Q3 - Solution

(a) Let $\underset{\sim}{X} = (X_1, \dots, X_n)$. λ is known.

$$f(\underset{\sim}{x} \mid \lambda, \theta) = \lambda^{-n} \exp\left\{-\frac{1}{\lambda}\Sigma(X_i - \theta)\right\} \prod_{i=1}^{n} I(X_i \geq \theta)$$

$$= \lambda^{-n} \exp\left\{-\frac{1}{\lambda}\Sigma(X_i - \theta)\right\} I(X_{(1)} \geq \theta)$$

Let $\theta_2 > \theta_1$. To show that this family has the MLR property in some statistic $T(\underset{\sim}{x})$, we consider the ratio

$$r(\underset{\sim}{x}) = \frac{f(\underset{\sim}{x} \mid \theta_2, \lambda)}{f(\underset{\sim}{x} \mid \theta_1, \lambda)} = \frac{\lambda^{-n} \exp\left\{-\frac{1}{\lambda}\Sigma(X_i - \theta_2)\right\} I(X_{(1)} \geq \theta_2)}{\lambda^{-n} \exp\left\{-\frac{1}{\lambda}\Sigma(X_i - \theta_1)\right\} I(X_{(1)} \geq \theta_1)}$$

$$= \exp\left\{\frac{1}{\lambda}(\theta_2 - \theta_1)\right\} \frac{I(X_{(1)} \geq \theta_2)}{I(X_{(1)} \geq \theta_1)}$$

$$= \begin{cases} \exp\left\{\frac{1}{\lambda}(\theta_2 - \theta_1)\right\} > 0 & \text{if } X_{(1)} \geq \theta_2 \\ 0 & \text{if } \theta_1 \leq X_{(1)} < \theta_2 \\ \text{undefined} & \text{if } X_{(1)} < \theta_1 \end{cases}$$

Thus, we see that $r(\underset{\sim}{X})$ is a monotone non-decreasing function in $X_{(1)}$, and thus, this family of distributions has the MLR property in $X_{(1)}$.

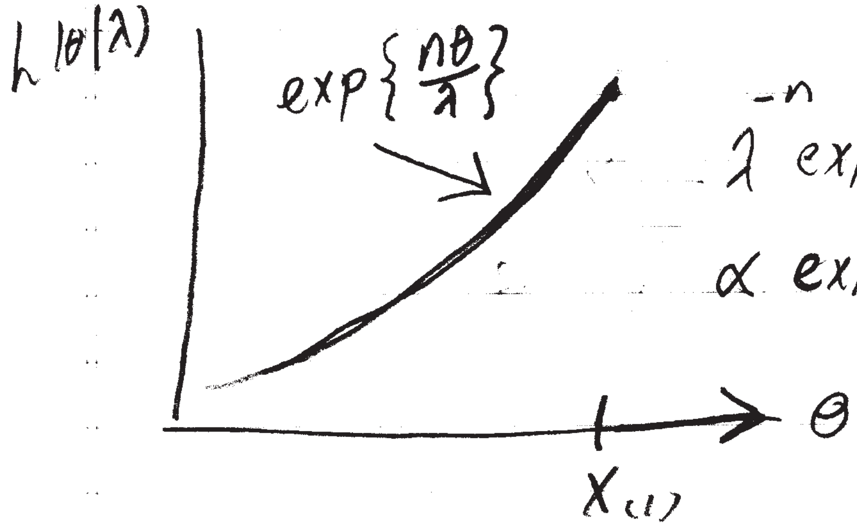
(b) The joint likelihood function of (θ, λ) is

$$L(\theta,\lambda) = \lambda^{-n} \exp\left\{-\frac{1}{\lambda}\sum_{i=1}^{n}(X_i-\theta)\right\} I(X_{(1)} \geq \theta)$$

$L(\theta,\lambda)$ is <u>not</u> differentiable in θ. However,
we can see that for <u>any</u> fixed $\lambda>0$, that

$L(\theta,\lambda)$ is a monotonic increasing function of θ,
whose monotonicity does not depend on λ.
$L(\theta,\lambda)=0$ for $\theta > X_{(1)}$ and

$$L(\theta,\lambda) = \lambda^{-n}\exp\left\{-\frac{1}{\lambda}\Sigma(X_i-\theta)\right\} \text{ for } \theta \leq X_{(1)}.$$



$$\lambda^{-n}\exp\left\{-\frac{1}{\lambda}\Sigma(X_i-\theta)\right\}$$
$$\propto \exp\left\{\frac{n\theta}{\lambda}\right\} \equiv L(\theta|\lambda)$$

Thus, since $L(\theta,\lambda)$ is monotone increasing in θ, <u>and</u> has global maximum at $\hat{\theta} = X_{(1)}$, for <u>any</u> fixed $\lambda>0$, it follows that the MLE of θ is $\hat{\theta} = X_{(1)}$.

