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UMVUE's

1) a) $\tilde{\alpha} = \frac{n+1}{n} X_{(n)}$ $\quad \tilde{\beta} = \frac{n+1}{n} Y_{(n)}$

$$f_{X_{(n)}}(x) = \frac{n}{\alpha^n} x^{n-1} \qquad f_{Y_{(n)}}(y) = \frac{n}{\beta^n} y^{n-1}$$

$$E\left(\frac{n+1}{n} X_{(n)}\right) = \alpha$$

$$E\left(\left(\frac{n+1}{n}\right)^2 X_{(n)}^2\right) = \left(\frac{n+1}{n}\right)^2 E\left(X_{(n)}^2\right) = \left(\frac{n+1}{n}\right)^2 \frac{n}{\alpha^n} \int_0^\alpha x^{n+1}\,dx$$

$$= \frac{(n+1)^2}{n\alpha^n} \left[\frac{1}{n+2} x^{n+2}\right]_0^\alpha = \frac{(n+1)^2 \alpha^{n+2}}{n\alpha^n (n+2)} = \frac{(n+1)^2 \alpha^2}{n(n+2)}$$

$$Var(\tilde{\alpha}) = Var\left(\frac{n+1}{n} X_{(n)}\right) = E\left[\left(\frac{n+1}{n}\right)^2 X_{(n)}^2\right] - E\left(\frac{n+1}{n} X_{(n)}\right)^2$$

$$= \frac{(n+1)^2 \alpha^2}{n(n+2)} - \alpha^2 = \alpha^2\left(\frac{(n+1)^2}{n(n+2)} - \frac{n(n+2)}{n(n+2)}\right)$$

$$= \alpha^2\left(\frac{\cancel{n^2} \cancel{+2n} + 1 \cancel{-n^2} \cancel{-2n}}{n(n+2)}\right) = \frac{\alpha^2}{n(n+2)}$$

$$Var(\tilde{\beta}) = \frac{\beta^2}{n(n+2)}$$

b) $\hat{\alpha} = X_{(n)}$ $\quad \hat{\beta} = Y_{(n)}$

let's look at $a_n(X_{(n)} - k)$

$$P(a_n(X_{(n)} - k) \le x) = P\left(X_{(n)} \le \frac{x}{a_n} + k\right) = \left[P\left(X_1 \le \frac{x}{a_n} + k\right)\right]^n$$

$$= \left[\frac{1}{\alpha}\left(\frac{x}{a_n} + k\right)\right]^n = \left(\frac{x/\alpha}{a_n} + \frac{k}{\alpha}\right)^n \quad \text{let } k=\alpha,\ a_n = n \Rightarrow \left(\frac{(x/\alpha)}{n} + 1\right)^n \longrightarrow e^{x/\alpha}$$

let's look @ $n(\alpha - X_{(n)})$

$$P(n(\alpha - X_{(n)}) \le x) = P\left(X_{(n)} \ge \alpha - \frac{x}{n}\right) = 1 - P\left(X_{(n)} \le \alpha - \frac{x}{n}\right)$$

$$= 1 - \left[P\left(X_1 \le \alpha - \frac{x}{n}\right)\right]^n = 1 - \left[\frac{1}{\alpha}\left(\alpha - \frac{x}{n}\right)\right]^n = 1 - \left(1 + \frac{(-x/\alpha)}{n}\right)^n$$

$$\longrightarrow 1 - e^{-x/\alpha}$$

Thus $n(\alpha - X_{(n)}) \xrightarrow{d} Exp(\alpha)$

& $n(\beta - Y_{(n)}) \xrightarrow{d} Exp(\beta)$

c) $\hat{\beta}$ $\hat{\alpha}$ $\longrightarrow$ $\hat{\theta} = \frac{\hat{\beta}}{\hat{\alpha}}$, $V = \hat{\alpha}$

Inverses: $\hat{\beta} = V\hat{\theta}$
$\hat{\alpha} = V$

Jacobian:
$$J = \begin{pmatrix} \frac{\partial \hat{\beta}}{\partial \hat{\theta}} & \frac{\partial \hat{\beta}}{\partial V} \\ \frac{\partial \hat{\alpha}}{\partial \hat{\theta}} & \frac{\partial \hat{\alpha}}{\partial V} \end{pmatrix}$$
$$= \begin{vmatrix} V & \hat{\theta} \\ 0 & 1 \end{vmatrix}$$
$$= V > 0$$

