2015 D2

1) a) $Y_{k\times 1} \sim N(\mu, \Sigma)$ Σ full rank & sym $\Rightarrow \Sigma$ is pos def

A sym

b/c Σ is full rank $\Rightarrow \Sigma^{-1}$ exists & is symmetric & so does $\Sigma^{-1/2}$

$$Y^T A Y = (Y^T \Sigma^{-1/2}) \Sigma^{1/2} A \Sigma^{1/2} (\Sigma^{-1/2} Y)$$

let $Z = \Sigma^{-1/2} Y \sim N(\Sigma^{-1/2}\mu, I)$

$$\Rightarrow Y^T A Y = Z^T \Sigma^{1/2} A \Sigma^{1/2} Z$$

let $B = \Sigma^{1/2} A \Sigma^{1/2}$
$= (\Sigma^{1/2} A^{1/2})(A^{1/2} \Sigma^{1/2})$
$= (A^{1/2}\Sigma^{1/2})^T (A^{1/2}\Sigma^{1/2}) \Rightarrow B$ is symmetric

So we can use SD:

let $B = P \Lambda P'$ where $\Lambda = \text{diag}(\lambda_i)$ $\lambda_1 \le \lambda_2 \le \dots \le \lambda_k$ are the eigenvalues of B

& P is the orthonormal matrix of eigenvalues of B

$$\Rightarrow Y^T A Y = Z^T B Z = Z^T P \Lambda P^T Z$$
$$= (P^T Z)^T \Lambda (P^T Z)$$

let $X = P^T Z \sim N(\mu^* = P^T \Sigma^{-1/2}\mu, I)$ b/c $Cov(P^T Z) = P^T Cov(Z) P = P^T I P$
$= P^T P = I$ orthonormal

let $X = (X_1, \dots, X_k)^T$ $\mu^* = (\mu_1^*, \dots \mu_k^*)$

$$Y^T A Y = X^T \Lambda X = (X_1, \dots, X_k) \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_k \end{pmatrix} \begin{pmatrix} X_1 \\ \vdots \\ X_k \end{pmatrix} = \sum_{i=1}^{k} \lambda_i X_i^2$$

Since $X_i \sim N(\mu_i^*, 1) \Rightarrow X_i^2 \sim \chi^2(1, \tfrac{1}{2}\mu_i^{*2})$
$Cov(X_i, X_j) = 0$ $i \neq j$ & since X_i normal, $X_i \perp\!\!\!\perp X_j \Rightarrow X_i^2 \perp\!\!\!\perp X_j^2$

let $W_i = X_i^2$

$\Rightarrow$ $Y^T A Y = \sum \lambda_i W_i$
where $W_i \sim \chi^2(d_i = 1, \delta_i = \tfrac{1}{2}\mu_i^{*2})$
& λ_i are the eigenvalues of $\Sigma^{1/2} A \Sigma^{1/2}$

useful for part c??

* λ_i of $T^{-1} C T$ same as λ_i of C

$$T = \Sigma^{1/2} \Rightarrow T^{-1} C T = \Sigma^{-1/2} \Sigma^{1/2} A \Sigma^{1/2} \Sigma^{1/2} = A\Sigma$$

2015 d2

1 a) proof for thought

λ is an eigenvalue of D

$\Longrightarrow$ is eigenvalue of $B^{-1}DB$ for invertible B

λ eigenvalue of D $\Longrightarrow$

$D\underline{x} = \lambda \underline{x}$ for eigenvector $\underline{x}$

$$DBB^{-1}\underline{x} = \lambda \underline{x}$$

$$\Rightarrow B^{-1}DB(\underbrace{B^{-1}\underline{x}}_{\underline{y}}) = \lambda\Big[\underbrace{B^{-1}\underline{x}}_{\underline{y}}\Big]$$

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b) $Y^T A Y = \sum \lambda_i w_i$, $w_i = x_i^2$, $x_i \sim N(\mu_i^*, 1)$

$$mgf = E\left[e^{t\sum_{i=1}^{k}\lambda_i w_i}\right] = E\left[e^{t\sum_{i=1}^{k}\lambda_i x_i^2}\right] = \prod_{i=1}^{k} E\left[e^{t\lambda_i x_i^2}\right]$$

$$* E\left[e^{t\lambda_i x_i^2}\right] = \int_{-\infty}^{\infty} e^{t\lambda_i x_i^2} \cdot f_{x_i}\, \partial x_i = \int_{-\infty}^{\infty} e^{t\lambda_i x_i^2} (2\pi(1))^{-1/2} e^{-\frac{1}{2(1)}(x_i - \mu_i^*)^2}\, \partial x_i$$

$$= \int (2\pi)^{-1/2} \exp\left\{-\frac{1}{2}\left(x_i^2 - 2x_i\mu_i^* + \mu_i^{*2}\right) + x_i^2\lambda_i t\right\} dx_i$$

$$= \int (2\pi)^{-1/2} \exp\left\{-\frac{1}{2}\left(x_i^2(1-2\lambda_i t) - 2x_i\mu_i^* + \mu_i^{*2}\right)\right\} dx_i$$

$$= \int (2\pi)^{-1/2} \exp\left\{-\frac{1-2\lambda_i t}{2}\left(x_i^2 - \frac{2x_i\mu_i^*}{1-2\lambda_i t} + \frac{\mu_i^{*2}}{1-2\lambda_i t}\right)\right\} dx_i$$

$$= \int (2\pi)^{-1/2} \exp\left\{\frac{-(1-2\lambda_i t)}{2}\left(x_i^2 - 2x_i\left(\frac{\mu_i^*}{1-2\lambda_i t}\right) + \left(\frac{\mu_i^*}{1-2\lambda_i t}\right)^2 - \left(\frac{\mu_i^*}{1-2\lambda_i t}\right)^2 + \frac{\mu_i^{*2}}{1-2\lambda_i t}\right\} dx_i$$

