2014 Qual, Section 1

(1).

1. We consider two groups of independent observations: $X_1, ..., X_n$ are i.i.d from $Unif(0, \alpha)$ and $Y_1, ..., Y_n$ are i.i.d from $Unif(0, \beta)$, where both α and β are unknown parameter assumed to be positive. For comparison, we are interested in inference on $\theta = \beta/\alpha$.

   (a) Derive the UMVUEs for α and β and calculate their respective variances.

   (b) Calculate the MLEs for α and β, denoted as $\hat{\alpha}$ and $\hat{\beta}$ respectively. Derive the asymptotic distributions for $\hat{\alpha}$ and $\hat{\beta}$ are some normalization.

   (c) The MLE for θ is then $\hat{\theta} = \hat{\beta}/\hat{\alpha}$. Derive the asymptotic distribution of $\hat{\theta}$ after normalization. Construct an asymptotic 95% confidence interval for θ based on the observations.

   (d) We wish to test the hypothesis $H_0 : \alpha = \beta$ versus $H_a : \alpha \neq \beta$. What is the likelihood ratio test static. Derive the exact distribution of this test statistic.

   (e) Note $E[X_k] = \alpha/2$ and $E[Y_k] = \beta/2$. Thus, a simple estimator for θ is $\bar{Y}_n/\bar{X}_n$. Derive the asymptotic distribution of this estimator after normalization. What is the asymptotic relative efficiency of this estimator with respect to $\hat{\theta}$, $2\bar{Y}_n/\hat{\alpha}$ and $\hat{\beta}/(2\bar{X}_n)$?

UNC Biostatistics Qualifying Exam Solutions

Ann Marie Weideman

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1 2014 Theory

1.1 Section 1

1.1.1 Problem 1 Similar to 2011 760 final.

A) Derive the UMVUEs for α and β and calculate their respective variances.

Topics:

1. **Methods for finding UMVUE:**
   - Method 1 for finding UMVUE of θ:
     (a) Find a complete and sufficient statistic $T(x)$ for θ (justify why it is a CSS).
     (b) Find a function $g(T(x))$ such that $E[g(T(x))] = \theta$, then $g(T(x))$ is the UMVUE for θ.
   - Method 2 for finding UMVUE of θ:
     (a) Find a complete and sufficient statistic $T(x)$ for θ (justify why it's a CSS).
     (b) Find an unbiased estimator for θ, denoted $\widetilde{T}(x)$
     (c) Calculate $E[\widetilde{T}(x)|T(x)]$ to yield the UMVUE for θ.
2. **Factorization theorem $\Rightarrow$ sufficient statistic**:
   Let $f(x|\theta)$ be the joint pdf or pmf of X. Then, T(X) is a sufficient statistic for θ iff $\exists g(T(x)|\theta)$ and $h(x) \ni f(x|\theta) = h(x)g(T(x)|\theta)$.
3. **Showing completeness**:
   Let $\{f(t|\theta) : \theta \in \Theta\}$ be a family of pdfs or pmfs for $T(x)$.
   The family is called complete if $E_\theta(g(T)) = 0 \Rightarrow P_\theta(g(T) = 0) = 1, \forall\theta \in \Theta$.
4. **Pdf of order statistic**:
   (a) $f_{x_{(1)}}(x) = nf(x)\{1 - F(x)\}^{n-1}$
   (b) $f_{x_{(n)}}(x) = nf(x)\{F(x)\}^{n-1}$

Given $x_i \sim Unif(0, \alpha) \Rightarrow f(\boldsymbol{x}) = (\frac{1}{\alpha})^n \mathbf{1}_{(0 \le x_i \le \alpha)} = (\frac{1}{\alpha})^n \mathbf{1}_{(0 \le x_{(1)})} \mathbf{1}_{(x_{(n)} \le \alpha)}$
Given $y_i \sim Unif(0, \beta) \Rightarrow f(\boldsymbol{y}) = (\frac{1}{\beta})^n \mathbf{1}_{(0 \le y_i \le \beta)} = (\frac{1}{\beta})^n \mathbf{1}_{(0 \le y_{(1)})} \mathbf{1}_{(y_{(n)} \le \beta)}$

To show sufficient:

Notice that the first indicator in each expression does not involve the unknown parameter, α and β. This will be our $h(x)$ and $h(y)$.

Thus, $f(\boldsymbol{x}) = \underbrace{\mathbf{1}_{(0 \leq x_{(1)})}}_{h(x)} \underbrace{(\frac{1}{\alpha})^n \mathbf{1}_{(x_{(n)} \leq \alpha)}}_{g(T(x)|\theta)}$ and $f(\boldsymbol{y}) = \underbrace{\mathbf{1}_{(0 \leq y_{(1)})}}_{h(y)} \underbrace{(\frac{1}{\beta})^n \mathbf{1}_{(y_{(n)} \leq \beta)}}_{g(T(y)|\theta)}$

This factorization allows us to see that the sufficient statistics are $T(x) = X_{(n)}$ and $T(y) = Y_{(n)}$.

