2014 Theory 1

1. We consider two groups of independent observations: $X_1, ..., X_n$ are i.i.d from $Unif(0, \alpha)$ and $Y_1, ..., Y_n$ are i.i.d from $Unif(0, \beta)$, where both α and β are unknown parameter assumed to be positive. For comparison, we are interested in inference on $\theta = \beta/\alpha$.

   (a) Derive the UMVUEs for α and β and calculate their respective variances.

   (b) Calculate the MLEs for α and β, denoted as $\hat{\alpha}$ and $\hat{\beta}$ respectively. Derive the asymptotic distributions for $\hat{\alpha}$ and $\hat{\beta}$ are some normalization.

   (c) The MLE for θ is then $\hat{\theta} = \hat{\beta}/\hat{\alpha}$. Derive the asymptotic distribution of $\hat{\theta}$ after normalization. Construct an asymptotic 95% confidence interval for θ based on the observations.

   (d) We wish to test the hypothesis $H_0 : \alpha = \beta$ versus $H_a : \alpha \neq \beta$. What is the likelihood ratio test static. Derive the exact distribution of this test statistic.

   (e) Note $E[X_k] = \alpha/2$ and $E[Y_k] = \beta/2$. Thus, a simple estimator for θ is $\bar{Y}_n/\bar{X}_n$. Derive the asymptotic distribution of this estimator after normalization. What is the asymptotic relative efficiency of this estimator with respect to $\hat{\theta}$, $2\bar{Y}_n/\hat{\alpha}$ and $\hat{\beta}/(2\bar{X}_n)$?

1. We consider 2 groups of independent observations | 2014 Theory 1

$X_1, \dots, X_n \overset{iid}{\sim} \text{Unif}(0,\alpha)$ and $Y_1, \dots, Y_n \overset{iid}{\sim} \text{Unif}(0,\beta)$ where both α and β are unknown parameters assumed to be positive. For comparison, we are interested in inference on $\theta = \beta/\alpha$.

(a) Derive the UMVUE for α and β and calculate their respective variances

$f_{X_1,\dots,X_n}(x_1,\dots,x_n) = \prod_{i=1}^{n} \frac{1}{\alpha} I(0<x_i<\alpha) = \left(\frac{1}{\alpha}\right)^n \prod_{i=1}^{n} I(0<x_i<\alpha) = \left(\frac{1}{\alpha}\right)^n I(0<X_{(1)}<X_{(n)}<\alpha)$ $X_{(n)}$ is the ss. by factorization

we will start w/ $X_{(n)}$ & check if it is a complete, sufficient statistic.

$P(X_{(n)} \le x) = P(X_1,\dots,X_n < x) = \frac{x^n}{\alpha^n} I(x \in (0,\alpha))$

$\Rightarrow f_{X_{(n)}}(x) = \frac{n x^{n-1}}{\alpha^n} I(x \in (0,\alpha))$

$E[g(X_{(n)})] = \int_0^\alpha \frac{n x^{n-1}}{\alpha^n} g(x)\,dx = 0 \quad \forall \alpha > 0$

$\Rightarrow \int_0^\alpha x^{n-1} g(x)\,dx = 0 \quad \forall \alpha>0$

take $\frac{d}{d\alpha}$ $\quad \alpha^{n-1} g(\alpha) = 0 \xrightarrow{\text{since } \alpha, n > 0} g(\alpha) = 0 \Rightarrow X_{(n)}$ is a complete sufficient statistic for α

$E[f(X_{(n)})] = \alpha$

$\int_0^\alpha \frac{n x^{n-1}}{\alpha^n} f(x)\,dx = \alpha$

$\frac{n}{\alpha^n} \int_0^\alpha x^{n-1} f(x)\,dx = \alpha \Rightarrow \int_0^\alpha x^{n-1} f(x)\,dx = \frac{\alpha^{n+1}}{n}$

$\frac{d}{d\alpha} \int_0^\alpha x^{n-1} f(x)\,dx = \frac{d}{d\alpha} \frac{\alpha^{n+1}}{n} \Rightarrow \alpha^{n-1} f(\alpha) = \frac{(n+1)\alpha^n}{n}$

$f(\alpha) = \frac{(n+1)\alpha}{n} = \frac{n+1}{n}\alpha$

$\Rightarrow \frac{n+1}{n} X_{(n)}$ is the UMVUE for α.

Similarly, $\frac{n+1}{n} Y_{(n)}$ is the UMVUE for β.

1.(a) con't

$$var\left(\frac{n+1}{n} X_{(n)}\right) = \left(\frac{n+1}{n}\right)^2 var(X_{(n)})$$

$$E[X_{(n)}] = \int_0^\alpha x \frac{nx^{n-1}}{\alpha^n} dx = \frac{n}{\alpha^n}\int_0^\alpha x^n dx = \frac{n}{\alpha^n}\left[\frac{1}{n+1}x^{n+1}\right]_0^\alpha = \frac{n}{n+1}\frac{\alpha^{n+1}}{\alpha^n} = \frac{n}{n+1}\alpha$$

$$E[X_{(n)}^2] = \int_0^\alpha x^2 \frac{nx^{n-1}}{\alpha^n} dx = \frac{n}{\alpha^n}\int_0^\alpha x^{n+1} dx = \frac{n}{\alpha^n}\left[\frac{1}{n+2}x^{n+2}\right]_0^\alpha = \frac{n}{n+2}\frac{\alpha^{n+2}}{\alpha^n} = \frac{n}{n+2}\alpha^2$$

$$var(X_{(n)}) = E[X_{(n)}^2] - E[X_{(n)}]^2 = \frac{n}{n+2}\alpha^2 - \left(\frac{n}{n+1}\right)^2\alpha^2 = \left[\frac{n}{n+2} - \frac{n^2}{(n+1)^2}\right]\alpha^2$$

$$= \left[\frac{n(n+1)^2 - n^2(n+2)}{(n+1)^2(n+2)}\right]\alpha^2 = \left[\frac{\cancel{n^3} + \cancel{2n^2} + n - (\cancel{n^3} + \cancel{2n^2})}{(n+1)^2(n+2)}\right]\alpha^2$$

$$= \frac{n}{(n+1)^2(n+2)}\alpha^2$$

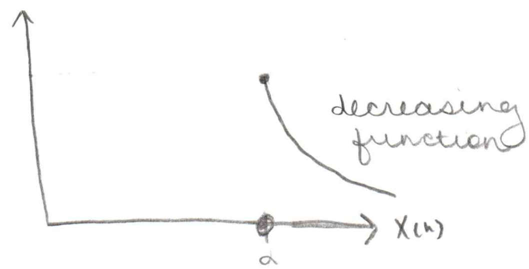
$$\Rightarrow var\left(\frac{n+1}{n} X_{(n)}\right) = \left(\frac{n+1}{n}\right)^2\left(\frac{n}{(n+1)^2(n+2)}\right)\alpha^2 = \frac{1}{n(n+2)}\alpha^2$$

Similarly, $var\left(\frac{n+1}{n} Y_{(n)}\right) = \frac{1}{n(n+2)}\beta^2$

1(b) Calculate the MLEs for α and β, $\hat{\alpha}$ & $\hat{\beta}$ respectively. | 2014 Theory 1

Derive the asymptotic distributions for $\hat{\alpha}$ and $\hat{\beta}$ after normalization.

$L(\alpha) = \prod_{i=1}^{n} \frac{1}{\alpha} I(0 < x_i < \alpha) = \frac{1}{\alpha^n} I(0 < X_{(1)} < X_{(n)} < \alpha)$



$\hat{\alpha} = X_{(n)}$, similarly $\hat{\beta} = Y_{(n)}$

we know $\sqrt{n}(\hat{\alpha} - \alpha) \xrightarrow{d} N(0, I_1(\alpha)^{-1})$ except we can't take derivatives so we must continue in another way.

consider the following sequence

$a_n(X_{(n)} - c_n)$ where a_n and c_n are sequences of constants

$P(a_n(X_{(n)} - c_n) \leq x) = P(X_{(n)} \leq \frac{x}{a_n} + c_n) = (P(X_1 \leq \frac{x}{a_n} + c_n))^n = \left(\frac{c_n + \frac{x}{a_n}}{\alpha}\right)^n$

let $c_n = \alpha$ and $a_n = -n$

$\Rightarrow \left(\frac{\alpha + \frac{x}{n}}{\alpha}\right)^n = \left(1 + \frac{x}{n\alpha}\right)^n \to e^{x/\alpha}$ as $n \to \infty$

consider $n(\alpha - X_{(n)})$

$P(n(\alpha - X_{(n)}) \leq x) = P(X_{(n)} \geq \alpha - \frac{x}{n}) = 1 - P(X_{(n)} \leq \alpha - \frac{x}{n}) = 1 - \left(\frac{\alpha - x/n}{\alpha}\right)^n$

$= 1 - \left(1 - \frac{(x/n)}{\alpha}\right)^n \to 1 - e^{-x/\alpha}$ as $n \to \infty$

$\Rightarrow n(\alpha - X_{(n)}) = n(\alpha - \hat{\alpha}) \xrightarrow{d} \exp(\alpha)$

Similarly, $n(\beta - Y_{(n)}) = n(\beta - \hat{\beta}) \xrightarrow{d} \exp(\beta)$

1(c). The MLE for θ is then $\hat{\theta} = \hat{\beta}/\hat{\alpha}$ Derive the asymptotic distribution of $\hat{\theta}$ after normalization. construct an asymptotic 95% CI for θ based on the observations.

