2019, Section 1, Qual. AMW

2. Let $X_1, \ldots, X_n$ be iid samples from a distn. w/ density function

$f(x) = \theta^{-1} e^{(a-x)/\theta} I(x>a)$ where $\theta > 0$

a) For a known, derive the UMP test of size α for testing $H_0: \theta \geq \theta_0$ vs. $\theta < \theta_0$,

where θ_0 is a known constant.

① Write in exponential family form.

$$f(\underset{\sim}{x}|\theta) = \prod_{i=1}^{n} \theta^{-1} e^{(a-x_i)/\theta} I(x_i > a) = \theta^{-n} e^{\sum_{i=1}^{n}(a-x_i)/\theta} I(x_{(1)} > a)$$

$$= \theta^{-n} \exp\left\{ \frac{1}{\theta} \sum_{i=1}^{n} (a-x_i) \right\} I(x_{(1)} > a)$$

$$= \theta^{-n} \exp\left\{ \frac{na}{\theta} - \frac{1}{\theta}\sum_i x_i \right\} I(x_{(1)} > a)$$

$\Rightarrow T(x) = \sum_i x_i$

Recall:

If $X_i \sim Exp(\theta) \Rightarrow \sum_i X_i \sim Gamma(n, \theta)$

If $X_i \sim Exp(\theta) \Rightarrow \frac{1}{n}\sum_i X_i \sim Gamma(n, \theta/n)$

② Write form of UMP level-α test

$$\phi(x) = \begin{cases} 1, & T(x) < K \quad \leftarrow \text{direction of alternative} \\ 0, & T(x) > K \end{cases} = \boxed{\begin{cases} 1, & \sum_i x_i < K \\ 0, & \sum_i x_i > K \end{cases}}$$

③ Find K w.r.t α

where $E_0[T(x) < K] = \alpha \Rightarrow \alpha = P_0(\sum_i x_i < K) = P_0(\sum_i (x_i - a) < K - na)$

$= F_G(K-na)$ where F_G is the cdf of a Gamma(n, θ_0)

$\Rightarrow F_G(K-na) = \alpha \Rightarrow K - na = F_G^{-1}(\alpha) \Rightarrow \boxed{K = F_G^{-1}(\alpha) + na}$

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2b) when a is known, derive the asymptotic distribution of the MLE of θ.

① First find $\hat{\theta}$

② Then, find the limiting distn. of MLE, $\sqrt{n}(\hat{\theta}-\theta) \xrightarrow{d} N(0, I_1^{-1}(\theta))$

① $\mathcal{L}(\theta \mid \underset{\sim}{X}) = \theta^{-n} e^{\sum_i (a-x_i)/\theta} I(x_{(1)} > a)$

$\Rightarrow \ell(\theta \mid \underset{\sim}{X}) = -n \log(\theta) + \sum_i (a-x_i)/\theta \ , \quad x_{(1)} > a$

$\Rightarrow \frac{\partial \ell}{\partial \theta} = \frac{-n}{\theta} - \frac{\sum_i (a-x_i)}{\theta^2} \overset{set}{=} 0 \Rightarrow \hat{\theta} = \frac{1}{n} \sum_i (a-x_i) = a - \bar{x}$

$E[x_i] = \theta + a$ by memoryless property

② $\frac{\partial^2 \ell}{\partial \theta^2} = \frac{n}{\theta^2} + \frac{2\sum_i (a-x_i)}{\theta^3} \Rightarrow -E\left[\frac{\partial^2 \ell}{\partial \theta^2}\right] = \frac{-n}{\theta^2} + \frac{2\sum_i (a-(\theta+a))}{\theta^3} = \frac{-n}{\theta^2} + \frac{2n\cancel{\theta}}{\theta^{\cancel{3}2}}$

$= \frac{n}{\theta^2}$

$\Rightarrow I_1(\theta)^{-1} = n\, I(\theta)^{-1} = \cancel{n} \cdot \frac{\theta^2}{\cancel{n}} = \theta^2$

Thus, $\boxed{\sqrt{n}(\hat{\theta}-\theta) \xrightarrow{d} N(0, \theta^2)}$

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2. In the rest of the questions, assume $a=\theta$, i.e., the density is

$$f(x)=\frac{1}{\theta}e^{(\theta-x)/\theta}\, I(x>\theta)$$

c) Prove that $\bar{X}/\theta$ and $X_{(1)}/\theta$

To show $\bar{X}/\theta$ is a pivotal quantity

Since $\frac{X_i-\theta}{\theta} \sim Exp(1)$, then $\frac{\overbrace{\frac{1}{n}\sum_i x_i}^{\bar{x}}-\theta}{\theta} \sim Gamma(n, 1/n)$.

Notice that $Gamma(n,1/n) \perp\!\!\!\perp \theta \Rightarrow \frac{\bar{X}-\theta}{\theta}=\frac{\bar{X}}{\theta}-1$ is a pivotal quantity.

