762 HW 11 uses different pdf.

Q2, Practice 5(a)

1. (25 points) Suppose that $y_1, \ldots, y_n$ are positive and independent random variables, where

$$p(y_i|\mu_i) = \frac{1}{\mu_i}\exp(-y_i/\mu_i), \quad \mu_i > 0, \tag{1}$$

where $E(y_i|\mu_i) = \mu_i$, $i = 1, \ldots n$. Let $\theta_i = 1/\mu_i$.

(a) (3 points) Suppose that θ_i is random with $\theta_i \sim \text{Gamma}(a_i, b_i)$, where $a_i/b_i = \exp(-x_i'\beta)$ and $a_i = 3$. Further assume $\text{Var}(\theta_i) = \tau \exp(x_i'\beta)$. Here, x_i is a $p \times 1$ vector of covariates and β is a $p \times 1$ vector of regression coefficients, and β is unknown. Derive the **marginal** mean and variance of y_i, that is, compute $E(y_i)$ and $\text{Var}(y_i)$.

(b) (3 points) Under the same assumptions as part (a), derive the marginal distribution of y_i.

(c) (7 points) Under the same assumptions as part (a), derive the score test for testing $H_0 : \tau = 0$ and give its asymptotic distribution under the null hypothesis.

(d) Now suppose we take μ_i to be a **fixed and unknown parameter** and we incorporate over-dispersion by taking $\text{Var}(y_i) = \sigma^2(v_i + \mu_i)$ where v_i is the variance function of the GLM in (1). Let $\mu_i = \exp\{x_i'\beta\}$.

(i) (5 points) Derive the quasi-likelihood score equations for β and a moment estimator for σ^2.

(ii) (7 points) Let $\hat{\beta}_P$ denotes the quasi-likelihood estimate of β. Derive the asymptotic covariance matrix for $\hat{\beta}_P$.

$$\begin{pmatrix} a & & 0 \\ & b & \\ 0 & c & d \end{pmatrix}\begin{pmatrix} c & f & g \\ h & i & j \\ k & l & m \\ n & o & p \end{pmatrix}$$

a e

1. Suppose that $y_1, \dots, y_n$ are positive & independent RVs, where $\boxed{\text{2016 Theory 2}}$

$p(y_i|\mu_i) = \frac{1}{\mu_i} \exp(-y_i/\mu_i)$ $\mu_i > 0$ where $E[y_i|\mu_i] = \mu_i$ $i = 1, \dots, n$

Let $\theta_i = 1/\mu_i \Rightarrow p(y_i|\theta_i) = \theta_i \exp(-y_i \theta_i)$

(a) Suppose that θ_i is random with $\theta_i \sim \text{Gamma}(a_i, b_i)$

where $\frac{a_i}{b_i} = \exp\{-x_i'\beta\}$ and $a_i = 3$, $\text{Var}(\theta_i) = \tau \exp(x_i'\beta)$

Derive the marginal mean & variance of y_i.

$E[y_i] = E[E[y_i|\theta_i]] = E[1/\theta_i]$

$= \int_0^\infty \frac{1}{\theta_i} \frac{b_i^3}{\Gamma(3)} \theta_i^{3-1} \exp\{-\theta_i b_i\} d\theta_i = \int_0^\infty \frac{b_i^3}{\Gamma(3)} \theta_i^{2-1} \exp\{-\theta_i b_i\} d\theta_i$

$= \frac{b_i^3}{\Gamma(3)} \frac{\Gamma(2)}{b_i^2} \int_0^\infty \frac{b_i^2}{\Gamma(2)} \theta_i^{2-1} \exp\{-\theta_i b_i\} d\theta_i = \frac{b_i}{2}$

$\text{Var}(y_i) = \text{Var}(E[y_i|\theta_i]) + E[\text{var}(y_i|\theta_i)]$

$= \text{var}(\frac{1}{\theta_i}) + E[\frac{1}{\theta_i^2}] = E[1/\theta_i^2] - E[1/\theta_i]^2 + E[1/\theta_i^2] = 2E[1/\theta_i^2] - E[1/\theta_i]^2$

$E[1/\theta_i^2] = \int_0^\infty \frac{1}{\theta_i^2} \frac{b_i^3}{\Gamma(3)} \theta_i^{3-1} \exp\{-\theta_i b_i\} d\theta_i = \int_0^\infty \frac{b_i^3}{\Gamma(3)} \theta_i^{1-1} \exp\{-\theta_i b_i\} d\theta_i$

$= \frac{b_i^3}{\Gamma(3)} \frac{\Gamma(1)}{b_i} \int \frac{b_i}{\Gamma(1)} \theta_i^{1-1} \exp\{-\theta_i b_i\} d\theta_i = \frac{b_i^2}{2}$

$\text{var}(y_i) = 2E[1/\theta_i^2] - E[1/\theta_i]^2 = 2\left(\frac{b_i^2}{2}\right) - \left(\frac{b_i}{2}\right)^2 = b_i^2 - \frac{b_i^2}{4} = \frac{3b_i^2}{4}$

1. (b) Under the same assumptions as part (a), derive the marginal distribution of y_i.

