1. (25 points) Let $X_1, ..., X_n$ be i.i.d from the following distribution

$$\begin{cases} 0 & \text{with probability } p, \\ \text{Uniform}[0, \theta] & \text{with probability } 1-p. \end{cases}$$

First, we assume that p is a known constant in $(0, 1)$ and that $\theta > 0$ is the only parameter of interest.

(a) (5 points) Based on only one observation X_1, find all the unbiased estimators for θ and calculate their variances. Does UMVUE exist for θ? Justify your answer.

(b) (3 points) Based on n observations $X_1, ..., X_n$, let $X_{(n)} = \max\{X_1, ..., X_n\}$ be the maximal observation. Show that $(X_{(n)}, \sum_{i=1}^n I(X_i > 0))$ is a sufficient statistic for θ. Furthermore, show that $\widehat{\theta} = X_{(n)}$ maximizes the observed likelihood function.

(c) (5 points) What is the exact distribution of $\widehat{\theta}$? Compute $E[\widehat{\theta}]$ and $Var(\widehat{\theta})$ and show that $\widehat{\theta}$ is consistent for θ.

(d) (6 points) Derive the asymptotic distribution of $n(\widehat{\theta} - \theta)$.

Now assume that both p and θ are unknown.

(e) (6 points) Calculate the maximum likelihood estimator for p to obtain the maximum likelihood estimator for $E[X_1]$. Derive the asymptotic distribution for the latter after proper normalization.

1. Let $X_1, \dots, X_n \overset{iid}{\sim} \begin{cases} 0 & \text{with probability } p \\ U[0,\theta] & \text{with probability } (1-p) \end{cases}$

2016 Theory 1

First, we assume p is a known constant in $(0,1)$ & $\theta > 0$ is the only parameter of interest

(a) Based on only 1 obs. X_1, find all the unbiased estimators for θ & calculate their variances. Does UMVUE exist for θ? Justify.

$$E[X] = 0(p) + \frac{\theta}{2}(1-p) = \frac{\theta(1-p)}{2} \Rightarrow \frac{2X_1}{1-p} \text{ is an unbiased estimator for } \theta.$$

$$\text{var}\left(\frac{2X}{1-p}\right) = \frac{4}{(1-p)^2}\text{var}(X) = \frac{\cancel{4}}{(1-p)^{\cancel{2}}}\frac{\cancel{(1-p)}\theta^2}{\cancel{12}\,3}(4-3(1-p)) = \frac{\theta^2(4-3(1-p))}{3(1-p)}$$

$$\text{var}(X) = \cancel{E[X^2] - E[X]^2}$$

see below

$$\cancel{= \left[0^2(p) + \left(\frac{\theta^2}{12} + \frac{\theta^2}{4}\right)(1-p)\right] - \left(\frac{\theta(1-p)}{2}\right)^2}$$

$$\cancel{= \frac{4\theta^2}{12}(1-p) - \frac{\theta^2(1-p)^2}{4} = \frac{(1-p)}{12}[4\theta^2 - 3\theta^2(1-p)] = \frac{(1-p)}{12}[\theta^2 + 3p\theta^2]}$$

$$\cancel{= \frac{(1-p)\theta^2}{12}(1+3p)}$$

~~$f_X(x) = p\,I(x=0)\left(\frac{1-p}{\theta}\right)I(0<x<\theta)$~~

~~$I(0 \le x \le \theta)$ by factorization X is a sufficient statistic; If X is a css,~~

~~$E\left[\frac{2X}{1-p} \,\middle|\, X\right] = \frac{2X}{1-p} \Rightarrow \frac{2X}{1-p}$ is the UMVUE?~~

I'm going to guess a UMVUE does not exist? But not sure why - come back to.

$$E[X^2] = \int_0^\theta x^2\left(\frac{1-p}{\theta}\right)dx = \frac{1-p}{\theta}\int_0^\theta x^2\,dx = \frac{1-p}{\theta}\left[\frac{1}{3}x^3\right]_0^\theta = \frac{1-p}{\theta}\left(\frac{\theta^3}{3} - 0\right)$$

$$= \frac{(1-p)\theta^2}{3}$$

$$\text{var}(X) = \frac{(1-p)\theta^2}{3} - \left(\frac{\theta(1-p)}{2}\right)^2 = \theta^2\left(\frac{1-p}{3} - \frac{(1-p)^2}{4}\right) = \theta^2\left(\frac{4(1-p) - 3(1-p)^2}{12}\right)$$

$$= \theta^2(1-p)\left(\frac{4-3(1-p)}{12}\right)$$

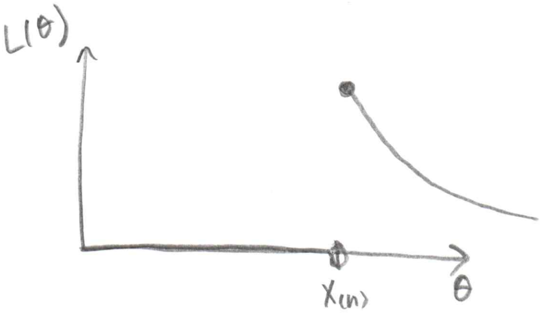
1(b). Based on n obs, $X_1, \dots, X_n$ let $X_{(n)} = \max\{X_1, \dots X_n\}$
Show that $\left(X_{(n)}, \sum_{i=1}^{n} I(X_i > 0)\right)$ is a S.S. for θ. Furthermore, show that $\hat{\theta} = X_{(n)}$ maximizes the observed likelihood function.

