Theory Exam Section I 2016

1). $X_1, \dots, X_n$ iid $\sim \begin{cases} 0 & \text{w/ prob } p \\ \text{Unif}(0,\theta) & \text{w/ prob } 1-p \end{cases}$

p Known constant $\in (0,1)$

$\theta > 0$ (parameter of interest).

(a) (i) Based on only one obs. X_1, find all the unbiased estimators for θ & calculate their variances.

(ii) Does the UMVUE exist for θ? (Justify answer)

(i) $E[X] = 0(p) + (\theta/2)(1-p) = \dfrac{\theta(1-p)}{2}$

$\Rightarrow E\left[\dfrac{2X}{1-p}\right] = \theta$ $\quad = E[E[X \mid \text{Dist}]] = E[X \mid \text{Dist } 1]P(\text{Dist } 1) + E[X \mid \text{Dist } 2]\,P(\text{Dist } 2)$

$\Rightarrow$ One unbiased estimator is $\dfrac{2X}{1-p} = Y_1$

$Var(Y_1) = \dfrac{4}{(1-p)^2} Var(X)$

$Var(X) = E(X^2) - (E(X))^2$

$= \underbrace{\left[0^2(p) + \left(Var(\text{Unif}(0,\theta)) + E(\text{Unif}(0,\theta))\right)(1-p)\right]}_{E[X^2]} - \left(\dfrac{\theta(1-p)}{2}\right)^2$

$= \dfrac{(1-p)\theta^2}{3} - \dfrac{\theta^2(1-p)^2}{4} = \dfrac{(1-p)}{12}\left(4\theta^2 - 3\theta^2(1-p)\right)$

$= \dfrac{(1-p)}{12}\left(\theta^2 + 3p\theta^2\right) = \dfrac{\theta^2(1-p)(1+3p)}{12}$

$Var(Y_1) = \dfrac{4}{(1-p)^2}\dfrac{\theta^2(1-p)}{12}(1+3p) = \dfrac{\theta^2(1+3p)}{3(1-p)}$

$E[X^2] = E[E[X^2 \mid \text{Dist}]] = E[X^2 \mid \text{Dist } 1]P(\text{Dist } 1) + E[X^2 \mid \text{Dist } 2]P(\text{Dist } 2)$

$= 0(p) + \left[Var(X \mid \text{Dist } 2) + (E(X \mid \text{Dist } 2))^2\right](1-p)$

$= (1-p)\left[\dfrac{\theta^2}{12} + \left(\dfrac{\theta}{2}\right)^2\right] = \dfrac{(1-p)}{12}\left(\theta^2 + 3\theta^2\right) = \dfrac{(1-p)\theta^2}{3}$

If $X_i \sim \text{Bern}(p) \Rightarrow X_i \sim \begin{cases} 0 & \text{prob } p \\ 1 & \text{prob } 1-p \end{cases}$

$\Rightarrow f(x_i) = p^x (1-p)^{1-x}$

Let $Y \sim \text{Bern}(p)$
$U \sim \text{Unif}(0, \theta)$

$X | Y = \begin{cases} 0 & Y = 1 \\ U & Y = 0 \end{cases}$

$f(x) = f(x|y) g(y)$
$= I(Y=1) f(x|Y=1) g(Y=1)$
$+ I(Y=0) f(x|Y=0) g(Y=0)$
$= I(Y=1)\, p\, I(x=0) + I(Y=0)(1-p) \frac{1}{\theta} I(0<x<\theta)$

UMVUE – Based on complete sufficient statistic

$f(x) = \left(f(x|Y=0) g(Y=0)\right)^{I(Y=0)} \cdot \left(f(x|Y=1) g(Y=1)\right)^{I(Y=1)}$
$= (p)^{I(x=0) I(Y=0)} \left((1-p)(1/\theta)\right)^{I(0<x<\theta) I(Y=1)} I(0 \le x \le \theta)$

$f(x) = p^{I(x=0)} \left((1-p)\theta^{-1}\right)^{I(0<x\le\theta)} I(0 \le x \le \theta)$

By the factorization rule, X = sufficient statistic
Show X = complete sufficient statistic?

If X complete sufficient statistic
$\Rightarrow E[Y_1 | X] = \text{UMVUE}$
$\Rightarrow E\left[\frac{2x}{1-p} \middle| x\right] = \frac{2x}{1-p}$ ✓

(b) $x_1, \dots, x_n$ obs

$x_{(n)} = \max(\underset{\sim}{x})$

(i) Show $(x_{(n)}, \sum_{i=1}^n I(x_i > 0))$ = suff stat for θ

(ii) Show $\hat{\theta} = x_{(n)}$ = MLE of obs. likelihood.

