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2. Suppose that $Y_1, \ldots, Y_n$ are $\perp\!\!\!\perp$ RVs and each $Y_i \sim Exp(\mu_i) \equiv Exp(\beta x_i)$

where $x_1, \ldots, x_n$ are known positive constants $\neq 0$ and $\beta > 0$ is an unknown parameter.

a) i) Find an explicit expression for the MLE $\hat{\beta}$ of β.

ii) Also, find the large sample distn. of $\sqrt{n}(\hat{\beta} - \beta)$.

$$\ulcorner \quad \mathcal{L}(\beta \mid \underset{\sim}{y}) = \prod_{i=1}^{n} \frac{1}{\beta x_i} e^{-y_i/\beta x_i} \Rightarrow \ell(\beta \mid y_i) = \sum_{i=1}^{n} \left[\log(\beta x_i) - y_i/\beta x_i \right]$$

$$\Rightarrow \frac{\partial \ell}{\partial \beta} = \sum_{i=1}^{n} \frac{-\cancel{x_i}}{\beta \cancel{x_i}} + \frac{y_i \cancel{x_i}}{(\beta x_i)^2} \overset{set}{=} 0 \Rightarrow \beta^2 \frac{n}{\beta} = + \sum_{i=1}^{n} \frac{y_i x_i}{x_i \cancel{{}^2}}$$

$$\Rightarrow \boxed{\hat{\beta} = + \frac{1}{n} \sum_{i=1}^{n} y_i / x_i}$$

$$\text{Since } \frac{\partial^2 \ell}{\partial \beta^2} = \sum_{i=1}^{n} \frac{1}{\beta^2} - \frac{2 y_i x_i^2}{(\beta x_i)^3} = \sum_{i=1}^{n} \frac{1}{\beta^2} - \frac{2 y_i \cancel{x_i^2}}{\beta^3 x_i \cancel{{}^2}} = \frac{n}{\beta^2} - \frac{2}{\beta^3} \sum_{i=1}^{n} y_i / x_i$$

$$\text{Then, } -E\left[\frac{\partial^2 \ell}{\partial \beta^2}\right] = -E\left[\frac{n}{\beta^2} - \frac{2}{\beta^3} \sum_{i=1}^{n} y_i / x_i\right] = -\frac{n}{\beta^2} + \frac{2}{\beta^3} \sum_{i=1}^{n} \frac{E[y_i]}{x_i}$$

$$= \frac{-n}{\beta^2} + \frac{2}{\beta^3} \sum_{i=1}^{n} \frac{\beta \cancel{x_i}}{\cancel{x_i}} = \frac{-n}{\beta^2} + \frac{2n\cancel{\beta}}{\cancel{\beta^3}\beta^2} = \frac{-n + 2n}{\beta^2} = \frac{n}{\beta^2}$$

$$\Rightarrow I_1(\beta)^{-1} = \cancel{n} \left(\frac{\beta^2}{\cancel{n}} \right)$$

Since, by properties of MLE, $\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{d} N(0, I_1(\beta)^{-1})$,

then, $\boxed{\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{d} N(0, \beta^2)}$ $\lrcorner$

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2b) Find a $\underbrace{\text{pivotal quantity}}_{\text{i)}}$ & use it to construct an $\underbrace{\text{exact 95\% CI for}}_{\text{ii)}}$ β.

Method 1: (Easier method)

i) Note that each Y_i are $\perp\!\!\!\perp$ but not necessary identically distributed since $Y_i \sim Exp(\beta X_i)$

where each X_i depends on the choice of i.

However, $Y_i/X_i \sim Exp(\beta)$ since the exponential distn. is member of scale family.

$\Rightarrow Y_i/\beta X_i \sim Exp(1) \perp\!\!\!\perp \beta$

Thus, $\boxed{\frac{Y_i}{\beta X_i} \text{ is a pivotal quantity}}$ since the resulting exponential distn. w/ mean 1 is parameter free.

ii) Then, a 95% CI(β) = $\left\{ \beta : \underbrace{a}_{\substack{0.025 \\ \text{quantile of} \\ Exp(1) \text{ distn.}}} \le \frac{Y_i}{\beta X_i} \le \underbrace{b}_{\substack{0.975 \\ \text{quantile of} \\ Exp(1) \text{ distn.}}} \right\}$

where $F_{Exp}(a) = 1 - e^{-a} = 0.025$

$\Rightarrow -e^{-a} = 0.025 - 1$

$\Rightarrow e^{-a} = 0.975$

$\Rightarrow -a = \log(0.975)$

$\Rightarrow a = -\log(0.975)$

$F_{Exp}(b) = 1 - e^{-b} = 0.975$

$\Rightarrow -e^{-b} = 0.975 - 1$

$\Rightarrow e^{-b} = 0.025$

$\Rightarrow -b = \log(0.025) \Rightarrow b = -\log(0.025)$

$$= \left\{ \beta : -\log(0.975) \le \frac{Y_i}{\beta X_i} \le -\log(0.025) \right\}$$

