... Poisson distributed with parameter $0 < \lambda < \infty$, and let $Z_1, Z_2, \ldots$
... sequence of exponential random variables with mean $1/\mu$, where $0 < \mu < \infty$,

1. Given N Poisson w/ parameter $0 < \lambda < \infty$ and let $Z_1, Z_2, \ldots$ be an iid sequence of exponential RV w/ mean $1/\mu$, where $0 < \mu < \infty$, and which are indep of N.

Let $X = 1\{N>0\} \max_{1 \le j \le N} Z_j$

where $1\{A\}$ is the indicator of A. Let $x_1, \ldots, x_n$ be iid realizations of X and define $\hat{\alpha}_n = \frac{1}{n}\sum_{i=1}^{n} 1\{x_i = 0\}$ and $\hat{\beta}_n = \frac{1}{n}\sum_{i=1}^{n} 1\{x_i \le 1\}$.

a) Show that $P(X \le t) = \exp(-\lambda e^{-\mu t}) = e^{-\lambda e^{-\mu t}} \quad \forall\, 0 \le t < \infty$.

$P(X \le t) = P(1\{N>0\} \max_{1 \le j \le N} Z_j \le t)$

Ignore. Was thinking

Know the maximum order statistic has pdf $f_{X_{(n)}}(t) = \frac{1}{n} f(x)\{F(x)\}^{n-1}$

Also given that $Z_j \perp\!\!\!\perp N$

$F_{X_{(n)}}(t) = \int_0^\infty \frac{1}{n} f(x)\{F(x)\}^{n-1} dx$

Know $P(X \le t) = P(1\{N>0\} Z_{(n)} \le t) = P(Z_{(n)} \le t \mid N=0) \cdot P(N=0)$

$+ P(Z_{(n)} \le t \mid N>0) \cdot P(N>0) = [P(Z \le t \mid N=0)]^n \cdot P(N=0)$

$+ [P(Z \le t \mid N>0)]^n \cdot P(N>0)$

$\perp\!\!\!\perp$ $\quad\quad$ $\perp\!\!\!\perp$

$= P(Z \le t)^n \cdot P(N=0) + P(Z \le t)^n \cdot P(N>0)$

$= \underbrace{(1-e^{-\mu t})}_{\text{cdf of poisson}(1/\mu)}{}^n \cdot \frac{e^{-\lambda} \cdot 0^\lambda}{0!} + (1-e^{-\mu t})^n \cdot \left(\underbrace{\sum_{k=0}^{\infty} \frac{e^{-\lambda} k^\lambda}{k!}}_{=1} - \frac{e^{-\lambda} 0^\lambda}{0!}\right)$

$= (1-e^{-\mu t})^n \cdot e^{-\lambda} + (1-e^{-\mu t})^n \cdot (1-e^{-\lambda})$

$= (1-e^{-\mu t})^n \cdot [\cancel{e^{-\lambda}} + 1 - \cancel{e^{-\lambda}}] = (1-e^{-\mu t})^n \overset{?}{=} \exp(-\lambda e^{-\mu t})$

Guessing I was supposed to use the defn. of exponential function, but I don't have the correct form and there is no lambda in mine.

b) Show that

Code: E

$$\sqrt{n}\begin{pmatrix}\hat{\alpha}_n - \alpha \\ \hat{\beta}_n - \beta\end{pmatrix} \xrightarrow{d} N\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{bmatrix}\alpha(1-\alpha) & \alpha(1-\beta) \\ \alpha(1-\beta) & \beta(1-\beta)\end{bmatrix}\right)$$

where $\alpha = e^{-\lambda}$ and $\beta = \exp(-\lambda e^{-\mu})$ as $n \to \infty$.

Take $\hat{\alpha}_n = \frac{1}{n}\sum_{i=1}^{n} 1\{X_i = 0\}$. By SLLN, $\frac{1}{n}\sum_{i=1}^{n} 1\{X_i=0\} \xrightarrow{p} E\left[\frac{1}{n}\sum_{i=1}^{n} 1\{X_i=0\}\right]$

$$= \frac{1}{n}\sum_{i=1}^{n} E[1\{X_i=0\}] = \frac{1}{n}\sum_{i=1}^{n} P(X_i=0)$$

$$= \frac{1}{n}\sum_{i=1}^{n} P(1\{N>0\}\max_{1\le j\le N} Z_j = 0)$$

$$= \frac{1}{n}\sum_{i=1}^{n} \frac{e^{-\lambda}\lambda^0}{0!}$$

$$= \frac{1}{n}\sum_{i=1}^{n} e^{-\lambda} = e^{-\lambda}$$

Thus, $\hat{\alpha}_n \xrightarrow{a.s.} \alpha = e^{-\lambda}$ by SLLN. (as $n \to \infty$)