(i) $L(\underset{\sim}{x}) = \prod_{i=1}^n f(x_i) = \cancel{p^{\sum_{i=1}^n I(x_i=0)} [(1-p)\theta^{-1}]^{\sum_{i=1}^n I(0<x_i\le\theta)}}$

$= p^{\sum_{i=1}^n I(x_i=0)} [(1-p)\theta^{-1}]^{\sum_{i=1}^n I(0<x_i\le\theta)} I(0 \le x_1, \dots, x_n \le \theta)$

$= p^{\sum_{i=1}^n I(x_i=0)} [(1-p)\theta^{-1}]^{\sum_{i=1}^n I(0<x_i\le\theta)} I(0 \le x_{(1)}) I(x_{(n)} \le \theta)$

By the factorization rule, if we can write

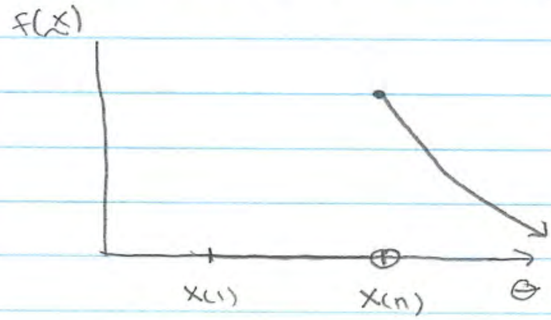
$f(\underset{\sim}{x}) = g(\theta, T(\underset{\sim}{x})) h(\underset{\sim}{x}) \Rightarrow T(x)$ = suff stat

In this case, $T(\underset{\sim}{x}) = (\sum_{i=1}^n I(0<x_i\le\theta), x_{(n)})$

since $g_1(\theta, T(\underset{\sim}{x})) = [(1-p)\theta^{-1}]^{\sum_{i=1}^n I(0<x_i\le\theta)}$

$g_2(\theta, T(\underset{\sim}{x})) = I(x_{(n)} \le \theta)$

(ii) Show $\hat{\theta} = x_{(n)}$ maximizes obs. likelihood



If $\theta < x_{(n)} \Rightarrow$ all $I(0<x_i\le\theta) = 0$

If $\theta \ge x_{(n)} \Rightarrow I(0<x_i\le\theta) = 1$

As $\theta \uparrow$, $\theta^{-1} \downarrow$

Therefore, $x_{(n)}$ maximizes $f(\underset{\sim}{x})$

ⓒ (i) What is the exact dist of $\hat{\theta}$?

(ii) Compute $E(\hat{\theta})$ + $Var(\hat{\theta})$

(iii) Show $\hat{\theta}$ consistent for θ.

(i) Dist of $X_{(n)} = \hat{\theta}$:

$$F_{X_{(n)}}(z) = P(X_{(n)} \le z)$$
$$= P(X_1, \ldots, X_n \le z)$$
$$= [P(X_i \le z)]^n \quad (iid)$$
$$= [F_X(z)]^n$$

$$\cancel{F_X(x) = \int_x f_X(x)\,dx}$$

$$\binom{n}{m} p^{n-m}(1-p)^m \quad m = \#\text{ non-zero} \rightarrow \text{pdf of Bern}(p)$$

~~m uniform~~

$$\rightarrow \binom{n}{m} p^{n-m}(1-p)^m \underbrace{[P(U \le t)]^m}$$

$$P(X_{(n)} \le t)$$
$$= \left(\frac{t}{\theta}\right)^m$$

Let $Y \sim Bern(p)$, $U \equiv X_{(n)}$ for $X \sim Unif(0,\theta)$

CDF: $P(Y \le \_, U \le t)$

$$\sum_{m=0}^{n} \binom{n}{m} p^{n-m}(1-p)^m \left(\frac{t}{\theta}\right)^m$$

$$= \sum_{m=0}^{n} \binom{n}{m} p^n \left(\frac{1-p}{p}\left(\frac{t}{\theta}\right)\right)^m$$

$$= p^n \sum_{m=0}^{n} \binom{n}{m} \left(\frac{1-p}{p}\left(\frac{t}{\theta}\right)\right)^m (1)^{n-m} = p^n\left(1 + \frac{1-p}{p} t/\theta\right)^n$$

$$= \boxed{(p + (1-p)\, t/\theta)^n}$$

$t = x_{(n)}$

$$F_t(t) = (p + (1-p)\,t/\theta)^n$$

$$f_t(t) = \frac{n(1-p)}{\theta}\,(p + (1-p)t/\theta)^{n-1}$$

$$E[t] = \int_0^\infty \frac{n(1-p)t}{\theta}\,(p + (1-p)t/\theta)^{n-1}\,dt$$

$$u = \frac{(1-p)t}{\theta} \qquad du = \frac{(1-p)}{\theta}\,dt \Rightarrow dt = \frac{\theta}{1-p}\,du$$

$$= \int_0^\infty n u\,(p+u)^{n-1}\left(\frac{\theta}{1-p}\right)du$$

...

