2017 Section 1

1. (25 points) Let N be Poisson distributed with parameter $0 < \lambda < \infty$, and let $X_1, X_2, \ldots$ be an i.i.d. sequence of positive random variables, independent of N, with $E\log(X_1) = \mu$, $\text{var}\,[\log(X_1)] = \sigma^2$, $|\mu| < \infty$, $0 < \sigma^2 < \infty$, and $M(\delta) = EX_1^\delta < \infty$ for some $\delta > 0$. Let $Y = \prod_{i=1}^N X_i$, where $\prod_{i=1}^0$ is defined as 1. Do the following:

   (a) (4 points) Show that $E\log Y = \lambda\mu$ and $\text{var}\,[\log Y] = \lambda(\sigma^2 + \mu^2)$. log(Y) is not poisson E ≠ Var

   (b) (5 points) Show that $EY^t = \exp(\lambda[M(t) - 1])$, for all $0 \le t \le \delta$.

   (c) (7 points) Show that $Y^{1/\lambda} \to_p e^\mu$, as $\lambda \to \infty$.

   (d) (9 points) Letting $\tau^2 = \lambda(\sigma^2 + \mu^2)$, show that

$$\left(e^{-\lambda\mu} Y\right)^{1/\tau} \to_d e^Z,$$

   as $\lambda \to \infty$, where $Z \sim N(0, 1)$.

1. Let N be Poisson distribution w/ parameter $0<\lambda<\infty$ and let $X_1, X_2, \ldots$ be an iid sequence of positive random variables, independent of N, with $E[\log(X_1)] = \mu$, $var(\log(X_1)) = \sigma^2$, $|\mu|<\infty, 0<\sigma^2<\infty$ and $M(\delta) = EX_1^\delta < \infty$ for some $\delta > 0$. Let $Y = \prod_{i=1}^N X_i$ where $\prod_{i=1}^0$ is defined as 1.

(a) Show that $E[\log(Y)] = \lambda\mu$ and $var(\log(Y)) = \lambda(\sigma^2+\mu^2)$

$$\log(Y) = \log\left(\prod_{i=1}^N X_i\right) = \sum_{i=1}^N \log(X_i)$$

$$E[\log(Y)] = E\left[\sum_{i=1}^N \log X_i\right] = E_N\left[E_X\left[\sum_{i=1}^N \log X_i \mid N\right]\right] = E_N\left[E_X[N\log(X_i)\mid N]\right]$$

$$= E_N\left[N\, E_X[\log(X_i)\mid N]\right] = E_N[N\mu] = \mu E_N[N] = \lambda\mu$$

$X_i \perp N \Rightarrow E[\log(X_i)\mid N] = E[\log(X_i)]$

$N \sim Poisson(\lambda) \Rightarrow E[N] = \lambda$

$$var(\log(Y)) = var\left(\sum_{i=1}^N \log X_i\right) = E\left[var\left(\sum_{i=1}^N \log X_i \mid N\right)\right] + var\left(E[\log(X_i)\mid N]\right)$$

For iid RV, $Var(\Sigma X_i) = \Sigma\, Var(X_i)$

$$= E\left[\sum_{i=1}^N var(\log X_i \mid N)\right] + var(N\mu) = E[N\, var(\log X_i \mid N)] + var(N\mu)$$

$$= E[N\sigma^2] + var(N\mu) = \sigma^2 E[N] + \mu^2 Var(N)$$

$$= \sigma^2\lambda + \mu^2\lambda$$

$N \sim Poisson(\lambda) \Rightarrow Var(N) = \lambda$

(b) Show that $E[Y^t] = \exp(\lambda[M(t)-1])$ $\forall\ 0 \le t \le \delta$

$$Y^t = \left(\prod_{i=1}^N X_i\right)^t = \prod_{i=1}^N X_i^t$$

$$E[Y^t] = E\left[\prod_{i=1}^N X_i^t\right] = E_N\left[E_X\left[\prod_{i=1}^N X_i^t \mid N\right]\right] = E_N\left[E\left[(X_i^t)^N \mid N\right]\right]$$

