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(a) This hypothesis test boils down to testing variances in a two-sample problem. The joint density of $X=(X_1,\dots,X_n)$ and $Y=(Y_1,\dots,Y_n)$ is

$$f(x,y) = C(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2)$$

$$\times \exp\left\{-\frac{1}{2\sigma_1^2}\sum X_i^2 - \frac{1}{2\sigma_2^2}\sum Y_i^2 + \frac{n\mu_1}{\sigma_1^2}\bar{X} + \frac{n\mu_2}{\sigma_2^2}\bar{Y}\right\}$$

where $\bar{X} = \frac{1}{n}\sum X_i$, $\bar{Y} = \frac{1}{n}\sum Y_i$.

This is an exponential family with the four parameters

$$\theta = -\frac{1}{2\sigma_2^2}, \quad \xi_1 = -\frac{1}{2\sigma_1^2}, \quad \xi_2 = \frac{n\mu_2}{\sigma_2^2}, \quad \xi_3 = \frac{n\mu_1}{\sigma_1^2}$$

and the sufficient statistics

$$U = \sum Y_i^2, \quad T_1 = \sum X_i^2, \quad T_2 = \bar{Y}, \quad T_3 = \bar{X}.$$

This multiparameter exponential family can be equivalently expressed in (Lemma 4.4.1 of Scheffé, page 123) terms of the parameters

$$\theta^* = -\frac{1}{2\sigma_2^2} + \frac{1}{2\Delta_0\sigma_1^2}, \quad \xi_i^* = \xi_i, \quad i=1,2,3.$$

and the statistics

$U^* = \sum Y_i^2$, $T_1^* = \sum X_i^2 + \frac{1}{\Delta_0} \sum Y_i^2$, $T_2^* = \bar{Y}$, $T_3^* = \bar{X}$.

The hypothesis $H_0: \frac{\sigma_2^2}{\sigma_1^2} = \Delta_0$ is therefore equivalent to $H_0: \theta^* = 0$, and thus our test is

$$\boxed{H_0: \theta^* = 0 \quad \text{vs.} \quad H_1: \theta^* \neq 0}$$

Now we use our theorem on multiparameter exponential families to find the UMPU test. We need to construct a statistic $V^* = h(U^*, T^*)$ which is independent of $T^* = (T_1^*, T_2^*, T_3^*)$ and $h(\cdot)$ needs to be linear in U^*.

Consider the statistic

$$V^* = \frac{\sum (Y_i - \bar{Y})^2 / \Delta_0}{\sum (X_i - \bar{X})^2 + \frac{1}{\Delta_0} \sum (Y_i - \bar{Y})^2}$$

We can write

$$V^* = \frac{(n-1)F}{(n-1) + (n-1)F}, \quad F = \frac{S_Y^2 / \Delta_0}{S_X^2}$$

and

$$S_Y^2 = \frac{1}{n-1}\sum(Y_i-\bar{Y})^2, \quad S_X^2 = \frac{1}{n-1}\sum(X_i-\bar{X})^2.$$

Under H_0, $\sigma_2^2 = \Delta_0\sigma_1^2$, and

$$F \sim F(n-1, n-1) \quad \text{and} \quad V^* \sim Beta\left(\frac{n-1}{2}, \frac{n-1}{2}\right)$$

and thus V^* is independent of T^*. Moreover, V^* is clearly linear in U^* since V^* can be written as

$$V^* = \frac{(U - nT_2^{*2})/\Delta_0}{T_1^* - nT_2^{*2} - nT_3^{*2}}$$

and the coefficient of U is > 0.

Thus, the rejection region of the UMPU test is

$$\boxed{\phi(V^*) = \begin{cases} 1 & V^* < c_1 \text{ or } V^* > c_2 \\ 0 & \text{otherwise} \end{cases}}$$

Where c_1 and c_2 are determined by

$$E_{H_0}(\phi) = \alpha \quad \text{and} \quad E_{H_0}[V^*\phi] = \alpha E_{H_0}(V^*)$$

Now

$$E_{H_0}(\phi) = \alpha \iff$$

$$\alpha = 1 - \int_{c_1}^{c_2} f(v^*)\, dv^*, \quad f(v^*) = \text{Beta}\left(\tfrac{n-1}{2}, \tfrac{n-1}{2}\right) \text{ density}$$

$$\Rightarrow \boxed{1-\alpha = F_{V^*}(c_2) - F_{V^*}(c_1)}$$

where $F_{V^*} = \text{beta}\left(\frac{n-1}{2}, \frac{n-1}{2}\right)$ cdf

also, since $E(V^*) = \dfrac{\frac{n-1}{2}}{2\left(\frac{n-1}{2}\right)} = \frac{1}{2}$

the second equation implies

$$\frac{\alpha}{2} = 1 - \int_{c_1}^{c_2} v^* f(v^*)\, dv^*$$

$$\iff \boxed{\int_{c_1}^{c_2} v^* f(v^*)\, dv^* = 1 - \frac{\alpha}{2}}$$

(b)

