2021. Sec 2. Q1.

see 762. mid 1. Solution given by Ibrahim.

3. Consider two linear models

$$Y = X_1\gamma_1 + X_2\gamma_2 + X_3\gamma_3 + \epsilon_1,$$

and

$$Y = X_1\beta_1 + X_3\beta_3 + \epsilon_2,$$

where Y is $n \times 1$, X_3 is a single column but X_1 and X_2 may have multiple columns, $E(\epsilon_k) = 0$, $\mathrm{Cov}(\epsilon_k) = \sigma^2 I$, ϵ_k are *iid* for $k = 1, 2$, and all parameters are unknown. Further, let J_n denote the $n \times 1$ vector of ones, and assume $J_n \in C(X_1)$, where $C(X_1)$ denotes the column space of X_1. Further, let M_1 denote the orthogonal projection operator onto $C(X_1)$, and M_{12} denotes the orthogonal projection operator onto $C(X_1, X_2)$.

(a) (7 points) Researchers are interested in knowing when the least squares estimates have $\mathrm{sign}(\hat{\beta}_3) \neq \mathrm{sign}(\hat{\gamma}_3)$. Show that $\mathrm{sign}(\hat{\beta}_3) \neq \mathrm{sign}(\hat{\gamma}_3)$ if and only if $\mathrm{sign}[X_3'(I-M_1)Y] \neq \mathrm{sign}[X_3'(I-M_{12})Y]$.

(b) Let M_3 denote the orthogonal projection operator onto the orthogonal complement of X_1 with respect to $C(X_1, X_2)$.

(i) (6 points) Express M_3 *only* in terms of M_1 and M_{12}.

(ii) (4 points) Define $r = \frac{X_3'(I-M_1)M_3(I-M_1)Y}{\sqrt{X_3'(I-M_1)M_3(I-M_1)X_3}\sqrt{Y'(I-M_1)M_3(I-M_1)Y}}$. Show that $r = \frac{X_3'M_3Y}{\sqrt{X_3'M_3X_3}\sqrt{Y'M_3Y}}$.

(c) (5 points) Assuming $\hat{\beta}_3$ is positive, show that $\mathrm{sign}(\hat{\beta}_3) \neq \mathrm{sign}(\hat{\gamma}_3)$ if and only if $X_3'(I-M_1)Y < X_3'M_3Y$.

(d) Now consider the more general case in which $Y \sim N_n(\mu, \sigma^2 I_n)$, where I_n is the $n \times n$ identity matrix and Y is $n \times 1$. Under the null model, $\mu = J_n\alpha$, and under the alternative model, $\mu = J_n\alpha + X\beta$, where X is $n \times p$ of rank $p < n$ that has been centered so that $X'J_n = 0_p$. Furthermore, if a random variable $W \sim \chi^2(a)$, then the density of W is given by $f(w) = \frac{w^{a/2-1}\exp(-w/2)}{\Gamma(a/2)2^{a/2}}$, $\quad w > 0$, and if $Z \sim \mathrm{Beta}(b, c)$, then $f(z) = \frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)} z^{b-1}(1-z)^{c-1}$, $\quad 0 < z < 1$.

(i) (7 points) Define $R^2 = \frac{Y'X(X'X)^{-1}X'Y}{\|Y - \frac{1}{n}J_nJ_n'Y\|^2}$, where $\|a\|^2 = a'a$ for any vector a. Show that under the null model, R^2 has a beta distribution and find its expected value.

(ii) (4 points) Under the setup of part (i), if p is increasing with n such that $\frac{p}{n-1-p} = \lambda$, with $0 < \lambda < \infty$, find the limit of R^2 as both $n, p \to \infty$ under the null model.

3

(a) Consider the model

$$
\begin{aligned}
Y &= X_1\beta_1 + X_3\beta_3 + \varepsilon \\
&= X_1\beta_1 + M_1X_3\beta_3 + (I-M_1)X_3\beta_3 + \varepsilon \\
&= X_1\left(\beta_1 + (X_1'X_1)^{-1}X_3\beta_3\right) + (I-M_1)X_3\beta_3 + \varepsilon \\
&= X_1\alpha + (I-M_1)X_3\beta_3 + \varepsilon .
\end{aligned}
$$

Since $X_1'(I-M_1)X_3 = 0$, this implies

Now $C(X_1) \perp C((I-M_1)X_3)$.