MGF definition

$$= E_N\left[M(t)^N\right] = E\left[\exp(N\log(M(t)))\right] = E\left[e^{N\log M(t)}\right] = E\left[e^{NP}\right] \quad P = \log M(t)$$

$$= \exp(\lambda(e^P - 1)) = \exp(\lambda(e^{\log M(t)} - 1))$$

$$= \exp(\lambda(M(t) - 1))$$

1.(c) Show that $Y^{1/\lambda} \xrightarrow[P]{} e^{\mu}$ as $\lambda \to \infty$

WTS $\lim_{\lambda \to \infty} P(|Y^{1/\lambda} - e^{\mu}| > \varepsilon) = 0$

$Y^{1/\lambda} = \left(\prod_{i=1}^{N} X_i\right)^{1/\lambda} = \prod_{i=1}^{N} X_i^{1/\lambda}$ $\quad$ $Y^{1/\lambda} = exp\{\frac{1}{\lambda}\log(Y)\} = exp\{\frac{1}{\lambda}\sum_{i=1}^{N}\log(X_i)\}$

$\therefore$ WTS $\lim_{\lambda \to \infty} P(|\exp\{\frac{1}{\lambda}\sum_{i=1}^{N}\log(X_i)\} - e^{\mu}| > \varepsilon) = 0$

$= \lim_{\lambda \to \infty} P(|\frac{1}{\lambda}\sum_{i=1}^{N}\log(X_i) - \mu| > \varepsilon) = 0$

From (a) we know $E[\log(Y)] = E[\sum_{i=1}^{N}\log X_i] = \lambda\mu$, $var(\log Y) = \lambda(\sigma^2 + \mu^2)$

$\therefore$ by Chebychev's Inequality

$$P(|\sum_{i=1}^{N}\log X_i - \lambda\mu| > \varepsilon) \le \frac{\lambda(\sigma^2+\mu^2)}{\varepsilon} \to \infty \text{ as } \lambda \to \infty$$

but, $E[\frac{1}{\lambda}\log(Y)] = \frac{1}{\lambda}E[\log Y] = \frac{\lambda\mu}{\lambda} = \mu$

$var(\frac{1}{\lambda}\log(Y)) = \frac{1}{\lambda^2}var(\log(Y)) = \frac{\lambda(\sigma^2+\mu^2)}{\lambda^2} = \frac{\sigma^2+\mu^2}{\lambda}$

$\therefore$ Again by Chebychev's

$$P(|\frac{1}{\lambda}\sum_{i=1}^{N}\log(X_i) - \mu| > \varepsilon) \le \frac{\frac{\sigma^2+\mu^2}{\lambda}}{\varepsilon}$$

$$P(|\frac{1}{\lambda}\sum_{i=1}^{N}\log(X_i) - \mu| > \varepsilon) \le \frac{\sigma^2+\mu^2}{\lambda\varepsilon} \to 0 \text{ as } \lambda \to \infty$$

$\Rightarrow \frac{1}{\lambda}\sum_{i=1}^{N}\log(X_i) \xrightarrow[P]{} \mu$

by CMT $e^{\frac{1}{\lambda}\sum\log(X_i)} \xrightarrow[P]{} e^{\mu} \equiv Y^{1/\lambda} \xrightarrow[P]{} e^{\mu}$

1.(d) Let $\tau^2 = \lambda(\sigma^2+\mu^2)$. Show that

$$(e^{-\lambda\mu} Y)^{1/\tau} \xrightarrow[d]{} e^{Z} \quad \text{as } \lambda \to \infty, \text{ where } Z \sim N(0,1)$$

$$\equiv \frac{1}{\tau}\log(e^{-\lambda\mu} Y) \xrightarrow[d]{} N(0,1)$$

$$\frac{1}{\tau}\log(e^{-\lambda\mu}Y) = \frac{1}{\tau}(\log e^{-\lambda\mu} + \log Y) = \underbrace{\frac{1}{\tau}}_{\sim \frac{1}{sd(\log(Y))}}(\log(Y) - \underbrace{\lambda\mu}_{\sim E(\log Y)})$$

let $\log(Y) = V$ $\therefore \frac{1}{\sqrt{var(V)}}(V - E[V]) \to N(0,1)$

$$\text{let } \frac{1}{\tau}\log(e^{-\lambda\mu}Y) = U = \log(e^{-\frac{\lambda\mu}{\tau}} Y^{1/\tau}) = -\frac{\lambda\mu}{\tau} + \log(Y^{1/\tau})$$

