2009 Theory II #3

3a) $$L_n(\theta) = \prod_{k=1}^{2}\prod_{i=1}^{n} \frac{\lambda_{ik}^{y_{ik}} e^{-\lambda_{ik}}}{y_{ik}!} (2\pi\sigma_k^2)^{-1/2} \exp\left\{-\frac{1}{2\sigma_k^2} x_{ik}^2\right\}$$

$$= \prod_{k=1}^{2}\prod_{i=1}^{n} \frac{(2\pi\sigma_k^2)^{-1/2}}{y_{ik}!} \exp\left\{ y_{ik}(\alpha_k+\beta x_{ik}) - e^{\alpha_k+\beta x_{ik}} - \frac{1}{2\sigma_k^2} x_{ik}^2 \right\}$$

$$\ell_n(\theta) = \sum_{k=1}^{2}\sum_{i=1}^{n} \left\{ -\frac{1}{2}\log(2\pi) - \log(y_{ik}!) - \frac{1}{2}\log(\sigma_k^2) + y_{ik}(\alpha_k+\beta x_{ik}) - e^{\alpha_k+\beta x_{ik}} - \frac{1}{2\sigma_k^2} x_{ik}^2 \right\}$$

Need to find $I_n(\theta)$

$$\frac{\partial}{\partial\alpha_1}\ell_n(\theta) = \frac{\partial}{\partial\alpha_1}\sum_{i=1}^{n}\left\{ y_{i1}(\alpha_1+\beta x_{i1}) - e^{\alpha_1+\beta x_{i1}} \right\}$$

$$= \sum_{i=1}^{n}\left\{ y_{i1} - e^{\alpha_1+\beta x_{i1}} \right\}$$

$$\frac{\partial}{\partial\alpha_2}\ell_n(\theta) = \sum_{i=1}^{n}\left\{ y_{i2} - e^{\alpha_2+\beta x_{i2}} \right\}$$

$$\frac{\partial}{\partial\beta}\ell_n(\theta) = \frac{\partial}{\partial\beta}\sum_{k=1}^{2}\sum_{i=1}^{n}\left\{ y_{ik}(\alpha_k+\beta x_{ik}) - e^{\alpha_k+\beta x_{ik}} \right\}$$

$$= \sum_{k=1}^{2}\sum_{i=1}^{n}\left\{ x_{ik}y_{ik} - x_{ik}e^{\alpha_k+\beta x_{ik}} \right\}$$

$$\frac{\partial}{\partial\sigma_1^2}\ell_n(\theta) = \frac{\partial}{\partial\sigma_1^2}\sum_{i=1}^{n}\left\{ -\frac{1}{2}\log(\sigma_1^2) - \frac{1}{2\sigma_1^2}x_{ik}^2 \right\}$$

$$= \sum_{i=1}^{n}\left\{ -\frac{1}{2\sigma_1^2} + \frac{1}{2(\sigma_1^2)^2}x_{ik}^2 \right\} = -\frac{n}{2\sigma_1^2} + \frac{1}{2(\sigma_1^2)^2}\sum_{i=1}^{n} x_{ik}^2$$

$$\frac{\partial}{\partial \sigma_2^2} \ell_n(\theta) = -\frac{n}{2\sigma_2^2} + \frac{1}{2(\sigma_2^2)^2} \sum_{i=1}^{n} X_{ik}^2$$

Now the second-order partial derivatives:

$$\frac{\partial^2}{\partial \alpha_1^2} \ell_n(\theta) = -\sum_{i=1}^{n} e^{\alpha_1 + \beta X_{i1}}$$

$$\frac{\partial^2}{\partial \alpha_2^2} \ell_n(\theta) = -\sum_{i=1}^{n} e^{\alpha_2 + \beta X_{i2}}$$

$$\frac{\partial^2}{\partial \beta^2} \ell_n(\theta) = -\sum_{k=1}^{2} \sum_{i=1}^{n} X_{ik}^2 e^{\alpha_k + \beta X_{ik}}$$

$$\frac{\partial^2}{\partial (\sigma_1^2)^2} \ell_n(\theta) = \frac{n}{2(\sigma_1^2)^2} - \frac{1}{(\sigma_1^2)^3} \sum_{i=1}^{n} X_{ik}^2$$

$$\frac{\partial^2}{\partial (\sigma_2^2)^2} \ell_n(\theta) = \frac{n}{2(\sigma_2^2)^2} - \frac{1}{(\sigma_2^2)^3} \sum_{i=1}^{n} X_{ik}^2$$

$$\frac{\partial^2}{\partial \alpha_1 \partial \beta} \ell_n(\theta) = -\sum_{i=1}^{n} X_{i1} e^{\alpha_1 + \beta X_{i1}}$$

0 for all others

$$\frac{\partial^2}{\partial \alpha_2 \partial \beta} \ell_n(\theta) = -\sum_{i=1}^{n} X_{i2} e^{\alpha_2 + \beta X_{i2}}$$