Therefore,

$$
\begin{aligned}
\hat{\beta}_3 &= \left(X_3'(I-M_1)'(I-M_1)X_3\right)^{-1} \cdot X_3'(I-M_1)'Y \\
&= C_1 X_3'(I-M_1)Y, \text{ where}
\end{aligned}
$$

$$
C_1 = \begin{cases} X_3'(I-M_1)'(I-M_1)X_3 = \|(I-M_1)X_3\|^2, & \text{if } (I-M_1)X_3 \neq 0 \\ 0, & \text{if } (I-M_1)X_3 = 0. \end{cases}
$$

$\Rightarrow C_1 \geq 0$.

This implies $\text{sign}(\hat{\beta}_3) = \text{sign}(X_3'(I-M_1)Y)$

Similarly by replacing X_1 with (X_1, X_2), we know

$$\text{sign}(\hat{\gamma}_3) = \text{sign}(X_3'(I-M_{12})Y)$$

Therefore,

$$\boxed{\text{sign}(\hat{\beta}_3) \neq \text{sign}(\hat{\gamma}_3) \text{ iff } \text{sign}(X_3'(I-M_1)Y) \neq \text{sign}(X_3'(I-M_{12})Y)}$$

(b) (i)

(p.3)

Claim: $M_3 = M_{12} - M_1$

Proof:

① $M_3' = (M_{12}-M_1)' = M_{12}' - M_1' = M_{12} - M_1 = M_3$.

② $$\begin{aligned}(M_{12}-M_1)^2 &= M_{12}^2 - M_{12}M_1 - M_1M_{12} + M_1^2 \\ &= M_{12} - M_1 - M_1 + M_1 \\ &= M_{12} - M_1 = M_3.\end{aligned}$$

③ For any $Z \in R^n$, we have a unique decomposition

$Z = x + y$, $x \in C(X_1, X_2)$ and $y \in C(X_1, X_2)^\perp$

Further, $x = a + b$,

$a \in C(X_1)$ and $b \in C(X_1)^\perp \cap C(X_1, X_2)$

Note that

(p.4)

$Z = (I - M_{12})Z + M_{12}Z$, where $(I-M_{12})Z \in C(X_1, X_2)^\perp$

and $M_{12}Z \in C(X_1, X_2)$.

This implies

$$x = M_{12}Z, \quad y = (I - M_{12})Z$$

$$\begin{aligned}x = M_{12}Z &= M_1M_{12}Z + (I - M_1)M_{12}Z \\ &= M_1Z + (M_{12} - M_1)Z\end{aligned}$$

$\Rightarrow a = M_1Z$, $b = (M_{12} - M_1)Z$

$\Rightarrow Z = (I - M_{12})Z + M_1Z + (M_{12} - M_1)Z$
$= (y + a) + b$.

$b \in C(X_1)^\perp \cap C(X_1, X_2)$

and $(y+a) \perp b$.

Now

$$\begin{aligned}M_3Z &= (M_{12}-M_1)(y+a+b) = (M_{12}-M_1)\left[(I-M_{12})Z + M_1Z + (M_{12}-M_1)Z\right] \\ &= (M_{12} - M_{12} - M_1 + M_1)Z + (M_1 - M_1)Z + (M_{12}-M_1)^2Z \\ &= (M_{12} - M_1)Z = b.\end{aligned}$$

$\Rightarrow M_3Z = b$ for any $Z = (y+a) + b$,

where $b \in C(X_1)^\perp \cap C(X_1, X_2)$ and $(y+a) \perp b$.

Therefore, $M_3 = M_{12} - M_1$ is the orthogonal projection operator as desired.

