(a) $$f(x,y|\alpha,\beta) = c(\alpha,\beta)\exp(-\alpha x-\beta y)\sum_{j=0}^{\infty}\frac{x^j y^j}{(j!)^2} \quad (1).$$

Any multiparameter exponential family can be written as

$$f(z|\theta) = h(z)\,g(\theta)\exp\{\eta^T(\theta)\,T(z)\}$$

where θ is vector valued. For (1) above, we have

$z = (x, y)$

$\theta = (\alpha, \beta)$

$g(\theta) = c(\alpha,\beta)$

$$h(z) = \sum_{j=0}^{\infty}\frac{x^j y^j}{(j!)^2}$$

$$\exp\{\eta^T(\theta)\,T(z)\} = \exp\{-\alpha x-\beta y\}$$

so that $\eta^T(\theta) = (-\alpha, -\beta)$, $T(z) = \begin{pmatrix} x \\ y \end{pmatrix}$

The rank of this multivariate exponential family is 2

$$\iint f(x,y|\alpha,\beta)\,dx\,dy = 1$$

which implies that

$$c(\alpha,\beta) \int_0^\infty \int_0^\infty \exp(-\alpha x - \beta y) \sum_{j=0}^\infty \frac{x^j y^j}{(j!)^2}\, dx\, dy = 1$$

$$\Rightarrow c(\alpha,\beta) \sum_{j=0}^\infty \int_0^\infty \int_0^\infty \exp(-\alpha x - \beta y) \frac{x^j y^j}{(j!)^2}\, dx\, dy = 1$$

$$\Rightarrow c(\alpha,\beta) \sum_{j=0}^\infty \left(\int_0^\infty \frac{x^{j+1-1} e^{-\alpha x}}{\Gamma(j+1)}\, dx \right) \left(\int_0^\infty \frac{y^{j+1-1} e^{-\beta y}}{\Gamma(j+1)}\, dy \right) = 1$$

$$\Rightarrow c(\alpha,\beta) \sum_{j=0}^\infty \alpha^{-(j+1)} \beta^{-(j+1)} = 1$$

gamma integral

$$\Rightarrow c(\alpha,\beta)\, \alpha^{-1} \beta^{-1} \sum_{j=0}^\infty \left(\frac{1}{\alpha\beta}\right)^j = 1$$

$$c(\alpha,\beta)\, \alpha^{-1} \beta^{-1} \frac{1}{1-\frac{1}{\alpha\beta}} = 1, \quad \text{for } \alpha\beta > 1$$

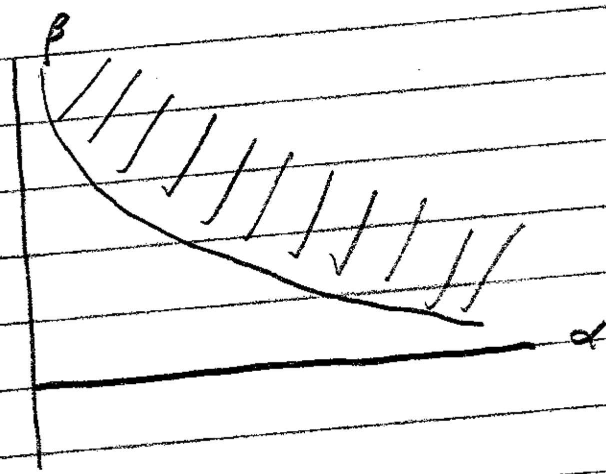
$$\Rightarrow c(\alpha,\beta) \left(\frac{1}{\alpha\beta - 1}\right) = 1$$

$$\Rightarrow c(\alpha,\beta) = \alpha\beta - 1, \quad \text{where } \alpha\beta > 1.$$

The

The parameter space is

$$\{(\alpha,\beta): \ \alpha\beta > 1,\ \alpha > 0,\ \beta > 0\}$$



(b) $f_X(x) = \int_0 f(x,y)\,dy$

$$= c(\alpha,\beta)\left(\sum_{j=0}^{\infty}\frac{x^j}{j!}\right) e^{-\alpha x}\int_0^{\infty} e^{-\beta y}\frac{y^{j+1-1}}{\Gamma(j+1)}\,dy$$

$$c(\alpha,\beta)\, e^{-\alpha x}\sum_{j=0}^{\infty}\frac{x^j}{j!}\left(\frac{1}{\beta^{j+1}}\right)$$

$$= c(\alpha,\beta)\, e^{-\alpha x}\beta^{-1}\sum_{j=0}^{\infty}\frac{(x/\beta)^j}{j!}$$

$$= c(\alpha,\beta)\, e^{-\alpha x}\beta^{-1} e^{x/\beta}$$

$$= (\alpha\beta-1)\beta^{-1} e^{-x(\alpha-\frac{1}{\beta})} = \left(\frac{\alpha\beta-1}{\beta}\right) e^{-x\left(\frac{\alpha\beta-1}{\beta}\right)}$$

