2017 Section 1

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1. (25 points) Let N be Poisson distributed with parameter $0 < \lambda < \infty$, and let $X_1, X_2, \ldots$ be an i.i.d. sequence of positive random variables, independent of N, with $E\log(X_1) = \mu$, $\text{var}[\log(X_1)] = \sigma^2$, $|\mu| < \infty$, $0 < \sigma^2 < \infty$, and $M(\delta) = EX_1^\delta < \infty$ for some $\delta > 0$. Let $Y = \prod_{i=1}^N X_i$, where $\prod_{i=1}^0$ is defined as 1. Do the following:

   (a) (4 points) Show that $E\log Y = \lambda\mu$ and $\text{var}[\log Y] = \lambda(\sigma^2 + \mu^2)$.

   (b) (5 points) Show that $EY^t = \exp(\lambda[M(t) - 1])$, for all $0 \le t \le \delta$.

   (c) (7 points) Show that $Y^{1/\lambda} \to_p e^\mu$, as $\lambda \to \infty$.

   (d) (9 points) Letting $\tau^2 = \lambda(\sigma^2 + \mu^2)$, show that

   $$\left(e^{-\lambda\mu}Y\right)^{1/\tau} \to_d e^Z,$$

   as $\lambda \to \infty$, where $Z \sim N(0, 1)$.

2. (25 points) Let F and G be two distinct known cumulative distribution functions on the real line and X be a single observation from the cumulative distribution function $\theta F(x) + (1-\theta)G(x)$, where $\theta \in [0,1]$ is unknown.

(a) (4 points) Given $0 < \theta_0 < 1$, derive a Uniformly Most Powerful (UMP) test of size α for testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$. You need to specify how the rejection region can be calculated .

(b) (6 points) Given $0 < \theta_1 < \theta_2 < 1$, derive a UMP test of size α for testing $H_0 : \theta \in [0, \theta_1] \cup [\theta_2, 1]$ versus $H_1 : \theta \in (\theta_1, \theta_2)$.

(c) (6 points) Show that a UMP test does not exist for testing $H_0 : \theta \in [\theta_1, \theta_2]$ versus $\theta \notin [\theta_1, \theta_2]$.

(d) (5 points) Obtain a Uniformly Most Powerful Unbiased (UMPU) test of size α for the problem in part (c).

(e) (4 points) Given $0 < \theta_1 < \theta_2 < 1$, derive the likelihood ratio test statistic for testing $H_0 : \theta \in [\theta_1, \theta_2]$ versus $\theta \notin [\theta_1, \theta_2]$.

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1. a) Show that $E[\log(Y)] = \lambda\mu$ and $Var[\log(Y)] = \lambda(\sigma^2+\mu^2)$.

Γ Given $Y = \prod_{i=1}^{N} X_i \Rightarrow \log(Y) = \sum_{i=1}^{N} \log(X_i) \Rightarrow E[\log(Y)] = E\left[E\left[\sum_{i=1}^{N} \log(X_i) \mid N\right]\right]$

$$= \sum_{i=1}^{E[N]} E[\log(X_i) \mid N] = E[N]\cdot\mu = \boxed{\lambda\mu}$$

& $Var(\log(Y)) = E\left[Var\left(\sum_{i=1}^{N} \log(X_i) \mid N\right)\right] + Var\left[E\left(\sum_{i=1}^{N} \log(X_i) \mid N\right)\right]$

$$= E\left[\sum_{i=1}^{N} Var(\log(X_i) \mid N)\right] + Var\left(\sum_{i=1}^{N} \underbrace{E[\log(X_i) \mid N]}_{\mu}\right)$$

$$= E[N]\cdot\sigma^2 + Var(N\mu) = \lambda\cdot\sigma^2 + \mu^2 Var(N) = \lambda\cdot\sigma^2 + \mu^2\cdot\lambda = \boxed{\lambda(\sigma^2+\mu^2)}$$ $\lrcorner$

1. b) Show that $E[Y^t] = \exp(\lambda[M(t)-1])$ $\forall$ $0 \le t \le \delta$

Γ $E[Y^t] = E\left[\left(\prod_{i=1}^{N} X_i\right)^t\right] = E\left[\left(\prod_{i=1}^{N} X_i^t\right)\right] = E\left[E\left(\prod_{i=1}^{N} X_i^t \mid N\right)\right] = E\left[\prod_{i=1}^{N} E(X_i^t \mid N)\right] = E\left[(E(X_i^t))^N\right]$

$$= E\left[(M(t))^N\right] = E\left[\underbrace{\exp(N \log M(t))}_{e^{N\log(M(t))}}\right]$$

Know, in general, the MGF of a RV Z is $M_Z(t) = E[e^{Zt}]$.