(ii)

$$M_3(I-M_1) = (M_{12}-M_1)(I-M_1)$$
$$= M_{12}-M_1-M_1+M_1 = M_{12}-M_1 = M_3.$$
$$\Rightarrow (I-M_1)M_3(I-M_1) = (I-M_1)M_3 = M_3.$$

Therefore,

$$r = \frac{X_3'(I-M_1)M_3(I-M_1)Y}{\left(\sqrt{X_3'(I-M_1)M_3(I-M_1)X_3}\right)\sqrt{Y'(I-M_1)M_3(I-M_1)Y}}$$

$$= \frac{X_3'M_3Y}{\sqrt{X_3'M_3X_3}\sqrt{Y'M_3Y}}.$$

(c) From part (a), we know that

$$0 < \hat{\beta}_3 = C_1X_3'(I-M_1)Y$$
$$= C_1X_3'(I-M_{12})Y + C_1X_3'(M_{12}-M_1)Y$$
$$= C_1X_3'(I-M_{12})Y + C_1X_3'M_3Y.$$

Since $C_1 > 0$, we have

$$0 < X_3'(I-M_1)Y = X_3'(I-M_{12})Y + X_3'M_3Y$$

Therefore,

$$X_3'(I-M_1)Y < X_3'M_3Y$$
$$\Leftrightarrow X_3'(I-M_{12})Y < 0$$
$$\Leftrightarrow \text{sign}(X_3'(I-M_1)Y) \neq \text{sign}(X_3'(I-M_{12})Y)$$
$$\Leftrightarrow \text{sign}(\hat{\beta}_3) \neq \text{sign}(\hat{\gamma}_3).$$

(d)

(i) For the full model $Y = J_n\alpha + X\beta + \varepsilon$, $\varepsilon \sim N_n(0, \sigma^2 I_n)$,

Let M_1 be the orthogonal projection operator onto $C(J_n)$

and M_2 be the orthogonal projection operator onto $C(X)$.

Since $X'J_n = 0$, we have $J_n \perp X$.

Thus, the orthogonal projection operator onto $C(J_n, X)$ is $M = M_1 + M_2$.

We have

$$Y'X(X'X)^{-1}X'Y = Y'M_2Y$$

and

$$Y - \frac{1}{n}J_nJ_n'Y = Y - J_n(J_n'J_n)^{-1}J_n'Y$$
$$= (I - M_1)Y.$$

Also, <u>under the null model,</u>

$$E(M_2Y) = M_2J_n\alpha = 0$$

$$Cov(M_2Y) = M_2(\sigma^2 I_n)M_2' = \sigma^2 M_2.$$

$$\Rightarrow Y'M_2Y = (M_2Y)'(M_2Y) \sim \chi^2_p$$

under null model

since rank$(M_2) = p$

Similarly, following identical steps, we have

$$Y'(I - M_1 - M_2)Y \sim \sigma^2\chi^2_{n-p-1}.$$

under null model

Further

$$cov((I - M_1 - M_2)Y, M_2Y)$$
$$= \sigma^2(I - M_1 - M_2)M_2 = \sigma^2(M_2 - M_2) = 0$$

and thus

$$(I - M_1 - M_2)Y \perp\!\!\!\perp M_2Y.$$

Now

$$R^2 = \frac{Y'M_2Y}{\|Y - M_1Y\|^2} = \frac{Y'M_2Y}{\|(I - M_1)Y\|^2}$$
$$= \frac{Y'M_2Y}{Y'(I - M_1)Y}$$
$$= \frac{Y'M_2Y}{Y'(I - M_1 - M_2)Y + Y'M_2Y}$$
$$= \frac{\chi^2_p}{\chi^2_{n-p-1} + \chi^2_p}$$

independent

P.9

We now claim that R^2 has a beta distribution, and this is easily shown by a change of variables learned in Bios 661.

Let $W = \chi^2_p$

$Z = \chi^2_{n-p-1}$

$W \perp\!\!\!\perp Z$

We have

$$R^2 = \frac{W}{W+Z} \quad \Rightarrow \quad W = VR^2$$

Let $V = W + Z \quad \Rightarrow \quad Z = V(1-R^2)$

$$f_{(R^2,V)}(R^2, V) = f_{(W,Z)}\left(VR^2, V(1-R^2)\right) |J|$$

$$= f_W(VR^2) f_Z(V(1-R^2)) |J| \quad , \quad W \perp\!\!\!\perp Z$$

$$= \left(\frac{2^{-p/2}}{\Gamma(p/2)} (VR^2)^{p/2-1} \exp\left(\frac{-VR^2}{2}\right)\right)\left(\frac{2^{-\frac{(n-p-1)}{2}}}{\Gamma\left(\frac{n-p-1}{2}\right)} \left(V(1-R^2)\right)^{\frac{n-p-1}{2}-1} \cdot \exp\left(\frac{-V(1-R^2)}{2}\right)\right)$$

$$J = \det\begin{bmatrix} \frac{dW}{dR^2} & \frac{dW}{dV} \\ \frac{dZ}{dR^2} & \frac{dZ}{dV} \end{bmatrix} = \det\begin{bmatrix} V & R^2 \\ -V & 1-R^2 \end{bmatrix} = V - VR^2 + VR^2 = V \quad \cdot |J|$$

