i}{1+x_i}\right)$$

$$\Rightarrow \boxed{\left[\frac{1}{n}\sum_{i=1}^{n}\log\left(1+\frac{y_i}{1+x_i}\right)\right]^{-1} - 1 = \tilde{\alpha}}$$

Uniqueness:

$$\frac{\partial^2}{\partial\alpha^2} l_n(\alpha|x) = \frac{-n}{(\alpha+1)^2} < 0 \quad \forall \alpha \;\&\; n$$

$\Rightarrow$ $\tilde{\alpha}$ must be a unique maximum

(ii) Show $\hat{\alpha}_n \xrightarrow{a.s.} \alpha_0$

If we can show $\hat{\mu}_n \xrightarrow{a.s.} \frac{1}{\alpha+1}$

$\Rightarrow (\hat{\mu}_n)^{-1} \xrightarrow{a.s.} \alpha+1$ by continuous mapping thm

$(\hat{\mu}_n > 0 \ \forall x, y)$

$\Rightarrow (\hat{\mu}_n)^{-1} - 1 \xrightarrow{a.s.} \alpha$

$\Rightarrow \tilde{\alpha} \xrightarrow{a.s.} \alpha$ ✓

By the strong law of large #s,

$\frac{1}{n}\sum_{i=1}^{n} \log(1+x_i+y_i) \xrightarrow{a.s.} E[\log(1+x_i+y_i)]$

- Expectation wrt $Y_i|X_i$ dist

$E[\log(1+x_i+y_i)] = ?$ (*) - Alternative integration method →

$z = \log(1+x_i+y_i) \Rightarrow y = e^z - 1 - w$

$w = x \qquad\qquad x = w$ (after next pg)

$f_{z,w}(z,w) = f_{x,y}(x=w, y=e^z-1-w)\,|J|$

$|J| = e^z$ like before

$= (\alpha+1)(1+x)^{-1}\left(\frac{1+x+y}{1+x}\right)^{-(\alpha+2)}|J| \Big|_{x=w,\ y=e^z-1-w}$

$= (\alpha+1)(1+w)^{-1}\left(\frac{e^z}{1+w}\right)^{-(\alpha+2)} e^z$

$= (\alpha+1)(1+w)^{\alpha+1}\exp(-z(\alpha+1))$

$f_z(z) = \int_0^{e^z-1} f_{z,w}(z,w)\,dz$

$= (\alpha+1)\exp(-z(\alpha+1))\int_0^{e^z-1}(1+w)^{\alpha+1}\,dw$

→

$E[\log(1+y/_{1+x})]$ given x:

$$\int_0^\infty \frac{(\alpha+1)}{1+x}\log(1+y/_{1+x})\left(1+\frac{y}{1+x}\right)^{-(\alpha+2)}dy$$

$$u = \log\left(1+y/_{1+x}\right)$$
$$\Rightarrow e^u = 1 + y/_{1+x}$$

$$du = \frac{1}{1+y/_{1+x}}\left(\frac{1}{1+x}\right)dy$$
$$= \frac{1}{e^u(1+x)}dy \Rightarrow e^u(1+x)\,du = dy$$

$$= \int_0^\infty \frac{(\alpha+1)}{(1+\cancel{x})}(e^u(1+\cancel{x}))\,u\exp(-(\alpha+2)u)\,du$$

$$= \int_0^\infty (\alpha+1)\,u\exp(-(\alpha+1)u)\,du$$

$$v = (\alpha+1)u$$
$$dv = (\alpha+1)du \Leftrightarrow du = \frac{1}{\alpha+1}dv$$

$$= (\alpha+1)\left(\frac{\Gamma(2)}{(\alpha+1)^2}\right)\underbrace{\int_0^\infty \frac{(\alpha+1)^2}{\Gamma(2)}u^{2-1}\exp(-u(\alpha+1))\,du}_{\text{pdf} = 1}$$

$$= \frac{1}{\alpha+1}$$ ✓

OR

$$= \int_0^\infty \frac{v}{\alpha+1}e^{-v}\,dv$$

$$= \frac{1}{\alpha+1}$$ ✓

(iii) $\sqrt{n}(\tilde{\alpha} - \alpha_0) \xrightarrow{d} N(0, \sigma^2 = h(\alpha_0))$