From (b) we know $n(\alpha - X_{(n)}) \xrightarrow{d} Exp(\alpha)$ and $n(\beta - Y_{(n)}) \xrightarrow{d} Exp(\beta)$

$\Rightarrow n(X_{(n)} - \alpha) \xrightarrow{d} -Exp(\alpha)$ and $n(Y_{(n)} - \beta) \xrightarrow{d} -Exp(\beta)$

we know $Y_{(n)}$ & $X_{(n)}$ are independent, thus

$$n\left(\begin{pmatrix} X_{(n)} \\ Y_{(n)} \end{pmatrix} - \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right) \xrightarrow{d} \begin{pmatrix} -Exp(\alpha) \\ -Exp(\beta) \end{pmatrix}$$

let $g(a,b) = \frac{b}{a} \Rightarrow \dot{g}(a,b) = \begin{pmatrix} -b/a^2 \\ 1/a \end{pmatrix}$

$\therefore$ By delta-method,

$$n\left(Y_{(n)}/X_{(n)} - \beta/\alpha\right) \xrightarrow{d} \begin{pmatrix} -\beta/\alpha^2 & 1/\alpha \end{pmatrix} \begin{pmatrix} -Exp(\alpha) \\ -Exp(\beta) \end{pmatrix} \equiv \frac{\beta}{\alpha^2} Exp(\alpha) - \frac{1}{\alpha} Exp(\beta)$$

$$\equiv Exp(\beta/\alpha) - Exp(\beta/\alpha) = \beta/\alpha\left(Exp_1(1) - Exp_2(1)\right) = \theta\left(Exp_1(1) - Exp_2(1)\right)$$

Note: These exponential distributions are still <u>independent</u>. They are separate distributions

We use the MGF to find the distribution of $\beta/\alpha\,(Exp_1(1) - Exp_2(1))$

let $U = Exp(1)$, $V = Exp(1)$, $U \perp V$

$$M_{\beta/\alpha(U-V)}(t) = E\left[e^{\beta/\alpha (U-V)t}\right] = E\left[e^{\beta/\alpha Ut}\right]E\left[e^{-\beta/\alpha Vt}\right]$$

$$= \int_0^\infty e^{\beta/\alpha ut} e^{-u} du \int_0^\infty e^{-\beta/\alpha vt} e^{-v} dv$$

$$= \int_0^\infty e^{(\beta/\alpha t - 1)u} du \int_0^\infty e^{-(\beta/\alpha + 1)v} dv = \left[\frac{1}{\beta/\alpha t - 1} e^{(\beta/\alpha t - 1)u}\right]_{u=0}^\infty \left[\frac{1}{-(\frac{\beta}{\alpha}t + 1)} e^{-(\beta/\alpha t + 1)v}\right]_{v=0}^\infty$$

$$= \frac{1}{\beta/\alpha t - 1}\left(\lim_{u\to\infty}\left(e^{(\beta/\alpha t - 1)u}\right) - e^0\right)\frac{1}{-(\frac{\beta}{\alpha}t + 1)}\left(\lim_{v\to\infty}\left(e^{-(\beta/\alpha t + 1)v} - e^0\right)\right)$$

$$= \frac{-1}{(\beta/\alpha t - 1)(\beta/\alpha t + 1)}\underbrace{\left(\lim_{u\to\infty}\left(e^{(\beta/\alpha t - 1)u}\right) - 1\right)}_{=0 \text{ when } \alpha/\beta > t}\underbrace{\left(\lim_{v\to\infty}\left(e^{-(\beta/\alpha t + 1)v}\right) - 1\right)}_{=0 \text{ when } -\alpha/\beta < t}$$

when $|t| > \alpha/\beta$ $\quad M_{\beta/\alpha(U-V)}(t) = \frac{-1}{\beta^2/\alpha^2 t^2 - 1} = \frac{1}{1 - (\beta/\alpha)^2 t^2}$

$\therefore \beta/\alpha\,(U - V) \sim Laplace(\mu = 0, b = \beta/\alpha) \equiv Laplace(\mu = 0, b = \theta)$

1(c) con't $\Rightarrow n\left(\frac{Y_{(n)}}{X_{(n)}} - \beta/\alpha\right) = n(\hat{\theta}-\theta) \xrightarrow{d} \text{Laplace}(\mu=0, b=\theta)$

let $F(z)$ be the CDF of a Laplace $\underset{\uparrow}{}^{(0,1)}$ distribution:

$$F(z) = \begin{cases} \frac{1}{2}e^{z} & \text{if } z \le 0 \\ 1-\frac{1}{2}e^{-z} & \text{if } z > 0 \end{cases}$$

let $Z = \theta(U-V) \sim \text{Laplace}(0,\theta)$

Laplace is a location-scale family, thus $\frac{1}{\theta}Z \sim \text{Laplace}(0,1)$

$\therefore Z/\theta \perp\!\!\!\perp \theta$ & is a pivotal quantity $\quad \hookrightarrow \frac{n}{\theta}\left(\frac{Y_{(n)}}{X_{(n)}} - \theta\right)$

$F(c_L) = .025 \Rightarrow \frac{1}{2}e^{c_L} = .025 \Rightarrow e^{c_L} = .05 \Rightarrow c_L = \log(.05)$

$F(c_u) = .975 \Rightarrow 1-\frac{1}{2}e^{-c_u} = .975 \Rightarrow \frac{1}{2}e^{-c_u} = .025 \Rightarrow e^{-c_u} = .05 \Rightarrow c_u = -\log(.05)$

A 95% CI for θ is:

$$\left\{\theta : \log(.05) < \frac{n}{\theta}(\hat{\theta}-\theta) < -\log(.05)\right\}$$

$$= \left\{\theta : \log(.05) < \frac{n\hat{\theta}}{\theta} - n < -\log(.05)\right\}$$

$$= \left\{\theta : \log(.05) + n < \frac{n\hat{\theta}}{\theta} < n - \log(.05)\right\}$$

$$= \left\{\theta : \frac{n+\log(.05)}{n\hat{\theta}} < \frac{1}{\theta} < \frac{n-\log(.05)}{n\hat{\theta}}\right\}$$

$$= \left\{\theta : \frac{n\hat{\theta}}{n-\log(.05)} < \theta < \frac{n\hat{\theta}}{n+\log(.05)}\right\}$$

1.(d) $H_0: \alpha=\beta$ vs $H_1: \alpha \neq \beta$

Find the LRT statistic, derive the exact distribution of Λ.

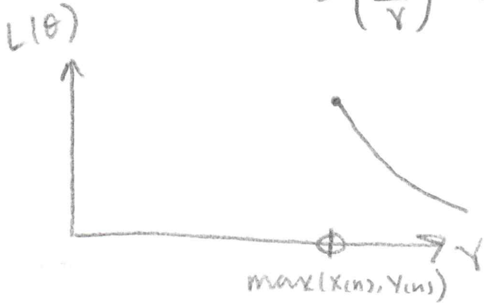
$$L(\alpha,\beta)=\prod_{i=1}^{n} \frac{1}{\alpha} I(0<x_i<\alpha)\frac{1}{\beta} I(0<y_i<\beta)$$

$$=\left(\frac{1}{\alpha\beta}\right)^n I(0<X_{(1)})I(X_{(n)}<\alpha)I(0<Y_{(1)})I(Y_{(n)}<\beta)$$

under $H_0: \alpha=\beta=\gamma$

$$L(\alpha=\gamma,\beta=\gamma) \propto \left(\frac{1}{\gamma}\right)^{2n} I(X_{(n)}<\gamma)I(Y_{(n)}<\gamma)$$

$$=\left(\frac{1}{\gamma}\right)^{2n} I(\max(X_{(n)},Y_{(n)})<\gamma)$$



a decreasing function of γ when $\gamma > \max(X_{(n)},Y_{(n)})$

$\Rightarrow$ MLE of γ $\Rightarrow$ $\hat{\gamma}=\max(X_{(n)},Y_{(n)})$

We know unrestricted $\hat{\alpha}=X_{(n)}$ $\hat{\beta}=Y_{(n)}$

$$\therefore \Lambda=\frac{\sup_{\theta\in\Theta_0} L(\alpha,\beta)}{\sup_{\theta\in\Theta_1} L(\alpha,\beta)}=\frac{L(\hat{\gamma})}{L(\hat{\alpha},\hat{\beta})} \propto \frac{\left(\frac{1}{\max(X_{(n)},Y_{(n)})}\right)^{2n}}{\left(\frac{1}{X_{(n)}Y_{(n)}}\right)^n}=\frac{(X_{(n)}Y_{(n)})^n}{(\max(X_{(n)},Y_{(n)}))^{2n}}$$