$$f_X(x) = p^{I(x=0)} \left(\frac{1-p}{\theta}\right)^{I(0<x\le\theta)} I(0 \le x \le \theta)$$

$$f_{\underline{X}}(x) = \prod_{i=1}^{n} f_X(x_i) = p^{\sum I(x_i=0)} \left(\frac{1-p}{\theta}\right)^{\sum I(0\le x_i \le \theta)} I(0 < X_{(1)} \le \cdots < X_{(n)} < \theta)$$

$$= p^{\sum I(X_i=0)} \left(\frac{1-p}{\theta}\right)^{\sum I(0 \le X_i) I(X_i < \theta)} I(0 < X_{(1)} < \cdots < X_{(n)} < \theta)$$

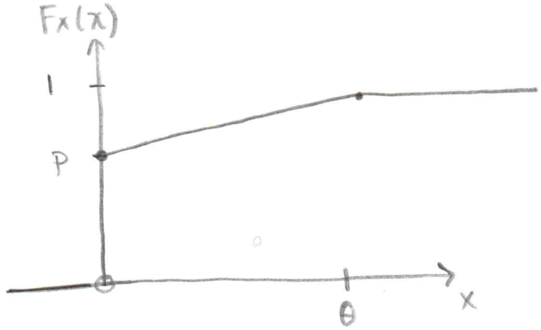
$\therefore$ By factorization thm, $\left(X_{(n)}, \sum I(0 \le X_i)\right)$ is a S.S. for θ.



$L(\theta)$ is a decreasing function WRT θ when $\theta \ge X_{(n)}$, and 0 when $\theta < X_{(n)}$
Thus, $X_{(n)}$ maximizes the likelihood.

1.(c) What is the exact distribution of $\hat{\theta}$?
Compute $E[\hat{\theta}]$ and $var(\hat{\theta})$ & show that $\hat{\theta}$ is consistent for θ.

CDF of X



$$\begin{cases} 0 & \text{if } x<0 \\ p & \text{if } x=0 \\ p+\frac{1-p}{\theta}x & \text{if } 0<x<\theta \\ 1 & \text{if } x>\theta \end{cases}$$

$$F_X(x) = pI(x=0) + \left(p+\tfrac{1-p}{\theta}x\right)I(0<x<\theta) + I(x\geq\theta)$$

$$= pI(0\leq x<\theta) + \tfrac{1-p}{\theta}x\, I(0<x<\theta) + I(x\geq\theta)$$

$$F_{X_{(n)}}(x) = \left[pI(0\leq x<\theta) + \tfrac{1-p}{\theta}x\, I(0<x<\theta) + I(x\geq\theta)\right]^n$$

$$F_{X_{(n)}}(x) = \left(p+\tfrac{1-p}{\theta}x\right)^n \implies f_{X_{(n)}}(x) = \frac{n(1-p)}{\theta}\left(p+\tfrac{1-p}{\theta}x\right)^{n-1}$$

$$E[X_{(n)}] = \int_0^\theta x\,\frac{n(1-p)}{\theta}\left(p+\tfrac{1-p}{\theta}x\right)^{n-1}dx = \frac{n(1-p)}{\theta}\int_0^\theta x\left(p+\tfrac{1-p}{\theta}x\right)^{n-1}dx = (*)$$

let $u = p+\frac{1-p}{\theta}x \implies \frac{u-p}{(1-p)}\theta = x \qquad du = \frac{1-p}{\theta}dx \implies \frac{\theta}{1-p}du = dx \qquad u\in(p,1)$

$$(*) = \frac{n\cancel{(1-p)}}{\cancel{\theta}}\int_p^1 \frac{u-p}{1-p}\theta\, u^{n-1}\frac{\cancel{\theta}}{\cancel{1-p}}du = \frac{n\theta}{1-p}\int_p^1 (u-p)u^{n-1}du = \frac{n\theta}{1-p}\int_p^1 u^n - pu^{n-1}du$$

$$= \frac{n\theta}{1-p}\int_p^1 u^n du - \int_p^1 pu^{n-1}du = \frac{n\theta}{1-p}\left[\frac{1}{n+1}u^{n+1}\Big|_p^1 - \frac{p}{n}u^n\Big|_p^1\right]$$

$$= \frac{n\theta}{1-p}\left[\frac{1}{n+1} - \frac{p^{n+1}}{n+1} - \frac{p}{n} + \frac{p^{n+1}}{n}\right] = \frac{\theta}{1-p}\left[\frac{n}{n+1} - \frac{np^{n+1}}{n+1} - p + p^{n+1}\right]$$

$$= \frac{\theta}{1-p}\left[\underbrace{\frac{n}{n+1}}_{\to 1} - \underbrace{\frac{np^{n+1}}{n+1}}_{\to 0} - \underbrace{p(1-\underbrace{p^n}_{\downarrow 0})}_{\to p}\right] \xrightarrow{p} \frac{\theta}{1-p}(1-p) = \theta, \text{ by Slutsky's}$$

1.(c) con't

$$E\left[X_{(n)}^2\right] = \int_0^\theta x^2 \frac{n(1-p)}{\theta}\left(p+\frac{1-p}{\theta}x\right)^{n-1}dx = \frac{n(1-p)}{\theta}\int_0^\theta x^2\left(p+\frac{1-p}{\theta}x\right)^{n-1}dx$$

using same substitution

$$= \frac{n(1-p)}{\theta}\int_p^1 \left(\frac{u-p}{1-p}\right)^2\theta^2 u^{n-1}\frac{\theta}{1-p}\,du = \frac{n(1-p)}{\theta}\int_p^1 \frac{(u-p)^2}{(1-p)^3}\theta^3 u^{n-1}\,du = \frac{n\theta^2}{(1-p)^2}\int_p^1 (u-p)^2 u^{n+1}\,du$$

