AMW

2. a) Given $T_1 \sim Pois(\lambda_1)$, $T_2 \sim Pois(\lambda_2)$, $U \sim Pois(\psi)$ (indep.)

Then, $P(X=x, Y=y) = P(U+T_1=x,\ U+T_2=y)$

$$= \sum_u P(U+T_1=x,\ U+T_2=y \mid U=u)\cdot P(U=u)$$

$$= \sum_u P(T_1=x-u)\cdot P(T_2=y-u)\cdot P(U=u)$$

$$= \sum_{u=0}^{\min(x,y)} \frac{\lambda_1^{(x-u)}}{(x-u)!}\exp(-\lambda_1)\cdot\frac{\lambda_2^{(y-u)}}{(y-u)!}\exp(-\lambda_2)\cdot\frac{\psi^u}{u!}\exp(-\psi)$$

$$= e^{-(\psi+\lambda_1+\lambda_2)}\lambda_1^x\lambda_2^y\sum_{u=0}^{\min(x,y)}\left(\frac{\psi}{\lambda_1\lambda_2}\right)^u\frac{1}{u!(x-u)!(y-u)!}$$

b) $$l(\psi,\lambda_1,\lambda_2\mid x,y) = -(\psi+\lambda_1+\lambda_2)+x\log(\lambda_1)+y\log(\lambda_2)+\log\left[\sum_{u=0}^{\min(x,y)}\left(\frac{\psi}{\lambda_1\lambda_2}\right)^u\cdot\frac{1}{u!(x-u)!(y-u)!}\right]$$

$\left(\frac{\psi}{\lambda_1\lambda_2}\right)^u$ or $\psi^u(\lambda_1\lambda_2)^{-u}$

ψ: $$\frac{\partial l}{\partial\psi} = -1+\frac{\left[\sum_{u=0}^{\min(x,y)} u\left(\frac{\psi}{\lambda_1\lambda_2}\right)^{u-1}\cdot\frac{1}{(\lambda_1\lambda_2)}\cdot\frac{1}{u!(x-u)!(y-u)!}\right]}{\left[\sum_{u=0}^{\min(x,y)}\left(\frac{\psi}{\lambda_1\lambda_2}\right)^u\cdot\frac{1}{u!(x-u)!(y-u)!}\right]}$$

$$= -1+\frac{\left[\sum_{u=0}^{\min(x,y)} u\,\psi^{u-1}\left[\frac{1}{\lambda_1\lambda_2}\right]^u\cdot\frac{1}{u!(x-u)!(y-u)!}\right]}{\left[\sum_{u=0}^{\min(x,y)}\psi^u\cdot\left[\frac{1}{\lambda_1\lambda_2}\right]^u\cdot\frac{1}{u!(x-u)!(y-u)!}\right]}$$

Assuming $0^0=1$ (not undefined), then have $$\left.\frac{\partial l}{\partial\psi}\right|_{\psi=0} = -1+\frac{\cancel{1}\cdot 0^0\cdot\left[\frac{1}{\lambda_1\lambda_2}\right]^1\cdot\frac{1}{\cancel{1!(x-1)!(y-1)!}}}{0^0\left[\cancel{\frac{1}{\lambda_1\lambda_2}}\right]^0\cdot\frac{1}{0!\cancel{(x)!(y)!}}}$$ $= \frac{1}{xy}$

$$= \frac{xy}{\lambda_1\lambda_2}-1$$

λ_1: $$\frac{\partial l}{\partial\lambda_1} = -1+\frac{x}{\lambda_1}+\frac{\sum_{u=0}^{\min(x,y)}\psi^u(-u)(\lambda_1\lambda_2)^{-u-1}\cdot\lambda_2\,\frac{1}{u!(x-u)!(y-u)!}}{\sum_{u=0}^{\min(x,y)}\left(\frac{\psi}{\lambda_1\lambda_2}\right)^u\cdot\frac{1}{u!(x-u)!(y-u)!}} \Rightarrow \left.\frac{\partial l}{\partial\lambda_1}\right|_{\psi=0}$$

