2013 Theory I #2

2a) By construction, of T_{θ_0}, $P_{\theta_0}(Y \in A(\theta_0)) \geq 1-\alpha.$

But $\theta \in C(y) \iff y \in A(\theta)$ for each θ so that

$$P_{\theta_0}(\theta_0 \in C(Y)) = P_{\theta_0}(Y \in A(\theta_0)) \geq 1-\alpha$$

Def'n: $C(X)$ is said to be a confidence set with level of significance $1-\alpha$ if $\inf_{\theta \in \Theta} P(\theta \in C(X)) \geq 1-\alpha$

2b) $$p(\underset{\sim}{x}) = (2\pi\sigma^2)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2\right\}$$

$$= (2\pi\gamma\mu^2)^{-n/2} \exp\left\{-\frac{1}{2\gamma\mu^2}\sum(x_i-\mu)^2\right\}$$

$$= (2\pi\gamma\mu^2)^{-n/2} \exp\left\{-\frac{1}{2\gamma}\left[\sum\frac{x_i^2}{\mu^2} - 2\frac{n\bar{x}}{\mu} + n\right]\right\}$$

$$l_n(\mu,\gamma) = -\frac{n}{2}\log\gamma - \frac{n}{2}\log\mu^2 - \frac{1}{2\gamma}\left[\sum\frac{x_i^2}{\mu^2} - 2\frac{n\bar{x}}{\mu} + n\right] - \frac{n}{2}\log(2\pi)$$

$$\frac{\partial}{\partial\mu} l_n(\mu,\gamma_0) = -\frac{n}{\mu} + \frac{1}{\gamma_0\mu^3}\sum x_i^2 - \frac{n}{\gamma_0\mu^2}\bar{x} \overset{set}{=} 0$$

$$\Rightarrow -n\gamma_0\tilde{\mu}^2 + \sum x_i^2 - \tilde{\mu}n\bar{x} = 0$$

$$\Rightarrow n\gamma_0\tilde{\mu}^2 + n\bar{x}\tilde{\mu} - \sum x_i^2 = 0$$

So $\tilde{\mu}$ is the choice of the following that yields the larger $l_n(\tilde{\mu},\gamma_0)$ among $\left\{\frac{-n\bar{x} \pm \sqrt{n^2\bar{x}^2 + n\gamma_0\sum x_i^2}}{2n\gamma_0}\right\}$

Under unrestricted model

$$\ell_n(\mu,\gamma) \propto -\frac{n}{2}\log\gamma - \frac{n}{2}\log\mu^2 - \frac{1}{2\gamma\mu^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$\frac{\partial}{\partial\gamma}\ell_n(\mu,\gamma) = -\frac{n}{2\gamma} + \frac{1}{2\gamma^2\mu^2}\sum_{i=1}^{n}(x_i-\mu)^2 \overset{set}{=} 0$$

$$\Rightarrow -n\hat{\gamma} + \frac{1}{\hat{\mu}^2}\sum_{i=1}^{n}(x_i-\hat{\mu})^2 = 0 \quad \Rightarrow \hat{\gamma} = \frac{1}{\hat{\mu}^2}\cdot\frac{1}{n}\sum_{i=1}^{n}(x_i-\hat{\mu})^2$$

from before

then plugging into expression for $\hat{\mu}$ we have

$$\sum_{i=1}^{n}(x_i-\hat{\mu})^2 + n\bar{x}\hat{\mu} - \sum x_i^2 = 0$$

$$\Rightarrow \cancel{\sum x_i^2} - 2n\bar{x}\hat{\mu} + n\hat{\mu}^2 + n\bar{x}\hat{\mu} - \cancel{\sum x_i^2} = 0$$

$$\Rightarrow \hat{\mu}^2 - \bar{x}\hat{\mu} = 0 \quad \Rightarrow \hat{\mu}(\hat{\mu} - x) = 0$$

$$\Rightarrow \hat{\mu} = \bar{x} \quad \text{since } \mu \neq 0 \text{ by assumption}$$