$$= \frac{n\theta^2}{(1-p)^2}\int_p^1 (u^2-2pu+p^2)u^{n-1}\,du = \frac{n\theta^2}{(1-p)^2}\left[\int_p^1 u^{n+1}du - 2p\int_p^1 u^n du + p^2\int_p^1 u^{n-1}du\right]$$

$$= \frac{n\theta^2}{(1-p)^2}\left[\frac{1}{n+2}u^{n+1}\Big|_p^1 - \frac{2p}{n+1}u^{n+1}\Big|_p^1 + \frac{p^2}{n}u^n\Big|_p^1\right]$$

$$= \frac{n\theta^2}{(1-p)^2}\left[\frac{1}{n+2} - \frac{p^{n+1}}{n+2} - \frac{2p}{n+1} + \frac{2p^{n+2}}{n+1} + \frac{p^2}{n} - \frac{p^{n+2}}{n}\right]$$

$$= \frac{n\theta^2}{(1-p)^2}\left[\frac{1}{n+2}\right.$$

(d) Derive the asymptotic distribution of $n(\hat{\theta}-\theta)$.

2016 Theory I

$P(n(\hat{\theta}-\theta) \le z)$

$= P(\hat{\theta}-\theta \le \frac{z}{n}) = P(\hat{\theta} \le \theta + z/n) = F_{X_{(n)}}(\theta + z/n)$

$= \left(p + \frac{1-p}{\theta}(\theta + \frac{z}{n})\right)^n$ found $F_{X_{(n)}}(x)$ in (c)

$= \left(p + 1 - p + \frac{(1-p)z}{\theta n}\right)^n$

$= \left(1 + \frac{(1-p)z/\theta}{n}\right)^n \xrightarrow[n\to\infty]{} \exp\left(\frac{(1-p)z}{\theta}\right)$

$\therefore\ n(\hat{\theta}-\theta) \xrightarrow[d]{} -\mathrm{Exp}\left(\frac{(1-p)}{\theta}\right)$

1.(e) Calculate the MLE for p to obtain the MLE for $E[x_i]$. [2016 Theory 1]
Derive the asymptotic distribution for the latter after proper norm..

$$L(p,\theta|\vec{x}) = p^{\sum I(x_i=0)} \left(\frac{1-p}{\theta}\right)^{\sum I(x_i \in (0,\theta))} I(0 \le x_{(1)} \le \cdots \le x_{(n)} \le 1)$$

$$l(p,\theta) = \sum I(x_i=0)\log(p) + \sum I(x_i \in (0,\theta))[\log(1-p) - \log(\theta)]$$

$$\frac{\partial l}{\partial p} = \frac{\sum I(x_i=0)}{p} - \frac{\sum I(x_i \in (0,\theta))}{1-p} \overset{set}{=} 0 \Rightarrow p\sum I(x_i \in (0,\theta)) = 1-p \sum I(x_i=0),$$

$$p\left(\sum I(x_i \in (0,\theta)) + \sum I(x_i = 0)\right) = \sum I(x_i=0) \Rightarrow \hat{p} = \frac{\sum I(x_i=0)}{\sum I(x_i=0) + \sum I(x_i \in (0,\theta))} = \frac{\sum I(x_i=0)}{n}$$

by MLE properties, we know $\hat{p} \xrightarrow{p} p$ as MLEs are consistent estimators

$$\Rightarrow \hat{p} - p \xrightarrow{p} 0 \Rightarrow n(\hat{p}-p) \xrightarrow{p} 0 \quad E[x_i] = \frac{1-p}{2}\theta \Rightarrow \frac{1-\hat{p}}{2}\hat{\theta}$$

$$n\begin{pmatrix} \hat{\theta} - \theta \\ \hat{p} - p \end{pmatrix} \xrightarrow{d} \begin{pmatrix} -\text{Exp}\left(\frac{1-p}{\theta}\right) \\ 0 \end{pmatrix}$$

let $g(a,b) = (1-b)\frac{a}{2}$ $\quad \nabla g(a,b) = \left(\frac{(1-b)}{2}, -\frac{a}{2}\right)$

$$n\Big(\frac{1-\hat{p}}{2}\hat{\theta} - \underbrace{\frac{1-p}{2}\theta}_{E[x_i]}\Big) \xrightarrow{d} \left(\frac{(1-p)}{2}, -\frac{\theta}{2}\right)\begin{pmatrix} -\text{Exp}\left(\frac{1-p}{\theta}\right) \\ 0 \end{pmatrix} \equiv -\frac{(1-p)}{2}\text{Exp}\left(\frac{1-p}{\theta}\right)$$

$$\equiv -\text{Exp}\left(\frac{(1-p)^2}{2\theta}\right)$$

1.(e) Now assume both p and θ are unknown. | 2016 Theory 1

Calculate the MLE for p to obtain the MLE for $E[X_1]$.
Derive the asymptotic dist. for the latter after proper normalization.

$$f_X(x) = p^{I(x=0)} \left(\frac{1-p}{\theta}\right)^{I(0<x\le\theta)} I(0\le x\le\theta)$$

$$l_X(x) = I(x=0)\log(p) + I(0<x<\theta)\log(1-p) - I(0<x<\theta)\log(\theta) + \log(I(0\le x<\theta))$$

$$\propto I(x=0)\log(p) + I(0<x<\theta)\log(1-p)$$

$$\frac{\partial l}{\partial p} = \frac{I(x=0)}{p} + \frac{I(0<x<\theta)}{1-p} \overset{set}{=} 0$$

$$I(x=0) - pI(x=0) + pI(0<x<\theta) = 0$$

$$I(x=0) = p(I(0<x\le\theta) - I(x=0))$$

$$\hat{p} = \frac{I(x=0)}{I(0<x\le\theta) - I(x=0)} = \begin{cases} 0 & \text{if } 0<x\le\theta \\ -1 & \text{if } x=0 \end{cases}$$

$$L(p,\theta|\vec{x}) = p^{\sum I(x_i=0)} \left(\frac{1-p}{\theta}\right)^{\sum I(x_i\in(0,\theta))} I(0\le x_{(1)}<\dots<x_{(n)}<\theta)$$