It is easily shown that the MLE of

$(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2)$ is $(\bar{X}, \bar{Y}, \hat{\sigma}_1^2, \hat{\sigma}_2^2)$

Where $\hat{\sigma}_1^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$, $\hat{\sigma}_2^2 = \frac{1}{n}\Sigma(Y_i - \bar{Y})^2$.

Under H_0, the MLE of $(\mu_1, \mu_2, \sigma_1^2)$ is

$(\bar{X}, \bar{Y}, \tilde{\sigma}_1^2)$ where

$$\tilde{\sigma}_1^2 = \frac{\Delta_0 \sum_{i=1}^{n}(X_i - \bar{X})^2 + \sum_{i=1}^{n}(Y_i - \bar{Y})^2}{2n\Delta_0}$$

Substituting these into the likelihood ratio, we obtain that the likelihood ratio statistic is

$$\lambda = \left(\frac{1}{1+F}\right)^{n/2}\left(\frac{F}{1+F}\right)^{n/2}$$

where F is defined in part (a).

Under H_0 $F \sim F(n-1, n-1)$. Since λ is unimodal in F, the LR test is equivalent to

one that $\boxed{\text{rejects } H_0 \text{ when } F < c_1 \text{ or } F > c_2}$
for some positive constants $c_1 < c_2$ chosen so that

$P(F < c_1) + P(F > c_2) = \alpha$ under H_0

Note that the UMPU test has the Same test Statistic as the LR test, but different critical values. The UMPU test has the additional requirement of being unbiased, that is, (c_1, c_2) must satisfy

$$P(B < c_1) + P(B > c_2) = \alpha$$

where

$$B \sim Beta\left(\frac{n-1}{2}, \frac{n-1}{2}\right).$$

(d) The sample correlation coefficient R is given by

$$R = \frac{\Sigma(X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\Sigma(X_i - \bar{X})^2 \, \Sigma(Y_i - \bar{Y})^2}}$$

$$\lambda = \frac{\sup L(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2)}{\sup L(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \sigma_{12})}$$

$=$

$$= \frac{(2\pi)^{-n} |\hat{\Sigma}_0|^{-1/2} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}(Z_i-\hat{\mu})'\hat{\Sigma}_0^{-1}(Z_i-\hat{\mu})\right\}}{(2\pi)^{-n} |\hat{\Sigma}|^{-1/2} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}(Z_i-\hat{\mu})\hat{\Sigma}^{-1}(Z_i-\hat{\mu})\right\}}$$

Where $Z_i = (X_i, Y_i)$

$\hat{\mu} = (\bar{X}, \bar{Y})$

$\hat{\sigma}_1^2 = \frac{1}{n}\Sigma(X_i-\bar{X})^2, \quad \hat{\sigma}_2^2 = \frac{1}{n}\Sigma(Y_i-\bar{Y})^2$

$$\hat{\Sigma}_0 = \begin{pmatrix} \hat{\sigma}_1^2 & 0 \\ 0 & \hat{\sigma}_2^2 \end{pmatrix}$$

$$\hat{\Sigma} = \begin{pmatrix} \hat{\sigma}_1^2 & \hat{\sigma}_{12} \\ \hat{\sigma}_{12} & \hat{\sigma}_2^2 \end{pmatrix}, \quad \hat{\sigma}_{12} = \frac{\sum_{i=1}^{n}(X_i-\bar{X})(Y_i-\bar{Y})}{n-1}$$