Since adding/subtracting a constant doesn't affect the nature of the pivotal quantity

$\Rightarrow$ $\boxed{\frac{\bar{X}}{\theta} \text{ is a pivotal quantity}}$ ✓

To show $X_{(1)}/\theta$ is a pivotal quantity

Note: Recall for order statistics:

Min order: $f_{X_{(1)}}(x)=nf(x)\{1-F(x)\}^{n-1}$

Max order: $f_{X_{(n)}}(x)=nf(x)\{F(x)\}^{n-1}$

(Note for this problem not required)

1) Find CDF of $X_{(1)}$

$$F(x)=\int_0^x \frac{1}{\theta}e^{(\theta-t)/\theta}dt = \frac{1}{\theta}\int_0^x e^{1-t/\theta}dt = \frac{1}{\cancel{\theta}}\cdot\left(\frac{-\cancel{\theta}}{1}\right)e^{1-t/\theta}\Big|_0^x = 1-e^{1-x/\theta}, \quad x>0$$

(2) Find CDF of $T=\frac{X_{(1)}-\theta}{\theta}=\frac{X_{(1)}}{\theta}-1$

$$F(t)=F\left(\frac{X_{(1)}-\theta}{\theta}<t\right)=F(X_{(1)}<\theta t+\theta)=F(X_{(1)}<\theta(t+1))=1-F(X_{(1)}>\theta(t+1))$$

$$=1-[F(X>\theta(t+1))]^n=1-[1-F(X\le\theta(t+1))]^n=1-[1-(1-e^{1-\cancel{\theta}(t+1)/\cancel{\theta}})]^n$$

$$=1-[e^{1-(t+1)}]^n=1-e^{-nt}, \quad t>0 \Rightarrow f(t)=\frac{d}{dt}(1-e^{-nt})=ne^{-nt}, \quad t>0$$

$\perp\!\!\!\perp \theta$

$\rightarrow T=\underbrace{\frac{X_{(1)}-\theta}{\theta}}_{\frac{X_{(1)}}{\theta}-1 \text{ is a pivotal quantity}} \sim Exp(1/n) \perp\!\!\!\perp \theta$

Since adding/subtracting a constant doesn't affect the nature of the pivotal quantity

$\Rightarrow$ $\boxed{\frac{X_{(1)}}{\theta} \text{ is a pivotal quantity}}$ ✓

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2 d) Obtain two CIs, each with confidence coeff $1-\alpha$ for θ, based on the two pivotal quantities in c).

From part c), showed $\frac{\bar{X}-\theta}{\theta} \sim \text{Gamma}(n, 1/n)$, so $\frac{\bar{X}-\theta}{\theta}$ a pivotal quantity.

and $\frac{X_{(1)}-\theta}{\theta} \sim \text{Exp}(n)$, so $\frac{X_{(1)}-\theta}{\theta}$ a pivotal quantity.

Take $\underbrace{a_1}_{\alpha/2 \text{ quantile of Gamma}(n,1/n)} < \frac{\bar{X}-\theta}{\theta} < \underbrace{b_1}_{1-\alpha/2 \text{ quantile of Gamma}(n,1/n)}$ where $F_G(a_1) = \alpha/2 \Rightarrow a_1 = F^{-1}_G(\alpha/2)$

$F_G(b_1) = 1-\alpha/2 \Rightarrow b_1 = F^{-1}_G(1-\alpha/2)$

where $F_G(\cdot)$ is the cdf of a Gamma$(n, 1/n)$.

$$\Rightarrow F^{-1}_G(\alpha/2) < \frac{\bar{X}-\theta}{\theta} < F^{-1}_G(1-\alpha/2)$$

$$\Rightarrow F^{-1}_G(\alpha/2) < \frac{\bar{X}}{\theta} - 1 < F^{-1}_G(1-\alpha/2) \Rightarrow 1+F^{-1}_G(\alpha/2) < \frac{\bar{X}}{\theta} < 1+F^{-1}_G(1-\alpha/2)$$

$$\Rightarrow \frac{\bar{X}}{1+F^{-1}_G(1-\alpha/2)} < \theta < \frac{\bar{X}}{1+F^{-1}_G(\alpha/2)}$$ where $F_G(\cdot)$ is the cdf of a Gamma$(n, 1/n)$

