2016 Day 2 Problem 1

a) $E(Y_i) = EE(Y_i|\theta_i) = E[1/\theta_i]$

$= \int_0^\infty \theta_i^{-1} \frac{1}{\Gamma(3) b_i^3} \theta_i^2 e^{-\theta_i/b_i} d\theta_i$

$= \frac{1}{2b_i^3}\int_0^\infty \theta e^{-\theta/b_i} d\theta = \frac{1}{2b_i^3}\int_0^\infty \frac{\Gamma(2)b_i^2}{\Gamma(2)b_i^2}\theta e^{-\theta/b_i} d\theta$

$= \frac{1}{2b_i} = \frac{1}{6}e^{-x_i'\beta}$

$Var\, Y_i = E(Var(Y_i|\theta_i)) + Var(E(Y_i|\theta_i))$

$= E((\frac{1}{\theta_i})^2) + Var(\frac{1}{\theta_i})$

$= E(\theta_i^{-2}) + E(\theta_i^{-2}) - E(\theta_i^{-1})^2$

$= 2E(\theta_i^{-2}) - \frac{1}{36}e^{-2x_i'\beta}$

$E(\theta_i^{-2}) = \int_0^\infty \frac{1}{\Gamma(3)b_i^3}\theta_i^{-2}\theta_i^2 e^{-\theta_i/b_i} d\theta_i$

$= \frac{1}{2b_i^2}\int_0^\infty \frac{1}{b_i} e^{-\theta_i/b_i} d\theta_i = \frac{1}{2b_i^2} = \frac{1}{2}\left(\frac{1}{b_i}\right)^2 = \frac{1}{2}\left(\frac{1}{3}e^{-x_i'\beta}\right)^2$

$= \frac{1}{18}e^{-2x_i'\beta}$

$Var\, Y_i = \left(\frac{1}{9} - \frac{1}{36}\right)e^{-2x_i'\beta} = \frac{1}{12}e^{-2x_i'\beta}$

$b = ae^{x_i'\beta} = 3e^{x_i'\beta}$

$ab^2 = 3b^2$
$= 9e^{2x_i'\beta}$

b) $f(y) = \int_0^\infty f(\theta, y)\, d\theta$

$= \int_0^\infty \theta e^{-y\theta} \frac{1}{\Gamma(3)b^3}\theta^2 e^{-\theta/b} d\theta$

$= \frac{1}{2b^3}\int_0^\infty \theta^3 e^{-\theta(y+1/b)} d\theta = \frac{1}{2b^3}\int_0^\infty \frac{\Gamma(4)(y+1/b)^4}{\Gamma(4)(y+1/b)^4}\theta^3 e^{-\theta(y+1/b)} d\theta$

$= \frac{3}{b^3(y+1/b)^4} = \frac{3}{b^3\frac{1}{b^4}(by+1)^4} = \frac{3b}{(by+1)^4}$

$f(y_i) = \frac{3b_i}{(b_iy_i+1)^4} = \frac{9e^{x_i'\beta}}{(3y_ie^{x_i'\beta}+1)^4}$

c) $\sigma_\tau^2 = I_{\tau\tau} - I_{\tau\beta}I_{\beta\beta}^{-1}I_{\beta\tau}$ $\alpha = (\beta, \tau)$

$SC = \frac{\partial_\tau l_n(\alpha)^2}{\sigma_\tau^2}\Big|_{\tau=0} \sim_{H_0} 0.5\chi_0^2 + 0.5\chi_1^2$

I don't even see τ in l_n??

score-test-example.pdf

formulas: (cooked up) $E(\theta_i) = k(x_i'\beta)$ $Var\,\theta_i = \tau f_i(x_i'\beta)$

$w_{1i} = \ddot{b}(\cdot)$ $w_{2i} = 0.5 f_i \dddot{b}(\cdot)$ $W_1 = diag(w_{1i})$ $W_2 = diag(w_{2i})$

$\partial_\tau l_n(\alpha) = \sum_{i=1}^n \frac{1}{2} f_i\{(y_i-\mu_i)^2 - \ddot{b}(\cdot)\}$

$I_{\beta\tau} = D_\theta(\beta)' W_2 1_n$ $D_\theta(\beta) = \left(\frac{\partial\theta_i}{\partial\beta_j}\right)$

$I_{\tau\tau} = \sum_{i=1}^n \frac{1}{4}f_i^2\{2(\ddot{b}(\cdot))^2 + b^{(4)}(\cdot)\}$ $I_{\beta\beta} = D_\theta(\beta)' W_1 D_\theta(\beta)$

2016 Day 2 Problem 1 part c

Score Test

$E(\theta_i) = a_i b_i = 3b_i = 9e^{x_i'\beta} \quad \frac{a_i}{b_i} = e^{-x_i'\beta} \Rightarrow b_i = 3e^{x_i'\beta}$

$Var\ \theta_i = \tau \exp(x_i'\beta) \quad f_i = e^{x_i'\beta} \rightarrow \theta_i = 6e^{x_i'\beta}$

$\dot{b}(\theta_i) = \mu_i = EY_i = 1/\theta_i = \frac{1}{6}e^{-x_i'\beta} \quad b^{(4)}(\theta_i) = \frac{-6}{\theta_i^4}$

$b(\theta_i) = \log \theta_i \quad \ddot{b}(\theta_i) = -\frac{1}{\theta_i^2} \quad \dddot{b}(\theta_i) = \frac{2}{\theta_i^3}$