Thus, we reject Ho When

$$\lambda = \frac{(\hat{\sigma}_1^2\hat{\sigma}_2^2)^{-n/2}\exp\{-n\}}{(\hat{\sigma}_1^2\hat{\sigma}_2^2-\hat{\sigma}_{12}^2)^{-n/2}\exp\{-n\}} < C_1$$

$$\lambda_1 < c_1 \iff \left[\frac{\hat{\sigma}_1^2\hat{\sigma}_2^2 - \hat{\sigma}_{12}^2}{\hat{\sigma}_1^2\hat{\sigma}_2^2}\right]^{n/2} < c_1$$

$$\iff 1 - \frac{\hat{\sigma}_{12}^2}{\hat{\sigma}_1^2\hat{\sigma}_2^2} < c_2$$

$$\iff R^2 > c_3, \quad R^2 = \frac{\hat{\sigma}_{12}^2}{\hat{\sigma}_1^2\hat{\sigma}_2^2}$$

$$\iff \boxed{|R| > c}$$

Now c is chosen so that under H_0, $(\rho = 0)$

$$P(|R| > c) = \alpha$$

(ii) Without loss of generality, assume $\mu_1 = \mu_2 = 0$ and $\sigma_1^2 = \sigma_2^2 = 1$. Let

$$X = (X_1, \ldots, X_n)', \quad Y = (Y_1, \ldots, Y_n)'$$

Let A_Y be an n-vector whose i^{th} component is

$$\frac{Y_i - \bar{Y}}{\sqrt{n-1}\, S_Y}, \quad S_Y = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2}$$

Note that $(n-1)S_X^2 - (A_Y' X)^2 = X' B_Y X$

where $B_Y = I_n - \frac{1}{n}JJ' - A_Y A_Y'$, where I_n is the $n \times n$ identity matrix and J_n is the n-vector of ones. Since $A_Y'A_Y = 1$ and $J'A_Y = 0$, $B_Y A_Y = 0$, $B_Y^2 = B_Y$ and $tr(B_Y) = n-2$. Consequently, when Y is considered to be a fixed vector, $X'B_Y X$ and $A_Y'X$ are independent, $A_Y'X \sim N(0,1)$, $X'B_Y X \sim \chi^2_{n-2}$ and $\dfrac{\sqrt{n-2}\, A_Y'X}{\sqrt{X'B_Y X}} \sim t_{n-2}$

Since $R = \dfrac{A_Y'X}{\sqrt{n-1}\, S_X}$, we have

$$P(R \le r) = E\left[P(R \le r \mid Y)\right]$$

$$= E\left[P\left(\frac{A_Y'X}{\sqrt{X'B_Y X + (A_Y'X)^2}} \le r \;\middle|\; Y\right)\right]$$

$$= E\left[P\left(\frac{A_Y'X}{\sqrt{X'B_Y X}} \le \frac{r}{\sqrt{1-r^2}} \;\middle|\; Y\right)\right]$$

$$= E\left[P\left(t_{n-2} \le \frac{r\sqrt{n-2}}{\sqrt{1-r^2}}\right)\right]$$

$$= \frac{\Gamma\left(\frac{n-1}{2}\right)}{\left[(n-2)\pi\, \Gamma\left(\frac{n-2}{2}\right)\right]^{1/2}} \int_0^{\frac{r\sqrt{n-2}}{\sqrt{1-r^2}}} \left(1 + \frac{u^2}{n-2}\right)^{-\frac{(n-1)}{2}} du$$

where t_{n-2} denotes a random variable having the t-distribution with $n-2$ df.

The third equality follows from the fact that

$$\frac{a}{\sqrt{a^2+b^2}} \le r \text{ iff } \frac{a}{\sqrt{b^2}} \le \frac{r}{\sqrt{1-r^2}} \text{ for}$$

real numbers a and b and $r \in (0,1)$.

Thus R has the density

$$f(r) = \frac{d}{dr} P(R \le r) = \frac{\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-2}{2}\right)} (1-r^2)^{\frac{n-4}{2}},$$

for $-1 \le r \le 1$ and 0 otherwise.

Thus c is determined such that

$$\boxed{\int_{-c}^{c} f(r)\,dr = 1-\alpha}$$

(iii) let $T = \sqrt{n}\, R$. It is easily seen that the density of T is given by

$$f(t) = \frac{\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{n\pi}\;\Gamma\left(\frac{n-2}{2}\right)} \left(1 - \frac{t^2}{n}\right)^{\frac{(n-4)}{2}} I(-\sqrt{n} \le t \le \sqrt{n})$$

Where $I(\cdot)$ denotes the indicator function

Now $\frac{\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{n\pi}\;\Gamma\left(\frac{n-2}{2}\right)} \to \frac{1}{\sqrt{2\pi}}$ as $n \to \infty$ by

Stirlings formula and thus

$$\lim_{n \to \infty} f(t) = \frac{1}{\sqrt{2\pi}} \cdot e^{-t^2/2}\; I(-\infty < t < \infty)$$

thus $\sqrt{n}\, R \xrightarrow{d} N(0,1)$

as $n \to \infty$.