$$=\left(\frac{X_{(n)}Y_{(n)}}{\max(X_{(n)},Y_{(n)})^2}\right)^n=\left[\frac{\min(X_{(n)},Y_{(n)})}{\max(X_{(n)},Y_{(n)})}\right]^n \in [0,1]$$

$$P(\Lambda<z)=P\left((\hat{\theta}^n<z \cap Y_{(n)}<X_{(n)}) \cup (\hat{\theta}^{-n}<z \cap X_{(n)}<Y_{(n)})\right)$$

$$=P(\hat{\theta}^n<z \cap Y_{(n)}<X_{(n)})+P(\hat{\theta}^{-n}<z \cap X_{(n)}<Y_{(n)})$$

$$P(\hat{\theta}^n<z, Y_{(n)}<X_{(n)})=P\left(\frac{Y_{(n)}}{X_{(n)}}<z^{1/n}, Y_{(n)}<X_{(n)}\right)=P\left(Y_{(n)}<z^{1/n}X_{(n)}, Y_{(n)}<X_{(n)}\right)$$

$$=P\left(Y_{(n)}<\min(z^{1/n}X_{(n)}, X_{(n)})\right)$$

$$=E\left[I(Y_{(n)}<\min(z^{1/n}X_{(n)},X_{(n)}))\right]=E_{X_{(n)}}\left[E_{Y_{(n)}}\left[Y_{(n)}<\min(z^{1/n}X_{(n)},X_{(n)}) \mid X_{(n)}\right]\right]$$

$$=\int_0^\alpha \int_0^{\min(z^{1/n}x,x)} \frac{nx^{n-1}}{\alpha^n}\frac{ny^{n-1}}{\beta^n}\,dy\,dx=\int_0^\alpha \frac{n^2x^{n-1}}{(\alpha\beta)^n}\int_0^{\min(z^{1/n}x,x)} y^{n-1}\,dy\,dx=\int_0^\alpha \frac{n^2x^{n-1}}{(\alpha\beta)^n}\left[\frac{1}{n}y^n\right]_0^{\min(z^{1/n}x,x)}dx$$

$$=\int_0^\alpha \frac{nx^{n-1}}{(\alpha\beta)^n}(\min(z^{1/n}x,x))^n\,dx$$ we know $z\in[0,1] \Rightarrow z^{1/n}\in[0,1] \Rightarrow z^{1/n}x \le x \ \forall z\in[0,1]$

$\Rightarrow \min(z^{1/n}x,x)=z^{1/n}x$

$$=\int_0^\alpha \frac{nx^{n-1}zx^n}{(\alpha\beta)^n}\,dx=\frac{zn}{(\alpha\beta)^n}\int_0^\alpha x^{2n-1}\,dx=\frac{zn}{(\alpha\beta)^n}\left[\frac{1}{2n}x^{2n}\right]_0^\alpha=\frac{z}{2}\frac{\alpha^{2n}}{\alpha^n\beta^n}=\frac{z}{2}\left(\frac{\alpha}{\beta}\right)^n$$

1.1(d) con't.

Similarly, $P(\hat{\theta}^{-n} < z, X_{(n)} < Y_{(n)}) = \frac{z}{2}\left(\frac{\beta}{\alpha}\right)^n$

$\therefore P(\Lambda < z) = \frac{z}{2}\left(\frac{\alpha}{\beta}\right)^n + \frac{z}{2}\left(\frac{\beta}{\alpha}\right)^n = \frac{z}{2}\left(\left(\frac{\alpha}{\beta}\right)^n + \left(\frac{\beta}{\alpha}\right)^n\right) \Rightarrow \Lambda \sim \text{Unif}\left(0, \frac{2}{\left(\frac{\alpha}{\beta}\right)^n + \left(\frac{\beta}{\alpha}\right)^n}\right)$

Thus, under $H_0: \alpha = \beta$ $\quad \Lambda \sim \text{Unif}(0,1)$

1.1(e) $E[X_k] = \frac{\alpha}{2}$ and $E[Y_k] = \frac{\beta}{2}$. Thus a simple estimator of θ is $\bar{Y}_n / \bar{X}_n$.
Derive the asy. dist. of θ after normalization. What is the ARE of θ WRT $\hat{\theta}$, $\frac{2\bar{Y}_n}{\hat{\alpha}}$, and $\frac{\hat{\beta}}{2\bar{X}_n}$

$\sqrt{n}\left(\begin{pmatrix}\bar{X}_n \\ \bar{Y}_n\end{pmatrix} - \begin{pmatrix}\alpha/2 \\ \beta/2\end{pmatrix}\right) \xrightarrow{d} N_2\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\alpha^2/12 & 0 \\ 0 & \beta^2/12\end{pmatrix}\right)$ let $g(a,b) = \frac{b}{a} \Rightarrow \nabla g(a,b) = \begin{pmatrix}-b/a^2 \\ 1/a\end{pmatrix}$

$\nabla g(\alpha/2, \beta/2) = \begin{pmatrix}-\beta/2 / \alpha^2/4 \\ 2/\alpha\end{pmatrix} = \begin{pmatrix}-2\beta/\alpha^2 \\ 2/\alpha\end{pmatrix} = 2/\alpha \begin{pmatrix}-\beta/\alpha \\ 1\end{pmatrix}$

$4/\alpha^2 \begin{pmatrix}-\beta/\alpha & 1\end{pmatrix}\begin{pmatrix}\alpha^2/12 & 0 \\ 0 & \beta^2/12\end{pmatrix}\begin{pmatrix}-\beta/\alpha \\ 1\end{pmatrix} = \frac{4}{\alpha^2}\begin{pmatrix}-\alpha\beta/12 & \beta^2/12\end{pmatrix}\begin{pmatrix}-\beta/\alpha \\ 1\end{pmatrix} = \frac{4}{\alpha^2}\left(\frac{\beta^2}{12} + \frac{\beta^2}{12}\right)$

$= \frac{8\beta^2}{12\alpha^2} = \frac{2}{3}\left(\frac{\beta}{\alpha}\right)^2 = \frac{2}{3}\theta^2$

$\sqrt{n}(\theta - \beta/\alpha) \xrightarrow{d} N(0, \frac{2}{3}\theta^2) \Rightarrow \text{var}\left(\frac{\bar{Y}_n}{\bar{X}_n}\right) = \frac{1}{n}\frac{2}{3}\theta^2$

We know $n(\hat{\theta} - \theta) \xrightarrow{d}$ Double Exponential $(0, \theta) \Rightarrow \text{var}(n(\hat{\theta} - \theta)) = 2\theta^2$

$\Rightarrow \text{var}(\hat{\theta}) = \frac{1}{n^2} 2\theta^2$

$P\left(\left|\frac{n+1}{n}\hat{\alpha} - \alpha\right| > \varepsilon\right) \le \frac{\alpha^2}{n(n+2)\varepsilon^2}$ $\quad \lim_{n\to\infty} P\left(\left|\frac{n+1}{n}\hat{\alpha} - \alpha\right| > \varepsilon\right) \le \lim_{n\to\infty} \frac{\alpha^2}{n(n+2)\varepsilon^2} = 0$

$\Rightarrow P\left(\left|\frac{n+1}{n}\hat{\alpha} - \alpha\right| > \varepsilon\right) \to 0 \Rightarrow \frac{n+1}{n}\hat{\alpha} \xrightarrow{P} \alpha \quad \frac{n+1}{n} \xrightarrow{P} 1 \Rightarrow \hat{\alpha} \xrightarrow{P} \alpha \Rightarrow \sqrt{n}(\hat{\alpha} - \alpha) \xrightarrow{P} 0$

This is not sufficient

Shown on next pg.

