a) $f(x,y \mid \alpha, \beta) = c(\alpha,\beta)\exp(-\alpha x - \beta y)\sum_{j=0}^{\infty} \frac{x^j y^j}{(j!)^2}$

$= \exp\{\sum_{k=1}^{2} \eta_k(\theta) T_k(x,y) + \log(c(\alpha,\beta))\} \underbrace{\sum_{j=0}^{\infty} \frac{x^j y^j}{(j!)^2}}_{h(x,y)}$

$\eta_1(\theta) = \alpha$

$\eta_2(\theta) = \beta$

$T_1(x,y) = -x$

$T_2(x,y) = -y$

rank = 2 ✓

$1 = \int_0^\infty \int_0^\infty c(\alpha,\beta)\exp(-\alpha x - \beta y)\sum_{j=0}^{\infty}\frac{x^j y^j}{(j!)^2}\, dy\,dx$

$\frac{1}{c(\alpha,\beta)} = \int_0^\infty \exp(-\alpha x)\int_0^\infty \exp(-\beta y) \lim_{n\to\infty}\sum_{j=0}^{n}\frac{x^j y^j}{(j!)^2}\,dy\,dx$

$= \lim_{n\to\infty}\int_0^\infty \exp(-\alpha x)\int_0^\infty \exp(-\beta y)\sum_{j=0}^{n}\frac{x^j y^j}{(j!)^2}\,dy\,dx$ $\boxed{\text{(MCT)}}$

$= \lim_{n\to\infty}\sum_{j=0}^{n}\int_0^\infty \frac{\exp(-\alpha x)x^j}{j!}\int_0^\infty \frac{\exp(-\beta y)\, y^j}{j!}\,dy\,dx$

$= \lim_{n\to\infty}\sum_{j=0}^{n}\frac{1}{(j!)^2}\int_0^\infty \exp(-\alpha x)\,x^j\left[\int_0^\infty \exp(-\beta y)\,y^j\,dy\right]dx$

let $u = \beta y$
$du = \beta\, dy$

$\frac{\beta^{-j}}{\beta}\underbrace{\int_0^\infty \exp(-u)\,u^j\,du}_{\Gamma(j+1)}$

$= \lim_{n\to\infty}\sum_{j=0}^{n}\frac{1}{(j!)^2}\,\alpha^{-(j+1)}\beta^{-(j+1)}\frac{(\Gamma(j+1))^2}{(j!)^2}$

$= \lim_{n\to\infty}\sum_{j=0}^{n}(\alpha\beta)^{-(j+1)} = \sum_{j=0}^{\infty}(\alpha\beta)^{-(j+1)} = \sum_{j=0}^{\infty}\left(\frac{1}{\alpha\beta}\right)^{j+1}$ $= \boxed{\frac{1}{\alpha\beta - 1}}$

$= \left(\frac{1}{\alpha\beta}\right)\sum_{j=0}^{\infty}\left(\frac{1}{\alpha\beta}\right)^j$

$= \left(\frac{1}{\alpha\beta}\right)\left[\frac{1}{1-\frac{1}{\alpha\beta}}\right]$

Thus $\boxed{c(\alpha,\beta) = \alpha\beta - 1}$ where $\alpha > 0$, $\beta > 0$ for gamma function.

$* \ \frac{1}{\alpha\beta} < 1$

b)

$$f_X(x) = \int_0^\infty f(x,y|\alpha,\beta)\,dy$$

$$= (\alpha\beta-1)\exp(-\alpha x)\int_0^\infty \exp(-\beta y)\sum_{j=0}^{\infty}\frac{x^j y^j}{(j!)^2}\,dy$$

$$= (\alpha\beta-1)\exp(-\alpha x)\int_0^\infty \exp(-\beta y)\lim_{n\to\infty}\sum_{j=0}^{n}\frac{x^j y^j}{(j!)^2}\,dy$$