Here, we have $E[e^{N \underbrace{\log(M(t))}}] = M_N(\log(M(t)))$

call this t for now

Know $N \sim Pois(\mu) \Rightarrow M_N(t) = \sum_{N=0}^{\infty} \frac{\lambda^N e^{-\lambda}}{N!} \cdot e^{Nt} = e^{-\lambda} \underbrace{\sum_{N=0}^{\infty} \frac{(\lambda e^t)^N}{N!}} = e^{-\lambda}\cdot e^{\lambda e^t}$

$$= e^{\lambda(e^t-1)}$$

if you memorized mgf of Poisson, then no need to do this. I did not!

in general, $\sum_{K=0}^{\infty} \frac{z^K}{K!} = e^z$

$$\Rightarrow E[Y^t] = M_N(\log(M(t))) = e^{\lambda(e^{\log(M(t))}-1)} = e^{\lambda(M(t)-1)}, \forall\, 0 \le t \le \delta.$$ $\lrcorner$

1 c) Show that $Y^{1/\lambda} \xrightarrow[p]{} e^{\mu}$ as $\lambda \to \infty$.

⌈ By Chebyshev ineq., in general, for a RV X w/ finite mean $E[X]$ and $Var[X]$, for $k \in \mathbb{R}$,

(thanks to Emily D. for this suggestion)

$$P(|X - E[X]| \geq k) \leq \frac{Var[X]}{k^2}.$$

Also, note that, by $\overbrace{\text{continuous mapping thm}}^{\text{CMT}}$, $1/\lambda \log(Y) \xrightarrow[p]{} \mu$ as $\lambda \to \infty$ is equivalent to showing that $Y^{1/\lambda} \xrightarrow[p]{} e^{\mu}$ as $\lambda \to \infty$.

Since $E[1/\lambda \log(Y)] = 1/\lambda \, E[\log(Y)] = 1/\lambda \cdot \overbrace{(\lambda\mu)}^{\text{part a)}} = \mu$ (finite)

& $Var[1/\lambda \log(Y)] = 1/\lambda^2 \, Var[\log(Y)] = 1/\lambda^2 \cdot \lambda(\sigma^2 + \mu^2) = \frac{1}{\lambda}(\sigma^2 + \mu^2)$ (finite since $Var[1/\lambda \log(Y)] = \frac{1}{\lambda}(\sigma^2+\mu^2) \to 0$ as $\lambda \to \infty$.)

then, by Chebyshev's Ineq.,

$$P(|1/\lambda \log(Y) - \underbrace{E[1/\lambda \log(Y)]}_{\lambda\mu}| \geq k) \leq \frac{Var[1/\lambda \log(Y)]}{k^2} = \frac{\sigma^2 + \mu^2}{\lambda k^2} \to 0 \text{ as } \lambda \to \infty$$

$$\Rightarrow \underbrace{P(|1/\lambda \log(Y) - \lambda\mu| \geq k) \to 0}_{\text{Defn. of convergence in probability.}} \text{ as } \lambda \to \infty.$$

Thus, $1/\lambda \log(Y) \xrightarrow[p]{} \mu$ as $\lambda \to \infty$

$\underset{\text{by CMT}}{\Rightarrow}$ $Y^{1/\lambda} \xrightarrow[p]{} e^{\mu}$ as $\lambda \to \infty$. ⌋

1 d) Letting $\tau^2 = \lambda(\sigma^2+\mu^2)$, show that $(e^{-\lambda\mu}Y)^{1/\tau} \xrightarrow{d} e^Z$ as $\lambda \to \infty$ where $Z \sim N(0,1)$.

⌈ By continuity theorem, if $\lim_{n\to\infty} M_n(t) = M(t)$ $\forall t$ around 0 (also works for CF),

then $X_n \xrightarrow{d} X$ as $n \to \infty$.

So, if we can show that, for $W = (e^{-\lambda\mu}Y)^{1/\tau}$, have $\lim_{\lambda\to\infty} M_{\log(W_\lambda)}(t) = M_{\log(W)}(t)$,

then we can claim that $\log(W) \xrightarrow{d} Z \underset{\text{by CMT}}{\Rightarrow} W \xrightarrow{d} e^Z$.