By MLE theory,

$\sqrt{n}(\tilde{\alpha} - \alpha_0) \xrightarrow{d} N(0, I^{-1}(\alpha))$

$$I(\alpha) = E\left[\frac{-\partial^2}{\partial \alpha^2} \ell(\alpha | x)\right]$$

$$E\left[-\frac{\partial}{\partial \alpha}\left(\frac{1}{\alpha+1} - \log(1 + x_i + y_i)\right)\right]$$

$$E\left[-\left(\frac{-1}{(\alpha+1)^2}\right)\right]$$

$$= \frac{1}{(\alpha+1)^2}$$

$\Rightarrow I^{-1}(\alpha) = (\alpha+1)^2$

$\Rightarrow h(\alpha_0) = (\alpha_0+1)^2$

(d) Asymptotic relative efficiency $\tilde{\alpha}_n$ to $\hat{\alpha}_n$:

$$\frac{I^{-1}(\alpha | X)}{I^{-1}(\alpha)}$$

$$= (\alpha_0 \cancel{+1})^2 \left(\frac{\alpha_0^2 + (\alpha_0+1)^2}{\alpha_0^2 \cancel{(\alpha_0+1)^2}} \right)$$

$$= \frac{\alpha_0^2 + (\alpha_0+1)^2}{\alpha_0^2}$$

$$= 1 + (1 + 1/\alpha_0)^2$$

$$> 1$$

$\Rightarrow$ $\tilde{\alpha}_n$ has greater asymptotic var

$\Rightarrow$ $\hat{\alpha}_n$ is more efficient (smaller asymptotic var).

2) (a) For each $\Theta_0 \in \Theta$, let T_{Θ_0} = test of $H_0: \Theta = \Theta_0$
w/ significance level α & acceptance region $A(\Theta_0)$

$$C(y) = \{\Theta : y \in A(\Theta)\}$$

Show $C(y)$ = level $1-\alpha$ confidence set

By defn of $C(y)$, $\Theta \in C(y) \iff y \in A(\Theta)$
Defn of confidence set:

$P(\Theta \in C(y)) \geq 1-\alpha$

— Prob of true $\Theta \in C(y) \geq 1-\alpha$

Since T_{Θ_0} is a test w/ significance level α,

$P(y \in A(\Theta)^c) \leq \alpha$ (prob of being in rejection region | Θ true)

$\iff 1 - P(y \in A(\Theta)^c) \geq 1-\alpha$

$\iff P(y \in A(\Theta)) \geq 1-\alpha$

$\iff P(\Theta \in C(y)) \geq 1-\alpha$ ✓

ⓑ Suppose $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

Note: random sample = iid

$\sigma^2 = \gamma\mu^2 \qquad -\infty < \mu < \infty, \quad \gamma > 0$

γ, μ both unknown, $\mu \neq 0$

Using part ⓐ, derive a confidence set for γ w/ confidence coefficient $1-\alpha$ by inverting the acceptance region of the likelihood ratio test for testing

$H_0: \gamma = \gamma_0$ vs. $H_A: \gamma \neq \gamma_0$

$$f(\underset{\sim}{x}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sqrt{\gamma\mu^2}} \exp\left(-\frac{1}{2}\frac{1}{\gamma\mu^2}(x_i-\mu)^2\right)$$

$$\log f(\underset{\sim}{x}) = \sum_{i=1}^{n}\left[-\frac{1}{2}\log 2\pi - \frac{1}{2}\log\gamma - \frac{1}{2}\log\mu^2 - \frac{1}{2}\frac{1}{\gamma\mu^2}(x_i-\mu)^2\right]$$

$$= -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\gamma - n\log\mu - \frac{1}{2}\left(\frac{1}{\gamma\mu^2}\right)\sum_{i=1}^{n}(x_i-\mu)^2$$