$$= (\alpha\beta-1)\exp(-\alpha x)\lim_{n\to\infty}\int_0^\infty \exp(-\beta y)\sum_{j=0}^{n}\frac{x^j y^j}{(j!)^2}\,dy \quad \boxed{(MCT)}$$

$$= (\alpha\beta-1)\exp(-\alpha x)\lim_{n\to\infty}\sum_{j=0}^{n}\int_0^\infty \exp(-\beta y)\frac{x^j y^j}{(j!)^2}\,dy$$

$$= (\alpha\beta-1)\exp(-\alpha x)\lim_{n\to\infty}\sum_{j=0}^{n}\frac{x^j}{(j!)^2}\left[\int_0^\infty \exp(-\beta y)\,y^j\,dy\right]$$

let $u = \beta y$

$du = \beta\,dy$

$$\frac{\beta^{-j}}{\beta}\int_0^\infty \underbrace{\exp(-u)\,u^j\,dy}_{\Gamma(j+1)}$$

$$= (\alpha\beta\cdot 1)\exp(-\alpha x)\lim_{n\to\infty}\sum_{j=0}^{n}\frac{x^j}{(j!)^2}\beta^{-(j+1)} \qquad \cancel{\frac{\Gamma(j+1)}{j!}}$$

$$= \frac{(\alpha\beta-1)\exp(-\alpha x)}{\beta}\underbrace{\sum_{j=0}^{\infty}\frac{(x/\beta)^j}{j!}}_{e^{x/\beta}}$$

$$= \frac{\alpha\beta-1}{\beta}\exp\left(x\left[-\alpha+\frac{1}{\beta}\right]\right)$$

$$= \left[\alpha-\frac{1}{\beta}\right]\exp\left(x\left[-\left(\alpha-\frac{1}{\beta}\right)\right]\right)$$

$$\Rightarrow X \sim \mathrm{Exp}\left(\frac{1}{\alpha-\frac{1}{\beta}}\right)$$

$$\Rightarrow E[X] = \frac{1}{\alpha-\frac{1}{\beta}} = \cancel{\frac{\alpha\beta-1}{\beta}}\ \frac{1}{\frac{\alpha\beta-1}{\beta}} = \boxed{\frac{\beta}{\alpha\beta-1}}$$

c) $f(x,y\mid\alpha,\beta) = \exp\left\{-\alpha x - \beta y - \log\left(\frac{1}{c(\alpha,\beta)}\right)\right\}\sum_{j=0}^{\infty}\frac{x^j y^j}{(j!)^2}$

$\theta = \begin{pmatrix}-\alpha\\-\beta\end{pmatrix} \quad x = \begin{pmatrix}x\\y\end{pmatrix}$

$b(\theta) = \log\left(\frac{1}{c(\alpha,\beta)}\right) = \log(s) \qquad t = \begin{pmatrix}t_1\\t_2\end{pmatrix}$

$\Rightarrow \psi_{X,Y}(t_1,t_2) = \exp\{b(\theta+t) - b(\theta)\}$

$= \exp\{\log(s(-\alpha+t_1, -\beta+t_2)) - \log(s(-\alpha,-\beta))\}$

$= s(-\alpha+t_1, -\beta+t_2)\, s^{-1}$

$= \dfrac{s^{-1}}{(t_1-\alpha)(t_2-\beta)-1}$

$= s^{-1}((t_1-\alpha)(t_2-\beta)-1)^{-1}$

$s(-\alpha,-\beta) = \dfrac{1}{c(\alpha,-\beta)} = \dfrac{1}{\alpha\beta-1} = s(\alpha,\beta)$

$E[X^jY^k] = \left.\dfrac{\partial^{j+k}\psi_{X,Y}(t_1,t_2)}{\partial t_1^j\,\partial t_2^k}\right|_{t_1=0,t_2=0}$

$\dfrac{\partial \psi_{X,Y}(t_1,t_2)}{\partial t_1} = -s^{-1}((t_1-\alpha)(t_2-\beta)-1)^{-2}(t_2-\beta)$