We observe that

$$\frac{1}{2\hat{\gamma}\hat{\mu}^2}\sum(x_i-\hat{\mu})^2 = \frac{1}{2\frac{1}{n}\sum(x_i-\hat{\mu})^2}\sum(x_i-\hat{\mu})^2 = \frac{n}{2}$$

Then under H_0, $\boxed{\text{Use answer on page 4?}}$

$$\Lambda = 2\left\{\frac{n}{2}\log\left(\frac{\gamma_0}{\hat{\gamma}}\right) + \frac{n}{2}\log\left(\frac{\tilde{\mu}}{\bar{x}}\right) + \frac{1}{2\gamma_0\tilde{\mu}^2}\sum(x_i-\tilde{\mu})^2 - \frac{n}{2}\right\} \xrightarrow{L} \chi_1^2$$

Define $A(\gamma) = \{\gamma > 0 : \Lambda < \chi_1^2(1-\alpha)\}$

Then $C(X)$ is a $1-\alpha$ confidence set for $C(x) = \{\gamma : x \in A(\gamma)\}$ under H_0

approximate

2c) Suppose μ_0, μ_1 are known quantities. Let $\gamma_1 > \gamma_0$, then a UMP test of $H_0: \gamma = \gamma_0$ vs. $H_1: \gamma = \gamma_1$ is given by

$$\phi(x) = \begin{cases} 1, & p(x;\mu_1,\gamma_1) > k\, p(x;\mu_0,\gamma_0) \\ 0, & \text{else} \end{cases}$$

where k satisfies $\alpha = E_{p_0}[\phi(x)]$. Now

$$\frac{p(x;\mu_1,\gamma_1)}{p(x;\mu_0,\gamma_0)} = \frac{(2\pi\gamma_1\mu_1^2)^{-n/2}\exp\left\{-\frac{1}{2\gamma_1\mu_1^2}\sum(x_i-\mu_1)^2\right\}}{(2\pi\gamma_0\mu_0^2)^{-n/2}\exp\left\{-\frac{1}{2\gamma_0\mu_0^2}\sum(x_i-\mu_0)^2\right\}}$$

$$= \left(\frac{\gamma_0\mu_0}{\gamma_1\mu_1}\right)^{n/2}\exp\left\{\frac{1}{2\gamma_0\mu_0^2}\sum(x_i-\mu_0)^2 - \frac{1}{2\gamma_1\mu_1^2}\sum(x_i-\mu_1)^2\right\}$$

so we reject when

$$k < \exp\left\{\frac{1}{2\gamma_0\mu_0^2}\sum(x_i-\mu_0)^2 - \frac{1}{2\gamma_1\mu_1^2}\sum(x_i-\mu_1)^2\right\}$$

Thus the rejection region depends on H_1 so no UMP test exists for known μ_0, μ_1. Suppose now we have to estimate μ_0, μ_1. We saw in (2b) that $\hat{\mu}_i$ depends on choice of γ_i, $i=1,2$ so that the rejection region still depends on the alternative hypothesis and still no UMP test exists.

2b alternative)

$$\Lambda = \frac{(2\pi\gamma_0\tilde{\mu}^2)^{-n/2}\exp\left\{-\frac{1}{2\gamma_0\tilde{\mu}^2}\sum(y_i-\tilde{\mu})^2\right\}}{(2\pi\hat{\gamma}\hat{\mu}^2)^{-n/2}\exp\left\{-\frac{1}{2\hat{\gamma}\hat{\mu}^2}\sum(y_i-\hat{\mu})^2\right\}}$$

$$= \left(\frac{\hat{\sigma}^2}{\gamma_0\tilde{\mu}}\right)^{n/2}\exp\left\{-\frac{1}{2\gamma_0\tilde{\mu}^2}\sum(y_i-\tilde{\mu})^2+\frac{n}{2}\right\}$$

where $\hat{\sigma}^2 = \frac{1}{n}\sum(x_i-\bar{x})^2$. Then an α-level test may be constructed of the form

$$\phi(x) = \begin{cases} 1, & \Lambda < k_{\gamma_0} \\ 0, & \text{else} \end{cases}$$

where k_{γ_0} satisfies $\alpha = E_{H_0}[\phi(X)]$. Let $A(\gamma_0) = \{x \in \mathbb{R}^n : \Lambda > k_{\gamma_0}\}$.
Then $C(x) = \{\gamma : x \in A(\gamma)\}$ is a $1-\alpha$ C.I. for γ.

