1. (25 points) Consider the linear model

$$Y = X\beta + \epsilon, \tag{1}$$

where Y is $n \times 1$, X is an $n \times p$ matrix of fixed covariates with rank $r < p$, β is $p \times 1$, and $\epsilon \sim N_n(0, \Sigma)$, where Σ is a known positive definite matrix.

(a) Derive the distribution of

$$U = (Y - X\beta)'\Sigma^{-1}(Y - X\beta),$$

and derive the mean and variance of U.

Note: You are not allowed to simply state the result of a theorem to give your answer. You must *derive* the results.

(b) Formally derive the set of all possible least squares solutions of β.

Note: You are not allowed to simply state a result or a formula for your answer. You must *derive* the result.

(c) Show that $\lambda'\beta$ is estimable if and only if

$$\lambda'(X'\Sigma^{-1}X)^{-}(X'\Sigma^{-1}X) = \lambda',$$

where a "-" denotes generalized inverse.

(d) Assume X has rank p. Show that the BLUE of β is equal to $(X'X)^{-1}X'Y$ if and only if there exists a non-singular $p \times p$ matrix F such that $\Sigma X = XF$.

(e) Assume X has rank p. Let s^2 be defined as

$$s^2 = \frac{Y'(I - M)Y}{n - p}$$

where M denotes the orthogonal projection operator onto the column space of X. Show that

$$E(s^2) \leq \frac{1}{n-p}\sum_{i=1}^{n}\sigma_{ii},$$

where σ_{ii} denotes the ith diagonal element of Σ, $i = 1, \ldots, n$. Can the upper bound on $E(s^2)$ be attained? Justify your answer.

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Points: (a) 5; (b) 5; (c) 5; (d) 5; (e) 5.

5+

12-14 points

2) [2019 Qual Problem]

$$Y = X\beta + \varepsilon \quad ; \quad Y_{n\times 1}, \quad X_{n\times p}, \quad r(X) = r < p, \quad \beta_{p\times 1}, \quad \varepsilon \sim N_n(0, \Sigma)$$

Σ known +DEF matrix

(a) <u>Derive</u> the distribution of $U = (Y - X\beta)'\Sigma^{-1}(Y - X\beta)$

We Know Σ is + DEF, so Write $\Sigma^{-1} = \Sigma^{-1/2}\Sigma^{-1/2}$ ← could also write $\Sigma = QQ'$, where Q nonsingular

$$\text{Then } U = (Y - X\beta)'\Sigma^{-1/2}\Sigma^{-1/2}(Y - X\beta)$$

$$= \left(\Sigma^{-1/2}(Y - X\beta)\right)'\left(\Sigma^{-1/2}(Y - X\beta)\right)$$

Since $Y \sim N_n(X\beta, \Sigma)$, then $Y - X\beta \sim N_n(0, \Sigma)$ and $\Sigma^{-1/2}(Y - X\beta) \sim N_n(0, I_n)$

The square of $N_n(0, I_n)$ distribution is χ^2_n, so $U = \left(\Sigma^{-1/2}(Y - X\beta)\right)'\left(\Sigma^{-1/2}(Y - X\beta)\right) \sim \chi^2_n$

<u>PDF of χ^2_n</u>: $f_U = \dfrac{1}{\Gamma(\frac{n}{2})\, 2^n} u^{\frac{n}{2}-1} \exp\left(-u/2\right)$ for $u > 0$, or 0 otherwise.