Redo this problem

$$l(p,\theta|\vec{x}) \propto \sum I(x_i=0)\log(p) + \sum I(x_i\in(0,\theta))\log(1-p) - \sum I(x_i\in(0,\theta))\log(\theta)$$

$$\propto \sum I(x_i=0)\log(p) + \sum I(x_i\in(0,\theta))\log(1-p)$$

$$\frac{\partial l}{\partial p} = \frac{\sum I(x_i=0)}{p} + \frac{\sum I(x_i\in(0,\theta))}{1-p} \overset{set}{=} 0$$

$$\sum I(x_i=0) - p\sum I(x_i=0) + p\sum I(x_i\in(0,\theta)) = 0$$

$$\sum I(x_i=0) = p\left(\sum I(x_i=0) - \sum I(x_i\in(0,\theta))\right)$$

$$\hat{p} = \frac{\sum I(x_i=0)}{\sum(I(x_i=0)+I(x_i\in(0,\theta)))} = \frac{\sum I(x_i=0)}{\sum(I(x_i=0)-(1-I(x_i=0)))}$$

when $x\in[0,\theta)$, $I(x_i\in(0,\theta)) = 1 - I(x_i=0)$

?

$$= \frac{\sum I(x_i=0)}{n}$$

$$E(X_1) = \frac{(1-p)}{2}\theta \Rightarrow \frac{(1-\hat{p})}{2}\hat{\theta} = x_{(n)}\,(\sum I(x_i>0)$$

rework distributions!

$$n\begin{pmatrix}\hat{\theta}-\theta \\ \hat{p}-p\end{pmatrix} \xrightarrow{d} \begin{pmatrix} \sim \\ \sim \end{pmatrix}$$

2. (25 points) Suppose that $y_1, \ldots, y_n$ are independent binary random variables, where

$$P(y_i = 1|\beta_0, \beta_1, x_i) = \frac{\exp(\beta_0 + \beta_1 x_i)}{1 + \exp(\beta_0 + \beta_1 x_i)},$$

where $x_1, \ldots, x_n$ are fixed covariates and they are not all equal.

(a) (6 points) Suppose that (β_0, β_1) are both unknown and suppose we wish to test

$$H_0 : \beta_1 = 0 \text{ versus } H_1 : \beta_1 \neq 0.$$

Derive the Uniformly Most Powerful Unbiased (UMPU) α level test for this hypothesis and express the rejection region and critical value in the simplest possible form. Please note that there need not be a closed form for the distribution of the test statistic.

(b) (5 points) Using the UMPU conditional test from part (a), compute an explicit closed form for its conditional mean and conditional variance under the null hypothesis to find an explicit form for an asymptotically correct approximation to the UMPU test. You are allowed to assume that the conditional test statistic is asymptotically normal.

(c) (4 points) Derive the score test for the hypothesis in part (a), and compare its form to the approximate UMPU test derived in part (b).

(d) (6 points) Now consider the more general problem in which we have p covariates, and

$$P(y_i = 1|\boldsymbol{\beta}, \boldsymbol{x}_i) = \frac{\exp(\boldsymbol{x}_i'\boldsymbol{\beta})}{1 + \exp(\boldsymbol{x}_i'\boldsymbol{\beta})},$$

where $\boldsymbol{x}_i = (x_{i1}, \ldots, x_{ip})'$ is a $p \times 1$ vector of covariates, and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)'$ is a $p \times 1$ vector of regression coefficients. Suppose we wish to test

$$H_0 : \boldsymbol{\ell}'\boldsymbol{\beta} = \theta_0 \text{ versus } H_1 : \boldsymbol{\ell}'\boldsymbol{\beta} \neq \theta_0,$$

where θ_0 is a specified scalar and $\boldsymbol{\ell}$ is a specified and non-zero $p \times 1$ vector. Derive the UMPU size α test for this hypothesis and express the rejection region and critical value in the simplest possible form.

(e) (4 points) Describe in detail a non-parametric bootstrap algorithm for computing the exact p-value based on the UMPU test of part (d).

2. Suppose that $y_1, \dots, y_n$ are independent binary RV where $P(y_i = 1 \mid \beta_0, \beta_1, x_i) = \dfrac{\exp(\beta_0 + \beta_1 x_i)}{1 + \exp(\beta_0 + \beta_1 x_i)}$ | 2016 Theory 1

where $x_1, \dots, x_n$ are covariates and are not all equal.

(a) Suppose that (β_0, β_1) are both unknown & suppose we wish to test $H_0: \beta_1 = 0$ vs. $H_1: \beta_1 \neq 0$.