ⓘ MLE under general case:

$$\frac{\partial}{\partial\gamma}\log f(\underset{\sim}{x}) = -\frac{n}{\cancel{2}}\left(\frac{1}{\gamma}\right) - \frac{\cancel{1}}{2}\left(-\frac{1}{\gamma^2}\right)\left(\frac{1}{\mu^2}\right)\Sigma_{i=1}^{n}(x_i-\mu)^2 \overset{set}{=} 0$$

$$\Rightarrow -n\gamma + \frac{1}{\mu^2}\Sigma_{i=1}^{n}(x_i-\mu)^2 = 0$$

$$\Rightarrow \frac{1}{n\mu^2}\Sigma_{i=1}^{n}(x_i-\mu)^2 = \gamma$$

Find MLE of μ ($\hat{\mu}$) and

$$\hat{\gamma} = \frac{1}{n\hat{\mu}^2}\Sigma_{i=1}^{n}(x_i-\hat{\mu})^2$$

$$\frac{\partial}{\partial \mu} \log f(\underset{\sim}{x}) = \frac{-n}{\mu} - \frac{1}{\cancel{2}\gamma}\left[\sum_{i=1}^{n}(x_i-\mu)^2\left(\frac{\cancel{-2}}{\mu^3}\right) + \left(\frac{1}{\mu^2}\right)(\cancel{2})\sum_{i=1}^{n}(x_i-\mu)(\cancel{-1})\right] \stackrel{set}{=} 0$$

$$\Rightarrow -n\mu^2 + \frac{1}{\gamma}\left[\sum_{i=1}^{n}(x_i-\mu)^2 + \mu\sum_{i=1}^{n}(x_i-\mu)\right] = 0$$

$$\Rightarrow -n\mu^2 + \frac{1}{\gamma}\left[\sum_{i=1}^{n}x_i^2 - 2\mu\sum_{i=1}^{n}x_i + \cancel{n\mu^2} + \mu\sum_{i=1}^{n}x_i - \cancel{n\mu^2}\right] = 0$$

$$\Rightarrow -n\mu^2 + \frac{1}{\gamma}\sum_{i=1}^{n}x_i^2 - \frac{\mu}{\gamma}\sum_{i=1}^{n}x_i = 0$$

$$\Rightarrow -n\mu^2\gamma + \sum_{i=1}^{n}x_i^2 - \mu\sum_{i=1}^{n}x_i = 0$$

$$\Rightarrow \mu^2(-n\gamma) + \mu\left(-\sum_{i=1}^{n}x_i\right) + \sum_{i=1}^{n}x_i^2 = 0$$

$$\Rightarrow \mu = \frac{-\left(-\sum_{i=1}^{n}x_i\right) \pm \sqrt{\left(\sum_{i=1}^{n}x_i\right)^2 - 4(-n\gamma)\left(\sum_{i=1}^{n}x_i^2\right)}}{2(-n\gamma)}$$

$$= \left(\frac{-b \pm \sqrt{b^2-4ac}}{2a}\right)$$

$$\boxed{\hat{\mu} = \frac{\sum_{i=1}^{n}x_i \pm \sqrt{\left(\sum_{i=1}^{n}x_i\right)^2 + 4n\gamma\sum_{i=1}^{n}x_i^2}}{-2n\gamma}}$$

(ii) Under H_0 case:

$\hat{\gamma}_0 = \gamma_0$

$$\hat{\mu}_0 = \frac{\sum_{i=1}^n x_i \pm \sqrt{(\sum_{i=1}^n x_i)^2 + 4n\gamma_0 \sum_{i=1}^n x_i^2}}{-2n\gamma_0}$$

(from work in part (a) (i), but plug in $\gamma = \gamma_0$ instead of $\gamma = \hat{\gamma}$)

$$LRT = \Lambda = \frac{\sup_{\theta \in \Theta_0} L(\mu, \gamma)}{\sup_{\theta \in \Theta} L(\mu, \gamma)}$$

$$= \frac{L(\hat{\mu}_0, \gamma_0)}{L(\hat{\mu}, \hat{\gamma})}$$

reject H_0 when $\Lambda < K$

(d) $x_1, \ldots, x_n \overset{iid}{\sim} N(\mu, \sigma^2)$

μ, σ^2 scalar parameters.

$\theta = (\mu, \phi) \quad (\phi = \sigma^2) \quad \theta$ unknown

Derive $1-\alpha$ asymptotically correct confidence sets for μ by: inverting acceptance regions for

(i) the likelihood ratio test
(ii) the Wald test
(iii) the score test.