(d) Asymptotic dist of $n(\hat{\theta}-\theta)$?

$$
\begin{aligned}
& F(n(t-\theta))(z) \\
&= P(n(t-\theta) \le z) \\
&= P(t-\theta \le z/n) \\
&= P(t \le z/n+\theta) \\
&= F_t(z/n+\theta) \\
&= \left(p + \frac{(1-p)}{\theta}\left(\frac{z}{n}+\theta\right)\right)^n \\
&= \left(p + (1-p) + \frac{(1-p)}{\theta}\frac{z}{n}\right)^n \\
&= \left(1 + \left(\frac{(1-p)z}{\theta}\right)\frac{1}{n}\right)^n
\end{aligned}
$$

$$\lim_{n\to\infty} (\text{above}) = \exp\left(\frac{(1-p)z}{\theta}\right) \checkmark$$

(e) Both p + θ unknown

$$f(x) = p^{I(x=0)} \left[(1-p)\,\theta^{-1} \right]^{I(0<x\le\theta)} I(0 \le x \le \theta)$$

$$\log f(x) = I(x=0)\log p + I(0<x\le\theta)\log(1-p) + c(\theta)$$

$$\frac{\partial}{\partial p}\log f(x) = \frac{I(x=0)}{p} - \frac{I(0<x\le\theta)}{1-p} \overset{set}{=} 0$$

Note: $I(0<x\le\theta) = 1 - I(x=0)$ for $0 \le x \le \theta$

$$\Rightarrow \frac{I(x=0)}{p} - \frac{(1-I(x=0))}{1-p}$$

$$\log f(\underset{\sim}{x}) = \sum_{i=1}^{n} \log f(x_i)$$

$$\frac{\partial}{\partial p}\log f(\underset{\sim}{x}) = \sum_{i=1}^{n} \frac{\partial}{\partial p}\log f(x_i)$$

$$= \frac{\sum_{i=1}^{n} I(x_i=0)}{p} - \frac{\left(n - \sum_{i=1}^{n} I(x_i=0)\right)}{1-p} \overset{set}{=} 0$$

$$\Rightarrow (1-\cancel{p})\sum_{i=1}^{n} I(x_i=0) - np + \cancel{p\sum_{i=1}^{n} I(x_i=0)} \overset{set}{=} 0$$

$$\Rightarrow np = \sum_{i=1}^{n} I(x_i=0)$$

$$\Rightarrow \hat{p} = \frac{\sum_{i=1}^{n} I(x_i=0)}{n}$$

From work in (a), $E[X_1] = \dfrac{(1-p)\theta}{2}$

$$\Rightarrow \text{MLE of } E[X_1] = \left.\frac{(1-p)\theta}{2}\right|_{\hat{\theta},\hat{p}}$$

$$= \frac{(1-\hat{p})\hat{\theta}}{n} \quad \Delta$$

plug in $\hat{p} = \dfrac{\sum_{i=1}^{n} I(x_i=0)}{n}$

and $\hat{\theta} = X_{(n)}$

Derive the asymptotic dist of MLE of $E(X_1)$ after proper normalization.

$$\frac{(1-\hat{p})\hat{\theta}}{n} = \left(\frac{\sum_{i=1}^{n} I(0 < x_i)}{n}\right)(X_{(n)})$$

By MLE theory, can find

$$\sqrt{n}\left(\begin{bmatrix}\hat{p}\\ \hat{\theta}\end{bmatrix} - \begin{bmatrix}p\\ \theta\end{bmatrix}\right) \xrightarrow{d} N(\underset{\sim}{0}, I^{-1}(p,\theta))$$

if $\log P(p,\theta)$ (twice) continuously differentiable

— Not the case here

But if goes to $N(0, \Sigma)$

$$\Rightarrow \sqrt{n}\,(g(\hat{p},\hat{\theta}) - g(p,\theta)) \xrightarrow{d} N(\underset{\sim}{0}, \nabla g(p,\theta)\, \Sigma\, \nabla g(p,\theta)^T)$$

$$g(p,\theta) = \frac{(1-p)\theta}{2}$$

$$\frac{\partial}{\partial p} g(p,\theta) = \frac{-\theta}{2} \qquad \frac{\partial}{\partial \theta} g(p,\theta) = \frac{(1-p)}{2}$$

Problem: $n(\hat{\theta} - \theta)$ will not have same asymptotic dist as $\sqrt{n}(\hat{p} - p)$

Will probably have to derive from first principles instead.

Want to find dist $U|T$

$U = \sum_{i=1}^n y_i x_i$

$T = \sum_{i=1}^n y_i$

$$P(U|T) = \frac{P(U,T)}{P(T)} = \frac{\text{joint dist } f(\underset{\sim}{y})}{P(T)}$$

- under the null hypothesis
  - use the boundary value (SOB)

$$B_1 = 0 \Rightarrow P(y_i = 1) = \frac{\exp(B_0)}{1+\exp(B_0)}$$

$$\Rightarrow Y_i \sim \text{Bern}\left(\frac{\exp(B_0)}{1+\exp(B_0)}\right)$$

$$\sum_{i=1}^n Y_i \sim \text{Bin}\left(n, \exp(B_0)/1+\exp(B_0)\right)$$

$$P(U,T|\Theta_B) = P(\overset{T}{\sum_{i=1}^n y_i}, \overset{U}{\sum_{i=1}^n y_i x_i} \mid B_1 = 0)$$