$$= \int (2\pi)^{-1/2} \exp\left\{\frac{-1}{2/(1-2\lambda_i t)}\left(x_i^2 - 2x_i\left(\frac{\mu_i^*}{1-2\lambda_i t}\right) + \left(\frac{\mu_i^*}{1-2\lambda_i t}\right)^2\right)\right\} \exp\left\{\frac{-(1-2\lambda_i t)}{2}\left(\frac{\mu_i^{*2}}{1-2\lambda_i t} - \left(\frac{\mu_i^*}{1-2\lambda_i t}\right)^2\right)\right\} dx_i$$

$$= \cancel{(2\pi)^{-1/2}} \exp\left\{\frac{-\cancel{(1-2\lambda_i t)}\mu_i^{*2}}{2\cancel{(1-2\lambda_i t)}}\left(1 - \frac{1}{1-2\lambda_i t}\right)\right\}\left(\cancel{2\pi}\frac{1}{(1-2\lambda_i t)}\right)^{1/2} \underbrace{\int \left(2\pi\left(\frac{1}{1-2\lambda_i t}\right)\right)^{-1/2} \exp\left\{\frac{-1}{2/(1-2\lambda_i t)}\left(x_i - \frac{\mu_i^*}{1-2\lambda_i t}\right)^2\right\} dx_i}_{= 1 \text{ b/c } N\left(\frac{\mu_i^*}{1-2\lambda_i t}, \frac{1}{1-2\lambda_i t}\right) \text{ dsn}}$$

$$= \left(\frac{1}{1-2\lambda_i t}\right)^{1/2} \exp\left\{\frac{-\mu_i^{*2}}{2}\left(\frac{\cancel{1}-2\lambda_i t - \cancel{1}}{1-2\lambda_i t}\right)\right\}$$

$$= \left(\frac{1}{1-2\lambda_i t}\right)^{1/2} \exp\left\{\frac{\mu_i^{*2}/2\,(2\lambda_i t)}{1-2\lambda_i t}\right\} = \frac{\exp\left\{\frac{\delta_i^*(2\lambda_i t)}{1-2\lambda_i t}\right\}}{(1-2\lambda_i t)^{1/2}}$$ ← confirmed via wiki (they used $\gamma_i/2 = \delta_i$)

$$\Rightarrow M_{Y^TAY}(t) = \frac{\exp\left\{\sum_{i=1}^{k}\frac{\delta_i^*(2\lambda_i t)}{(1-2\lambda_i t)^{1/2}}\right\}}{\prod_{i=1}^{k}(1-2\lambda_i t)^{1/2}}$$

$$(1-2\lambda_i t) > 0 \Rightarrow t < \frac{1}{2\lambda_i}$$

$Y'AY = X'\Lambda X, \quad X \sim N(\mu^* = P'\Sigma^{-1/2}\mu, I)$

2015 D2

$\lambda_1 + \lambda_2 = \lambda_1^2 + \lambda_2^2$

$\Rightarrow \lambda_1^2 - \lambda_1 = -\lambda_2^2 + \lambda_2$

1) c) $tr(A\Sigma) = tr[(A\Sigma)^2]$

$\Rightarrow \sum_{i=1}^{k} \lambda_i = \sum_{i=1}^{k} \gamma_i$

where λ_i are eigenvalues of $A\Sigma$

γ_i are eigenvalues of $(A\Sigma)^2$

$tr(A\Sigma) = tr(\underbrace{\Sigma^{1/2} A \Sigma^{1/2}}_{\text{sym from (a)}})$

$tr([\Sigma^{1/2} A \Sigma^{1/2}]^2) = tr([\Sigma^{1/2} A \Sigma^{1/2}][\Sigma^{1/2} A \Sigma^{1/2}])$

$= tr(\Sigma^{1/2} A \Sigma A \Sigma^{1/2})$

$= tr(A\Sigma A \Sigma^{1/2}\Sigma^{1/2}) = tr(A\Sigma A\Sigma) = tr((A\Sigma)^2)$

Since $\Sigma^{1/2} A \Sigma^{1/2}$ is symmetric, let λ_i = eigenvals of $\Sigma^{1/2} A \Sigma^{1/2}$

$\Rightarrow tr((\Sigma^{1/2} A \Sigma^{1/2})^2) = \sum_{i=1}^{k} \lambda_i^2$

$\Rightarrow tr((A\Sigma)^2) = \sum_{i=1}^{k} \lambda_i^2$

but $tr((A\Sigma)^2) = tr(A\Sigma) = r$

$\Rightarrow \sum_{i=1}^{k} \lambda_i^2 = \sum_{i=1}^{k} \lambda_i = r$

$\Rightarrow \lambda_i \in \{0,1\} \ \forall i$ & since $\sum_{i=1}^{k} \lambda_i = r$ exactly r of the λ's are 1

WLOG, assume $\lambda_1 = \cdots = \lambda_r = 1$ & $\lambda_{r+1} = \cdots = \lambda_k = 0$

from a

$\Rightarrow Y'AY = \sum_{i=1}^{k} \lambda_i W_i = \sum_{i=1}^{r} \lambda_i W_i + \cancel{\sum_{i=r+1}^{k} \lambda_i W_i} \to 0$

$= \sum_{i=1}^{r} W_i \sim \chi^2(r, \Sigma \delta_i)$

b/c $W_i \overset{\text{indep}}{\sim} \chi^2(1, \delta_i)$

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