To show complete:

Also know $X_{(n)}$ complete since

$E_\alpha[g(X_{(n)})] = n(\frac{1}{\alpha})^n \int_0^\alpha g(t)t^{n-1}dt = 0, \forall \alpha > 0$
$\Rightarrow 0 = \int_0^\alpha g(t)t^{n-1}dt$ and $0 = \frac{d}{d\alpha}\int_0^\alpha g(t)t^{n-1}dt = g(\alpha)\alpha^{n-1}, \forall \alpha > 0$
$\Rightarrow g(\alpha) = 0, \forall t > 0 \Rightarrow P_\alpha(g(T) = 0) = 1, \forall \alpha > 0.$

A similar proof holds to show that Y_n is complete.
Thus, $X_{(n)}$ and $Y_{(n)}$ are CSS.

To find an unbiased estimator:

Since $E[X_{(n)}] = n(\frac{1}{\alpha})^n \int_0^\alpha x^n dx = \frac{n}{n+1}\alpha$, then $E[\frac{n+1}{n}X_{(n)}] = \alpha \Rightarrow \frac{n+1}{n}X_n$ is an unbiased estimator of α.
Similarly, $\frac{n+1}{n}Y_{(n)}$ is an unbiased estimator of β.

UMVUEs:

Thus, $\boxed{\widetilde{\alpha} = \frac{n+1}{n}X_{(n)} \text{ and } \widetilde{\beta} = \frac{n+1}{n}Y_{(n)}}$ are the UMVUEs for α and β, respectively.

Variances of UMVUEs:

$Var(\widetilde{\alpha}) = Var(\frac{n+1}{n}X_{(n)}) = E[(\frac{n+1}{n})^2 X_{(n)}^2] - (E[\frac{n+1}{n}X_{(n)}])^2 = \frac{(n+1)^2}{n}(\frac{1}{\alpha})^n \int_0^\alpha x^{n+1}dx - \alpha^2 = \frac{(n+1)^2}{n(n+2)}\alpha^2 - \alpha^2 = \frac{(n+1)^2\alpha^2 - n(n+2)\alpha^2}{n(n+2)} = \frac{\alpha^2}{n(n+2)}$

Similarly, $Var(\widetilde{\beta}) = Var(\frac{n+1}{n}Y_{(n)}) = \frac{\beta^2}{n(n+2)}$.

Thus, $\boxed{Var(\widetilde{\alpha}) = \frac{\alpha^2}{n(n+2)} \text{ and } Var(\widetilde{\beta}) = \frac{\beta^2}{n(n+2)}.}$

B) Calculate the MLEs for α and β, denoted as $\hat{\alpha}$ and $\hat{\beta}$. Derive the asymptotic distributions of $\hat{\alpha}$ and $\hat{\beta}$ after normalization.

> **Topics:**
>
> **Exponential Function:**
> $(1+\frac{c}{n})^n \to e^c$ as $n \to \infty$
> $(1-\frac{c}{n})^n \to e^{-c}$ as $n \to \infty$

Given $x_i \sim Unif(0,\alpha) \Rightarrow L(\alpha|\boldsymbol{x}) = (\frac{1}{\alpha})^n \mathbf{1}_{(0 \le x_i \le \alpha)} = (\frac{1}{\alpha})^n \mathbf{1}_{(0 \le x_{(1)})} \mathbf{1}_{(x_{(n)} \le \alpha)}$

We want to find α such that $(\frac{1}{\alpha})^n$ is maximized. Since $(\frac{1}{\alpha})^n$ is a decreasing function of α, for $\alpha \ge X_{(n)} \Rightarrow L(\alpha|\boldsymbol{x})$ is maximized at $\alpha = X_{(n)}$.

Similarly, $L(\beta|\boldsymbol{y})$ is maximized at $\beta = Y_{(n)}$.

Thus, $\boxed{\hat{\alpha} = X_{(n)} \text{ and } \hat{\beta} = Y_{(n)}.}$

By asymptotic distribution of $X_{(n)}$, we mean sequences k_n and a_n, along with nondegenerate RV $T \ni k_n(X_{(n)} - a_n) \xrightarrow{d} T$.

Then, $F(t) = P(k_n(X_{(n)} - a_n) \le t) = P(X_{(n)} \le \frac{t}{k_n} + a_n) = [P(X \le \frac{t}{k_n} + a_n)]^n = [\frac{1}{\alpha}(\frac{t}{k_n} + a_n)]^n = [\frac{t/\alpha}{k_n} + \frac{a_n}{\alpha}]^n$.

Let $a_n = \alpha$ and $k_n = n$. Then, $F(t) = [\frac{t/\alpha}{n} + 1]^n \xrightarrow{d} e^{t/\alpha}$ for $0 \le \frac{t/\alpha}{n} + 1 \le 1 \Rightarrow -1 \le \frac{t/\alpha}{n} \le 0 \Rightarrow t \le 0$.

