2016 Qual, Section 1 Ann Marie Weideman

1. (25 points) Let $X_1, ..., X_n$ be i.i.d from the following distribution

$$\begin{cases} 0 & \text{with probability } p, \\ \text{Uniform}[0, \theta] & \text{with probability } 1 - p. \end{cases}$$

First, we assume that p is a known constant in $(0, 1)$ and that $\theta > 0$ is the only parameter of interest.

(a) (5 points) Based on only one observation X_1, find all the unbiased estimators for θ and calculate their variances. Does UMVUE exist for θ? Justify your answer.

(b) (3 points) Based on n observations $X_1, ..., X_n$, let $X_{(n)} = \max\{X_1, ..., X_n\}$ be the maximal observation. Show that $(X_{(n)}, \sum_{i=1}^n I(X_i > 0))$ is a sufficient statistic for θ. Furthermore, show that $\widehat{\theta} = X_{(n)}$ maximizes the observed likelihood function.

(c) (5 points) What is the exact distribution of $\widehat{\theta}$? Compute $E[\widehat{\theta}]$ and $Var(\widehat{\theta})$ and show that $\widehat{\theta}$ is consistent for θ.

(d) (6 points) Derive the asymptotic distribution of $n(\widehat{\theta} - \theta)$.

Now assume that both p and θ are unknown.

(e) (6 points) Calculate the maximum likelihood estimator for p to obtain the maximum likelihood estimator for $E[X_1]$. Derive the asymptotic distribution for the latter after proper normalization.

1 a) 2016 Qual, Section 1 AMW

Given $X_1, \dots, X_n \overset{iid}{\sim} \begin{cases} 0 & \text{w/ prob } p \\ \text{Unif}[0,\theta] & \text{w/ prob } 1-p \end{cases}$

$p \in (0,1)$ known.

$\theta > 0$ is the only parameter of interest.

Let $Y \sim Bin(p)$. Then, $f(x|y) = \begin{cases} 1 & \text{if } Y=1 \\ \frac{1}{\theta} & \text{if } Y=0 \end{cases}$

Then, $E[X] = E[E[X|Y]] = E\left[E[X|Y=1]\, I(Y=1) + E[X|Y=0]\, I(Y=0) \right]$

$= E\left[\underbrace{X \cdot P(X|Y=1)}_{0} \cdot I(Y=1) + \left[\int_0^\theta x \underbrace{P(X|Y=0)}_{\frac{1}{\theta}} dx \right] I(Y=0) \right]$

$= E\left[\left[\int_0^\theta x \cdot \frac{1}{\theta} dx \right] I(Y=0) \right] = E\left[\left(\frac{1}{2\theta} x^2 \Big|_0^\theta \right) I(Y=0) \right] = E\left[\frac{\theta}{2} I(Y=0) \right]$

$= \frac{\theta}{2} P(Y=0) = \frac{\theta \cdot (1-p)}{2}$

Since $E[X] = \frac{\theta(1-p)}{2} \Rightarrow E\left[\frac{2X}{(1-p)}\right] = \theta \Rightarrow$ $\frac{2X}{(1-p)}$ is an unbiased estimator of θ. for $x \in [0,\theta]$

$Var(\hat{\theta}) = Var\left(\frac{2X}{(1-p)}\right) = \frac{4}{(1-p)^2} Var(X)$ ← find this (*)

$Var(X) = E[X^2] - E[X]^2 = E[E[X^2|Y]] - E[E[X|Y]]^2$

$= E\left[E[X^2|Y=1]\, I(Y=1) + E[X^2|Y=0]\, I(Y=0) \right]$

$- \left\{ E\left[E[X|Y=1] I(Y=1) + E[X|Y=0] I(Y=0) \right] \right\}^2$

$= E\left[\overset{0}{\cancel{X^2 \cdot P(X|Y=1)}} I(Y=1) + \left(\int_0^\theta x^2 \cdot P(X|Y=0)\, dx \right) I(Y=0) \right]$

$- \underbrace{\left\{ \frac{\theta(1-p)}{2} \right\}^2}_{\text{from first part}} = E\left[\frac{1}{3} x^3 \cdot \frac{1}{\theta} \Big|_0^\theta I(Y=0) \right] - \frac{\theta^2 (1-p)^2}{4}$

$= \frac{1}{3}\theta^2 P(Y=0) - \frac{\theta^2(1-p)^2}{4} = \frac{\theta^2(1-p)}{3} - \frac{\theta^2(1-p)^2}{4}$

$= \frac{\theta^2(1-p)(4-3(1-p))}{12}$

$\overset{(*)}{\Rightarrow} Var(\theta) = \frac{\cancel{4}}{(1-p)^{\cancel{2}}} \cdot \frac{\theta^2 \cancel{(1-p)}(4-3(1-p))}{\underset{3}{\cancel{12}}} = \frac{4\theta^2 - 3(1-p)\theta^2}{3(1-p)}$

$= \frac{4\theta^2 - 3\theta^2 + 3p\theta^2}{3(1-p)} = \boxed{\frac{\theta^2(1+3p)}{3(1-p)}}$

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1a) cont'd

Even if an unbiased estimator exists, this <u>does not</u> guarantee that a UMVUE exists.