$H_0: \mu = \mu_0$

$H_A: \mu \neq \mu_0$

Invert test of these hypotheses

(i) the likelihood ratio test

$-2 \log \Lambda \xrightarrow{d} \chi^2(r)$

$r = 1$ since only interested in 1 parameter.

$$\Lambda = \frac{\sup_{\theta \in \Theta_0} L(\underset{\sim}{x})}{\sup_{\theta \in \Theta} L(\underset{\sim}{x})}$$

Under general case:

$$\hat{\mu} = \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \hat{\mu})^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

(could show work elsewhere)

Under H_0 case:

$$\hat{\mu}_0 = \mu_0$$

$$\hat{\sigma}_0^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu_0)^2$$

Plug in generic μ instead of a hypothesized value μ_0

$$\mu, \quad \tilde{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu)^2$$

$$\ell(\underset{\sim}{x}) = \log f(\underset{\sim}{x}) = -\frac{1}{2}\log 2\pi - \frac{1}{2}\log \sigma^2 - \frac{1}{2}\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$-2\log\Lambda = -2\Big[-\tfrac{1}{2}\log\tilde{\sigma}^2 - \tfrac{1}{2}\tfrac{1}{\tilde{\sigma}^2}\textstyle\sum_{i=1}^{n}(x_i-\mu)^2 + $$
$$- \left(-\tfrac{1}{2}\log\hat{\sigma}^2 - \tfrac{1}{2}\tfrac{1}{\hat{\sigma}^2}\textstyle\sum_{i=1}^{n}(x_i-\hat{\mu})^2\right)\Big]$$

$$= \log\tilde{\sigma}^2 + \frac{1}{\tilde{\sigma}^2}\sum_{i=1}^{n}(x_i-\mu)^2 - \log\hat{\sigma}^2 - \frac{1}{\hat{\sigma}^2}\sum_{i=1}^{n}(x_i-\bar{x})^2$$

$$\tilde{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2 \qquad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2$$

$$= \log\left(\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2\right) + n - \log\left(\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\right) - n$$

$$= \log\left(\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2\right) - \log\left(\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\right)$$

$$= \log\left(\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right)$$

$$= \log\left(\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2 + n(\mu-\bar{x})^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right)$$

$$= \log\left(1 + \frac{n(\mu-\bar{x})^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right)$$

Since $-2\log\Lambda \xrightarrow{d} \chi^2(1)$

$\Rightarrow (1-\alpha)100\%$ CR is the following:

$$\left\{\mu : \log\left(1 + \frac{n(\mu-\bar{x})^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right) \le \chi^2(1, 1-\alpha)\right\}$$

(ii) Wald test:

$$W_n = \cancel{(\hat{\mu}-\mu_0)^T I_n(\hat{\mu})(\hat{\mu}-\mu_0)} \xrightarrow{d} \chi^2(1)$$

$$= (R\hat{\theta} - R\theta_0)^T (R I_n(\hat{\theta})^{-1} R^T)^{-1} (R\hat{\theta} - R\theta) \xrightarrow{d} \chi^2(1)$$

$$R = [1 \ 0] \quad \theta = \begin{bmatrix} \mu \\ \sigma^2 \end{bmatrix}$$

$\hat{\mu} = \bar{x}$ like before (MLE under general case)

$$I_n(\hat{\theta}) = E[-\partial^2/\partial\theta^2 \, l_n(\underset{\sim}{x})]$$

$$l_n(\underset{\sim}{x}) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\sigma^2 - \frac{1}{2}\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$\frac{\partial}{\partial\mu} l(\underset{\sim}{x}) = -\frac{1}{\cancel{2}}\frac{1}{\sigma^2}(-\cancel{2})\sum_{i=1}^{n}(x_i-\mu)$$

$$\frac{\partial^2}{\partial\mu^2} l(\underset{\sim}{x}) = \frac{1}{\sigma^2}(-n)$$

$$\Rightarrow -E[\partial^2/\partial\mu^2 \, l_n(\underset{\sim}{x})] = \frac{n}{\sigma^2}$$

$$\frac{\partial^2}{\partial\mu\partial\sigma^2} l_n(\underset{\sim}{x}) = \left(\frac{-1}{\sigma^4}\right)\sum_{i=1}^{n}(x_i-\mu)$$

$$E[-\partial^2/\partial\mu\partial\sigma^2] = \frac{1}{\sigma^2}\sum_{i=1}^{n} E(x_i-\mu) = 0$$