$\dfrac{\partial^2 \psi_{X,Y}(t_1,t_2)}{\partial t_1^2} = 2\, s^{-1}((t_1-\alpha)(t_2-\beta)-1)^{-3}(t_2-\beta)^2$

$\vdots$

$\dfrac{\partial^j \psi_{X,Y}(t_1,t_2)}{\partial t_1^j} = j!\,(-1)^j s^{-1}((t_1-\alpha)(t_2-\beta)-1)^{-(j+1)}(t_2-\beta)^j$

$\dfrac{\partial^{j+1} \psi_{X,Y}(t_1,t_2)}{\partial t_1^j\,\partial t_2} = j!\,(-1)^j s^{-1}(-(j+1))((t_1-\alpha)(t_2-\beta)-1)^{-(j+2)}(t_2-\beta)^j(t_1-\alpha)$

$\vdots$

$\dfrac{\partial^{j+k} \psi_{X,Y}(t_1,t_2)}{\partial t_1^j\,\partial t_2^k} = (j+k)!\,(-1)^{j+k} s^{-1}((t_1-\alpha)(t_2-\beta)-1)^{-(j+k+1)}(t_2-\beta)^j(t_1-\alpha)^k$

$\Rightarrow E[X^jY^k] = (j+k)!\,(-1)^{j+k}s^{-1}(\underbrace{(-\alpha)(-\beta)}_{\alpha\beta}-1)^{-(j+k+1)}(-\beta)^j(-\alpha)^k$

$= (j+k)!\,(-1)^{j+k}s^{-1}(\alpha\beta-1)^{-(j+k+1)}\beta^j\alpha^k(-1)^{j+k}$

$= (j+k)!\,s^{-1}(\alpha\beta-1)^{-(j+k+1)}\beta^j\alpha^k$

c) continued

Now $S = (\alpha\beta - 1)^{-1}$

$$\frac{\partial s}{\partial \alpha} = -(\alpha\beta - 1)^{-2}(\beta)$$

$\vdots$

$$\frac{\partial^j s}{\partial \alpha^j} = (-1)^j (\alpha\beta - 1)^{-(j+1)} (j!) (\beta)^j$$

$$\frac{\partial^{j+1} s}{\partial \alpha^j \partial \beta} = (-1)^j (j!)(-(j+1)) (\alpha\beta - 1)^{-(j+2)} \beta^j \alpha$$

$$\frac{\partial^{j+2} s}{\partial \alpha^j \partial \beta^2} = (-1)^j j! (-(j+1))(-(j+2)) (\alpha\beta - 1)^{-(j+3)} \beta^j \alpha^2$$

$\vdots$

$$\frac{\partial^{j+k} s}{\partial \alpha^j \partial \beta^k} = (-1)^j \cancel{j!} (-1)^k \frac{(j+k)!}{\cancel{j!}} (\alpha\beta - 1)^{-(j+k+1)} \beta^j \alpha^k$$

$$= (-1)^{j+k} (j+k)! (\alpha\beta - 1)^{-(j+k+1)} \alpha^k \beta^j$$

which is equal to $E[X^j Y^k]$ except for $(-1)^{j+k}$ term & S^{-1} term ✓

c) continued

$$\frac{\partial^2}{\partial t_1^2}\left[\left((\alpha+t_1)(\beta+t_2)-1\right)^{-1}\right] = 2\left((\alpha+t_1)(\beta+t_2)-1\right)^{-3}(\beta+t_2)^2$$

$$\vdots$$

$$\frac{\partial^j}{\partial t_1^j}\left[\left((\alpha+t_1)(\beta+t_2)-1\right)^{-1}\right] = (-1)^{j+k}(j+k)!\left((\alpha+t_1)(\beta+t_2)-1\right)^{-(j+k+1)}(\beta+t_2)^j(\alpha+t_1)^k$$

$$\frac{\partial^{j+k}}{\partial_{t_1}^j \partial_{t_2}^k} S(\alpha+t_1, \beta+t_2)$$

$$\Rightarrow E[X^j Y^k] = S^{-1}(-1)^{j+k}(j+k)!(\alpha\beta-1)^{-(j+k+1)}\beta^j\alpha^k$$