Next,

$E[X_{ik}^2] = \sigma_k^2$

$$E\left[X_{ik}^2 e^{\alpha_k + \beta X_{ik}}\right] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_k^2}} x^2 e^{\alpha_k + \beta x} \exp\left\{-\frac{1}{2\sigma_k^2} x^2\right\} dx$$

$$= e^{\alpha_k} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_k^2}} x^2 \exp\left\{-\frac{1}{2\sigma_k^2} x^2 + \beta x\right\} dx = e^{\alpha_k} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_k^2}} x^2 \exp\left\{-\frac{1}{2\sigma_k^2}\left[x^2 - 2\sigma^2\beta x + \sigma^4\beta^2 - \sigma^4\beta^2\right]\right\} dx$$

$$= \exp\left\{\alpha_k + \tfrac{1}{2}\sigma_k^2\beta^2\right\} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_k^2}}\, x^2 \exp\left\{-\frac{1}{2\sigma_k^2}\left(x - \sigma_k^2\beta\right)^2\right\} dx$$

$\leftarrow N(\beta\sigma_k^2, \sigma_k^2)$ kernal

$$= \exp\left\{\alpha_k + \tfrac{1}{2}\sigma_k^2\beta^2\right\}\left[\sigma_k^2 + (\sigma_k^2\beta)^2\right]$$

Then

$$I_n(\alpha_1,\alpha_2,\beta,\sigma_1^2,\sigma_2^2) = \begin{array}{c|ccccc} & \alpha_1 & \alpha_2 & \beta & \sigma_1^2 & \sigma_2^2 \\ \hline \alpha_1 & ne^{\alpha_1+\frac{1}{2}\sigma_1^2\beta^2} & & \beta\sigma_1^2 e^{\alpha_1+\frac{1}{2}\beta^2\sigma_1^2} & & \\ \alpha_2 & & ne^{\alpha_2+\frac{1}{2}\sigma_2^2\beta^2} & \beta\sigma_2^2 e^{\alpha_2+\frac{1}{2}\beta\sigma_2^2} & & \\ \beta & \beta\sigma_1^2 e^{\alpha_1+\frac{1}{2}\beta^2\sigma_1^2} & \beta\sigma_2^2 e^{\alpha_2+\frac{1}{2}\beta\sigma_2^2} & n\sum_{k=1}^{2}\left[\sigma_k^2+(\sigma_k^2\beta)^2\right]\exp\left\{\alpha_k+\frac{1}{2}\sigma_k^2\beta^2\right\} & & \\ \sigma_1^2 & & & & \frac{n}{2(\sigma_1^2)^2} & \\ \sigma_2^2 & & & & & \frac{n}{2(\sigma_2^2)^2} \end{array}$$

$$E\left[e^{\alpha_k+\beta X_{ik}}\right] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_k^2}}\, e^{\alpha_k+\beta x}\exp\left\{-\frac{1}{2\sigma^2}x^2\right\}dx = e^{\alpha_k}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma_k^2}}\exp\left\{-\frac{1}{2\sigma^2}x^2+\beta x\right\}dx$$

$$= e^{\alpha_k}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma_k^2}}\exp\left\{-\frac{1}{2\sigma^2}\left[x^2 - 2\sigma^2\beta x + \sigma_k^4\beta^2 - \sigma_k^4\beta^2\right]\right\}dx$$

$$= e^{\alpha_k+\sigma_k^2\beta^2/2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma_k^2}}\exp\left\{-\frac{1}{2\sigma_k^2}\left(x-\sigma_k^2\beta\right)^2\right\}dx = e^{\alpha_k+\frac{1}{2}\sigma_k^2\beta^2}$$

$$E\left[X_{ik}e^{\alpha_k+\beta X_{ik}}\right] = \ldots = \beta\sigma_k^2 e^{\alpha_k+\frac{1}{2}\beta^2\sigma_k^2}$$

Use next page!

$$
\begin{array}{c|ccccc}
 & \sigma_1 & \sigma_2 & \alpha_1 & \alpha_2 & \beta \\
\hline
\sigma_1 & \frac{n}{2(\sigma_1^2)^2} & 0 & 0 & 0 & 0 \\
\sigma_2 & 0 & \frac{n}{2(\sigma_2^2)^2} & 0 & 0 & 0 \\
\alpha_1 & 0 & 0 & ne^{\alpha_1+\frac{1}{2}\sigma_1^2\beta} & 0 & n\beta\sigma_1^2 e^{\alpha_1+\frac{1}{2}\beta^2\sigma_1^2} \\
\alpha_2 & 0 & 0 & 0 & ne^{\alpha_2+\frac{1}{2}\sigma_2^2\beta} & n\beta\sigma_2^2 e^{\alpha_2+\frac{1}{2}\beta^2\sigma_2^2} \\
\beta & 0 & 0 & n\beta\sigma_1^2 e^{\alpha_1+\frac{1}{2}\beta^2\sigma_1^2} & n\beta\sigma_2^2 e^{\alpha_2+\frac{1}{2}\beta^2\sigma_2^2} & n\sum_{k=1}^{2}\left[\sigma_k^2+(\sigma_k^2\beta)^2\right]\exp\left\{\alpha_k+\frac{1}{2}\sigma_k^2\beta^2\right\}
\end{array}
$$

