1)

a) (i) $\frac{d}{d\alpha} l_n = \frac{n}{\alpha} + \frac{n}{\alpha+1} - \sum \log(1+x_i+y_i) = 0$

$\frac{1}{\hat{\alpha}_n+1} + \frac{1}{\hat{\alpha}_n} = \frac{1}{n}\sum \log(1+x_i+y_i) = \hat{\mu}_n$

$\Rightarrow g(\hat{\alpha}_n) = \hat{\mu}_n$

$\Rightarrow g^{-1}(\hat{\mu}_n) = \hat{\alpha}_n$

hopefully $E[\log(1+X_1+Y_1)] = g(\alpha)$

(ii) Show $\hat{\alpha}_n \xrightarrow[a.s]{} \alpha_0$

(want to show $\hat{\mu}_n \xrightarrow[a.s]{} g(\alpha)$)

let $Z_1 = \log(1+X_1+Y_1)$ $\Rightarrow$ $X_1 = W_1$, $Y_1 = e^{Z_1} - 1 - W_1$ $\rightarrow W_1 = e^{Z_1} - 1 - Y_1$

$W_1 = X_1$

$\Rightarrow \begin{pmatrix} 0 & 1 \\ e^{Z_1} & -1 \end{pmatrix} = -e^{Z_1}$ $\Rightarrow |-e^{Z_1}| = e^{Z_1}$

$f_{Z_1,W_1} = f(w_1, e^{z_1}-1-w_1)e^{z_1} = \alpha(\alpha+1)(\cancel{1} + \cancel{w_1} + e^{z_1} \cancel{-1} \cancel{-w_1})^{-(\alpha+2)} e^{z_1}$

$= \alpha(\alpha+1)(e^{z_1})^{-(\alpha+2)} \cdot e^{z_1} = \alpha(\alpha+1)e^{-z_1(\alpha+1)}$

$f_{Z_1} = \alpha(\alpha+1)e^{-z_1(\alpha+1)} \int_0^{e^{z_1}-1} 1 \, dw_1 = \alpha(\alpha+1)e^{-z(\alpha+1)}[w_1]_0^{e^{z_1}-1} = \alpha(\alpha+1)e^{-z(\alpha+1)}(e^{z_1}-1)$ $\quad z>0$

$= \alpha(\alpha+1)[e^{-z\alpha} - e^{-z_1(\alpha+1)}]$

$E[Z_1] = \alpha(\alpha+1)\left[\int_0^\infty z_1 e^{-z_1\alpha} dz_1 - \int_0^\infty z_1 e^{-z_1(\alpha+1)} dz_1\right]$

$= \alpha(\alpha+1)\left[\frac{\Gamma(2)}{\alpha^2} \underbrace{\int_0^\infty \frac{\alpha^2}{\Gamma(1)} z_1^{2-1} e^{-z_1\alpha} dz_1}_{Gam(2,\alpha)} - \frac{\Gamma(2)}{(\alpha+1)^2} \underbrace{\int_0^\infty \frac{(\alpha+1)^2}{\Gamma(2)} z_1^{2-1} e^{-z_1(\alpha+1)} dz_1}_{Gam(2,\alpha+1)}\right]$

$= \alpha(\alpha+1)\left[\frac{1}{\alpha^2} - \frac{1}{(\alpha+1)^2}\right] = \frac{\alpha+1}{\alpha} - \frac{\alpha}{\alpha+1} = \frac{\alpha}{\alpha} + \frac{1}{\alpha} - \frac{\alpha}{\alpha+1}$

$= \frac{1}{\alpha} + 1 - \frac{\alpha}{\alpha+1} = \frac{1}{\alpha} + \frac{\alpha+1-\alpha}{\alpha+1} = \frac{1}{\alpha} + \frac{1}{\alpha+1} = g(\alpha) < \infty$