$\Rightarrow \sqrt{n}\begin{pmatrix}\bar{Y} - \beta/2 \\ \hat{\alpha} - \alpha\end{pmatrix} \xrightarrow{d} \begin{pmatrix}N(0, \beta^2/12) \\ 0\end{pmatrix}$ not a problem w/ it being 0 as long as it never goes in the denom.

let $g(a,b) = \frac{2a}{b} \Rightarrow \nabla g(a,b) = \begin{pmatrix}2/b \\ -\frac{2a}{b^2}\end{pmatrix}$

$\nabla g(\beta/2, \alpha) = \begin{pmatrix}2/\alpha \\ -\beta/\alpha^2\end{pmatrix}$ $\quad \begin{pmatrix}\frac{2}{\alpha}, & -\frac{\beta}{\alpha^2}\end{pmatrix}\begin{pmatrix}N(0, \beta^2/12) \\ 0\end{pmatrix} = N\left(0, \frac{4}{12}\frac{\beta^2}{\alpha^2}\right) \equiv N\left(0, \frac{1}{3}\theta^2\right)$

$\Rightarrow \sqrt{n}\left(\frac{2\bar{Y}}{\hat{\alpha}} - \theta\right) \xrightarrow{d} N(0, 1/3\,\theta^2)$

$\Rightarrow \text{var}\left(\frac{2\bar{Y}}{\hat{\alpha}}\right) = \frac{\theta^2}{3n}$

1(c) con't | 2014 Theory 1

Similarly, $\hat{\beta} \xrightarrow{p} \beta \Rightarrow \sqrt{n}(\hat{\beta}-\beta) \xrightarrow{p} 0$ by Slutsky's Thm.

$$\sqrt{n}\begin{pmatrix} \bar{X}-\alpha/2 \\ \hat{\beta}-\beta \end{pmatrix} \xrightarrow{d} \begin{pmatrix} N(0,\alpha^2/12) \\ 0 \end{pmatrix}$$ let $g(a,b)=\frac{b}{2a}$ $\nabla g(a,b)=\left(-\frac{b}{2a^2}, \frac{1}{2a}\right)=\frac{1}{2a}\left(-\frac{b}{a},1\right)$

$$\nabla g(\alpha/2,\beta)=\frac{1}{2(\frac{\alpha}{2})}\left(-\frac{\beta}{\alpha/2},1\right)=\frac{1}{\alpha}\left(-\frac{2\beta}{\alpha},1\right)$$

$$\nabla g(\alpha/2,\beta)\begin{pmatrix} N(0,\alpha^2/12) \\ 0 \end{pmatrix}=\left(-\frac{2\beta}{\alpha^2},\frac{1}{\alpha}\right)\begin{pmatrix} N(0,\alpha^2/12) \\ 0 \end{pmatrix}=N\left(0,\frac{4\beta^2}{12\alpha^2}\right)\equiv N\left(0,\tfrac{1}{3}\theta^2\right)$$

$$\Rightarrow \sqrt{n}\left(\frac{\hat{\beta}}{2\bar{X}_n}-\theta\right)\xrightarrow{d} N\left(0,\tfrac{1}{3}\theta^2\right) \Rightarrow var\left(\frac{\hat{\beta}}{2\bar{X}_n}\right)=\frac{\theta^2}{3n}$$

$\therefore$ $ARE(\theta,\hat{\theta})=\frac{var(\hat{\theta})}{var(\theta)}=\frac{1/n^2\, 2\theta^2}{1/n\,\frac{2}{3}\theta^2}=\frac{3}{n} \Rightarrow \hat{\theta}$ is infinitely more asy. efficient than θ.

$ARE\left(\theta,\frac{2\bar{Y}_n}{\hat{\alpha}}\right)=\frac{var\left(\frac{2\bar{Y}_n}{\hat{\alpha}}\right)}{var(\theta)}=\frac{\theta^2/3n}{2\theta^2/3n}=\frac{1}{2} \Rightarrow \frac{2\bar{Y}_n}{\hat{\alpha}}$ is twice as efficient as θ.

$ARE\left(\theta,\frac{\hat{\beta}}{2\bar{X}_n}\right)=\frac{var(\hat{\beta}/2\bar{X}_n)}{var(\theta)}=\frac{\theta^2/3n}{2\theta^2/3n}=\frac{1}{2} \Rightarrow \frac{\hat{\beta}}{2\bar{X}_n}$ is twice as efficient as θ.

If $X_n \to X_{RV}$ $Y_n \to y_{constant}$ $g(X_n,Y_n)\to g(X,y)$ by CMT

If $\begin{pmatrix} X_n \\ Y_n \end{pmatrix} \rightsquigarrow \begin{pmatrix} X \\ y \end{pmatrix}$ you no longer need to worry about dependence in asymptotics, because that constant is completely independent.

Show $\sqrt{n}\left(\frac{\beta}{\hat{\alpha}_n}-\frac{\beta}{\alpha}\right)=0$ to show $\sqrt{n}\left(\frac{2\bar{Y}_n}{\hat{\alpha}_n}-\frac{\beta}{\hat{\alpha}_n}+\frac{\beta}{\hat{\alpha}_n}-\frac{\beta}{\alpha}\right)\rightsquigarrow \frac{Z}{\alpha}+0$

(*) $\sqrt{n}(\hat{\alpha}-\alpha)=\frac{1}{\sqrt{n}}\,n(\hat{\alpha}_n-\alpha)=\frac{1}{\sqrt{n}}O_p(1)=o_p(1)$

2. Consider a decision problem with a parameter space Θ having a finite number of values, $\theta_1, \ldots, \theta_l, l < \infty$.

   (a) Show that a Bayes rule d_B with respect to a prior distribution Λ on Θ having positive probabilities $\lambda_1, \ldots, \lambda_l > 0$ is admissible.

   (b) The result in part (a) conflicts with other results for continuous parameter spaces where Bayes rules may not be admissible, eg, James-Stein estimation. In the discrete case described above, show that if $\lambda_i = 0$, some $i = 1, \ldots, l$, then the resulting Bayes rule d_B may not be admissible.

   (c) Suppose that the frequentist risk of d_B in part (b) is finite and constant on those θ_i's having $\lambda_i > 0$. Show that this decision rule is minimax, that is, minimizes the maximum risk, on those θ_i's with $\lambda_i > 0$.

   (d) Can anything be said about whether or not d_B in part (b) is minimax on $\theta_i, i = 1, \ldots, l$? Discuss.

In (e), (f), and (g), consider the following classification problem. Suppose that X is an observation from the density

$$p(x|\theta) = \theta^{-1} I(0 < x < \theta),$$

where $I(.)$ denotes the indicator function and the parameter space is $\Theta = \{1, 2, 3\}$. It is desired to classify X as arising from $p(x|1)$, $p(x|2)$, or $p(x|3)$, under a 0-1 loss function (zero loss for a correct decision, a loss of one for an incorrect decision).

   (e) Find the form of the Bayes rule for this problem.

   (f) Find the decision rule which minimizes the maximum risk over Θ and the corresponding least favorable prior distribution.

   (g) Find the decision rule which minimizes the maximum risk over $\theta = 1$ and $\theta = 2$ and the corresponding least favorable prior distribution. Is this minimax rule the same as that in (f)? Explain.

2. Consider a decision problem w/ a parameter space Θ having a finite number of values $\theta_1, \dots, \theta_\ell$, $\ell < \infty$

2014 Theory 1

(a) Show that a Bayes rule d_B WRT a prior distribution Λ on Θ having positive probabilities $\lambda_1, \dots, \lambda_\ell > 0$ is admissible.

Thm 1.9 in notes

Assume d_B is a Bayes Rule WRT Λ s.t. $\mathcal{R}(\Lambda, d_B) = \inf_{d \in \mathcal{D}} \mathcal{R}(\Lambda, d)$

& assume d_B is inadmissible $\therefore \exists$ a rule $d^* \in \mathcal{D}$ s.t.

$R(\theta, d^*) \leq R(\theta, d_B)$ $\forall \theta$ w/ strict inequality at at least 1 θ.

$$\therefore \mathcal{R}(\Lambda, d_B) = E[R(\theta, d_B)] = \sum_{i=1}^{\ell} R(\theta_i, d_B)\lambda_i > \sum_{i=1}^{\ell} R(\theta_i, d^*)\lambda_i = \mathcal{R}(\Lambda, d^*)$$

which is a contradiction of d_B being the Bayes Rule, thus d_B must be admissible.

(b) Show that if $\lambda_i = 0$ for some $i = 1, \dots, \ell$, then the resulting Bayes rule d_B may not be admissible.

Assume d_B is a Bayes Rule and d^* is an admissible rule.

$\therefore R(\theta, d^*) \leq R(\theta, d_B)$ $\forall \theta_i \in \Theta$ w/ strict inequality for some θ_i.

let $\Theta_A = \{\theta_i : R(\theta_i, d^*) < R(\theta_i, d_B)\}$ and $\Theta_B = \{\theta_i : R(\theta_i, d^*) = R(\theta_i, d_B)\}$

Reorder θ s.t. $(\theta_1, \dots, \theta_k) = \Theta_A$ and $(\theta_{k+1}, \dots, \theta_\ell) = \Theta_B$

Suppose $\lambda_1, \dots, \lambda_k = 0$, $\lambda_{k+1}, \dots, \lambda_\ell > 0$

$$\mathcal{R}(\theta, d^*) = \sum_{i=1}^{\ell} R(\theta_i, d^*)\lambda_i = \sum_{i \in A} R(\theta_i, d^*)\lambda_i + \sum_{i \in B} R(\theta_i, d^*)\lambda_i = \sum_{i \in B} R(\theta_i, d^*)\lambda_i$$

$$\mathcal{R}(\theta, d_B) = \sum_{i=1}^{\ell} R(\theta_i, d_B)\lambda_i = \sum_{i \in A} R(\theta_i, d_B)\lambda_i + \sum_{i \in B} R(\theta_i, d_B)\lambda_i = \sum_{i \in B} R(\theta_i, d_B)\lambda_i$$

$\rightarrow \mathcal{R}(\Lambda, d^*) = \mathcal{R}(\Lambda, d_B)$ $\therefore$ we have found a case where d_B is a Bayes Rule and inadmissible.