then joint pdf of $(\hat{\theta}, V)$ is
$$f_{\hat{\theta},V}(\hat{\theta}, V) = f_{\hat{\alpha},\hat{\beta}}(V, V\hat{\theta})\,|J|$$
$$= f_{\hat{\alpha}}(V) f_{\hat{\beta}}(V\hat{\theta})\,|J| \text{ by } \hat{\alpha} \perp \hat{\beta}$$
$$= \frac{n}{\alpha^n} V^{n-1} \cdot \frac{n}{\beta^n} (V\hat{\theta})^{n-1}\, V$$
$$= \frac{n^2}{\alpha^n \beta^n} V^{n-1+n-1+1}\, \hat{\theta}^{n-1}$$
$$= \frac{n^2}{\alpha^n \beta^n} V^{2n-1}\, \hat{\theta}^{n-1} \quad \text{for } 0 < V < \alpha,\ 0 < \hat{\theta} < \infty$$

then pdf of $\hat{\theta}$ is
$$f_{\hat{\theta}}(\theta) = \int_0^\alpha \frac{n^2}{\alpha^n \beta^n} V^{2n-1} \theta^{n-1}\, dV$$
$$= \frac{n^2}{\alpha^n \beta^n} \theta^{n-1} \int_0^\alpha V^{2n-1}\, dV$$

$\theta = w$ small

$$= \frac{n^2}{\alpha^n \beta^n} \theta^{n-1} \left[\frac{1}{2n} V^{2n} \Big|_0^\alpha \right]$$
$$= \frac{n^2}{\alpha^n \beta^n} w^{n-1} \cdot \frac{1}{2n} \alpha^{2n} = \frac{n}{2} \left(\frac{\alpha}{\beta}\right)^n w^{n-1} \quad \text{for } 0 < w < \infty$$

$\frac{\alpha}{\beta} = \frac{1}{\theta}$

$= \frac{n w^{n-1}}{2\theta^n}$

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1) c) cont....

$$F_{\hat{\theta}}(\omega) = \frac{n}{2\theta^n}\int_0^\omega x^{n-1}\,dx = \frac{\cancel{n}}{2\theta^n}\left[\frac{1}{\cancel{n}}x^n\right]_0^\omega = \frac{\omega^n}{2\theta^n} = \frac{1}{2}\left(\frac{\omega}{\theta}\right)^n$$

look at $an(\hat{\theta} - k)$

$$P(an(\hat{\theta}-k) \le x) = P\left(\hat{\theta} \le \frac{x}{an} + k\right) = \frac{1}{2}\left[\frac{1}{\theta}\left(\frac{x}{an} + k\right)\right]^n = \frac{1}{2}\left[\frac{(x/\theta)}{an} + \frac{k}{\theta}\right]^n$$

let $k=\theta$, $an = n$ $\longrightarrow \frac{1}{2} e^{x/\theta}$

$$P(n(\theta - \hat{\theta}) \le k) = 1 - P\left(\hat{\theta} \le \frac{-x}{n} + \theta\right) = 1 - \frac{1}{2}\left[\frac{1}{\theta}\left(\theta - \frac{x}{n}\right)\right]^n$$

$$= 1 - \frac{1}{2}\left[1 + \frac{(-x/\theta)}{n}\right]^n \longrightarrow 1 - \frac{1}{2}e^{-x/\theta}$$

Pivotal quantity

$$n\left(1 - \frac{\hat{\theta}}{2^{1/n}\theta}\right) \xrightarrow{d} Exp(1)$$

use quantiles of $Exp(1)$ to get 95% CI

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1) c) let $\underset{\hat{\theta}}{Z} = \frac{Y_{(n)}}{X_{(n)}} \Rightarrow \begin{matrix} X_{(n)} = W \\ Y_{(n)} = ZW \end{matrix} \rightarrow \begin{vmatrix} 1 & 0 \\ Z & W \end{vmatrix} = W$

$W = X_{(n)}$

$f_{ZW}(z,\omega) = w\, f_{X_{(n)}}(\omega) f_{Y_{(n)}}(z\omega)$

$= \frac{n^2}{\alpha^n \beta^n} \omega^{n-1} (z\omega)^{n-1}\omega \qquad 0<z<\theta, \quad 0<\omega<\alpha$

$= \frac{n^2}{\alpha^n \beta^n} \omega^{2n-1} z^{n-1}$

$f_z(z) \propto \frac{n^2}{\alpha^n\beta^n} z^{n-1} \int_0^\alpha \omega^{2n-1} d\omega = \frac{n^2}{\alpha^n\beta^n} z^{n-1}\left[\frac{1}{2n}\omega^{2n}\right]_0^\alpha = \frac{n z^{n-1}}{2\alpha^n\beta^n}\alpha^{2n} = \frac{n z^{n-1}\alpha^n}{2\beta^n} = \frac{n z^{n-1}}{2\theta^n}$

$1 = c\frac{n}{2\theta^n}\int_0^\theta z^{n-1} dz = \frac{c\cancel{n}}{2\theta^n}\left[\frac{1}{\cancel{n}} z^n\right]_0^\theta = \frac{c}{2} \Rightarrow c = 2$