Derive the UMPU - α level test, express rejection region & critical value in simplest form

$$\therefore Y \mid \beta_0, \beta_1, x_i \sim \text{Bernoulli}\Big(\overbrace{\frac{\exp(\beta_0+\beta_1 x_i)}{1+\exp(\beta_0+\beta_1 x_i)}}^{p}\Big)$$

$$f_{Y_i \mid x_i}(y_i) = \left(\frac{\exp(\beta_0+\beta_1 x_i)}{1+\exp(\beta_0+\beta_1 x_i)}\right)^{y_i}\left(\frac{1}{1+\exp(\beta_0+\beta_1 x_i)}\right)^{1-y_i}$$

$$\ell(y) = y_i \log\left(\frac{\exp(\beta_0+\beta_1 x_i)}{1+\exp(\beta_0+\beta_1 x_i)}\right) + (1-y_i)\log\left(\frac{1}{1+\exp(\beta_0+\beta_1 x_i)}\right)$$

$$= y_i \log\left(\frac{\exp(\beta_0+\beta_1 x_i)}{1+\exp(\beta_0+\beta_1 x_i)}(1+\exp(\beta_0+\beta_1 x_i))\right) + \log\left(\frac{1}{1+\exp(\beta_0+\beta_1 x_i)}\right)$$

$$= y_i \log(\exp(\beta_0+\beta_1 x_i)) + \log\left(\frac{1}{1+\exp(\beta_0+\beta_1 x_i)}\right)$$

$$\ell(\beta_0, \beta_1 \mid \vec{y}, \vec{x}) = \sum_{i=1}^{n}\{y_i \log(\exp(\beta_0+\beta_1 x_i)) - \log(1+\exp(\beta_0+\beta_1 x_i))\}$$

$$= \sum_{i=1}^{n} y_i(\beta_0+\beta_1 x_i) - \sum_{i=1}^{n}\log[1+\exp(\beta_0+\beta_1 x_i)]$$

$$f_{\vec{Y}}(\vec{y} \mid \vec{x}) = \frac{\exp(\beta_0 \sum y_i + \beta_1 \sum y_i x_i)}{\prod_{i=1}^{n} 1+\exp(\beta_0+\beta_1 x_i)} \quad \therefore \text{member of multiparameter exponential family}$$

$$h(y)c(\theta) = \left[\prod_{i=1}^{n} 1+\exp(\beta_0+\beta_1 x_i)\right]^{-1} \quad \eta_0 = \beta_0 \quad T = \sum y_i \quad \eta_1 = \beta_1 \quad U = \sum y_i x_i$$

$\therefore$ CSS for β_0 is $\sum y_i$ (nuisance)

CSS for β_1 is $\sum y_i x_i$

2.(a) con't

we know the UMPU is of the form

$$\phi(u) = \begin{cases} 1 & u < c_1(t) \text{ OR } u > c_2(t) \\ \gamma_i & u = c_1(t) \text{ or } u = c_2(t) \\ 0 & c_1(t) < u < c_2(t) \end{cases}$$

where (i) $E_{\theta_0}[\phi(u) \mid T] = \alpha$ and (ii) $E_{\theta_0}[u\phi(u) \mid T] = \alpha E_{\theta_0}[u \mid T]$ (iii)

let $A = \{Y : \sum y_i = t\}$, $B(c_1, c_2) = \{y : \sum y_i x_i \notin [c_1, c_2]\}$

$B_1 = \{y : \sum x_i y_i = c_1\}$, $B_2 = \{y : \sum x_i y_i = c_2\}$

(i) $E_{\theta_0}[\phi(u) \mid T = t] = E_{\theta_0}[\phi(\sum x_i y_i) \mid \sum y_i = t]$

$$= 1 P_{\theta_0}[\sum x_i y_i \notin [c_1(t), c_2(t)] \mid \sum y_i = t] + \gamma_1 P_{\theta_0}[\sum x_i y_i = c_1 \mid \sum y_i = t] + \gamma_2 P_{\theta_0}[\sum x_i y_i = c_2 \mid \sum y_i = t]$$

$$= \frac{P_{\theta_0}[y \in (A \cap B)] + \gamma_1 P_{\theta_0}[y \in (A \cap B_1)] + \gamma_2 P_{\theta_0}[y \in (A \cap B_2)]}{P_{\theta_0}[y \in A]}$$

$$= \frac{\sum_{y \in A} c(\theta \mid \beta_1 = 0) e^{\cancelto{0}{\beta_1 \sum x_i y_i} + \beta_0 \sum y_i} [I(y \in B) + \gamma_1 I(y \in B_1) + \gamma_2 I(y \in B_2)]}{\sum_{y \in A} c(\theta \mid \beta_1 = 0) e^{\cancelto{0}{\beta_1 \sum x_i y_i} + \beta_0 \sum y_i}}$$

where $c(\beta_0)$

$$= \frac{\cancel{c(\theta \mid \beta_1 = 0) e^{\beta_0 \sum y_i}} \sum_{y \in A} [I(y \in B) + \gamma_1 I(y \in B_1) + \gamma_2 I(y \in B_2)]}{\cancel{c(\theta \mid \beta_1 = 0) e^{\beta_0 \sum y_i}} \sum_{y \in A} J_n}$$

where $J_n = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_{n \times 1}$

note: can pull out of summation only because we are conditioning on $\sum y_i$

2(a) con't

(ii) $E_{\theta_0}[U\phi(U)|T] = E_{\theta_0}\left[\sum_{i=1}^{n} x_i y_i \,\phi(\sum x_i y_i) \,\middle|\, \sum y_i = t\right]$

$$= \sum_{Y\in A} \sum_{i=1}^{n} x_i y_i \,\phi(\sum x_i y_i)\, P_{\theta_0}(Y=y \mid \sum y_i = t)$$

$$= \frac{\sum_{Y\in A} \left(\sum_{i=1}^{n} x_i y_i\right)\left[I(Y\in B) + \gamma_1 I(Y\in B_1) + \gamma_2 I(Y\in B_2)\right]}{\sum_{Y\in A} J_n}$$

(iii) $E_{\theta_0}[U|T] = E_{\theta_0}\left[\sum x_i y_i \mid \sum y_i = t\right]$

$$= \sum_{y\in A} \sum_{i=1}^{n} x_i y_i \, P_{\theta_0}[Y=y \mid \sum y_i = t]$$

$$= \sum_{y\in A} \left(\sum_{i=1}^{n} x_i y_i\right) \frac{P_{\theta_0}(Y=y, \sum y_i = t)}{P_{\theta_0}(y\in A)}$$