2d.i) Consider the test T_{θ_0} of $H_0: \mu = \mu_0$ vs. $H_1: \mu \neq \mu_0$.
Then under H_0, $\tilde{\sigma}^2 = \frac{1}{n}\sum(x_i-\mu_0)^2$, and in the unrestricted model
$\hat{\mu} = \bar{x}$, $\hat{\sigma}^2 = \frac{1}{n}\sum(x_i-\bar{x})^2$.

$\boxed{1:\ LRT_n}$ $\boxed{\ell_n(\mu,\sigma^2) \propto -\frac{n}{2}\log\sigma^2 - \underbrace{\frac{1}{2\sigma^2}\sum(x_i-\mu)^2}_{}}$

$= -\frac{n}{2}$ for both models

$$LRT_n \overset{(*)}{=} 2\left[\ell_n(\hat{\mu},\hat{\sigma}^2) - \ell_n(\mu_0,\tilde{\sigma}^2)\right]$$

$$= 2\left\{\frac{n}{2}\log\left(\frac{\tilde{\sigma}^2}{\hat{\sigma}^2}\right)\right\} = n\log\left\{\frac{\sum(x_i-\mu_0)^2}{\sum(x_i-\bar{x})^2}\right\} \xrightarrow{L} \chi^2_1$$

under H_0.

Use exact results instead?

Let $A(\mu_0) = \{x \in \mathbb{R}^n : LRT_n(x) < \chi^2_1(1-\alpha)\}$. Then $C(x) = \{\mu : x \in A(\mu)\}$ is an asymptotic $1-\alpha$ confidence set for μ.

$\boxed{\text{2: Wald test}}$

just do for X_1 next time (iid)

Have to do some calculations

$$\frac{\partial}{\partial \mu} \ell_n(\mu, \sigma^2) = \frac{1}{\sigma^2} \sum (X_i - \mu) = \frac{n}{\sigma^2}(\bar{X} - \mu)$$

$$\frac{\partial^2}{\partial \mu^2} \ell_n(\mu, \sigma^2) = -\frac{n}{\sigma^2}$$

$$\frac{\partial}{\partial \sigma^2} \ell_n(\mu, \sigma^2) = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum (X_i - \mu)^2$$

$$\frac{\partial^2}{\partial (\sigma^2)^2} \ell_n(\mu, \sigma^2) = \frac{n}{2\sigma^4} - \frac{1}{\sigma^6} \sum (X_i - \mu)^2$$

$$\frac{\partial^2}{\partial \mu \partial \sigma^2} \ell_n(\mu, \sigma^2) = -\frac{n}{\sigma^4}(\bar{X} - \mu)$$

$$E\left[\frac{\partial^2}{\partial (\sigma^2)^2} \ell_n(\mu, \sigma^2)\right] = \frac{n}{2\sigma^4} - \frac{1}{\sigma^6} n\sigma^2 = -\frac{n}{2\sigma^4}$$

$$E\left[\frac{\partial^2}{\partial \mu \partial \sigma^2} \ell_n(\mu, \sigma^2)\right] = 0$$

Then

$$I_n(\mu, \sigma^2) = \begin{bmatrix} \frac{n}{\sigma^2} & 0 \\ 0 & \frac{n}{2\sigma^4} \end{bmatrix}$$

