2015 D2

Q3) a) $\mu(X) =$

(i) $E_{Y_1|X_1}(Y_1|X_1) = E_{D_1|X_1}(E(Y_1|X_1,D_1))$

$= E_{D|X}(E(Y_1|X_1,D_1=1)I(D_1=1|X_1)$

$+ E(Y_1|X_1,D_1=0)I(D_1=0|X_1))$

$= E(Y_1|X_1,D_1=1)E_{D|X}(I(D_1=1|X_1))$

$+ E(Y_1|X_1,D_1=0)E_{D|X}(I(D_1=0|X))$

$= E(Y_1|X_1,D_1=1)P(D_1=1|X_1) + E(Y_1|X_1,D_1=0)P(D_1=0|X))$

(ii) $\{\mu(X)\} + (D - P(D=1|X))\gamma(X)$

by (i) $= \{\tilde{\mu}(X,0)P(D=0|X) + \tilde{\mu}(X,1)P(D=1|X)\} + (D - P(D=1|X))\gamma(X)$

$= \tilde{\mu}(X,0)[1-P(D=1|X)] + \tilde{\mu}(X,1)P(D=1|X) + (D - P(D=1|X))\gamma(X)$

$= P(D=1|X)[\tilde{\mu}(X,1) - \tilde{\mu}(X,0) - \gamma(X)] + \tilde{\mu}(X,0) + D\gamma(X)$

let $\gamma = \tilde{\mu}(X,1) - \tilde{\mu}(X,0)$

$= \cancel{P(D=1|X)[\gamma(X) - \gamma(X)]} \to 0 + \tilde{\mu}(X,0) + D(\tilde{\mu}(X,1) - \tilde{\mu}(X,0))$

$= \tilde{\mu}(X,0)(1-D) + \tilde{\mu}(X,1)D$

$= \tilde{\mu}(X,D)$

$E(YS|D,X)$

$= E(Y|D,X)E(S|D)$

$$f_{Y,S|D,X} = \frac{f_{Y,S,X,D}}{f_{X,D}}$$

$$= \frac{f_{Y,S,X|D}f_D}{f_{X,D}}$$

$$= \frac{f_{Y,X|D}f_{S|D}\cancel{f_D}}{f_{X|D}\cancel{f_D}}$$

$$= \frac{f_{Y,X|D}f_{S|D}}{f_{X|D}} = \frac{f_{Y|X,D}\cancel{f_{X|D}}f_{S|D}}{\cancel{f_{X|D}}}$$

$$= f_{Y|X,D}f_{S|D}$$

b) $p(X) = Pr(D=1 \mid X) \approx 0$

$\gamma(X) = (1, X')\gamma_0$ ← parameter vector

1) Show $\tilde{\mu}(X, D) = E(Y \mid X, D, S=1) = E(Y \mid X, D)$,

can be approx. be a model linear in $\xi_0 = (\beta_0, \gamma_0)'$ ← parameter vector

$+ E(Y_i \mid X_i) = \mu(X_i) = (1, X')\beta_0$

from a)

$$\tilde{\mu}(X, D) = \mu(X) + \left(D - Pr(D=1 \mid X)\right)\gamma(X)$$

$$\approx \mu(X) + D\gamma(X)$$

$$= (1, X')\beta_0 + D(1, X')\gamma_0$$

$$= \left((1, \underset{1\times k}{X'}), \underset{1\times k}{(D, DX')}\right)\begin{pmatrix} \beta_{0,1\times k} \\ \gamma_{0,k\times 1} \end{pmatrix}$$

$$Y_i^* \equiv \hat{\mu}(X_i, D)$$

$$Y^* = \begin{pmatrix} Y_1^* \\ \vdots \\ Y_n^* \end{pmatrix} = \begin{pmatrix} (1, X_1'), (D_1, D_1 X_1') \\ (1, X_2'), (D_2, D_2 X_2') \\ \vdots \\ (1, X_n'), (D_n, D_n X_2') \end{pmatrix}\begin{pmatrix} \beta_0 \\ \gamma_0 \end{pmatrix}$$

assume

$$= X^* \xi_0$$

2015 D2

3c) $\pi(x) = P(D=1|X, S=1)$ $\tilde{\pi} = P(D=1|S=1)$

$p(x) = P(D=1|X)$ $\tilde{p} = P(D=1)$

WTS $\log\left(\frac{p(x)}{1-p(x)}\right) = \log\left[\left(\frac{\pi(x)}{1-\pi(x)}\right)\left(\frac{\tilde{p}}{1-\tilde{p}}\right)\left(\frac{1-\tilde{\pi}}{\tilde{\pi}}\right)\right]$

$\Leftrightarrow \frac{p(x)}{1-p(x)} = \underset{①}{\left(\frac{\pi(x)}{1-\pi(x)}\right)}\underset{②}{\left(\frac{\tilde{p}}{1-\tilde{p}}\right)}\underset{③}{\left(\frac{1-\tilde{\pi}}{\tilde{\pi}}\right)}$

① $\frac{\pi(x)}{1-\pi(x)} = \frac{P(D=1|X,S=1)}{P(D=0|X,S=1)} = \frac{\left[\frac{P(D=1,X,S=1)}{\cancel{P(X,S=1)}}\right]}{\left[\frac{P(D=0,X,S=1)}{\cancel{P(X,S=1)}}\right]} = \frac{P(D=1,X,S=1)}{P(D=0,X,S=1)}$

② $\frac{\tilde{p}}{1-\tilde{p}} = \frac{P(D=1)}{P(D=0)}$

③ $\frac{1-\tilde{\pi}}{\tilde{\pi}} = \frac{P(D=0|S=1)}{P(D=1|S=1)} = \frac{\left[\frac{P(D=0,S=1)}{\cancel{P(S=1)}}\right]}{\left[\frac{P(D=1,S=1)}{\cancel{P(S=1)}}\right]} = \frac{P(D=0,S=1)}{P(D=1,S=1)}$

$\frac{\pi(x)}{1-\pi(x)}\frac{\tilde{p}}{1-\tilde{p}}\frac{1-\tilde{\pi}}{\tilde{\pi}} = \frac{P(D=1,X,S=1)}{P(D=0,X,S=1)}\frac{P(D=1)}{P(D=0)}\frac{P(D=0,S=1)}{P(D=1,S=1)}$

$= \frac{P(D=1,X,S=1)}{P(D=0,X,S=1)}\left(\frac{1}{P(S=1|D=1)}\right)\left(P(S=1|D=0)\right)$

$= \frac{P(D=1,X,S=1)}{P(D=0,X,S=1)}\frac{P(S=1|D=0,X)}{P(S=1|D=1,X)}$ b/c $S \perp\!\!\!\perp X | D$

$*\begin{cases} \text{for } i=0,1 \\ P(S=1|D=i,X) = \frac{P(D=i,S=1,X)}{P(D=i,X)} \\ \Rightarrow P(D=i,X) = \frac{P(D=i,S=1,X)}{P(S=1|D=i,X)} \end{cases}$

$\overset{*}{=} \frac{P(D=1,X)}{P(D=0,X)}$

$= \frac{P(D=1|X)\cancel{f(X)}}{P(D=0|X)\cancel{f(X)}} = \frac{P(D=1|X)}{1-P(D=1|X)} = \frac{p(x)}{1-p(x)}$