Similarly, $\hat{\beta}_n = \frac{1}{n}\sum_{i=1}^{n} 1\{X_i \le 1\}$. By LLN, $\frac{1}{n}\sum_{i=1}^{n} 1\{X_i \le 1\} \xrightarrow{p} E\left[\frac{1}{n}\sum_{i=1}^{n} 1\{X \le 1\}\right]$

$$= \frac{1}{n}\sum_{i=1}^{n} E[1\{X_i \le 1\}] = \frac{1}{n}\sum_{i=1}^{n} P(X_i \le 1)$$

part a)

$$= \frac{1}{n}\sum_{i=1}^{n} e^{-\lambda e^{-\mu(1)}} = \frac{1}{n}\sum_{i=1}^{n} e^{-\lambda e^{-\mu}} = e^{-\lambda e^{-\mu}}$$

Thus, $\hat{\beta}_n \xrightarrow{a.s.} \beta = \exp(-\lambda e^{-\mu})$ by SLLN. (as $n \to \infty$)

Know by multivariate CLT that,

$$\sqrt{n}\begin{pmatrix}\hat{\alpha}_n - \alpha \\ \hat{\beta}_n - \beta\end{pmatrix} \xrightarrow{d} N\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}Var(1\{X_i=0\}) & Cov(1\{X_i=0\}, 1\{X_i \le 1\}) \\ \text{''} \quad \text{''} & Var(1\{X_i \le 1\})\end{pmatrix}\right)$$

where $Var(1\{X_i=0\}) = E[\underbrace{1\{X_i=0\}^2}_{\substack{\text{an indicator squared} \\ \text{is itself} \Rightarrow 1\{X_i=0\}^2 = 1\{X_i=0\}}}] - E[1\{X_i=0\}]^2 = E[1\{X_i=0\}] - E[1\{X_i=0\}]^2$

$$= P(X_i=0) - P(X_i=0)^2 = \exp(-\lambda e^{-\mu(0)}) - [\exp(-\lambda e^{-\mu \cdot 0})]^2$$

cont'd $\longrightarrow$

$$= \exp(-\lambda) - \exp(-2\lambda) = \alpha - \alpha^2 = \alpha(1-\alpha) \checkmark$$

b) cont'd

Code: E

Where $Var(1\{X_i \le 1\}) = E[1\{X_i \le 1\}^2] - E[1\{X_i \le 1\}]^2$

$$= E[1\{X_i \le 1\}] - E[1\{X_i \le 1\}]^2$$

$$= \exp(-\lambda e^{-\mu}) - \exp(-2\lambda e^{-\mu})$$

$$= \beta - \beta^2 = \beta(1-\beta) \checkmark$$

and finally $Cov(1\{X_i = 0\}, 1\{X_i \le 1\}) = \underbrace{E[1\{X_i = 0\} \cdot 1\{X_i \le 1\}]}_{E[1\{X_i = 0\}]} - \overbrace{E[1\{X_i = 0\}]}^{\alpha} \cdot \overbrace{E[1\{X_i \le 1\}]}^{\beta}$

$$= \alpha - \alpha\beta = \alpha(1-\beta) \checkmark$$

In conclusion, by multivariate CLT, have:

$$\boxed{\sqrt{n}\begin{pmatrix} \hat{\alpha}_n - \alpha \\ \hat{\beta}_n - \beta \end{pmatrix} \xrightarrow{d} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha(1-\alpha) & \alpha(1-\beta) \\ \alpha(1-\beta) & \beta(1-\beta) \end{pmatrix}\right)}$$

1 c) Let $\hat{\lambda}_n = -\log(\hat{\alpha}_n)$ and $\hat{\mu}_n = -\log[-\log(\hat{\beta}_n)/\hat{\lambda}_n]$. Code: E

Show that $\hat{\lambda}_n$ and $\hat{\mu}_n$ converge a.s. to λ and μ, respectively, as $n \to \infty$

Want to show $\hat{\lambda}_n = -\log(\hat{\alpha}_n) \xrightarrow{a.s.} \lambda$ (i)

& $\hat{\mu}_n = -\log[-\log(\hat{\beta}_n)/\hat{\lambda}_n] \xrightarrow{a.s.} \mu$ (ii)

(i) Do the first and then use that result in the 2nd (through a consequence of the continuous mapping theorem).