$\Rightarrow \boxed{f_z(z) = 2\left(\frac{n z^{n-1}}{2\theta^n}\right) = \frac{n z^{n-1}}{\theta^n}} \qquad 0<z<\theta$

$F_{\hat{\theta}}(z) = \frac{n}{\theta^n}\int_0^z x^{n-1}dx = \frac{\cancel{n}}{\theta^n}\left[\frac{1}{\cancel{n}}x^n\right]_0^z = \left(\frac{z}{\theta}\right)^n$

look at: $an(k-\hat{\theta})$ (b/c we know from (b))

$P(an(k-\hat{\theta}) \le x) = P(-\hat{\theta} \le \frac{x}{an} - k) = P(\hat{\theta} \ge k - \frac{x}{an}) = 1 - P(\hat{\theta} \le k - \frac{x}{an})$

$= 1 - \left[\frac{1}{\theta}(k - \frac{x}{an})\right]^n = 1 - \left[\frac{k}{\theta} - \frac{(x/\theta)}{an}\right]^n \underset{k=\theta}{\overset{\text{let } an=n}{=\!=}} 1 - \left[1 + \frac{(-x/\theta)}{n}\right]^n \rightarrow 1 - e^{-x/\theta} \leftarrow Exp(\theta)$

$*$ So $n(\theta - \hat{\theta}) \rightarrow_d Exp(\theta)$

95% CI : pivotal quantity: $\frac{n}{\theta}(\theta - \hat{\theta}) \rightarrow_d Exp(1)$

let $C_{0.025}, C_{0.975}$ denote respective quantiles

95% CI: $\left\{\theta : C_{0.025} \le n(1 - \frac{\hat{\theta}}{\theta}) \le C_{0.975}\right\}$

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1) d. $L(\alpha, \beta \mid \underset{\sim}{x}, \underset{\sim}{y}) = \alpha^{-n} I(X_{(n)} < \alpha)\, \beta^{-n} I(Y_{(n)} < \beta)$

Unrestricted MLE (H_A): indpt. $\Rightarrow \hat{\alpha} = X_{(n)}, \; \hat{\beta} = Y_{(n)}$

Restricted MLE (H_0): $H_0: \alpha = \beta = \gamma \quad L(\alpha, \beta) = \gamma^{-2n} I(\max\{X_{(n)}, Y_{(n)}\} < \gamma)$

$\Longrightarrow \hat{\gamma}_R = \hat{\alpha}_R = \hat{\beta}_R = \max\{X_{(n)}, Y_{(n)}\}$

$$\Lambda = \left(\frac{\hat{\alpha}\hat{\beta}}{\hat{\gamma}_R^2}\right)^n > k_1 \Rightarrow \left(\frac{\hat{\alpha}\hat{\beta}}{\hat{\gamma}_R^2}\right) > k_2$$

$-2\log\Lambda \longrightarrow \chi^2_1$ (idk how to do exact....)

e) $\sqrt{n}\begin{pmatrix} \bar{X}_n - \frac{\alpha}{2} \\ \bar{Y}_n - \frac{\beta}{2} \end{pmatrix} \to_d N\left(0, \begin{pmatrix} \alpha^2/12 & 0 \\ 0 & \beta^2/12 \end{pmatrix}\right)$

$g(a,b) = b/a \Rightarrow \nabla g(a,b) = \begin{pmatrix} -b/a^2 \\ 1/a \end{pmatrix} = \frac{1}{a}\begin{pmatrix} -b/a \\ 1 \end{pmatrix}$

$\nabla g\left(\frac{\alpha}{2}, \frac{\beta}{2}\right) = \frac{1}{\alpha/2}\begin{pmatrix} -(\beta/2)/(\alpha/2) \\ 1 \end{pmatrix} = \frac{2}{\alpha}\begin{pmatrix} -\beta/\alpha \\ 1 \end{pmatrix}$

$V = \frac{2}{\alpha}\begin{pmatrix} -\frac{\beta}{\alpha} & 1 \end{pmatrix} \frac{1}{12}\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}^2 \frac{2}{\alpha}\begin{pmatrix} -\beta/\alpha \\ 1 \end{pmatrix} = \frac{4}{12\alpha^2}\left(\cdots \right) = \frac{2}{3}\left(\frac{\beta}{\alpha}\right)^2 = \frac{2}{3}\theta^2$

$\hat{\theta}$'s asymp var is θ^2

since $n(\theta - \hat{\theta}) \to_d Exp(\theta)$

$* \; ARE(\tilde{\theta}, \hat{\theta}) = \frac{2/3\,\theta^2}{\theta^2} = \frac{2}{3}$ so $\tilde{\theta}$ has smaller var

might not be able to use b/c $\hat{\theta}$ isn't asymp normal

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