2016 Day 2 Problem 2

$$u' = (\beta_1+\beta_2-\beta_3,\ \beta_2+\beta_3,\ -\beta_2-\beta_3,\ -\beta_1-\beta_2+\beta_3)$$

$$= \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = X\beta$$

$$X' = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$$

a) $\hat{\beta} = (X'X)^{-}X'Y \qquad \hat{\mu} = X\hat{\beta} = X(X'X)^{-}X'Y_{4\times 1}$

X is less than full rank $\rightarrow$ use X^*

$$X^* = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow$$ can't be delete rows b/c dimensions must be delete columns

$$X'X = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 4 & 0 \\ -2 & 0 & 4 \end{bmatrix} \rightarrow$$ proving to myself that LTFR is a problem

$$\left[\begin{array}{ccc|ccc} 2 & 2 & -2 & 1 & 0 & 0 \\ 2 & 4 & 0 & 0 & 1 & 0 \\ -2 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1/2 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1/2 & 0 \\ -1 & 0 & 2 & 0 & 0 & 1/2 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1/2 & 0 & 0 \\ 0 & 1 & 1 & -1/2 & 1/2 & 0 \\ 0 & 1 & 1 & 1/2 & 0 & 1/2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1/2 & 0 & 0 \\ 0 & 1 & 1 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & -1/2 & 1/2 \end{array}\right]$$

$$X^* = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \\ -1 & -1 \end{bmatrix} \qquad X^{*\prime}X^* = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}$$

$$(X^{*\prime}X^*)^{-1} = \frac{1}{8-4}\begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

$$X^*(X^{*\prime}X^*)^{-1}X^{*\prime} = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1/2 \\ 1/2 & -1/2 \\ -1/2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 0 & 0 & -1/2 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ -1/2 & 0 & 0 & 1/2 \end{bmatrix} = M$$

$$\hat{\mu}' = \left(\tfrac{1}{2}Y_1 - \tfrac{1}{2}Y_4,\ \tfrac{1}{2}Y_2 - \tfrac{1}{2}Y_3,\ -\tfrac{1}{2}Y_2 + \tfrac{1}{2}Y_3,\ -\tfrac{1}{2}Y_1 + \tfrac{1}{2}Y_4\right)$$

b) $\lambda'\beta = (0 \quad 1 \quad -1)\beta = \beta_2 - \beta_3$ is estimable

$\iff \lambda \in C(X')$ that is $\exists \rho : \rho' X = \lambda'$ aka $X'\rho = \lambda$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 \end{array}\right] \to$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right] \text{※} \quad 0 \neq -2 \Rightarrow \lambda \notin C(X')$$

$\therefore \beta_2 - \beta_3$ is not estimable

$\beta_2+\beta_3=0$
$-\beta_2=-\beta_3$

c) $E_0 = \{u : u' = (\beta_1 + 2\beta_2, 0, 0, -\beta_1 - 2\beta_2)\}$

$C(X)^\perp = N(X')$ in general

$$\begin{matrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -2 & 0 \end{matrix}$$

$N(X') = \{v : v_1 - v_4 = 0\}$

seems to not change anything.…

$E \cap E_0^\perp = \{u : u' = (\beta_1 + \beta_2 - \beta_3, \beta_2 + \beta_3, -\beta_2 - \beta_3, -\beta_1 - \beta_2 + \beta_3), -\beta_2 \neq \beta_3\}$

from looking at solutions:

$E_0 = \text{span}\{(1 \ 0 \ 0 \ -1)'\}$

$E = \text{span}\{(1 \ 0 \ 0 \ -1)', (0 \ 1 \ -1 \ 0)'\}$ ← orthogonal

$E \cap E_0^\perp = \text{span}\{(0 \ 1 \ -1 \ 0)'\}$

d) $$F = \frac{Y'(M-M_0)Y/(2-1)}{Y'(I-M)Y/(n-2)} \sim F\left(1, n-2, \frac{\|(M-M_0)X\beta\|^2}{2\sigma^2}\right)$$

$n=4$

$$M_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$M - M_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$(M-M_0)Y = (0 \quad Y_2 - Y_3 \quad -Y_2 + Y_3 \quad 0)'\frac{1}{2}$

$$Y'(M-M_0)Y = \frac{1}{2}Y_2^2 - Y_2Y_3 + \frac{1}{2}Y_3^2$$

$$I - M = \begin{bmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix}$$

$(I-M)Y = \frac{1}{2}(Y_1+Y_4, Y_2+Y_3, Y_2+Y_3, Y_1+Y_4)'$

$$Y'(I-M)Y = \frac{1}{2} \cancel{\sum_{i=1}^{4}\sum_{j=1}^{4} Y_iY_j}$$

$Y_1(Y_1+Y_4)$
$+Y_2(Y_2+Y_3)$
$+Y_3(Y_2+Y_3)$
$+Y_4(Y_1+Y_4)$
$=(Y_1+Y_4)^2+(Y_2+Y_3)^2$

$$F = \frac{\left(\frac{1}{2}Y_2^2 - Y_2Y_3 + \frac{1}{2}Y_3^2\right)(2)}{\frac{1}{2}\left[(Y_1+Y_4)^2 + (Y_2+Y_3)^2\right]} \sim F\Bigg(1, 2, \underbrace{\frac{\|(M-M_0)X\beta\|^2}{2\sigma^2}}_{=0 \text{ under } H_0}\Bigg)$$

e) $Y \sim N(\mu, \sigma^2 I)$

$$F = \frac{(\lambda'\beta)'(\lambda\cancel{X}(X'X)^{-}\cancel{X}\lambda)^{-}(\lambda'\beta)}{Y'(I-M)Y/(n-2)} \qquad \rho' = (0\ 1\ 0\ 0)$$

$\lambda' = (0\ 1\ 1)$ $\qquad \lambda'\hat{\beta} = \rho' X\hat{\beta} = \rho' MY \qquad n=4$

$$\left\{\beta : \frac{(\lambda'\hat{\beta}-\lambda'\beta)'(\lambda'(X'X)^{-}\lambda)^{-}(\lambda'\hat{\beta}-\lambda'\beta)}{Y'(I-M)Y/(n-2)} \leq F(1, n-2, 1-\alpha)\right\}$$

$$= \left\{\beta : \frac{(\rho'MY-\lambda'\beta)'(\lambda'(X'X)^{-}\lambda)^{-}(\rho'MY-\lambda'\beta)}{Y'(I-M)Y/(n-2)} \leq F(1, n-2, 1-\alpha)\right\}$$

$\rho'MY = \frac{1}{2}Y_2 - \frac{1}{2}Y_3 \qquad \lambda'\beta = \beta_2 + \beta_3$

$(\lambda'(X'X)^{-}\lambda)^{-} = 2$

95% CI for $\beta_2+\beta_3$ is

$$\left\{\beta : \frac{(\frac{1}{2}Y_2-\frac{1}{2}Y_3-\beta_2-\beta_3)'2(\frac{1}{2}Y_2-\frac{1}{2}Y_3-\beta_2-\beta_3)}{\frac{1}{2}[(Y_1+Y_4)^2+(Y_2+Y_3)^2]/2} \leq F(1, 4, 1-\alpha)\right\}$$