$$f_Y(y_i) = \int_0^\infty f_{Y,\theta}(y,\theta)\,d\theta = \int_0^\infty f_{Y|\theta}(y,\theta) f_\theta(\theta)\,d\theta$$

$$= \int_0^\infty \theta_i \exp\{-y_i\theta_i\} \frac{b_i^3}{\Gamma(3)} \theta_i^{3-1} \exp\{-\theta_i b_i\}\,d\theta_i$$

$$= \frac{b_i^3}{\Gamma(3)} \int_0^\infty \theta_i^{4-1} \exp\{-\theta_i(y_i+b_i)\}\,d\theta_i$$

$$= \frac{b_i^3}{\Gamma(3)} \frac{\Gamma(4)}{(y_i+b_i)^4} \underbrace{\int_0^\infty \frac{(y_i+b_i)^4}{\Gamma(4)} \theta_i^{4-1} \exp\{-\theta_i(y_i+b_i)\}\,d\theta_i}_{\text{Gamma}(4,\, y_i+b_i) = 1}$$

$$= \frac{\Gamma(4)}{\Gamma(3)} \frac{b_i^3}{(y_i+b_i)^4} = \frac{3!}{2!} \frac{b_i^3}{(y_i+b_i)^4} = 3\frac{b_i^3}{(y_i+b_i)^4}$$

1.(c) Under the same assumptions as part (a), derive the score test for testing $H_0: \tau = 0$ and give its asymptotic distribution under the null.

2016 Theory 2

$$p(y_i|\mu_i) = \frac{1}{\mu_i}\exp\{-y_i\frac{1}{\mu_i}\} = \exp\{-y_i/\mu_i - \log(\mu_i)\}$$

$var(\theta_i) = \tau\exp\{x_i'\beta\}$
$\Rightarrow f_i = \exp\{x_i'\beta\}$

$\theta_i = -\frac{1}{\mu_i} \quad b(\mu_i) = \log(\mu_i) \Rightarrow \mu_i = -\frac{1}{\theta_i} \Rightarrow b(\theta_i) = \log(-\frac{1}{\theta_i})$

$\log\mu_i$

$\dot{b}(\theta_i) = -\frac{1}{\theta_i} \quad \ddot{b}(\theta_i) = \frac{1}{\theta_i^2} \quad \dddot{b}(\theta_i) = \frac{-2}{\theta_i^3} \quad b^{(4)}(\theta_i) = \frac{6}{\theta_i^4}$

$\theta_i = -\theta_i$
gamma distrib. w/ $E[\theta_i] = \exp\{-x_i'\beta\}$

$w_{1i} = \ddot{b}(\cdot) = \frac{1}{\theta_i^2} = \frac{1}{(-\frac{1}{\mu_i})^2} = \mu_i^2 = \exp\{2x_i'\beta\}$

$w_{2i} = .5f_i\dddot{b}(\cdot) = \frac{1}{2}\exp\{x_i'\beta\}(-\frac{2}{\theta_i^3}) = \exp\{x_i'\beta\}(-\frac{1}{(-1/\mu_i)^3}) = \exp\{x_i'\beta\}\mu_i^3 = \exp\{4x_i'\beta\}$

$$D_\theta'(\beta) = \left(\frac{\partial\theta}{\partial\beta}\right)' = \begin{pmatrix} \frac{\partial\theta_1}{\partial\beta_1} & \cdots & \frac{\partial\theta_n}{\partial\beta_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial\theta_1}{\partial\beta_p} & \cdots & \frac{\partial\theta_n}{\partial\beta_p} \end{pmatrix} = \begin{pmatrix} e^{x_1'\beta}x_{11} & \cdots & e^{x_n'\beta}x_{n1} \\ \vdots & \ddots & \vdots \\ e^{x_1'\beta}x_{1p} & \cdots & e^{x_n'\beta}x_{np} \end{pmatrix}_{p\times n}$$

$-\theta_i \quad -\exp\{-x_i'\beta\}$
$\frac{1}{\theta_i} \Rightarrow \exp\{x_i'\beta\}$

$W_1 = diag(\exp\{2x_1'\beta\}, \ldots, \exp\{2x_n'\beta\}) \quad W_2 = diag(\exp\{4x_1'\beta\}, \ldots, \exp\{4x_n'\beta\})$

$I_{\beta\tau} = D_\theta(\beta)'W_2 1_n \quad I_{\beta\beta} = D_\theta(\beta)'W_1 D_\theta(\beta) \quad I_{\tau\tau} = \sum_{i=1}^n \frac{1}{4}f_i^2[2\ddot{b}(\theta_i)^2 + b^{(4)}(\theta_i)]$

$$\partial_\tau \ell_n(\alpha) = \sum_{i=1}^n \frac{1}{2}f_i[(y_i-\mu_i)^2 - \ddot{b}(\theta_i)] = \frac{1}{2}\sum_{i=1}^n \exp\{x_i'\beta\}(y_i^2 - 2y_i\exp\{x_i'\beta\} + \exp\{2x_i'\beta\} - \exp\{2x_i'\beta\})$$

$$= \frac{1}{2}\sum_{i=1}^n \exp\{x_i'\beta\}(y_i^2 - 2y_i\exp\{x_i'\beta\})$$

$$= \frac{1}{2}\sum_{i=1}^n y_i^2\exp\{x_i'\beta\} - 2y_i\exp\{2x_i'\beta\}$$

$$S_\tau = \frac{\partial_\tau \ell_n(\tilde{\alpha})}{\sigma_\tau^2} I[\partial_\tau \ell(\tilde{\alpha}) > 0] \xrightarrow{d} 0.5\chi_0^2 + 0.5\chi_1^2$$ where $\tilde{\alpha} = (\tilde{\beta}, \tilde{\tau})$ is the MLE under $\tau = 0 \Rightarrow \tilde{\alpha} = (\tilde{\beta}, 0)$

