$\beta_1^* = \beta_1 + (X_1^T X_1)^{-1} X_1^T X_3 \beta_3$ ✓

3) a) * Reparametrize $Y = X_1\beta_1 + X_3\beta_3 + \varepsilon_2 \iff Y = X_1\beta_1^* + (I - M_1)X_3\beta_3 + \varepsilon$

now $\begin{pmatrix} X_1^T \\ X_3^T(I-M_1) \end{pmatrix} (X_1, (I-M_1)X_3) \begin{pmatrix} \hat\beta_1^* \\ \hat\beta_3 \end{pmatrix} = \begin{pmatrix} X_1^T Y \\ (I-M_1)X_3^T Y \end{pmatrix}$

thus $\boxed{X_3^T(I-M_1)X_3\hat\beta_3 = X_3^T(I-M_1)\cancel{X_3^T}Y}$ ✓

* Reparametrize $Y = (X_1, X_2)\begin{pmatrix} \gamma_1^* \\ \gamma_2^* \end{pmatrix} + (I - M_{12})X_3\gamma_3 + \varepsilon_1$ ✓ and by normal equations we obtain

$\boxed{X_3^T(I-M_{12})X_3\hat\gamma_3 = X_3^T(I-M_{12})\cancel{X_3^T}Y}$ ✓ We prove the negation statement

* $\text{sign}(\hat\beta_3) = \text{sign}(\hat\gamma_3) \iff \text{sign}[X_3^T(I-M_1)] = \text{sign}[X_3^T(I-M_{12})]$

$\Rightarrow$: If $\text{sign}(\hat\beta_3) = \text{sign}(\hat\gamma_3)$ WLOG assume are both <u>positive</u> then since $X_3^T(I-M_{12})X_3$ and $X_3^T(I-M_1)X_3$ is both quadratic forms then $\text{sign}(X_3^T(I-M_{12})X_3) = \text{sign}(X_3^T(I-M_1)X_3) > 0$ ✓ thus we must have $\text{sign}(X_3^T(I-M_1)Y) > 0$ and $\text{sign}(X_3^T(I-M_{12})Y) > 0$.

$\Leftarrow$: If $\text{sign}(X_3^T(I-M_{12})Y) = \text{sign}(X_3^T(I-M_1)Y) \overset{\text{WLOG}}{>} 0$ and $\text{sign}(X_3^T(I-M_1)X_3) = \text{sign}(X_3^T(I-M_{12})X_3) > 0$ then we must have that $\text{sign}(\hat\beta_3) > 0$ and $\text{sign}(\hat\gamma_3) > 0$ ✓ thus $\text{sign}(\hat\beta_3) = \text{sign}(\hat\gamma_3)$.

b) (i) M_{12} is the opo onto $C(X_1, X_2)$

now $C(X_1,X_2) = C(X_1,X_2) \cap C(X_1) + C(X_1, X_2) \cap C(X_1)^{\perp}$

$\Leftrightarrow C(X_1,X_2) = C(X_1) + C(X_1,X_2) \cap C(X_1)^{\perp}$

thus $M_{12} = M_1 + M_3 \Rightarrow \boxed{M_3 = M_{12} - M_1}$ ✓

(ii) We must show $\boxed{(I - M_1) M_3 = M_3}$ indeed

$(I - M_1)(M_{12} - M_1) = M_{12} - M_1 - M_1 M_{12} + M_1^2 =$ since $C(X_1) \subset C(X_1,X_2)$

$M_{12} - M_1 - M_1 + M_1 = M_{12} - M_1 = M_3$ ✓ as required.

c) $X_3^T (I - M_1) Y - X_3^T M_3 Y < 0 \Leftrightarrow X_3^T (I - M_1) Y - X_3^T (M_{12} - M_1) Y < 0$

$\Leftrightarrow X_3^T (I - M_1 - M_{12} + M_1) Y < 0 \Leftrightarrow X_3^T (I - M_{12}) Y < 0$ ✓

thus $\text{sign}(\hat{\gamma}_3) < 0$ ✓+ by using part (a) using

part (a) Thus $\text{sign}(\hat{\gamma}_3) < 0$ and $\text{sign}(\hat{\beta}_3) > 0$ thus

$\text{sign}(\hat{\gamma}_3) \neq \text{sign}(\hat{\beta}_3)$ ✓

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$M_0 = \frac{1}{n} J_n J_n^T$

d) (i) $R^2 = \frac{Y^T M Y}{Y^T (I - M_0) Y} = \frac{Y^T M Y}{Y^T M Y + Y^T (I - M_0 - M) Y}$ ✓

Now let $X = Y^T M Y \sim \chi^2(p)$ ✓ and $Y^T (I - M_0 - M) Y \sim \chi^2(n-1-p)$ ✓

Now $M(I - M_0 - M) = M - MM_0 - M^2 = M - MM_0 - M = -MM_0$

$MM_0 = \frac{1}{n} X (X^T X)^{-1} \underbrace{X^T J_n}_{\to 0} J_n^T = 0$ thus X is independendt from Y ✓

thus $R^2 = \frac{X}{X+Y} \sim \text{Beta}\left(\frac{p}{2}, \frac{n-1-p}{2}\right)$ ✓

* $E(R^2) = \frac{\frac{p}{2}}{\frac{n-1-p}{2} + \frac{p}{2}} = \frac{\frac{p}{2}}{\frac{n-1}{2}} = \boxed{\frac{p}{n-1}}$ ✓

(ii) $\frac{n-1-p}{p} = \frac{1}{\lambda} \Rightarrow \frac{n-1}{p} - 1 = \frac{1}{\lambda} \Rightarrow \frac{n-1}{p} = \frac{1}{\lambda} + 1 \Rightarrow$

$\frac{p}{n-1} = \frac{1}{\frac{1+\lambda}{\lambda}} = \frac{\lambda}{1+\lambda}$

let $R = R^2$

$\boxed{E(R^2) = \frac{p}{n-1}}$

let $b = \frac{p}{2}$

let $c = \frac{n-1-p}{2}$

* Now $Var(R) = E(R^2) - E(R)^2$

$E(R^2) = \int \frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)} x^{b+1} (1-x)^{c-1} dx \to$ kernel Beta$(b+2, c)$

(−1/2)

thus $E(R^2) = \frac{\Gamma(b+c)\,\Gamma(b+2)}{\Gamma(b)\,\Gamma(b+c+2)} = \frac{(b+1)(b)}{(b+c+1)(b+c)} = \frac{\left(\frac{p}{2}+1\right)\left(\frac{p}{2}\right)}{\left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}+1\right)} = \left(\frac{\lambda}{1+\lambda}\right)^2$ ✗

Thus $Var(R) = \left(\frac{\lambda}{1+\lambda}\right)^2 - \left(\frac{\lambda}{1+\lambda}\right)^2 = 0$

Thus the limit of R^2 is the expectation

$$\frac{P}{n-1} = \boxed{\frac{\lambda}{1+\lambda}}$$

In fact R^2 converges to a dirac delta function with point mass given at $\frac{\lambda}{1+\lambda}$.