$$\frac{\partial^2 l_n(\underset{\sim}{x})}{\partial(\sigma^2)^2} = -B$$

$$E[-\partial^2/\partial(\sigma^2)^2 \, l_n(\underset{\sim}{x})] = E(B)$$

$$I_n(\theta) = \begin{bmatrix} n/\sigma^2 & 0 \\ 0 & E(B) \end{bmatrix}$$

$$I_n(\theta)^{-1} = \begin{bmatrix} \sigma^2/n & 0 \\ 0 & E(B)^{-1} \end{bmatrix}$$

$$(R I_n(\theta)^{-1} R^T)^{-1} = (\sigma^2/n)^{-1} = \frac{n}{\sigma^2}$$

$$W_n = (\bar{x} - \mu_0)^2 \left(\frac{n}{\hat{\sigma}^2} \right)$$

$$= (\bar{x} - \mu_0)^2 \left(\frac{n}{\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2} \right)$$

$1-\alpha$ level CR:

$$\left\{ \mu : (\bar{x} - \mu)^2 \left(\frac{n}{\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2} \right) \le \chi^2(1, 1-\alpha) \right\}$$

– Could rearrange if time

(iii) Score test:

$$Sc_n = \dot{\ell}(\tilde{\mu}, \tilde{\sigma}^2)^T I_n^{-1}(\tilde{\mu}, \tilde{\sigma}^2)\, \dot{\ell}(\tilde{\mu}, \tilde{\sigma}^2)$$

$$\frac{\partial}{\partial(\sigma^2)} l_n(\underline{x}) = -\frac{n}{2} \frac{1}{\sigma^2} - \frac{1}{2}\left(\frac{-1}{\sigma^4} \right) \sum_{i=1}^{n} (x_i - \mu)^2$$

$$\frac{\partial^2}{\partial(\sigma^2)^2} l_n(\underline{x}) = -\frac{n(-1)}{2} \frac{1}{\sigma^4} + \frac{1}{\cancel{2}}\left(\frac{-\cancel{2}}{\sigma^6} \right) \sum_{i=1}^{n} (x_i - \mu)^2$$

$$E\left[-\partial^2 / \partial(\sigma^2)^2\, l_n(\underline{x}) \right] =$$

$$= -\frac{n}{2} \frac{1}{\sigma^4} + \frac{1}{\sigma^6} n \sigma^2$$

$$= -\frac{n}{2} \frac{1}{\sigma^4} + \frac{n}{\sigma^4}$$

$$= \frac{n}{2} \left(\frac{1}{\sigma^4} \right)$$

$$I_n^{-1}(\mu, \sigma^2) = \begin{bmatrix} \sigma^2/n & 0 \\ 0 & 2\sigma^4/n \end{bmatrix}$$

$\tilde{\mu}, \tilde{\sigma}^2$ = MLE under H_0

Just let $\tilde{\mu} = \mu$

Find $\tilde{\sigma}^2$

$$\tilde{\sigma}^2 = \frac{1}{n} \Sigma_{i=1}^{n} (x_i - \mu)^2$$

- solves $\frac{\partial}{\partial \sigma^2} \ln(\underset{\sim}{x}) = 0$

$$\dot{\ell}(\mu, \tilde{\sigma}^2) = \begin{bmatrix} \partial/\partial\mu \ \ln(\underset{\sim}{x}) \ \big|_{\sigma^2 = \tilde{\sigma}^2} \\ \partial/\partial\sigma^2 \ \ln(\underset{\sim}{x}) \ \big|_{\sigma^2 = \tilde{\sigma}^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\tilde{\sigma}^2} \sum_{i=1}^{n} (x_i - \mu) \\ \\ 0 \end{bmatrix}$$

$$\dot{\ell}(\mu, \tilde{\sigma}^2)^T I_n(\mu, \tilde{\sigma}^2)^{-1} \dot{\ell}(\mu, \tilde{\sigma}^2)$$

$$= \left(\frac{1}{\tilde{\sigma}^2} \sum_{i=1}^{n} (x_i - \mu)\right)^2 \left(\frac{\tilde{\sigma}^2}{n}\right)$$

$$= \frac{\left(\Sigma_{i=1}^{n} (x_i - \mu)\right)^2}{\tilde{\sigma}^2/n}$$

$$= \frac{n\left(\Sigma_{i=1}^{n} (x_i - \mu)\right)^2}{\frac{1}{n} \Sigma_{i=1}^{n} (x_i - \mu)^2}$$

$1 - \alpha$ level CR:

$$\left\{ \mu : \frac{n\left(\Sigma_{i=1}^{n} (x_i - \mu)\right)^2}{\frac{1}{n} \Sigma_{i=1}^{n} (x_i - \mu)^2} \leq \chi^2(1, 1-\alpha) \right\}$$

Are these sets always intervals?