2. (c) Suppose that the frequentist risk of d_B in (b) is finite & constant on those θ_is having $\lambda_i > 0$. Show that this decision rule is minimax, that is, minimizes the maximum risk, on those θ_i's with $\lambda_i > 0$. | 2014 Theory I

Assume $R(\theta_i, d_B) = c \quad \forall\ \theta_i$ s.t. $\lambda_i > 0$

let $\lambda_1, \dots, \lambda_K = 0$ and $\lambda_{K+1}, \dots, \lambda_\ell > 0$ $\therefore$ only consider the space $\Theta_B = \{\theta_{K+1}, \dots, \theta_\ell\}$

Assume d_B is not minimax over Θ_B.

$\therefore \exists\ d^*$ s.t. $\sup_{\theta \in \Theta_B} R(\theta, d^*) < \sup_{\theta \in \Theta_B} R(\theta, d_B) = c$

$$\mathcal{R}(\Lambda, d^*) = \sum_{i=1}^{\ell} R(\theta_i, d^*)\lambda_i = \sum_{i=1}^{K} R(\theta_i, d^*)\lambda_i + \sum_{i=K+1}^{\ell} R(\theta_i, d^*)\lambda_i$$

$$= \sum_{i=K+1}^{\ell} R(\theta_i, d^*)\lambda_i < \sum_{i=K+1}^{\ell} R(\theta_i, d_B)\lambda_i = \sum_{i=1}^{\ell} R(\theta_i, d_B)$$

$$= \mathcal{R}(\Lambda, d_B)$$

which is a contradiction that d_B is Bayes.

Thus d_B is minimax over Θ_B.

(d) Can anything be said about whether or not d_B in part (b) is minimax on θ_i, $i = 1, \dots, \ell$? Discuss.

~~No, assume ... talk about on Wednesday~~

If d_B is not admissible, so $\exists$ a rule d s.t. $R(\theta_i, d) \le R(\theta_i, d_B)$ $\forall i = 1, \dots, \ell$ w/ some strict <

$\Rightarrow \sup_{\theta \in \Theta} R(\theta, d) \le \sup_{\theta \in \Theta} R(\theta, d_B) \rightarrow d_B$ is NOT minimax.

If d_B is admissible, then it could be minimax as long as it's uniq.

But in (b), we showed that if d_B is assumed admissible, then we can have another Bayes Rule s.t. $\mathcal{R}(\Lambda, d_B) = \mathcal{R}(\Lambda, d^*)$

So d_B cannot be admissible & unique, hence NOT minimax.

2(e). Consider the following classification problem. Suppose that X is an observation from the density

$$p(x|\theta) = \frac{1}{\theta} I(0<x<\theta) \quad \text{Unif } [0, \theta]$$

and the parameter space is $\Theta = \{1, 2, 3\}$

It is desired to classify X as arising from $p(x|1), p(x|2), p(x|3)$, under 0-1 loss

1(e) Find the form of the Bayes Rule for this problem

Under 0-1 loss the Bayes Rule is the posterior mode.

Let $\Lambda = \{\lambda_1, \lambda_2, \lambda_3\}$ be the prior assumed w/ $\sum_{i=1}^{3} \lambda_i = 1$

$p(\theta_i|x) \propto p(x|\theta)\lambda_i \quad \therefore \quad p(1|x) = I(0<x<1)\lambda_1$

$$p(2|x) = \frac{1}{2} I(0<x<2)\lambda_2$$

$$p(3|x) = \frac{1}{3} I(0<x<3)\lambda_3$$

If $0<x<1$

$p(1|x) = \lambda_1, \; p(2|x) = \frac{\lambda_2}{2}, \; p(3|x) = \frac{\lambda_3}{3}$

$$\therefore \phi(x) = \begin{cases} \theta_1 & \lambda_1 > \frac{\lambda_2}{2}, \frac{\lambda_3}{3} \\ \theta_2 & \lambda_2 > \lambda_1, \lambda_3/3 \\ \theta_3 & \lambda_3 > \lambda_1, \lambda_2/2 \end{cases}$$

If $1<x<2$

$p(1|x) = 0, \; p(2|x) = \frac{\lambda_2}{2}, \; p(3|x) = \frac{\lambda_3}{3}$

$$\phi(x) = \begin{cases} \theta_2 & \lambda_2/2 > \lambda_3/3 \\ \theta_3 & \lambda_3/3 > \lambda_2/2 \end{cases}$$

If $2<x<3$

$p(1|x) = 0, \; p(2|x) = 0, \; p(3|x) = \frac{\lambda_3}{3}$

$$\phi(x) = \{\theta_3$$

2(e). con't

If $2<x<3$

$\phi_3(x)=1$, $\phi_1(x)=0$, $\phi_2(x)=0$

If $1<x<2$

$$\phi_3(x)=\begin{cases}1 & \text{if } \frac{\lambda_3}{3}>\frac{\lambda_2}{2}\\ \gamma_0 & \frac{\lambda_3}{3}=\frac{\lambda_2}{2}\\ 0 & \text{o.w.}\end{cases}$$

$$\phi_2(x)=\begin{cases}1 & \text{if } \frac{\lambda_2}{2}>\frac{\lambda_3}{3}\\ \gamma_0 & \text{if } \frac{\lambda_2}{2}=\frac{\lambda_2}{2}\\ 0 & \text{o.w.}\end{cases} \qquad \phi_1(x)=0$$

If $0<x<1$

$\lambda_1>\frac{\lambda_2}{2}>\frac{\lambda_3}{3}$ / $\phi_1(x)=1$, $\phi_2=0$, $\phi_3=0$

$\lambda_1>\frac{\lambda_3}{3}>\frac{\lambda_2}{2}$ / $\phi_1(x)=1$, $\phi_2=0$, $\phi_3=0$

$\lambda_1>\frac{\lambda_2}{2}=\frac{\lambda_3}{2}$ / $\phi_1(x)=1$, $\phi_2=0$, $\phi_3=0$

$\frac{\lambda_2}{2}>\lambda_1>\frac{\lambda_3}{3}$ / $\phi_1(x)=0$, $\phi_2(x)=1$, $\phi_3(x)=0$

$\frac{\lambda_2}{2}>\frac{\lambda_3}{3}>\lambda_1$ / $\phi_1(x)=0$, $\phi_2(x)=1$, $\phi_3(x)=0$

$\frac{\lambda_2}{2}>\frac{\lambda_3}{3}=\lambda_1$ / $\phi_1(x)=0$, $\phi_2(x)=1$, $\phi_3(x)=0$

$\frac{\lambda_3}{3}>\lambda_1>\frac{\lambda_2}{2}$ / $\phi_1(x)=0$, $\phi_2(x)=0$, $\phi_3(x)=1$

$\frac{\lambda_3}{3}>\frac{\lambda_2}{2}>\lambda_1$ / " "

$\frac{\lambda_3}{3}>\lambda_1=\frac{\lambda_2}{2}$ / " "

$\frac{\lambda_3}{3}=\frac{\lambda_2}{2}=\lambda_1$ $\quad \phi_1(x)=\gamma_1$, $\phi_2(x)=\gamma_2$, $\phi_3(x)=\gamma_3=1-\gamma_1-\gamma_2$, $\gamma_1+\gamma_2+\gamma_3=1$

$\lambda_1=\frac{\lambda_2}{2}>\lambda_3/3$ $\quad \phi_1(x)=\gamma_4$ $\phi_2(x)=1-\gamma_4$, $\phi_3(x)=0$ $\quad 0<\gamma_4<1$

$\lambda_1=\frac{\lambda_3}{3}>\frac{\lambda_2}{2}$ $\quad \phi_1(x)=\gamma_5$ $\phi_2(x)=0$ $\phi_3(x)=1-\gamma_5$ $\quad 0<\gamma_5<1$

$\frac{\lambda_2}{2}=\frac{\lambda_3}{3}>\lambda_1$ $\quad \phi_1(x)=0$ $\phi_2(x)=\gamma_6$ $\phi_3(x)=1-\gamma_6$ $\quad 0<\gamma_6<1$

$\phi_1 = \left[I\left(\lambda_1 > \frac{\lambda_2}{2}, \lambda_1 > \frac{\lambda_3}{3}\right) + \gamma_1 I\left(\lambda_1 = \frac{\lambda_2}{2} = \frac{\lambda_3}{3}\right) + \gamma_4 I\left(\lambda_1 = \frac{\lambda_2}{2} > \frac{\lambda_3}{3}\right) + \right.$
$\left. \gamma_5 I\left(\lambda_1 = \frac{\lambda_3}{3} > \frac{\lambda_2}{2}\right)\right] I(0<x<1)$

