2017 Qualifying Exam Section 2 Question 1

February 21, 2019

(a) We can write

$$\begin{aligned} A &= X(X'V^{-1}X)^{-}X'V \\ &= V^{1/2}V^{-1/2}X(X'V^{-1/2}V^{-1/2}X)^{-}X'V^{-1/2}V^{-1/2} \\ &= V^{1/2}B(B'B)^{-}B'V^{-1/2} \end{aligned}$$

Note that the term involving the B's is an orthogonal projection operator, so it is symmetric and idempotent.

Now,

$$\begin{aligned} V^{-1}(I-A) &= V^{-1/2}V^{-1/2}(I - V^{1/2}B(B'B)^{-}B'V^{-1/2}) \\ &= V^{-1/2}(V^{-1/2} - B(B'B)^{-}B'V^{-1/2}) \\ &= V^{-1/2}(I - B(B'B)^{-}B')V^{-1/2} \end{aligned}$$

It can be easily shown that $V^{-1}(I-A) = (I-A)'V^{-1}$. Once we have this result,

$$\begin{aligned} (I-A)'V^{-1}(I-A) &= (I-A)'V^{-1}VV^{-1}(I-A) \\ &= V^{-1/2}(I - B(B'B)^{-}B')V^{-1/2}VV^{-1/2}(I - B(B'B)^{-})V^{-1/2} \\ &= V^{-1/2}(I - B(B'B)^{-}B')V^{-1/2} \\ &= V^{-1}(I-A) \end{aligned}$$

The third equality follows because $V^{-1/2}VV^{-1/2} = I$, and $(I - B(B'B)^{-}B)$ is an orthogonal projection operator (hence it is idempotent).

(b) Note that since V is positive definite, we can write $V = Q'Q$ where Q is nonsingular. Let P be the orthogonal projection operator onto $C(Q^{-1}X)$. Then

$$\begin{aligned} P &= Q^{-1}X(X'Q'^{-1}Q^{-1}X)^{-}X'Q'^{-1} \\ &= Q^{-1}X(X'(QQ')^{-1}X)^{-}X'Q'^{-1} \\ &= Q^{-1}X(X'V^{-1}X)^{-}X'Q'^{-1} \end{aligned}$$

By definition of a projection, we have

$$\begin{aligned} P(Q^{-1}X) = Q^{-1}X& \\ &\iff Q^{-1}X(X'V^{-1}X)^{-}X'Q'^{-1}Q^{-1}X = Q^{-1}X \\ &\iff Q^{-1}X(X'V^{-1}X)^{-}X'V^{-1}X = Q^{-1}X \\ &\iff Q^{-1}AX = Q^{-1}X \\ &\iff AX = X \end{aligned}$$

Again by definition of a projection, we must have that A is *a* projection operator onto $C(X)$. If we define the inner product between to vectors to be $x'V^{-1}y$, then A is the orthogonal projection operator onto $C(X)$.