Thus $\hat{\mu}_n \xrightarrow[a.s]{} g(\alpha)$ by SLLN ($E|Z_1| < \infty$)

let $h(x) = g^{-1}(x)$ $\Rightarrow$ h is continuous $\quad h(g(x)) = x$

By CMT $h(g(\hat{\mu}_n)) = \hat{\alpha}_n \xrightarrow[a.s]{} h(g(\alpha_0)) = \alpha_0$

a) (iii) use mle propertes

b) skipped: $f(y_i|X_i) = (\alpha+1)((1+X_i)^{-1})(1+\frac{y_i}{1+X_i})^{-(\alpha+2)}$

c) (i)

$$\ell(\alpha) \propto n\log(\alpha+1) - \alpha \sum \log\left(1+\frac{y_i}{1+X_i}\right)$$

$$\frac{d}{d\alpha}\ell n = \frac{n}{\alpha+1} - \sum \log\left(1+\frac{y_i}{1+X_i}\right) \overset{set}{=} 0$$

$$\Rightarrow \tilde{\alpha}_n = \left[\frac{1}{n}\sum \log\left(1+\frac{y_i}{1+X_i}\right)\right]^{-1} - 1$$

(ii) Show $\tilde{\alpha}_n \to_{a.s} \alpha_0$ $\left\{\text{Consider } U_i = \frac{Y_i}{1+X_i}\right\}$

$$U_i = \frac{Y_i}{1+X_i} \Rightarrow Y_i = U_i(1+X_i) \qquad \frac{d}{dU_i}Y_i = 1+X_i$$

$$f_{U_i|X_i} = f_{U_i} = f_{Y_i|X_i}(U_i(1+X_i)) = (\alpha+1)\cancel{(1+X_i)}^{-1}\left(1+\frac{U_i\cancel{(1+X_i)}}{\cancel{(1+X_i)}}\right)^{-(\alpha+2)}\cancel{(1+X_i)}$$

$$= (\alpha+1)(1+U_i)^{-(\alpha+2)} \qquad U_i \geq 0 \quad \boxed{\Rightarrow * \ U_i \text{ iid} *}$$

let $W_i = \log(1+U_i)$

$dV_i = (1+U_i)^{-(\alpha+2)}dU_i$

$\Rightarrow dW_i = \frac{1}{1+U_i}dU_i$

$V_i = \frac{-1}{\alpha+1}(1+U_i)^{-(\alpha+1)}$

$$E[\log(1+U_i)] = (\alpha+1)\int_0^\infty \log(1+U_i)(1+U_i)^{-(\alpha+2)}dU_i$$

$$= (\alpha+1)\left\{\left[\frac{-1}{\alpha+1}\log(1+U_i)(1+U_i)^{-(\alpha+1)}\right]_0^\infty + \frac{1}{\alpha+1}\int_0^\infty (1+U_i)^{-(\alpha+1)}\left(\frac{1}{1+U_i}dU_i\right)\right\}$$

$$= \left[-\log(1+U_i)(1+U_i)^{-(\alpha+1)}\right]_0^\infty + \int_0^\infty (1+U_i)^{-(\alpha+2)}dU_i$$

$$= [-0+0] + \left[\frac{-1}{\alpha+1}(1+U_i)^{-(\alpha+1)}\right]_0^\infty = \frac{1}{\alpha+1} < \infty$$

$\Rightarrow$ by SLLN $[E(U_i)<\infty]$ $\quad \frac{1}{n}\sum U_i \to_{a.s.} E(U_i) = \frac{1}{\alpha+1}$

let $g(z) = [z]^{-1} - 1$ cont

by CMT, $\quad g\left(\frac{1}{n}\sum U_i\right) = \tilde{\alpha}_n \to_{as} g(E(U_i)) = \alpha_0$

(iii) for refrence, Brady got: $E[(\log(1+U_i))^2] = \frac{2}{(\alpha+1)^2}$

$Var[\log(1+U_i)] = \frac{1}{(\alpha+1)^2}$

by CLT $\sqrt{n}\left(\frac{1}{n}\sum \log(1+U_i) - \frac{1}{\alpha_0+1}\right) \to_d N\left(0, \frac{1}{(\alpha+1)^2}\right)$

use $g(x) = \frac{1}{x} - 1$

$\Rightarrow$ by delta $\sqrt{n}(\tilde{\alpha}_n - \alpha_0) \to_d N(0, (\alpha_0+1)^2)$