Similarly, take $\underbrace{a_2}_{\alpha/2 \text{ quantile of Exp}(n)} < \frac{X_{(1)}-\theta}{\theta} < \underbrace{b_2}_{1-\alpha/2 \text{ quantile of Exp}(n)}$ where $F_E(a_2) = 1-e^{-n(a_2)} = \alpha/2$

$$\Rightarrow e^{-n(a_2)} = 1-\alpha/2$$

$$\Rightarrow -n(a_2) = \log(1-\alpha/2)$$

$$\Rightarrow a_2 = -\tfrac{1}{n}\log(1-\alpha/2)$$

$$F_E(b_2) = 1-e^{-n(b_2)} = 1-\alpha/2$$

$$\Rightarrow e^{-n(b_2)} = \alpha/2$$

$$\Rightarrow -n(b_2) = \log(\alpha/2)$$

$$\Rightarrow b_2 = -\tfrac{1}{n}\log(\alpha/2)$$

$$\Rightarrow -\tfrac{1}{n}\log(1-\alpha/2) < \frac{X_{(1)}-\theta}{\theta} < -\tfrac{1}{n}\log(\alpha/2)$$

$$\Rightarrow -\tfrac{1}{n}\log(1-\alpha/2) < \frac{X_{(1)}}{\theta} - 1 < -\tfrac{1}{n}\log(\alpha/2)$$

$$\Rightarrow 1-\tfrac{1}{n}\log(1-\alpha/2) < \frac{X_{(1)}}{\theta} < 1-\tfrac{1}{n}\log(\alpha/2)$$

$$\Rightarrow \frac{X_{(1)}}{1-\frac{1}{n}\log(\alpha/2)} < \theta < \frac{X_{(1)}}{1-\frac{1}{n}\log(1-\alpha/2)}$$

Thus,
$$\boxed{\begin{aligned} CI_1 &= \left\{\theta : \frac{\bar{X}}{1+F^{-1}_G(1-\alpha/2)} < \theta < \frac{\bar{X}}{1+F^{-1}_G(\alpha/2)}\right. \\ CI_2 &= \left\{\theta : \frac{X_{(1)}}{1-\frac{1}{n}\log(\alpha/2)} < \theta < \frac{X_{(1)}}{1-\frac{1}{n}\log(1-\alpha/2)}\right\} \end{aligned}}$$

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2e) When n is sufficiently large, which of the two CIs has shorter length? Justify.

For CI_1:

$$\text{Width}(CI_1) = \frac{\bar{X}}{1+F^{-1}_G(\alpha/2)} - \frac{\bar{X}}{1+F^{-1}_G(1-\alpha/2)} = \bar{X}\left[\frac{1}{1+F^{-1}_G(\alpha/2)} - \frac{1}{1+F^{-1}_G(1-\alpha/2)}\right]$$

$$= \bar{X}\left[\frac{\cancel{1}+F^{-1}_G(1-\alpha/2) - \cancel{1} - F^{-1}_G(\alpha/2)}{(1+F^{-1}_G(\alpha/2))(1+F^{-1}_G(1-\alpha/2))}\right] = \bar{X}\left[\frac{F^{-1}_G(1-\alpha/2) - F^{-1}_G(\alpha/2)}{(1+F^{-1}_G(\alpha/2))(1+F^{-1}_G(1-\alpha/2))}\right]$$

From CLT, $\lim_{n\to\infty} \sqrt{n}\left(F^{-1}_G(1-\alpha/2) - F^{-1}_G(\alpha/2)\right) > 0$ assuming $\alpha < 1/2$.

(Know the asymptotic distn. for the pth sample quantile is $N\left(p, \frac{p(1-p)}{nf(x_p)^2}\right)$ where $f(x_p)$ is the value of the distn. density at the pth quantile.) See wiki "quantile"

For CI_2:

$$\text{Width}(CI_2) = \frac{X_{(1)}}{1-\frac{1}{n}\log(1-\alpha/2)} - \frac{X_{(1)}}{1-\frac{1}{n}\log(\alpha/2)} = X_{(1)}\left[\frac{1}{1-\frac{1}{n}\log(1-\alpha/2)} - \frac{1}{1-\frac{1}{n}\log(\alpha/2)}\right]$$

$$= X_{(1)}\left[\frac{1-1/n\log(\alpha/2) - 1 + \frac{1}{n}\log(1-\alpha/2)}{(1-\frac{1}{n}\log(1-\alpha/2))(1-\frac{1}{n}\log(\alpha/2))}\right] = X_{(1)}\left[\frac{\frac{1}{n}[\log(1-\alpha/2) - \log(\alpha/2)]}{(1-\frac{1}{n}\log(1-\alpha/2))(1-\frac{1}{n}\log(\alpha/2))}\right]$$

Notice that
$$X_{(1)}\left[\frac{\overbrace{\frac{1}{n}[\log(1-\alpha/2) - \log(\alpha/2)]}^{\to 0}}{\underbrace{(1-\frac{1}{n}\log(1-\alpha/2))}_{\to 1}\underbrace{(1-\frac{1}{n}\log(\alpha/2))}_{\to 1}}\right] \to 0 \text{ as } n\to\infty$$

Thus, since $\text{Width}(CI_1) \not\to 0$ as $n\to\infty$

but $\text{width}(CI_2) \to 0$ as $n \to \infty$

$$\Rightarrow \boxed{\text{Width}(CI_2) < \text{width}(CI_1) \text{ for large } n.}$$