P.10

Thus

$$f(R^2, V) = \left(\frac{2^{\frac{-n+1}{2}}}{\Gamma(p/2)\Gamma\left(\frac{n-p-1}{2}\right)}\right)(R^2)^{p/2-1} V^{p/2-1} \cdot V^{\frac{n-p-1}{2}-1} \cdot V$$

$$(1-R^2)^{\frac{n-p-1}{2}-1} \exp\left(\frac{-VR^2}{2}\right) \exp\left(\frac{-V(1-R^2)}{2}\right)$$

$$= \left(\frac{2^{\frac{-n+1}{2}}}{\Gamma(p/2)\Gamma\left(\frac{n-p-1}{2}\right)}\right)(R^2)^{p/2-1}(1-R^2)^{\frac{n-p-1}{2}-1} V^{\frac{n-1}{2}-1} \exp\left(\frac{-V}{2}\right)$$

$$f(R^2) = \int f(R^2, v)\, dv$$

$$= \left(\frac{2^{-\frac{n+1}{2}}}{\Gamma(p/2)\,\Gamma\left(\frac{n-p-1}{2}\right)} \right) (R^2)^{p/2-1} (1-R^2)^{\frac{n-p-1}{2}-1} \underbrace{\int_0^\infty v^{\frac{n-1}{2}-1} \exp\left(\frac{-v}{2}\right) dv}_{= \frac{\Gamma\left(\frac{n-1}{2}\right)}{\left(\frac{1}{2}\right)^{\frac{n-1}{2}}}}$$

$$= \left(\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma(p/2)\,\Gamma\left(\frac{n-p-1}{2}\right)} \right) (R^2)^{p/2-1} (1-R^2)^{\frac{n-p-1}{2}-1}\; 2^{-\frac{n+1}{2}}\; 2^{n/2-1/2}$$

$$= \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma(p/2)\,\Gamma\left(\frac{n-p-1}{2}\right)} (R^2)^{p/2-1} (1-R^2)^{\frac{n-p-1}{2}-1}, \quad 0<R^2<1.$$

$$= \text{Beta}\left(\frac{p}{2}, \frac{n-p-1}{2}\right) \text{ density.}$$

P.11

$$E(R^2) = \int_0^1 \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma(p/2)\,\Gamma\left(\frac{n-p-1}{2}\right)} R^2 (R^2)^{p/2-1} (1-R^2)^{\frac{n-p-1}{2}-1}\, dR^2$$

$$= \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma(p/2)\,\Gamma\left(\frac{n-p-1}{2}\right)} \int_0^1 (R^2)^{p/2+1-1} (1-R^2)^{\frac{n-p-1}{2}-1}\, dR^2$$

beta function

$$= \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma(p/2)\,\Gamma\left(\frac{n-p-1}{2}\right)} B\left(p/2+1, \frac{n-p-1}{2}\right)$$

$$= \left[\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{n-p-1}{2}\right)} \right] \left[\frac{\Gamma(p/2+1)\,\Gamma\left(\frac{n-p-1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \right]$$

$$= \left(\frac{p}{2}\right)\left(\frac{2}{n-1}\right) = \boxed{\frac{p}{n-1}}.$$

We use the fact that $\Gamma(x+1) = x\,\Gamma(x)$

$$\Gamma(p/2+1) = p/2\; \Gamma(p/2)$$

$$\Gamma\left(\frac{n+1}{2}\right) = \Gamma\left(\frac{n-1}{2}+1\right) = \left(\frac{n-1}{2}\right)\Gamma\left(\frac{n-1}{2}\right).$$

(ii) Since $\frac{p}{n-p-1} = \lambda$,

this implies $\frac{n}{p} - \frac{1}{p} - 1 = \frac{1}{\lambda}$

$$\Rightarrow \frac{n-1}{p} = \frac{1}{\lambda} + 1 = \frac{\lambda+1}{\lambda}$$