$\phi_2 = \left[I\left(\frac{\lambda_2}{2} > \frac{\lambda_3}{3}, \frac{\lambda_2}{2} > \lambda_1\right) + \gamma_2 I\left(\lambda_1 = \frac{\lambda_2}{2} = \frac{\lambda_3}{3}\right) + (1-\gamma_4) I\left(\lambda_1 = \frac{\lambda_2}{2} > \frac{\lambda_3}{3}\right)\right.$
$\left. + \gamma_6 I\left(\frac{\lambda_2}{2} = \frac{\lambda_3}{3} > \lambda_1\right)\right] I(0<x<1) + I(1<x<2)\left[I\left(\frac{\lambda_2}{2} > \frac{\lambda_3}{3}\right) + \gamma_7 I\left(\frac{\lambda_2}{2} = \frac{\lambda_3}{3}\right)\right]$

$\phi_3 = \left[I\left(\frac{\lambda_3}{3} > \frac{\lambda_2}{2}, \frac{\lambda_3}{3} > \lambda_1\right) + \gamma_3 I\left(\lambda_1 = \frac{\lambda_2}{2} = \frac{\lambda_3}{3}\right) + (1-\gamma_5) I\left(\lambda_1 = \frac{\lambda_3}{3} > \frac{\lambda_2}{2}\right)\right.$
$\left. + (1-\gamma_6) I\left(\frac{\lambda_2}{2} = \frac{\lambda_3}{3} > \lambda_1\right)\right] I(0<x<1) + I(1<x<2)\left[I\left(\frac{\lambda_3}{3} > \frac{\lambda_2}{2}\right) + (1-\gamma_7) I\left(\frac{\lambda_2}{2} = \frac{\lambda_3}{3}\right)\right]$
$+ I(2<x<3)$

$\therefore$ The form of the minimax rule is $\phi = (\phi_1, \phi_2, \phi_3)$ where ϕ_1, ϕ_2, ϕ_3 are defined above.

(f) Find the decision rule which minimizes the maximum risk over Θ & the corresponding least favorable prior

Bayes $\Rightarrow$ Minimax: need either constant frequentist risk or $R(\Lambda, d_\Lambda) = \sup_\theta (R(\theta, d_\Lambda))$

We will look for a prior which gives constant risk

$R(\theta_i, \phi) = \sum_{i=1}^{3} L(\theta_i, a_j) E_{\phi_j}(X) = 1 - E_{\theta_i}(\phi_i(x))$

$\Rightarrow R(\theta_i, \phi)$ is constant $\Leftrightarrow E_{\theta_i}(\phi_i(x))$

$E_{\theta_1}(\phi_1(x)) = E_{\theta_2}(\phi_2(x)) = E_{\theta_3}(\phi_3(x))$
when $x \in (0,1)$

$R(1, d_M) = R(2, d_M) = R(3, d_M) \Rightarrow \lambda_1 = \frac{\lambda_2}{2} = \frac{\lambda_3}{3}$ $\quad \sum_{i=1}^{3} \lambda_i = 1 \Rightarrow$

$\lambda_1 + \lambda_2 + \lambda_3 = 1 \qquad \lambda_1 = \frac{\lambda_2}{2} = \frac{\lambda_3}{3}$

$\lambda_1 = \frac{1 - \lambda_1 - \lambda_3}{2} \qquad \lambda_1 + 2\lambda_1 + 3\lambda_1 = 1$

$2\lambda_1 = 1 - \lambda_1 - \lambda_3 \qquad 6\lambda_1 = 1$

$3\lambda_1 = 1 - \lambda_3 \qquad \lambda_1 = 1/6$

$\lambda_1 = \frac{1-\lambda_3}{3} \Rightarrow \frac{1}{6} = \frac{1-\lambda_3}{3} \Rightarrow \frac{1}{2} = 1 - \lambda_3 \Rightarrow \lambda_3 = 1/2$

$1 - 1/2 - 1/6 = \frac{6-3-1}{6} = \frac{2}{6} = \frac{1}{3} = \lambda_2$

$\Lambda = (1/6, 1/3, 1/2)$

under $\Lambda = (1/6, 1/3, 1/2)$

$\phi_1(x) = \gamma_1 I(0<x<1)$

$\phi_2(x) = \gamma_2 I(0<x<1) + \gamma_7 I(1 \le x < 2)$

$\phi_3(x) = \gamma_3 I(0<x<1) + (1-\gamma_7) I(1 \le x<2) + I(2 \le x<3)$

$E_{\theta_1}(\phi_1(x)) = \gamma_1 P(0<x<1) = \gamma_1$

$E_{\theta_2}(\phi_2(x)) = \gamma_2 P_2(0<x<1) + \gamma_7 P_2(1 \le x<2) = \gamma_2/2 + \gamma_7/2$

$E_{\theta_2}(\phi_3(x)) = \gamma_3 P_3(0<x<1) + (1-\gamma_7) P_3(1 \le x<2) + P_3(2 \le x<3) = \gamma_3/3 + (1-\gamma_7)/3 + 1/3$

$\therefore$ constant risk s.t.

$2\gamma_1 = \gamma_2 + \gamma_7$ & $3\gamma_1 = 1 + \gamma_3 + (1-\gamma_7) = 2 - \gamma_3 - \gamma_7$

$\therefore$ the minimax rule is any satisfying the above & the least favorable prior is $\Lambda = (1/6, 1/3, 1/2)$

(g). Find the decision rule which minimizes the maximum risk over $\theta = 1$, and $\theta = 2$ and the corresponding least favorable prior. Is this minimax rule the same as in (f)?

Need $E_{\theta_1}(\phi_1) = E_{\theta_2}(\phi_2)$ & can ignore any case where $\theta_i = 3$

<u>$0<x<1$:</u>

$\lambda_1 = \frac{\lambda_2}{2} \Rightarrow \phi_1(x) = \gamma_0,\ \phi_2(x) = 1-\gamma_0$

$\lambda_1 > \frac{\lambda_2}{2} \Rightarrow \phi_1(x) = 1,\ \phi_2(x) = 0$

$\lambda_2/2 > \lambda_1 \Rightarrow \phi_1(x) = 0,\ \phi_2(x) = 1$

<u>$1<x<2$:</u> $\phi_1(x) = 0,\ \phi_2(x) = 1$

$\phi_1(x) = I(0<x<1)\left[\gamma_0 I(\lambda_1 = \frac{\lambda_2}{2}) + I(\lambda_1 > \lambda_2/2)\right]$

$\Rightarrow \phi_2(x) = I(0<x<1)\left[(1-\gamma_0) I(\lambda_1 = \frac{\lambda_2}{2}) + I(\lambda_2/2 > \lambda_1)\right] + I(1<x<2)$

Λ: $E_{\theta_1}(\phi_1) = E_{\theta_2}(\phi_2) \Leftrightarrow \lambda_1 = \frac{\lambda_2}{2}$

2014
Theory 1.

3. Suppose that (X, Y) are two random variables with joint distribution

$$f(x, y|\alpha, \beta) = c(\alpha, \beta) \exp(-\alpha x - \beta y) \sum_{j=0}^{\infty} \frac{x^j y^j}{(j!)^2} \tag{1}$$

for $x > 0, y > 0$. Also, let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be a random sample from (X, Y), and let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$.

(a) Show that the joint distribution of (X, Y) in (1) is in the multiparameter exponential family and identify the rank, show that $c(\alpha, \beta) = \alpha\beta - 1$, and find the parameter space of (α, β).

(b) Derive the marginal distribution of X from (1) and show that $E(X) = \frac{\beta}{\alpha\beta - 1}$.

(c) From (1), show that $E(X^j Y^k) = (-1)^{j+k} S^{-1} \frac{\partial^{j+k} S}{\partial \alpha^j \beta^k}$, where $S \equiv S(\alpha, \beta) = 1/c(\alpha, \beta)$.

(d) Show that the conditional distribution of $Y|X = x$ depends on β but is free of α, and derive the asymptotic distribution of $\bar{Y}|\bar{X} = \bar{x}$, properly normalized.

(e) Based on a sample of size n, derive a UMPU size α^* test for $H_0 : \beta = 2$ against $H_1 : \beta > 2$ and obtain an explicit expression for the critical value of the test.

(f) Based on a sample of size n, derive an exact 95% confidence interval for β.

(g) Derive the score test for testing $H_0 : \beta = 2$ and obtain its asymptotic distribution.

(h) Based on a sample of size n, under squared error loss, derive the generalized Bayes estimator of $\theta = \frac{\alpha}{\beta}$ assuming the joint prior $\pi(\alpha, \beta) \propto \alpha^{-1}\beta^{-1}$ and determine whether the generalized Bayes estimator is admissible.