forgot 0

$$= -1+\frac{x}{\lambda_1}+\frac{0^0\cancel{(0)}(\lambda_1\lambda_2)^{0-1}+0^1\cancel{(1)}(\lambda_1\lambda_2)^{1-1}+\sum_{u=2}^{\min(x,y)}0^u(-u)(\lambda_1\lambda_2)^{-u-1}\cdot\lambda_2}{\sum_{u=0}^{\min(x,y)}\left(\frac{\psi}{\lambda_1\lambda_2}\right)^u\cdot\frac{1}{u!(x-u)!(y-u)!}} = \frac{x}{\lambda_1}-1$$

λ_2: Similar to above

$$\left.\frac{\partial l}{\partial\lambda_2}\right|_{\psi=0} = \frac{x}{\lambda_2}-1$$

$$\Rightarrow \frac{\partial l}{\partial(\psi,\lambda_1,\lambda_2)} = \begin{pmatrix}\frac{xy}{\lambda_1\lambda_2}-1\\ \frac{x}{\lambda_1}-1\\ \frac{y}{\lambda_2}-1\end{pmatrix}$$

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2b) cont'd

To obtain expected info matrix, also evaluated @ $\psi = 0$.

$$\text{Let } A = \sum_{u=0}^{\min(x,y)} \psi^u \cdot \left[\frac{1}{\lambda_1 \lambda_2}\right]^u \cdot \frac{1}{u!(x-u)!(y-u)!}$$

Then,

ψ: $$\frac{\partial^2 l}{\partial \psi^2} = \frac{\partial}{\partial \psi}\left(-1 + \frac{A'}{A}\right) = \left(\frac{1}{A}\right)' A' + \left(\frac{1}{A}\right) A'' = -\frac{1}{A^2} A' A' + \frac{1}{A} A'' = -\frac{(A')^2}{A^2} + \frac{A''}{A}$$

$$= -\frac{\left(\sum_{u=0}^{\min(x,y)} u\,\psi^{u-1}\left[\frac{1}{\lambda_1\lambda_2}\right]^u \cdot \frac{1}{u!(x-u)!(y-u)!}\right)^2}{\left(\sum_{u=0}^{\min(x,y)} \psi^u \cdot \left[\frac{1}{\lambda_1\lambda_2}\right]^u \cdot \frac{1}{u!(x-u)!(y-u)!}\right)^2} + \frac{\sum_{u=0}^{\min(x,y)} u(u-1)\,\psi^{u-2}\left[\frac{1}{\lambda_1\lambda_2}\right]^u \cdot \frac{1}{u!(x-u)!(y-u)!}}{\left(\sum_{u=0}^{\min(x,y)} \psi^u \left[\frac{1}{\lambda_1\lambda_2}\right]^u \cdot \frac{1}{u!(x-u)!(y-u)!}\right)}$$

$$\Rightarrow \left.\frac{\partial^2 l}{\partial \psi^2}\right|_{\psi=0} = \frac{-\left(1\cdot 0^0\left[\frac{1}{\lambda_1\lambda_2}\right]^1 \cdot \frac{1}{1!(x-1)!(y-1)!}\right)^2}{\left[0^0\left[\frac{1}{\lambda_1\lambda_2}\right]^0 \frac{1}{(0!\,x!\,y!)}\right]^2} + \frac{\left(\cancel{2(2-1)}\,0^0\left[\frac{1}{\lambda_1\lambda_2}\right]^2 \cdot \frac{1}{\cancel{2}(x-\cancel{2})!\,(y-\cancel{2})!}\right)}{\left(1\cdot\left[\cancel{\frac{1}{\lambda_1\lambda_2}}\right]^0 \frac{1}{0!(x)!(y)!}\right)}$$