Take $M_{\log(W_\lambda)}(t) = E[e^{t\log(W_\lambda)}] = E[e^{t(1/\tau(-\lambda\mu) + 1/\tau \log(Y))}] = e^{\frac{-\lambda\mu t}{\tau}} E[e^{t\log(Y)/\tau}]$

$$= e^{-\lambda\mu t/\tau} E[Y^{t/\tau}] \underset{\text{part c)}}{=} e^{-\lambda\mu t/\tau} e^{\lambda\underbrace{[M(t/\lambda)-1]}_{f(x)}}$$

$$\underset{\substack{\text{Taylor series expansion} \\ \text{around 0 (using 0 b/c of contny theorem)}}}{=} e^{-\lambda\mu t/\tau} \cdot e^{\lambda[\underbrace{M(0)-1}_{f(a)} + \underbrace{\dot{M}(0)}_{f'(a)}\underbrace{(t/\tau - 0)}_{(x-a)} + \underbrace{\ddot{M}(0)}_{f''(a)}\underbrace{\frac{(t/\tau-0)^2}{2}}_{\frac{(x-a)^2}{2!}} + o(t^2/\tau^2)]}$$

Thanks Euphy!

Note that: $M[\delta] = E[X_1^\delta] = E[e^{\delta\log(X_1)}] = M_{\log(X_1)}(\delta)$

Then, $M[0] = E[X_1^0] = M_{\log(X_1)}(0) = E[\log(X_1)^0] = 1$

$\dot{M}[0] = E[X_1^1] = \dot{M}_{\log(X_1)}(0) = E[\log(X_1)^1] = \mu$

$\ddot{M}[0] = E[X_1^2] = \ddot{M}_{\log(X_1)}(0) = E[\log(X_1)^2] = Var[\log(X_1)] + E[\log(X_1)]^2 = \sigma^2 + \mu^2$

$$\Rightarrow M_{\log(W_\lambda)}(t) = e^{-\lambda\mu t/\tau} \cdot e^{\lambda[\cancel{1}-\cancel{1} + \mu t/\tau + (\sigma^2+\mu^2)t^2/2\tau^2 + o(t^2/\tau^2)]}$$

$$= e^{\cancel{-\lambda\mu t/\tau} + \cancel{\lambda\mu t/\tau} + \frac{\cancel{\lambda(\sigma^2+\mu^2)} t^2}{2\cancel{\lambda(\sigma^2+\mu^2)}} + o\left(\frac{\cancel{\lambda} t^2}{\cancel{\lambda}(\sigma^2+\mu^2)}\right)}$$

$$\xrightarrow[\text{as } \lambda\to\infty]{} e^{t^2/2}$$, which is the MGF of a standard normal random variate.

Thus, by continuity thm, $\log(W) \xrightarrow{d} Z \underset{\text{by CMT}}{\Rightarrow} W \xrightarrow{d} e^Z$ as $\lambda \to \infty$

where $Z \sim N(0,1)$. ⌋

2. a) Given $0 < \theta_0 < 1$, derive a UMP test of size α for testing $H_0: \theta \leq \theta_0$ vs. $H_1: \theta > \theta_0$.

You need to specify how the rejection region can be calculated.

[Take $f(x)$ to be the Radon-Nikodym derivative of $F(x)$ w.r.t. $F(x) + G(x)$.

$g(x)$ to be the Radon-Nikodym derivative of $G(x)$ w.r.t. $F(x) + G(x)$.

$\Rightarrow h(x) = \theta f(x) + (1-\theta) g(x)$ is the pdf of X.

Take $0 < \theta_1 < \theta_2 < 1$. Then,

$$\frac{p(\theta_2)}{p(\theta_1)} = \frac{\theta_2 f(x) + (1-\theta_2) g(x)}{\theta_1 f(x) + (1-\theta_1) g(x)} = \frac{\theta_2 f(x)/g(x) + (1-\theta_2)}{\theta_1 f(x)/g(x) + (1-\theta_1)} = \frac{\theta_2 y(x) + 1 - \theta_2}{\theta_1 y(x) + 1 - \theta_1} = \frac{\theta_2 (y(x)-1) + 1}{\theta_1 (y(x)-1) + 1}$$

where $y(x) = \frac{f(x)}{g(x)}$. Can see that this ratio $\frac{p(\theta_2)}{p(\theta_1)}$ has MLR property in $Y(x)$.

Thus, the UMPU level α test has the form

$$\phi(x) = \begin{cases} 1, & Y(x) > c \quad \leftarrow \text{direction of alternative} \\ \gamma, & Y(x) = c \\ 0, & Y(x) < c \end{cases}$$

where $\alpha = E_0[\phi(x)] = P_0(Y(x) > c) + \gamma P_0(Y(x) = c)$ can be used to find c and γ.]