Now

$$\frac{\partial^{j+k} S}{\partial\alpha^j\partial\beta^k} = \frac{\partial^{j+k}}{\partial\alpha^j\partial\beta^k}(\alpha\beta-1)^{-1}$$

$$= (-1)^{j+k}(\alpha\beta-1)^{-(j+k+1)}(j+k)!\,\alpha^k\beta^j \quad \checkmark$$

$$\Rightarrow E[X^j Y^k] = (-1)^{j+k} S^{-1}\frac{\partial^{j+k} S}{\partial\alpha^j\beta^k}.$$

d)

$$f_{Y|X} = \frac{f_{X,Y}}{f_X} = \frac{(\alpha\beta-1)\exp(-\alpha x-\beta y)\sum_{j=0}^{\infty}\frac{x^j y^j}{(j!)^2}}{(\alpha-\frac{1}{\beta})\exp\left(x\left[-(\alpha-\frac{1}{\beta})\right]\right)}$$

$$= \frac{(\alpha\beta-1)}{\frac{\alpha\beta-1}{\beta}}\exp\left\{-\alpha x-\beta y+\alpha x-\frac{1}{\beta}x\right\}\sum_{j=0}^{\infty}\frac{x^j y^j}{(j!)^2}$$

depends on β ✓
but free of α.

$$= \beta\exp\left\{-\beta y-\frac{1}{\beta}x\right\}\sum_{j=0}^{\infty}\frac{x^j y^j}{(j!)^2}$$

$$= \beta\exp\left\{-\beta y-\frac{1}{\beta}x\right\}\left\{\sum_{j=0}^{\infty}\frac{x^j}{j!}\right\}\left\{\sum_{j=0}^{\infty}\frac{y^j}{j!}\right\} \text{ independent}$$

$$= \beta\exp\left\{-\beta y-\frac{1}{\beta}x\right\} e^x e^y$$

d) continued

$$\left[\begin{array}{l} \text{CLT} \\ \bar{Y}|\bar{X}=\bar{x} \xrightarrow{d} N(\mu, \sigma^2/n) \end{array}\right]$$

$$\mu = E[Y|X]$$

$$= \int_0^\infty y\,\beta \exp\left(-\beta y - \tfrac{1}{\beta}x\right) \sum_{j=0}^{\infty} \frac{x^j y^j}{(j!)^2}\, dy$$

$$= \beta \exp\left(-\tfrac{1}{\beta}x\right) \int_0^\infty y \exp(-\beta y) \sum_{j=0}^{\infty} \frac{x^j y^j}{(j!)^2}\, dy$$

$$= \beta \exp\left(-\tfrac{1}{\beta}x\right) \int_0^\infty y \exp(-\beta y) \lim_{n\to\infty} \sum_{j=0}^{n} \frac{x^j y^j}{(j!)^2}\, dy$$

$$= \beta \exp\left(-\tfrac{1}{\beta}x\right) \lim_{n\to\infty} \int_0^\infty y \exp(-\beta y) \sum_{j=0}^{n} \frac{x^j y^j}{(j!)^2}\, dy \qquad \text{(MCT)}$$

$$= \beta \exp\left(-\tfrac{1}{\beta}x\right) \lim_{n\to\infty} \sum_{j=0}^{n} \frac{x^j}{(j!)^2} \underbrace{\left[\int_0^\infty y^{j+1} \exp(-\beta y)\, dy\right]}_{\substack{\text{let } u = \beta y \\ du = \beta dy}}$$

$$\frac{\beta^{-(j+1)}}{\beta} \underbrace{\int_0^\infty \exp(-u)\, u^{j+1}\, du}_{\Gamma(j+2)}$$

$$= \beta \exp\left(-\tfrac{1}{\beta}x\right) \lim_{n\to\infty} \sum_{j=0}^{n} \frac{x^j}{(j!)^2} \beta^{-(j+2)} (j+1)!$$