But, we need $t \ge 0$ to define an exponential distribution, so try inverting the sign to get $n(\alpha - X_{(n)})$. Then, $P(n(\alpha - X_{(n)}) \le t) = P(X_{(n)} \ge \alpha - \frac{t}{n}) = 1 - P(X_{(n)} \le \alpha - \frac{t}{n}) = 1 - [P(X \le \alpha - \frac{t}{n}]^n = 1 - [\frac{1}{\alpha}(\alpha - \frac{t/\alpha}{n})]^n = 1 - [1 - \frac{t}{n}]^n \xrightarrow{d} 1 - e^{-t/\alpha}$.

Thus, $\boxed{n(\alpha - X_{(n)}) \xrightarrow{d} Exp(\alpha)}$ and, similarly, $\boxed{n(\beta - Y_{(n)}) \xrightarrow{d} Exp(\beta).}$

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1 c) The MLE for θ is $\hat{\theta} = \hat{\beta}/\hat{\alpha}$. Derive the asymptotic distn. of $\hat{\theta}$ after normalization. Construct a 95% CI (θ).

[Part b), $n\left(\begin{pmatrix}\alpha\\ \beta\end{pmatrix} - \begin{pmatrix}X_{(n)}\\ Y_{(n)}\end{pmatrix}\right) \xrightarrow{d} \begin{pmatrix}Exp(\alpha)\\ Exp(\beta)\end{pmatrix}$ which Dr. K says is equivalent to $n\left(\begin{pmatrix}X_{(n)}\\ Y_{(n)}\end{pmatrix} - \begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right) \xrightarrow{d} \begin{pmatrix}-Exp(\alpha)\\ -Exp(\beta)\end{pmatrix}$ (b/c exp distn. is a member of scale family.)

Then, by Delta method, since $g(\alpha,\beta) = \beta/\alpha$ is differentiable w/

$$g'(\alpha,\beta) = \begin{pmatrix}\frac{\partial}{\partial\alpha}(\beta/\alpha)\\ \frac{\partial}{\partial\beta}(\beta/\alpha)\end{pmatrix} = \begin{pmatrix}-\beta/\alpha^2\\ 1/\alpha\end{pmatrix}$$ and is non-zero valued, then

$$n(\underbrace{g(X_{(n)}, Y_{(n)})}_{MLE} - \underbrace{g(\alpha,\beta)}_{truth}) \xrightarrow{d} g'(\alpha,\beta)\begin{pmatrix}-Exp(\alpha)\\ -Exp(\beta)\end{pmatrix}$$

$$\Rightarrow n\left(\hat{\beta}/\hat{\alpha} - \beta/\alpha\right) \xrightarrow{d} \left(-\beta/\alpha^2 \quad 1/\alpha\right)\begin{pmatrix}-Exp(\alpha)\\ -Exp(\beta)\end{pmatrix} = \beta/\alpha^2\, Exp(\alpha) - 1/\alpha\, Exp(\beta)$$

$$= \frac{\beta\cdot\alpha}{\alpha^2} Exp(1) - \frac{\beta}{\alpha} Exp(1)$$

$$= \frac{\beta}{\alpha}\left[Exp(1) - Exp(1)\right]$$

Thanks to Taylor K, since we want to find distn. of $\underbrace{Exp(1)}_{X_1} - \underbrace{Exp(1)}_{X_2}$

NOTE: In general,
$P(A) = E[I(A)]$
$P(A|B) = E[I(A)|B]$
$P(A) = E_B[p(A|B)]$

Then, $P(X_1 - X_2 < t) = E_{X_2}[E[I(X_1 < t + X_2) | X_2]]$

$= E_{X_2}[P(X_1 < t + X_2 | X_2)] = E_{X_2}[1 - e^{-t-X_2}]$

$= \int_0^\infty (1 - e^{-t-X_2})e^{-X_2}dX_2 = \int_0^\infty e^{-X_2} - e^{-t-2X_2}dX_2$

$= -e^{-X_2} + \frac{1}{2}e^{-t-2X_2}\Big|_0^\infty = \lim_{X_2\to\infty}(-e^{-X_2} + \frac{1}{2}e^{-t-2X_2}) - (\overset{-1}{\cancel{-e^0}} + \frac{1}{2}e^{-t-2(0)}) = 1 - \frac{1}{2}e^{-t}$

Recall that the cdf of a Laplace is $F(x) = \begin{cases}\frac{1}{2}\exp\left(\frac{x-\mu}{b}\right), & x \le \mu\\ 1 - \frac{1}{2}\exp\left(-\frac{x-\mu}{b}\right), & x \ge \mu\end{cases}$

So, can recognize this derived CDF as that of a Laplace ($\mu=0$, $b=1$).