$$= \left\{ \beta : \frac{Y_i}{-\log(0.025)} \le \beta X_i \le \frac{Y_i}{-\log(0.975)} \right\}$$

$$= \boxed{\left\{ \beta : \frac{Y_i}{-\log(0.025)\, X_i} \le \beta \le \frac{Y_i}{-\log(0.975)\, X_i} \right\}}$$

see next pg. for alternate method
$\longrightarrow$

2b cont'd

Method 2 (Harder Method):

i) ① Find a SS for β:

$$f(y_i|\beta) = \frac{1}{\beta x_i}\exp\{-y_i/\beta x_i\} = \underbrace{\frac{1}{x_i}}_{h(x)} \cdot \underbrace{\frac{1}{\beta}\exp\{-\frac{1}{\beta}\frac{y_i}{x_i}\}}_{g(T(x)|\theta)}$$

$\Rightarrow$ y_i/x_i is a SS for β by factorization thm.

② Find CDF of y_i:

Since $y_i \sim Exp(\beta x_i) \Rightarrow F(y_i) = 1 - \exp\{-y_i/\beta x_i\}$

③ Find CDF for SS:

$$F(t) = P(y_i/x_i \le t) = P(y_i \le t x_i) = 1 - \exp\{-t\cancel{x_i}/\beta \cancel{x_i}\} = 1 - \exp\{-t/\beta\}$$

④ Find CDF of CDF of SS:

$$F_Z(F(t)) = P(F(t) \le z) = P(1 - \exp\{-y_i/\beta x_i\} \le z) = P(-\exp\{-y_i/\beta x_i\} \le z-1) = P(\exp\{-y_i/\beta x_i\} \ge 1-z)$$

$$= P(-y_i/\beta x_i \ge \log(1-z)) = P(y_i \le -\beta x_i \log(1-z)) = 1 - \exp\{\cancel{\beta x_i}\log(1-z)/\cancel{\beta x_i}\}$$

$$= 1 - (1-z) = z \quad , \quad 0 < z < 1$$

Since $F_Z(z) = z$, $0<z<1$ is parameter free (i.e. free of β) $\Rightarrow$ $\boxed{1-\exp\{-y_i/\beta x_i\} \text{ is a pivot.}}$

⑤ Construct a 95% CI for β around the pivotal quantity:

Take $\underbrace{a}_{\text{0.025 quantile of a Unif(0,1)}} \le 1 - e^{-y_i/\beta x_i} \le \underbrace{b}_{\text{0.975 quantile of a Unif(0,1)}}$, where $F_{unif}(a) = a = 0.025$, $F_{unif}(b) = b = 0.975$

$\Rightarrow 0.025 \le 1 - e^{-y_i/\beta x_i} \le 0.975$

$\Rightarrow 0.025 - 1 \le -e^{-y_i/\beta x_i} \le 0.975 - 1$

$\Rightarrow -0.975 \le -e^{-y_i/\beta x_i} \le -0.025$

$\Rightarrow 0.025 \le e^{-y_i/\beta x_i} \le 0.975 \Rightarrow \log(0.025) \le -y_i/\beta x_i \le \log(0.975) \Rightarrow -\log(0.025) \ge y_i/\beta x_i \ge -\log(0.975)$

$$\Rightarrow \frac{y_i}{-\log(0.025)} \le \beta x_i \le \frac{y_i}{-\log(0.975)} \Rightarrow \frac{y_i}{-\log(0.025)x_i} \le \beta \le \frac{y_i}{-\log(0.975)x_i}$$

$$\Rightarrow \boxed{95\%\ CI(\beta) = \left\{\beta : \frac{y_i}{-\log(0.025)x_i} \le \beta \le \frac{y_i}{-\log(0.975)x_i}\right\}}$$

2c) Consider the following estimator of β: $\tilde{\beta} = (\sum_i Y_i)/\sum_i x_i$.

Show that the finite sample efficiency of $\tilde{\beta}$ relative to $\hat{\beta}$ is less than 1.

$\ulcorner$ Need to compute variance (finite sample) of both estimators & compare.

$\tilde{\beta}$: $Var[(\sum_i Y_i)/\sum_i x_i] = \sum_i Var(Y_i)/(\sum_i x_i)^2 = \sum_i \beta^2 x_i^2/(\sum_i x_i)^2 = \beta^2 \dfrac{\sum_i x_i^2}{(\sum_i x_i)^2}$

$\hat{\beta}$: $Var(\hat{\beta}) = \beta^2$ since $\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{d} N(0, \beta^2)$.

Then, $FSE(\tilde{\beta}, \hat{\beta}) = \dfrac{Var(\hat{\beta})}{Var(\tilde{\beta})} = \dfrac{\cancel{\beta^2}}{\cancel{\beta^2} \dfrac{\sum_i x_i^2}{(\sum x_i)^2}} = \dfrac{(\sum_i x_i)^2}{\sum_i x_i^2}$

By Jensen's ineq. for f a convex fcn, $f(\sum_i p_i x_i) \leq \sum_i p_i f(x_i)$.

