DISCLAIMER: The solutions are a little messy because I show the process of trying something but it being the wrong approach

1.) $T_i \sim Exp(1/\lambda) = \frac{1}{\lambda}\exp(-\frac{1}{\lambda}x)$

$\lambda = E[T_i]$

$R_i | u_i = I(T_i \in [0, u_i]) \sim Bern(p_i)$

$p_i = P(T_i \in [0, u_i])$

$= \int_0^{u_i} \frac{1}{\lambda}\exp(-\frac{1}{\lambda}x)\,dx$

$= \frac{1}{\lambda}\int_0^{u_i} \exp(-\frac{1}{\lambda}x)dx$

$= \frac{1}{\lambda}\left[-\lambda\exp(-\frac{1}{\lambda}x)\right|_0^{u_i}$

$= \frac{1}{\lambda}\left[-\lambda\exp(-\frac{u_i}{\lambda}) + \lambda\right]$

$= 1-\exp(-\frac{u_i}{\lambda})$

observed (R_i, T_i)

$$f(R_i, u_i) = \underbrace{p_i^{R_i}(1-p_i)^{1-R_i}}_{f(R_i|u_i)} \cdot \underbrace{1}_{f(u_i)}$$

$$= \left[1-\exp(-\frac{u_i}{\lambda})\right]^{R_i}\left[\exp(-\frac{u_i}{\lambda})\right]^{1-R_i}$$

$R_i = I(T_i \in [0, u_i])$

$1-R_i = I(T_i \notin [0, u_i])$

$= I(T_i \in [u_i, \infty))$

$$L(R_i, u_i) = \prod_{i=1}^{n}\left[1-\exp(-\frac{u_i}{\lambda})\right]^{R_i}\left[\exp(-\frac{u_i}{\lambda})\right]^{1-R_i}$$

$$\ell(R_i, u_i) = \sum_{i=1}^{n}\left\{R_i\left(\log(1-\exp(-\frac{u_i}{\lambda}))\right) - (1-R_i)\left(\frac{u_i}{\lambda}\right)\right\}$$

$$\frac{\partial \ell}{\partial \lambda} = \sum_{i=1}^{n}\left\{R_i\left[\frac{\exp(-\frac{u_i}{\lambda})(\frac{u_i}{\lambda^2})}{1-\exp(-\frac{u_i}{\lambda})}\right. + (1-R_i)\left(\frac{u_i}{\lambda^2}\right)\right\}$$

$E[R_i] = E_{u_i}[E_{R_i}[R_i|u_i]] = E_{u_i}[1-\exp(-u_i/\lambda)]$

$= 1 - E[\exp(-u_i/\lambda)] = 1 - \int_0^1 \exp(-\frac{u_i}{\lambda})\,du_i$

$= 1+\lambda\left[\exp(-\frac{u_i}{\lambda})\right|_0^1 = 1+\lambda\exp(-\frac{1}{\lambda}) - \lambda$

(b) $\theta = \frac{1}{\lambda}$

$$l(R_i, u_i \mid \theta) = \sum_{i=1}^{n} \{ R_i [\log(1 - \exp(-u_i\theta))] - (1 - R_i)(u_i\theta) \}$$

$$\frac{\partial l}{\partial \theta} = \sum_{i=1}^{n} \left\{ R_i \frac{(-\exp(-u_i\theta))(-u_i)}{1 - \exp(-u_i\theta)} - (1 - R_i)(u_i) \right\} \overset{set}{=} 0$$

$$\Rightarrow \sum_{i=1}^{n} \left\{ \frac{R_i u_i \exp(-u_i\theta)}{1 - \exp(-u_i\theta)} - (1 - R_i)(u_i) \right\} = 0$$

$\sum \frac{x_i}{y_i} = 1$

$\not\Rightarrow \sum_i x_i = \sum_i y_i$

$\sum_i (x_i - y_i) = 0$

$\Rightarrow \sum_i x_i = \sum_i y_i$

can't separate product within summation

The solution of

"$\sum_{i=1}^{n} \frac{R_i u_i}{\exp(u_i\theta) - 1} = \sum_i (1 - R_i)(u_i)$" is $\hat{\theta}_{MLE}$

Assuming regularity conditions hold:

$$\sqrt{n}(\hat{\lambda}-\lambda) \xrightarrow[d]{} N(0, I_1(\lambda)^{-1})$$

$$I_n(\lambda) = E\left[-\frac{\partial^2 l}{\partial \lambda^2}\right] =$$

$$\frac{\partial l}{\partial \lambda} = \frac{\partial \theta}{\partial \lambda} \cdot \frac{\partial l}{\partial \theta}$$

$$\frac{\partial^2 l}{\partial \lambda^2} = \frac{\partial}{\partial \lambda}\left(\frac{\partial \theta}{\partial \lambda} \cdot \frac{\partial l}{\partial \theta}\right) = \left(\frac{\partial}{\partial \lambda}\frac{\partial \theta}{\partial \lambda}\right)\frac{\partial l}{\partial \theta} + \left(\frac{\partial}{\partial \lambda}\frac{\partial l}{\partial \theta}\right)\frac{\partial \theta}{\partial \lambda}$$

$$= \underbrace{\frac{\partial^2 \theta}{\partial \lambda^2}}_{\frac{2}{\lambda^3}} \underbrace{\frac{\partial l}{\partial \theta}} + \underbrace{\frac{\partial^2 l}{\partial \lambda \partial \theta}} \underbrace{\frac{\partial \theta}{\partial \lambda}}_{= \left(-\frac{1}{\lambda^2}\right)}$$

$$\theta = \frac{1}{\lambda}$$

$$\frac{\partial \theta}{\partial \lambda} = -\frac{1}{\lambda^2}$$

$$\frac{\partial^2 \theta}{\partial \lambda^2} = \frac{2}{\lambda^3}$$

will have Expectation 0 due to regularity conditions

$$\frac{\partial^2 l}{\partial \lambda \partial \theta} = \frac{\partial}{\partial \lambda}\left(\frac{\partial l}{\partial \theta}\right)$$

~~fx~~ of θ

$$= \frac{\partial \theta}{\partial \lambda} \cdot \frac{\partial^2 l}{\partial \theta^2}$$

$$\sum_{i=1}^{n}\left\{\frac{R_i u_i \exp(-u_i\theta)}{\underbrace{1-\exp(-u_i\theta)}_{=R_i u_i \cdot (\exp(u_i\theta)-1)^{-1}}} - (1-R_i)(u_i)\right\} = \frac{\partial l}{\partial \theta}$$

$$\frac{\partial^2 l}{\partial \theta^2} = \sum_i R_i u_i \left\{-\frac{\exp(u_i\theta) u_i}{(\exp(u_i\theta)-1)^2}\right\}$$

$$\mathbb{E}\left[\frac{\partial l}{\partial \theta}\right]$$

$$= n \cdot \mathbb{E}\left[\frac{R_i u_i}{\exp(u_i\theta)-1} - (1-R_i)u_i\right]$$

$$= n \cdot \mathbb{E}\left[\mathbb{E}\left(\text{''} \mid u_i\right)\right]$$

$$= n \cdot \mathbb{E}\left[\frac{\mathbb{E}(R_i \mid u_i) \cdot u_i}{\exp(u_i\theta)-1} - (1-\mathbb{E}(R_i \mid u_i)) \cdot u_i\right]$$

$$= n \cdot \mathbb{E}\left[\underbrace{\frac{p_i u_i}{\exp(u_i\theta)-1}}_{= p_i \cdot \exp(u_i\theta)} - (1-p_i) u_i\right]$$

$$p_i = 1-\exp\left(-\frac{u_i}{\lambda}\right) = 1-\exp(-u_i\theta)$$

$$= n \cdot \mathbb{E}\left[\frac{u_i}{\exp(u_i\theta)} - \frac{u_i}{\exp(u_i\theta)}\right] = 0$$