= joint dist under boundary value.

$$= \frac{\left(\frac{\exp(B_0 \sum_{i=1}^n y_i + 0)}{\prod_{i=1}^n (1+\exp(B_0+0))}\right)}{\binom{n}{\sum_{i=1}^n y_i}\left(\frac{\exp(\sum_{i=1}^n y_i(B_0))}{(1+\exp(B_0))^{\sum_{i=1}^n y_i}}\right)\left(\frac{1}{(1+\exp(B_0))^{n-\sum_{i=1}^n y_i}}\right)}$$

$$= \frac{1}{\binom{n}{\sum_{i=1}^n y_i}} = \frac{(\sum_{i=1}^n y_i)!(n-\sum_{i=1}^n y_i)!}{n!} \quad ?$$

Instead of finding the dist of $U|T$, use other approach.

UMPU test will be:

$$\phi(u) = \begin{cases} 1 & u < c_1(t) \text{ or } u > c_2(t) \\ \gamma_i & u = c_i(t) \quad i=1,2 \\ 0 & \text{otherwise} \end{cases}$$

Under the conditions that

$E_{\theta_0}[\phi(U) \mid T] = \alpha$ and

$E_{\theta_0}[U\phi(U) \mid T] = \alpha E_{\theta_0}[U \mid T]$

2). $y_1, \ldots, y_n$ indep. Binary vars

$$P(y_i = 1 \mid B_0, B_1, x_i) = \frac{\exp(B_0 + B_1 x_i)}{1 + \exp(B_0 + B_1 x_i)}$$

$x_1, \ldots, x_n$ fixed covariates

(a) (B_0, B_1) both unknown

$H_0: B_1 = 0$ vs $H_A: B_1 \neq 0$

Derive the UMPU α level test

- rejection region & critical value in simplest possible form.

$$f(\underline{y}) = \prod_{i=1}^{n} f(y_i) = \prod_{i=1}^{n} P(y_i=1)^{I(y_i=1)} P(y_i=0)^{I(y_i=0)}$$

$$= \prod_{i=1}^{n} \left[\left(\frac{\exp(B_0 + B_1 x_i)}{1 + \exp(B_0 + B_1 x_i)} \right)^{y_i} \left(\frac{1}{1 + \exp(B_0 + B_1 x_i)} \right)^{1-y_i} \right]$$

Note: y_i binary $\Rightarrow I(y_i = 1) = y_i$

$$= \prod_{i=1}^{n} \left(\frac{\exp(y_i B_0 + y_i x_i B_1)}{1 + \exp(B_0 + B_1 x_i)} \right)$$

$$= \frac{\exp\left(B_0 \sum_{i=1}^{n} y_i + B_1 \sum_{i=1}^{n} y_i x_i\right)}{\prod_{i=1}^{n} \left(1 + \exp(B_0 + B_1 x_i)\right)}$$

This is in exponential family form

$$h(y)\, c(\theta) \exp\left(\sum_{k=1}^{2} \theta_k T_k(y)\right)$$

$h(y)\, c(\theta) = \prod_{i=1}^{n} \left(1 + \exp(B_0 + B_1 x_i)\right)$

$\theta_1 = B_0$ $\qquad\qquad \theta_2 = B_1$

$T_1(y) = \sum_{i=1}^{n} y_i$ $\qquad T_2(y) = \sum_{i=1}^{n} y_i x_i$

$\Rightarrow$ complete Sufficient statistic for nuisance B_0 is $\sum_{i=1}^{n} y_i$,

complete suff stat for param of interest B_1 is $\sum_{i=1}^{n} x_i y_i$

(b) UMPU test based on $U < c_1(t)$, $U > c_2(t)$ for rejection region

$U = \sum_{i=1}^n y_i x_i$

$T = \sum_{i=1}^n y_i$

Find Asymptotic Conditional dist on boundary $B_1 = 0$

(Liapunov CLT ⇒ assume condition true)

$$\frac{\sum_{i=1}^n Y_{ni} - \sum_{i=1}^n E[Y_{ni}]}{\sqrt{\sum_{i=1}^n Var(Y_{ni})}} \xrightarrow{d} N(0,1) \Rightarrow \frac{\sum_{i=1}^n y_i x_i - \sum_{i=1}^n E[y_i x_i \mid \sum_{i=1}^n y_i]}{\sqrt{\sum_{i=1}^n Var[y_i x_i \mid \sum_{i=1}^n y_i]}} \xrightarrow{d} N(0,1)$$

