2016 - 2 - 2 pg. 6

$$E = \left\{ u : u^{\circ} = \begin{pmatrix} \beta_1 + \beta_2 - \beta_3 \\ \beta_2 + \beta_3 \\ -\beta_2 - \beta_3 \\ -\beta_1 - \beta_2 + \beta_3 \end{pmatrix} \right\}$$

$$u = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$$

$cov(Y) = \sigma^2 I_{4\times4}$

a) $\hat{\mu} = MY$ $\qquad M = X(X'X)^{-}X'$

$$X = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ -1 & -1 & 1 \end{bmatrix}_{4\times3}$$

$c1 \quad c2 \quad c3$ rank = 3

$c3 = c2 - 2c1$

$\Rightarrow$ keep $c1$ & $c2$

$$X^* = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \\ -1 & -1 \end{bmatrix} \qquad \Rightarrow M = X^*(X^{*\prime}X^*)^{-1}X^{*\prime}$$

$$X^{*\prime}X^* = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\Rightarrow (X^{*\prime}X^*)^{-1} = \frac{1}{8-4}\begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$X^*(X^{*\prime}X^*)^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}$$

$$X^*(X^{*\prime}X^*)^{-1}X^{*\prime} = \underset{4\times2}{\begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}} \underset{2\times4}{\begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

a) cont'd

$$\hat{\mu} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}_{4\times 4} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}_{4\times 1}$$

$$= \begin{bmatrix} \frac{1}{2}(y_1 - y_4) \\ \frac{1}{2}(y_2 - y_3) \\ \frac{1}{2}(y_3 - y_2) \\ \frac{1}{2}(y_4 - y_1) \end{bmatrix}. \quad \checkmark$$

b) BLUE of $\beta_2 - \beta_3$

$$(\beta_2 - \beta_3) = \lambda' \beta = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}' \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} =$$

est. if $\exists \rho$ s.t.

$$\underset{1\times 4}{\rho'} \, \underset{4\times 3}{X} = \underset{1\times 3}{\lambda'}$$

$$(p_1 \quad p_2 \quad p_3 \quad p_4) \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ -1 & -1 & 1 \end{bmatrix} = (0 \quad 1 \quad -1)$$

$$0 = p_1 - p_4$$

$$1 = p_1 + p_2 - p_3 - p_4$$

$$-1 = -p_1 + p_2 - p_3 + p_4$$

$$\Rightarrow \quad p_2 - p_3 = 1 \qquad -(p_1 - p_4) = -p_1 + p_4 = 0$$

$$\Rightarrow \quad 0 + 1 \neq -1$$

$\boxed{\text{not estimable}}$

2016 - 2 - 2 pg. 8

c) $H_0: \beta_2 + \beta_3 = 0$ vs. $H_1: \beta_2 + \beta_3 \neq 0$

$\beta_2 + \beta_3 = 0 \Rightarrow \beta_3 = -\beta_2$

E_0 is E where H_0 true

E_0 & $E \cap E_0^{\perp}$ where $E_0^{\perp}$ orthogonal complement of E_0

$E_0 = \{u : u' = (\beta_1 + \beta_2 + \beta_2,\ \beta_2 - \beta_2,\ -\beta_2 + \beta_2,\ -\beta_1 - \beta_2 - \beta_2)\}$

$\beta_1 + 2\beta_2,\ 0,\ 0,\ -\beta_1 - 2\beta_2$

$\Rightarrow E_0 = \{u : u' = (\beta_1 + 2\beta_2, 0, 0, -\beta_1 - 2\beta_2)\}$

$= \text{span}\left\{\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}\right\}$

Gram Schmidt

$E = \text{span}\left\{\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}, \begin{bmatrix}-1\\1\\-1\\1\end{bmatrix}\right\}$

$= \text{span}\left\{\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}\right\}$

$\begin{matrix}0\\-1\\1\\0\end{matrix}$

$[1\ \ 0\ \ 0\ -1]\begin{bmatrix}a\\b\\c\\d\end{bmatrix} = a - d = 0$

$\Rightarrow a = d$

$\Rightarrow E \cap E_0^{\perp} = \text{span}\left\{\begin{bmatrix}0\\-1\\1\\0\end{bmatrix}\right\}$.

c) $H_0: \beta_2 + \beta_3 = 0$ $\qquad$ $\beta_2 + \beta_3 = 0 \Rightarrow \cancel{\beta_2 = \beta_3}$

vs.