3. Suppose (X,Y) are 2 RVs with joint distribution

$$f(x,y\mid\alpha,\beta) = c(\alpha,\beta)\exp(-\alpha x-\beta y)\sum_{j=0}^{\infty}\frac{x^j y^j}{(j!)^2} \qquad (1)$$

for $x>0, y>0$. Let $(X_1,Y_1),\dots,(X_n,Y_n)$ be a random sample from (X,Y) and let $\bar{X}=\frac{1}{n}\sum_{i=1}^{n}X_i$ and $\bar{Y}=\frac{1}{n}\sum_{i=1}^{n}Y_i$

(a) Show that the joint distribution of (X,Y) in (1) is in the multiparameter exponential family & identify its rank. Show that $c(\alpha,\beta)=\alpha\beta-1$ & find the parameter space of (α,β).

$$f(x,y\mid\alpha,\beta) = c(\alpha,\beta)\sum_{j=0}^{\infty}\frac{x^j y^j}{(j!)^2}\exp\{-\alpha x+(-\beta)y\} = \sum_{j=0}^{\infty}\frac{x^j y^j}{(j!)^2}\exp\{-\alpha x+(-\beta)y+(-\log(c(\alpha,\beta)))\}$$

$$\eta=\begin{pmatrix}-\alpha\\-\beta\end{pmatrix} \quad T(x)=\begin{pmatrix}x\\y\end{pmatrix} \quad \psi(\theta)=-\log(c(\alpha,\beta))$$

$\therefore$ $f(x,y)$ is a rank 2 multiparameter exponential family distribut.

$$1=\int_0^\infty\int_0^\infty c(\alpha,\beta)\exp(-\alpha x-\beta y)\sum_{j=0}^{\infty}\frac{x^j y^j}{(j!)^2}\,dy\,dx$$

$$=c(\alpha,\beta)\sum_{j=0}^{\infty}\int_0^\infty\frac{\exp(-\alpha x)x^j}{j!}\,dx\int_0^\infty\frac{\exp(-\beta y)y^j}{j!}\,dy$$

$$=c(\alpha,\beta)\sum_{j=0}^{\infty}\int_0^\infty\frac{x^{(j+1)-1}e^{-\alpha x}}{\Gamma(j+1)}\,dx\int_0^\infty\frac{y^{(j+1)-1}e^{-\beta y}}{\Gamma(j+1)}\,dy$$

$$=c(\alpha,\beta)\sum_{j=0}^{\infty}\alpha^{-(j+1)}\beta^{-(j+1)} = \frac{c(\alpha,\beta)}{\alpha\beta}\sum_{j=0}^{\infty}\left(\frac{1}{\alpha\beta}\right)^j \underset{\text{when } \alpha\beta>1}{=} \frac{c(\alpha,\beta)}{\alpha\beta}\,\frac{1}{1-\frac{1}{\alpha\beta}} = \frac{c(\alpha,\beta)}{\alpha\beta-1}=1$$

$\Rightarrow$ $c(\alpha,\beta)=\alpha\beta-1$, where $\alpha\beta>1$

$\Rightarrow$ The parameter space of (α,β) is: $\{\alpha,\beta: 0<\beta<\alpha\}$

3.(b) Derive the marginal distribution of X from (1) & show that $E[X] = \frac{\beta}{\alpha\beta - 1}$

$$f(x,y|\alpha,\beta) = (\alpha\beta - 1)\exp(-\alpha x - \beta y)\sum_{j=0}^{\infty} \frac{x^j y^j}{(j!)^2}$$

$$\Rightarrow f_X(x) = \int_0^\infty (\alpha\beta-1)\exp(-\alpha x - \beta y)\sum_{j=0}^{\infty}\frac{x^j y^j}{(j!)^2}\,dy$$

$$= (\alpha\beta-1)\exp(-\alpha x)\sum_{j=0}^{\infty}\frac{x^j}{j!}\int_0^\infty \frac{y^{(j+1)-1}e^{-\beta y}}{\Gamma(j+1)}\,dy$$

$$= (\alpha\beta-1)\exp(-\alpha x)\sum_{j=0}^{\infty}\frac{x^j}{j!}\beta^{-(j+1)}$$

$$= \frac{\alpha\beta-1}{\beta}e^{-\alpha x}\sum_{j=0}^{\infty}\frac{(x/\beta)^j}{j!} = \frac{\alpha\beta-1}{\beta}e^{-\alpha x}e^{x/\beta}$$

$$= \frac{\alpha\beta-1}{\beta}e^{-x(\alpha - 1/\beta)} = \frac{\alpha\beta-1}{\beta}e^{-x\left(\frac{\alpha\beta-1}{\beta}\right)}, \quad \alpha>0,\ \beta>0,\ \alpha>\beta,\ x>0$$

$$\Rightarrow X \sim \exp\left(\frac{\alpha\beta-1}{\beta}\right) \Rightarrow E[X] = \frac{\beta}{\alpha\beta-1}$$ by properties of exponential distribution

3(c). From (1), show $E[X^j Y^k] = (-1)^{j+k} S^{-1} \frac{\partial^{j+k} S}{\partial \alpha^j \beta^k}$

where $S \equiv S(\alpha,\beta) = \frac{1}{c(\alpha,\beta)} = \frac{1}{\alpha\beta - 1}$

We want to work with the MGF here

$$\Psi_{X,Y}(s,t) = E\left[e^{sX+tY}\right]$$

$$= \int_0^\infty \int_0^\infty e^{sx+ty} (\alpha\beta-1) e^{(-\alpha x - \beta y)} \sum_{j=0}^{\infty} \frac{x^j y^j}{(j!)^2}\, dy\, dx$$

$$= (\alpha\beta-1)\sum_{j=0}^{\infty} \int_0^\infty \frac{e^{-x(\alpha-s)} x^{(j+1)-1}}{\Gamma(j+1)}\, dx \int_0^\infty \frac{e^{-y(\beta-t)} y^{(j+1)-1}}{\Gamma(j+1)}\, dy$$

$$= (\alpha\beta-1)\sum_{j=0}^{\infty} \left(\frac{1}{\alpha-s}\right)^{j+1}\left(\frac{1}{\beta-t}\right)^{j+1}$$

$$= \frac{(\alpha\beta-1)}{(\alpha-s)(\beta-t)} \sum_{j=0}^{\infty} \left(\frac{1}{(\alpha-s)(\beta-t)}\right)^j = \frac{(\alpha\beta-1)}{(\alpha-s)(\beta-t)}\left(\frac{1}{1-\frac{1}{(\alpha-s)(\beta-t)}}\right)$$

$$= \frac{\alpha\beta-1}{(\alpha-s)(\beta-t)-1} = \frac{S((\alpha-s),(\beta-t))}{S(\alpha,\beta)}$$

$$E[X^j Y^k] = \frac{\partial^{j+k} \Psi_{X,Y}(s,t)}{\partial s^j \partial t^k} = \frac{\partial^{j+k}}{\partial s^j \partial t^k} \frac{S(\alpha-s,\beta-t)}{S(\alpha,\beta)} = S^{-1}(\alpha,\beta) \frac{\partial^{j+k}}{\partial s^j \partial t^k} S(\alpha-s,\beta-t)$$

$$= S^{-1}(\alpha,\beta) \frac{\partial^{j+k}}{\partial s^j \partial t^k} \left((\alpha-s)(\beta-t)-1\right)^{-1}\Big|_{(s,t)=(0,0)}$$

$$= (-1)^{j+k} \frac{\partial^{j+k}}{\partial \alpha^j \partial \beta^k} S(\alpha-s,\beta-t)\Big|_{(s,t)=(0,0)}$$

$$= (-1)^{j+k} \frac{\partial^{j+k}}{\partial \alpha^j \partial \beta^k} S(\alpha,\beta)$$

3(d) Show that the conditional distribution of $Y|X=x$ depends on β but is free from α. Derive the asymptotic distribution of $\bar{Y}|\bar{X}=\bar{x}$, properly normalized.

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{(\alpha\cancel{\beta-1})\sum_{j=0}^{\infty}\frac{x^j y^j}{(j!)^2}\exp\{-\alpha x-\beta y\}}{\frac{\alpha\cancel{\beta-1}}{\beta}\exp\{-x(\frac{\alpha\beta-1}{\beta})\}}$$

$$= \beta\exp\{-\alpha x-\beta y+x(\tfrac{\alpha\beta-1}{\beta})\}\sum_{j=0}^{\infty}\frac{x^j y^j}{(j!)^2}$$

$$= \beta\exp\{-\beta y+x(\tfrac{\alpha\beta-1-\alpha\beta}{\beta})\}\sum_{j=0}^{\infty}\frac{x^j y^j}{(j!)^2}$$

$$= \beta\exp\{-\tfrac{1}{\beta}x-\beta y\}\sum_{j=0}^{\infty}\frac{x^j y^j}{(j!)^2}$$

By CLT we know

$\bar{Y}|\bar{X}=\bar{x} \xrightarrow{d} N(\mu,\sigma^2/n)$ where $\mu=E[Y|X=x]$ and $\sigma^2=Var(Y|X=x)$

will come back and solve for these at the end

3. (e) Based on a sample size of n, derive a UMPU size α^* test for $H_0: \beta = 2$ vs $H_1: \beta > 2$ & obtain an explicit expression for the critical value of the test.