Here $f(\cdot) = (\cdot)^2$ which is a convex fcn. and $p_i = 1$, $\ni (\sum_i x_i)^2 \leq \sum_i x_i^2$

$\Rightarrow \boxed{FSE(\tilde{\beta}, \hat{\beta}) = \dfrac{(\sum_i x_i)^2}{\sum_i x_i^2} \leq 1.}$ ✓ $\lrcorner$

d) Now, consider a different model for the mean, specifically $\frac{1}{\mu_i} = \alpha + \gamma x_i$ where α and γ are unknown parameters. Find a minimal sufficient statistic for (α, γ).

$$\mathcal{L}(\alpha,\gamma \mid \underset{\sim}{y}) = \prod_{i=1}^{n} (\alpha+\gamma x_i) \exp\{(\alpha+\gamma x_i) y_i\} = \exp\{\textstyle\sum_i \log(\alpha+\gamma x_i) - (\alpha+\gamma x_i) y_i\}$$

$$= \exp\{\underbrace{\textstyle\sum_i \log(\alpha+\gamma x_i)}_{-b(\theta)} - \underbrace{\alpha \textstyle\sum_i y_i - \gamma \textstyle\sum_i x_i y_i}_{T(y)\theta}\} \quad \text{where } \theta = \begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}$$

The general form of a multiparameter exp. family is:

$$f(\theta \mid y) = \exp\{\phi(T(y)\theta - b(\theta) - c(y)) - \tfrac{1}{2} S(y, \phi)\}$$

So, the above likelihood has this form where

$\phi = 1$

$\theta = -(\alpha, \gamma)$

$T(y) = (\sum_i y_i, \sum_i x_i y_i)$

$b(\theta) = -\sum_i \log(\alpha + \gamma x_i)$

$c(y) = 0$

$S(y, \phi) = 0$

Thus, $T(y) = (\sum_i y_i, \sum_i x_i y_i)$ is a CSS for $\theta = (\alpha, \gamma)$ since it also contains an open set in $\mathbb{R}^2$.

A CSS is also minimal $\Rightarrow$ $\boxed{T(y) = (\sum_i y_i, \sum_i x_i y_i) \text{ is a MSS for } \theta = (\alpha, \gamma).}$

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2e) By appropriate conditioning, obtain the conditional score fcn for γ (eliminating α). You don't need to simplify.

⌈ From part d), know that $f(\underset{\sim}{y} \mid \alpha, \gamma) = \exp\{ \sum_i \log(\alpha+\gamma x_i) - \alpha \sum_i y_i - \gamma \sum_i x_i y_i \}$

Note that y_i are not iid since each $y_i \sim Exp(\alpha+\gamma x_i)$ where $i=1,\dots,n$ changes the value of x_i.

In order to eliminate α, we need to condition on $\sum_i y_i$, the SS for α.

Since finding an explicit expression for $\sum_i y_i$ is difficult, write conditional likelihood as,

$$\mathcal{L}_c(\gamma) = \frac{\exp\{ \sum_i \cancel{\log(\alpha+\gamma x_i)} - \alpha \sum_i y_i - \gamma \sum_i x_i y_i \}}{\sum_{\tilde{y} \in S} \exp\{ \sum_i \cancel{\log(\alpha+\gamma x_i)} - \alpha \sum_i \tilde{y}_i - \gamma \sum_i x_i \tilde{y}_i \}}$$

$$= \frac{\exp\{ \cancel{-\alpha \sum_i y_i} - \gamma \sum_i x_i y_i \}}{\sum_{\tilde{y} \in S(s_0)} \exp\{ \cancel{-\alpha \sum_i \tilde{y}_i} - \gamma \sum_i x_i \tilde{y}_i \}} = \frac{\exp\{-\gamma \sum_i x_i y_i\}}{\sum_{\tilde{y} \in S(s_0)} \exp\{-\gamma \sum_i x_i \tilde{y}_i\}} \quad \text{for } S(s_0) = \{ \tilde{y}_i : \sum_i \tilde{y}_i = s_0 \}$$

$$\Rightarrow \ell_c(\gamma) = -\gamma \sum_i x_i y_i - \log\left(\sum_{\tilde{y} \in S(s_0)} \exp\{-\gamma \sum_i x_i \tilde{y}_i\} \right)$$

$$\Rightarrow \frac{\partial \ell_c}{\partial \gamma} = -\sum_i x_i y_i + \frac{\sum_{\tilde{y} \in S(s_0)} \sum_i x_i \tilde{y}_i \cancel{\exp\{-\gamma \sum_i x_i \tilde{y}_i\}}}{\sum_{\tilde{y} \in S(s_0)} \cancel{\exp\{-\gamma \sum_i x_i \tilde{y}_i\}}} = \boxed{-\sum_i x_i y_i + \frac{\sum_{\tilde{y} \in S(s_0)} \sum_i x_i \tilde{y}_i}{|S(s_0)|}}$$

The size (or cardinality) of set $|S(s_0)|$ ⌋