(c) Now complete data is

(R_i, T_i, u_i)

observed: (R_i, u_i)

$E[\log f(R_i, t_i, u_i); \lambda \mid (R_i, u_i); \lambda^{(k)}]$

$\cancel{P(R_i, T_i, u_i) = P(T_i)\, P(R_i, u_i \mid T_i)}$ → Don't know this

$\cancel{P(R_i, u_i \mid T_i) = \dfrac{P(R_i, u_i, T_i)}{P(T_i)}}$

$P(T_i \mid R_i, u_i) = \dfrac{P(T_i, R_i, u_i)}{P(R_i, u_i)}$ → we have this

$P(T_i, R_i, u_i) = P(T_i \mid R_i, u_i)\, P(R_i, u_i)$

$P(A \mid B) = \begin{cases} P(A \mid B=0), & B=0 \\ P(A \mid B=1), & B=1 \end{cases}$

$= P(A \mid B=0)\, I(B=0) + P(A \mid B=1)\, I(B=1)$

Need $P(T_i \mid R_i, u_i)$

$= I(R_i=1)\, P(T_i \mid R_i=1, u_i) + I(R_i=0)\, P(T_i \mid R_i=0, u_i)$

(Density)

$P(T_i \mid R_i=1, u_i)$

$= \dfrac{\partial}{\partial t} \Pr(T_i \le t \mid R_i=1, u_i)$

(CDF)

$x_1, \dots, x_n \sim N(\theta, 1)$

$\ell(\theta) = \prod_{i=1}^{n} \dfrac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x_i - \theta)^2\right)$

pdf — cdf (= prob)

∵ X_i : conti

$P(A \mid B) = \dfrac{P(A, B)}{P(B)}$ Real

||

$\frac{\partial}{\partial t} \Pr(A \le t \mid B)$

$\Pr(T_i \le t \mid R_i = 1, u_i)$

$\Pr(T_i \le t \mid T_i \in [0, u_i], u_i]$

$t \le u_i$? – Don't know

$= \Pr(T_i \le t \wedge u_i \mid T_i \in [0, u_i], u_i)$

$\int_0^{t \wedge u_i} f_{T_i \mid T_i \in [0, u_i], u_i}(t_i)\, dt_i$

$= \int_0^{t \wedge u_i} \frac{1}{\lambda} \exp\left(-\frac{t}{\lambda}\right) dt$

$f_{T_i}(t)$: cond'l density of T_i |

$= Pr(T_i \le t \wedge u_i \mid u_i)$

$T_i \mid T_i \in [0, u_i]$

$Pr(A \mid B, C) = \dfrac{Pr(A, B \mid C)}{Pr(B \mid C)}$ $\to \frac{ABC}{C}$, $\to \frac{BC}{C}$

$\frac{ABC}{BC}$

$T_i \in \mathbb{R}^+$

$Pr(T_i \le t \mid T_i \in [0, u_i], u_i)$

$= \dfrac{Pr(T_i \le t, T_i \in [0, u_i] \mid u_i)}{Pr(T_i \in [0, u_i] \mid u_i)}$

$= \dfrac{Pr(T_i \in [0, t \wedge u_i] \mid u_i)}{Pr(T_i \in [0, u_i] \mid u_i)}$ $= \int_0^{t \wedge u_i} \frac{1}{\lambda} \exp(-\frac{u}{\lambda})\, du$ / $= \int_0^{u_i} \underbrace{\frac{1}{\lambda} \exp(-\frac{u}{\lambda})}_{f_{T_i}(u)} du$

$f_{T_i \mid T_i \in [0, u_i], u_i}(t)$

$= \dfrac{\frac{1}{\lambda} \exp(-\frac{t \wedge u_i}{\lambda})}{\int_0^{u_i} \frac{1}{\lambda} \exp(-\frac{u}{\lambda})\, du}$

Density

$P(T_i = t \mid R_i = 0, u_i)$

$= \frac{\partial}{\partial t} Pr(T_i \le t \mid R_i = 0, u_i)$

$= \frac{\partial}{\partial t} Pr(T_i \le t \mid T_i \notin [0, u_i], u_i)$

$= \frac{\partial}{\partial t} Pr(T_i \le t \mid T_i > u_i, u_i)$

$= \frac{\partial}{\partial t} \left[\dfrac{Pr(T_i \le t, T_i > u_i \mid u_i)}{Pr(T_i > u_i \mid u_i)} \right]$

$= \frac{\partial}{\partial t} \left[\dfrac{\int_{u_i}^{t} \frac{1}{\lambda} \exp(-\frac{s}{\lambda})\, ds}{\int_{u_i}^{\infty} \frac{1}{\lambda} \exp(-\frac{s}{\lambda})\, ds} \right]$

$= \frac{\partial}{\partial t} \left[\dfrac{1 - \exp(-\frac{s}{\lambda}) \Big|_{u_i}^{t}}{1 - \exp(-\frac{s}{\lambda}) \Big|_{u_i}^{\infty}} \right] = \frac{\partial}{\partial t} \left[\dfrac{(1 - \exp(-\frac{t}{\lambda})) - (1 - \exp(-\frac{u_i}{\lambda}))}{(1 - (1 - \exp(-\frac{u_i}{\lambda}))} \right]$

$$= \frac{\partial}{\partial t}\left[\frac{-\exp(-\frac{t}{\lambda}) + \exp(-\frac{u_i}{\lambda})}{\exp(-\frac{u_i}{\lambda})}\right]$$

$$= \frac{\frac{1}{\lambda}\exp(-\frac{t}{\lambda})}{\exp(-\frac{u_i}{\lambda})} = \frac{1}{\lambda}\exp\left(-\frac{t}{\lambda}+\frac{u_i}{\lambda}\right) = \frac{1}{\lambda}\exp\left(\frac{u_i-t}{\lambda}\right)$$

density

$$p(T_i, R_i, U_i) = p(T_i \mid R_i, U_i)\cdot p(R_i \mid U_i)\cdot p(U_i)$$

$$p(T_i \mid R_i, U_i) = I(R_i = 1)\cdot p(T_i \mid R_i = 1, U_i) + I(R_i = 0)\cdot p(T_i$$

(c)

E-step:

$E[\log f(R_i, u_i, T_i); \lambda \mid (R_i, u_i); \lambda^{(k)}]$

Need complete density, $Pr(R_i, u_i, T_i)$

Have $Pr(R_i, u_i)$

$$P(R_i, u_i \mid T_i) = \frac{P(R_i, u_i, T_i)}{P(T_i)}$$

$$\Rightarrow P(R_i, u_i, T_i) = \underbrace{P(R_i, u_i \mid T_i)}_{\text{Can't compute this}} Pr(T_i)$$

Try #2: $P(T_i \mid R_i, u_i) = \dfrac{P(T_i, R_i, u_i)}{P(R_i, u_i)}$

$$\Rightarrow P(T_i, R_i, u_i) = \underbrace{P(T_i \mid R_i, u_i)}_{\text{Try to find this}} \underbrace{P(R_i, u_i)}_{\text{Have this}}$$

$$P(T_i \mid R_i, u_i) = \frac{\partial}{\partial t} \underbrace{Pr(T_i \le t \mid R_i, u_i)}$$

$$Pr(T_i \le t \mid R_i, u_i) = Pr(T_i \le t \mid R_i = 1, u_i) I(R_i = 1) + Pr(T_i \le t \mid R_i = 0, u_i) I(R_i = 0)$$

$$= P(T_i \le t \mid T_i \in [0, u_i], u_i) I(T_i \in [0, u_i]) + P(T_i \le t \mid T_i > u_i, u_i) I(T_i > u_i)$$

$$= I(T_i \in [0, u_i]) \frac{P(T_i \le t, T_i \in [0, u_i] \mid u_i)}{P(T_i \in [0, u_i] \mid u_i)} + I(T_i > u_i) \frac{P(T_i \le t, T_i > u_i \mid u_i)}{P(T_i > u_i \mid u_i)}$$

$$= I(T_i \in [0, u_i]) \frac{P(T_i \le t \wedge u_i \mid u_i)}{P(T_i \in [0,u_i] \mid u_i)} + I(T_i > u_i) \frac{P(u_i < T_i \le t \mid u_i)}{P(T_i > u_i \mid u_i)}$$

consider 2 cases

1) $t \ge u_i$

$P(T_i \le u_i \mid u_i) = 1$

why? B.c. of indicator

2) $t \le u_i$

compute

$$= I(T_i \in [0, u_i]) \frac{\int_0^t \frac{1}{\lambda} \exp(-\frac{s}{\lambda})\, ds}{\int_0^{u_i} \frac{1}{\lambda} \exp(-\frac{s}{\lambda})\, ds} + I(T_i > u_i) \frac{\int_{u_i}^t \frac{1}{\lambda} \exp(-\frac{s}{\lambda})\, ds}{\int_{u_i}^{\infty} \frac{1}{\lambda} \exp(-\frac{s}{\lambda})\, ds}$$

$$= I(R_i = 1) \frac{\left\{ \frac{1}{\lambda} [-\lambda \exp(-\frac{s}{\lambda}) \Big|_0^t \right\}}{\left\{ \frac{1}{\lambda} [-\lambda \exp(-\frac{s}{\lambda}) \Big|_0^{u_i} \right\}} + I(R_i = 0) \frac{\left\{ \frac{1}{\lambda} (-\lambda \exp(-\frac{s}{\lambda}) \Big|_{u_i}^t \right\}}{\left\{ \frac{1}{\lambda} (-\lambda \exp(-\frac{s}{\lambda}) \Big|_{u_i}^{\infty} \right\}}$$

$$= \frac{1}{\lambda}\left[-\lambda \exp(-\frac{t}{\lambda}) + \lambda \exp(-\frac{u_i}{\lambda})\right]$$

$$= \frac{1}{\lambda}\left[-\lambda + \lambda \exp(-\frac{u_i}{\lambda})\right]$$

$$= I(R_i = 1) \left[\frac{1 - \exp(-\frac{t}{\lambda})}{1 - \exp(-\frac{u_i}{\lambda})} \right] + I(R_i = 0) \frac{\left[\exp(-\frac{u_i}{\lambda}) - \exp(-\frac{t}{\lambda})\right]}{\exp(-\frac{u_i}{\lambda}) - 1}$$

$$= \left\{ \frac{[1-\exp(-\frac{t}{\lambda})]}{1-\exp(-\frac{u_i}{\lambda})} \right\}^{R_i} \left\{ \frac{[\exp(-\frac{u_i}{\lambda}) - \exp(-\frac{t}{\lambda})]}{\exp(-\frac{u_i}{\lambda})} \right\}^{1-R_i}$$

$$\frac{\partial}{\partial t}[\quad]$$

$$= \frac{\partial}{\partial t}\left\{ \frac{[1-\exp(-\frac{t}{\lambda})]^{R_i}\,[\exp(-\frac{u_i}{\lambda})-\exp(-\frac{t}{\lambda})]^{1-R_i}}{[1-\exp(-\frac{u_i}{\lambda})]^{R_i}\,[\exp(-\frac{u_i}{\lambda})]^{1-R_i}} \right\} \quad ??$$

$$= \frac{1}{\boxed{\quad}} \Big\{ R_i \cdot \left(1-\exp(-\tfrac{t}{\lambda})\right)^{R_i-1} \cdot \frac{1}{\lambda} \exp(-\tfrac{t}{\lambda}) \cdot \left[\exp(-\tfrac{u_i}{\lambda}) - \exp(-\tfrac{t}{\lambda})\right]^{1-R_i}$$
$$+ \left[1-\exp(-\tfrac{t}{\lambda})\right]^{R_i} \cdot (1-R_i) \cdot \left[\exp(-\tfrac{u_i}{\lambda}) - \exp(-\tfrac{t}{\lambda})\right]^{-R_i} \cdot \frac{1}{\lambda} \cdot \exp(-\tfrac{t}{\lambda}) \Big\}$$

$R_i = 0$ or 1

$$= \frac{1}{\boxed{\quad}} \left\{ \frac{1}{\lambda} \exp(-\tfrac{t}{\lambda}) \right\}^{R_i} \cdot \left\{ \frac{1}{\lambda} \exp(-\tfrac{t}{\lambda}) \right\}^{1-R_i}$$

$$= \frac{1}{\boxed{\quad}} \frac{1}{\lambda} \exp(-\tfrac{t}{\lambda})$$

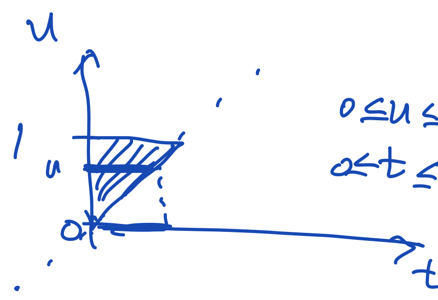
$$P(T_i, U_i, R_i) = \underbrace{\frac{1}{\lambda}\exp(-\tfrac{T_i}{\lambda})}_{} \cdot \overbrace{I(T_i \in [0,\infty))\, I(R_i \in \{0,1\})\, I(U_i \in [0,1])}^{p(T_i) \quad \to 1 \qquad \to 1}$$