Thus

$$f(x) = \begin{cases} \left(\frac{\alpha\beta - 1}{\beta}\right) e^{-x\left(\frac{\alpha\beta-1}{\beta}\right)}, & x > 0,\ \alpha > 0,\ \beta > 0, \\ & \alpha\beta > 1 \\ 0 & \text{otherwise} \end{cases}$$

$$X \sim \text{exponential}\left(\frac{\alpha\beta-1}{\beta}\right), \quad \alpha > 0,\ \beta > 0,\ \alpha\beta > 1$$

$$E(X) = \frac{\beta}{\alpha\beta - 1}, \quad \text{since if } X \sim \exp(\lambda),\ E(X) = \frac{1}{\lambda}$$

(c) Let $\Psi_{X,Y}(s,t) = E\left(e^{sX+tY}\right)$ = MGF of (X,Y).

$$= \frac{\partial^{j+k} \Psi_{X,Y}(s,t)}{\partial s^j \, \partial t^k}\Bigg|_{(s,t)=(0,0)} = E(X^j Y^k)$$

$$\Psi_{X,Y}(s,t) = C(\alpha,\beta) \sum_{j=0}^{\infty} \int_0^{\infty}\int_0^{\infty} \frac{e^{-x(\alpha-s)} x^{j+1-1}}{\Gamma(j+1)} \, dx \; \frac{e^{-y(\beta-t)}\, y^{j+1-}}{\Gamma(j+1)} \, d

$$= C(\alpha,\beta) \sum_{j=0}^{\infty} \left(\frac{1}{\alpha-s}\right)^{j+1} \left(\frac{1}{\beta-t}\right)^{j+1}$$

$$= C(\alpha,\beta) \left[(\alpha-s)(\beta-t)\right]^{-1} \sum_{j=0}^{\infty} \left(\frac{1}{(\alpha-s)(\beta-t)}\right)^{j}$$

$$= C(\alpha,\beta) \left[(\alpha-s)(\beta-t)\right]^{-1} \left(\frac{1}{1-(\alpha-s)(\beta-t)}\right)$$

$$= C(\alpha,\beta) \left(\frac{1}{(\alpha-s)(\beta-t)-1}\right)$$

$$= C(\alpha,\beta) \left[(\alpha-s)(\beta-t)-1\right]^{-1} = S^{-1}\varphi(s,t)$$

where $S^{-1} = C(\alpha,\beta)$,

$$\varphi(s,t) = \left[(\alpha-s)(\beta-t)-1\right]^{-1} = S(\alpha-s,\beta-t)$$

Now note that since when can interchange α with $-s$ and β with $-t$, we have

by chain rule

so that $$\left.\frac{\partial^{j+k}\varphi(s,t)}{\partial s^j \partial t^k}\right|_{(s,t)=(0,0)} = (-1)^{j+k} \left.\frac{\partial S(\alpha-s,\beta-t)}{\partial \alpha^j \partial \beta^k}\right|_{(s,t)=}$$

$$= (-1)^{j+k} \frac{\partial^{j+k}}{\partial \alpha^j \partial \beta^k} S(\alpha, \beta)$$

$$= (-1)^{j+k} \frac{\partial^{j+k}}{\partial \alpha^j \partial \beta^k} (\alpha\beta - 1)^{-1} = (-1)^{j+k} \frac{\partial^{j+k}}{\partial \alpha^j \partial \beta^k} (S)$$

Thus

$$E(X^j Y^k) = S^{-1} (-1)^{j+k} \frac{\partial^{j+k}}{\partial \alpha^j \partial \beta^k} (S)$$

(d)

$$f(y|x) = \frac{f(x,y)}{f(x)}$$

$$= \frac{c(\alpha,\beta) \exp(-\alpha x - \beta y) \sum_{j=0}^{\infty} \frac{x^j y^j}{(j!)^2}}{\left(\frac{c(\alpha,\beta)}{\beta}\right) e^{-x\left(\frac{c(\alpha,\beta)}{\beta}\right)}}$$

$$= \left(\sum_{j=0}^{\infty} \frac{x^j y^j}{(j!)^2} \right) \left(\beta e^{-\beta y} e^{-\alpha x} e^{x\left(\frac{c(\alpha,\beta)}{\beta}\right)} \right)$$

$$= \sum_{j=0}^{\infty} \frac{x^j y^j}{(j!)^2} \left(\beta e^{-\beta y} e^{-x\left(\alpha - \frac{(\alpha\beta - 1)}{\beta}\right)} \right)$$

$$= \left(\sum_{j=0}^{\infty} \frac{x^j y^j}{(j!)^2} \right) \beta e^{-\beta y} e^{-x/\beta}$$