Recall that the necessary-sufficient condition for an unbiased estimator to be the UMVUE says that the unbiased estimator <u>MUST</u> be uncorrelated w/ every unbiased estimator of 0.

However, $T(x) = \frac{2X}{1-p}$ for $X \in [0, \theta]$ represents all unbiased estimators of θ. Thus, a necessary and sufficient condition for $T(x)$ to be the UMVUE of θ is that $T(X)$ must be uncorrelated w/ every unbiased estimator of 0.

However $E[T(X)] = 0 \Rightarrow X = 0$ and $T(x) = \frac{2X}{1-p} \not\perp X$, so not uncorrelated w/ estimator of 0.

Thus, a UMVUE does not exist for θ. ┘

Amw

1 b) Based on n obs, $X_1, \dots, X_n$, let $X_{(n)} = \max\{X_1, \dots, X_n\}$ be the max obs

i) Show that $\left(X_{(n)}, \sum_i I(X_i > 0)\right)$ is a SS for θ.

ii) Show that $\hat{\theta} = X_{(n)}$ maximizes the observed likelihood fcn.

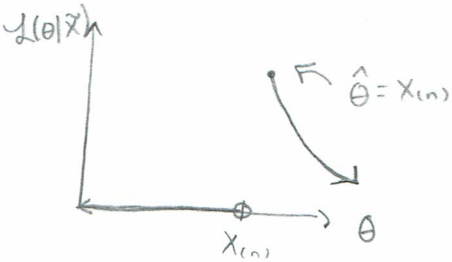
(i)
$$\mathcal{L}(\theta \mid \tilde{x}) = \prod_{i=1}^{n} p^{I(x_i=0)} \left[(1-p)\frac{1}{\theta}\right]^{I(0<x_i\le\theta)} I(0 \le x_i \le \theta)$$

$$= p^{\sum_i I(x_i=0)} \left[(1-p)\frac{1}{\theta}\right]^{\sum_i I(0<x_i\le\theta)} I(0 \le x_{(1)})\, I(x_{(n)} \le \theta)$$

$$= \underbrace{p^{\sum_i I(x_i=0)} I(0 \le x_{(1)})}_{h(\underset{\sim}{x})} \cdot \underbrace{\left[(1-p)/\theta\right]^{\sum_i I(0<x_i\le\theta)} I(x_{(n)} \le \theta)}_{g(\theta, \underbrace{T(\mathbf{x})}_{SS})}$$

$$\Rightarrow \boxed{T(\underset{\sim}{x}) = \left(X_{(n)}, \sum_i I(0<x_i)\right) \text{ is a SS for } \theta \text{ by factorization thm.}}$$

(ii)



As θ increases, $\mathcal{L}(\theta|\tilde{x})$ decreases.

For $\theta < X_{(n)} \Rightarrow I(X_{(n)} \le \theta) = 0$ so the whole likelihood is 0.

For $X_{(n)} \le \theta \Rightarrow I(0 < X_i \le \theta) = 1$ so $\mathcal{L}(\theta|\tilde{x}) > 0$ and decreasing as a fcn of θ.

Thus, $\boxed{\hat{\theta} = X_{(n)}}$

2016 Qual, Section 1

AMW

1.c) i) What is the exact distn. of $\hat{\theta}$?

ii) Compute $E[\hat{\theta}]$ and $Var[\hat{\theta}]$ and show that $\hat{\theta}$ is consistent for θ.

i) From b), had $\hat{\theta} = X_{(n)}$

Want to find pdf of $X_{(n)}$.