$\Rightarrow \underbrace{\frac{\beta}{\alpha}}_{\theta}\left[Exp(1) - Exp(1)\right] \sim Laplace(\mu=0, b=\frac{\beta}{\alpha})$ (Laplace is member of location-scale family.)

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1 c) cont'd

From previous pg., showed $n(\hat{\theta}-\theta) \xrightarrow{d} \overbrace{\frac{\beta}{\alpha}}^{\theta}[Exp(1) - Exp(1)]$

$\equiv \text{Laplace}(\mu=0, b=\beta/\alpha)$

* Correction from Emily D. Thanks, Emily!

Since the Laplace distribution is a member of the location-scale family, then $\frac{n}{\theta}(\hat{\theta}-\theta) \xrightarrow{d} \text{Laplace}(\mu=0, b=1)$. $\perp\!\!\!\perp \theta$, so $\frac{n}{\theta}(\hat{\theta}-\theta)$ is a pivotal quantity.

Thus, for $\underbrace{a}_{\substack{0.025 \text{ quantile} \\ \text{of Laplace}(0,1)}} \le \frac{n}{\theta}(\hat{\theta}-\theta) \le \underbrace{b}_{\substack{0.975 \text{ quantile} \\ \text{of Laplace}(0,1)}}$

The CDF: $F(t) = \begin{cases} \frac{1}{2}\exp(t), & t \le 0 \\ 1 - \frac{1}{2}\exp(-t), & t > 0 \end{cases}$

<u>Lower Bound</u>: If $F(a) = 0.025$, then use $F(t) = \frac{1}{2}\exp(t)$, since $F(t) = \frac{1}{2}\exp(t)$ for $t \le 0 \iff F(t) \le \underbrace{F(0)}_{0.5}$ since F monotone for fixed t.

$\Rightarrow 0.025 = \frac{1}{2}\exp(a) \Rightarrow 0.05 = \exp(a) \Rightarrow a = \log(0.05)$

<u>Upper Bound</u>: If $F(b) = 0.975$, then use $F(t) = 1 - \frac{1}{2}\exp(-t)$ since $F(t) = 1 - \frac{1}{2}\exp(-t)$ for $t > 0 \iff F(t) > \underbrace{F(0)}_{0.5}$ since F monotone for fixed t.

$\Rightarrow 0.975 = 1 - \frac{1}{2}\exp(-b) \Rightarrow -0.025 = -\frac{1}{2}\exp(-b) \Rightarrow 0.05 = \exp(-b) \Rightarrow \log(0.05) = -b$

$\Rightarrow b = -\log(0.05)$.

approximate since using $\hat{\theta}$

Thus, a 95% CI $(\theta) \approx \left\{\theta : \log(0.05) \le \frac{n}{\theta}(\hat{\theta}-\theta) \le -\log(0.05)\right\}$

$= \left\{\theta : \log(0.05) \le \frac{n\hat{\theta}}{\theta} - n \le -\log(0.05)\right\}$

$= \left\{\theta : \log(0.05) + n \le \frac{n\hat{\theta}}{\theta} \le n - \log(0.05)\right\}$

$= \left\{\theta : \frac{1}{n - \log(0.05)} \le \frac{\theta}{n\hat{\theta}} \le \frac{1}{n + \log(0.05)}\right\}$

$$= \boxed{\left\{\theta : \frac{n\hat{\theta}}{n - \log(0.05)} \le \theta \le \frac{n\hat{\theta}}{n + \log(0.05)}\right\}}$$

AMW

1 d)

<u>Likelihood</u>: $\mathcal{L}(\alpha,\beta|\underset{\sim}{X},\underset{\sim}{Y}) = \alpha^{-n} I(X_{(n)}<\alpha)\,\beta^{-n} I(Y_{(n)}<\beta)$

<u>Unrestricted MLE under H_A</u>: $\hat{\alpha} = X_{(n)}$ & $\hat{\beta} = Y_{(n)}$

<u>Restricted MLE under H_0</u>: $\gamma = \alpha = \beta \Rightarrow \mathcal{L}(\gamma|\underset{\sim}{X},\underset{\sim}{Y}) = \gamma^{-2n} I(\max\{X_{(n)},Y_{(n)}\}<\gamma)$