$$E[y_i x_i \mid \sum_{i=1}^n y_i] = x_i E[y_i \mid \sum_{i=1}^n y_i]$$

On boundary $B_1 = 0$

$\Rightarrow \sum_{i=1}^n y_i = Z_n \sim Bin(n, \exp(B_0)/1+\exp(B_0))$

$$E[y_i \mid \sum_{i=1}^n y_i] = \sum y_i P(y_i \mid \sum_{i=1}^n y_i)$$

$$= \sum y_i \frac{P(y_i, \sum_{i=1}^n y_i)}{P(\sum_{i=1}^n y_i)}$$

$$= (1) \frac{P(y_i = 1, \sum_{j \neq i} y_j = m-1)}{P(\sum_{i=1}^n y_i = m)} + 0$$

$$P(y_i = 1, \sum_{j \neq i} y_j = m-1)$$

$$= P(y_i = 1) P(\sum_{j \neq i} y_i = m-1)$$

let $\frac{\exp(B_0)}{1+\exp(B_0)} = p$

$$= p \binom{n}{m-1} p^{m-1} (1-p)^{n-m+1}$$

$$\frac{\text{above}}{P(\sum_{i=1}^n y_i = m)} = \longrightarrow$$

$$= \frac{\binom{n}{m-1} \cancel{p^m} (1-p)^{\cancel{(n-m)}+1}}{\binom{n}{m} \cancel{p^m} (1-p)^{\cancel{(n-m)}}}$$

$$= \frac{\cancel{n!}}{(m-1)!\,(n-m+1)!} \cdot \frac{m!\,(n-m)!}{\cancel{n!}} \cdot (1-p)$$

$$= \frac{m}{n-m+1} (1-p)$$

$$Var(y_i x_i \mid \Sigma_{i=1}^n y_i = m)$$
$$= x_i^2 \, Var(y_i \mid \Sigma_{i=1}^n y_i = m)$$
$$= x_i^2 \left[E[y_i^2 \mid \Sigma_{i=1}^n y_i = m] - (E(y_i \mid \Sigma_{i=1}^n y_i = m))^2 \right]$$

$E[y_i^2 \mid T] = E[y_i \mid T]$ since y_i binary

$$= x_i^2 \, E[y_i \mid \Sigma_{i=1}^n y_i = m] \left(1 - E[y_i \mid \Sigma_{i=1}^n y_i = m]\right)$$

By Liapunov CLT (w/ condition)

$$\frac{\sqrt{n}\,(S - E(S))}{\sqrt{Var(S)}} \sim N(0,1)$$

$$S = \Sigma_{i=1}^n y_i x_i$$

$$E[S \mid T] = \sum_{i=1}^n \frac{x_i m (1-p)}{n-m+1} \quad \left(p = \frac{\exp(\beta_0)}{1+\exp(\beta_0)} \right)$$

$$Var(S \mid T) = \sum_{i=1}^n x_i^2 \left(\frac{\cancel{x_i} m (1-p)}{n-m+1} \right) \left(1 - \frac{\cancel{x_i} m (1-p)}{n-m+1} \right)$$

Therefore, Can choose cut-off values based on

$$\frac{\sqrt{n}\,(U - E[U \mid T])}{\sqrt{Var(U \mid T)}} > z_{1-\alpha/2}$$

area to the left

and $$\frac{\sqrt{n}\,(U - E[U \mid T])}{\sqrt{Var(U \mid T)}} < z_{\alpha/2}$$

$N(0,1)$ symmetric,

so $z_{\alpha/2} = -z_{1-\alpha/2}$

$$\phi(y) = \begin{cases} 1 & \left| \frac{\sqrt{n}\,(u - E[u \mid T])}{\sqrt{Var(u \mid T)}} \right| > z_{1-\alpha/2} \\ 0 & \text{otherwise.} \end{cases}$$

$$E[u \mid T] = \sum_{i=1}^{n} \frac{x_i\, m(1-p)}{n-m+1} \quad \text{where } p = \frac{\exp(\beta_0)}{1+\exp(\beta_0)}$$

$$Var[u \mid T] = \sum_{i=1}^{n} x_i^2 \left(\frac{m(1-p)}{n-m+1} \right) \left(1 - \frac{m(1-p)}{n-m+1} \right)$$

(c) Derive the score test for the hypothesis in part (a) & compare its form to the approximate UMPU test derived in (b).

$$SC_n = \dot{\ell}_n(\tilde{B})^T I_n^{-1}(\tilde{B})\, \dot{\ell}_n(\tilde{B})$$

$\dot{\ell}_n(B) = \frac{\partial}{\partial B} \ell_n(B)$ → evaluate at MLE under H_0 → $\tilde{B}$

$I_n^{-1}(B) = \left[E\{-\partial^2/\partial B\, \ell_n(B)\} \right]^{-1}$

- evaluate at $\hat{B}$ as well.