$\boxed{\text{* Take } \lim_{n\to\infty}\frac{1}{n} \text{ first next time}}$

$$\left(\left[I_n(\theta)\right]^{-1}\right)_{5,5} = \left\{C - B'A^{-1}B\right\}^{-1}$$

$$= \left\{ n\sum_{k=1}^{2}\left[\sigma_k^2+(\sigma_k^2\beta)^2\right]\exp\left\{\alpha_k+\frac{1}{2}\sigma_k^2\beta^2\right\} - \frac{\left(n\beta\sigma_1^2 e^{\alpha_1+\frac{1}{2}\beta\sigma_1^2}\right)^2}{ne^{\alpha_1+\frac{1}{2}\sigma_1^2\beta}} - \frac{\left(n\beta\sigma_2^2 e^{\alpha_2+\frac{1}{2}\beta\sigma_2^2}\right)^2}{ne^{\alpha_2+\frac{1}{2}\sigma_2^2\beta}} \right\}^{-1}$$

$$= \left\{ n\sum_{k=1}^{2}\sigma_k^2 e^{\alpha_k+\frac{1}{2}\sigma_k^2\beta^2} \right\}^{-1}$$

Then

$$\sqrt{2m}\,(\hat\beta-\beta) \xrightarrow{L} N(0,\tau) \text{ where}$$

$$\tau = \left[\lim_{n\to\infty}\frac{1}{2n}\left(\left[I_n(\theta)\right]\right)_{5,5}\right]^{-1} = \left\{\frac{1}{2}\sum_{k=1}^{2}\sigma_k^2 e^{\alpha_k+\frac{1}{2}\sigma_k^2\beta^2}\right\}^{-1}$$

$k=1,2$

2b.i) Same process as before, now using only the data from one center.

See Joe's answer. For an estimator, use $\left[\lim_{n\to\infty} \frac{1}{n} I_n(\theta)\right]^{-1} \Big|_{\theta=\hat{\theta}}$

When is this consistent?

2b.ii) Under the MLE theory regularity conditions and since the data from the two centers is independent we obtain

$$\sqrt{n}\left(\begin{bmatrix}\hat{\beta}_1\\ \hat{\beta}_2\end{bmatrix} - \begin{bmatrix}\beta\\ \beta\end{bmatrix}\right) \xrightarrow{L} N\left(0, \begin{bmatrix} V_1 & 0 \\ 0 & V_2\end{bmatrix}\right)$$

where V_1, V_2 as found in (2b.i)

So that by the delta method for g as specified we have

$$\sqrt{n}\left(g(\hat{\beta}_1, \hat{\beta}_2) - \beta\right) \xrightarrow{L} N\left(0, \left(\frac{\partial g_1(\beta,\beta)}{\partial \beta}\right)^2 V_1 + \left(\frac{\partial g_2(\beta,\beta)}{\partial \beta}\right)^2 V_2\right)$$

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derivative is continuous

2b.iii) g^*_{opt} is ~~continuously~~ differentiable, scalar, and $g^*_{opt}(\beta,\beta) = \beta$. How to show it is optimum?

minimize $a^2 V_1 + b^2 V$ subject to

$g(\beta,\beta) = \beta$

$\dot{g}_1(\beta) = a$

$\dot{g}_2(\beta) = b$

3b. iv) $\frac{\partial}{\partial(x,y)} g^*_{opt}(x,y) = \left(\frac{V_2}{V_1+V_2}, \frac{V_1}{V_1+V_2} \right)$

Then $\dot{g}^*_{opt}(\beta,\beta) \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} [\dot{g}^*_{opt}(\beta,\beta)]' = \frac{\overbrace{V_1^2 V_2 + V_2^2 V_1}^{V_1 V_2 (V_1 + V_2)}}{(V_1+V_2)^2} = \frac{V_1 V_2}{V_1+V_2}$

so

$\sqrt{2n}\,(g^*_{opt}(\hat\beta_1,\hat\beta_2) - \beta) \xrightarrow{L} N\left(0, \frac{2V_1V_2}{V_1+V_2}\right)$

~~However~~ ~~so~~ Therefore,

$$ARE(\hat{g}^*_{opt}(\hat\beta_1,\hat\beta_2), \hat\beta) = \left. \frac{\frac{2V_1V_2}{V_1+V_2}}{\frac{2}{V_1^{-1}+V_2^{-1}}} \right|_{\hat V_1, \hat V_2} = \left. \frac{V_1 V_2}{V_1+V_2} \Big/ \frac{1}{\frac{1}{V_1} + \frac{1}{V_2}} \right|_{\hat V_1, \hat V_2}$$

$$= \left. \frac{V_1 V}{V_1+V_2} \left(\frac{1}{V_1} + \frac{1}{V_2} \right) \right|_{\hat V_1, \hat V_2} = \left. \frac{V_1 V_2}{V_1+V_2} \cdot \frac{V_1+V_2}{V_1 V_2} \right|_{\hat V_1, \hat V_2} = 1$$

3c) Same approach as in (1), see Joe's answer