$x(\cancel{x-1})! = x(\cancel{x-1})!$ $\quad y(\cancel{y-1})!\; y(\cancel{y-1})!$ $\quad x(x-1)(\cancel{x-2})!$ $\quad y(y-1)(\cancel{y-2})!$

$$= -\frac{x^2y^2}{\lambda_1^2\lambda_2^2\,\cancel{(x-1)!(x-1)!(y-1)!(y-1)!}} + \frac{x(x-1)\,y(y-1)}{\lambda_1^2\lambda_2^2} = \frac{-x^2y^2 + (x^2-x)(y^2-y)}{\lambda_1^2\lambda_2^2}$$

$$= \frac{\cancel{-x^2y^2} + \cancel{x^2y^2} - x^2y - y^2x + xy}{\lambda_1^2\lambda_2^2} = \boxed{\frac{-xy(x+y-1)}{\lambda_1^2\lambda_2^2}}$$

λ_1: $$\frac{\partial^2 l}{\partial \lambda_1^2} = \frac{\partial}{\partial \lambda_1}\left(-1 + \frac{x}{\lambda_1} + \frac{A'}{A}\right) = \frac{-x}{\lambda_1^2} - \frac{(A')^2}{A^2} + \frac{A''}{A}$$

$$= -\frac{x}{\lambda_1^2} - \frac{\left(\sum_{u=0}^{\min(x,y)} \psi^u (-u)(\lambda_1\lambda_2)^{-u-1}\right)^2}{\left(\sum_{u=0}^{\min(x,y)} \psi^u \left[\frac{1}{\lambda_1\lambda_2}\right]^u \cdot \frac{1}{u!(x-u)!(y-u)!}\right)^2} + \frac{\left(\sum_{u=0}^{\min(x,y)} \psi^u (-u)(-u-1)(\lambda_1\lambda_2)^{-u-2}\right)}{\left(\sum_{u=0}^{\min(x,y)} \psi^u\left[\frac{1}{\lambda_1\lambda_2}\right]^u \cdot \frac{1}{u!(x-u)!(y-u)!}\right)}$$

$$\Rightarrow \left.\frac{\partial^2 l}{\partial \lambda_1^2}\right|_{\psi=0} = -\frac{x}{\lambda_1^2}$$

λ_2: Similarly, $\left.\frac{\partial^2 l}{\partial \lambda_2^2}\right|_{\psi=0} = -\frac{y}{\lambda_2^2}$

Cont'd next pg. $\longrightarrow$

2b) cont'd.

$\underline{\psi\lambda_1}$: $$\frac{\partial^2 l}{\partial\lambda_1\partial\psi} = \frac{\partial}{\partial\lambda_1}\left(-1+\frac{\left(\sum_{u=0}^{\min(x,y)} u\,\psi^{u-1}\left[\frac{1}{\lambda_1\lambda_2}\right]^u \cdot \frac{1}{u!(x-u)!(y-u)!}\right)}{\left(\sum_{u=0}^{\min(x,y)} \psi^u \cdot \left[\frac{1}{\lambda_1\lambda_2}\right]^u \cdot \frac{1}{u!(x-u)!(y-u)!}\right)}\right)$$

$$= +\frac{\left(\sum_{u=0}^{\min(x,y)} u\,\psi^{u-1}\left[\frac{1}{\lambda_1\lambda_2}\right]^u \cdot \frac{1}{u!(x-u)!(y-u)!}\right)\left(\sum_{u=0}^{\min(x,y)} u\psi^u\,[\lambda_1\lambda_2]^{-u-1}\lambda_2 \cdot \frac{1}{u!(x-u)!(y-u)!}\right)}{\left(\sum_{u=0}^{\min(x,y)} \psi^u\left[\frac{1}{\lambda_1\lambda_2}\right]^u \frac{1}{u!(x-u)!(y-u)!}\right)^2}$$

$$-\frac{\left(\sum_{u=0}^{\min(x,y)} u^2\,\psi^{u-1}[\lambda_1\lambda_2]^{-u-1}\lambda_2 \cdot \frac{1}{u!(x-u)!(y-u)!}\right)}{\left(\sum_{u=0}^{\min(x,y)} \psi^u\left[\frac{1}{\lambda_1\lambda_2}\right]^u \cdot \frac{1}{u!(x-u)!(y-u)!}\right)}$$

$$\Rightarrow \left.\frac{\partial^2 l}{\partial\lambda_1\partial\psi}\right|_{\psi=0} = 0 - \frac{\frac{1}{\lambda_1^2\lambda_2^{\cancel{2}}} \cdot \frac{\cancel{\lambda_2}}{(x-1)!(y-1)!}}{\frac{1}{(x)!(y)!}} = \boxed{\frac{-xy}{\lambda_1^2\lambda_2}}$$