$\longrightarrow P(Y_1, \dots, y_n, \sum y_i) = P(Y_1, \dots, y_n)$ as $\sum y_i$ contains no add'l information.

$$= \sum_{y\in A} \sum_{i=1}^{n} x_i y_i \frac{P_{\theta_0}(Y=y)}{P_{\theta_0}(y\in A)}$$

$$= \sum_{y\in A} \frac{(\sum x_i y_i) c(\beta_0) e^{\beta_0 \sum y_i}}{\sum_{y\in A} c(\beta_0) e^{\beta_0 \sum y_i}} = \frac{c(\beta_0) e^{\beta_0 \sum y_i}}{c(\beta_0) e^{\beta_0 \sum y_i}} \sum_{y\in A} \frac{\sum x_i y_i}{\sum_{y\in A} J_n} = \frac{\sum_{y\in A} \sum_i x_i y_i}{\sum_{y\in A} J_n}$$

$\therefore$ choose $c_1, c_2, \gamma_1, \gamma_2$ s.t.

$E_{\theta_0}[\phi(U)|T] = \alpha$ and $E_{\theta_0}[U\phi(U)|T] = \alpha E_{\theta_0}[U|T]$

where $(\gamma_1, \gamma_2) \in [0,1]\times[0,1]$ for a chosen level of α.

Enumerate all possible vectors Y & then restrict to all $Y\in A$

vary c_1 & c_2 within $Y\in A$ to determine appropriate cutpoints.

2.(b) Using the UMPU conditions from (a), compute an explicit closed form for its conditional mean & variance under the null to find an explicit form for an asymptotically correct approx. to the UMPU test. You are allowed to assume that the conditional test statistic is asy. normal.

In plain words: Normalize the conditional test statistic $U|T = \Sigma x_i y_i \mid \Sigma y_i = t$ which will allow us to remove γ_1 & γ_2 from the test w/ continuity & simplify the calculations for c_1 & c_2.

$$E_{\theta_0}[U|T] = \frac{\sum_{y \in A} \sum_{i=1}^{n} x_i y_i}{\sum_{y \in A} J_n} = \mu$$

$$var_{\theta_0}[U|T] = E_{\theta_0}[U^2|T] - \mu^2$$

$$var_{\theta_0}(U|T) = \frac{\sum_{y \in A} \left(\sum_{i=1}^{n} x_i y_i\right)^2}{\sum_{y \in A} J_n} - \left(\frac{\sum_{y \in A} \left(\sum_{i=1}^{n} x_i y_i\right)}{\sum_{y \in A} J_n}\right)^2 = \sigma^2$$

$\therefore$ The asymptotic UMPU α-level test is of the form:

$$\phi(u) = \begin{cases} 1 & \Sigma x_i y_i < c_1(\Sigma y_i) \text{ or } \Sigma x_i y_i > c_2(\Sigma y_i) \\ 0 & \end{cases}$$

where $\alpha = E_{\theta_0}[\phi(\Sigma x_i y_i) \mid \Sigma Y_i = t]$ and $E_{\theta_0}[u\phi(u)|T] = \alpha E[U|T]$

$$E_{\theta_0}[\phi(\Sigma x_i y_i) \mid \Sigma Y_i = t] = P(\Sigma x_i y_i < c_1(\Sigma y_i) \text{ OR } \Sigma x_i y_i > c_2(\Sigma y_i) \mid \Sigma Y_i = t)$$

$$= 1 - P(c_1(\Sigma y_i) < \Sigma x_i y_i < c_2(\Sigma y_i) \mid \Sigma Y_i = t)$$

$$= 1 - \int_{c_1}^{c_2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(z-\mu)^2/2\sigma^2} dz$$

$$= 1 - (\Phi(c_2) - \Phi(c_1))$$ where $\Phi(\cdot)$ is the CDF of a $N(\mu, \sigma^2)$

2(b) con't

$$\alpha E_{\theta_0}[U|T] = \alpha\mu = E_{\theta_0}[U\phi(U)|T] = E\left[\sum x_i y_i\, \phi(\sum x_i y_i) \,\middle|\, \sum y_i = t\right]$$

$$= E[z\phi(z)] = E[zI(z<c_1) + zI(z>c_2)]$$

$$= E[zI(z<c_1)] + E[zI(z>c_2)]$$

$$= \int_0^{c_1} z f_z\, dz + \int_{c_2}^{\infty} z f_z\, dz = \int_{-\infty}^{\infty} z f_z\, dz - \int_{c_1}^{c_2} z f_z\, dz$$

$$= \mu - \int_{c_1}^{c_2} \frac{z}{\sqrt{2\pi\sigma^2}} . e^{-(z-\mu)^2/2\sigma^2} dz$$

$$\alpha\mu = \mu - \Phi(c_2) + \Phi(c_1) \quad \text{where } \Phi(\cdot) \text{ is the CDF for } N(\mu, \sigma^2)$$

3. (25 points) Suppose $S \sim \text{Binomial}(n, p)$ and conditional on $S = s$, let $X_1, \ldots, X_{s+1}$ be iid from a $N(\mu, 1)$ distribution. The value of n is known whereas (p, μ) are both unknown, $0 < p < 1$, and $-\infty < \mu < \infty$. We observe $(S, X_1, \ldots, X_{S+1})$ and we wish to test $H_0 : \mu \leq 0$ versus $H_1 : \mu > 0$ at level α.

   (a) (5 points) Write out the joint density of $(S, X_1, \ldots, X_{S+1})$ and show that it belongs to a full-rank exponential family, and find the two dimensional complete sufficient statistic. Do the same thing for the special case that $\mu = 0$.