$\Rightarrow \hat{\gamma} = \max\{X_{(n)},Y_{(n)}\}$

<u>LRT statistic</u>:

$$\Lambda = \frac{\sup_{\theta\in\Theta_0} L(\theta|X)}{\sup_{\theta\in\Theta} L(\theta|X)} = \frac{L(\hat{\gamma}|\underset{\sim}{X},\underset{\sim}{Y})}{L(\hat{\alpha},\hat{\beta}|\underset{\sim}{X},\underset{\sim}{Y})} = \frac{\hat{\gamma}^{-2n} I(\max\{X_{(n)},Y_{(n)}\}<\gamma)}{\hat{\alpha}^{-n} I(X_{(n)}<\alpha)\,\hat{\beta}^{-n} I(Y_{(n)}<\beta)}$$

$$= \frac{\hat{\gamma}^{-2n}}{\hat{\alpha}^{-n}\hat{\beta}^{-n}} = \left(\frac{\hat{\alpha}\hat{\beta}}{\hat{\gamma}^2}\right)^n = \left[\frac{X_{(n)}Y_{(n)}}{\max\{X_{(n)},Y_{(n)}\}^2}\right]^n = \Big(\underbrace{\frac{Y_{(n)}}{X_{(n)}}}_{\hat{\theta}}\Big)^n \underbrace{I(X_{(n)}>Y_{(n)})}_{I(1>\hat{\theta})} + \Big(\underbrace{\frac{X_{(n)}}{Y_{(n)}}}_{\hat{\theta}^{-1}}\Big)^n \underbrace{I(X_{(n)}\le Y_{(n)})}_{I(1\le\hat{\theta})}$$

$= \hat{\theta}^n I(\hat{\theta}<1) + \hat{\theta}^{-n} I(\hat{\theta}\ge 1)$, $0<\Lambda\le 1$ since $\hat{\gamma} = \max\{X_{(n)},Y_{(n)}\}$

$0 < \left(\frac{X_{(n)}Y_{(n)}}{\max\{X_{(n)},Y_{(n)}\}^2}\right)^n \le 1$.

Then, $F_\Lambda(z) = P(\Lambda\le z) = P(\hat{\theta}^n\le z|\hat{\theta}<1)P(\hat{\theta}<1) + P(\hat{\theta}^{-n}\le z|\hat{\theta}\ge 1)P(\hat{\theta}\ge 1)$

$$= \underbrace{P(\hat{\theta}^n\le z, \hat{\theta}<1)}_{A} + \underbrace{P(\hat{\theta}^{-n}\le z, \hat{\theta}\ge 1)}_{B} \quad \text{for } z\in(0,1]$$

$A = P(\hat{\theta}^n\le z, \hat{\theta}<1) = P(\hat{\theta}\le z^{1/n}, \hat{\theta}\le 1) = P\left(\frac{Y_{(n)}}{X_{(n)}}\le z^{1/n}, \frac{Y_{(n)}}{X_{(n)}}\le 1\right)$

$= P\Big(\underbrace{\frac{Y_{(n)}}{X_{(n)}}}_{U}\le z^{1/n}\Big)$ (can drop off second piece of joint b/c $z\in(0,1] \Rightarrow z^{1/n}\in(0,1] \Rightarrow \frac{Y_{(n)}}{X_{(n)}}\le z^{1/n} \Rightarrow \frac{Y_{(n)}}{X_{(n)}}\le 1$)

Let $U = \frac{Y_{(n)}}{X_{(n)}}$, $V = X_{(n)}$ $\Rightarrow$ $X_{(n)} = V$, $Y_{(n)} = UV$ $\Rightarrow |J| = \begin{vmatrix} 0 & 1 \\ v & u \end{vmatrix} = |v|$

$\Rightarrow f_{U,V}(u,v) = n\left(\frac{1}{\alpha}\right)\left(\frac{v}{\alpha}\right)^{n-1}\cdot n\left(\frac{1}{\beta}\right)\left(\frac{uv}{\beta}\right)^{n-1}\cdot|v| = n^2\left(\frac{1}{\alpha\beta}\right)^n u^{n-1}v^{2n-1}$, $0<u\le 1$, $0<v<\alpha$

$\Rightarrow f_U(u) = \int_v f_{U,V}(u,v)\,dv = \int_0^\alpha n^2\left(\frac{1}{\alpha\beta}\right)^n u^{n-1}v^{2n-1}\,dv = \frac{n^{\cancel{2}}}{2\cancel{n}}\left(\frac{1}{\alpha\beta}\right)^n u^{n-1}\alpha^{2n} = \frac{n}{2}\left(\frac{\alpha}{\beta}\right)^n u^{n-1}$

cont'd →

AHW

1.d) cont'd

$$\Rightarrow A = P\Big(\underbrace{\frac{Y_{(n)}}{X_{(n)}}}_{u} \le z^{1/n}\Big) = \int_0^{z^{1/n}} \frac{n}{2}\left(\frac{\alpha}{\beta}\right)^n u^{n-1}\,du = \frac{\cancel{n}}{2}\left(\frac{\alpha}{\beta}\right)^n \cdot \frac{1}{\cancel{n}} u^n \Big|_0^{z^{1/n}} = \frac{z}{2}\left(\frac{\alpha}{\beta}\right)^n$$