Find $\tilde{B}$ = MLE under H_0

→ $\tilde{B}_1 = 0$

$\tilde{B}_0 = ?$

$$\ell_n(B) = \sum_{i=1}^{n} \left[y_i B_0 + y_i x_i B_1 - \log(1 + \exp(B_0 + B_1 x_i)) \right]$$

$$\frac{\partial}{\partial B_0} \ell_n(B) = \sum_{i=1}^n y_i - \sum_{i=1}^n \frac{\exp(B_0 + B_1 x_i)}{1 + \exp(B_0 + B_1 x_i)} \overset{set}{=} 0$$

when $B_1 = 0$

$$\Rightarrow \sum_{i=1}^n y_i = \sum_{i=1}^n \frac{\exp(B_0)}{1 + \exp(B_0)} = \frac{n \exp(B_0)}{1 + \exp(B_0)} = \frac{n}{1 + \exp(-B_0)}$$

$$\Rightarrow \frac{1}{n} \sum_{i=1}^n y_i = \frac{1}{1 + \exp(-B_0)}$$

$$\Rightarrow (1 + \exp(-B_0)) \frac{1}{n} \sum_{i=1}^n y_i = 1$$

$$\Rightarrow \exp(-B_0) = \frac{1 - \frac{1}{n}\sum_{i=1}^n y_i}{\frac{1}{n}\sum_{i=1}^n y_i}$$

$$\Rightarrow B_0 = -\log\left(\frac{1 - \frac{1}{n}\sum_{i=1}^n y_i}{\frac{1}{n}\sum_{i=1}^n y_i} \right)$$

$$\Rightarrow \boxed{\tilde{B}_0 = \log\left(\frac{\frac{1}{n}\sum_{i=1}^n y_i}{1 - \frac{1}{n}\sum_{i=1}^n y_i} \right)}$$

$$\frac{\partial}{\partial \beta_0} l_n(\beta) \Big|_{\tilde{\beta}_0} = 0$$

$$\frac{\partial}{\partial \beta_1} l_n(\beta) \Big|_{\tilde{\beta}} = \sum_{i=1}^{n} y_i x_i - \sum_{i=1}^{n} \frac{x_i \exp(\beta_0 + \beta_1 x_i)}{1 + \exp(\beta_0 + \beta_1 x_i)} \Big|_{\beta_1 = 0,\ \beta_0 = \tilde{\beta}_0}$$

$$= \sum_{i=1}^{n} y_i x_i - \frac{\exp(\beta_0)}{1 + \exp(\beta_0)} \sum_{i=1}^{n} x_i \Big|_{\beta_0 = \tilde{\beta}_0}$$

$$= \sum_{i=1}^{n} y_i x_i - \frac{\exp(\tilde{\beta}_0)}{1 + \exp(\tilde{\beta}_0)} \sum_{i=1}^{n} x_i$$

→ plug in $\hat{\beta}_0$ from last pg

$$\boxed{\dot{l}_n(\tilde{\beta}) = \begin{bmatrix} 0 \\ \sum_{i=1}^{n} y_i x_i - \left(\exp(\tilde{\beta}_0)/1 + \exp(\tilde{\beta}_0)\right) \sum_{i=1}^{n} x_i \end{bmatrix}}$$

$$\frac{\partial^2}{\partial \beta_0^2} l_n(\beta) \Big|_{\tilde{\beta}} =$$

$$\exp(\beta_0)\left(1 + \cancel{\exp(\beta_0)} - \cancel{\exp(\beta_0)}\right)$$

$$= -\sum_{i=1}^{n} \frac{(1 + \exp(\beta_0))\exp(\beta_0) - \exp(\beta_0)\exp(\beta_0)}{(1 + \exp(\beta_0))^2} \Big|_{\beta_0 = \tilde{\beta}}$$

$$= \frac{-n \exp(\tilde{\beta}_0)}{(1 + \exp(\hat{\beta}_0))^2}$$

$$E\left[-\partial^2/\partial \beta_0^2\, l_n(\beta)\right] \Big|_{\tilde{\beta}} = \frac{n \exp(\tilde{\beta}_0)}{(1 + \exp(\hat{\beta}_0))^2}$$

$$\frac{\partial^2}{\partial \beta_1^2} l_n(\beta) =$$

3). $S \sim Bin(n, p)$

$X_1, \dots, X_{S+1} | S \sim iid\ N(\mu, 1)$

n known

(p, μ) unknown.

obs $(S, X_1, \dots, X_{S+1})$

$H_0: \mu \le 0$ vs. $H_A: \mu > 0$ at level α

(a) Write out the joint density of $(S, X_1, \dots, X_{S+1})$

- show it belongs to full rank exponential family
- find two dimensional complete suff stat
- Do same for special case $\mu = 0$

$$P(s, \underset{\sim}{x}) = P(\underset{\sim}{x} | s) P(s)$$

$$= \left(\prod_{i=1}^{S+1} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x_i - \mu)^2\right) \right) \binom{n}{s} p^s (1-p)^{n-s}$$

$$= \binom{n}{s} (2\pi)^{-\frac{(S+1)}{2}} \exp\left(-\frac{1}{2}\left(\sum_{i=1}^{S+1} x_i^2 - 2\mu \sum_{i=1}^{n} x_i + n\mu^2 \right) + S \log p + (n-s)\log(1-p) \right)$$