1.(d) Suppose μ_i is a fixed, unknown parameter and we incorperate overdispersion by taking $var(y_i) = \sigma^2(\nu_i + \mu_i)$ where ν_i is the variance function of the GLM in (1). Let $\mu_i = exp\{x_i'\beta\}$

(i) Derive the quasi-likelihood score equations for β & a moment estimator for σ^2

$\hat{\mu}_i = e^{x_i'\hat{\beta}}$

(ii) Taylor Expansion of score equation around true β, $\tilde{\beta}$

1. Suppose that $y_1, \dots, y_n$ are positive & independent RVs, where $p(y_i|\mu_i) = \frac{1}{\mu_i} \exp(y_i/\mu_i)$ $\quad \mu_i > 0$

where $E[y_i|\mu_i] = \mu_i$ $\quad i = 1, \dots, n$. Let $\theta_i = 1/\mu_i$

(a) Suppose $\theta_i \sim \text{Gamma}(a_i, b_i)$ where $a_i/b_i = \exp(-x_i'\beta)$ and $a_i = 3$

Further assume $var(\theta_i) = \tau \exp\{-x_i'\beta\}$ $\quad x_i$ is a $p \times 1$ vector of covariates

$\beta_{p\times 1}$ vector of regression coefficients, β unknown.

Derive the marginal mean and variance of y_i

$$E[y_i] = E_\mu[E_y[y_i|\mu_i]] = E_\mu[\mu_i] = E[1/\theta_i]$$

$$a_i = 3 \Rightarrow \frac{3}{b_i} = \exp\{-x_i'\beta\} = b_i = \frac{3}{\exp\{-x_i'\beta\}} = 3\exp\{x_i'\beta\}$$

$$E[1/\theta_i] = \int_0^\infty \frac{1}{\theta_i} \frac{1}{\Gamma(3)\, b_i^3} \theta_i^{3-1} \exp\Big\{\frac{\theta_i}{\underbrace{3\exp\{x_i'\beta\}}_{b_i}}\Big\} d\theta_i$$

$$= \frac{1}{\Gamma(3)\, b_i^3} \int_0^\infty \theta_i^{2-1} \exp\Big\{\frac{\theta_i}{\underbrace{3\exp\{x_i'\beta\}}_{b_i}}\Big\} d\theta_i$$

$$= \frac{1}{\Gamma(3)\, b_i^3} \Gamma(2)\, b_i^2 \underbrace{\int_0^\infty \frac{1}{\Gamma(2)\, b_i^2} \exp\Big\{\frac{\theta_i}{b_i}\Big\} d\theta_i}_{\text{Gamma}(2, b_i)}$$

$$= \frac{(2-1)!\, b_i^2}{(3-1)!\, b_i^3} = \frac{1!}{2!} \frac{1}{b_i} = \frac{1}{2b_i} = \frac{1}{6\exp\{x_i'\beta\}}$$

$$var(y_i) = E[var(y_i|\mu_i)] + var(E[y_i|\mu_i])$$
$$= E[\mu_i^2] + var(\mu_i)$$
$$= var(\mu_i) + E(\mu_i)^2 + var(\mu_i) = 2var(\mu_i) + E(\mu_i)^2$$
$$= 2E[\mu_i^2] - E[\mu_i]^2$$

$$var(\mu_i) = E[\mu_i^2] - E[\mu_i]^2$$

$$E[\mu_i^2] = E[1/\theta_i^2] = \int_0^\infty \frac{1}{\theta_i^2} \frac{1}{\Gamma(3)\, b_i^3} \theta_i^{3-1} \exp\Big\{\frac{\theta_i}{b_i}\Big\} d\theta_i$$

$$= \frac{1}{\Gamma(3) b_i^3} \Gamma(1)\, b_i \underbrace{\int_0^\infty \frac{1}{\Gamma(1)\, b_i} \theta_i^{1-1} \exp\Big\{\frac{\theta_i}{b_i}\Big\} d\theta_i}_{\text{Gamma}(1, b_i)}$$

$$= \frac{(1-1)!\, b_i}{(3-1)!\, b_i^3} = \frac{1}{2b_i^2}$$

$$var(y_i) = 2\Big(\frac{1}{2b_i^2}\Big) - \Big(\frac{1}{2b_i}\Big)^2 = \frac{1}{b_i^2} - \frac{1}{4b_i^2} = \frac{3}{4b_i^2} = \frac{3}{4(9\exp\{2x_i'\beta\})} = \frac{1}{12\exp\{2x_i'\beta\}}$$

1.(b) Under the same assumptions as part (a), derive the marginal distribution of y_i $\boxed{\text{2016 Theory 2}}$

$\theta_i \sim \text{Gamma}(3, b_i) \quad 1/\mu_i = \theta_i \Rightarrow \mu_i = 1/\theta_i \quad |J| = \frac{1}{\mu_i^2}$

$$f_\mu(\mu_i) = f_{\theta_i}(1/\mu_i)\frac{1}{\mu_i^2} = \frac{1}{\Gamma(3) b_i^3}(1/\mu_i)^{3-1}\exp\left\{\frac{-1/\mu_i}{b_i}\right\}\frac{1}{\mu_i^2}$$

$$= \frac{1}{\Gamma(3) b_i^3}(1/\mu_i)^{3+1}\exp\left\{\frac{-1/\mu_i}{b_i}\right\}$$

$$= \frac{1}{\Gamma(3) b_i^3}\mu_i^{-3-1}\exp\left\{-\frac{1}{\mu_i b_i}\right\}$$