→ Derive MGF to determine mean & var. of U.

$$\Psi_U(t) = E[e^{tu}] = \int_0^\infty e^{tu} \frac{1}{\Gamma(\frac{n}{2})\, 2^{n/2}} u^{\frac{n}{2}-1} e^{-u/2}\, du = \int_0^\infty \frac{1}{\Gamma(\frac{n}{2})\, 2^{n/2}} u^{\frac{n}{2}-1} e^{-u(\frac{1}{2}-t)}\, du$$

$$\text{Let } v = u\left(\tfrac{1}{2} - t\right) \rightarrow dv = \left(\tfrac{1}{2} - t\right)du \longrightarrow \int_0^\infty \frac{1}{\Gamma(\frac{n}{2})\, 2^{n/2}} \left(\frac{v}{1/2 - t}\right)^{\frac{n}{2}-1} e^{-v} \left(\tfrac{1}{2} - t\right)^{-1} dv$$

$$= \frac{1}{2^{n/2}\left(\frac{1}{2} - t\right)^{n/2}} \underbrace{\int_0^\infty \frac{1}{\Gamma(\frac{n}{2})} v^{\frac{n}{2}-1} e^{-v}\, dv}_{\text{pdf Gamma}(\frac{n}{2}, 1)} = 2^{-n/2}\left(\tfrac{1}{2} - t\right)^{-n/2} = (1 - 2t)^{-n/2}$$

So, $\Psi_U(t) = (1 - 2t)^{-n/2}$

→ 1st Moment: $\Psi_U'(t) = -\frac{n}{2}(1 - 2t)^{-\frac{n}{2}-1}(-2) = n(1 - 2t)^{-\frac{n}{2}-1}$, so $\Psi_U'(0) = n$

→ 2nd Moment: $\Psi_U''(t) = n\left(-\frac{n}{2} - 1\right)(1 - 2t)^{-\frac{n}{2}-2}(-2)$, so $\Psi_U''(0) = \left(-\frac{n^2}{2} - n\right)(-2) = n^2 + 2n$

Mean: $E(U) = n$

$\mathrm{Var}(U) = E(U^2) - E(U)^2 = n^2 + 2n - n^2 = 2n$

(b) Formally <u>derive</u> the set of all possible LS solutions of β.

Take the transformed model, $Q^{-1}Y = Q^{-1}X\beta + Q^{-1}\varepsilon$, where $\Sigma = QQ'$ since it is + DEF

Since $Y \sim N_n(X\beta, \Sigma)$, then $Q^{-1}Y \sim N_n(X\beta, Q^{-1}\Sigma Q^{-1'})$

and $Q^{-1}\Sigma Q^{-1'} = Q^{-1}(QQ')Q^{-1'} = I_n$. Thus $Q^{-1}Y \sim N_n(X\beta, I_n)$

Call this model $\tilde{Y} = \tilde{X}\beta + \tilde{\varepsilon}$, where $\tilde{Y} = Q^{-1}Y$, $\tilde{X} = Q^{-1}X$, $\tilde{\varepsilon} = Q^{-1}\varepsilon$.

Let $\tilde{M}$ be the OPO onto $C(\tilde{X})$: $\tilde{M} = \tilde{X}(\tilde{X}'\tilde{X})^{-}\tilde{X}$

We know any LSE for β must satisfy: $(\tilde{Y} - \tilde{X}\hat{\beta})'(\tilde{Y} - \tilde{X}\hat{\beta}) = \min_{\beta} (\tilde{Y} - \tilde{X}\beta)'(\tilde{Y} - \tilde{X}\beta)$

$$(\tilde{Y} - \tilde{X}\beta)'(\tilde{Y} - \tilde{X}\beta) = ((\tilde{Y} - \tilde{M}\tilde{Y}) + (\tilde{M}\tilde{Y} - \tilde{X}\beta))'((\tilde{Y} - \tilde{M}\tilde{Y}) + (\tilde{M}\tilde{Y} - \tilde{X}\beta))$$

$$= ((\tilde{Y} - \tilde{M}\tilde{Y}) - (\tilde{X}\beta - \tilde{M}\tilde{Y}))'((\tilde{Y} - \tilde{M}\tilde{Y}) - (\tilde{X}\beta - \tilde{M}\tilde{Y}))$$