2 b) Given $0 < \theta_1 < \theta_2 < 1$, derive a UMP test of size α for testing $H_0: \theta \in [0, \theta_1] \cup [\theta_2, 1]$ vs. $H_1: \theta \in (\theta_1, \theta_2)$.

[Take power $= \beta(\theta) = \int \phi [\theta f(x) + (1-\theta) g(x)] d(F+G) = \theta \int \phi f(x) d(F+G) + \int \phi g(x) d(F+G)$

$- \theta \int \phi g(x) d(F+G) = \underbrace{\theta \int \phi [f(x) + g(x)] d(F+G)}_{m} + \underbrace{\int \phi g(x) d(F+G)}_{b}$

$= m\theta + b \Rightarrow \beta(\theta)$ is a linear function of θ.

Thus, $\beta(\theta)$ satisfies $\underbrace{E_\theta[\phi(x)]}_{\beta} \leq \alpha \quad \forall \theta \in \Theta_0$.

Since $\phi(x)$ is level α and $\beta = \alpha$ is max power $\Rightarrow \phi(x) \equiv \alpha$ is UMP.]

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2 c) Show that a UMP test doesn't exist for testing $H_0: \theta \in [\theta_1, \theta_2]$ vs. $\theta \notin [\theta_1, \theta_2]$.

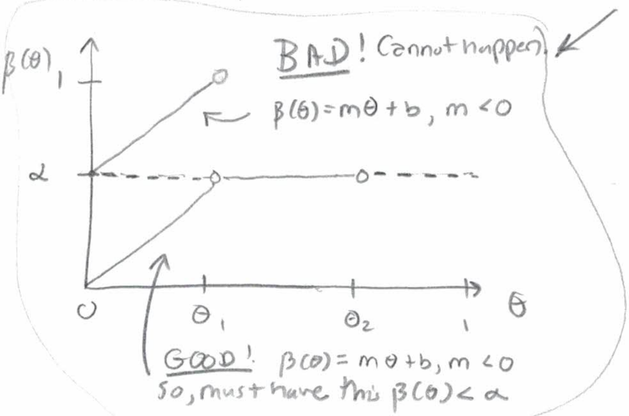
Suppose $\phi^*(x)$ is UMP

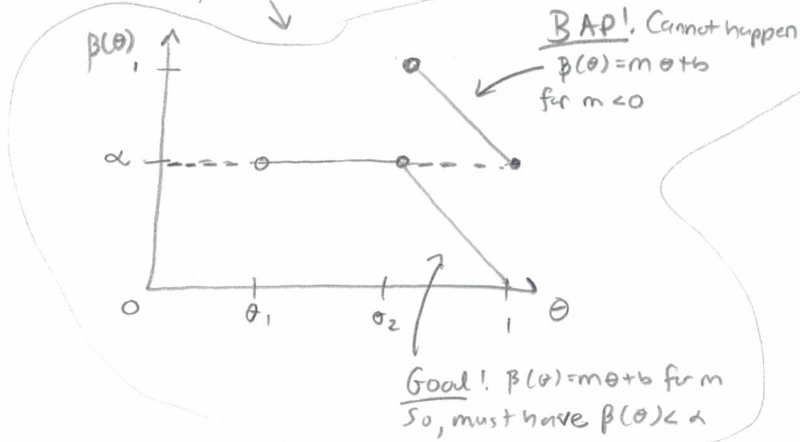
From b), showed $\beta_{\phi^*}(x) = m\theta + b$, a linear fn of θ.

If $m > 0 \Rightarrow$ for $\theta < \theta_1$ must have $\beta_{\phi^*}(x) < \alpha$ (power can't be greater than α, b/c that would imply that power is increasing up from α at $\theta = 0$, which couldn't happen).

If $m < 0 \Rightarrow$ for $\theta > \theta_2$, must have $\beta_{\phi^*}(x) < \alpha$ (power can't be greater than α b/c that would force power to always be > 0).

Here is a picture of why $\beta_{\phi^*}(x) \not> \alpha$ for $\theta < \theta_1$ and $\beta_{\phi^*}(x) \not> \alpha$ for $\theta > \theta_2$.





Notice that $\phi^*(x)$ is not as powerful as $\phi(x) \equiv \alpha$ for $m < 0$ or $m > 0$.