$f_{Y|\mu}(y|\mu_i) = \frac{1}{\mu_i}\exp\{-y_i/\mu_i\} \quad \mu_i > 0 \Rightarrow f_{Y|\theta}(y_i|\theta_i) = \theta_i \exp\{-y_i\theta_i\} \quad \theta_i > 0$

$$f_{Y,\theta}(y,\theta) = f_{Y|\theta}(y_i,\theta_i) f_\theta(\theta_i) = \theta_i\exp\{-y_i\theta_i\}\frac{1}{\Gamma(3) b_i^3}\theta_i^{3-1}\exp\{\theta_i/b_i\}$$

$$= \frac{1}{\Gamma(3) b_i^3}\theta_i^{4-1}\exp\{\theta_i(1/b_i + y_i)\}$$

$$f_Y(y_i) = \int_0^\infty \frac{1}{\Gamma(3) b_i^3}\theta_i^{4-1}\exp\{\theta_i(1/b_i + y_i)\}\,d\theta_i$$

$$= \frac{1}{\Gamma(3) b_i^3}\Gamma(4)\left(\frac{1}{b_i}+y_i\right)^{-4}\underbrace{\int_0^\infty \frac{1}{\Gamma(4)(1/b_i+y_i)^{-4}}\theta_i^{4-1}\exp\{-\theta(1/b_i+y_i)\}\,d\theta}_{\text{Gamma}(4,\,(1/b_i+y_i)^{-1})}$$

$$= \frac{(4-1)!}{(3-1)!}\frac{1}{b_i^3\left(\frac{1}{b_i}+y_i\right)^4} = \cancel{\frac{3}{b_i^2 + b_i^3 y_i}}$$

$$f_{Y,\mu}(y_i,\mu_i) = \frac{1}{\mu_i}\exp\{-y_i/\mu_i\}\frac{1}{\Gamma(3) b_i^3}\left(\frac{1}{\mu_i}\right)^{3+1}\exp\left\{-\frac{1}{\mu_i b_i}\right\}$$

$$= \frac{1}{\Gamma(3) b_i^3}\left(\frac{1}{\mu_i}\right)^{4+1}\exp\left\{-\frac{1}{\mu_i}\left(y_i+\frac{1}{b_i}\right)\right\}$$

$$f_Y(y_i) = \int_0^\infty \frac{1}{\Gamma(3) b_i^3}\left(\frac{1}{\mu_i}\right)^{4+1}\exp\left\{-\frac{1}{\mu_i}(y_i+1/b_i)\right\}d\mu_i$$

$$\frac{1}{\Gamma(3) b_i^3}\Gamma(4)(y_i+1/b_i)^{-4}\underbrace{\int_0^\infty \frac{1}{\Gamma(4)(y_i+1/b_i)^{-4}}\left(\frac{1}{\mu_i}\right)^{4+1}\exp\left\{-\frac{1}{\mu_i}(y_i+1/b_i)\right\}d\mu_i}_{\text{IGamma}(4,\,(y_i+1/b_i)^{-1})}$$

1. $y_1, \dots, y_n > 0$, independent $\quad p(y_i|\mu_i) = \frac{1}{\mu_i} \exp\left\{\frac{-y_i}{\mu_i}\right\} \quad \mu_i > 0$

$E[y_i|\mu_i] = \mu_i \quad i = 1, \dots, n \quad$ Let $\theta_i = \frac{1}{\mu_i}$

Given $\theta_i \sim \text{Gamma}(a_i, b_i)$ where $a_i/b_i = \exp(-x_i'\beta)$ and $a_i = 3$

assume $Var(\theta_i) = \tau \exp(x_i'\beta)$

Derive the marginal mean & variance of y_i

$$E[y_i] = E[E[y_i|\mu_i]] = E[\mu_i] = E[1/\theta_i]$$

$$E[1/\theta_i] = \int_0^\infty \frac{1}{\theta_i} \frac{b_i^3}{\Gamma(3)} \theta_i^{3-1} \exp\{-b_i\theta_i\}$$

1.(c) under the same assumptions as in (a), derive the score test for testing $H_0: \tau = 0$ and give its asymptotic distribution under the null. | 2016 Theory 2

$H_0: \tau = 0$ vs $H_0: \tau > 0$ on boundary, since var > 0.

Score will follow $.5\chi^2_1 + .5(0)$

We know $var(\theta_i) = \tau\, exp(x_i'\beta)$

general form of variance: $var(\theta_i) = f(x_i'\beta) \Rightarrow f(\cdot) = exp(\cdot)$

and $E[\theta_i] = \frac{a_i}{b_i} = exp\{-x_i'\beta\} \Rightarrow E[\mu_i] = exp\{x_i'\beta\}$

① Write $y_i|\mu_i$ in exponential family form

$p(y_i|\mu_i) = \theta_i\, exp(-y_i\theta_i) = exp\{-y_i\theta_i + \log\theta_i\}$ $\quad exp\{\phi(\theta y - b(\theta) - c(y)) - \frac{1}{2}S(y,\phi)\}$

let $\phi = 1$ $\quad \theta = -\theta_i$ $\quad y = -y_i$ $\quad b(\theta_i) = -\log\theta_i \Rightarrow b(\theta) = -\log(-\theta)$

$\dot{b}(\theta) = -\frac{1}{\theta}$ $\quad \ddot{b}(\theta) = \frac{1}{\theta^2}$ $\quad b^{(3)}(\theta) = -\frac{2}{\theta^3}$ $\quad b^{(4)}(\theta) = \frac{6}{\theta^4}$