Similarly, $B = P(\hat{\theta}^{-n} \le z, \hat{\theta} \ge 1) = P(\hat{\theta}^{-1} \le z^{1/n}, \underbrace{\hat{\theta}}_{\frac{Y_{(n)}}{X_{(n)}}} \ge 1)$

$$= P\Big(\frac{X_{(n)}}{Y_{(n)}} \le z^{1/n}, \frac{X_{(n)}}{Y_{(n)}} \le 1\Big) = P\Big(\frac{X_{(n)}}{Y_{(n)}} \le z^{1/n}\Big)$$ since $z \in (0,1] \Rightarrow \frac{X_{(n)}}{Y_{(n)}} \le 1$

same process as in A

$$= \frac{z}{2}\left(\frac{\beta}{\alpha}\right)^n$$

$$\Rightarrow F_\Lambda(z) = \frac{z}{2}\left(\frac{\alpha}{\beta}\right)^n + \frac{z}{2}\left(\frac{\beta}{\alpha}\right)^n = \frac{z}{2}\left[\left(\frac{\alpha}{\beta}\right)^n + \left(\frac{\beta}{\alpha}\right)^n\right]$$

$$\Rightarrow z \sim \text{Unif}\left(0, \frac{2}{\left[\left(\frac{\alpha}{\beta}\right)^n + \left(\frac{\beta}{\alpha}\right)^n\right]}\right)$$

Under H_0, $\alpha = \beta \Rightarrow z \overset{H_0}{\sim} \text{Unif}\left(0, \frac{2}{[1^n + 1^n]}\right) \equiv \text{Unif}(0,1)$. $\lrcorner$

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1. e) Note $E[X_k] = \alpha/2$ and $E[Y_k] = \beta/2$. Thus, a simple estimator for θ is $\bar{Y}_n/\bar{X}_n$. Derive the asymptotic distn. of this estimator after normalization.

What is the ARE (asymptotic relative efficiency) of this estimator w.r.t $\hat{\theta}$, $2\bar{Y}_n/\hat{\alpha}$, and $\hat{\beta}/2\bar{X}_n$?

TOPICS : Delta Method: $\boxed{\text{Univariate}}$ - For a sequence of RV X_n satisfying $\sqrt{n}(X_n - \theta) \xrightarrow{d} X$ for X having some distribution, then $\sqrt{n}\,(g(X_n) - g(\theta)) \xrightarrow{d} \nabla g(\theta) \cdot X$ where g has a derivative at θ and is non-zero valued.

$\boxed{\text{Multivariate}}$ - For a vector of sequences of RV $\begin{pmatrix} X_n \\ Y_n \end{pmatrix}$ satisfying $\sqrt{n}\left(\begin{pmatrix} X_n \\ Y_n \end{pmatrix} - \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right) \xrightarrow{d} \begin{pmatrix} X \\ Y \end{pmatrix}$ for (X, Y) having some joint distribution, then

$$\sqrt{n}\,(g(X_n, Y_n) - g(\alpha, \beta)) \xrightarrow{d} \nabla g(\theta)^T \begin{pmatrix} X \\ Y \end{pmatrix}$$

Asymptotic Relative Efficiency (ARE) :

The efficiency of some estimators of θ, $\hat{\theta}$ w.r.t. $\tilde{\theta}$, is the ratio of their asymptotic variances,

i.e. $ARE(\hat{\theta}, \tilde{\theta}) = \dfrac{a_1 \tilde{\sigma}^2}{a_2 \hat{\sigma}^2} = \dfrac{Var(\tilde{\theta})}{Var(\hat{\theta})}$. If $ARE(\hat{\theta}, \tilde{\theta}) < 1$, then $\tilde{\theta}$ is the more efficient estimator.

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1. e) cont'd,

By CLT, $\dfrac{\bar{X}_n - E(\bar{X}_n)}{\sqrt{Var(\bar{X}_n)}} \xrightarrow{d} N(0,1) \Rightarrow \dfrac{\bar{X}_n - \frac{1}{n}\sum_{i=1}^{n} E(X_i)}{\sqrt{\frac{1}{n^2}\sum_{i=1}^{n} Var(X_i)}} = \dfrac{\bar{X}_n - \alpha/2}{\sqrt{\frac{1}{n}\frac{\alpha^2}{12}}}$

$= \dfrac{\sqrt{n}\,(\bar{X}_n - \alpha/2)}{\sqrt{\frac{\alpha^2}{12}}} \xrightarrow{d} N(0,1) \Rightarrow \sqrt{n}\,(\bar{X}_n - \alpha/2) \xrightarrow{d} N(0, \alpha^2/12)$

Similarly, $\sqrt{n}\,(\bar{Y}_n - \beta/2) \xrightarrow{d} N(0, \beta^2/12)$

0 since $Cov(\bar{X}_n, \bar{Y}_n) = 0$ since $X_i \perp\!\!\!\perp Y_i$

Thus, $\sqrt{n}\left(\begin{pmatrix}\bar{X}_n \\ \bar{Y}_n\end{pmatrix} - \begin{pmatrix}\alpha/2 \\ \beta/2\end{pmatrix}\right) \xrightarrow{d} N\left(\begin{pmatrix}0\\0\end{pmatrix}, \underbrace{\begin{pmatrix}\alpha^2/12 & 0 \\ 0 & \beta^2/12\end{pmatrix}}_{\Sigma}\right)$