From a), $f(x|y) = \begin{cases} 1 & \text{if } y=1 \\ \frac{1}{\theta} & \text{if } y=0 \end{cases}$ for $Y \sim Bern(p)$

$$\Rightarrow F(X|Y) = \begin{cases} 1 & \text{if } y=1 \\ \frac{x}{\theta} & \text{if } y=0 \end{cases} \quad (*)$$

Then, $F_{X_{(n)}}(x) = P(X_{(n)} < x) = [P(X_1 < x)]^n = [F(x_1)]^n = [P(0 \le X_1 \le x)]^n$

$= [P(0 \le X_1 \le x \mid Y_1 = 0) P(Y_1 = 0) + P(0 \le X_1 \le x \mid Y_1 = 1) \cdot P(Y_1 = 1)]^n$

$\overset{(*)}{=} \left[\frac{x}{\theta}(1-p) + 1 \cdot p\right]^n, \quad 0 \le x \le \theta$

$\Rightarrow f_{X_{(n)}}(x) = \frac{d}{dx}\left[\frac{x}{\theta}(1-p) + p\right]^n = \frac{n(1-p)}{\theta}\left[\frac{x}{\theta}(1-p)+p\right]^{n-1}, \quad 0 \le x \le \theta$

ii) To compute $E[\underbrace{\hat{\theta}}_{X_{(n)}}]$ and $Var[\underbrace{\hat{\theta}}_{X_{(n)}}]$, will use distribution just derived above in (i).

$E[\hat{\theta}] = E[X_{(n)}] = \int_0^\theta \frac{n(1-p)}{\theta}\left[\frac{x}{\theta}(1-p)+p\right]^{n-1} x \, dx$

Let $u = \frac{x}{\theta}(1-p) + p$

$\Rightarrow du = \frac{1}{\theta}(1-p)\,dx$

$\Rightarrow \frac{\theta\, du}{(1-p)} = dx$

$\therefore \frac{\theta(u-p)}{(1-p)} = x$

Lower Bound: $u = \frac{0}{\theta}(1-p) + p = p$

Upper Bound: $u = \frac{\theta}{\theta}(1-p) + p = 1$

$\overset{\text{sub}}{=} \int_p^1 \frac{n\cancel{(1-p)}}{\cancel{\theta}} u^{n-1} \underbrace{\frac{\cancel{\theta}(u-p)}{(\cancel{1-p})}}_{x} \cdot \underbrace{\frac{\theta\, du}{(1-p)}}_{dx}$

$= \frac{n\theta}{(1-p)} \int_p^1 u^{n-1}(u-p)\,du$

$= \frac{n\theta}{(1-p)}\left[\frac{1}{n+1}u^{n+1} - \frac{p}{n}u^n\right]\Big|_p^1$

$E[\hat{\theta}]$

$= \boxed{\frac{n\theta}{(1-p)}\left[\frac{1}{n+1} - \frac{p}{n} - \frac{p^{n+1}}{n+1} + \frac{p^{n+1}}{n}\right]} = \frac{\theta}{(1-p)}\left[\underbrace{\frac{n}{n+1}}_{\xrightarrow{P} 1} - p - \underbrace{\frac{n}{n+1}}_{\xrightarrow{P} 1} \underbrace{p^{n+1}}_{\xrightarrow{P} 0} + \underbrace{p^{n+1}}_{\xrightarrow{P} 0}\right]$

(since $\cancel{0} < p < 1$)

$\xrightarrow{P} \frac{\theta}{(1-p)}[1-p] = \theta$ by Slutsky's

$= \boxed{\frac{n\theta}{(1-p)}\left[\frac{1-p^{n+1}}{n+1} + \frac{p(p^n-1)}{n}\right]}$

2016 Qual, Section 1

1.c) ii)

$$E[\hat{\theta}^2] = E[X_{(n)}^2] = \int_0^\theta \frac{n(1-p)}{\theta}\left[\frac{x}{\theta}(1-p)+p\right]^{n-1} x^2 \, dx$$

Let $u = \frac{x}{\theta}(1-p)+p$

$\Rightarrow du = \frac{1}{\theta}(1-p)\,dx$

$\Rightarrow \frac{\theta\, du}{(1-p)} = dx$

& $\frac{\theta(u-p)}{(1-p)} = x \Rightarrow \frac{\theta^2(u-p)^2}{(1-p)^2} = x^2$

Lower Bound: $u = \frac{0}{\theta}(1-p)+p = p$

Upper Bound: $u = \frac{\theta}{\theta}(1-p)+p = 1$

$$\overset{\text{sub}}{=} \int_p^1 \frac{n\cancel{(1-p)}}{\cancel{\theta}} u^{n-1} \underbrace{\frac{\theta^2(u-p)^2}{(1-p)^2}}_{x^2} \cdot \underbrace{\frac{\cancel{\theta}\, du}{\cancel{(1-p)}}}_{dx}$$

$$= \frac{n\theta^2}{(1-p)^2}\int_p^1 u^{n-1}(u-p)^2\,du = \frac{n\theta^2}{(1-p)^2}\int_p^1 u^{n-1}(u^2-2up+p^2)\,du$$