② General form for score test: def on 903

$$S_\tau = \frac{\partial_\alpha \ln(\alpha)^2}{\sigma_\tau^2} I\{\partial_\tau \ln(\alpha) > 0\}\Big|_{\tilde{\alpha}} \xrightarrow{d} 0.5\chi^2_0 + 0.5\chi^2_1$$

so we need $\ln(\alpha)$ & the derivatives

$$\partial_\tau \ln(\alpha) = \sum_{i=1}^n \frac{1}{2} f_i\{(y_i - \mu_i)^2 - \ddot{b}(\theta_i)\} = \sum_{i=1}^n \frac{1}{2} exp(x_i'\beta)\{(y_i-\mu_i)^2 - \mu_i^2\}$$

$$= \sum_{i=1}^n \frac{1}{2} exp(x_i'\beta)\{y_i^2 - 2\mu_i y_i + \cancel{\mu_i^2} - \cancel{\mu_i^2}\} = \sum_{i=1}^n \frac{1}{2} exp(x_i'\beta)\{y_i^2 - 2\,exp(x_i'\beta) y_i\}$$

$$= \frac{1}{2}\sum_{i=1}^n exp(y_i^2 - 2exp(x_i'\beta)y_i)$$

$$\partial_\tau \ln(\alpha)^2 = \frac{1}{4}\left[\sum_{i=1}^n exp(y_i^2 - 2exp\{x_i'\beta\}y_i)\right]^2$$

1.(c) $\sigma_\tau^2 = I_{\tau\tau} - I_{\tau\beta} I_{\beta\beta}^{-1} I_{\beta\tau}$

canonical link θ

$W_{1i} = \ddot{b}(\cdot)$ $\quad w_{2i} = \frac{1}{2} f_i \dddot{b}(\cdot)$ $\quad I_{\beta\tau} = D_\theta(\beta)^T W_2 1_n$

$I_{\tau\tau} = \frac{1}{4} \sum f_i^2 \{2(\dddot{b}(\cdot))^2 + b^{(4)}(\cdot)\}$ $\quad I_{\beta\beta} = D_\theta(\beta)^T W_1 D_\theta(\beta)$

$W_1 = diag(w_{11}, \ldots, w_{1n})$ $\quad W_2 = diag(w_{21}, \ldots, w_{2n})$ $\quad D_\theta(\beta) = \frac{\partial \theta}{\partial \beta}$

$w_{1i} = \ddot{b}(\cdot)$ $\quad w_{2i} = \frac{1}{2} f_i \dddot{b}(\cdot)$

canonical link θ, not our θ_i

$D_\theta(\beta) = diag(\exp(-x_i'\beta)$

Conclude Thus reject H_0 if $S_\tau > .5\chi_0^2(1-\alpha) + .5\chi_1^2(1-\alpha)$

1(d). Now suppose we take μ_i to be a fixed & unknown parameter & we incorporate over-dispersion by taking $Var(y_i) = \sigma^2(\nu_i + \mu_i)$ where ν_i is the variance function of the GLM in (1).

Let $\mu_i = exp\{x_i'\beta\}$

(i) Derive the quasilikelihood score equations for β & a moment estimator for σ^2.

Quasilikelihood : $\sum_{i=1}^{n} \frac{\partial \mu_i}{\partial \beta} \frac{(y_i - \mu_i)}{Var(y_i)} \overset{set}{=} 0$ $\qquad \nu_i = var_{(1)}(y_i) = \mu_i^2$

$\frac{\partial \mu_i}{\partial \beta} = \frac{\partial}{\partial \beta} exp\{x_i'\beta\} = x_i\, exp\{x_i'\beta\}$

$\therefore$ Quasilikelihood : $\sum_{i=1}^{n} \frac{x_i\, exp\{x_i'\beta\}(y_i - \mu_i)}{\sigma^2(\mu_i^2 + \mu_i)}$ $\qquad$ where $\mu_i = exp\{x_i'\beta\}$

$\mu_i^2 = exp\{2x_i'\beta\}$

$$= \sum_{i=1}^{n} \frac{x_i\, exp\{x_i'\beta\}(y_i - exp\{x_i'\beta\})}{\sigma^2(exp\{2x_i'\beta\} + exp\{2x_i\beta\})}$$

Moment estimator :

$$E\left[\sum_{i=1}^{n} \frac{(y_i - \mu_i)^2}{Var(y_i)}\right] \approx n\sigma^2 \quad ?$$

2. (25 points) Suppose that Y is a 4×1 vector with $E(Y) = \mu$, $\mu \in E$, where E is the set $E = \{u : u' = (\beta_1 + \beta_2 - \beta_3, \beta_2 + \beta_3, -\beta_2 - \beta_3, -\beta_1 - \beta_2 + \beta_3)\}$, where the β_i are real numbers, $i = 1, 2, 3$ and a $'$ denotes matrix (vector) transposition. Further assume that $\mathrm{Cov}(Y) = \sigma^2 I_{4\times4}$, where σ^2 is unknown.

(a) (5 points) Derive $\hat{\mu}$, the ordinary least squares estimate of μ, by carrying out the appropriate projection. X^* method

(b) (4 points) Find the BLUE of $\beta_2 - \beta_3$ or show that it is nonestimable.

(c) (4 points) Consider testing $H_0 : \beta_2 + \beta_3 = 0$ versus $H_1 : \beta_2 + \beta_3 \neq 0$. Let E_0 denote the set E assuming that H_0 is true. Explicitly give the sets E_0 and $E \cap E_0^{\perp}$, where $E_0^{\perp}$ denotes the orthogonal complement of E_0.