   (b) (3 points) Derive the joint MLE's of (p, μ), denoted by $(\hat{p}, \hat{\mu})$.

   (c) (5 points) Assuming that standard MLE theory applies, derive the joint asymptotic distribution of $(\hat{p}, \hat{\mu})$, properly normalized.

   (d) (6 points) Let $\phi(S, X_1, \ldots, X_{S+1})$ be *any* unbiased level α test of H_0 versus H_1. Write out what unbiasedness means for the power function $\beta(p, \mu)$ of such a test, and explain in detail why unbiasedness implies that $\beta(p, 0) = \alpha$ for all p.

   (e) (6 points) Find the complete form of the UMPU test of H_0 versus H_1, including specification of the rejection region in terms of the sample mean of the X_i's and the $1 - \alpha$ quantile of a well known distribution.

3. Suppose $S \sim \text{Binomial}(n,p)$ and conditional on $S=s$, let $\boxed{\text{2016 Theory 1}}$ $X_1, \dots, X_{S+1}$ be iid from a $N(\mu, 1)$. n known, (p,μ) unknown. $p \in (0,1)$, $\mu \in (-\infty, \infty)$. We observe $(S, X_1, \dots, X_{S+1})$ and want to test $H_0: \mu \le 0$ vs. $H_1: \mu > 0$ at level α.

(a) Write out the joint density of $(S, X_1, \dots, X_{S+1})$ & show it belongs to a full rank exp. family. Find the 2-D CSS. Do the same thing for the special case that $\mu = 0$

$$f_{S,X}(s,x) = f_{X|S}(x|s) f_S(s) = (2\pi)^{-\frac{s+1}{2}} \exp\left\{-\frac{\sum_{i=1}^{s+1}(x_i-\mu)^2}{2}\right\}\binom{n}{p} p^s (1-p)^{n-s}$$

$$l(p,\mu) = -\frac{s+1}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^{s+1}(x_i-\mu)^2 + \log\binom{n}{p} + s\log(p) + (n-s)\log(1-p)$$

$$= -\frac{s+1}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^{s+1}(x_i^2 - 2x_i\mu + \mu^2) + \log\binom{n}{p} + s\log\left(\frac{p}{1-p}\right) + n\log(1-p)$$

$$= -\frac{s+1}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^{s+1}x_i^2 + \mu\sum_{i=1}^{s+1}x_i - (s+1)\mu^2 + \log\binom{n}{p} + s\log\left(\frac{p}{1-p}\right) + n\log(1-p)$$

$$f_{S,X}(s,x) = \exp\left\{\underbrace{-\frac{s+1}{2}\log(2\pi)}\underbrace{-\frac{1}{2}\sum_{i=1}^{s+1}x_i^2} + \mu\sum_{i=1}^{s+1}x_i - (s+1)\mu^2 + \log\binom{n}{p} + s\log\left(\frac{p}{1-p}\right) + n\log(1-p)\right\}$$

$$(2\pi)^{-\frac{s+1}{2}} \exp\left(-\frac{1}{2}\sum_{i=1}^{s+1}x_i^2\right) \exp\left\{+\mu\sum_{i=1}^{s+1}x_i - (s+1)\mu^2 + s\log\left(\frac{p}{1-p}\right) - \left(-\log\binom{n}{p} - n\log(1-p)\right)\right\}$$

$$h(x) = \left(2\pi^{-\frac{s+1}{2}}\right)\exp\left(-\frac{1}{2}\sum_{i=1}^{s+1}x_i^2\right)$$

$\eta_1(\theta) = \mu$ $\quad \eta_2(\theta) = \mu^2$ $\quad \eta_3(\theta) = \log\left(\frac{p}{1-p}\right)$ $\quad B(\theta) = -\log\binom{n}{p} - n\log(1-p)$

$T_1(x) = \sum_{i=1}^{s+1} x_i$ $\quad T_2(x) = -(s+1)$ $\quad T_3(x) = s$

$\therefore$ The 2-dimensional CSS for (μ, p) is: $\left(\sum_{i=1}^{s+1} x_i, s\right)$

<u>When $\mu = 0$:</u>

$$f_{S,X}(s,x) = (2\pi)^{-\frac{s+1}{2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{s+1}(x_i)^2\right\}\binom{n}{p} p^s (1-p)^{n-s}$$

$$= (2\pi)^{-\frac{s+1}{2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{s+1}(x_i)^2 + \log\binom{n}{p} + s\log(p) + (n-s)\log(1-p)\right\}$$

$$= \underbrace{(2\pi)^{-\frac{s+1}{2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{s+1}x_i^2\right\}}_{h(x)} \exp\left\{s\log\left(\frac{p}{1-p}\right) - \left(-\log\binom{n}{p} - n\log(1-p)\right)\right\}$$

$\eta_1(\theta) = \log\left(\frac{p}{1-p}\right)$

$T_1(x) = s$

The CSS for p is (s).