Let $h(\mu, \sigma^2) = \mu - \mu_0$, then $\dot{h}(\mu, \sigma^2) = [1 \quad 0]$. Then for $\hat{\theta} = (\hat{\mu}, \hat{\sigma}^2)$

$$W_n(x) = h(\hat{\theta}) \left\{ \dot{h}(\hat{\theta}) [I_n(\hat{\theta})]^{-1} [\dot{h}(\hat{\theta})]' \right\}^{-1} h(\hat{\theta})$$

$$= \frac{(\bar{x} - \mu_0)^2}{\hat{\sigma}^2/n} \sim \frac{1}{n/(n-1)} F_{1,n-1}$$

under H_0. Let $A(\mu_0) = \{x \in \mathbb{R}^n : W_n(x) < \frac{n-1}{n} F_{1,n-1}\}$. Then $C(x) = \{\mu : x \in A(\mu)\}$ is a ~~exact~~ $(1-\alpha)$-level confidence set for μ.

$\boxed{3: R_n}$

Define

$$s_n(\theta) = \begin{bmatrix} \frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i - \mu) \\ -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^{n} (x_i - \mu)^2 \end{bmatrix}$$

Let $\tilde{\theta} = (\mu_0, \tilde{\sigma}^2)$. Then

$$R_n(x) = [s_n(\tilde{\theta})]' [I_n(\tilde{\theta})]^{-1} s_n(\tilde{\theta})$$

$$= \begin{bmatrix} \frac{n}{\tilde{\sigma}^2}(\bar{x} - \mu_0) & 0 \end{bmatrix} \begin{bmatrix} \frac{\tilde{\sigma}^2}{n} & 0 \\ 0 & \frac{2\tilde{\sigma}^4}{n} \end{bmatrix} \begin{bmatrix} \frac{n}{\tilde{\sigma}^2}(\bar{x} - \mu_0) \\ 0 \end{bmatrix}$$

$$= \left[\frac{n}{\tilde{\sigma}^2}(\bar{x} - \mu_0)\right]^2 \cdot \frac{\tilde{\sigma}^2}{n} = \frac{(\bar{x} - \mu_0)^2}{\tilde{\sigma}^2/n} \sim F_{1,n}$$

under H_0. Let $A(\mu_0) = \{x \in \mathbb{R}^n : R_n(x) < F_{1,n}^{(1-\alpha)}\}$. Then $C(x) = \{\mu : x \in A(\mu)\}$ is an $1-\alpha$ confidence set for μ.

2d.ii) $\boxed{\text{LRT:}}$ The confidence set is given by

$$c(x) = \left\{\mu:\ n\log\left(\frac{\sum(x_i-\mu)^2}{\sum(x_i-\bar{x})^2}\right) < \chi_1^2(1-\alpha)\right\}$$

$$= \left\{\mu:\ \frac{\sum(x_i-\mu)^2}{\sum(x_i-\bar{x})^2} < \exp\left\{\frac{1}{n}\chi_1^2(1-\alpha)\right\}\right\}$$

$$\boxed{\begin{aligned}\sum(x_i-\mu)^2 = \sum(x_i-\bar{x}+\bar{x}-\mu)^2 &= \sum\left\{(x_i-\bar{x})^2 + 2(x_i-\bar{x})(\bar{x}-\mu) + (\bar{x}-\mu)^2\right\}\\ &= \sum(x_i-\bar{x})^2 + n(\bar{x}-\mu)^2\end{aligned}}$$

$$= \left\{\mu:\ \frac{\sum(x_i-\bar{x})^2 + n(\bar{x}-\mu)^2}{\sum(x_i-\bar{x})^2} < \exp\left\{\frac{1}{n}\chi_1^2(1-\alpha)\right\}\right\}$$

$$= \left\{\mu:\ 1 + (\bar{x}-\mu)^2/\hat{\sigma}^2 < \exp\left\{\frac{1}{n}\chi_1^2(1-\alpha)\right\}\right\}$$

$$= \left\{\mu:\ (\bar{x}-\mu)^2 < \hat{\sigma}^2\left[e^{n^{-1}\chi_1^2(1-\alpha)} - 1\right]\right\}$$

