1. $Y_1, \dots, Y_n \overset{ind}{\sim} f_{Y_i}(y_i \mid \mu_i)$

$f_{Y_i \mid \mu_i}(y_i \mid \mu_i) = \frac{1}{\mu_i} \exp\{-\frac{y_i}{\mu_i}\}$

$\theta_i \sim \text{Gamma}(a_i, b_i)$, $\frac{a_i}{b_i} = \exp\{\underset{\sim}{x}_i' \beta\}$, $a_i = 3$

$Var(\theta_i) = \tau \exp\{\underset{\sim}{x}_i' \beta\}$

a) $E(Y_i) = \frac{b_i}{a_i - 1}$

$Var(Y_i) = 2 b_i^2 \frac{\Gamma(a_i - 2)}{\Gamma(a_i)} - \left[\frac{b_i}{a_i - 1}\right]^2$

b) * Using $\frac{b^a}{\Gamma(a)}$ gamma pdf

$f_Y(y_i) = 3 \frac{b_i^3}{(y_i + b_i)^4}$ $\quad y_i > 0$

d) Suppose μ_i are fixed, known ω_i is variance fn. of GLM

$Var(Y_i) = \sigma^2(\omega_i + \mu_i)$ where

$Var(Y_i \mid \mu_i) = \frac{1}{\phi} \underbrace{\frac{\partial^2 b(\theta_i)}{\partial \theta_i^2}}_{\text{variance fn.}}$

$f_{Y_i \mid \mu_i}(y_i \mid \mu_i) = \frac{1}{\mu_i} \exp\{-\frac{1}{\mu_i} y_i\}$

$= \exp\{\ln(\frac{1}{\mu_i}) - \frac{1}{\mu_i} y_i\} = \exp\{y_i(-\frac{1}{\mu_i}) - (-\ln(\frac{1}{\mu_i}))\}$

$\Rightarrow \theta_i = -\frac{1}{\mu_i}$, $b(\theta_i) = -\ln(-\theta_i)$, $\phi = 1$

$\frac{\partial b(\theta_i)}{\partial \theta_i} = -\frac{1}{-\theta_i}(-1) = -\frac{1}{\theta_i}$

$\frac{\partial^2 b(\theta_i)}{\partial \theta_i^2} = -(-1)\frac{1}{\theta_i^2} = \frac{1}{\theta_i^2}$ at θ_i

$= \mu_i^2 = \omega_i$

2016 Day 2. P1

1 d cont...

Quasi-log likelihood

$$l_q(\underset{\sim}{\mu}) = \sum_{i=1}^{n} q_i(\mu_i; y_i), \quad q_i(\mu_i; y_i) = \int_{y_i}^{\mu_i} \frac{y_i - t}{\sigma^2 V_i(t)} dt$$

Quasi-score

$$\frac{\partial}{\partial \beta_j} l_q = \frac{\partial}{\partial \underset{\sim}{\mu}} l_q(\underset{\sim}{\mu}) \cdot \frac{\partial \underset{\sim}{\mu}}{\partial \beta_j}$$

$$\frac{\partial}{\partial \mu_i} l_q(\mu_i) = \frac{\partial}{\partial \mu_i} q_i(\mu_i; y_i) = \frac{y_i - \mu_i}{\sigma^2 V_i(\mu_i)}$$

$$\frac{\partial \mu_i}{\partial \beta_j} = \exp\{\underset{\sim}{x}_i' \underset{\sim}{\beta}\} x_{ij}$$