Similarly, $$\left.\frac{\partial^2 l}{\partial\lambda_2\partial\psi}\right|_{\psi=0} = \boxed{\frac{-xy}{\lambda_1\lambda_2^2}}$$

$$\frac{\partial^2 l}{\partial\lambda_1\partial\lambda_2} = \frac{\partial}{\partial\lambda_2}\left(-1+\frac{x}{\lambda_1}+\frac{\sum_{u=0}^{\min(x,y)} \psi^u(-u)(\lambda_1\lambda_2)^{-u-1}\lambda_2 \cdot \frac{1}{u!(x-u)!(y-u)!}}{\sum_{u=0}^{\min(x,y)}\left(\frac{\psi}{\lambda_1\lambda_2}\right)^u \cdot \frac{1}{u!(x-u)!(y-u)!}}\right)$$

$$= +\frac{\left(\sum_{u=0}^{\min(x,y)} \psi^u u(\lambda_1\lambda_2)^{-u-1}\lambda_2 \cdot \frac{1}{u!(x-u)!(y-u)!}\right)\left(\sum_{u=0}^{\min(x,y)} \psi^u u(\lambda_1\lambda_2)^{-u-1}\lambda_1 \cdot \frac{1}{u!(x-u)!(y-u)!}\right)}{\sum_{u=0}^{\min(x,y)} (\psi)^u(\lambda_1\lambda_2)^{-u}\frac{1}{u!(x-u)!(y-u)!}}$$

$$-\frac{\left(\sum_{u=0}^{\min(x,y)} \psi^u(u)(-u-1)(\lambda_1\lambda_2)^{-u-2}\lambda_2^2 \cdot \frac{1}{u!(x-u)!(y-u)!}\right.}{\sum_{u=0}^{\min(x,y)} \psi^u(\lambda_1\lambda_2)^{-u}\frac{1}{u!(x-u)!(y-u)!}}$$

$$\Rightarrow \left.\frac{\partial^2 l}{\partial\lambda_1\partial\lambda_2}\right|_{\psi=0} = 0$$

$$\Rightarrow \ddot{l}_{\psi=0} = -\begin{pmatrix} \frac{xy(x+y-1)}{\lambda_1^2\lambda_2^2} & \frac{xy}{\lambda_1^2\lambda_2} & \frac{xy}{\lambda_1\lambda_2^2} \\ \frac{xy}{\lambda_1^2\lambda_2} & \frac{x}{\lambda_1^2} & 0 \\ \frac{xy}{\lambda_1\lambda_2^2} & 0 & \frac{y}{\lambda_2^2} \end{pmatrix}$$

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2b) cont'd.

Then, to find the components of $-E(\ddot{\ell}_{\psi=0})$

Recall $\psi = 0$

$$+E\left(\frac{xy(x+y-1)}{\lambda_1^2\lambda_2^2}\right) = \frac{1}{\lambda_1^2\lambda_2^2} E[xy(x+y-1)] = \frac{1}{\lambda_1^2\lambda_2^2}\left\{E[x^2y] + E[xy^2] - E[xy]\right\}$$

$$= \frac{1}{\lambda_1^2\lambda_2^2}\left\{E[x^2]\cdot E[y] + E[x]\cdot E[y^2] - E[x]\cdot E[y]\right\}$$