$$= \left\{\mu:\ |\bar{x}-\mu| < \sqrt{\hat{\sigma}^2\left(e^{n^{-1}\chi_1^2(1-\alpha)} - 1\right)}\right\}$$

$$= \left\{\mu:\ \bar{x} - \sqrt{\hat{\sigma}^2\left(e^{n^{-1}\chi_1^2(1-\alpha)} - 1\right)} < \mu < \bar{x} + \sqrt{\hat{\sigma}^2\left(e^{n^{-1}\chi_1^2(1-\alpha)} - 1\right)}\right\}$$

$\boxed{\text{Wald:}}$

$$c(x) = \left\{\mu:\ \frac{(\bar{x}-\mu)^2}{\hat{\sigma}^2/n} < \frac{n-1}{n}F_{1,n-1}(1-\alpha)\right\}$$

$$= \left\{\mu:\ |\bar{x}-\mu| < \sqrt{(n-1)\hat{\sigma}^2 F_{1,n-1}(1-\alpha)}\right\}$$

$$= \left\{ \mu : \bar{x} - \sqrt{(n-1)\hat{\sigma}^2 F_{1,n-1}(1-\alpha)} < \mu < \bar{x} + \sqrt{(n-1)\hat{\sigma}^2 F_{1,n-1}(1-\alpha)} \right\}$$

$\boxed{\text{Score:}}$

$$c(x) = \left\{ \mu : \frac{n(\bar{x}-\mu_0)^2}{\frac{1}{n}\sum(x_i-\mu)^2} < F_{1,n}(1-\alpha) \right\}$$

$$= \left\{ \mu : \frac{n(\bar{x}-\mu)^2}{\frac{1}{n}\sum(x_i-\bar{x})^2 + (\bar{x}-\mu)^2} < F_{1,n}(1-\alpha) \right\}$$

$$= \left\{ \mu : \frac{n}{F_{1,n}(1-\alpha)} < \frac{\hat{\sigma}^2 + (\bar{x}-\mu)^2}{(\bar{x}-\mu)^2} \right\}$$

$$= \left\{ \mu : \frac{n}{F_{1,n}(1-\alpha)} < \frac{\hat{\sigma}^2}{(\bar{x}-\mu)^2} + 1 \right\} = \left\{ \mu : \frac{n - F_{1,n}(1-\alpha)}{F_{1,n}(1-\alpha)} < \frac{\hat{\sigma}^2}{(\bar{x}-\mu)^2} \right\}$$

→ after multiplying by nonnegative #

$$= \left\{ \mu : \frac{n - F_{1,n}(1-\alpha)}{F_{1,n}(1-\alpha)} < \frac{\hat{\sigma}^2}{(\bar{x}-\mu)^2} \right\} = \left\{ \mu : (\bar{x}-\mu)^2 < \hat{\sigma}^2 \frac{F_{1,n}(1-\alpha)}{n - F_{1,n}(1-\alpha)} \right\}$$

$$= \left\{ \mu : \bar{x} - \sqrt{\hat{\sigma}^2 \frac{F_{1,n}(1-\alpha)}{n - F_{1,n}(1-\alpha)}} < \mu < \bar{x} + \sqrt{\hat{\sigma}^2 \frac{F_{1,n}(1-\alpha)}{n - F_{1,n}(1-\alpha)}} \right\}$$

notice that we have to choose α so that $F_{1,n}(1-\alpha) < n$. For if not then $c(x) \supset \left\{ \mu : \frac{(\bar{x}-\mu_0)^2}{\hat{\sigma}^2 + (\bar{x}-\mu_0)^2} < 1 \right\} = \mathbb{R}$

2e) By CLT

$$\sqrt{n}\left(\begin{bmatrix}\frac{1}{n}\sum X_i \\ \frac{1}{n}\sum X_i^2\end{bmatrix} - \begin{bmatrix}\mu_1 \\ \mu_2\end{bmatrix}\right) \xrightarrow{L} N\Bigg(0, \overbrace{\begin{bmatrix}\sigma^2 & \mu_3 - \mu_1\mu_2 \\ \mu_3 - \mu_1\mu_2 & \mu_4 - \mu_2^2\end{bmatrix}}^{\Sigma}\Bigg)$$