$$f_{X,Y}(x,y) = c(\alpha,\beta)\exp\{-\alpha x - \beta y\}\sum_{j=0}^{\infty}\frac{x^j y^j}{(j!)^2}$$

$$\therefore L(\alpha,\beta) = \prod_{i=1}^{n} c(\alpha\beta)\exp\{-\alpha x_i - \beta y_i\}\sum_{j=0}^{\infty}\frac{x_i^j y_i^j}{(j!)^2}$$

$$= c(\alpha,\beta)^n \exp\{-\alpha\sum_{i=1}^{n}x_i - \beta\sum_{i=1}^{n}\}\prod_{i=1}^{n}\sum_{j=0}^{\infty}\frac{x_i^j y_i^j}{(j!)^2}$$

$\theta = \beta$, $U(Y) = -\Sigma y_i$, $T_1(X) = -\Sigma x_i$, $\xi_1 = \alpha$ $\quad k=1$ (notation on pg 63 of chapter 2)

$\therefore$ By thm 2.5,

$$\phi(x,y) = \begin{cases} 1 & \text{if } -\sum_{i=1}^{n} y_i > c(t) \\ 0 & \text{if } -\sum_{i=1}^{n} y_i < c(t) \end{cases} \equiv \begin{cases} 1 & \text{if } \sum_{i=1}^{n} y_i < c(t) \\ 0 & \text{if } \sum_{i=1}^{n} y_i > c(t) \end{cases}$$

where $E_{\theta_0}[\phi(x,y) \mid T_1 = t_1] = \alpha^*$

$$E_{\beta=2}[\phi(x,y) \mid T_1 = t_1] = E\left[\sum_{i=1}^{n} y_i < c(t) \mid \beta = 2, \Sigma x_i = t\right] \qquad X \perp\!\!\!\perp Y$$

$$\Rightarrow E\left[\sum_{i=1}^{n} y_i < c(t) \mid \beta = 2\right] = \int_0^{c(t)} f(\Sigma y_i \mid \beta = 2)\, d\Sigma y_i = \alpha^*$$

solve the above equation for $c(t)$

$$f_Y(y) = \int_0^{\infty}(\alpha\beta-1)\exp(-\alpha x - \beta y)\sum_{j=0}^{\infty}\frac{x^j y^j}{(j!)^2}\,dx$$

$$= \sum_{j=0}^{\infty}(\alpha\beta-1)\frac{e^{-\beta y}y^j}{j!}\int_0^{\infty}\frac{e^{-\alpha x}x^j}{j!}\,dx = \sum_{j=0}^{\infty}(\alpha\beta-1)\frac{e^{-\beta y}y^j}{j!}\int\frac{e^{-\alpha x}x^{(j+1)-1}}{\Gamma(j+1)}\,dx$$

$$= (\alpha\beta-1)e^{-\beta y}\sum_{j=0}^{\infty}\frac{y^j}{j!}\alpha^{-(j+1)} = \frac{\alpha\beta-1}{\alpha}e^{-\beta y}\sum_{j=0}^{\infty}\frac{(y/\alpha)^j}{j!} = \frac{\alpha\beta-1}{\alpha}e^{-\beta y}e^{y/\alpha}$$

$$= \frac{\alpha\beta-1}{\alpha}e^{-\left(\frac{\alpha\beta-1}{\alpha}\right)}$$

$\Rightarrow Y \sim \exp\left(\frac{\alpha\beta-1}{\alpha}\right)$

$\Rightarrow \Sigma y_i \sim \text{Gamma}\left(n, \frac{\alpha\beta-1}{\alpha}\right)$

under null, $\overset{\Rightarrow \beta=2}{\Sigma y_i} \sim \text{Gamma}(n, 2 - 1/\alpha)$

not a pivotal, unsure how to continue.

3(e) con't $\Sigma y_i \mid \beta=2 \sim Gamma(n, 2^{-1/\alpha})$ let $\Sigma y_i = A$

$$f_A(a \mid \beta=2) = \frac{1}{\Gamma(n)(2^{-1/\alpha})^n} a^{n-1} e^{-a/2^{-1/\alpha}}$$

$$\int_0^{c(t)} \frac{1}{\Gamma(n)(2^{-1/\alpha})^n} a^{n-1} e^{-a/(2^{-1/\alpha})} da = \frac{1}{\Gamma(n)(2^{-1/\alpha})^n} \int_0^{c(t)} a^{n-1} e^{-a/(2^{-1/\alpha})} da = \alpha^*$$

3(f) Based on a sample of size n, derive an exact 95% CI for β.
We can base this CI off of the 2-sided test for $\beta=2$ vs $\beta \neq 2$
where

$$\phi_2(x,y) = \begin{cases} 1 & c_1(t) < \Sigma y_i < c_2(t) \\ 0 & o.w. \end{cases}$$

where $.05 = E[1-\phi^* \mid T=t, \beta=2]$

and $E[\Sigma y_i(1-\phi^*) \mid T=t, \beta=2] = .05\, E[\Sigma y_i \mid T=t, \beta=2]$

3(g) Derive the score test for testing $\beta=2$ & obtain its asymptotic distribution. | 2014 Theory I

From (e) we know

$$L(\alpha,\beta) = (\alpha\beta-1)^n \exp\{-\alpha\sum X_i - \beta\sum Y_i\} \prod_{i=1}^{n}\sum_{j=0}^{\infty} \frac{x_i^j y_i^j}{(j!)^2}$$

$$\ell(\alpha,\beta) \propto n\log(\alpha\beta-1) - \alpha\sum X_i - \beta\sum Y_i$$

$$\frac{\partial \ell(\alpha,\beta)}{\partial\alpha} = \frac{n\beta}{\alpha\beta-1} - \sum X_i \qquad \frac{\partial \ell(\alpha,\beta)}{\partial\beta} = \frac{n\alpha}{\alpha\beta-1} - \sum Y_i$$

$$\frac{\partial^2 \ell(\alpha,\beta)}{\partial\alpha^2} = \frac{(\alpha\beta-1)(0) - n\beta(\beta)}{(\alpha\beta-1)^2} = -\frac{n\beta^2}{(\alpha\beta-1)^2} \qquad \frac{\partial^2 \ell(\alpha,\beta)}{\partial\beta^2} = \frac{(\alpha\beta-1)0 - n\alpha(\alpha)}{(\alpha\beta-1)^2} = \frac{-n\alpha^2}{(\alpha\beta-1)^2}$$

$$\frac{\partial^2 \ell(\alpha,\beta)}{\partial\alpha\partial\beta} = \frac{(\alpha\beta-1)n - n\beta(\alpha)}{(\alpha\beta-1)^2} = -\frac{n}{(\alpha\beta-1)^2}$$

$$\Rightarrow I_n(\alpha,\beta) = \frac{n}{(\alpha\beta-1)^2}\begin{pmatrix} \beta^2 & 1 \\ 1 & \alpha^2 \end{pmatrix}$$

$$\frac{\partial \ell(\alpha,\beta)}{\partial\alpha} = \frac{n\beta}{\alpha\beta-1} - \sum X_i = 0 \Rightarrow \frac{n\beta}{\sum X_i} = \alpha\beta - 1 \Rightarrow 1 + \frac{n\beta}{\sum X_i} = \alpha\beta \Rightarrow \frac{1}{\beta} + \frac{n}{\sum X_i} = \alpha$$

under H_0, $\beta = 2 \Rightarrow \tilde{\alpha} = \frac{1}{2} + \frac{1}{\bar{X}}$ where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

$$\therefore S_n = \left(0, \frac{n\tilde{\alpha}}{2\tilde{\alpha}-1} - \sum Y_i\right)\frac{n}{(2\tilde{\alpha}-1)^2}\begin{pmatrix} 4 & 1 \\ 1 & \tilde{\alpha}^2 \end{pmatrix}\begin{pmatrix} 0 \\ \frac{n\tilde{\alpha}}{2\tilde{\alpha}-1} - \sum Y_i \end{pmatrix}$$

$$= \frac{n}{(2\tilde{\alpha}-1)^2}\left(\frac{n\tilde{\alpha}}{2\tilde{\alpha}-1} - \sum Y_i, \frac{n\tilde{\alpha}^3}{2\tilde{\alpha}-1} - \tilde{\alpha}^2\sum Y_i\right)\begin{pmatrix} 0 \\ \frac{n\tilde{\alpha}}{2\tilde{\alpha}-1} - \sum Y_i \end{pmatrix}$$

$$= \frac{n}{(2\tilde{\alpha}-1)^2}\left(\frac{n\tilde{\alpha}^3}{2\tilde{\alpha}-1} - \tilde{\alpha}^2\sum Y_i\right)\left(\frac{n\tilde{\alpha}}{2\tilde{\alpha}-1} - \sum Y_i\right)$$

$$= \frac{n\tilde{\alpha}^2}{(2\tilde{\alpha}-1)}\left(\frac{n\tilde{\alpha}}{2\tilde{\alpha}-1} - \sum Y_i\right)^2 \xrightarrow{d} \chi^2_1 \text{ under } H_0, \text{ as } n\to\infty$$