- No
- LRT & Wald test $1-\alpha$ level CR's can easily be made into intervals
- Score test $1-\alpha$ CR would be difficult to convert into a CI & should just be left as a region.

Wald:

$$\frac{n(\bar{x}-\mu)^2}{\frac{1}{n}\sum_{i=1}^n (x_i-\bar{x})^2} \xrightarrow{d} \chi^2(1) \quad , \quad \frac{n(\bar{x}-\mu)^2}{\frac{1}{n}\sum_{i=1}^n (x_i-\bar{x})^2} \leq \chi^2(1,1-\alpha)$$

$$\Leftrightarrow \frac{\sqrt{n}(\bar{x}-\mu)}{\sqrt{\frac{1}{n}\sum_{i=1}^n (x_i-\bar{x})^2}} \xrightarrow{d} N(0,1) \quad , \quad \Leftrightarrow \left|\frac{\sqrt{n}(\bar{x}-\mu)}{\sqrt{\frac{1}{n}\sum_{i=1}^n (x_i-\bar{x})^2}}\right| \leq \sqrt{\chi^2(1,1-\alpha)}$$

$$\Leftrightarrow \mu : \bar{x} \pm (s/\sqrt{n})\, z_{1-\alpha/2} \quad , \quad \Leftrightarrow -\sqrt{\chi^2(1,1-\alpha)} \leq \frac{\bar{x}-\mu}{s/\sqrt{n}} \leq \sqrt{\chi^2(1,1-\alpha)}$$

$$\Leftrightarrow \bar{x} - (s/\sqrt{n})\sqrt{\chi^2(1,1-\alpha)} \leq \mu \leq \bar{x} + (s/\sqrt{n})\sqrt{\chi^2(1,1-\alpha)}$$

(e) $X_1, \ldots, X_n$ iid $E(X_i) = \mu$, $Var(X_i) = \sigma^2$

Interested in CI for $\theta = \mu/\sigma$

By Multivariate CLT,

$$\sqrt{n}\left(\begin{bmatrix} \frac{1}{n}\sum_{i=1}^{n} X_i \\ \frac{1}{n}\sum_{i=1}^{n} X_i^2 \end{bmatrix} - \begin{bmatrix} E(X_i) \\ E(X_i^2) \end{bmatrix}\right) \xrightarrow{d} N\left(\underset{\sim}{0}, \begin{bmatrix} Var(X_i) & Cov(X_i, X_i^2) \\ Cov(X_i, X_i^2) & Var(X_i^2) \end{bmatrix}\right.$$

$$E[X_i] = \mu \quad = A$$

$$E[X_i^2] = Var(X_i) + (E(X_i))^2$$
$$= \sigma^2 + \mu^2 = B$$

$$Cov(X_i, X_i^2) = E[X_i^3] - E(X_i)E(X_i^2)$$
$$= E[X_i^3] - \mu(\sigma^2 + \mu^2)$$
$$= \gamma$$

$$Var(X_i^2) = E(X_i^4) - (E(X_i^2))^2$$
$$= E(X_i^4) - (\sigma^2 + \mu^2)^2$$
$$= \delta$$

Since 4th moment is finite,

$\Rightarrow$ $E(X_i^4)$ & $E(X_i^3)$ finite

$\Rightarrow$ $Cov(X_i, X_i^2)$ finite & $Var(X_i^2)$ finite

Delta method:

$$\sqrt{n}\left(g\left(\bar{X}, \tfrac{1}{n}\sum X_i^2\right) - g(\overset{A}{\mu}, \overset{B}{\sigma^2 + \mu^2})\right) \xrightarrow{d} N(0, \nabla g \, \Sigma \, \nabla g^T)$$

$$g(\mu, \sigma^2 + \mu^2) = \frac{\mu}{\sigma}$$

$$= \frac{\mu}{\sqrt{(\sigma^2 + \mu^2) - \mu^2}} = \frac{A}{\sqrt{B - A^2}}$$

$$\frac{\partial}{\partial A} g(A,B) = (1)\frac{1}{\sqrt{B-A^2}} + A\left(-\tfrac{1}{2}\right)(B-A^2)^{-3/2}(-2A)$$