$$= (\tilde{Y} - \tilde{M}\tilde{Y})'(\tilde{Y} - \tilde{M}\tilde{Y}) - \underbrace{(\tilde{Y} - \tilde{M}\tilde{Y})'(\tilde{X}\beta - \tilde{M}\tilde{Y})} - \underbrace{(\tilde{X}\beta - \tilde{M}\tilde{Y})'(\tilde{Y} - \tilde{M}\tilde{Y})} + (\tilde{X}\beta - \tilde{M}\tilde{Y})'(\tilde{X}\beta - \tilde{M}\tilde{Y})$$

$$\tilde{Y}'\tilde{X}\beta - \tilde{Y}'\tilde{M}\tilde{Y} - \tilde{Y}'\tilde{M}'\tilde{X}\beta + \tilde{Y}'\tilde{M}'\tilde{M}\tilde{Y}$$

$$\cancel{\tilde{Y}'\tilde{X}\beta} - \cancel{\tilde{Y}'\tilde{M}\tilde{Y}} - \cancel{\tilde{Y}'\tilde{X}\beta} + \cancel{\tilde{Y}'\tilde{M}\tilde{Y}}$$

$$0$$

Similarly, this term is 0

$$= (\tilde{Y} - \tilde{M}\tilde{Y})'(\tilde{Y} - \tilde{M}\tilde{Y}) + (\tilde{X}\beta - \tilde{M}\tilde{Y})'(\tilde{X}\beta - \tilde{M}\tilde{Y})$$

This is clearly minimized when $\tilde{X}\beta - \tilde{M}\tilde{Y} = 0 \Rightarrow \tilde{X}\beta = \tilde{M}\tilde{Y}$

So, to find LSE for β, we can solve $\tilde{X}\beta = \tilde{M}\tilde{Y}$

Since X not full rank, we claim the solution to this is: $\hat{\beta} = (\tilde{X}'\tilde{X})^{-}\tilde{X}'Y - (I - \tilde{X}'(\tilde{X}'\tilde{X})^{-}X)Z$,

where $Z \in \mathbb{R}^p$.

To prove this is the solution

First, we knote $\tilde{M}\tilde{Y} \in C(\tilde{X})$, and $\tilde{M}\tilde{Y} = \tilde{X}(\tilde{X}'\tilde{X})^{-}\tilde{X}'\tilde{Y}$

Thus, for $\tilde{X}\beta = \tilde{M}\tilde{Y} = \tilde{X}(\tilde{X}'\tilde{X})^{-}\tilde{X}'\tilde{Y}$, we can see that

$\hat{\beta} = (\tilde{X}'\tilde{X})^{-}\tilde{X}'\tilde{Y}$ is a solution

$\rightarrow$

Second, since X not full rank, we know any $\hat{\beta}$ satisfying $\tilde{X}\hat{\beta} = \tilde{M}\tilde{Y}$ is LSE along with $\hat{\beta} + m$ for $m \in N(\tilde{X})$

Note: $C(M) = C(X)$
So $C(X')^{\perp}$ is
$C(I - M_{X'})$

Since $N(\tilde{X}) = C(\tilde{X}')^{\perp}$, then $N(\tilde{X})$ has the same column space as $C(\tilde{X}')^{\perp}$,

which is $I - \tilde{M}_{\tilde{X}'} = I - \tilde{X}'(\tilde{X}\tilde{X}')^{-}\tilde{X}$

So, for $m \in N(\tilde{X})$, we know $\forall\, z \in \mathbb{R}^p$, $m = (I - \tilde{M}_{\tilde{X}'})z$

$$\Rightarrow m = \left(I - \tilde{X}'(\tilde{X}\tilde{X}')^{-}\tilde{X}\right)z$$

So, putting this all together,

$$\hat{\beta} = (\tilde{X}'\tilde{X})^{-}\tilde{X}'\tilde{Y} + \left(I - \tilde{X}'(\tilde{X}\tilde{X}')^{-}\tilde{X}\right)z$$

(c) Show $\lambda'\beta$ is estimable <u>iff</u> $\lambda'(X'\Sigma^{-1}X)^{-}(X'\Sigma^{-1}X) = \lambda'$

· First, assume $\lambda'\beta$ is estimable.

We know $\lambda'\beta$ estimable in WLS model iff it is estimable in transformed model.