(d) (6 points) Assuming normality for Y, construct the simplest possible expression for the F statistic for the hypothesis $H_0 : \mu \in E_0$ versus $H_1 : \mu \notin E_0$, where E_0 is specified in part (c), and give the distribution of the F statistic under the null and alternative hypotheses.

(e) (6 points) Assuming normality for Y, construct an exact 95% confidence interval for $\beta_2 + \beta_3$.

2. Suppose $Y_{4\times 1}$ vector with $E[Y]=\mu$ $\mu \in E$ where

$E = \{\mu : \mu' = (\beta_1+\beta_2-\beta_3,\ \beta_2+\beta_3,\ -\beta_2-\beta_3,\ -\beta_1-\beta_2+\beta_3)\}$ where the $\beta_i \in \mathbb{R}$ $i=1,2,3$

Assume $Cov(Y) = \sigma^2 I_{4\times4}$ where σ^2 is unknown

(a) Derive $\hat{\mu}$, the OLS estimate of μ, by carrying out the appropriate projection

$$E(Y) = \underset{4\times3}{X}\underset{3\times1}{\beta} = \begin{bmatrix} \beta_1+\beta_2-\beta_3 \\ \beta_2+\beta_3 \\ -\beta_2-\beta_3 \\ -\beta_1-\beta_2+\beta_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

$$\mu = \Lambda'\beta = X\beta = IX\beta \Rightarrow P' = I = P$$

$$\Lambda'\beta = X\beta \Rightarrow \Lambda' = X \Rightarrow \Lambda' \in C(X) \Rightarrow \Lambda \in C(X')$$
$$\Rightarrow \Lambda'\hat{\beta} \text{ is estimable}$$

LSE of $\Lambda'\beta = P'MY = MY$

$M = X(X'X)^{-1}X' = X^*(X^{*\prime}X^*)^{-1}X^{*\prime}$ where X^* is the lineary indpt columns of X.

$$X_3 = X_2 - 2X_1 \Rightarrow X^* = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \\ -1 & -1 \end{bmatrix}$$

$$X^{*\prime}X^* = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix} \quad (X^{*\prime}X^*)^{-1} = \frac{1}{8-4}\begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$M = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 1 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\Lambda'\beta = MY = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} Y_1 - Y_4 \\ Y_2 - Y_3 \\ -Y_2 + Y_3 \\ -Y_1 + Y_4 \end{bmatrix}$$

2(b). Find the BLUE of $\beta_2-\beta_3$ or show that it is not estimable. 2016 Theory 2

Estimable if $\lambda'\beta = \rho' X\beta$ $\quad \lambda = X'\rho$

$$X' = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix} \quad \text{Span}(X') = \left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$$

$$\lambda'\beta = \beta_2 - \beta_3 \Rightarrow \lambda' = (0 \ 1 \ -1)$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{array}\right]$$

$\therefore$ we have no solution

Thus $\lambda \notin C(X')$ & $\beta_2-\beta_3$ is <u>NOT</u> estimable.

(c) consider testing $H_0: \beta_2+\beta_3 = 0$ vs $H_1: \beta_2+\beta_3 \neq 0$

Let E_0 denote the set E assuming H_0 is true

Explicitly give the sets E_0 and $E \cap E_0^\perp$, where $E_0^\perp$ is the orthogonal complement to E_0.

$$E = \left\{ u : u = \begin{bmatrix} \beta_1+\beta_2-\beta_3 \\ \beta_2+\beta_3 \\ -\beta_2-\beta_3 \\ -\beta_1-\beta_2+\beta_3 \end{bmatrix} \right\} \quad \text{under } H_0: E_0 = \left\{ u : u = \begin{bmatrix} \beta_1+(\beta_2+\beta_3)-2\beta_3 \\ \beta_2+\beta_3 \\ -(\beta_2+\beta_3) \\ -\beta_1-(\beta_2+\beta_3)+2\beta_3 \end{bmatrix}, \beta_2+\beta_3=0 \right\}$$

$$E_0 = \left\{ u : u = \begin{bmatrix} \beta_1-2\beta_3 \\ 0 \\ 0 \\ -\beta_1+2\beta_3 \end{bmatrix} \right\} = \left\{ u : u = (\beta_1-2\beta_3)\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\} \Rightarrow \text{Span}(E_0) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right)$$

$$E = \left\{ u : u = (\beta_1+\beta_2-\beta_3)\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + (\beta_1+\beta_2)\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

$$[1 \ 0 \ 0 \ -1]\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} = 0 \Rightarrow E_0^\perp \cap E = \left\{ u : (\beta_1+\beta_2)\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

2(d). Assuming normality for Y, construct the simplest possible expression for the F statistic for the hypothesis