$$\Rightarrow tU = \frac{t}{\tau}\log(e^{-\lambda\mu}Y) = -\frac{\lambda t\mu}{\tau} + \log(Y^{t/\tau})$$

$$E[e^{tU}] = E\left[\exp\left(-\frac{t\lambda\mu}{\tau} + \log(Y^{t/\tau})\right)\right]$$

$$= \exp\left(-\frac{t\lambda\mu}{\tau} + E[\log(Y^{t/\tau})]\right) = \exp\left(-\frac{t\lambda\mu}{\tau} + (\lambda[M(t/\tau)-1]\right)$$

2nd order taylor expansion around $t/\tau = 0$.

$$= \exp\left(-\frac{t\lambda\mu}{\tau} + \lambda\left(M(0) - 1 + \dot{M}(0)\frac{t}{\tau} + \ddot{M}(0)\frac{t^2}{2\tau^2} + o_p(t^2/\tau^2)\right)\right)$$

$$= \exp\left(-\frac{t\lambda\mu}{\tau} + \lambda\left(1 - 1 + \frac{\mu t}{\tau} + \frac{(\sigma^2+\mu^2)t^2}{2\tau^2} + o_p(t^2/\tau^2)\right)\right)$$

$$= \exp\left(-\frac{\lambda\mu}{\cancel{\tau}}t + \frac{\lambda\mu}{\cancel{\tau}}t + \frac{\cancel{(\sigma^2+\mu^2)}t^2}{2\cancel{(\sigma^2+\mu^2)}} + o_p(t^2/\tau^2)\right)$$

$$= \exp(t^2/2) \quad \text{as } \lambda \to \infty$$

$M(0) = 1$

$\dot{M}(0) = E[X] = \mu$

$\ddot{M}(0) = E[X^2] =$

$var(X) + E[X]^2 = \sigma^2 + \mu^2$

By uniqueness of MGF / CFs, we know

$$U = \frac{1}{\tau}\log(e^{-\lambda\mu}Y) \xrightarrow[d]{} N(0,1) \equiv Z$$

$$\Rightarrow \text{by CMT } e^U = (e^{-\lambda\mu}Y)^{1/\tau} \xrightarrow[d]{} e^Z$$

$M(t) = E[X_1^t] = E[$

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we did this for 761 review

2. (25 points) Let F and G be two distinct known cumulative distribution functions on the real line and X be a single observation from the cumulative distribution function $\theta F(x) + (1-\theta)G(x)$, where $\theta \in [0, 1]$ is unknown.

   (a) (4 points) Given $0 < \theta_0 < 1$, derive a Uniformly Most Powerful (UMP) test of size α for testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$. You need to specify how the rejection region can be calculated .

   (b) (6 points) Given $0 < \theta_1 < \theta_2 < 1$, derive a UMP test of size α for testing $H_0 : \theta \in [0, \theta_1] \cup [\theta_2, 1]$ versus $H_1 : \theta \in (\theta_1, \theta_2)$.

come back to see if solution is correct

   (c) (6 points) Show that a UMP test does not exist for testing $H_0 : \theta \in [\theta_1, \theta_2]$ versus $\theta \notin [\theta_1, \theta_2]$.

   (d) (5 points) Obtain a Uniformly Most Powerful Unbiased (UMPU) test of size α for the problem in part (c).

   (e) (4 points) Given $0 < \theta_1 < \theta_2 < 1$, derive the likelihood ratio test statistic for testing $H_0 : \theta \in [\theta_1, \theta_2]$ versus $\theta \notin [\theta_1, \theta_2]$.

2\. Let F and G be two distinct CDFs on the real line and X be a <u>single observation</u> from the CDF $\theta F(x) + (1-\theta) G(x)$ where $\theta \in [0,1]$ is unknown.