Where we write $\mu_k = EX_1^k$, $k=2,3,4$. Note that all moments below the fourth exist by an application of Jensen's (or Hölder's) inequality. Now let $g(a,b) = \frac{a}{\sqrt{b-a^2}}$. Then

$$\frac{\partial}{\partial a} g(a,b) = \frac{(b-a^2)^{1/2} - a \cdot \frac{1}{2}(b-a^2)^{-1/2}(-2a)}{b-a^2}$$

$$= \frac{(b-a^2)^{1/2} + a^2(b-a^2)^{-1/2}}{b-a^2} = \frac{b}{(b-a^2)^{3/2}}$$

$$\frac{\partial}{\partial b} g(a,b) = -\frac{a}{2(b-a^2)^{3/2}}$$

Now

$$g(\mu_1, \mu_2) = \mu/\sigma$$

$$\dot{g}(\mu_1, \mu_2) = \begin{bmatrix}\frac{\mu_2}{(\mu_2-\mu_1^2)^{3/2}} & -\frac{\mu_1}{2(\mu_2-\mu_1^2)^{3/2}}\end{bmatrix} = \begin{bmatrix}\frac{\mu_2}{\sigma^3} & -\frac{\mu_1}{2\sigma^3}\end{bmatrix}$$

Then

$$\sqrt{n}\left(g\left(\frac{1}{n}\sum X_i, \frac{1}{n}\sum X_i^2\right) - \mu/\sigma\right) \xrightarrow{L} N(0, \nu)$$

Where

$$\tau = \dot{g}(\mu_1, \mu_2)\, \Sigma\, [\dot{g}(\mu_1, \mu_2)]'$$

$$= \begin{bmatrix} \frac{\mu_2}{\sigma^3} & -\frac{\mu_1}{2\sigma^3} \end{bmatrix} \begin{bmatrix} \sigma^2 & \mu_3 - \mu_1\mu_2 \\ \mu_3 - \mu_1\mu_2 & \mu_4 - \mu_2^2 \end{bmatrix} \begin{bmatrix} \frac{\mu_2}{\sigma^3} \\ -\frac{\mu_1}{2\sigma^3} \end{bmatrix}$$

$$= \left(\frac{\mu_2}{\sigma^3}\right)^2 \sigma^2 - 2\frac{\mu_2}{\sigma^3} \cdot \frac{\mu_1}{2\sigma^3}(\mu_3 - \mu_1\mu_2) + \left(\frac{\mu_1}{2\sigma^3}\right)^2 (\mu_4 - \mu_2^2)$$

$$= \frac{\mu_2^2}{\sigma^4} - \frac{\mu_1\mu_2}{\sigma^6}(\mu_3 - \mu_1\mu_2) + \frac{\mu_1^6}{2\sigma^6}(\mu_4 - \mu_2)$$

Write $Y_n = g\left(\frac{1}{n}\sum X_i, \frac{1}{n}\sum X_i^2\right)$. Then

$$c(y_n) = \left\{\theta : -\sqrt{\tau}\, z_{1-\alpha/2} < \sqrt{n}(y_n - \theta) < \sqrt{\tau}\, z_{1-\alpha/2}\right\}$$

$$= \left\{\theta : -y_n - \sqrt{\tau/n}\, z_{1-\alpha/2} < -\theta < -y_n + \sqrt{\tau/n}\, z_{1-\alpha/2}\right\}$$

$$= \left\{\theta : y_n - \sqrt{\tau/n}\, z_{1-\alpha/2} < \theta < y_n + \sqrt{\tau/n}\, z_{1-\alpha/2}\right\}$$

is an asymptotic $1-\alpha$ C.I. for $\theta = \mu/\sigma$