$H_0: \mu \in E_0$ vs. $H_1: \mu \notin E_0$ & give the distribution under the null & alternative hypotheses

$$F = \frac{Y'(M-M_0)Y \,/\, r(M-M_0)}{Y'(I-M)Y \,/\, r(I-M)}$$

$M_0 = X_0(X_0'X_0)^{-1}X_0'$ where $X_0 = M - M_{MP}$, we know $M-M_0$ is the OPO onto $E \cap E_0^{\perp}$ $\Rightarrow$

$$M - M_0 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \left(\begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad r(M-M_0) = 1$$

$$Y'(M-M_0)Y = \begin{bmatrix} Y_1 & Y_2 & Y_3 & Y_4 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} = \begin{bmatrix} 0, & Y_2 - Y_3, & -Y_2 + Y_3, & 0 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} \frac{1}{2}$$

$$= \frac{1}{2}\left(Y_2(Y_2 - Y_3) + Y_3(-Y_2 + Y_3) \right) = \frac{1}{2}(Y_2^2 - Y_2Y_3 - Y_2Y_3 + Y_3^2)$$

$$= \frac{1}{2}(Y_2 - Y_3)^2$$

$$I - M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix} \qquad r(I-M) = 2$$

$$Y'(I-M)Y = \frac{1}{2} \begin{bmatrix} Y_1 & Y_2 & Y_3 & Y_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} Y_1 + Y_4, & Y_2 + Y_3, & Y_2 + Y_3, & Y_1 + Y_4 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix}$$

$$= \frac{1}{2}\left[Y_1(Y_1 + Y_4) + Y_2(Y_2 + Y_3) + Y_3(Y_2 + Y_3) + Y_4(Y_1 + Y_4) \right]$$

$$= \frac{1}{2}\left[Y_1^2 + Y_1Y_4 + Y_2^2 + Y_2Y_3 + Y_2Y_3 + Y_3^2 + Y_1Y_4 + Y_4^2 \right]$$

$$= \frac{1}{2}\left[Y_1^2 + 2Y_1Y_4 + Y_4^2 + Y_2^2 + 2Y_2Y_3 + Y_3^2 \right] = \frac{1}{2}\left[(Y_1 + Y_4)^2 + (Y_2 + Y_3)^2 \right]$$

2(d) con't

$$F = \frac{Y'(M-M_0)Y / r(M-M_0)}{Y'(I-M)Y / r(I-M)} = \frac{\frac{1}{2}(Y_2-Y_3)^2 / 1}{\frac{1}{2}[(Y_1+Y_4)^2 + (Y_2+Y_3)^2] / 2}$$

$$= \frac{2(Y_2-Y_3)^2}{(Y_1+Y_4)^2+(Y_2+Y_3)^2}$$

under H_0: $F \sim F(1,2)$

under H_0: $F \sim F(1,2,\delta)$ where δ is the non-centrality parameter

$$\delta = \frac{E[Y]'(M-M_0)E[Y]}{2\sigma^2} = \frac{\beta'X'(M-M_0)X\beta}{2\sigma^2}$$

2(e) Assuming normality for Y, construct an exact 95% CI for $\beta_2+\beta_3$

$$\lambda'\beta = [0 \ 1 \ 1]\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix} \Rightarrow \lambda = \begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$\lambda' = \rho' X = [\rho_1 \ \rho_2 \ \rho_3 \ \rho_4]\begin{bmatrix}1 & 1 & -1\\ 0 & 1 & 1\\ 0 & -1 & -1\\ -1 & -1 & 1\end{bmatrix} = (\rho_1-\rho_4,\ \rho_1+\rho_2-\rho_3-\rho_4,\ -\rho_1+\rho_2-\rho_3+\rho_4)$$

$\Rightarrow \rho_1-\rho_4 = 0$ (A)

$\rho_1+\rho_2-\rho_3-\rho_4 = 1$ (B)

$-\rho_1+\rho_2-\rho_3+\rho_4 = 1$ (C)

$A \Rightarrow (B) \Rightarrow \rho_2-\rho_3 = 1$

$A \Rightarrow (C)$

$\therefore$ a possible $\rho = \begin{pmatrix}1\\1\\0\\1\end{pmatrix}$ s.t. $\lambda'\hat{\beta} = \rho' MY$

$$\rho' MY = (1 \ 1 \ 0 \ 1)\frac{1}{2}\begin{pmatrix}Y_1-Y_4\\ Y_2-Y_3\\ -Y_2+Y_3\\ -Y_1+Y_4\end{pmatrix} = \frac{1}{2}((Y_1-Y_4)+(Y_2-Y_3)+(-Y_1+Y_4))$$

$$= \frac{1}{2}(\cancel{Y_1}-\cancel{Y_4}+Y_2-Y_3-\cancel{Y_1}+\cancel{Y_4}) = \frac{1}{2}(Y_2-Y_3)$$

$$F = \frac{Y'MP(P'MP)^{-1}P'MY / r(\Lambda)}{Y'(I-M)Y / r(I-M)} = \frac{(P'MY)'(P'MP)^{-1}P'MY / r(\Lambda)}{Y'(I-M)Y / r(I-M)}$$

$$p'Mp = (1\ 1\ 0\ 1)\ \tfrac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = (0\ 1\ -1\ 0)\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}\tfrac{1}{2} = \tfrac{1}{2}(2) = 1$$

$$\Rightarrow (p'Mp)^{-1} = 1$$

$$F = \frac{\frac{1}{2}(Y_2 - Y_3)\frac{1}{2}(Y_2 - Y_3)\ /1}{(Y_1+Y_4)^2 + (Y_2+Y_3)^2/2} = \cancel{\frac{1}{4}\frac{(Y_2-Y_3)^2}{(Y_1+Y_4)^2+(Y_2+Y_3)^2}}$$

$\therefore$ a $(1-\alpha)100\%$ CI for $\beta_2 + \beta_3$

$$\left\{ (\beta_2, \beta_3) : \frac{\left(\frac{1}{2}(Y_2 - Y_3) - \beta_2 + \beta_3\right)^2}{\frac{1}{2}\left[(Y_1+Y_4)^2 + (Y_2+Y_3)^2\right]} \le F_{(1,2,1-\alpha)} \right\}$$