Again, since Σ + DEF, let $\Sigma = QQ'$, Q nonsingular

Then, we have tranformed model $Q^{-1}Y = Q^{-1}X\beta + Q^{-1}\varepsilon$; $Q^{-1}Y \sim N_n(X\beta, I_n)$

(as shown in (a) & (b))

So, $\lambda'\beta$ estimable $\Rightarrow \lambda'\beta \in C(Q^{-1}X) \Rightarrow \lambda = (Q^{-1}X)'b$ for some $b \in \mathbb{R}^r$

$\Rightarrow \lambda' = b'(Q^{-1}X)$

Then, $\lambda'(X'\Sigma^{-1}X)^{-}(X'\Sigma^{-1}X) = (b'(Q^{-1}X))(X'(QQ')^{-1}X)^{-}(X'(QQ')^{-1}X)$

$$= b'Q^{-1}X(X'Q'^{-1}Q^{-1}X)^{-}(X'Q'^{-1}Q^{-1}X)$$

$$= b'\underbrace{(Q^{-1}X)((Q^{-1}X)'(Q^{-1}X))^{-}(Q^{-1}X)'}_{M_{Q^{-1}X}}(Q^{-1}X)$$

$$= b'M_{Q^{-1}X}(Q^{-1}X)$$

$$= b'(Q^{-1}X) = \underline{\underline{\lambda'}} \quad ✓$$

· Second, assume $\lambda'(X'\Sigma^{-1}X)^{-}(X'\Sigma^{-1}X) = \lambda'$

Transpose both sides : $\lambda = (X'\Sigma^{-1}X)(X'\Sigma^{-1}X)^{-}\lambda$

Again, with $\Sigma = QQ'$ $\rightarrow$ $\lambda = (X'(QQ')^{-1}X)(X'(QQ')^{-1}X)^{-}\lambda$

$$= (Q^{-1}X)'\underbrace{(Q^{-1}X)((Q^{-1}X)'(Q^{-1}X))^{-}\lambda}_{\text{call this } P}$$

$$= (Q^{-1}X)'P \quad \Rightarrow \lambda \in C((Q^{-1}X)')$$

$\Rightarrow \lambda$ is estimable in the transformed model, and hence in the WLS model

(d) $r(X) = p$. Show the BLUE of $\beta = (X'X)^{-1}X'Y$ <u>iff</u> $\exists$ a non-singular $p \times p$ F st $\Sigma X = XF$

- First, assume the BLUE of β is $\hat{\beta} = (X'X)^{-1}X'Y$

Now that X full rank, we know from (b) that any LSE for β must satisfy

$$\begin{aligned}\hat{\beta} &= (\tilde{X}'\tilde{X})^{-1}\tilde{X}'\tilde{Y} \\ &= ((Q^{-1}X)'(Q^{-1}X))^{-1}(Q^{-1}X)'Q^{-1}Y \\ &= (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}Y\end{aligned}$$

Since X full rank, β is estimable, and by Gauss-Markov, $\hat{\beta} = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}Y$ is BLUE of β

Thus, with our assumption, we have: $(X'X)^{-1}X'Y = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}Y$

$$\Leftrightarrow (X'X)^{-1}X' = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}$$

Transposing $\Rightarrow X(X'X)^{-1} = \Sigma^{-1}X(X'\Sigma^{-1}X)^{-1}$

$$\Leftrightarrow \Sigma X(X'X)^{-1} = X(X'\Sigma^{-1}X)^{-1}$$

$$\Leftrightarrow \Sigma X = X(X'\Sigma^{-1}X)^{-1}(X'X)$$

$\Leftrightarrow \Sigma X = XF$, where $F = (X'\Sigma^{-1}X)^{-1}(X'X)$, which is non-singular, $p \times p$ since $(X'\Sigma^{-1}X)^{-1}$ and $(X'X)$ are both, bases for $\mathbb{R}^p$ and $p \times p$ invertible.