=

$$= \exp\left(-\tfrac{x}{\beta}\right) \lim_{n\to\infty} \sum_{j=0}^{n} \frac{x^j \beta^{-(j+1)} (j+1)}{j!}$$

$$= \beta^{-1} \exp\left(-\tfrac{x}{\beta}\right) \lim_{n\to\infty} \left[\sum_{j=0}^{n} \frac{x^j \beta^{-j} \cancel{j}}{\cancel{j}\,(j-1)!} + \sum_{j=0}^{n} \frac{x^j \beta^{-j}}{j!} \right]$$

$$= \beta^{-1} \exp\left(-\tfrac{x}{\beta}\right) \lim_{n\to\infty} \left[\underbrace{\frac{x}{\beta} \sum_{j=0}^{n} \frac{(x/\beta)^j}{j!}}_{\text{change of variables}} + \sum_{j=0}^{n} \frac{(x/\beta)^j}{j!} \right]$$

$$= \beta^{-1} \exp\left(-\tfrac{x}{\beta}\right) \left[\tfrac{x}{\beta} e^{x/\beta} + e^{x/\beta} \right] = \beta^{-1}\left[\tfrac{x}{\beta} + 1\right] = \boxed{\frac{x}{\beta^2} + \frac{1}{\beta}}$$

d) continued

$$\sigma^2 = \text{var}(Y|x) = E[Y^2|x] - (E[Y|x])^2$$

$$E[Y^2|x] = \int_0^\infty y^2 \beta \exp(-\beta y - \tfrac{1}{\beta}x) \sum_{j=0}^{\infty} \frac{x^j y^j}{(j!)^2}\, dy$$

$$= \beta \exp(-\tfrac{1}{\beta}x) \lim_{n\to\infty} \sum_{j=0}^{n} \frac{x^j}{(j!)^2} \left[\underbrace{\int_0^\infty y^{j+2} \exp(-\beta y)\, dy}_{\frac{\beta^{-(j+2)}}{\beta} \underbrace{\int_0^\infty \exp(-u) u^{j+2}\, du}_{\Gamma(j+3)}} \right]$$

let $u = \beta y$

$du = \beta dy$

$$= \beta \exp(-\tfrac{1}{\beta}x) \lim_{n\to\infty} \sum_{j=0}^{n} \frac{x^j}{(j!)^2} \beta^{-(j+3)} (j+2)!$$

$$= \exp(-x/\beta) \lim_{n\to\infty} \sum_{j=0}^{n} \frac{x^j \beta^{-(j+2)} (j+2)(j+1)}{j!}$$

$(j+2)(j+1) \rightarrow j^2 + 3j + 2$

$$= \beta^{-2} \exp(-x/\beta) \lim_{n\to\infty} \left[\sum_{j=0}^{n} \frac{x^j \beta^{-j} j^2}{j!} + \sum_{j=0}^{n} \frac{x^j \beta^{-j} (3j)}{j!} + \sum_{j=0}^{n} \frac{x^j \beta^{-j} (2)}{j!} \right]$$

$$= \beta^{-2} \exp(-x/\beta) \left[\sum_{j=0}^{\infty} \frac{(x/\beta)^j j^2}{j!} + 3\left(\frac{x}{\beta} e^{x/\beta}\right) + 2 e^{x/\beta} \right]$$

$$\downarrow$$

$$\sum_{j=0}^{\infty} \frac{(x/\beta)^j j}{(j-1)!}$$

$$\sum_{\ell=0}^{\infty} \frac{(x/\beta)^{\ell+1}}{\ell!} (\ell+1)$$

$$= \frac{x}{\beta} \left[\sum_{\ell=0}^{\infty} \frac{(x/\beta)^\ell \ell}{\ell!} + \sum_{\ell=0}^{\infty} \frac{(x/\beta)^\ell}{\ell!} \right]$$