$T_i \sim Exp(\frac{1}{\lambda})$: $f(T_i) = \frac{1}{\lambda}\exp(-\frac{T_i}{\lambda})\, I(T_i \in (0,\infty))$

$U_i \sim unif[0,1]$: $f(U_i) = I(U_i \in [0,1])$

$R_i = I(T_i \in [0, U_i])$ $\quad R_i \sim Ber(p)$

$p = \mathbb{E} R_i = P(T_i \in [0, u_i])$



$0 \le u \le 1$

$0 \le t \le u$

$$= \int_{t\in[0,u]} f_{T_i,u_i}(t,u)\,dt\,du$$

$$= \int_0^1 \int_0^u \frac{1}{\lambda}\exp(-\frac{t}{\lambda})\cdot dt\,du$$

$$= \int_0^1 \left[-\exp(-\frac{t}{\lambda})\right]_0^u du$$

$$= \int_0^1 -\exp(-\frac{u}{\lambda}) + 1 \, du$$

$$= \left[+\lambda\exp(-\frac{u}{\lambda}) + u\right]_0^1$$

$$= \lambda\exp(-\frac{1}{\lambda}) + 1 - \lambda$$

$$f(R_i) = (\lambda\exp(-\frac{1}{\lambda}) + 1 - \lambda)^{R_i}\,(\lambda - \lambda\exp(-\frac{1}{\lambda}))^{1-R_i} \cdot I(R_i \in \{0,1\})$$

$$f(R_i, u_i) = f(R_i | u_i)\cdot f(u_i)$$

$$= (1-\exp(-\frac{u_i}{\lambda}))^{R_i}\cdot \exp(-\frac{u_i}{\lambda})^{1-R_i} \cdot I(R_i \in \{0,1\}) \cdot I(u_i \in [0,1])$$

$R_i | u_i \sim Ber(P(u_i))$

$$P(u_i) = \mathbb{E}(R_i | u_i) = P(T_i \in [0, u_i] | u_i) = \int_0^{u_i} \frac{1}{\lambda}\exp(-\frac{t}{\lambda})\,dt = 1 - \exp(-\frac{u_i}{\lambda})$$

$$f(T_i, R_i, u_i) = f(T_i | R_i, u_i) \cdot f(R_i, u_i)$$

$$\iiint f(T_i, U_i, R_i)\, dT_i\, dU_i\, dR_i = 1$$

(d) $L(u_i) = \prod_{i: T_i \in [0, u_i]} \left(\frac{1}{\lambda} \exp\{-\frac{u_i}{2\lambda}\} \right)$ $\quad T_i = \begin{cases} 0 & T_i > u_i \\ \frac{u_i}{2} & T_i \in [0, u_i] \end{cases}$

$= \exp\left(\sum_i \{-\frac{u_i}{2\lambda} - \log(\lambda)\} \right)$

$\ell(u_i) = \sum_i \left(-\frac{u_i}{2\lambda} - \log(\lambda)\right)$ $\quad m = \sum_{i=1}^n I(T_i \in [0, u_i])$

$= \sum_{i=1}^n \left(I(T_i \in [0, u_i]) \left(-\frac{u_i}{2\lambda} - \log(\lambda)\right) \right)$

$\frac{\partial \ell}{\partial \lambda} = \sum_{i=1}^n I(T_i \in [0, u_i]) \left(\frac{u_i}{2\lambda^2}\right) - \frac{\sum_{i=1}^n I(T_i \in [0, u_i])}{\lambda} \overset{set}{=} 0$

$\sum_{i=1}^n u_i I(T_i \in [0, u_i]) = 2\lambda \sum_{i=1}^n I(T_i \in [0, u_i])$

$\tilde{\lambda} = \frac{\sum_{i=1}^n u_i R_i}{2 \sum_{i=1}^n R_i}$

Asymptotics

$\tilde{\lambda} = \frac{\frac{1}{n} \sum_{i=1}^n u_i R_i}{2\left(\frac{1}{n} \sum_{i=1}^n R_i\right)}$

By LLN, $\frac{1}{n} \sum_{i=1}^n u_i R_i \xrightarrow{p} E(u_1 R_1)$

" " $\frac{1}{n} \sum_{i=1}^n R_i \xrightarrow{p} E(R_1) =$

$1 + \lambda \exp(-\frac{1}{\lambda}) - \lambda$

$E_{u_1}[E[u_1 R_1 | u_1]] = E_{u_1}[u_1 E(R_1 | u_1)] = E[u_1 (1 - \exp(-\frac{u_1}{\lambda}))]$

$= E[u_1 - u_1 \exp(-\frac{u_1}{\lambda})] = \frac{1}{2} - \int_0^1 u_1 \exp(-\frac{u_1}{\lambda}) \, du_1$

$$\int u_1 \exp\left(-\frac{u_1}{\lambda}\right) du_1 = c_1\left[u_1 \exp\left(-\frac{u_1}{\lambda}\right)\right] + c_2\left[\exp\left(-\frac{u_1}{\lambda}\right)\right]$$

$$u_1 \exp\left(-\frac{u_1}{\lambda}\right) = c_1\left[\exp\left(-\frac{u_1}{\lambda}\right) - \frac{u_1}{\lambda}\exp\left(-\frac{u_1}{\lambda}\right)\right] + c_2\left[-\frac{1}{\lambda}\exp\left(-\frac{u_1}{\lambda}\right)\right]$$

$$c_1 = -\lambda \qquad c_2 = -\lambda^2$$

$$= \left[\cancel{-\lambda \exp\left(-\frac{u_1}{\lambda}\right)} + u_1 \exp\left(-\frac{u_1}{\lambda}\right)\right]$$

$$\cancel{+ \lambda \exp\left(-\frac{u_1}{\lambda}\right)}$$

$$\int_0^1 u_1 \exp\left(-\frac{u_1}{\lambda}\right) du_1 = -\lambda u_1 \exp\left(-\frac{u_1}{\lambda}\right) - \lambda^2 \exp\left(-\frac{u_1}{\lambda}\right) \Big|_0^1$$

$$= -\lambda \exp\left(-\frac{1}{\lambda}\right) - \lambda^2 \exp\left(-\frac{1}{\lambda}\right) \quad -$$

$$\left[-\lambda^2\right] = \lambda^2\left(1 - \exp\left(-\frac{1}{\lambda}\right)\right) - \lambda \exp\left(-\frac{1}{\lambda}\right)$$

$$= \lambda^2 + \exp\left(-\frac{1}{\lambda}\right)(-\lambda - \lambda^2)$$

By Slutsky's Thm:

$$\tilde{\lambda} = \frac{\frac{1}{n}\sum_{i=1}^{n} u_i R_i}{2\left(\frac{1}{n}\sum_{i=1}^{n} R_i\right)} \xrightarrow{P} \frac{\lambda^2 + \exp\left(-\frac{1}{\lambda}\right)(-\lambda - \lambda^2)}{2\left(1 + \lambda \exp\left(-\frac{1}{\lambda}\right) - \lambda\right)}$$

$$= \frac{\lambda^2 - \lambda^2 \exp\left(-\frac{1}{\lambda}\right) - \lambda \exp\left(-\frac{1}{\lambda}\right)}{2 + 2\lambda \exp\left(-\frac{1}{\lambda}\right) - 2\lambda}$$

$$= \frac{\lambda\left(\lambda - \lambda \exp\left(-\frac{1}{\lambda}\right) - \exp\left(-\frac{1}{\lambda}\right)\right)}{2 + 2\lambda \exp\left(-\frac{1}{\lambda}\right) - 2\lambda}$$