Thus, $\Sigma X = XF$, F non-singular $p \times p$.

$\rightarrow$

· Now, assume $\exists$ F non-singular $p \times p$ st $\Sigma X = XF$.

Then, $\Sigma X = XF$

$$\Leftrightarrow X = \Sigma^{-1} X F \Leftrightarrow X F^{-1} = \Sigma^{-1} X$$

So, $\hat{\beta} = (X' \Sigma^{-1} X)^{-1} X' \Sigma^{-1} Y$ (from results in (b))

$$= ((XF^{-1})' X)^{-1} (XF^{-1})^T Y$$

$$= (F^{-T} X' X)^{-1} (XF^{-1})^T Y$$

$$= (X'X)^{-1} F^T F^{-T} X' Y$$

$$= (X'X)^{-1} X' Y \quad \checkmark$$

(e) $r(X)=p$, $s^2 = \dfrac{Y'(I-M)Y}{n-p}$, M is OPO onto $C(X)$

Show $E(s^2) \le \dfrac{1}{n-p}\sum_{i=1}^{n} \sigma_{ii}$; σ_{ii} is i^{th} diagonal of Σ

- $$E(s^2) = E\left(\frac{Y'(I-M)Y}{n-p}\right) = \frac{1}{n-p}E\left(Y'(I-M)Y\right) = \frac{1}{n-p}\left[E(Y)'(I-M)E(Y) + tr((I-M)\Sigma)\right]$$

$$= \frac{1}{n-p}\left[(X\beta)'\underbrace{(I-M)(X\beta)}_{X\beta - MX\beta = X\beta - X\beta = 0} + tr((I-M)\Sigma)\right] = \frac{1}{n-p}tr((I-M)\Sigma) = \frac{1}{n-p}\left[tr(\Sigma) - tr(M\Sigma)\right]$$

$$= \frac{1}{n-p}tr(\Sigma) - \frac{1}{n-p}tr(M\Sigma)$$

Note that $tr(\Sigma) = \sum_{i=1}^{n}\sigma_{ii}$, so $E(S^2) = \left(\frac{1}{n-p}\sum_{i=1}^{n}\sigma_{ii}\right) - \frac{1}{n-p}tr(M\Sigma)$

Thus, we must show $\frac{1}{n-p}tr(M\Sigma) > 0$

- $tr(M\Sigma) = tr(M^2\Sigma) = tr(M'M\Sigma) = tr(M\Sigma M')$

For the matrix $M\Sigma M'$ and $\forall\, z \in \mathbb{R}^n$, $z'M\Sigma M'z = \overbrace{(M'z)'}^{\text{Scalar}}\Sigma\overbrace{(M'z)}^{\text{Scalar}} \ge 0$ since Σ +DEF (by definition)

$\Rightarrow M\Sigma M'$ is + Def $\left\{\text{since } z'\underline{M\Sigma M'}z \ge 0\right\}$, so $\lambda_i \ge 0$, where λ_i's are eigenvalues of $M\Sigma M'$.

$tr(M\Sigma M') = \Sigma\lambda_i \ge 0$, and clearly $\frac{1}{n-p} > 0$

Putting this all together, $E(S^2) = \frac{1}{n-p}\sum_{i=1}^{n}\sigma_{ii} - \frac{1}{n-p}tr(M\Sigma M')$

$$= \frac{1}{n-p}\sum_{i=1}^{n}\sigma_{ii} - \frac{1}{n-p}\Sigma_i\lambda_i$$

$$\le \frac{1}{n-p}\sum_{i=1}^{n}\sigma_{ii}$$

✓

- Can the upper bound of $E(s^2)$ be obtained?

The only way this can be attained is if $\frac{1}{n-p}\sum_{i=1}^{n}\lambda_i = 0 \Leftrightarrow \sum_{i=1}^{n}\lambda_i = 0$,

and since $\lambda_i \ge 0$, this would mean all $\lambda_i = 0\ \forall i$.

Actually cannot achieve because lamda_i >0 for all i, since M\SigmaM' is +DEF