$$\left[\frac{x}{\beta} e^{x/\beta} \quad + \quad e^{x/\beta} \right]$$

$$\Rightarrow E[Y^2|x] = \beta^{-2} e^{-x/\beta} \left[\frac{x}{\beta} e^{x/\beta} \left[\frac{x}{\beta} + 1\right] + 3\frac{x}{\beta} e^{x/\beta} + 2 e^{x/\beta} \right]$$

$\Rightarrow$ Replace x by $\bar{x}$ & have distribution.

e) $H_0: \beta = 2$ vs. $H_1: \beta > 2$

$$f(x, y \mid \alpha, \beta) = \prod_{i=1}^{n} c(\alpha, \beta) \exp(-\alpha x_i - \beta y_i) \sum_{j=0}^{\infty} \frac{x^j y^j}{(j!)^2}$$

$$= [c(\alpha, \beta)]^{n} \exp\left(-\alpha \sum_{i=1}^{n} x_i - \beta \sum_{i=1}^{n} y_i\right) \times \prod_{i=1}^{n} \left\{ \sum_{j=0}^{\infty} \frac{x^j y^j}{(j!)^2} \right\}$$

$\Rightarrow$ $\theta = \beta$

$u(x) = -\sum_{i=1}^{n} y_i$

$\Sigma = \alpha$

$T(x) = -\sum_{i=1}^{n} x_i$

$$\Rightarrow \quad \phi(x) = \begin{cases} 1 & U \geq c(t) \\ 0 & U < c(t) \end{cases}$$

where

$$\alpha = E_{H_0}[\phi(u) \mid T = t]$$

$$= P_{H_0}(U \geq c(t) \mid T = t)$$

$$= P_{H_0}\left(-\sum_{i=1}^{n} Y_i \geq c(t) \,\Big|\, -\sum_{i=1}^{n} X_i = t\right)$$

$$= P_{H_0}\left(\sum_{i=1}^{n} Y_i \leq -c(t) \,\Big|\, \sum_{i=1}^{n} X_i = -t\right)$$

$-\sum y_i \leq c_1(t)$ or $-\sum y_i \geq c_2(t)$

$\Rightarrow \sum y_i \geq c_1(t)$ or $\sum y_i \leq c_2(t)$

f) $H_0: \beta = 2$ $H_1: \beta \neq 2$

$c_1(t) \leq \sum y_i \leq c_2(t)$

$$\phi(x) = \begin{cases} 1 & U \leq c_1(t) \text{ or } U \geq c_2(t) \\ 0 & \text{o.w.} \end{cases}$$

$\alpha = E_{\beta=2}[\phi(U) \mid T = t]$ & $\alpha E_{\beta=2}[U \mid T = t] = E_{\beta=2}[U \phi(U) \mid T = t]$

9) $H_0: \beta = 2$

$$L(\alpha,\beta) = \prod_{i=1}^{n} (\alpha\beta-1)\exp(-\alpha x_i - \beta y_i) \sum_{j=0}^{\infty} \frac{x_i^j y_i^j}{(j!)^2}$$

$$= (\alpha\beta-1)^n \exp\left(-\alpha\sum x_i - \beta\sum y_i\right) \prod_{i=1}^{n} \left\{ \sum_{j=0}^{\infty} \frac{x_i^j y_i^j}{(j!)^2} \right\}$$

$$\Rightarrow \ell(\alpha,\beta) \propto n\log(\alpha\beta-1) - \alpha\sum x_i - \beta\sum y_i$$

$$\Rightarrow \frac{\partial \ell(\alpha,\beta)}{\partial\alpha} = n\frac{\beta}{\alpha\beta-1} - \sum x_i \cancel{\overset{set}{=} 0} = n\beta(\alpha\beta-1)^{-1} - \sum x_i$$

