2012 Qualifying Exam Section 1 Question 3

February 21, 2019

In this part, let T_0 be an unbiased estimator of an unknown parameter and consider the properties of T_0 under squared error loss.

(a)

(i.)

Show that $T_0 + c$ is not a minimax estimator under squared error loss, where $\neq 0$ is a known constant.

Proof:
We want to find an estimator d such that $\sup_\theta R(\theta, T_0 + c) > \sup_\theta R(\theta, d)$ We can conjecture that $T_0 + c$ is a "bad" estimator, since it adds a nonzero constant to an unbiased estimator. Consider the risk of T_0 under squared loss. That is,

$$\begin{aligned} R(\theta, T_0) &= \mathbb{E}\left(L(\theta, T_0)\right) \\ &= \mathbb{E}(\theta - T_0)^2 \\ &= \mathbb{E}(T_0 - \theta)^2 \\ &= \mathrm{Var}(T_0|\theta) \end{aligned}$$

The risk for $T_0 + c$ is

$$\begin{aligned} R(\theta, T_0 + c) &= \mathbb{E}\left(L(\theta, T_0 + c)\right) \\ &= \mathbb{E}(\theta - T_0 - c)^2 \\ &= \mathbb{E}(T_0 - \theta) + c^2 \\ &= \mathrm{Var}(T_0|\theta) + c^2 \end{aligned}$$

Hence,

$$\sup_\theta R(\theta, T_0 + c) = \sup_\theta \mathrm{Var}(T_0|\theta) + c^2 > \sup_\theta \mathrm{Var}(T_0|\theta) = \sup_\theta R(\theta, T_0)$$

Hence, $T_0 + c$ is not minimax because the supremum risk of T_0 is lower than that of $T_0 + c$. (Note however it is minimax under the case in problem (ii), since all estimators are minimax.

(ii.)

Show that the estimator cT_0 is not minimax under squared error loss unless $\sup_\theta R_T(\theta) = \infty$ for any estimator T of θ, where $c \in (0, 1)$ is a known constant and $R_T(\theta)$ is the frequentist risk function for T.

Proof:

$$\begin{aligned} R(\theta, cT_0) &= \mathbb{E}_\theta(cT_0 - \theta)^2 \\ &= \mathbb{E}_\theta(c(T_0 - \theta) - \theta(1 - c))^2 \\ &= \mathrm{Var}_\theta(T_0) + (1 - c)^2\theta^2 \end{aligned}$$

Now, if $c \geq 1$, then by part (i) we have

$$R(\theta, cT_0) \geq Var(T_0|\theta) = R(\theta, T_0)$$

with equality holding if and only if $c = 1$. Hence, under this case, cT_0 is inadmissible for $c > 1$ and hence not minimax.

If $c \leq 0$, then

$$\begin{aligned} R(\theta, cT_0) &> \theta^2 \\ \rightarrow \sup_\theta R(\theta, cT_0) &= \infty \end{aligned}$$

Moreover, the risk function is decreasing over $(-\infty, 0]$, so the trivial estimator 0 minimizes the risk over this function and hence in this scenario, cannot be minimax. $c = 0$ cannot be unique minimax because it has infinite supremum risk.

Under the last case, all estimators are minimax, so the problem is trivial.