3. (25 points) Consider n independent observations $(\mathbf{y}_i, \mathbf{x}_i)$ satisfying a Multivariate Linear Model (MLM) given by

$$\mathbf{y}_i = \mathbf{B}^T\mathbf{x}_i + \mathbf{e}_i, \tag{2}$$

where $\mathbf{y}_i$ is a $q \times 1$ response vector, $\mathbf{x}_i$ is a $p \times 1$ vector of covariates, and $\mathbf{B} = (\beta_{jl})$ is a $p \times q$ coefficient matrix with $\text{rank}(\mathbf{B}) = r^\star \leq \min(p, q)$. Moreover, the error term $\mathbf{e}_i \sim N(\mathbf{0}, \Sigma_R)$ for all i, where Σ_R is a $q \times q$ positive definite matrix, and the $\mathbf{x}_i$ are independently and identically distributed (i.i.d) with $E(\mathbf{x}_i) = \mu_x$ and $\text{Cov}(\mathbf{x}_i) = \Sigma_X$. Our problem of interest is to perform hypothesis testing on $\mathbf{B}$ as follows:

$$H_0 : \mathbf{CB} = \mathbf{B}_0 \quad \text{v.s.} \quad H_1 : \mathbf{CB} \neq \mathbf{B}_0, \tag{3}$$

where $\mathbf{C}$ is an $r \times p$ matrix and $\mathbf{B}_0$ is an $r \times q$ matrix. For simplicity, Σ_R is assumed to be known.

(a) (3 points) Consider a Projection Regression Modeling (PRM) given by

$$\mathbf{w}^T\mathbf{y}_i = (\mathbf{Bw})^T\mathbf{x}_i + \mathbf{w}^T\mathbf{e}_i = \boldsymbol{\beta}_{\mathbf{w}}^T\mathbf{x}_i + \varepsilon_i, \tag{4}$$

where $\mathbf{w}$ is a $q \times 1$ direction vector such that $\mathbf{w}^T\mathbf{w} = 1$. For a fixed vector $\mathbf{w}$, that is independent of data, please derive the maximum likelihood estimate of $\boldsymbol{\beta}_{\mathbf{w}}$, denoted as $\widehat{\boldsymbol{\beta}}_{\mathbf{w}}$ and its distribution.

(b) (3 points) Consider the following hypotheses:

$$H_{0W} : \mathbf{C}\boldsymbol{\beta}_{\mathbf{w}} = \mathbf{b}_0 \quad \text{v.s.} \quad H_{1W} : \mathbf{C}\boldsymbol{\beta}_{\mathbf{w}} \neq \mathbf{b}_0, \tag{5}$$

where $\mathbf{C}\boldsymbol{\beta}_{\mathbf{w}} = \mathbf{CBw}$ and $\mathbf{b}_0 = \mathbf{B}_0\mathbf{w}$. We define four spaces associated with the null and alternative hypotheses of (3) and (5) as follows:

$$S_{H_0} = \{\mathbf{B} : \mathbf{CB} = \mathbf{B}_0\}, \qquad S_{H_{0W}} = \{\mathbf{B} : \mathbf{C}\boldsymbol{\beta}_{\mathbf{w}} = \mathbf{b}_0\},$$
$$S_{H_1} = \{\mathbf{B} : \mathbf{CB} \neq \mathbf{B}_0\}, \qquad S_{H_{1W}} = \{\mathbf{B} : \mathbf{C}\boldsymbol{\beta}_{\mathbf{w}} \neq \mathbf{b}_0\}.$$

Show $S_{H_0} \subset S_{H_{0W}}$ and $S_{H_{1W}} \subset S_{H_1}$ for any $\mathbf{w}$ with unit norm.

(c) (5 points) For a given $\mathbf{w}$, derive the Wald test statistic $T_n(\mathbf{w})$, its null distribution, and its mean and variance under H_{1W} conditional on $\mathbf{x}_i$s, based on model (5). Hint: for $\mathbf{u} \sim N(\mu, \Sigma_0)$, the mean and variance of $\mathbf{u}^T\Lambda\mathbf{u}$ are, respectively, given by $\text{tr}[\Lambda\Sigma_0] + \mu^T\Lambda\mu$ and $2\text{tr}[\Lambda\Sigma_0\Lambda\Sigma_0] + 4\mu^T\Lambda\Sigma_0\Lambda\mu$, where Λ is a symmetric matrix.

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(d) (5 points) Show that conditional on $\mathbf{x}_i$s,

$$\mathrm{SNR}(\mathbf{w}) = \{E_{H_1}[T_n(\mathbf{w})] - E_{H_0}[T_n(\mathbf{w})]\}/\sqrt{\mathrm{Var}_{H_0}[T_n(\mathbf{w})]}$$

is an increasing function of $\mathrm{HR}(\mathbf{w}) = \mathbf{w}^T\widehat{\Sigma}_C\mathbf{w}/\mathbf{w}^T\Sigma_R\mathbf{w}$. Please derive the explicit form of $\widehat{\Sigma}_C$ and its limit.

(e) (5 points) For $r = 1$, derive $\widehat{\mathbf{w}} = \mathrm{argmax}_{\mathbf{w},\mathbf{w}^T\mathbf{w}\ \ 1}\mathrm{HR}(\mathbf{w})$ and its limit.

(f) (4 points) Calculate $T_n(\widehat{\mathbf{w}})$ and simplify its expression as much as possible.