$$= \binom{n}{s} (2\pi)^{-(S+1)/2} \exp\left(-\frac{1}{2} \sum_{i=1}^{S+1} x_i^2 \right) \exp\left(-\frac{1}{2} n\mu^2 \right) \cdot \exp\left(n \log(1-p) \right)$$

$$\cdot \exp\left(\mu \sum_{i=1}^{S+1} x_i + S \log\left(\frac{p}{1-p}\right) \right)$$

$$= h(s, \underset{\sim}{x})\, c(p, \mu) \exp\left(\sum_{k=1}^{2} \theta_k T_k \right)$$

$$h(s, \underset{\sim}{x}) = \binom{n}{s} (2\pi)^{-(S+1)/2} \exp\left(-\frac{1}{2} \sum_{i=1}^{S+1} x_i^2 \right)$$

$$c(p, \mu) = \exp\left(-\frac{1}{2} n\mu^2 \right) \exp\left(n \log(1-p) \right)$$

$\theta_1 = \mu \quad T_1 = \sum_{i=1}^{S+1} x_i$

$\theta_2 = \log\left(\frac{p}{1-p}\right) \quad T_2 = S$

} two terms for two unknown parameters ✓ full rank (rank 2)

$\underset{\sim}{T} = [S, \sum_{i=1}^{S+1} x_i]$ = complete suff stat.

Special case $\mu = 0$:

$h(s, \underset{\sim}{x})$ same as before

$$c(p) = \exp\left(n \log(1-p)\right)$$

$$\exp(\theta T(\underset{\sim}{x}, s)) = \exp\left(s \log\left(p/1-p\right)\right)$$

(b) Derive the joint MLE's of (p, μ) denoted $(\hat{p}, \hat{\mu})$

$$\log P(s, \underset{\sim}{x})$$
$$= \log g(s, \underset{\sim}{x}) - \frac{1}{2} n\mu^2 + n \log(1-p) + \mu \Sigma_{i=1}^{s+1} x_i + s \log\left(\frac{p}{1-p}\right)$$

$$\frac{\partial}{\partial \mu} \log P(s, \underset{\sim}{x}) = -n\mu + \Sigma_{i=1}^{s+1} x_i \overset{set}{=} 0$$

$$\Rightarrow \boxed{\hat{\mu} = \frac{1}{n} \sum_{i=1}^{s+1} x_i}$$

$$\frac{\partial}{\partial p} \log P(s, \underset{\sim}{x}) = \frac{-n}{1-p} + \frac{s}{p} - \frac{s}{1-p}(-1) \overset{set}{=} 0$$

$$\Rightarrow +\frac{(n-s)}{1-p} = \frac{s}{p}$$

$$\Rightarrow p(n-s) = s(1-p)$$

$$\Rightarrow p(n-s+s) = s$$

$$\Rightarrow \boxed{\hat{p} = \frac{s}{n}}$$

Note:

$$\frac{\partial^2}{\partial \mu^2} \log P(s, \underset{\sim}{x}) = -n < 0 \checkmark$$

$$\frac{\partial^2}{\partial p^2} \log P(s, \underset{\sim}{x}) = \frac{-n}{(1-p)^2} - \frac{s}{p^2} + \frac{s}{(1-p)^2}$$

$$= \frac{-(n-s)}{(1-p)^2} - \frac{s}{p^2} < 0 \checkmark$$

$$\frac{\partial^2}{\partial \mu \partial p} \log P(s, \underset{\sim}{x}) = 0$$

$$\Rightarrow \begin{vmatrix} \partial^2/\partial \mu^2 & \partial^2/\partial \mu \partial p \\ \partial^2/\partial \mu \partial p & \partial^2/\partial p^2 \end{vmatrix} > 0 \checkmark$$

(c) Assuming std MLE theory applies, derive the joint asymptotic dist of $(\hat{p}, \hat{\mu})$ properly normalized

$$\sqrt{n}\left(\begin{bmatrix}\hat{p}\\ \hat{\mu}\end{bmatrix} - \begin{bmatrix}p\\ \mu\end{bmatrix}\right) \xrightarrow{d} N\left(\begin{bmatrix}0\\0\end{bmatrix}, (I(p,\mu))^{-1}\right)$$

$$I(p,\mu) = E\left[-\partial^2/\partial\varepsilon^2 \log P(p,\mu)\right]$$

$$\frac{\partial^2}{\partial\mu^2} \log P(S, \underset{\sim}{x}) = -n$$

$$E\left[-\lim_{n\to\infty} \frac{1}{n}(-n)\right] = 1$$

$$\frac{\partial^2}{\partial p^2} \log P(S, \underset{\sim}{x}) = \frac{-(n-S)}{(1-p)^2} - \frac{S}{p^2}$$

$$E\left[-\lim_{n\to\infty}\frac{1}{n}\left(\frac{-(n-S)}{(1-p)^2} - \frac{S}{p^2}\right)\right]$$

