Part 2 Theory Exam: Question 2

(a) We can write the model as

$$Y = X\beta + \varepsilon$$

$$E(\varepsilon) = 0 \quad \mathrm{cov}(\varepsilon) = \sigma^2 I_4$$

$$\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}_{4\times 1}, \quad X = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix}_{4\times 3}$$

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}_{4\times 1}$$

We see that X is <u>not</u> of full rank, and

$\mathrm{rank}(X) = 2$.

$$C(X) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \right\}$$

Let $X^* = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \\ -1 & -1 \end{pmatrix}$. $\quad C(X) = C(X^*)$

Then the projection onto $C(X)$ is

$$M = X^{*}(X^{*\prime}X^{*})^{-1}X^{*\prime}$$

$$(X^{*\prime}X^{*}) = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \\ -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix} = 2\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$(X^{*\prime}X^{*})^{-1} = \frac{1}{2}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

So

$$\hat{\mu} = MY = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} Y$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} Y$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} y_1 - y_4 \\ y_2 - y_3 \\ y_3 - y_2 \\ y_4 - y_1 \end{pmatrix}$$

Thus

$$\boxed{\hat{\mu} = MY = \frac{1}{2}\begin{pmatrix} y_1 - y_4 \\ y_2 - y_3 \\ y_3 - y_2 \\ y_4 - y_1 \end{pmatrix}}$$

(b) We can write

$$\beta_2 - \beta_3 = \lambda'\beta = (0\ 1\ -1)\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$$

$$\lambda' = (0\ 1\ -1)$$

$\beta_2 - \beta_3$ is estimable iff $\lambda' = \rho' X$ for some $\rho_{4\times 1}$. Let's check.

$$(0\ 1\ -1) = \underset{1\times 4}{(\rho_1, \rho_2, \rho_3\ \rho_4)}\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix}_{4\times 3}$$

$$\Rightarrow \quad 0 = \rho_1 - \rho_4 \qquad -1 = -\rho_1 + \rho_2 - \rho_3 + \rho_4$$
$$1 = \rho_1 + \rho_2 - \rho_3 - \rho_4$$

$\rho_1 = \rho_4 \Rightarrow$

$1 = \rho_4 + \rho_2 - \rho_3 - \rho_4$

$$\left.\begin{array}{l} \Rightarrow 1 = \rho_2 - \rho_3 \\ \text{also} \\ -1 = \rho_2 - \rho_3 \end{array}\right\} \text{contradiction}$$

$\boxed{\text{Thus } \beta_2 - \beta_3 \text{ is NOT estimable}}$

(c)

$H_0: \beta_2 + \beta_3 = 0$

$H_1: \beta_2 + \beta_3 \neq 0$

$$\lambda' \beta = (0 \ 1 \ 1) \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \beta_2 + \beta_3$$

$\beta_2 + \beta_3$ is estimable since

$$(0 \ 1 \ 1) = (\rho_1, \rho_2, \rho_3 \ \rho_4) \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{aligned} 0 &= \rho_1 - \rho_4 \\ 1 &= \rho_1 + \rho_2 - \rho_3 - \rho_4 \\ 1 &= -\rho_1 + \rho_2 - \rho_3 + \rho_4 \end{aligned}$$

$$\Rightarrow \left.\begin{aligned} 1 &= \rho_2 - \rho_3 \\ 1 &= \rho_2 - \rho_3 \end{aligned}\right\} \Rightarrow \beta_2 + \beta_3 \text{ is estimable.}$$

The estimation space under H_0 is denoted by E_0. $\beta_2 + \beta_3 = 0 \Rightarrow \beta_2 = -\beta_3$

$$E_0 = \{u : u' = (\beta_1 - 2\beta_3, 0, 0, 2\beta_3 - \beta_1)\}$$

$$= \{u : u' = (\beta_1 - 2\beta_3)(1, 0, 0, -1)\}$$

Thus
$$E_0 = \text{span}\left\{\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}\right\}$$

which is a 1 dimensional subspace of R^4.

$$E = C(X) = \text{Span}\left\{\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}\right\}$$

Now $\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$ is orthogonal to the vector $\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$

since their inner product is 0. But notice that

$$\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = -\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

That is, it can be written as a linear combination of vectors comprising $\mathcal{C}(X)$.