Thus

$$f(y|x) = \left(\beta e^{-\beta y} e^{-x/\beta} \sum_{j=0}^{\infty} \frac{x^j y^j}{(j!)^2} \right), \quad y > 0, \ \beta > 0$$

Which is free of α. Say $Y|X \sim D(\beta)$

Now, By the CLT

$$\bar{Y} | \bar{X} = \bar{x} \xrightarrow{d} N(\mu, \sigma^2/n)$$

$$\mu = E(Y|X),$$

$$\sigma^2 = var(Y|X)$$

Given $(X_1, .., X_n)$, $(Y_1, .., Y_n)$ are independent, so we can use CLT.

$$E(Y|X) = \sum_{j=0}^{\infty} \frac{x^j}{j!} e^{-x/\beta} \beta \int_0^{\infty} e^{-\beta y} \frac{y^{j+2-1}}{\Gamma(j+1)} dy$$

$$= \sum_{j=0}^{\infty} \frac{x^j}{j!} e^{-x/\beta} \beta \frac{1}{\Gamma(j+1)} \left(\frac{\Gamma(j+2)}{\beta^{j+2}} \right)$$

$$= e^{-x/\beta} \sum_{j=0}^{\infty} \frac{x^j}{j!} \frac{(j+1)\Gamma(j+1)}{\beta^{j+1}\Gamma(j+1)}$$

$$= e^{-x/\beta} \sum_{j=0}^{\infty} \frac{x^j (j+1)}{j!\, \beta^{j+1}}$$

$$= \beta^{-1} e^{-x/\beta} \left[\sum_{j=0}^{\infty} \frac{j x^j}{j!\, \beta^j} + \sum_{j=0}^{\infty} \frac{x^j}{j!\, \beta^j} \right]$$

$$= \beta^{-1} e^{-x/\beta} \left[\sum_{j=0}^{\infty} \frac{j (x/\beta)^j}{j!} + \sum_{j=0}^{\infty} \frac{(x/\beta)^j}{j!} \right]$$

$$= \beta^{-1} e^{-x/\beta} \left[\frac{x}{\beta} e^{x/\beta} + e^{x/\beta} \right]$$

$$= \beta^{-1} \left[\frac{x}{\beta} + 1 \right] = \boxed{\frac{1}{\beta} + \frac{x}{\beta^2}}$$

Note that If $X \sim Pois(\lambda)$

$$\lambda = E(X) = \sum_{k=0}^{\infty} \frac{k \lambda^k e^{-\lambda}}{k!} \Rightarrow \lambda e^{\lambda} = \sum_{k=0}^{\infty} \frac{k \lambda^k}{k!}$$

Now [Replace x by $n\bar{X}$ in expression for μ]

$$E(Y^2|X) = \sum_{j=0}^{\infty} \frac{x^j}{j!} e^{-x/\beta} \beta \int_0^{\infty} e^{-\beta y} \frac{y^{j+3-1}}{j!} dy$$

$$= \sum_{j=0}^{\infty} \frac{x^j}{j!} e^{-x/\beta} \beta \frac{1}{\Gamma(j+1)} \left(\frac{\Gamma(j+3)}{\beta^{j+3}} \right)$$

$$= e^{-x/\beta} \beta^{-2} \sum_{j=0}^{\infty} \frac{x^j}{j!} \frac{(j+2)(j+1)}{\beta^j}$$

$$= e^{-x/\beta} \beta^{-2} \sum_{j=0}^{\infty} \frac{(x/\beta)^j}{j!} \left[j^2 + 3j + 1 \right]$$

$$= e^{-x/\beta} \beta^{-2} \left[\sum_{j=0}^{\infty} j^2 \frac{(x/\beta)^j}{j!} + 3 \sum_{j=0}^{\infty} j \frac{(x/\beta)^j}{j!} + \sum_{j=0}^{\infty} \frac{(x/\beta)^j}{j!} \right]$$

$$= e^{-x/\beta} \beta^{-2} \left[\sum_{j=0}^{\infty} j^2 \frac{(x/\beta)^j}{j!} + 3 \left(\frac{x}{\beta} e^{x/\beta} \right) + e^{x/\beta} \right]$$