Take $\hat{\lambda}_n = -\log(\hat{\alpha}_n)$, Know from proof in part b) that $\hat{\alpha}_n \xrightarrow{a.s.} \alpha = e^{-\lambda}$ by SLLN.

Thus, by CMT, know $\log(\hat{\alpha}_n) \xrightarrow{a.s.} \log(e^{-\lambda}) = -\lambda$

By another application of CMT, know $\boxed{\underbrace{-\log(\hat{\alpha}_n)}_{\hat{\lambda}_n} \xrightarrow{a.s.} \lambda \text{ as } n \to \infty}$ ✓

(ii) Now, to show 2nd convergence, will apply result from first convergence.

By (i), know $\hat{\lambda}_n \xrightarrow{a.s.} \lambda$

Also, know from proof in part b) that $\hat{\beta}_n \xrightarrow{a.s.} \beta = \exp(-\lambda e^{-\mu})$

$\Rightarrow \log(\hat{\beta}_n) \xrightarrow{a.s.} -\lambda e^{-\mu}$ by CMT

$\Rightarrow -\log(\hat{\beta}_n) \xrightarrow{a.s.} \lambda e^{-\mu}$ by another application of CMT

Know that for two sequences X_n and Y_n where $X_n \xrightarrow{a.s.} X$ and $Y_n \xrightarrow{a.s.} Y$ that

$X_n/Y_n \xrightarrow{a.s.} X/Y$ (this was shown in Dr. Kosorok's review session and can be proved by a consequence of CMT AS LONG AS $Y_n \neq 0$ and as long as $Y_n \not\xrightarrow{a.s.} 0$.)

Thus, by the above $-\log(\hat{\beta}_n)/\hat{\lambda}_n \xrightarrow{a.s.} \dfrac{\lambda e^{-\mu}}{\lambda} = e^{-\mu}$ since $\lambda > 0$

Then, by two final applications of CMT, get $\log\left[\log(\hat{\beta}_n)/\hat{\lambda}_n\right] \xrightarrow{a.s.} \log(e^{-\mu}) = -\mu$

and $\boxed{-\log\left[\log(\hat{\beta}_n)/\hat{\lambda}_n\right] \xrightarrow{a.s.} \mu \text{ as } n \to \infty}$ ✓

1 d) Let $\theta = \lambda - \mu$ and $\hat{\theta}_n = \hat{\lambda}_n - \hat{\mu}_n$. Show that $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N(0, \sigma^2)$ as $n \to \infty$ and give the form of σ^2 in terms of λ and μ

Code: E

[Know from part b) that

$$\sqrt{n}\begin{pmatrix} \hat{\alpha}_n - \alpha \\ \hat{\beta}_n - \beta \end{pmatrix} \xrightarrow{d} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \overbrace{\begin{bmatrix} \alpha(1-\alpha) & \alpha(1-\beta) \\ \alpha(1-\beta) & \beta(1+\beta) \end{bmatrix}}^{\Sigma}\right)$$

Need to apply delta method to get the requested result.

(1st order) Delta Method says that, as long as the first derivative $g'(\theta) \neq 0$ and variance is finite then,

$$\sqrt{n}\left(g(\hat{\alpha}_n, \hat{\beta}_n) - g(\alpha, \beta)\right) \xrightarrow{d} N\left(0, \nabla g(\alpha,\beta)' \, \Sigma \, \nabla g(\alpha,\beta)\right)$$

two minuses made the +

Let $g(a,b) = -\log(a) + \log[-\log(b)/(-\log(a))] = -\log(a) + \log\left[\frac{\log(b)}{\log(a)}\right]$

$$\Rightarrow \nabla g(a,b) = \begin{bmatrix} \frac{\partial}{\partial a}\left\{-\log(a) + \log\left[\frac{\log(b)}{\log(a)}\right]\right\} \\ \frac{\partial}{\partial b}\left\{-\log(a) + \log\underbrace{\left[\frac{\log(b)}{\log(a)}\right]}_{u}\right\} \end{bmatrix} = \begin{bmatrix} \left(-1/a + \frac{\frac{-\log(b)}{a[\log(a)]^2}}{[\log(b)/\log(a)]}\right) \\ \frac{\frac{1}{b\log(a)}}{\log(b)/\log(a)} \end{bmatrix}$$