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(a) Given $0<\theta_0<1$, derive a UMP test of size α for testing $H_0: \theta \le \theta_0$ vs. $H_1: \theta > \theta_0$. You need to specify how the rejection region can be calculated.

let $f(x)$ and $g(x)$ be R-N (Radon-Nikodym) derivatives of $F(x), G(x)$ wrt $F(x)+G(x)$
Then, the density of X is:

$$\theta f(x) + (1-\theta) g(x).$$

Let $0<\theta_1<\theta_2<1 \Rightarrow \dfrac{\theta_2 f(x) + (1-\theta_2) g(x)}{\theta_1 f(x) + (1-\theta_1) g(x)} = \dfrac{\theta_2 \frac{f(x)}{g(x)} + (1-\theta_2)}{\theta_1 \frac{f(x)}{g(x)} + (1-\theta_1)}$

$\therefore$ X is MLR in $\dfrac{f(x)}{g(x)}$.

By an extension of the Neymann-Pearson lemma, a UMP test is given by

$$\phi_1(x) = \begin{cases} 1 & \text{if } \frac{f(x)}{g(x)} > c \\ \gamma & \text{if } f(x)/g(x) = c \\ 0 & \text{if } f(x)/g(x) < c \end{cases}$$

where c & γ are determined by $E_{\theta_0}[\phi(x)] = \alpha = P_0\left(\frac{f(x)}{g(x)} > c\right) + \gamma P_0\left(\frac{f(x)}{g(x)} = c\right)$

If you use $f(x)-g(x)$ you still have $+g(x)$ in both the numerator & denominator. What if $g(x) \to \infty$?

$$\frac{\theta_2 f(x) + (1-\theta_2) g(x)}{\theta_1 f(x) + (1-\theta_1) g(x)} > 1 \ ?$$

$$\frac{\theta_2 \frac{f(x)}{g(x)} + 1 - \theta_2}{\theta_1 \frac{f(x)}{g(x)} + 1 - \theta_1} > 1$$

$$\theta_2 \frac{f(x)}{g(x)} + 1 - \theta_2 > \theta_1 \frac{f(x)}{g(x)} + 1 - \theta_1$$

$$\theta_2 \left(\frac{f(x)}{g(x)} - 1\right) + 1 > \theta_1 \left(\frac{f(x)}{g(x)} - 1\right) + 1$$

$$\theta_2 \left(\frac{f(x)}{g(x)} - 1\right) > \theta_1 \left(\frac{f(x)}{g(x)} - 1\right)$$

$\theta_2 > \theta_1$ ✓ $\therefore$ MLR holds.

2(b). Given $0<\theta_1<\theta_2<1$, derive a UMP test of size α for testing $H_0: \theta \in [0,\theta_1] \cup [\theta_2, 1]$ vs. $H_1: \theta \in (\theta_1, \theta_2)$

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For any test $\phi(x)$, the power is as follows:

$$\beta_\phi(\theta) = E[\phi(x)] = \int \phi(x)[\theta f(x) + (1-\theta) g(x)]\, d(F+G)$$

$$= \int \phi(x)\big[\theta[f(x)-g(x)] + g(x)\big]\, d(F+G)$$

$$= \theta \int \phi(x)(f(x)-g(x))\, d(F+G) + \int \phi(x) g(x)\, d(F+G)$$

Thus $\beta(\theta)$ is a linear function of θ.

Since $\phi(x)$ is level α and $\beta_\phi(\theta)$ is linear,

$\beta_\phi(x) \le \alpha \quad \forall\ \theta \in [0,1]$, thus $\phi_2(x) \equiv \alpha$ is a UMP test for H_0.

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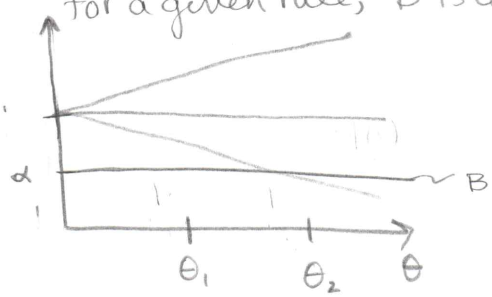
2. (c) Show that a UMP test does not exist for testing $H_0: \theta \in [\theta_1, \theta_2]$ vs. $H_1: \theta \notin [\theta_1, \theta_2]$