$$\sum_{j=0}^{\infty} j^2 \frac{(x/\beta)^j}{j!} = \sum_{j=1}^{\infty} j^2 \frac{(x/\beta)^j}{j!} = \sum_{j=1}^{\infty} j \cancel{j} \frac{(x/\beta)^j}{\cancel{j}(j-1)!}$$

$$= \sum_{j=1}^{\infty} j \frac{(x/\beta)^j}{(j-1)!} = \sum_{l=0}^{\infty} (l+1) \frac{(x/\beta)^{l+1}}{l!} \qquad l = j-1$$

$$= \frac{x}{\beta} \left[\sum_{l=0}^{\infty} \frac{(x/\beta)^l}{l!} + \sum_{l=0}^{\infty} l \frac{(x/\beta)^l}{l!} \right]$$

$$= \frac{x}{\beta} \left[e^{x/\beta} + \frac{x}{\beta} e^{x/\beta} \right]$$

$$= \frac{x}{\beta} \left(1 + \frac{x}{\beta} \right) e^{x/\beta}$$

[Replace x by $n\bar{x}$ in the formula above]

Thus

$$\boxed{E(Y^2|X) = e^{-X/\beta}\beta^{-2}\left[\frac{X}{\beta}\left(1+\frac{X}{\beta}\right)e^{X/\beta} + 3\frac{X}{\beta}e^{X/\beta} + e^{X/\beta}\right]}$$

Thus

$$\sigma^2 = E(Y^2|X) - \mu^2.$$

$$\frac{\sqrt{n}(\bar{Y}-\mu)}{\sigma} \xrightarrow{d} N(0,1)$$

Replace X by $n\bar{X}$ in the formula above

(e) $H_0: \beta = 2$

$H_1: \beta > 2$

Here, we can use Theorem 2.7 on page 330 of the Bios 761 notes.

We can write the multiparameter exponential family based on a sample of size n as

$$f(\underset{\sim}{x}, \underset{\sim}{y} | \alpha, \beta) = [c(\alpha,\beta)]^n \exp\left(-\alpha \Sigma X_i - \beta \Sigma Y_i\right) \times \prod_{i=1}^{n}\left\{\sum_{j=0}^{\infty} \frac{x_i^j y_i^j}{(j!)^2}\right\}$$

so that $\theta = \beta$, $U = \Sigma Y_i$, $T_1 = \Sigma X_i$, $\xi_1 = \alpha$, $k = 1$

from notation of Theorem 2.7

Thus, the rejection region of the unique test is

$$\phi(\underset{\sim}{x}, \underset{\sim}{y}) = \begin{cases} 1 & \text{if } \quad \Sigma Y_i < c(t) \\ 0 & \quad \Sigma Y_i > c(t) \end{cases}$$

$T = \Sigma X_i$

Thus, to make the test size α^*, we need to compute

$$\alpha^* = E_{\beta=2}\left[\phi(\underset{\sim}{x}, \underset{\sim}{y}) \mid T = t\right]$$

$$= P\left(\sum_{i=1}^{n} Y_i < c(t) \mid T = t, \beta = 2\right) = \int_0^{c(t)} f(R \mid t, \beta=2)\, dR$$

$R = \sum_{i=1}^{n} Y_i$

Note that given $X_1, \ldots, X_n$, $Y_1 \ldots Y_n$ are independent so that

The distribution of $R \mid T = t$ does not have an obvious closed form, $R = \sum_{i=1}^{n} Y_i$

Thus

$$\alpha^* = \int_0^{c(t)} f(R \mid t, \beta=2)\, dR$$

and we solve this equation for $c(t)$, given a value of α^*.

(f) A 95% CI for β can be found by inverting the two-sided test of $H_0: \beta = 2$ vs. $H_1: \beta \neq 2$.

Thus a 95% CI for β is the set of all β's in the interval

$$1 - \phi^* = \begin{cases} 1 & \text{if } c_1(t) < \Sigma Y_i < c_2(t) \\ 0 & \text{otherwise} \end{cases}$$

where

$$.05 = E_\beta\left[1 - \phi^* \mid T = t\right]$$

and $$E_\beta\left[\Sigma Y_i (1 - \phi^*) \mid T = t\right] = .05\, E_\beta\left[\Sigma Y_i \mid T = t\right]$$

(g) $H_0: \beta = 2$

Method 1: Score test is based on computing

$$R_n = i(\theta_0)\, I_n(\theta_0)^{-1}\, i(\theta_0)$$

We can use the conditional distribution of

$Y|X$ to computation (method 1)