(e) $\sqrt{n}(\tilde{\lambda}-\lambda) \xrightarrow[d]{} N(0,\sigma^2)$ $\quad p_1 = E(R_1|u_1) = 1-\exp(-\frac{u_1}{\lambda})$

(1) $\sqrt{n}(\frac{1}{n}\sum_i u_i R_i - E(u_1 R_1)) \xrightarrow[d]{} N(0, var(u_1 R_1))$

(2) $\sqrt{n}(\frac{1}{n}\sum_i R_i - E(R_1)) \xrightarrow[d]{} N(0, var(R_1))$

$$var(u_1 R_1) = E_{u_1}[var(u_1 R_1 | u_1)] + var_{u_1}(E[u_1 R_1 | u_1])$$

$$= E_{u_1}(p_1(1-p_1)u_1^2] + var_{u_1}(p_1 u_1)$$

(3) $\theta_n = \begin{pmatrix} \sum_i u_i R_i \\ \sum_i R_i \end{pmatrix}$ $\quad \theta = \begin{pmatrix} u_1 R_1 \\ R_1 \end{pmatrix}$

$$\sqrt{n}\left(\frac{1}{n}\theta_n - \theta\right) \xrightarrow[d]{} N(0,\sigma^2)$$

$$\sigma^2 = \begin{bmatrix} var(u_1 R_1) & E(u_1 R_1) - E(u_1 R_1)E(R_1) \\ " \quad " & var(R_1) \end{bmatrix}$$

Then by delta method, we want distn of $\dfrac{\sum_i u_i R_i}{\sum_i R_i}$

$$g(X,Y) = \frac{X}{Y} \qquad \frac{\partial g}{\partial x} = \frac{1}{Y} \qquad \frac{\partial g}{\partial Y} = -\frac{X}{Y^2}$$

$$g' = \begin{pmatrix} 1/Y \\ -X/Y^2 \end{pmatrix}_{2\times 1}$$

$$\sqrt{n}\left(\frac{1}{n}g\left(\sum_i u_i R_i, \sum_i R_i\right) - g(E(u_1 R_1), E(R_1))\right) \xrightarrow[d]{} N(0, \nabla g^T \sigma^2 \nabla g)$$

$$\nabla g^T \sigma^2 \nabla g = \begin{pmatrix} \frac{1}{E(R_1)} & -\frac{E(u_1 R_1)}{E(R_1)} \end{pmatrix} \begin{bmatrix} var(u_1 R_1) & E(u_1 R_1) - E(u_1 R_1)E(R_1) \\ " \quad " & var(R_1) \end{bmatrix} \begin{pmatrix} 1/E(R_1) \\ -E(u_1 R_1)/E(R_1) \end{pmatrix}$$

$$var(u_1 R_1) = E_{u_1}[var(u_1 R_1 | u_1)] + var_{u_1}[E[u_1 R_1 | u_1]]$$

$$= E_{u_1}[u_1^2 p_1(1-p_1)] + var_{u_1}(u_1 p_1)$$

p_1 from (a)
$1-\exp(-u_1/\lambda)$

$$= E[u_1^2 (1-\exp(-\frac{u_1}{\lambda}))(\exp(-\frac{u_1}{\lambda}))] +$$

$$var(u_1(1-\exp(-\frac{u_1}{\lambda})))$$

$$= E[u_1^2 \exp(-\frac{u_1}{\lambda}) - u_1^2 \exp(-\frac{2u_1}{\lambda})] +$$

$$var(u_1 - u_1 \exp(-\frac{u_1}{\lambda}))$$

$$\to E(u_1^2 R_1) = E(u_1 R_1)$$

$$cov(u_1 R_1, R_1) = E(u_1 R_1) - E(u_1 R_1) E(R_1)$$

$$= E(u_1 R_1)[1-\exp(-\frac{u_1}{\lambda})]$$

$$= E(u_1 R_1)[1-\exp(-\frac{u_1}{\lambda})]$$

$$E[u_1 R_1] = \lambda^2 + \exp(-\frac{1}{\lambda})(-\lambda-\lambda^2)$$

$$= [\lambda^2 + \exp(\frac{-1}{\lambda})(-\lambda-\lambda^2)][1-\exp(-u_1/\lambda)]$$

$$var(R_1) = E_{u_1}[var(R_1|u_1)] + var_{u_1}(E[R_1|u_1])$$

$$= var_{u_1}(p_1) + E_{u_1}[E(R_1^2|u_1) - E[R_1|u_1]^2]$$

$$= var_{u_1}(p_1) + E_{u_1}[\int_0^1 u_1^2 (1-\exp(-\frac{u_1}{\lambda})) du_1 - p_1^2]$$

so $\nabla g^T \sigma^2 \nabla g$ all terms defined

2.) $x_1, \dots x_n \sim N(0, \sigma^2)$ $\qquad \tau = \frac{1}{2\sigma^2}$

$$f(\tau) = \frac{1}{\Gamma(a) b^a} \tau^{a-1} e^{-\tau/b}$$

$E(\tau) = ab$

$var(\tau) = ab^2$

$$L(\sigma^2, d) = (d-\sigma^2)^2/\sigma^4$$

Find Bayes Rule - Minimizes posterior expected loss

Posterior

$$P(\tau \mid x) \propto P(\tau) P(x \mid \tau)$$

$$\propto \frac{1}{\Gamma(a) b^a} \tau^{a-1} e^{-\tau/b} \left\{ \prod_{i=1}^{n} \frac{1}{\sqrt{2\cancel{\pi}\sigma^2}} \exp\left(-\frac{x_i^2}{2\sigma^2}\right) \right\}$$

$$\propto \tau^{\alpha-1} e^{-\tau/b} (\tau^{n/2}) \exp(-\tau \sum_i x_i^2)$$

$$= \tau^{a+n/2-1} e^{-\tau/b - \tau \sum_i x_i^2}$$

$$= \tau^{\alpha+n/2-1} e^{-\tau(1/b + \sum_i x_i^2)}$$

$$\sim \Gamma\left(a+n/2, \ b + \frac{1}{\sum_i x_i^2}\right)$$

Mean

$$(a+n/2)\left(b + \frac{1}{\sum_i x_i^2}\right)$$

$$E[(d-\sigma^2)^2/\sigma^4]$$

$$= E\left[\left(d - \frac{1}{2\tau}\right)^2 / \frac{1}{4\tau^2}\right] = E\left[4\tau^2\left(d - \frac{1}{2\tau}\right)^2\right]$$