$H_1: \beta_2 + \beta_3 \neq 0$ $\qquad$ $\beta_3 = -\beta_2$

E_0 is E where H_0 true

E_0 & $E \cap E_0^{\perp}$ where $E_0^{\perp}$ orthogonal complement of E_0

$$E_0 = \{u : u' = (\beta_1 + \beta_2 + \beta_2,\ \beta_2 - \beta_2,\ -\beta_2 \overset{+\beta_2}{\cancel{+\beta_2}},\ -\beta_1 - \beta_2 - \beta_2)\}$$

$$\beta_1 + 2\beta_2,\ 0,\ 0,\ -\beta_1 - 2\beta_2$$

$$\Rightarrow E_0 = \{u : u' = (\beta_1 + 2\beta_2, 0, 0, -\beta_1 - 2\beta_2)\}$$

$$= \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}\right\}$$

Gram Schmidt

$$E = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right\}$$

$$= \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}\right\}$$

$$\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$[1\ \ 0\ \ 0\ \ -1]\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = a - d = 0$$

$$\Rightarrow a = d$$

$$\Rightarrow E \cap E_0^{\perp} = \text{span}\left\{\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}\right\}.$$

d) $\lambda = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$\cancel{[\lambda'\beta]^T (X'X)^- [\lambda'\beta]}$

$$\frac{(\lambda'\hat{\beta})^2}{MSE\ \lambda'(X'X)^-\lambda}$$

or $\lambda' = \rho' X$

$$M_{MP} = MP(P'MP)^- P'M$$

$$F = \frac{\|M_{MP}Y\|^2 / r(M_{MP})}{\|(I-M)Y\|^2 / r(I-M)} \sim F_{r(M_{MP}),\, r(I-M),\, \gamma}$$

$$\gamma = \frac{\|M_{MP}X\beta\|^2}{2\sigma^2}$$

$$(0 \quad 1 \quad 1) = (p_1 \quad p_2 \quad p_3 \quad p_4)\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

$\Rightarrow$ $0 = p_1 - p_4$

$1 = p_1 + p_2 - p_3 - p_4$

$1 = -p_1 + p_2 - p_3 + p_4$

$1 = p_2 - p_3$

$\Rightarrow 1 = p_2 - p_3$ ✓

let $p_1 = 0$

$p_4 = 0$

$p_2 = 1$

$p_3 = 0$

$$P'MP = (0 \quad 1 \quad 0 \quad 0)\,\frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{1}{2}(0 \quad 1 \quad 0 \quad 0)\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{2}[1] = \frac{1}{2}$$

$\Rightarrow (P'MP)^- = 2$

$\frac{}{4}[(y_1 + y_4) + \ldots]$

2016-2-2 pg. 10

d) cont'd

$$MP(P'MP)^- = \frac{1}{2}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}(2) = \begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$$

$$\exists\ MP(P'MP)^-P'M = \frac{1}{2}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\begin{pmatrix}0 & 1 & -1 & 0\end{pmatrix} =$$

$$= \frac{1}{2}\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 1 & -1 & 0\\ 0 & -1 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$\|M_{MP}Y\|^2$

$$= Y'M_{MP}Y = (y_1\ \ y_2\ \ y_3\ \ y_4)\frac{1}{2}\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 1 & -1 & 0\\ 0 & -1 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{pmatrix}y_1\\y_2\\y_3\\y_4\end{pmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}0\\ y_2-y_3\\ y_3-y_2\\ 0\end{bmatrix}'\begin{pmatrix}y_1\\y_2\\y_3\\y_4\end{pmatrix}$$

$$= \frac{1}{2}\left[y_2(y_2-y_3) + y_3(y_3-y_2)\right]$$

$$= \frac{1}{2}\left[y_2(y_2-y_3) - y_3(y_2-y_3)\right]$$

$$= \frac{1}{2}\left[(y_2-y_3)(y_2-y_3)\right]$$

$$= \frac{1}{2}(y_2-y_3)^2$$

$r(M_{MP}) = 1.$

$$I - M = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1 & 0 & 0 & -1\\ 0 & 1 & -1 & 0\\ 0 & -1 & 1 & 0\\ -1 & 0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}\frac{1}{2} & 0 & 0 & \frac{1}{2}\\ 0 & \frac{1}{2} & \frac{1}{2} & 0\\ 0 & \frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{2} & 0 & 0 & \frac{1}{2}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 1 & 0\\ 0 & 1 & 1 & 0\\ 1 & 0 & 0 & 1\end{bmatrix}$$

$r(I-M) = 2.$

d) cont'd 2016-2-2 pg. 11

$$Y'(1-M)Y = (y_1 \quad y_2 \quad y_3 \quad y_4)' \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} y_1 + y_4 \\ y_2 + y_3 \\ y_2 + y_3 \\ y_1 + y_4 \end{bmatrix}'\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

$$= \frac{1}{2}\Big[y_1(y_1+y_4) + y_4(y_1+y_4) + y_2(y_2+y_3) + y_3(y_2+y_3) \Big]$$

$$= \frac{1}{2}\Big[(y_1+y_4)^2 + (y_2+y_3)^2 \Big]$$

check

$$1-M = 1 + \frac{1}{2}\begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$\Rightarrow F = \frac{\frac{1}{2}(y_2-y_3)^2}{\frac{1}{4}\left[(y_1+y_4)^2 + (y_2+y_3)^2\right]} = \frac{2(y_2-y_3)^2}{(y_1+y_4)^2 + (y_2+y_3)^2}$$