$w_{1i} = -\frac{1}{\theta_i^2} = \frac{-1}{36e^{2x_i'\beta}}$

$w_{2i} = +0.5\, e^{x_i'\beta} \frac{2}{216 e^{3x_i'\beta}} = \frac{1}{216} e^{-2x_i'\beta}$

$D_\theta(\beta) = \left(\frac{\partial\theta_i}{\partial\beta_j}\right) \quad \frac{\partial\theta}{\partial\beta} = 6e^{x_i'\beta} x_i' \quad \frac{\partial\theta_i}{\partial\beta_j} = 6e^{x_i'\beta} x_{ij}$

$I_{\tau\tau} = \sum_{i=1}^n \frac{1}{4} e^{2x_i'\beta}\left[2\,\frac{}{36} e^{-2x_i'\beta} - \frac{1}{6^3} e^{-4x_i'\beta}\right]$

$= \sum_{i=1}^n \left(\frac{1}{54} - \frac{1}{4\cdot 216} e^{-2x_i'\beta}\right)$

$\partial_\tau l_n(\alpha) = \sum_{i=1}^n \frac{1}{2} e^{x_i'\beta}\left[\left(y_i - \frac{1}{6}e^{-x_i'\beta}\right)^2 + \frac{1}{36}e^{-2x_i'\beta}\right]$

$= \sum_{i=1}^n \left(\frac{1}{2}e^{x_i'\beta} y_i^2 - \frac{1}{6} y_i + \frac{1}{36} e^{-x_i'\beta}\right)$

questions: is $\mu_i = EY_i = \dot{b}(\theta_i) = \frac{1}{\theta_i} = \frac{1}{6}e^{-x_i'\beta}$ correct?

d) i) score equation: $\cancel{DX}^T(y-\mu)=0$ $D^T V^{-1}$

looked up:

$$0=S_n(\beta)=\sum_{i=1}^{n}\partial_\beta E(y_i)^T\,[\underbrace{Var(y_i)}_{\sigma^2(\mu_i^2+\mu_i)}]^{-1}(y_i-\underbrace{E(y_i)}_{\frac{1}{6}e^{-x_i'\beta}})$$

$$\partial_\beta E(y_i)=\partial_\beta\left[\tfrac{1}{6}e^{-x_i'\beta}\right]=-\tfrac{1}{6}x_i'e^{-x_i'\beta}$$

$$0=\sum_{i=1}^{n}\frac{1}{6}x_i e^{x_i'\beta}\frac{1}{\sigma^2 e^{x_i'\beta}(e^{x_i'\beta}+1)}\left(y_i-\frac{1}{6}e^{-x_i'\beta}\right)$$

$$=\sum_{i=1}^{n}\frac{1}{6\sigma^2}x_i\,\frac{y_i-\frac{1}{6}e^{-x_i'\beta}}{e^{x_i'\beta}+1}$$

moment estimator

$$\hat\sigma^2=\frac{1}{n-p}\sum_{i=1}^{n}\frac{(y_i-\hat\mu_i)^2}{V(\hat\mu_i)}=\frac{1}{n-p}\sum_{i=1}^{n}\frac{(y_i-\hat\mu_i)^2}{\hat\mu_i^2}$$

where $\hat\mu_i=y_i$?.

e) $Cov(\hat\beta_p)=\sigma^2(D'V^{-1}D)^{-1}$

$\hat\beta_p$ solves

? $\frac{1}{\sigma^2}D'V^{-1}(y-\mu)=0$ $\qquad \Sigma=Var(Y_i)$

$\sqrt{n}(\bar Y-\mu)\xrightarrow{d}N(0,\Sigma)$ if Y_i's are iid by CLT

$$\sqrt{n}(\hat\beta_p-\beta)\xrightarrow{d}(D'V^{-1}D)\,N(0,\Sigma)$$

$$Cov(\hat\beta_p)^{-1}=(D'V^{-1})\underbrace{\Sigma}_{\sigma^2 V}(V^{-1}D)'$$
$$=\sigma^2 D'V^{-1}\,V\,V^{-1}D$$
$$=\sigma^2 D'V^{-1}D$$

$\sigma^2 I$ should come in somewhere I think though Σ

$$-E[\partial_\beta S_n(\beta)]=E(S_n(\beta)^{\otimes 2})$$
$$=E[D'V^{-1}(y-\mu)(y-\mu)'V^{-1}D]$$
$$=D'V^{-1}\sigma^2 V V^{-1}D=\sigma^2 D'V^{-1}D$$

$$0=S_n(\hat\beta)\approx S_n(\beta)+\partial_\beta S_n(\beta)(\hat\beta-\beta)$$
$$-[\partial_\beta S_n(\beta)]^{-1}S_n(\beta)\approx\hat\beta-\beta$$
$$\sqrt{n}(\hat\beta-\beta)\approx-\underbrace{\sqrt{n}\tfrac{1}{n}S_n(\beta)}_{\xrightarrow{d}\,N(0,\,E(S_n(\beta)^{\otimes 2}))}\Big[\underbrace{\tfrac{1}{n}\partial_\beta S_n(\beta)}_{\xrightarrow{P}\,E(\partial_\beta S_n(\beta))}\Big]^{-1}\to -E(\partial_\beta S_n(\beta))^{-1}N(0,E(S_n(\beta)^{\otimes 2}))$$

$$Cov(\hat\beta_p)=E(\partial_\beta S_n(\beta))^{-1}$$
$$=\sigma^2(D'V^{-1}D)^{-1}$$