From part (i), we know that

$$Y'M_2Y \sim \sigma^2 \chi^2_p$$

and $Y'(I-M_1)Y \sim \sigma^2 \chi^2_{n-1}$

$$\Rightarrow \frac{Y'M_2Y}{p} \xrightarrow{P} 1 \text{ and } \frac{Y'(I-M_1)Y}{n-1} \xrightarrow{P} 1 \text{ by LLN.}$$

$$\Rightarrow R^2 = \frac{Y'M_2Y}{Y'(I-M_1)Y} = \frac{[Y'M_2Y/p]}{[Y'(I-M_1)Y/(n-1)]} \cdot \left(\frac{p}{n-1}\right) \xrightarrow{P} \lim \left(\frac{p}{n-1}\right) = \frac{\lambda}{\lambda+1},$$

By the continuous mapping theorem.

Thus, $\boxed{\lim_{n,p\to\infty} R^2 = \frac{\lambda}{\lambda+1}}$.

2021. Sec 2. Q2.

(a). $Y_1 \sim \text{binomial}(n_1, \theta)$. $Y_2 \sim \text{binomial}(n_2, \theta\lambda)$

$$P(Y_1 = y_1) = \binom{n_1}{y_1} \cdot \theta^{y_1} (1-\theta)^{n_1 - y_1} = \exp\left\{ y_1 \log\theta + (n_1 - y_1)\log(1-\theta) + \log\left[\binom{n_1}{y_1}\right] \right\}$$

$$P(Y = y_2) = \binom{n_2}{y_2} \cdot (\theta\lambda)^{y_2} \cdot (1-\theta\lambda)^{n_2 - y_2}.$$

$$P(Y_1 = y_1, Y_2 = y_2) = \exp\left\{ y_1 \cdot \log\theta + (n_1 - y_1) \cdot \log(1-\theta) + y_2 \cdot (\log\theta + \log\lambda) + (n_2 - y_2) \cdot \log(1-\theta\lambda) + \log\binom{n_1}{y_1} + \log\binom{n_2}{y_2} \right\}.$$

$$= \exp\left\{ y_1 \cdot \underbrace{\log\frac{\theta}{1-\theta}}_{\theta_1} + y_2 \cdot \underbrace{\log\frac{\theta\lambda}{1-\theta\lambda}}_{\theta_2.} + \underbrace{n_1 \log(1-\theta) + n_2 \log(1-\theta\lambda)}_{-b(\theta).} + \underbrace{\log\binom{n_1}{y_1} + \log\binom{n_2}{y_2}}_{-\frac{1}{2} S(y,\phi).} \right\}.$$

$$\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

(b). $n_1 = n_2 = n$. $\theta = \lambda$

$$l(n, \theta \mid y_1, y_2) = y_1 \cdot \log\frac{\theta}{1-\theta} + y_2 \cdot \log\frac{\theta^2}{1-\theta^2} + n \cdot \log(1-\theta) + n \cdot \log(1-\theta^2) + \log\binom{n}{y_1} + \log\binom{n}{y_2}.$$

$Y_1 \sim Bin(n, \theta)$ $Y_2 \sim Bin(n, \theta^2)$.

Consider $X_i \overset{iid}{\sim} \text{Bernoulli}(\theta)$. $Y_1 = \sum_{i=1}^{n} X_i = n \cdot \bar{X}$.

By CLT, $\sqrt{n}(\bar{X} - E(X_1)) \xrightarrow{d} N(0, Var(X_1))$

$$\Rightarrow \sqrt{n} \cdot \left(\frac{Y_1}{n} - \theta \right) \xrightarrow{d} N(0, \theta(1-\theta))$$

Consider $Z_i \overset{iid}{\sim} \text{Bernoulli}(\theta^2)$ $\quad Y_2 = \sum_{i=1}^{n} Z_i = n\cdot\bar{Z}$

By CLT, $\sqrt{n}(\bar{Z} - E(Z_1)) \xrightarrow{d} N(0, Var(Z_1))$

$$\Rightarrow \sqrt{n}\cdot\left(\frac{Y_2}{n} - \theta^2\right) \xrightarrow{d} N(0, \theta^2(1-\theta^2))$$

Consider $g(x) = x^{1/2}$. $\nabla g(x) = \frac{1}{2}x^{-\frac{1}{2}}$.