$$\frac{\partial \ell(\alpha,\beta)}{\partial\beta} = n\frac{\alpha}{\alpha\beta-1} - \sum y_i \cancel{\overset{set}{=} 0}$$

$$\Rightarrow \frac{\beta}{\alpha\beta-1} = \bar{x} \qquad \frac{\alpha}{\alpha\beta-1} = \bar{y}$$

$$\Rightarrow \beta = \frac{-\bar{x}}{1-\alpha\bar{y}} \qquad \Rightarrow \alpha = \alpha\beta\bar{y} - \bar{y}$$

$$\Rightarrow \alpha(1-\beta\bar{y}) = -\bar{y}$$

$$\Rightarrow \alpha = \frac{-\bar{y}}{1-\beta\bar{y}}$$

$$\Rightarrow \hat{\beta} = \frac{-\bar{x}}{1 + \frac{\bar{y}}{1-\hat{\beta}\bar{y}}\bar{y}}$$

$$\Rightarrow \hat{\beta} + \frac{\hat{\beta}\bar{y}^2}{1-\hat{\beta}\bar{y}} = -\bar{x}$$

Under $H_0: \beta = 2$

$$\Rightarrow \frac{n\,2}{2\alpha-1} = \sum x_i$$

$$\Rightarrow 2 = \bar{x}(2\alpha-1)$$

$$\Rightarrow \frac{2}{\bar{x}} + 1 = 2\alpha \Rightarrow \boxed{\frac{1}{\bar{x}} + \frac{1}{2} = \hat{\alpha}}$$

g) continued

$$\frac{\partial^2 \ell(\alpha,\beta)}{\partial \alpha^2} = -n\beta(\alpha\beta-1)^{-2}(\beta) = \frac{-n\beta^2}{(\alpha\beta-1)^2}$$

$$\frac{\partial^2 \ell(\alpha,\beta)}{\partial \beta^2} = \frac{-n\alpha^2}{(\alpha\beta-1)^2}$$

$$\frac{\partial^2 \ell(\alpha,\beta)}{\partial \alpha\partial \beta} = n\frac{(\alpha\beta-1)(1)-\beta(\alpha)}{(\alpha\beta-1)^2} = n\frac{\alpha\beta-1-\alpha\beta}{(\alpha\beta-1)^2} = \frac{-n}{(\alpha\beta-1)^2}$$

$$\Rightarrow I_n(\alpha,\beta) = \begin{pmatrix} \frac{n\beta^2}{(\alpha\beta-1)^2} & \frac{n}{(\alpha\beta-1)^2} \\ \frac{n}{(\alpha\beta-1)^2} & \frac{n\alpha^2}{(\alpha\beta-1)^2} \end{pmatrix}$$

$$\Rightarrow R_n = \begin{bmatrix} \frac{2n}{2\tilde{\alpha}-1} - \Sigma x_i \\ \frac{\tilde{\alpha} n}{2\tilde{\alpha}-1} - \Sigma y_i \end{bmatrix}^T \begin{pmatrix} \frac{4n}{(2\tilde{\alpha}-1)^2} & \frac{n}{(2\tilde{\alpha}-1)^2} \\ \frac{n}{(2\tilde{\alpha}-1)^2} & \frac{\tilde{\alpha}^2 n}{(2\tilde{\alpha}-1)^2} \end{pmatrix}^{-1} \begin{bmatrix} \frac{2n}{2\tilde{\alpha}-1} - \Sigma x_i \\ \frac{\tilde{\alpha} n}{2\tilde{\alpha}-1} - \Sigma y_i \end{bmatrix}$$

Now $\frac{2n}{2\tilde{\alpha}-1} - \Sigma x_i = \frac{2n}{\frac{2}{\bar{x}}+1-1} - \Sigma x_i$