$$= \frac{n\theta^2}{(1-p)^2}\int_p^1 (u^{n+1}-2pu^n+p^2u^{n-1})\,du = \frac{n\theta^2}{(1-p)^2}\left[\frac{1}{n+2}u^{n+2} - \frac{2p}{n+1}u^{n+1} + \frac{p^2}{n}u^n\right]\Big|_p^1$$

$$= \frac{n\theta^2}{(1-p)^2}\left[\left(\frac{1}{n+2} - \frac{2p}{n+1} + \frac{p^2}{n}\right) - \left(\frac{p^{n+2}}{n+2} - \frac{2p^{n+2}}{n+1} + \frac{p^{n+2}}{n}\right)\right]$$

$$= \frac{n\theta^2}{(1-p)^2}\left[\frac{1-p^{n+2}}{n+2} + \frac{2p(p^{n+1}-1)}{n+1} + \frac{p^2(1-p^n)}{n}\right]$$

Then, $Var(\hat{\theta}) = E[\hat{\theta}^2] - E[\hat{\theta}]^2 =$

$$\boxed{\frac{n\theta^2}{(1-p)^2}\left[\frac{1-p^{n+2}}{n+2} + \frac{2p(p^{n+1}-1)}{n+1} + \frac{p^2(1-p^n)}{n}\right] - \frac{n^2\theta^2}{(1-p)^2}\left[\frac{1-p^{n+1}}{n+1} + \frac{p(p^n-1)}{n}\right]^2}$$

where

$$\lim_{n\to\infty} Var(\hat{\theta}) = \lim_{n\to\infty}\left\{\frac{\theta^2}{(1-p)^2}\left[\overbrace{\frac{n}{n+2}}^{\xrightarrow{P} 1}(1-\overbrace{p^{n+2}}^{\xrightarrow{P} 0}) + \overbrace{\frac{n}{n+1}}^{\xrightarrow{P} 1} 2p(\overbrace{p^{n+1}}^{\xrightarrow{P} 0}-1) + p^2(1-\overbrace{p^n}^{\xrightarrow{P} 0})\right] - \frac{\theta^2}{(1-p)^2}\left[\overbrace{\frac{n}{n+1}}^{\xrightarrow{P} 1}(1-\overbrace{p^{n+1}}^{\xrightarrow{P} 0}) + p(\overbrace{p^n}^{\xrightarrow{P} 0}-1)\right]^2\right\}$$

since $0<p<1$

$$\longrightarrow \left\{\frac{\theta^2}{(1-p)^2}\left[1-2p+p^2\right] - \frac{\theta^2}{(1-p)^2}\left[1-p\right]^2\right\}$$

$$= \frac{\theta^2}{(1-p)^2}\left[1-p\right]^2 - \frac{\theta^2}{(1-p)^2}\left[1-p\right]^2 = 0$$ by Slutsky's

Thus, since $E[\hat{\theta}] \xrightarrow{P} \theta$ (previous page)

& $Var(\hat{\theta}) \xrightarrow{P} 0$ (above),

then $\hat{\theta}$ is a consistent estimator of θ.

2016 Qual, Section 1

AMW

1 d) Find the asymptotic distn. of $n(\hat{\theta}-\theta)$.

I know from part c) that $\hat{\theta} = X_{(n)}$ has cdf $F_{X_{(n)}}(x) = \left[\frac{x}{\theta}(1-p)+p\right]^n$, $0 \le x \le \theta$

Let's first find the asimptotic distn. of $n(\theta-\hat{\theta})$ (instead of $n(\hat{\theta}-\theta)$), and then we will manipulate this resulting distn. to get that for $n(\hat{\theta}-\theta)$.

Take $P(n(\theta-t) \le z) = P(\theta - t \le z/n) = P(-t \le z/n - \theta)$

$$= P(t \ge \theta - z/n) = 1 - P(t \le \theta - z/n) = 1 - F_{X_{(n)}}[\theta - z/n]$$

$$= 1 - \left[\frac{(\theta - z/n)}{\theta}(1-p)+p\right]^n = 1 - \left[\left(1-\frac{z}{n\theta}\right)(1-p)+p\right]^n = 1 - \left[1 - \cancel{p} - \frac{z}{n\theta} + \frac{zp}{n\theta} + \cancel{p}\right]^n$$

$$= 1 - \left[1 - \frac{z(1-p)/\theta}{n}\right]^n \xrightarrow{d} \underbrace{1 - e^{-(1-p)/\theta \, z}}$$