3 (b). Derive the joint MLEs, denoted by $(\hat{p}, \hat{\mu})$. 2016 Theory 1

From work in (a) we know

$$\ell(p,\mu) \propto \mu \sum_{i=1}^{S+1} x_i - \tfrac{1}{2}(S+1)\mu^2 + S\log(p) + (n-S)\log(1-p)$$

$$\frac{\partial \ell}{\partial p} = \frac{S}{p} - \frac{(n-S)}{1-p} \overset{\text{set}}{=} 0 \qquad \frac{\partial \ell}{\partial \mu} = \sum_{i=1}^{S+1} x_i - (S+1)\mu \overset{\text{set}}{=} 0$$

$$\frac{S}{p} = \frac{(n-S)}{1-p} \qquad \sum_{i=1}^{S+1} x_i = (S+1)\mu$$

$$S - \cancel{Sp} = np - \cancel{Sp} \qquad \hat{\mu} = \frac{\sum_{i=1}^{S+1} x_i}{(S+1)} = \bar{X} \quad \text{where } \bar{X} = \frac{1}{S+1}\sum_{i=1}^{S+1} x_i$$

$$\boxed{\frac{S}{n} = \hat{p}}$$

$$\cancel{\frac{\partial^2 \ell}{\partial p^2} = -\frac{S}{p^2} + \frac{n-S}{(1-p)^2} = \frac{-S(1-p)^2 + np^2 - Sp^2}{p^2(1-p)^2} < 0 \checkmark}$$

$$\cancel{\frac{\partial^2 \ell}{\partial \mu^2} = -2(S+1) < 0 \checkmark \qquad \frac{\partial^2 \ell}{\partial \mu \partial p} = 0}$$

(c) Assuming standard MLE theory applies, derive the joint asypt. distribution of $(\hat{p}, \hat{\mu})$

$$\sqrt{n}\left(\begin{pmatrix}\hat{p}\\ \hat{\mu}\end{pmatrix} - \begin{pmatrix}p\\ \mu\end{pmatrix}\right) \xrightarrow{d} N(0, I_1(p,\mu)^{-1}) \equiv N\left(0, \begin{pmatrix}\frac{1}{p(1-p)} & 0\\ 0 & p\end{pmatrix}^{-1}\right) \equiv N\left(0, \begin{pmatrix}p(1-p) & 0\\ 0 & 1/p\end{pmatrix}\right)$$

$$\frac{\partial^2 \ell}{\partial p^2} = -\frac{S}{p^2} - \frac{(n-S)}{(1-p)^2} \qquad E\left[-\frac{\partial^2 \ell}{\partial p^2}\right] = \frac{np}{p^2} + \frac{(n-np)}{(1-p)^2} = \frac{n}{p} + \frac{n(1-p)}{(1-p)^2} = \frac{n}{p} + \frac{n}{1-p}$$

$$= \frac{n - np + np}{p(1-p)} = \frac{n}{p(1-p)}$$

$$\frac{\partial^2 \ell}{\partial \mu^2} = -(S+1) \qquad E\left[-\frac{\partial^2 \ell}{\partial \mu^2}\right] = (np+1) = np+1$$

$$\frac{\partial^2 \ell}{\partial \mu \partial p} = 0 \quad \Rightarrow I_n(p,\mu) = \begin{bmatrix}\frac{n}{p(1-p)} & 0\\ 0 & np+1\end{bmatrix} \qquad I_1(p,\mu) = \lim_{n\to\infty}\begin{bmatrix}\frac{1}{p(1-p)} & 0\\ 0 & p+\frac{1}{n}\end{bmatrix}$$

$$I_1(p,\mu) = \begin{bmatrix}\frac{1}{p(1-p)} & 0\\ 0 & p\end{bmatrix} \qquad I_1(p,\mu)^{-1} = (1-p)\begin{pmatrix}p & 0\\ 0 & \frac{1}{p(1-p)}\end{pmatrix} \equiv \begin{pmatrix}p(1-p) & 0\\ 0 & 1/p\end{pmatrix}$$

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3(d). Let $\phi(S, X_1, \ldots, X_{S+1})$ be any unbiased level α test of H_0 vs. H_1. Write out what unbiasedness means for the power function $\beta(p, \mu)$ of such a test & explain in detail why unbiasedness $\Rightarrow \beta(p, 0) = \alpha \;\forall\, p$.

An unbiased test implies

$E_{\Theta_0}[$

look at Pittman efficiency

3(e) Find the complete form of the UMPU test of H_0 vs. H_1 including specification of the rejection region in terms of the sample mean of the X_is and the $1-\alpha$ quantile of a well known dist.

$H_0: \mu \le 0$ vs. $H_1: \mu > 0$

Write out extention. w/ exponential family $\Rightarrow$ MLR.

$$\phi(\vec{x}, s) = \begin{cases} 1 & \text{if } \sum_{i=1}^{s+1} x_i > c(t) \\ 0 & \text{o.w.} \end{cases}$$

where $c(t)$ is determined by $\alpha = E[\phi(\vec{x}, s) \mid S = s]$

$$\alpha = E_{\theta_0}\left[\sum_{i=1}^{s+1} x_i > c(t) \mid S = s\right]$$

$$= P_{\theta_0}\left(\sum_{i=1}^{s+1} x_i > c(t) / S = s\right)$$

we know $X / S \sim N(\mu, 1)$ & under $H_0: \mu \le 0$

Thus $\frac{1}{s+1}\sum_{i=1}^{s+1} x_i = \bar{x} \sim N(\mu, \frac{1}{s+1}) \Rightarrow \sqrt{s+1}\,\bar{X} \sim N(\mu, 1)$ & we use the boundary pt. of Θ_0 to determine cutoffs. thus under H_0 $\sqrt{s+1}\,\bar{X} \sim N(0,1) \equiv Z$

$$\alpha = P\left(Z > \frac{c_1(t)}{\sqrt{s+1}}\right) \Rightarrow \frac{c_1(t)}{\sqrt{s+1}} = Z(.95)$$ the 95th percentile of a $N(0,1)$.

$$\Rightarrow c(t) = \sqrt{s+1}\, Z_{.95}(x)$$

$$\therefore \phi(\vec{x}, s) = \begin{cases} 1 & \text{if } \frac{1}{s+1}\sum_{i=1}^{s+1} x_i > \sqrt{s+1}\, Z_{.95}(x) \\ 0 & \text{o.w.} \end{cases}$$