$$= \frac{1}{\sqrt{B-A^2}} + \frac{A^2}{(B-A^2)^{3/2}} \Bigg|_{A=\mu,\ B=\sigma^2+\mu^2}$$

$$= \frac{1}{\sigma} + \frac{\mu^2}{\sigma^3}$$

$$\frac{\partial}{\partial B} g(A,B) = \left(-\tfrac{1}{2}\right)(B-A^2)^{-3/2}(1)(A) \Bigg|_{A=\mu,\ B=\sigma^2+\mu^2}$$

$$= -\frac{1}{2} \frac{A}{(B-A^2)^{3/2}}$$

$$= \frac{-1}{2}\left(\frac{\mu}{\sigma^3}\right)$$

$$\nabla g(\mu,\sigma^2) = \left[\frac{1}{\sigma} + \frac{\mu^2}{\sigma^3},\ \frac{-\mu}{2\sigma^3}\right]$$

By the Delta Method,

$$\sqrt{n}\left(\frac{\frac{1}{n}\sum_{i=1}^{n} x_i}{\sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2}} - \frac{\mu}{\sigma}\right) \xrightarrow{d}$$

$$N\left(0,\ \left[\frac{1}{\sigma} + \frac{\mu^2}{\sigma^3},\ \frac{-\mu}{2\sigma^3}\right]\begin{bmatrix}\sigma^2 & \gamma \\ \gamma & \delta\end{bmatrix}\begin{bmatrix}1/\sigma + \mu^2/\sigma^3 \\ -\mu/2\sigma^3\end{bmatrix}\right)$$

where $\gamma = E(x_i^3) - \mu(\sigma^2+\mu^2)$
$\delta = E(x_i^4) - (\sigma^2+\mu^2)^2$

For notational purposes, let the variance be denoted α $(\alpha = \nabla g(\mu,\sigma^2)\, \Sigma\, \nabla g(\mu,\sigma^2)^T)$

Then $1-\alpha$ level CI:

$\sqrt{n}(Y-\theta) \xrightarrow{d} N(0,\alpha)$ (Y defined below, $\theta = \mu/\sigma^2$)

$\Rightarrow \dfrac{\sqrt{n}(Y-\theta)}{\sqrt{\alpha}} \xrightarrow{d} N(0,1)$

Replace μ + σ^2 terms in α with a consistent estimate for μ + σ^2

- $\hat{\mu} = \bar{x}$ = MLE of μ
- $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2$ = MLE of σ^2

Denote $\alpha \Big|_{\mu=\hat{\mu},\, \sigma=\hat{\sigma}} = \hat{\alpha}$

Then By Slutsky's thm, since $\hat{\mu}$ + $\hat{\sigma}^2$ consistent for $\hat{\mu}$ + $\hat{\sigma}^2$, $\Rightarrow$ $\hat{\alpha}$ consistent for α by the continuous mapping thm ($\alpha = h(\mu,\sigma^2)$)

($\Rightarrow \hat{\alpha} \xrightarrow{P} \alpha$)

$\Rightarrow \dfrac{\sqrt{n}(Y-\theta)}{\sqrt{\hat{\alpha}}} \xrightarrow{d} N(0,1)$ by Slutsky's thm

$\Rightarrow$ $1-\alpha$ level CI for θ: (asymptotic)

$$\boxed{Y - \sqrt{\hat{\alpha}/n}\, z_{1-\alpha/2} \le \mu \le Y + \sqrt{\hat{\alpha}/n}\, z_{1-\alpha/2}}$$

where $Y = \dfrac{\frac{1}{n}\sum_{i=1}^{n} x_i}{\sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2}}$

and $z_{1-\alpha/2}$ = $1-\alpha/2$ quantile of $N(0,1)$ dist

3). r-sided coin

side i w/ prob p_i such that $\sum_{i=1}^{r} p_i = 1$

Positive integers $n_1, n_2, \dots, n_r$ (given)

N_i = # flips required until side i has appeared for the n_i time

$N = \min(N_i)$

- N = # of flips required until some side i has appeared n_i times

(a) Derive the marginal dist of N_i $i = 1, \dots, r$

$N_i \sim$ neg bin (n_i, p_i)

- Total $= N_i$
- in $N_i - 1$ flips, have $n_i - 1$ successes
- in N_i^{th} flip, have success

$$f(N_i) = \binom{N_i - 1}{n_i - 1} p_i^{n_i} (1-p_i)^{N_i - n_i}$$

(b) Prove whether or not N_i = indep. RV's.