By Delta Method, $\sqrt{n}\,(\underset{\bar{Y}_n/\bar{X}_n}{g(\bar{X}_n, \bar{Y}_n)} - \underset{b/a}{g(a,b)}) \xrightarrow{d} N(0, \nabla g(a,b)^T \Sigma \nabla g(a,b))$

where $B = \beta/2$ and $a = \alpha/2$ and thus $\nabla g(a,b) = \begin{pmatrix}\frac{\partial}{\partial a}(b/a) \\ \frac{\partial}{\partial b}(b/a)\end{pmatrix} = \begin{pmatrix}-b/a^2 \\ 1/a\end{pmatrix} =$ ← sub back $b = \beta/2$ and $a = \alpha/2$

$= \frac{2}{\alpha}\begin{pmatrix}-\beta/\alpha \\ 1\end{pmatrix}$

Then, $\sqrt{n}\left(\dfrac{\bar{Y}_n}{\bar{X}_n} - \dfrac{\beta}{\alpha}\right) \xrightarrow{d} N\left(0, \underbrace{\frac{4}{\alpha^2}\begin{pmatrix}-\beta/\alpha & 1\end{pmatrix}\begin{pmatrix}\alpha^2/12 & 0 \\ 0 & \beta^2/12\end{pmatrix}\begin{pmatrix}-\beta/\alpha \\ 1\end{pmatrix}}\right)$

$= \frac{4}{\alpha^2}\begin{pmatrix}-\frac{\beta\alpha}{12} & \frac{\beta^2}{12}\end{pmatrix}\begin{pmatrix}-\beta/\alpha \\ 1\end{pmatrix} = \frac{4}{\alpha^2}\left(\frac{\beta^2}{12} + \frac{\beta^2}{12}\right)$

$= \dfrac{8\beta^2}{12\alpha^2} = \dfrac{2}{3}\dfrac{\beta^2}{\alpha^2}$

$\equiv N(0, \frac{2}{3}\theta^2)$

(1) Find $ARE(\tilde{\theta}, \hat{\theta})$ where $\tilde{\theta} = \dfrac{\bar{Y}_n}{\bar{X}_n}$ and $Var(\tilde{\theta}) = \frac{1}{n}\frac{2}{3}\theta^2$

From c), $n(\hat{\theta} - \theta) \xrightarrow{d} T$ where $T \sim Laplace(\mu = 0, b = \theta)$

$\Rightarrow Var[n(\hat{\theta} - \theta)] = n^2 Var(\hat{\theta} - \underbrace{\theta}_{\text{no variance}}) = n^2 Var(\hat{\theta}) \approx \underbrace{2\theta^2}_{\text{Variance of a Laplace}(\mu=0,\, b=\theta)}$

$\Rightarrow Var(\hat{\theta}) = \frac{2}{n^2}\theta^2$

Thus, $ARE(\tilde{\theta}, \hat{\theta}) = \dfrac{Var(\hat{\theta})}{Var(\tilde{\theta})} = \dfrac{\frac{2}{n^2}\cancel{\theta^2}}{\frac{1}{\cancel{n}}\frac{2}{3}\cancel{\theta^2}} = \dfrac{3}{n}$

cont'd next pg. →

Ann Marie Weideman

② Find $ARE\left(\tilde{\theta}, \frac{2\bar{Y}_n}{\hat{\alpha}}\right)$ where $\tilde{\theta} = \frac{\bar{Y}_n}{\bar{X}_n}$ and $Var(\tilde{\theta}) = \frac{1}{n} \cdot \frac{2}{3}\theta^2$.

Also, recall that $\hat{\alpha}$ is a biased estimator of α where $E[\hat{\alpha}] = \frac{n}{n+1} \cdot \alpha$

$\Rightarrow E\left[\frac{n+1}{n}\hat{\alpha}\right] = \alpha$.

For this problem, will need to show that $\frac{n+1}{n}\hat{\alpha} \xrightarrow{P} \alpha$

$\Leftrightarrow P\left(\left|\frac{n+1}{n}\hat{\alpha} - \alpha\right| > \epsilon\right) \longrightarrow 0$.

From Euphy ☺ Thanks Euphy!

By Chebyshev Ineq., $P(|X - \mu| \geq \epsilon) \leq \frac{Var(X)}{\epsilon^2}$.

In this case, $\mu = \alpha$ and

$Var\left(\frac{n+1}{n}\hat{\alpha}\right) = \frac{\alpha^2}{n(n+2)}$ (part a)

Thus, $P\left(\left|\frac{n+1}{n}\hat{\alpha} - \alpha\right| \geq \epsilon\right) \leq \frac{\alpha^2}{n(n+2)\epsilon^2} \longrightarrow 0$

$\Rightarrow \frac{n+1}{n}\hat{\alpha} \xrightarrow{P} \alpha$.