$NF(1, 2, \gamma)$

where under alt, $\gamma = \dfrac{\beta' X' M_{MP} X\beta}{2\sigma^2}$

$$M_{MP}X\beta = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 \\ 0 & -1 & +1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} \beta_1+\beta_2-\beta_3 \\ \beta_2+\beta_3 \\ -\beta_2-\beta_3 \\ -\beta_1-\beta_2+\beta_3 \end{pmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ +(\beta_2+\beta_3)+\beta_2+\beta_3 \\ -(\beta_2+\beta_3)+-\beta_2-\beta_3 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 \\ 2(\beta_2+\beta_3) \\ -2(\beta_2+\beta_3) \end{bmatrix}$$

d) cont'd

$$(X\beta)' M_{MP} X\beta = \begin{pmatrix} \beta_1+\beta_2-\beta_3 \\ \beta_2+\beta_3 \\ -\beta_2-\beta_3 \\ -\beta_1-\beta_2+\beta_3 \end{pmatrix}^T \cdot \frac{1}{2} \begin{bmatrix} 0 \\ 2(\beta_2+\beta_3) \\ -2(\beta_2+\beta_3) \\ 0 \end{bmatrix}$$

$$= \frac{1}{2}\left[2(\beta_2+\beta_3)^2 + 2(\beta_2+\beta_3)^2 \right]$$

$$= 2(\beta_2+\beta_3)^2$$

$\Rightarrow \gamma = \dfrac{(\beta_2+\beta_3)^2}{\sigma^2}$ under alt and 0 under null

e) $\lambda'\hat{\beta} = \rho' M Y$

$= (0 \; 1 \; 0 \; 0) \frac{1}{2} \begin{bmatrix} y_1-y_4 \\ y_2-y_3 \\ y_3-y_2 \\ y_4-y_1 \end{bmatrix}$ $\lambda = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$= \frac{1}{2}(y_2-y_3)$

(y_2-y_3).

$\lambda'\beta = \beta_2+\beta_3$

$\Rightarrow \{\beta: \frac{1}{4}(y_2-y_3)^2$

e) $\left\{ \beta : \dfrac{(\Lambda'\hat{\beta} - \Lambda'\beta)^T (\Lambda'(X'X)^- \Lambda)^- (\Lambda'\hat{\beta} - \Lambda'\beta) / r(\Lambda)}{MSE} \leq c_\alpha \right\}$

$c_\alpha = F(1-\alpha, r(\Lambda), r(I-M))$

Now

$MSE = \| (I-M)Y \|^2 / r(I-M) = \frac{1}{4}[(y_1+y_4)^2 + (y_2+y_3)^2]$

$(X'X)^- = (X^{*\prime}X^*)^{-1}$

$(X^{*\prime}X^*)^{-1} = \frac{1}{4}\begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}$

$\lambda' = (0 \ 1 \ 1) \rightarrow$ drop last column $\rightarrow (0 \ 1)$

$\Rightarrow \lambda'(X^{*\prime}X^*)^{-1} = \frac{1}{4}(0 \quad 1) \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}$

$= \frac{1}{4}[-2 \quad 2]$

$\Rightarrow \lambda'(X^{*\prime}X^*)^{-1}\lambda = \frac{1}{4}(-2 \quad 2)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$= \frac{1}{4}(2) = \frac{1}{2}$

$\Rightarrow (\Lambda'(X'X)^-\Lambda)^- = 2$

Now $\lambda'\beta = (0 \ 1 \ 1)\begin{pmatrix} \beta_2 \\ \beta_3 \end{pmatrix} = \beta_2 + \beta_3$

$\lambda'\hat{\beta} = \rho' M Y = (0 \ 1 \ 0 \ 0)\frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & +1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$

$= \frac{1}{2}(y_2 - y_3)$

$\Rightarrow (\lambda'\hat{\beta} - \lambda'\beta)^T = \frac{1}{2}(y_2 - y_3) - (\beta_2 + \beta_3)$

$\Rightarrow \left\{ \beta : \dfrac{(\frac{1}{2}(y_2-y_3) - (\beta_2+\beta_3))^2 \, 2/1}{\frac{1}{4}[(y_1+y_4)^2 + (y_2+y_3)^2]} \leq F(0.95, 1, 2) \right\}$