Thus $\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \in \mathcal{C}(X)$

and hence

$$\boxed{E \cap E_0^{\perp} = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}\right\}}$$

$$= \{u : u' = (0, \alpha, -\alpha, 0)\}$$

(d) $H_0: \mu \in E_0$ vs. $H_1: \mu \notin E_0$

$$F = \frac{\|(M-M_0)Y\|^2 / r(M-M_0)}{\|(I-M)Y\|^2 / r(I-M)}$$

$M - M_0$ = orthogonal projection operator onto $\text{Span}\left\{\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\right\}$

$$M - M_0 = \begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\left[(0\ 1\ -1\ 0)\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\right]^{-1}(0\ 1\ -1\ 0)$$

$$= \frac{1}{2}\begin{pmatrix}0&0&0&0\\0&1&-1&0\\0&-1&1&0\\0&0&0&0\end{pmatrix} = M - M_0$$

$$(M-M_0)Y = \frac{1}{2}\begin{pmatrix}0\\y_2-y_3\\y_3-y_2\\0\end{pmatrix}$$

$$\Rightarrow \|(M-M_0)Y\|^2 = \frac{1}{2}(y_2-y_3)^2$$

also $r(M-M_0) = 1$

$$(I-M)Y = Y - MY$$

$$= \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} y_1-y_4 \\ y_2-y_3 \\ y_3-y_2 \\ y_4-y_1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} y_1+y_4 \\ y_2+y_3 \\ y_2+y_3 \\ y_1+y_4 \end{pmatrix}$$

$$\|(I-M)Y\|^2 = \frac{1}{2}(y_1+y_4)^2 + \frac{1}{2}(y_2+y_3)^2$$

$r(I-M) = 2$, $MSE = \left[\frac{1}{2}(y_1+y_4)^2 + \frac{1}{2}(y_2+y_3)^2\right]/2$

Therefore

$$F = \frac{\frac{1}{2}(y_2-y_3)^2}{\left[\frac{1}{2}(y_1+y_4)^2 + \frac{1}{2}(y_2+y_3)^2\right]/2}$$

$$\boxed{F = \frac{2(y_2 - y_3)^2}{(y_1 + y_4)^2 + (y_2 + y_3)^2}} \overset{H_0}{\sim} F(1, 2)$$

Under H, $F \sim F(\delta, 1, 2)$

δ = non-centrality parameter

$$= \frac{\|(M - M_0)X\beta\|^2}{2\sigma^2}$$

$$(M - M_0)X\beta = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \beta_2 + \beta_3 \\ -\beta_2 - \beta_3 \\ 0 \end{pmatrix}$$

$$\|(M - M_0)X\beta\|^2 = 2(\beta_2 + \beta_3)^2$$

Thus

$$\delta = \frac{2(\beta_2 + \beta_3)^2}{2\sigma^2}$$

$$= \boxed{\frac{(\beta_2 + \beta_3)^2}{\sigma^2}}$$

(e)

$$\beta_2 + \beta_3 = (0\ 1\ 1)\begin{pmatrix}\beta_1 \\ \beta_2 \\ \beta_3\end{pmatrix}$$

$$= \lambda'\beta, \quad \lambda' = (0, 1, 1)$$

The estimate of $\lambda'\beta$ is

$$\lambda'\hat{\beta} = \rho' X\hat{\beta} = \rho'\hat{\mu}$$

from part (c), we can pick $\rho' = (1, 1, 0, 1)$

Thus $\lambda'\hat{\beta} = (1\ 1\ 0\ 1)\ \frac{1}{2}\begin{pmatrix} y_1 - y_4 \\ y_2 - y_3 \\ y_3 - y_2 \\ y_4 - y_1 \end{pmatrix}$

$$= \tfrac{1}{2}(y_1 - y_4) + \tfrac{1}{2}(y_2 - y_3) + \tfrac{1}{2}(y_4 - y_1)$$

$$= \tfrac{1}{2}(y_2 - y_3)$$

Now $[\lambda'(X'X)^{-}\lambda]^{-} = [\lambda^{*\prime}(X^{*\prime}X^{*})^{-1}\lambda^{*}]^{-1}$

$$= \left[(1 \;\; 1)\,\tfrac{1}{2}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]^{-1} = \left(\frac{1}{2}\right)^{-1} = 2$$

Thus the 95% CI for $\beta_2 + \beta_3$ is

given by the set of $(\beta_1, \beta_2, \beta_3)$ satisfying

$$\frac{(\lambda'\hat{\beta} - \lambda'\beta)'[\lambda'(X'X)^{-}\lambda]^{-}(\lambda'\hat{\beta} - \lambda'\beta)}{MSE}$$

$$\leq F(1-\alpha, 1, 2)$$

$$\frac{2\left(\frac{1}{2}(y_2-y_3)-(\beta_2+\beta_3)\right)^2}{\left[\frac{1}{2}(y_1+y_4)^2+\frac{1}{2}(y_2+y_3)^2\right]/2} \le F(1-\alpha,1,2)$$

$$\Leftrightarrow \boxed{\frac{8\left[\frac{1}{2}(y_2-y_3)-(\beta_2+\beta_3)\right]^2}{(y_1+y_4)^2+(y_2+y_3)^2} \le F(1-\alpha,1,2)}$$