By Delta method, $\sqrt{n}\left(\sqrt{\frac{Y_2}{n}} - \theta\right) \xrightarrow{d} N(0, \Sigma)$.

where $\Sigma = \frac{1}{2}\cdot(\theta^2)^{-\frac{1}{2}} \cdot \theta^2(1-\theta^2) \cdot \frac{1}{2}\cdot(\theta^2)^{-\frac{1}{2}} = \frac{1}{4}(1-\theta^2)$

When $\theta(1-\theta) > \frac{1}{4}(1-\theta^2)$, $\theta > \frac{1}{\cancel{5}\,3}$

Thus, $\frac{Y_1}{n}$ has larger asymptotic var if $\theta > \frac{1}{\cancel{5}\,3}$. Otherwise $\sqrt{\frac{Y_2}{n}}$ has larger var.

(c). Let $T = a\cdot\frac{Y_1}{n} + (1-a)\cdot\sqrt{\frac{Y_2}{n}}$. Thus $E(T) = a\cdot E\left(\frac{Y_1}{n}\right) + (1-a)\cdot E\left(\sqrt{\frac{Y_2}{n}}\right) = \theta$.

$a \in [0,1]$

$$var(T) = a^2 \cdot Var\left(\frac{Y_1}{n}\right) + (1-a)^2 \cdot Var\left(\sqrt{\frac{Y_2}{n}}\right).$$

$$= a^2 \cdot \frac{\theta(1-\theta)}{n} + (1-a)^2 \cdot \frac{\frac{1}{4}(1-\theta^2)}{n} = f(a).$$

$$\frac{\partial f(a)}{\partial a} = 2a\cdot\frac{\theta(1-\theta)}{n} + 2(a-1)\cdot\frac{\frac{1}{4}(1-\theta^2)}{n} \overset{set}{=} 0 \Rightarrow \hat{a} = \frac{\theta+1}{5\theta+1}$$

Thus, $T = a\cdot\frac{Y_1}{n} + (1-\hat{a})\cdot\sqrt{\frac{Y_2}{n}}$. $\quad a = \frac{\theta+1}{5\theta+1}$.

$$Var(T) = \left(\frac{\theta+1}{5\theta+1}\right)^2 \cdot \frac{\theta(1-\theta)}{n} + \left(\frac{4\theta}{5\theta+1}\right)^2 \cdot \frac{\frac{1}{4}(1-\theta^2)}{n}$$

$$\sqrt{n}(T - \theta) \xrightarrow{d} N(0, n\cdot Var(T))$$

We can use $\hat{\theta}_{MLE}$ to estimate θ. ?

(d). $\ell(\theta, \lambda \mid Y, n_1, n_2) \propto = y_1 \cdot \log \frac{\theta}{1-\theta} + y_2 \cdot \log \frac{\theta\lambda}{1-\theta\lambda} + n_1 \cdot \log(1-\theta) + n_2 \cdot \log(1-\theta\lambda)$

$$\frac{\partial \ell}{\partial \lambda} = y_2 \cdot \frac{1-\theta\lambda}{\theta\lambda} \cdot \frac{\theta(1-\theta\lambda)+\theta\cdot\theta\lambda}{(1-\theta\lambda)^2} + n_2 \cdot \frac{1}{1-\theta\lambda} \cdot (-\theta)$$

$$= y_2 \cdot \frac{1}{\lambda(1-\theta\lambda)} - \frac{n_2\theta}{1-\theta\lambda} \quad ✓ \qquad = \frac{y_2}{\lambda} - \frac{\theta n_2}{1-\theta\lambda} + \frac{y_2\theta}{1-\theta\lambda}$$

$$\frac{\partial \ell}{\partial \theta} = y_1 \cdot \frac{1-\theta}{\theta} \cdot \frac{1-\theta+\theta}{(1-\theta)^2} + y_2 \cdot \frac{1-\theta\lambda}{\theta\lambda} \cdot \frac{\lambda(1-\theta\lambda)+\lambda\cdot\theta\lambda}{(1-\theta\lambda)^2} + n_1 \cdot \frac{(-1)}{1-\theta} + n_2 \cdot \frac{(-\lambda)}{1-\theta\lambda}$$