Based on a sample of size n

$$L = \prod_{i=1}^{n} f(y_i|x_i)$$

$$= \beta^n e^{-\beta \Sigma Y_i} e^{-\Sigma X_i/\beta} \prod_{i=1}^{n} \left(\sum_{j=0}^{\infty} \frac{X_i^j Y_i^j}{(j!)^2} \right)$$

$\theta_0 = \beta_0 = 2$ in this case

$$\log L = n \log \beta - \beta \Sigma Y_i - \frac{\Sigma X_i}{\beta} + \sum_{i=1}^{n} \log (a_i)$$

$$\left.\frac{\partial \log L}{\partial \beta}\right|_{\beta=2} = \left.\frac{n}{\beta} - \Sigma Y_i + \frac{\Sigma X_i}{\beta^2}\right|_{\beta=2}$$

$$= \frac{n}{2} - \Sigma Y_i + \frac{\Sigma X_i}{4} = i(\theta_0)$$

$=$

$$I_n(\theta_0) = -E\left[\frac{\partial^2 \log L}{\partial \beta^2}\right]$$

$$\frac{\partial^2 \log L}{\partial \beta^2} = -\frac{n}{\beta^2} - \frac{2\sum X_i}{\beta^3}$$

$$-E\left(\frac{\partial^2 \log L}{\partial \beta^2}\right) = n/\beta^2 + \frac{2\sum X_i}{\beta^3}\Bigg|_{\beta=2} = n/4 + \frac{2\sum X_i}{8}$$

$$= I_n(\theta_0)$$

$$R_n = \left(n/2 - \sum Y_i + \frac{\sum X_i}{4}\right)^2 \left(n/4 + \frac{2\sum X_i}{8}\right)^{-1}$$

$$R_n \xrightarrow{d} \chi_1^2 \quad \text{as } n \to \infty.$$

<u>Method 2:</u> We can also compute the score test based on the joint distribution of $(X_1, Y_1), \dots, (X_n, Y_n)$

The log-likelihood based on n observations is

$$l_n = \log L = -\alpha \sum X_i - \beta \sum Y_i + n \log(\alpha\beta - 1)$$

$$\frac{\partial l_n}{\partial \alpha} = -\sum x_i + \frac{n\beta}{\alpha\beta-1}$$

$$\frac{\partial l_n}{\partial \beta} = -\sum y_i + \frac{n\alpha}{\alpha\beta-1}$$

$$\frac{\partial^2 l}{\partial \alpha^2} = \frac{-n\beta^2}{(\alpha\beta-1)^2}$$

$$\frac{\partial^2 l_n}{\partial \alpha \partial \beta} = \frac{-n}{(\alpha\beta-1)^2}$$

$$\frac{\partial^2 l_n}{\partial \beta^2} = \frac{-n\alpha^2}{(\alpha\beta-1)^2}$$

Therefore

$$I_n(\alpha,\beta) = \begin{pmatrix} \frac{n\beta^2}{(\alpha\beta-1)^2} & \frac{n}{(\alpha\beta-1)^2} \\ \frac{n}{(\alpha\beta-1)^2} & \frac{n\alpha^2}{(\alpha\beta-1)^2} \end{pmatrix}$$

$I_n(\alpha,\beta)$ = Fisher information under $H_0: \beta = 2$,

$$l_n = -\alpha \sum X_i - 2\sum Y_i + n \log(2\alpha - 1)$$

$$\frac{\partial l_n}{\partial \alpha} = 0 \iff -\sum X_i + \frac{2n}{2\alpha - 1} = 0$$

$$\Rightarrow \hat{\alpha} = \frac{n}{\sum X_i} + \frac{1}{2} = \boxed{\frac{1}{\bar{X}} + \frac{1}{2}}$$

Thus, the Score test is

$$R_n = \left(0, -\sum Y_i + \frac{n\hat{\alpha}}{2\hat{\alpha}}\right) \begin{pmatrix} \frac{4n}{(2\hat{\alpha}-1)^2} & \frac{n}{(2\hat{\alpha}-1)^2} \\ \frac{n}{(2\hat{\alpha}-1)^2} & \frac{n\hat{\alpha}^2}{(2\hat{\alpha}-1)^2} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ -\sum Y_i + \frac{n\hat{\alpha}}{2\hat{\alpha}-1} \end{pmatrix}$$

$$= \left(-\sum Y_i + \frac{n\hat{\alpha}}{2\hat{\alpha}-1}\right)^2 \frac{4n(2\hat{\alpha}-1)^2}{4n^2\hat{\alpha}^2 - n^2}$$

$$\hat{\alpha} = \frac{n}{\sum X_i} + \frac{1}{2}$$

$$R_n \xrightarrow{H_0} \chi^2_1 \text{ as } n \to \infty.$$