$$= \frac{1}{\lambda_1^2\lambda_2^2}\left\{\left[\underbrace{Var(x)}_{\lambda_1} + \underbrace{(E(x))^2}_{\lambda_1^2}\right]\underbrace{E(y)}_{\lambda_2} + \underbrace{E[x]}_{\lambda_1}\left[\underbrace{Var(y)}_{\lambda_2} + \underbrace{(E(y))^2}_{\lambda_2^2}\right] - \underbrace{E[x]\cdot E[y]}_{\lambda_1\cdot\lambda_2}\right\}$$

$$= \frac{1}{\lambda_1^2\lambda_2^2}\left\{\lambda_1\lambda_2 + \lambda_1^2\lambda_2 + \cancel{\lambda_1\lambda_2} + \lambda_1\lambda_2^2 - \cancel{\lambda_1\lambda_2}\right\} = \frac{\lambda_1\lambda_2(1+\lambda_1+\lambda_2)}{\lambda_1^2\lambda_2^2} = \frac{1+\lambda_1+\lambda_2}{\lambda_1\lambda_2}$$

$$+E\left(\frac{xy}{\lambda_1^2\lambda_2}\right) = \frac{1}{\lambda_1^2\lambda_2}E(x)\cdot E(y) = \frac{\cancel{\lambda_1}\cancel{\lambda_2}}{\lambda_1^{\cancel{2}}\cancel{\lambda_2}} = \frac{1}{\lambda_1}$$

$$+E\left(\frac{xy}{\lambda_1\lambda_2^2}\right) = \frac{1}{\lambda_1\lambda_2^2}E(x)\cdot E(y) = \frac{\lambda_1\cancel{\lambda_2}}{\cancel{\lambda_1}\lambda_2^{\cancel{2}}} = \frac{1}{\lambda_2}$$

$$+E\left(\frac{x}{\lambda_1^2}\right) = \frac{1}{\lambda_1^2}E(x) = \frac{1}{\lambda_1}$$

$$+E\left(\frac{y}{\lambda_2^2}\right) = \frac{1}{\lambda_2^2}E(y) = \frac{1}{\lambda_2}$$

Then,

$$\boxed{-E(\ddot{\ell}_{\psi=0}) = \begin{pmatrix} \frac{(1+\lambda_1+\lambda_2)}{\lambda_1\lambda_2} & \frac{1}{\lambda_1} & \frac{1}{\lambda_2} \\ \frac{1}{\lambda_1} & \frac{1}{\lambda_1} & 0 \\ \frac{1}{\lambda_2} & 0 & \frac{1}{\lambda_2} \end{pmatrix}}$$

2c) Want to test $H_0: \psi = 0$

First, let's review the 3 tests for this null: Want to test null of the form $H_0: R\xi = b_0$.

<u>Wald</u>: $W_n = [R\xi - b_0]'[R\, I_n(\xi)^{-1}R']^{-1}[R\xi - b_0]\Big|_{\xi=\hat{\xi}}$

<u>Score</u>: $SC_n = \dot{\ell}_n(\xi)'[I_n(\xi)]^{-1}\dot{\ell}_n(\xi)\Big|_{\xi=\tilde{\xi}}$

<u>LRT</u>: $LRT_n = 2[\underbrace{\ell_n(\hat{\xi})}_{} - \ell_n(\tilde{\xi})]$

($\ln(\hat{\xi})$ always $> \ln(\tilde{\xi})$ b/c unrestricted space has higher likelihood than restricted space and we want $LRT_n > 0$.)

In this case, asked to find score test & to identify its asymptotic distn. under H_0 $(n \to \infty)$

Notice here we have n obs.

① 1st, find score vector under the null. Need to find $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ to sub into score vector.

$$\left.\frac{\partial l_n}{\partial \lambda_1}\right|_{\psi=0} = \frac{\sum_i x_i}{\lambda_1} - n = 0 \Rightarrow \tilde{\lambda}_1 = \frac{1}{n}\sum_i x_i$$

$$\left.\frac{\partial l_n}{\partial \lambda_2}\right|_{\psi=0} = \frac{\sum_i y_i}{\lambda_2} - n = 0 \Rightarrow \tilde{\lambda}_2 = \frac{1}{n}\sum_i y_i$$

$$\Rightarrow \dot{\ell}_n^T = \left(\frac{n^2 \sum_i x_i y_i}{\sum_i x_i \sum_i y_i} - n,\ 0,\ 0\right)$$

② Now, compute the inverse of the Fisher info across n obs.