Then, since $\frac{n}{n+1} \xrightarrow{P} 1$, then by Slutsky's, $\frac{n}{n+1} \cdot \frac{n+1}{n}\hat{\alpha} \xrightarrow{P} 1 \cdot \alpha = \alpha$

$\Rightarrow \hat{\alpha} \xrightarrow{P} \alpha$.

Now

Take $\sqrt{n}\left(\frac{2\bar{Y}_n}{\hat{\alpha}} - \frac{\beta}{\alpha}\right) = \frac{\sqrt{n}}{\hat{\alpha}}\left(2\bar{Y}_n - \frac{\beta\hat{\alpha}}{\alpha}\right) = \frac{\sqrt{n}}{\hat{\alpha}}\left(\underbrace{2\bar{Y}_n - \beta}_{\text{split}} + \underbrace{\beta - \beta\frac{\hat{\alpha}}{\alpha}}\right) = \frac{\sqrt{n}}{\hat{\alpha}}(2\bar{Y}_n - \beta) + \frac{\sqrt{n}}{\hat{\alpha}}\left(\beta - \beta\frac{\hat{\alpha}}{\alpha}\right)$

$= \frac{2\sqrt{n}}{\hat{\alpha}}(\bar{Y}_n - \beta/2) + \frac{\beta\sqrt{n}}{\alpha\hat{\alpha}}(\alpha - \hat{\alpha})$

Know $\sqrt{n}(\bar{Y}_n - \beta/2) \xrightarrow{d} N(0, \beta^2/12)$ and $n(\alpha - \hat{\alpha}) \xrightarrow{d} Exp(\alpha)$

Also, since $\hat{\alpha} \xrightarrow{P} \alpha$ then $\frac{2}{\hat{\alpha}} \xrightarrow{P} \frac{2}{\alpha}$ by CMT

and $\frac{\beta}{\alpha\hat{\alpha}\sqrt{n}} \xrightarrow{P} 0$.

Then, by Slutsky's, $\underbrace{\frac{2}{\hat{\alpha}}}_{\xrightarrow{P} \frac{2}{\alpha}} \cdot \underbrace{\sqrt{n}(\bar{Y}_n - \beta/2)}_{\xrightarrow{d} N(0, \beta^2/12)} \xrightarrow{d} \frac{2}{\alpha} \cdot N(0, \beta^2/12)$

$\equiv N(0, \theta^2/3)$

Also, by Slutsky's, $\underbrace{\frac{\beta}{\alpha\hat{\alpha}\sqrt{n}}}_{\xrightarrow{P} 0} \cdot \underbrace{n(\alpha - \hat{\alpha})}_{\xrightarrow{d} Exp(\alpha)} \xrightarrow{d} 0 \cdot Exp(\alpha) \equiv 0$

Then, the sum $\frac{2}{\hat{\alpha}} \cdot \sqrt{n}(\bar{Y}_n - \beta/2) + \frac{\beta}{\alpha\hat{\alpha}\sqrt{n}} \cdot n(\alpha - \hat{\alpha}) \xrightarrow{d} N(0, \theta^2/3)$ by Slutsky's.

Then, $Var\left[\sqrt{n}\left(\frac{2\bar{Y}_n}{\hat{\alpha}} - \underbrace{\frac{\beta}{\alpha}}_{0 \text{ variance}}\right)\right] = n \cdot Var\left(\frac{2\bar{Y}_n}{\hat{\alpha}}\right) \approx \underbrace{\frac{\theta^2}{3}}_{\text{Variance of } N(0, \theta^2/3)} \Rightarrow Var\left(\frac{2\bar{Y}_n}{\hat{\alpha}}\right) \approx \frac{1}{n} \cdot \frac{\theta^2}{3}$

Ann Marie Weideman

1e) (2) cont'd.

Thus, the $ARE\left(\tilde{\theta}, \frac{2\bar{Y}_n}{\hat{\alpha}}\right) = \frac{Var(2\bar{Y}_n/\hat{\alpha})}{Var(\tilde{\theta})} = \frac{\left(\frac{1}{\cancel{n}} \cdot \frac{\cancel{\theta^2}}{3}\right)}{\left(\frac{1}{\cancel{n}}\, \frac{2}{3}\cancel{\theta^2}\right)} = \frac{1}{2}$

If we were asked (which we weren't), which is asymptotically more efficient, it would be $\frac{2\bar{Y}_n}{\hat{\alpha}}$, since $ARE\left(\tilde{\theta}, \frac{2\bar{Y}_n}{\hat{\alpha}}\right) = \frac{1}{2} < 1 \Rightarrow$ numerator asymptotically smaller than denominator $\Rightarrow$ asymptotic variance of numerator smaller than denominator.