$$= \frac{y_1}{\theta(1-\theta)} + \frac{y_2}{\theta(1-\theta\lambda)} - \frac{n_1}{1-\theta} - \frac{n_2\lambda}{1-\theta\lambda} \quad ✓$$

$$\frac{\partial^2 \ell}{\partial \lambda^2} = y_2 \cdot \frac{-(1-\theta\cdot 2\lambda)}{\lambda^2(1-\theta\lambda)^2} + n_2\theta \cdot \frac{-\theta}{(1-\theta\lambda)^2} = \frac{y_2(2\theta\lambda-1)}{\lambda^2(1-\theta\lambda)^2} - \frac{n_2\theta^2}{(1-\theta\lambda)^2}.$$

$$-E\left(\frac{\partial^2 \ell}{\partial \lambda^2}\right) = -\frac{2\theta\lambda-1}{\lambda^2(1-\theta\lambda)^2} \cdot n_2 \cdot \theta\lambda + \frac{n_2\theta^2}{(1-\theta\lambda)^2} = \frac{n_2\theta}{(1-\theta\lambda)^2} \cdot \left(\cancel{1}\,\theta - 2\theta + \frac{1}{\lambda}\right) = \frac{n_2\theta}{\lambda(1-\theta\lambda)}$$

$$\frac{\partial^2 \ell}{\partial^2 \theta^2} = y_1 \frac{-(1-2\theta)}{\theta^2(1-\theta)^2} + y_2 \cdot \frac{-(1-2\theta\lambda)}{\theta^2(1-\theta\lambda)^2} + n_1 \cdot \frac{(-1)}{(1-\theta)^2} + n_2\lambda \cdot \frac{(-\lambda)}{(1-\theta\lambda)^2}$$

$$-E\left(\frac{\partial^2 \ell}{\partial \theta^2}\right) = -n_2\cancel{\theta}\lambda \cdot \frac{(2\theta\lambda-1)}{\theta^{\cancel{2}}(1-\theta\lambda)^2} + \frac{n_1}{(1-\theta)^2} + \frac{n_2\lambda^2}{(1-\theta\lambda)^2} = \frac{n_1+n_2\lambda}{\theta} + \frac{n_1}{1-\theta} + \frac{\lambda^2 n_2}{1-\lambda\theta}.$$

$$\frac{\partial^2 \ell}{\partial\lambda\partial\theta} = \frac{y_2}{\cancel{\lambda}} \cdot \frac{\cancel{+\lambda}}{(1-\theta\lambda)^2} - n_2 \cdot \frac{(1-\theta\lambda)+\lambda\cdot\theta}{(1-\theta\lambda)^2} = \frac{y_2-n_2}{(1-\theta\lambda)^2}.$$

$$-E\left(\frac{\partial^2 \ell}{\partial\lambda\partial\theta}\right) = \frac{-n_2\theta\lambda}{(1-\theta\lambda)^2} + \frac{n_2}{(1-\theta\lambda)^2} = \frac{n_2}{1-\theta\lambda}$$

$$I(\xi) = \begin{pmatrix} \frac{n_1+n_2\lambda}{\theta} + \frac{n_1}{1-\theta} + \frac{\lambda^2 n_2}{1-\lambda\theta}. & \frac{n_2}{1-\theta\lambda} \\ \frac{n_2}{1-\theta\lambda} & \frac{n_2\theta}{\lambda(1-\theta\lambda)} \end{pmatrix}$$

(e). ① given $n_1 = n_2 = n$. derive $l_n(\theta, \lambda \mid y, n)$

② $\frac{\partial l_n}{\partial \theta} \overset{\text{set}}{=} 0$. use θ to replace λ in the equation.

③ derive $I_n^{-1}(\xi)$.

④ $SC_n = \frac{\partial l_n}{\partial \xi}\Big|_{\tilde{\xi}}^{T} I_n(\tilde{\xi})^{-1} \frac{\partial l_n}{\partial \xi}\Big|_{\tilde{\xi}} = \frac{\partial l_n}{\partial \theta}\Big|_{\tilde{\theta}}^{T} \cdot L'' \cdot \frac{\partial l_n}{\partial \theta}\Big|_{\tilde{\theta}}$.

(f). $R_1 = Y_1 - n\theta$. $R_2 = Y_2 - n\theta^2$. $Y_1 \sim Bin(n, \theta)$. $Y_2 \sim Bin(n, \theta^2)$.