$\frac{1}{2\sigma^2} = \tau$ $\qquad \sigma^2\tau = \frac{1}{2}$ $\qquad \sigma^2 = \frac{1}{2\tau}$

$4\tau^2 = \frac{1}{\sigma^4}$

$$= E\left[4\tau^2\left(d^2 - \frac{d}{\tau} + \frac{1}{4\tau^2}\right)\right]$$

$$= E(4\tau^2 d^2 - 4d\tau + 1] = 4d^2 E_{\tau|x}[\tau^2] - 4d E_{\tau|x}[\tau] + 1$$

$$d_n = \underset{d}{\operatorname{argmin}} \; 4d^2 \underset{\tau|x}{E}[\tau^2] - 4d\underset{\tau|x}{E}[\tau] + 1$$

$$E[\tau^2] = var(\tau) + (E(\tau)^2) \qquad var(\tau) = E(\tau^2) - (E(\tau))^2$$

$$= ab^2 + (ab)^2$$

$$= (a+n/2)\left(b + \frac{1}{\sum_i x_i^2}\right)^2 + (a+n/2)^2\left(b + 1/\sum_i x_i^2\right)^2$$

$$d_n = \underset{d}{\operatorname{argmin}} \; 4d^2 \underset{\tau|x}{E}[\tau^2] - 4d\underset{\tau|x}{E}[\tau] + 1$$

$$8d \underset{\tau|x}{E}[\tau^2] - 4E[\tau] \overset{set}{=} 0$$

$$d_n = \frac{E[\tau]}{2E[\tau^2]} = \frac{(a+n/2)\left(b + \frac{1}{\sum_i x_i^2}\right)}{2\left[(a+n/2)\left(b+\frac{1}{\sum_i x_i^2}\right)^2 + (a+n/2)^2\left(b + 1/\sum_i x_i^2\right)^2\right]}$$

$$= \frac{1}{2}\left[1 \Big/ \left(b + \frac{1}{\sum_i x_i^2}\right) + (a+n/2)\left(b + \frac{1}{\sum_i x_i^2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{\left(b + \frac{1}{\sum_i x_i^2}\right)\left(1 + a + \frac{n}{2}\right)}\right] = \hat{\tau}_{Bayes}$$

$$\Rightarrow \hat{\sigma}^2 = \frac{1}{2\hat{\tau}_{Bayes}} = \left[\left(b + \frac{1}{\sum_i x_i^2}\right)\left(1 + a + \frac{n}{2}\right)\right]$$

(ii) Defn of admissible rule:

No rule d' exists s.t.

$R(\theta, d') \le R(\theta, d) \quad \forall \theta \in \Theta$

FTSOC, assume $d = Y + \beta$ is admissible

$\therefore \quad R(\sigma^2, Y+\beta) \le R(\sigma^2, \alpha Y + \beta) \quad \forall \alpha, \beta \in \mathbb{R}$

$\Rightarrow E[\{(Y+\beta) - \sigma^2\}^2] \le E[\{(\alpha Y + \tilde{\beta}) - \sigma^2\}^2]$

$\Rightarrow E[Y^2 + 2Y\beta + \beta^2 - 2(Y+\beta)\sigma^2 + \cancel{\sigma^4}] \le$

$E[\alpha^2 Y^2 + 2\alpha\tilde{\beta}Y + \tilde{\beta}^2 - 2(\alpha Y + \tilde{\beta})\sigma^2 + \cancel{\sigma^4}]$

$\Rightarrow E(Y^2) + 2\beta E(Y) + \beta^2 - 2\beta\sigma^2 - 2\sigma^2 E(Y) \le$

$\alpha^2 E(Y^2) + 2\alpha\tilde{\beta}E(Y) + \tilde{\beta}^2 - 2\tilde{\beta}\sigma^2 - 2\alpha\sigma^2 E(Y)$

$Y = \frac{1}{n}\sum_i x_i^2 \qquad X_1, \dots, X_n \sim N(0, \sigma^2)$

$\frac{1}{\sigma} X_1, \dots, X_n \sim N(0,1)$

$\frac{1}{\sigma^2} X_i^2 \sim \chi^2_1$

$\frac{1}{\sigma^2} \sum_i X_i^2 \sim \chi^2_n \sim E(\chi^2_n) = n$

$var(\chi^2_n) = 2n$

$E(\sum_i X_i^2) = n\sigma^2 \qquad var(\sum_i X_i^2) = 2n\sigma^4$

$E(\frac{1}{n}\sum_i X_i^2) = \sigma^2 = E(Y) \qquad var(\frac{1}{n}\sum_i X_i^2) = \frac{2\sigma^4}{n}$

$E(Y^2) = var(Y) + (E(Y))^2$

$= \frac{2\sigma^4}{n} + \sigma^4 = \frac{\sigma^4(2+n)}{n}$

$\Rightarrow \frac{\sigma^4(2+n)}{n} + \cancel{2\beta\sigma^2} + \beta^2 - \cancel{2\beta\sigma^2} - 2\sigma^4 \le$

$\alpha^2\left[\frac{\sigma^4(2+n)}{n}\right] + 2\alpha\tilde{\beta}\sigma^2 - 2\alpha\sigma^4 + \tilde{\beta}^2 - 2\tilde{\beta}\sigma^2$

Take $\alpha = 1$
$\tilde{\beta} = 0$
$\beta = 1$

$$\Rightarrow \frac{\sigma^4(2+n)}{n} + \beta^2 - 2\sigma^4 \le \frac{\sigma^4(2+n)}{n} - 2\sigma^4$$

$$\Rightarrow \beta^2 \le 0$$

This is a contradiction b.c. $\beta \neq 0$

$\therefore$ $\gamma + \beta$ is an admissible rule

(b) $X_1, \dots, X_n \sim N(\mu, 1) = p(x|\mu)$

$p(\mu) \sim N(0,1)$

$$p(\mu|x) \propto p(\mu)\, p(x|\mu)$$

$$= \cancel{\frac{1}{\sqrt{2\pi}}} \exp\left(-\frac{\mu^2}{2}\right) \cancel{\frac{1}{\sqrt{2\pi}}}^{n} \prod_{i=1}^{n} \exp\left(-\frac{1}{2}(x_i-\mu)^2\right)$$

$$\propto \exp\left(-\frac{\mu^2}{2}\right) \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2\right\}$$

$$= \exp\left(-\frac{\mu^2}{2}\right) \exp\left(-\cancel{\frac{\sum_i x_i^2}{2}} + \mu \sum_i x_i - \frac{n\mu^2}{2}\right)$$

$$\propto \exp\left(-\frac{\mu^2}{2} + \mu \sum_i x_i - \frac{n\mu^2}{2}\right)$$

$$= \exp\left(-\frac{1}{2}\left(\mu^2(n+1) - 2\mu \sum_i x_i\right)\right)$$

$$= \exp\left(-\frac{(n+1)}{2}\left(\mu^2 - \frac{2\mu}{(n+1)} \sum_i x_i\right)\right)$$

$$\propto \exp\left(-\frac{(n+1)}{2}\left(\mu - \frac{\sum_i x_i}{(n+1)}\right)^2\right) \sim N\left(\frac{\sum_i x_i}{(n+1)}, \frac{1}{(n+1)}\right)$$

$$\tilde{p} = \frac{(n+1)}{\sqrt{2\pi}} \int_0^c \exp\left(-\left(\frac{n+1}{2}\right)\left(\mu - \frac{\sum_i x_i}{(n+1)}\right)^2\right) d\mu$$

$$= \Phi\left(\frac{c - \left(\sum_i x_i/(n+1)\right)}{\sqrt{1/(n+1)}}\right) - \Phi\left(\frac{0 - \left(\sum_i x_i/(n+1)\right)}{\sqrt{1/n+1}}\right)$$