A & B indep = $P(A)P(B) = P(A \cap B)$

Find joint dist of $\underset{\sim}{N_i}$

- is this $\prod P(N_i)$?

Logic approach:

Suppose wlog that in the first N_1 flips, there are n_1 successes in choosing side 1

If this is the case, then in the first N_1 flips, there can be at most $N_1 - n_1$ successes in any other side.

Suppose for side 2 our given $n_2 >$ the resulting $N_1 - n_1$

If $n_2 > N_1 - n_1$, then $N_1 \neq N_2$ even if all other flips in first N_1 flips are on side 2.

Since there is this restriction $\Rightarrow N_i$ not indep.

Now suppose flips performed at random times generated by a Poisson process w/ rate $\lambda = 1$

T_i = time until side i has appeared for the n_i time

$i = 1, \dots, r$

$T = \min(T_i)$

(c) Marginal dist of T_i

If flips occur by Pois($\lambda = 1$) process, then time between flips $\sim$ Exp($\lambda = 1$) $\Rightarrow$

$$f(t_j - t_{j-1}) = \frac{1}{\lambda} \exp(-(t_j - t_{j-1})/\lambda)$$

$$= \exp(-(t_j - t_{j-1}))$$

let $\Delta_j = t_j - t_{j-1}$ $\quad j = 1, \dots, r$

$\Rightarrow f(\Delta_j) = \exp(-\Delta_j)$

$$T_K = \sum_{j=1}^{K} (t_j - t_{j-1}) = \sum_{j=1}^{K} \Delta_j$$

$\Rightarrow$ if $\Delta_j \sim \text{Exp}(1) \equiv \text{Gamma}(1,1)$

$\Rightarrow T_K \sim \text{Gamma}(K, 1)$

$T_i \mid N_i \sim \text{Gamma}(N_i, 1)$

$P(T_i, N_i) = P(T_i \mid N_i) P(N_i)$

$\Rightarrow P(T_i) = \int_{N_i} P(T_i \mid N_i) P(N_i) \, dN_i$

$P(T_i|N_i)\, P(N_i)$

$$= \frac{1}{\Gamma(N_i)} T_i^{N_i-1} \exp(-T_i) \cdot \binom{N_i-1}{n_i-1} p_i^{n_i} (1-p_i)^{N_i-n_i}$$

$$\binom{N_i-1}{n_i-1} = \frac{(N_i-1)!}{(n_i-1)!\,(N_i-n_i)!} = \frac{\Gamma(N_i)}{\Gamma(n_i)\Gamma(N_i-n_i+1)}$$

$$= T_i^{N_i-1} \exp(-T_i) \frac{p_i^{n_i}(1-p_i)^{N_i-n_i}}{(n_i-1)!\,(N_i-n_i)!}$$

$$\sum_{N_i = n_i \leftarrow \min N_i}^{\infty} T_i^{N_i-1} \exp(-T_i) \frac{p_i^{n_i}(1-p_i)^{N_i-n_i}}{(n_i-1)!\,(N_i-n_i)!}$$

$$= \frac{T_i^{n_i-1}}{(n_i-1)!} \exp(-T_i)\, p_i^{n_i} \sum_{N_i-n_i=0}^{\infty} \frac{T_i^{N_i-n_i}(1-p_i)^{N_i-n_i}}{(N_i-n_i)!}$$

$$= \frac{T_i^{n_i-1}}{(n_i-1)!} \exp(-T_i)\, p_i^{n_i} \exp(T_i(1-p_i))$$

$$\left(\text{Note: } \sum_{x=0}^{\infty} \frac{\mu^x}{x!} = \exp(\mu) \right)$$

$$= \frac{p_i^{n_i}}{\Gamma(n_i)} T_i^{n_i-1} \exp(-T_i p_i)$$

$\Rightarrow T_i \sim \text{Gamma}(n_i, 1/p_i)$ ✓

(d) T_i indep?