[cdf of $\text{Exp}\left(\frac{\theta}{(1-p)}\right)$ assuming pdf of exponential is $f(x) = \lambda e^{-\lambda x}$ where $E(X) = 1/\lambda$]

Thus, since $n(\theta - \hat{\theta}) \xrightarrow{d} \text{Exp}\left(\frac{\theta}{(1-p)}\right)$, then $-n(\hat{\theta}-\theta) \xrightarrow{d} \text{Exp}\left(\frac{\theta}{(1-p)}\right)$

$\Rightarrow$ $\boxed{n(\hat{\theta}-\theta) \xrightarrow{d} -\text{Exp}\left(\frac{\theta}{(1-p)}\right) \equiv \text{Exp}\left(\frac{-\theta}{(1-p)}\right)}$ (since exponential distn. a member of scale family)

2016 Qual, Section 1

AMW

1. e) Now assume p & θ are unknown.

i) Calculate the MLE for p to obtain the MLE for $E[X_1]$.

ii) Derive the asymptotic distribution of the latter after proper normalization.

i) Know from part b) that

$$\mathcal{L}(\theta,p|\tilde{x}) = \prod_{i=1}^{n} p^{I(x_i=0)} \left[(1-p)\frac{1}{\theta}\right]^{I(0<x_i\le\theta)} I(0\le x_i\le\theta)$$

$$\Rightarrow \ell(\theta,p|\tilde{x}) = \sum_{i=1}^{n}\left[I(x_i=0)\log(p) + I(0<x_i\le\theta)\log(1-p) - I(0<x_i\le\theta)\log(\theta)\right]$$

$$\Rightarrow \frac{\partial \ell}{\partial p} = \sum_{i=1}^{n}\left[\frac{I(x_i=0)}{p} - \frac{I(0<x_i\le\theta)}{(1-p)}\right] \overset{set}{=} 0$$

$$\Rightarrow \frac{1}{p}\sum_{i=1}^{n} I(x_i=0) = \frac{1}{(1-p)}\sum_{i=1}^{n} I(0<x_i\le\theta)$$

$$\Rightarrow \frac{1-p}{p} = \frac{\sum_{i=1}^{n} I(0<x_i\le\theta)}{\sum_{i=1}^{n} I(x_i=0)}$$

$$\Rightarrow \frac{1}{p} \cancel{-1} = \frac{\sum_{i=1}^{n} I(0<x_i\le\theta)}{\sum_{i=1}^{n} I(x_i=0)} + 1$$

$$\Rightarrow \frac{1}{p} = \frac{\sum_{i=1}^{n} I(0<x_i\le\theta) + \sum_{i=1}^{n} I(x_i=0)}{\sum_{i=1}^{n} I(x_i=0)} \Rightarrow \hat{p} = \frac{\sum_{i=1}^{n} I(x_i=0)}{\underbrace{\sum_{i=1}^{n} I(0<x_i\le\theta) + \sum_{i=1}^{n} I(x_i=0)}_{n}}$$

$$\Rightarrow \boxed{\hat{p} = \frac{\sum_{i=1}^{n} I(x_i=0)}{n}} = E[I(x_i=0)] = P(x_i=0) = p$$

$X_{(n)}$

From part a), have $E[X_1] = \dfrac{\theta(1-p)}{2} \Rightarrow \boxed{\widehat{E[X_1]} = \dfrac{\hat{\theta}(1-\hat{p})}{2} = \dfrac{X_{(n)}(1-p)}{n}}$

ii) Now to derive the asymptotic distn. of $\widehat{E[X_1]}$ after proper normalization.

Know $n(\hat{\theta}-\theta) \xrightarrow{d} Exp\left(\frac{-\theta}{(1-p)}\right)$ from part d)

Also, since $\hat{p} = p \xrightarrow{P} p$, then $\frac{(1-\hat{p})}{2} \xrightarrow{P} \frac{(1-p)}{2}$ by CMT (conts. mapping thm).

Then, by Slutsky's, $\underbrace{n(\hat{\theta}-\theta)\frac{(1-\hat{p})}{2}} \xrightarrow{d} \frac{(1-p)}{2} Exp\left(\frac{-\theta}{(1-p)}\right) = \underbrace{Exp\left(-\frac{\theta}{2}\right)}$

$$n\left(\underbrace{\frac{\hat{\theta}(1-\hat{p})}{2}}_{\widehat{E[X_1]}} - \frac{\theta(1-\hat{p})}{2}\right)$$

b/c exponential distn. is member of scale family.

Thus, $\boxed{n\left(\widehat{E[X_1]} - \frac{\theta(1-p)}{2}\right) \xrightarrow{d} Exp\left(-\frac{\theta}{2}\right)}$