(ii) $\mu = c$

$$\tilde{p} = \underbrace{\Phi\left(\sqrt{n+1}\left(\mu - \frac{1}{n+1}\sum_i x_i\right)\right)}_{\sim \text{Unif}(0,1)} - \underbrace{\Phi\left(\sqrt{n+1}\left(-\frac{1}{n+1}\sum_i X_i\right)\right)}$$

By CLT, this converges in distn to $Z \sim N(0,1)$

$\Phi(Z) = W \quad g(X_n) \xrightarrow[d]{} g(W)$ by CMT

want $f(W)$

$$f(w) = f(z)\left|\frac{\partial w}{\partial z}\right|^{-1}$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)\phi(w)^{-1} = 1$$

$f(w) \sim \text{Unif}(0,1)$

$$\tilde{P} \xrightarrow[d]{} \text{Unif}(0,1)$$

$$'' \; \Phi\left(\sqrt{n+1}\left(\mu - \mu - \frac{1}{n+1}\sum_i X_i\right)\right)$$

$$= \Phi\left(\underbrace{\sqrt{n+1}\left(\mu - \frac{1}{n+1}\sum_i\right)} - \underbrace{\mu\sqrt{n+1}}_{\to\infty}\right)$$

$\xrightarrow[d]{} W, \quad W \sim N(0,1)$

$\to \Phi(-\infty) = 0$

(c) $X_1, \dots, X_n \sim N(\mu, 1)$

$p(\mu)$: $p(\mu=0) = \lambda$ $\quad$ $p(\mu) \sim \text{Bern}(1-\lambda)$

$p(\mu=1) = 1-\lambda$

$H_0: \mu = 0 \quad H_1: \mu = 1 \quad$ 0-1 loss

(a) $p(\mu|x) \propto p(x|\mu)\,p(\mu)$

(i)

$$\propto \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x_i-\mu)^2\right) (1-\lambda)^{\mu} (\lambda)^{1-\mu} I(\mu \in \{0,1\})$$

$$\propto \exp\left(-\frac{1}{2}\sum_i x_i^2 + \mu \sum_i x_i - n\frac{\mu^2}{2} + \mu \log(1-\lambda) + (1-\mu)\log\lambda + \log(I(\mu \in \{0,1\}))\right)$$

$$\propto \underbrace{\exp\left(-\frac{n}{2}\mu^2 + \mu\left(\sum_i x_i + \log(1-\lambda) - \log\lambda\right)\right) I(\mu \in \{0,1\})}_{h(\mu)}$$

$p(\mu|x) = \underline{c} \cdot h(\mu)$

$$\cancel{\propto \exp\left(-\frac{n}{2}\left(\mu - \frac{1}{n}\left(\sum x_i + \log(1-\lambda)\right)\right)^2\right)}$$

$$\cancel{\sim N\left(\frac{1}{n}\left(\sum_i x_i + \log(1-\lambda)\right), \frac{1}{n}\right)}$$

$$\cancel{\sim N\left(\bar{x} + \frac{1}{n}\log(1-\lambda), \frac{1}{n}\right)}$$

we will use form above instead

Not normal
Domain of μ restricted to $\{0,1\}$

$P(\mu=0|x)$ ✓ $= \dfrac{h(0)}{h(0)+h(1)}$

$P(\mu=1|x)$ ✓ $= \dfrac{h(1)}{h(0)+h(1)}$

$\mu = \{0,1\}$

$a = \{0,1\}$

$l(\mu, a) = I(\mu \neq a) = 1 - I(\mu = a)$

Posterior Expected Loss $= E_{\mu|x}[l(\mu, a) | X = x]$

$G(a)$ $= E_{\mu|x}[1 - I(\mu = a) | X = x]$

$= 1 - p_{\mu|x}(\mu = a | X = x)$

If $G(0) < G(1)$, then $\hat{a} = 0$ is Bayes Rule

" $G(1) < G(0)$ " $\hat{a} = 1$ "

① $G(0) = 1 - P_{\mu|x}(\mu = 0 | x = x)$

$$= 1 - \underset{\substack{\uparrow \\ \text{some proportionality} \\ \text{constant}}}{C}\exp\Big(\underbrace{\overset{0}{\cancel{-\tfrac{n}{2}\mu^2}}}_{0} + \mu\underbrace{\big(\textstyle\sum_i x_i + \log(1-\lambda) - \log\lambda\big)}_{0}\Big) I(\mu \in \{0,1\})$$

$$= 1 - C$$

② $G(1) = 1 - P_{u|x}(u = 1 \; x = x)$

$$= 1 - C \cdot$$

$$\exp\Big(\underbrace{-\tfrac{n}{2}\cancel{\mu^2}}_{-\frac{n}{2}} + \underbrace{\cancel{\mu}}_{1}\big(\textstyle\sum_i x_i + \log(1-\lambda) - \log\lambda\big)\Big) I(\mu \in \{0,1\})$$

$$= 1 - C\exp\Big(-\tfrac{n}{2} + \big(\textstyle\sum_i x_i + \log(1-\lambda) - \log(\lambda)\big)\Big)$$

$$= 1 - C\exp\Big(-\tfrac{n}{2} + \textstyle\sum_i x_i + \log\big(\tfrac{1-\lambda}{\lambda}\big)\Big)$$

we know

$$P(u=0 | x=x) + P(u=1 | x=x) = 1$$

$$C + C\exp\Big(-\tfrac{n}{2} + \textstyle\sum_i x_i + \log\big(\tfrac{1-\lambda}{\lambda}\big)\Big) = 1 \qquad C = \frac{1}{1+\exp\big(-\frac{n}{2} + \sum_i x_i + \log(\frac{1-\lambda}{\lambda})\big)}$$