Chain rule says $\frac{1}{u} \cdot \frac{\partial}{\partial b}\left(\frac{\log(b)}{\log(a)}\right)$

$$= \begin{bmatrix} -1/a & \frac{-\cancel{\log(b)}}{a[\log(a)]^{\cancel{2}}} \cdot \frac{\cancel{\log(a)}}{\cancel{\log(b)}} \\ \frac{1}{b\cancel{\log(a)}} \cdot \frac{\cancel{\log(a)}}{\log(b)} & \end{bmatrix} = \begin{bmatrix} -\frac{1}{a} & -\frac{1}{a\log(a)} \\ \frac{1}{b\log(b)} & \end{bmatrix}$$

assuming $a \neq 0$, $\log(a) \neq 0$ & $\log(b) \neq 0$

Sub back $a = \alpha$ and $b = \beta$

$$= \begin{bmatrix} -\frac{1}{\alpha} & -\frac{1}{\alpha\log(\alpha)} \\ \frac{1}{\beta\log(\beta)} & \end{bmatrix}$$

cont'd →

1 d) cont'd

Code : E

Then, to find final variance have

$$\underset{1\times2}{\nabla g(\alpha,\beta)'} \cdot \underset{2\times2}{\Sigma} \cdot \underset{2\times1}{\nabla g(\alpha,\beta)}$$

$$= \underset{1\times2}{\begin{bmatrix} \left(-\frac{1}{\alpha} - \frac{1}{\alpha\log(\alpha)}\right) & \frac{1}{\beta\log(\beta)} \end{bmatrix}} \underset{2\times2}{\begin{bmatrix} \alpha(1-\alpha) & \alpha(1-\beta) \\ \alpha(1-\beta) & \beta(1-\beta) \end{bmatrix}} \begin{bmatrix} \left(-\frac{1}{\alpha} - \frac{1}{\alpha\log(\alpha)}\right) \\ \frac{1}{\beta\log(\beta)} \end{bmatrix}$$

$$= \left\{\left[\left(-\frac{1}{\alpha} - \frac{1}{\alpha\log(\alpha)}\right)(\alpha(1-\alpha)) + \frac{1}{\beta\log(\beta)} \cdot \alpha(1-\beta)\right], \left[\left(-\frac{1}{\alpha} - \frac{1}{\alpha\log(\alpha)}\right)(\alpha(1-\beta)) + \frac{1}{\beta\log(\beta)}\beta(1-\beta)\right]\right\}_{1\times2}$$

$$\cdot \begin{bmatrix} \left(-\frac{1}{\alpha} - \frac{1}{\alpha\log(\alpha)}\right) \\ \frac{1}{\beta\log(\beta)} \end{bmatrix}_{2\times1}$$

$$\left.\begin{aligned} = &\left[\left(-\frac{1}{\alpha} - \frac{1}{\alpha\log(\alpha)}\right)(\alpha(1-\alpha)) + \frac{1}{\beta\log(\beta)}\alpha(1-\beta)\right]\left(-\frac{1}{\alpha} - \frac{1}{\alpha\log(\alpha)}\right) \\ + &\left[\left(-\frac{1}{\alpha} - \frac{1}{\alpha\log(\alpha)}\right)(\alpha(1-\beta)) + \frac{1}{\beta\log(\beta)}\beta(1-\beta)\right]\frac{1}{\beta\log(\beta)} \end{aligned}\right\} = \sigma^2$$

Thus, $\sqrt{n}(\underset{\hat\lambda_n - \hat\mu_n}{\hat\theta_n} - \underset{\lambda-\mu}{\theta}) \xrightarrow{d} N(0,\sigma^2)$ by delta method w/ variance, σ^2, as defined here.

Code: E

1. e) Construct an asymptotically valid hypothesis test of $H_0: \theta = 0$ vs. $H_1: \theta \neq 0$.

Know from part d) that $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N(0, \underbrace{\sigma^2}_{= I_1^{-1}(\theta)})$

We can construct a Wald test using the fact that $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N(0, I_1^{-1}(\theta))$

Let $W = (R\xi - d)'(R' I_1(\theta)^{-1} R)^{-1}(R\xi - d)$ (form from 762 that I remember)

Here since $H_0: \theta = 0 \Rightarrow \underbrace{R\xi}_{1} = \theta$ and $d = 0$.

Also, from d) since $I_1^{-1}(\theta) = \sigma^2$, then we can use these facts to construct our Wald test.

Sub to get $W = (\theta - 0)'(1 \cdot \sigma^2 \cdot 1)^{-1}(\theta - 0) = \theta^2/\sigma^2 \overset{H_0}{\sim} \chi^2_1$ for σ^2 as defined in e).

Thus, reject the null if $W > \underbrace{\chi^2_1(1-\alpha)}_{= 3.84 \text{ if } \alpha = 0.05}$ where $\chi^2_1(1-\alpha)$ represents the $1-\alpha$ quantile of a chi-squared distribution with 1 df.

$\Downarrow$

$H_0: \theta = 0$