$$= E\left[\frac{-1}{(1-p)^2}\left(\lim_{n\to\infty}\frac{S}{n}\right) + \frac{1}{(1-p)^2} + \frac{1}{p^2}\lim_{n\to\infty}\frac{S}{n}\right]$$

$$= \frac{-1}{(1-p)^2}\lim_{n\to\infty}\left(\frac{\cancel{n}p}{\cancel{n}}\right) + \frac{1}{(1-p)^2} + \frac{1}{p^2}\lim_{n\to\infty}\left(\frac{\cancel{n}p}{\cancel{n}}\right)$$

$$= \frac{1-p}{(1-p)^2} + \frac{p}{p^2} = \frac{1}{1-p} + \frac{1}{p} = \frac{p+1-p}{p(1-p)} = \frac{1}{p(1-p)}$$

$$\frac{\partial^2}{\partial\mu\,\partial p} \log P(S, \underset{\sim}{x}) = 0$$

$$I(p,\mu) = \begin{bmatrix} 1/p(1-p) & 0 \\ 0 & 1 \end{bmatrix}$$

$$I^{-1}(p,\mu) = \begin{bmatrix} p(1-p) & 0 \\ 0 & 1 \end{bmatrix}$$

$$\sqrt{n}\left(\begin{bmatrix}\hat{p}\\ \hat{\mu}\end{bmatrix} - \begin{bmatrix}p\\ \mu\end{bmatrix}\right) \xrightarrow{d} N\left(\underset{\sim}{0}, \begin{bmatrix} p(1-p) & 0 \\ 0 & 1 \end{bmatrix}\right) \checkmark$$

(d) $\phi(S, X_1, \ldots, X_{S+1})$ = any unbiased level α test of H_0 vs. H_1

(i) Write out what unbiasedness means for the power function $B(p, \mu)$ of such a test

Unbiasedness:

$B(p, \mu) \leq \alpha$ $\underbrace{\text{for } \theta \in \Theta_0}_{\mu \leq 0}$ (null H_0 case)

$B(p, \mu) \geq \alpha$ for $\underbrace{\theta \in \Theta_1}_{\mu > 0}$ (alternative H_1 case)

(ii) Explain why unbiasedness implies that $B(p, 0) = \alpha$ for all p.

Since we are dealing w/ an exponential family, the power function is continuous.

Let μ_{1n} = sequence of μ values that increases to + converges to 0 as $n \to \infty$

Let μ_{2n} = " " that decreases to + " " as $n \to \infty$

By def^n of unbiasedness,

$B(p, \mu_{1n}) \leq \alpha$ $\forall n$ since $\mu_{1n} < 0$

and $B(p, \mu_{2n}) \geq \alpha$ $\forall n$ since $\mu_{2n} > 0$

Since $B(p, \mu)$ continuous, the limits $\lim_{n \to \infty} B(p, \mu_{1n})$ + $\lim_{n \to \infty} B(p, \mu_{2n})$ exists

$\longrightarrow$

$$\lim_{n\to\infty} B(\rho, \mu_{1n}) = B(\rho, 0) \le \alpha$$

$$\lim_{n\to\infty} B(\rho, \mu_{2n}) = B(\rho, 0) \ge \alpha$$

Since $\alpha \le B(\rho, 0) \le \alpha$

$\Rightarrow \quad B(\rho, 0) = \alpha \quad \forall \rho$ ✓

(e) Find the complete form of the UMPU test of

$H_0: \mu \leq 0$ vs. $H_1: \mu > 0$

- specify rejection region in terms of $\frac{1}{S+1}\sum_{i=1}^{S+1} x_i = \bar{x}_S$ and the $1-\alpha$ quantile of a well known dist

Complete suff stat of $\mu = \sum_{i=1}^{S+1} x_i$ (or $\bar{x}_S$)

complete suff stat of nuisance $\rho = S$.

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \bar{x} > c(s) \\ \gamma & \bar{x} = c(s) \\ 0 & \bar{x} < c(s) \end{cases} \rightarrow x \mid s \text{ continuous} \rightarrow \gamma = 0$$

such that $E_{\Theta_0}[\bar{x} > c(s) \mid s] = \alpha$

We know the conditional dist of x_i's

$x_i \mid s \sim N(\mu, 1)$

$\frac{1}{S+1}\sum_{i=1}^{S+1} x_i \mid s \sim N(\mu, 1/S+1)$

Plug in the boundary value of $\mu \rightarrow \mu = 0$

$\Rightarrow \bar{x}_S \sqrt{S+1} \mid S \sim N(0,1)$

area to the left

$\Rightarrow$ Reject when $\bar{x}_S \sqrt{S+1} > z_{1-\alpha}$

where $z \sim N(0,1)$ [CDF $\Phi(z)$, pdf $\phi(z)$]

$\Rightarrow \bar{x}_S > \frac{z_{1-\alpha}}{\sqrt{S+1}}$

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \bar{x}_S > z_{1-\alpha}/\sqrt{S+1} \\ 0 & \bar{x}_S \leq z_{1-\alpha}/\sqrt{S+1} \end{cases}$$