<u>Formula for inverse of a 3x3</u>: $A^{-1}_{3\times3} = \frac{1}{|A|}\mathrm{Adj}(A)$

To compute $\mathrm{Adj}(A)$
① Transpose A
② Find determinants of each 2x2 minor matrix.
③ Multiply matrix from ② by $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

<u>Leave out n for now</u>

<u>First, det(A) = |A|</u>: $$|I(\psi)| = \begin{vmatrix} \frac{(1+\lambda_1+\lambda_2)}{\lambda_1\lambda_2} & 1/\lambda_1 & 1/\lambda_2 \\ 1/\lambda_1 & 1/\lambda_1 & 0 \\ 1/\lambda_2 & 0 & 1/\lambda_2 \end{vmatrix} = \left[\frac{(1+\lambda_1+\lambda_2)}{\lambda_1\lambda_2}\begin{vmatrix} 1/\lambda_1 & 0 \\ 0 & 1/\lambda_2 \end{vmatrix} - \frac{1}{\lambda_1}\begin{vmatrix} 1/\lambda_1 & 0 \\ 0 & 1/\lambda_2 \end{vmatrix} + \frac{1}{\lambda_2}\begin{vmatrix} 1/\lambda_1 & 1/\lambda_1 \\ 1/\lambda_2 & 0 \end{vmatrix}\right]$$

$$= \left[\frac{(1+\lambda_1+\lambda_2)}{\lambda_1\lambda_2}\left(\frac{1}{\lambda_1\lambda_2}\right) - \frac{1}{\lambda_1}\left(\frac{1}{\lambda_1\lambda_2}\right) - \frac{1}{\lambda_2}\left(\frac{1}{\lambda_1\lambda_2}\right)\right] = \left[\frac{1}{\lambda_1^2\lambda_2^2} + \cancel{\frac{\lambda_1}{\lambda_1^2\lambda_2^2}} + \cancel{\frac{\lambda_2}{\lambda_1^2\lambda_2^2}} - \cancel{\frac{1}{\lambda_1^2\lambda_2}} - \cancel{\frac{1}{\lambda_1\lambda_2^2}}\right] = \frac{1}{\lambda_1^2\lambda_2^2}$$

<u>2nd, Adj(A)</u> ① $$A^T = \begin{pmatrix} \frac{(1+\lambda_1+\lambda_2)}{\lambda_1\lambda_2} & 1/\lambda_1 & 1/\lambda_2 \\ 1/\lambda_1 & 1/\lambda_1 & 0 \\ 1/\lambda_2 & 0 & 1/\lambda_2 \end{pmatrix}$$

② (1,1) $\begin{vmatrix} 1/\lambda_1 & 0 \\ 0 & 1/\lambda_2 \end{vmatrix} = \frac{1}{\lambda_1\lambda_2}$, (1,2) $\begin{vmatrix} 1/\lambda_1 & 0 \\ 1/\lambda_2 & 1/\lambda_2 \end{vmatrix} = \frac{1}{\lambda_1\lambda_2}$, (1,3) $\begin{vmatrix} 1/\lambda_1 & 1/\lambda_1 \\ 1/\lambda_2 & 0 \end{vmatrix} = \frac{1}{\lambda_1\lambda_2}$

(2,1) $\begin{vmatrix} 1/\lambda_1 & 1/\lambda_2 \\ 0 & 1/\lambda_2 \end{vmatrix} = \frac{1}{\lambda_1\lambda_2}$, (2,2) $\begin{vmatrix} \frac{(1+\lambda_1+\lambda_2)}{\lambda_1\lambda_2} & 1/\lambda_2 \\ 1/\lambda_2 & 1/\lambda_2 \end{vmatrix} = \frac{(1+\lambda_1+\lambda_2)}{\lambda_1\lambda_2^2} - \frac{1}{\lambda_2^2}$

