b) let $Z_i \sim Gam(\alpha, \frac{1}{\alpha})$ for $\alpha > 0$. Calculate μ_{ij} & γ^2

write the likelihood for (β, α) & show it can be expressed in closed form via the gamma function

from a: $E(Y_{ij}|X_{ij}) = \mu_{ij} = e^{x_{ij}^T\beta} E(Z_i) = e^{x_{ij}^T\beta} \underbrace{(\alpha \tfrac{1}{\alpha})}_{E(Z_i)=1} = \boxed{e^{x_{ij}^T\beta} = \mu_{ij}}$

$var(Z_i) = \alpha \left(\frac{1}{\alpha}\right)^2 = \frac{1}{\alpha}$

$\Rightarrow \gamma^2 = \left(\frac{\sqrt{1/\alpha}}{1}\right)^2 = \left(\frac{1}{\sqrt{\alpha}}\right)^2 = \boxed{\frac{1}{\alpha} = \gamma^2}$

$$f(y_{ij}, y_{ik}, Z_i) = f(y_{ij}, y_{ik}|Z_i) f(Z_i) = f(y_{ij}|Z_i) f(y_{ik}|Z_i) f(Z_i)$$

$$= \left(\frac{\lambda_{ij}^{y_{ij}} e^{-\lambda_{ij}}}{y_{ij}!}\right)\left(\frac{\lambda_{ik}^{y_{ik}} e^{-\lambda_{ik}}}{y_{ik}!}\right)\left(\frac{\alpha^\alpha}{\Gamma(\alpha)} Z_i^{\alpha-1} e^{-\alpha Z_i}\right)$$

$$= \left(\frac{Z_i^{y_{ij}} e^{y_{ij} x_{ij}^T\beta} e^{-Z_i e^{x_{ij}^T\beta}}}{y_{ij}!}\right)\left(\frac{e^{y_{ik} x_{ik}^T\beta} Z_i^{y_{ik}} e^{-Z_i e^{x_{ik}^T\beta}}}{y_{ik}!}\right)\left(\frac{\alpha^\alpha}{\Gamma(\alpha)} Z_i^{\alpha-1} e^{-\alpha Z_i}\right)$$

$$= \frac{\alpha^\alpha e^{(y_{ij}x_{ij} + y_{ik}x_{ik})^T\beta}}{y_{ij}! y_{ik}! \Gamma(\alpha)} Z_i^{(\alpha + y_{ij} + y_{ik})-1} e^{-Z_i(\alpha + e^{x_{ij}^T\beta} + e^{x_{ik}^T\beta})}$$

$$\Rightarrow f(y_{ij}, y_{ik}) = \int_0^\infty f(y_{ij}, y_{ik}, Z_i)\, dZ_i = \frac{\alpha^\alpha e^{(y_{ij}x_{ij} + y_{ik}x_{ik})^T\beta}}{y_{ij}! y_{ik}! \Gamma(\alpha)} \int_0^\infty Z_i^{(\alpha + y_{ij} + y_{ik})-1} e^{-Z_i(\alpha + e^{x_{ij}^T\beta} + e^{x_{ik}^T\beta})} dZ_i$$

$$= \frac{\alpha^\alpha e^{(y_{ij}x_{ij} + y_{ik}x_{ik})^T\beta}}{y_{ij}! y_{ik}! \Gamma(\alpha)} \frac{\Gamma(\alpha + y_{ij} + y_{ik})}{(\alpha + e^{x_{ij}^T\beta} + e^{x_{ik}^T\beta})^{\alpha + y_{ij} + y_{ik}}} \underbrace{\int_0^\infty \frac{(\alpha + e^{x_{ij}^T\beta} + e^{x_{ik}^T\beta})^{\alpha + y_{ij} + y_{ik}}}{\Gamma(\alpha + y_{ij} + y_{ik})} Z_i^{(\alpha + y_{ij} + y_{ik})-1} e^{-Z_i(\alpha + e^{x_{ij}^T\beta} + e^{x_{ik}^T\beta})} dZ_i}_{1}$$

$$\Rightarrow L(\alpha, \beta) = \prod_{i=1}^m f(y_{ij}, y_{ik}) = \prod_{i=1}^m \frac{\alpha^\alpha \Gamma(\alpha + y_{ij} + y_{ik})\, e^{(y_{ij}x_{ij} + y_{ik}x_{ik})^T\beta}}{y_{ij}! y_{ik}! \Gamma(\alpha) (\alpha + e^{x_{ij}^T\beta} + e^{x_{ik}^T\beta})^{\alpha + y_{ij} + y_{ik}}}$$

*note $j=1, k=2$
(b/c $j=1,2$)

b. $\mu_{ij} = \exp\{x_{ij}^T\beta\} E[z_i] = \exp\{x_{ij}^T\beta\}$

$\gamma^2 = \frac{Var(z_i)}{E[z_i]^2} = \alpha\left(\frac{1}{\alpha}\right)^2 = \frac{1}{\alpha}$

$$p(y_{i1}, y_{i2}, z_i) = p(y_{i1}, y_{i2} \mid z_i)p(z_i) = \frac{\lambda_{i1}^{y_{i1}} e^{-\lambda_{i1}}}{y_{i1}!} \cdot \frac{\lambda_{i2}^{y_{i2}} e^{-\lambda_{i2}}}{y_{i2}!} \cdot \frac{\alpha^\alpha}{\Gamma(\alpha)} z_i^{\alpha-1} e^{-\alpha z_i}$$

$$= \frac{(\exp\{x_{i1}^T\beta\}z_i)^{y_{i1}} e^{-\exp\{x_{i1}^T\beta\}z_i}}{y_{i1}!} \cdot \frac{(\exp\{x_{i2}^T\beta\}z_i)^{y_{i2}} e^{-\exp\{x_{i2}^T\beta\}z_i}}{y_{i2}!} \cdot \frac{\alpha^\alpha}{\Gamma(\alpha)} z_i^{\alpha-1} e^{-\alpha z_i}$$

$$= \frac{e^{x_{i1}^T\beta y_{i1}}}{y_{i1}!} \cdot \frac{e^{x_{i2}^T\beta y_{i2}}}{y_{i2}!} \cdot \frac{\alpha^\alpha}{\Gamma(\alpha)} \cdot z_i^{y_{i1}+y_{i2}+\alpha-1} e^{(-\exp\{x_{i1}^T\beta\} - \exp\{x_{i2}^T\beta\} - \alpha)z_i}$$

So $$p(y_{i1}, y_{i2}) = \int p(y_{i1}, y_{i2}, z_i)\, dz_i = \frac{e^{x_{i1}^T\beta y_{i1}}}{y_{i1}!} \cdot \frac{e^{x_{i2}^T\beta y_{i2}}}{y_{i2}!} \cdot \frac{\alpha^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(y_{i1}+y_{i2}+\alpha)}{(\exp\{x_{i1}^T\beta\}+\exp\{x_{i2}^T\beta\}+\alpha)^{y_{i1}+y_{i2}+\alpha}}$$

So $$L(\alpha,\beta) = \prod_{i=1}^{n} \frac{\exp\{y_{i1}x_{i1}^T\beta + y_{i2}x_{i2}^T\beta\} \cdot \alpha^\alpha \cdot \Gamma(y_{i1}+y_{i2}+\alpha)}{y_{i1}!\, y_{i2}!\, \Gamma(\alpha) \left[\exp\{x_{i1}^T\beta\} + \exp\{x_{i2}^T\beta\} + \alpha\right]^{y_{i1}+y_{i2}+\alpha}}$$

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