$\hat{\alpha} = 0$ if

$$1 - C < 1 - C\exp\Big(-\tfrac{n}{2} + \textstyle\sum_i x_i + \log\big(\tfrac{1-\lambda}{\lambda}\big)\Big)$$

$$\Rightarrow 1 < -C\Big(\exp\Big(-\tfrac{n}{2} + \textstyle\sum_i x_i + \log\big(\tfrac{1-\lambda}{\lambda}\big)\Big) + 1\Big)$$

$$-\frac{1}{C} > \exp\Big(-\tfrac{n}{2} + \textstyle\sum_i x_i + \log\big(\tfrac{1-\lambda}{\lambda}\big)\Big) + 1$$

$$-1 - \exp\Big(-\tfrac{n}{2} + \textstyle\sum_i x_i + \log\big(\tfrac{1-\lambda}{\lambda}\big)\Big) > \exp\Big(-\tfrac{n}{2} + \textstyle\sum_i x_i + \log\big(\tfrac{1-\lambda}{\lambda}\big)\Big) + 1$$

$$-2 > 2\exp\Big(-\tfrac{n}{2} + \textstyle\sum_i x_i + \log\big(\tfrac{1-\lambda}{\lambda}\big)\Big)$$

$$1 < -\exp\left(-\frac{n}{2} + \sum_i x_i + \log\left(\frac{1-\lambda}{\lambda}\right)\right)$$

$$0 < \frac{n}{2} - \sum_i x_i - \log\left(\frac{1-\lambda}{\lambda}\right)$$

$$\sum_i x_i < \frac{n}{2} - \log\left(\frac{1-\lambda}{\lambda}\right) \qquad G(0) < G(1)$$

$$d_\lambda(x) = I\left(\sum_i x_i > \frac{n}{2} - \log\left(\frac{1-\lambda}{\lambda}\right)\right),$$

" $d_\lambda = 1$: reject H_0 (accept H_1)
$d_\lambda = 0$: accept H_0 "

<u>Accept</u> H_0 if $\sum_i x_i \leq \frac{n}{2} - \log\left(\frac{1-\lambda}{\lambda}\right)$
($\Leftrightarrow d_\lambda = 0$)

<u>Reject</u> H_0 if $\sum_i x_i > \frac{n}{2} - \log\left(\frac{1-\lambda}{\lambda}\right)$
($\Leftrightarrow d_\lambda = 1$)

(ii) Minimax rule is Bayes Rule with constant risk

we have d_λ from above

$\underline{\text{Risk}} = R(\mu, d_\lambda) = E_{\cancel{\mu}}[L(\mu, d_\lambda)]$ $\qquad L(\mu, a) = 1 - I(\mu = a)$

fn of "μ" $\in \{0,1\}$, "d_λ"

$R(\mu=0, d_\lambda)$: Accept null but $d_\lambda = 1$

$R(\mu=1, d_\lambda)$

$= E_\mu[1 - I(\mu = d_\lambda(x))]$

$= 1 - E_\mu[I(\mu = d_\lambda(x))]$

$= 1 - P_\mu(\mu = d_\lambda(x))$

$\mu \in \{0,1\} \quad d_\lambda \in \{0,1\}$

① $1 - P_X(d_\lambda(X) = 0)$ - Accept null

$= 1 - P_{\mu=0}\left(\sum_i X_i < \frac{n}{2} - \log\left(\frac{1-\lambda}{\lambda}\right)\right)$

$X_i \sim N(0,1)$ (under θ_0, $\mu = 0$)

$\sum_i X_i \sim N(0, n)$

$= P_X\left(\sum_i X_i > \frac{n}{2} - \log\left(\frac{1-\lambda}{\lambda}\right)\right)$

$= P_X\left(\frac{\sum_i X_i - \cancel{\mu}^{0}}{\sqrt{n}} > \frac{\frac{n}{2} - \log\left(\frac{1-\lambda}{\lambda}\right) - \cancel{\mu}^{0}}{\sqrt{n}}\right)$

$= 1 - \Phi\left(\frac{\frac{n}{2} - \log\left(\frac{1-\lambda}{\lambda}\right)}{\sqrt{n}}\right)$ constant (no μ) $= R(0, d_\lambda)$

$R(\overset{\mu_1}{1}, d_\lambda)$: $\quad \boxed{\sum X_i \sim N(n, n)}$ $(\because)_{\mu=1}$ $\quad X_i \sim N(1,1)$

② Reject null $\quad 1 - P_X(d_\lambda(X) = 1)$

$= 1 - P_{\mu=1}\left(\sum_i X_i > \frac{n}{2} - \log\left(\frac{1-\lambda}{\lambda}\right)\right)$

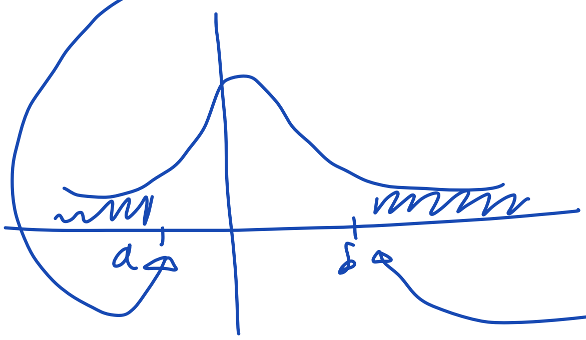
$= P_{\mu=1}\left(\sum_i X_i < \frac{n}{2} - \log\left(\frac{1-\lambda}{\lambda}\right)\right)$

$= \Phi\left(\frac{\frac{n}{2} - \log\left(\frac{1-\lambda}{\lambda}\right) - n}{\sqrt{n}}\right) = \Phi\left(\frac{-\frac{n}{2} - \log\left(\frac{1-\lambda}{\lambda}\right)}{\sqrt{n}}\right) = R(1, d_\lambda)$

constant

Need:

$$\Phi\Bigg(\underbrace{\frac{-\frac{n}{2}-\log\left(\frac{1-\lambda}{\lambda}\right)}{\sqrt{n}}}_{a}\Bigg) = 1-\Phi\Bigg(\underbrace{\frac{\frac{n}{2}-\log\left(\frac{1-\lambda}{\lambda}\right)}{\sqrt{n}}}_{b}\Bigg)$$



$$\Phi(a) = 1-\Phi(b)$$

$$\Leftrightarrow \quad a = -b$$

$$-\left(\frac{-\frac{n}{2}-\log\left(\frac{1-\lambda}{\lambda}\right)}{\sqrt{n}}\right) = \frac{\frac{n}{2}-\log\left(\frac{1-\lambda}{\lambda}\right)}{\sqrt{n}}$$

$$\Rightarrow \frac{n}{2}+\log\left(\frac{1-\lambda}{\lambda}\right) = \frac{n}{2}-\log\left(\frac{1-\lambda}{\lambda}\right)$$

$$\Rightarrow \log\left(\frac{1-\lambda}{\lambda}\right)+\log\left(\frac{1-\lambda}{\lambda}\right)=0$$

$$\log(1-\lambda)-\log(\lambda)+\log(1-\lambda)-\log(\lambda)=0$$

$$2\log(1-\lambda)=2\log(\lambda)$$

$\lambda = 0.5$ gives minimax rule

$$\log\left(\frac{1-0.5}{0.5}\right)=0$$

$d_{mini}(x) = I\left(\sum_i x_i > \frac{n}{2}\right)$, " $d_n = 1$: reject H_0 (accept H_1)
$d_n = 0$: accept H_0 "

<u>Accept</u> H_0 if $\sum_i x_i \le \frac{n}{2}$
($\Leftrightarrow d_{min} = 0$)

<u>Reject</u> H_0 if $\sum_i x_i > \frac{n}{2}$
($\Leftrightarrow d_{mini} = 1$)