$= n\bar{x} - \Sigma x_i = 0.$

$$\Rightarrow R_n = \begin{pmatrix} 0 \\ \frac{\tilde{\alpha} n}{2\tilde{\alpha}-1} - \Sigma y_i \end{pmatrix}^T \begin{pmatrix} \frac{4n}{(2\tilde{\alpha}-1)^2} & \frac{n}{(2\tilde{\alpha}-1)^2} \\ \frac{n}{(2\tilde{\alpha}-1)^2} & \frac{\tilde{\alpha}^2 n}{(2\tilde{\alpha}-1)^2} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \frac{\tilde{\alpha} n}{2\tilde{\alpha}-1} - \Sigma y_i \end{pmatrix}$$

g) continued

$$\begin{pmatrix} \frac{4n}{(2\tilde{\alpha}-1)^2} & \frac{n}{(2\tilde{\alpha}-1)^2} \\ \frac{n}{(2\tilde{\alpha}-1)^2} & \frac{\tilde{\alpha}^2 n}{(2\tilde{\alpha}-1)^2} \end{pmatrix}^{-1} = \frac{1}{\frac{4n^2\tilde{\alpha}^2}{(2\tilde{\alpha}-1)^4} - n^2} \begin{pmatrix} \frac{\tilde{\alpha}^2 n}{(2\tilde{\alpha}-1)^2} & \frac{-n}{(2\tilde{\alpha}-1)^2} \\ \frac{-n}{(2\tilde{\alpha}-1)^2} & \frac{4n}{(2\tilde{\alpha}-1)^2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\tilde{\alpha}^2 \cancel{n} (2\tilde{\alpha}-1)^2}{4n^{\cancel{2}}\tilde{\alpha}^2 - n^{\cancel{2}}} & \frac{-\cancel{n}(2\tilde{\alpha}-1)^2}{4n^{\cancel{2}}\tilde{\alpha}^2 - n^{\cancel{2}}} \\ \frac{-\cancel{n}(2\tilde{\alpha}-1)^2}{4n^{\cancel{2}}\tilde{\alpha}^2 - n^{\cancel{2}}} & \frac{4\cancel{n}(2\tilde{\alpha}-1)^2}{4n^{\cancel{2}}\tilde{\alpha}^2 - n^{\cancel{2}}} \end{pmatrix}$$

$$\Rightarrow R_n = \begin{pmatrix} \left(\frac{\tilde{\alpha} n}{2\tilde{\alpha}-1} - \Sigma y_i\right)\left(\frac{(2\tilde{\alpha}-1)^2}{4n\tilde{\alpha}^2 - n}\right) & \left(\frac{\tilde{\alpha} n}{2\tilde{\alpha}-1} - \Sigma y_i\right)\left(\frac{4(2\tilde{\alpha}-1)^2}{4n\tilde{\alpha}^2 - n}\right) \end{pmatrix} \begin{pmatrix} 0 \\ \frac{\tilde{\alpha} n}{2\tilde{\alpha}-1} - \Sigma y_i \end{pmatrix}$$

$$= \left(\frac{\tilde{\alpha} n}{2\tilde{\alpha}-1} - \Sigma y_i\right)^2 \frac{4(2\tilde{\alpha}-1)^2}{4n\tilde{\alpha}^2 - n} \qquad \tilde{\alpha} = \frac{1}{\bar{x}} + \frac{1}{2}$$

$$\to \chi^2_1 \quad \text{as} \quad n \to \infty.$$

h) $p(\alpha, \beta \mid x, y) \propto \pi(\alpha, \beta)\, p(x, y \mid \alpha, \beta)$

$$\propto \frac{1}{\alpha\beta} \prod_{i=1}^{n} (\alpha\beta - 1) \exp(-\alpha x_i - \beta y_i) \sum_{j=0}^{\infty} \frac{x^j y^j}{(j!)^2}$$

$$\propto \frac{(\alpha\beta-1)^n}{\alpha\beta} \exp(-\alpha \Sigma x_i - \beta \Sigma y_i)$$