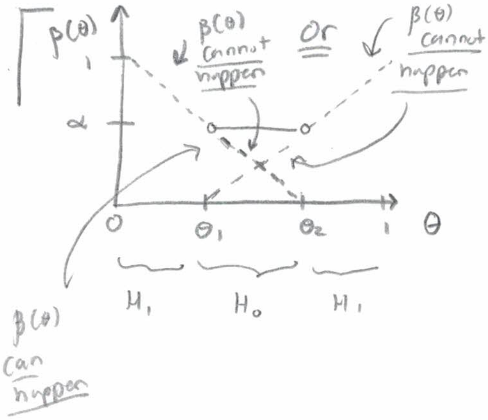
What about for $m = 0$? I.e., what about $\phi^*(x) = \phi(x) \equiv \alpha$. Is this possible?

Recall that, in part a), we showed that $\phi(x)$ is UMP level α for testing $H_0: \theta \le \theta_0$ vs. $H_1: \theta > \theta_0$.

Thus, the UMP test of size α for testing $H_0: \theta \le \theta_2$ vs. $H_1: \theta > \theta_2$ has power $> \alpha$ at $\theta_0 \in (\theta_2, 1]$. Thus, $\phi^*(x)$ is more powerful than $\phi(x) \equiv \alpha$ at θ_0.

Thus, $\phi^*(x)$ cannot be UMP.

2d) Obtain a UMPU test of size α for the problem in c).



Let $\tilde{T}(x)$ be an unbiased test.
In order to produce an unbiased test,

by defn. 2.2, must have $\beta_{\tilde{T}}(\theta) \leq \alpha$ under H_0 and $\beta_{\tilde{T}}(\theta) \geq \alpha$ under H_1.

We already showed the power function is linear. So, the slope of the power function, i.e. m, ← in $\beta(\theta) = m\theta + b$ must be 0. Otherwise, $m < 0$ for $\theta < \theta_1$ and $m > 0$ for $\theta > \theta_2$ to create an unbiased test, which isn't possible because $\beta(\theta)$ is linear.

Thus, $\tilde{T}(x) \equiv \alpha$ is UMPU.

2e) Given $0 < \theta_1 < \theta_2 < 1$, derive the LRT for testing $H_0 : \theta \in [\theta_1, \theta_2]$ vs. $H_1 : \theta \notin [\theta_1, \theta_2]$

LRT statistic: $\Lambda = \dfrac{\sup_{\theta \in \Theta_0} l(\theta)}{\sup_{\theta \in \Theta} l(\theta)}$, Here $l(\theta) = \theta f(x) + (1-\theta) g(x)$
$= \theta(f(x) - g(x)) + g(x)$

Under Θ (full space, $0 \leq \theta \leq 1$):

Notice that if $f(x) - g(x) \geq 0 \Rightarrow f(x) \geq g(x)$ for $0 \leq \theta \leq 1$, then $l(\theta) = \theta f(x) + (1-\theta) g(x)$ is at its max when $l(\theta) = f(x)$.

If $g(x) > f(x)$ for $0 \leq \theta \leq 1$, then $l(\theta) = \theta f(x) + (1-\theta) g(x)$ is at its max when $l(\theta) = g(x)$.

$$\Rightarrow \sup_{0 \leq \theta \leq 1} l(\theta) = \begin{cases} f(x), & f(x) \geq g(x) \\ g(x), & f(x) < g(x) \end{cases}$$

Under Θ_0 (null space, $\theta \in [\theta_1, \theta_2]$)

If $f(x) - g(x) \geq 0 \Rightarrow f(x) \geq g(x)$. For $\theta \in [\theta_1, \theta_2]$, want to maximize the first term in the likelihood $l(\theta) = \theta[\underbrace{f(x) - g(x)}_{>0}] + g(x)$, so will choose largest value in interval of θ_2.

If $f(x) < g(x)$, for $\theta \in [\theta_1, \theta_2]$, want to minimize the first term in $l(\theta) = \theta[\underbrace{f(x) - g(x)}_{<0}] + g(x)$ to maximize the likelihood. So, will choose smallest value in interval of θ_1.

$$\Rightarrow \sup_{\theta \in [\theta_1, \theta_2]} l(\theta) = \begin{cases} \theta_2[f(x) - g(x)] + g(x), & f(x) \geq g(x) \\ \theta_1[f(x) - g(x)] + g(x), & f(x) < g(x) \end{cases}$$

LRT statistic: Thus,

$$\boxed{\Lambda = \frac{\sup_{\theta \in \Theta_0} l(\theta)}{\sup_{\theta \in \Theta} l(\theta)} = \begin{cases} \dfrac{\theta_2[f(x) - g(x)] + g(x)}{f(x)}, & f(x) \geq g(x) \\ \dfrac{\theta_1[f(x) - g(x)] + g(x)}{g(x)}, & f(x) < g(x) \end{cases}}$$