(2,3) $\begin{vmatrix} \frac{(1+\lambda_1+\lambda_2)}{\lambda_1\lambda_2} & 1/\lambda_1 \\ 1/\lambda_1 & 0 \end{vmatrix} = \frac{1}{\lambda_1^2}$

(3,1) $\begin{vmatrix} 1/\lambda_1 & 1/\lambda_2 \\ 1/\lambda_1 & 0 \end{vmatrix} = \frac{1}{\lambda_1\lambda_2}$, (3,2) $\begin{vmatrix} \frac{(1+\lambda_1+\lambda_2)}{\lambda_1\lambda_2} & 1/\lambda_2 \\ 1/\lambda_1 & 0 \end{vmatrix} = \frac{1}{\lambda_1\lambda_2}$, (3,3) $\begin{vmatrix} \frac{(1+\lambda_1+\lambda_2)}{\lambda_1\lambda_2} & 1/\lambda_1 \\ 1/\lambda_1 & 1/\lambda_1 \end{vmatrix} = \frac{(1+\lambda_1+\lambda_2)}{\lambda_1^2\lambda_2} - \frac{1}{\lambda_1}$

Cont'd next pg.

2c) cont'd. apply $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

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Then, $$[I_n(\psi)]^{-1} = \frac{1}{n}\lambda_1^2\lambda_2^2 \begin{pmatrix} 1/\lambda_1\lambda_2 & -1/\lambda_1\lambda_2 & 1/\lambda_1\lambda_2 \\ -1/\lambda_1\lambda_2 & \left[\frac{(1+\lambda_1+\lambda_2)}{\lambda_1\lambda_2^2} - \frac{1}{\lambda_2^2}\right] & -1/\lambda_1^2 \\ 1/\lambda_1\lambda_2 & 1/\lambda_1\lambda_2 & \left[\frac{(1+\lambda_1+\lambda_2)}{\lambda_1^2\lambda_2} - \frac{1}{\lambda_1^2}\right] \end{pmatrix}$$

3 Then, to compute the score stat,

$$SC_n = \dot{\ell}_n(\tilde{\psi})' [I_n(\tilde{\psi})]^{-1} \dot{\ell}_n(\tilde{\psi})$$

$$= \left(\frac{n^2\sum_i x_i y_i}{\sum_i x_i \sum y_i} - n,\ 0,\ 0\right) \cdot \frac{1}{n}\lambda_1^2\lambda_2^2 \begin{pmatrix} 1/\lambda_1\lambda_2 & -1/\lambda_1\lambda_2 & 1/\lambda_1\lambda_2 \\ -1/\lambda_1\lambda_2 & \left[\frac{(1+\lambda_1+\lambda_2)}{\lambda_1\lambda_2^2} - \frac{1}{\lambda_2^2}\right] & -1/\lambda_1^2 \\ 1/\lambda_1\lambda_2 & 1/\lambda_1\lambda_2 & \left[\frac{(1+\lambda_1+\lambda_2)}{\lambda_1^2\lambda_2} - \frac{1}{\lambda_1^2}\right] \end{pmatrix} \begin{pmatrix} \frac{n^2\sum_i x_i y_i}{\sum_i x_i \sum_i y_i} - n \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{1}{n}\lambda_1^{\cancel{2}}\lambda_2^{\cancel{2}} \underbrace{\left[\frac{n^2\sum_i x_i y_i}{\sum_i x_i \sum y_i} - n\right]^2}_{\text{Factor out } n \text{ \& square}} \left(1/\lambda_1\lambda_2\right) = \boxed{n\lambda_1\lambda_2\left[\frac{n\sum_i x_i y_i}{\sum_i x_i \sum y_i} - 1\right]^2}$$ ✓

Under $H_0: \psi = 0 \Rightarrow \boxed{SC_n \xrightarrow{d} \chi^2_{(1)}}$ ✓