$S_n(\beta) = \sum_{i=1}^{2} \left(\frac{\partial}{\partial \beta} E(y_i)\right) \cdot \left(Var(y_i)\right)^{-1} \left(y_i - E(y_i)\right)$. $R_1 . R_2 = y_i - E(y_i)$.

$S_n(\theta \mid y_1) = n \cdot \frac{1}{n\theta(1-\theta)} \cdot R_1 = \frac{R_1}{\theta(1-\theta)}$

$S_n(\theta \mid y_2) = 2n\theta \cdot \frac{1}{n\theta^2(1-\theta^2)} R_2 = \frac{2R_2}{\theta(1-\theta^2)}$.

Let $U(\theta \mid y_1, y_2) = a \cdot \frac{R_1}{\theta(1-\theta)} + (1-a) \cdot \frac{2R_2}{\theta(1-\theta^2)}$. $\tilde{\theta}_n$ solves $U(\theta) = 0$.

Taylor Expansion of $u(\tilde{\theta}_n)$ at $u(\theta)$:

$0 = u(\tilde{\theta}_n) = u(\theta) + (\tilde{\theta}_n - \theta) \frac{\partial u(\bar{\theta})}{\partial \theta}$. where $\bar{\theta} = t\theta + (1-t)\tilde{\theta}_n$. $t \in (0,1)$.

$$\Rightarrow \sqrt{n}(\tilde{\theta}_n - \theta) = -\sqrt{n} \cdot u(\theta) \cdot \left(\frac{\partial u(\bar{\theta})}{\partial \theta}\right)^{-1}$$

$$= \frac{u(\theta)}{\sqrt{n}} \cdot \left(-\frac{1}{n} \frac{\partial u(\bar{\theta})}{\partial \theta}\right)^{-1}$$

$$= \left[-\frac{1}{n} \cdot E\left(\frac{\partial u(\theta)}{\partial \theta}\right)\right]^{-1} \cdot \frac{u(\theta)}{\sqrt{n}} + o_p(1)$$

Quasi-Likelihood Standard procedure

$$\Rightarrow cov\left(\sqrt{n}(\tilde{\theta} - \theta)\right) \doteq \left[-\frac{1}{n} \cdot E\left(\frac{\partial u(\theta)}{\partial \theta}\right)\right]^{-1} \cdot cov\left(\frac{u(\theta)}{\sqrt{n}}\right) \cdot \left[-\frac{1}{n} \cdot E\left(\frac{\partial u(\theta)}{\partial \theta}\right)\right]^{-1}$$

$$-\frac{1}{n} E\left(\frac{\partial U}{\partial \theta}\right) = -\frac{1}{n} \cdot E\left(\frac{-an}{\theta(1-\theta)} - \frac{4n(1-a)}{1-\theta^2}\right) = \frac{a}{\theta(1-\theta)} + \frac{4(1-a)}{1-\theta^2}$$

$$\mathrm{cov}(U(\theta)) = \frac{a^2}{\theta^2(\cancel{1-\theta})^2} \cdot n\theta\cancel{(1-\theta)} + \frac{4(1-a)^2}{\theta^2(\cancel{1-\theta^2})^2} \cdot n\theta^2\cancel{(1-\theta^2)}$$

$$= \frac{a^2 n}{\theta(1-\theta)} + \frac{4n(1-a)^2}{1-\theta^2}$$

Thus, $\mathrm{cov}\left(\sqrt{n}\left(\tilde{\theta}-\theta\right)\right) = \left(\frac{a}{\theta(1-\theta)} + \frac{4(1-a)}{1-\theta^2}\right)^{-2} \cdot \left(\frac{a^2}{\theta(1-\theta)} + \frac{4(1-a)^2}{1-\theta^2}\right)$

$a = \frac{1}{2}$.

$$= \left(\frac{\frac{1}{2}}{\theta(1-\theta)} + \frac{2}{1-\theta^2}\right)^{-2} \cdot \left(\frac{\frac{1}{4}}{\theta(1-\theta)} + \frac{1}{1-\theta^2}\right)$$

$$= \left(\frac{1}{\theta(1-\theta)} + \frac{4}{1-\theta^2}\right)^{-1}$$