Let $\phi^*(x)$ be a UMP test

$$\beta_{\phi^*}(\theta) = \theta \underbrace{\int \phi^*(x)(f(x)-g(x))\, dF+G}_{A} + \underbrace{\int \phi^*(x)\, g(x)\, dF+G}_{B} = \theta A + B$$

for a given rule, B is constant



If $A > 0$ $\beta_{\phi^*}(\theta) < \alpha$ for $\theta < \theta_1$

$\Rightarrow \beta_{\phi^*} < \beta_\phi$

If $A < 0$ $\beta_{\phi^*}(\theta) < \alpha$ for $\theta > \theta_2$

$\Rightarrow \beta_{\phi^*} < \beta_\phi$

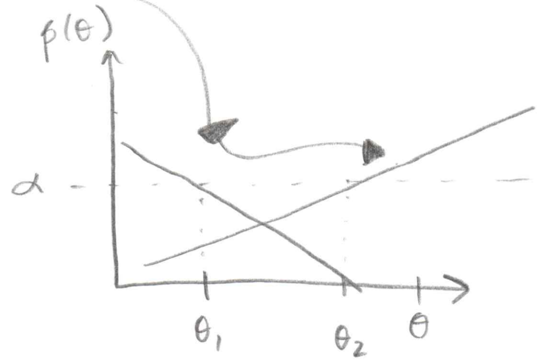
$\therefore$ ϕ^* cannot be UMP

can $\phi(x) \equiv \alpha$ be UMP?

From (a), we know the UMP for testing $H_0: \theta \le \theta_0$ vs. $H_1: \theta > \theta_0$ has a UMP test. let $\theta_0 = \theta_2$ $H_0: \theta \le \theta_2$ vs. $H_1: \theta > \theta_2$ has power $> \alpha$ when $\theta_0 \in [\theta_2, 1]$ $\Rightarrow$ more powerful than $\phi(x) \equiv \alpha$ in that region. Thus $\phi(x) \equiv \alpha$ is NOT UMP.

$\therefore$ No UMP test exists for the test $H_0: \theta \in [\theta_1, \theta_2]$ vs. $H_1: \theta \in [\theta_1, \theta_2]$

It may seem silly to use the 1-sided test for this hypothesis, but it is still a valid α-level test & is more powerful in the $[\theta_2, 1]$ region.



why is power $> \alpha$ when $\theta_0 \in [\theta_2, 1]$?

UMP $\Rightarrow$ UMPU

2.(d) Obtain a UMPU test of size α for the problem
$H_0: \theta \in [\theta_1, \theta_2]$ vs. $H_1: \theta \notin [\theta_1, \theta_2]$

Unbiased $\Rightarrow \beta(\theta) \le \alpha \ \forall \theta \in \Theta_0$ and $\beta(\theta) \ge \alpha \ \forall \theta \in \Theta_1$

$\Rightarrow \beta_\phi(\theta) \le \alpha \ \forall \theta \in [\theta_1, \theta_2]$ and $\beta_\phi(\theta) \ge \alpha \ \forall \theta \notin [\theta_1, \theta_2]$

Thus, when the power function is linear as shown in (b), tests w/ constant power can be unbiased.
Thus $\phi(x) \equiv \alpha$ is UMPU level α for H_0.

(e). Given $0 < \theta_1 \le \theta_2 < 1$ derive the LRT for testing $H_0: \theta \in [\theta_1, \theta_2]$ vs. $H_1: \theta \notin [\theta_1, \theta_2]$.

with only 1 observation,

$$L(\theta) = \theta f(x) + (1-\theta) g(x)$$

$$\sup_{0 \le \theta \le 1} L(\theta) = \begin{cases} \theta = 1 & f(x) \ge g(x) \\ \theta = 0 & f(x) < g(x) \end{cases} \Rightarrow \begin{cases} f(x) & f(x) \ge g(x) \\ g(x) & f(x) < g(x) \end{cases}$$

$$\sup_{0 \le \theta_1 < \theta_2 \le 1} L(\theta) = \begin{cases} \theta_2 f(x) + (1-\theta_2) g(x) & f(x) \ge g(x) \\ \theta_1 f(x) + (1-\theta_1) g(x) & f(x) < g(x) \end{cases}$$

$$\Lambda(x) = \begin{cases} \theta_2 + (1-\theta_2) \frac{g(x)}{f(x)} & f(x) \ge g(x) \\ \theta_1 \frac{f(x)}{g(x)} + (1-\theta_1) & f(x) < g(x) \end{cases}$$

$\Lambda(x)$ is the LRT test statistic for $H_0: \theta \in [\theta_1, \theta_2]$.