$$\Rightarrow \frac{\partial}{\partial \underset{\sim}{\mu}} l_q(\underset{\sim}{\mu}) = \begin{pmatrix} \frac{y_1 - \mu_1}{\sigma^2 V_1(\mu_1)} \\ \vdots \\ \frac{y_n - \mu_n}{\sigma^2 V_n(\mu_n)} \end{pmatrix}_{n \times 1} \qquad \frac{\partial \underset{\sim}{\mu}}{\partial \underset{\sim}{\beta}} = \begin{pmatrix} \frac{\partial \mu_1}{\partial \beta_1} & \cdots & \frac{\partial \mu_1}{\partial \beta_p} \\ \frac{\partial \mu_2}{\partial \beta_1} & & \frac{\partial \mu_2}{\partial \beta_p} \\ \vdots & & \vdots \\ \frac{\partial \mu_n}{\partial \beta_1} & & \frac{\partial \mu_n}{\partial \beta_p} \end{pmatrix}_{n \times p}$$

$$\Rightarrow \frac{\partial}{\partial \underset{\sim}{\beta}} l_q(\underset{\sim}{\mu}) = \begin{pmatrix} \frac{y_1 - e^{x_1'\beta}}{\sigma^2 [(e^{x_1'\beta})^2 + e^{x_1'\beta}]} \\ \vdots \end{pmatrix} \begin{pmatrix} \mu_1 x_{11} & \mu_1 x_{12} & \cdots \\ \mu_2 x_{21} & \mu_2 x_{22} & \\ \vdots & \vdots & \\ \mu_n x_{n1} & \mu_n x_{n2} & \end{pmatrix}$$

$$S_n(\beta) = \sum_{i=1}^{n} \frac{e^{x_i^T \beta} x_i (y_i - e^{x_i^T \beta})}{(e^{2x_i^T \beta} + e^{x_i^T \beta})}$$

$$E\left[\sum (y_i-\mu_i)^2\right] = \sigma^2 \sum V_i(\mu_i)$$

d) $_{4b}$ $E\left[\sum \frac{(y_i-\mu_i)^2}{(V_i+\mu_i)}\right] \approx n\sigma^2$ $\quad * V_i+\mu_i = \mu_i(\mu_i+1)$

$$\Rightarrow \hat{\sigma}^2 \approx \frac{1}{n-p}\sum_{i=1}^{n} \frac{(y_i-\mu_i)^2}{\mu_i(\mu_i+1)} \Rightarrow \frac{1}{n-p}\sum_{i=1}^{n}\frac{(y_i-e^{x_i'B})^2}{e^{x_i'B}(e^{x_i'B}+1)}$$

$$= E\left[\sum_{i=1}^{n}\left(\frac{Y_i-\mu_i}{\sqrt{\sigma^2 V_i(\mu_i)}}\right)^2\right] = \sum\left[E\left(\frac{Y_i-\mu_i}{\sqrt{\sigma^2 V_i(\mu_i)}}\right)^2\right]$$

$$= \frac{1}{\sigma^2}\sum\left[E\left(\frac{Y_i-\mu_i}{\sqrt{V_i(\mu_i)}}\right)^2\right]$$

$$= \frac{1}{\sigma^2}\sum_{i=1}^{n}\frac{1}{V_i(\mu_i)} E\left[(Y_i-\mu_i)^2\right]$$

$$= \frac{1}{\sigma^2}\sum_{i=1}^{n}\frac{1}{V_i(\mu_i)}\left(\sigma^2 V_i(\mu_i)\right)$$

$$= \frac{n}{\sigma^2}\cdot\sigma^2 = n$$

762 final
4b

$$E\left[\sum\frac{(Y_i-\mu_i)^2}{V_i(\mu_i)}\right] = n\sigma^2$$

$$\Rightarrow \sum\frac{(Y_i-\mu_i)^2}{V_i(\mu_i)} = n\sigma^2$$

$$\Rightarrow \sum\frac{(Y_i-\hat{\mu}_i)^2}{V_i(\hat{\mu}_i)} = (n-p)\hat{\sigma}^2$$

$$\Rightarrow \frac{1}{n-p}\sum\frac{(Y_i-\hat{\mu}_i)^2}{V_i(\hat{\mu}_i)} = \hat{\sigma}^2$$