To compute the MLE of λ, we note that that $L(\theta,\lambda)$ is defferentiable in λ.

$$\log L(\hat{\theta},\lambda) = -n\log\lambda - \frac{1}{\lambda}\Sigma(X_i-\hat{\theta})$$

$$= -n\log\lambda - \frac{1}{\lambda}\Sigma(X_i - X_{(1)})$$

$$\frac{\partial}{\partial\lambda}\log L(\hat{\theta},\lambda) = -\frac{n}{\lambda} + \frac{1}{\lambda^2}\Sigma(X_i - X_{(1)}) = 0$$

$$\Rightarrow \quad -n\lambda + \Sigma(X_i - X_{(1)}) = 0$$

$$\Rightarrow \quad \hat{\lambda} = \frac{1}{n}\Sigma(X_i - X_{(1)})$$

To see that this is a local maximum, note that

$$\frac{\partial^2 \log L(\hat{\theta},\lambda)}{\partial\lambda^2} = \frac{n}{\lambda^2} - \frac{2\,\Sigma(X_i - X_{(1)})}{\lambda^3}$$

evaluating at $\hat{\lambda}$, we get

$$\frac{n}{\hat{\lambda}^2} - \frac{2n\hat{\lambda}}{\hat{\lambda}^3} = \frac{n}{\hat{\lambda}^2} - \frac{2n}{\hat{\lambda}^2} = \frac{-n}{\hat{\lambda}^2} < 0$$

Thus $\hat{\lambda} = \frac{1}{n}\Sigma(X_i - X_{(1)})$ is the MLE of λ.

The joint MLE's of (θ,λ) are therefore $(X_{(1)}, \Sigma(X_i - X_{(1)}))$

(c)

Let us first find the density of $X_{(1)}$.

$$F_{X_{(1)}}(x) = P(X_{(1)} \geq x)$$

$$= P(X_1 \geq x, \ldots, X_n \geq x)$$

$$= \prod_{i=1}^{n} P(X_i \geq x) = \prod_{i=1}^{n} 1 - F_{X_i}(x)$$

$$= (1 - F(x))^n$$

$$f_{X_{(1)}}(x) = n(1 - F(x))^{n-1} f(x)$$

$$F(x) = \int_\theta^x \frac{1}{\lambda} \exp\{-\tfrac{1}{\lambda}(u-\theta)\}\, du$$

$$= -\exp\{-\tfrac{1}{\lambda}(u-\theta)\}\Big|_\theta^x$$

$$= 1 - \exp\{-\tfrac{1}{\lambda}(x-\theta)\}$$

$$f_{X_{(1)}}(x) = n\left[\exp\{-\tfrac{1}{\lambda}(x-\theta)\}\right]^{n-1}\left(\tfrac{1}{\lambda}\exp\{-\tfrac{1}{\lambda}(x-\theta)\}\right)$$

$$= \frac{n}{\lambda}\exp\{-\tfrac{n}{\lambda}(x-\theta)\}$$

Thus, we can recognize the density of $X_{(1)}$ as $E(\theta, \lambda/n)$.

Now to find the distribution of $\Sigma(X_i - X_{(1)})$ and to show independence, we consider the transformation

$$Y_1 = X_{(1)}$$

$$Y_i = (n-i+1)\left(X_{(i)} - X_{(i-1)}\right), \quad i=2,\ldots,n.$$

The joint density of the order statistics $\underset{\sim}{U} = (X_{(1)}, \ldots, X_{(n)})$ is given by

$$f_{\underset{\sim}{U}}(u_1, \ldots, u_n) = n! \prod_{i=1}^{n} f(u_i) \qquad u_1 < u_2 < \cdots < u_n$$

From the transformation above, we have

$$Y_1 = X_{(1)}, \quad Y_2 = (n-1)(X_{(2)} - X_{(1)})$$

$$Y_3 = (n-2)(X_{(3)} - X_{(2)}), \quad \ldots \quad Y_n = (X_{(n)} - X_{(n-1)})$$

This transformation is 1-1, and the inverse transformation is

$X_{(1)} = Y_1$

$X_{(2)} = Y_1 + \frac{y_2}{n-1}$

$X_{(3)} = Y_1 + \frac{y_2}{n-1} + \frac{y_3}{n-2}$

$\vdots$

$X_{(n)} = Y_1 + \frac{y_2}{n-1} + \frac{y_3}{n-2} + \frac{y_4}{n-3} + \cdots Y_n.$

The Jacobian of the transformation is

$$J = \frac{\partial(X_{(1)}, \ldots, X_{(n)})}{\partial(Y_1, \ldots, Y_n)} = \det\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & \frac{1}{n-1} & & \\ 1 & \frac{1}{n-1} & \frac{1}{n-2} & \\ & & \ddots & 1 \end{pmatrix}$$

$$= \frac{1}{(n-1)!}$$

So

$$f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) = \frac{n!}{(n-1)!}\left\{ f(y_1)\, f\left(y_1 + \frac{y_2}{n-1}\right) \cdot \times f\left(y_1 + \frac{y_2}{n-1} + \frac{y_3}{n-2}\right) \cdots \times f\left(y_1 + \sum_{i=2}^{n} \frac{y_i}{n-i+1}\right) \right\}$$

$$n \left\{ \frac{1}{\lambda} \exp\left\{-\frac{1}{\lambda}(y_1-\theta)\right\} \quad \frac{1}{\lambda}\exp\left\{-\frac{1}{\lambda}\left(y_1+\frac{y_2}{n-1}-\theta\right)\right\}\right.$$

$$\cdot \frac{1}{\lambda}\exp\left\{-\frac{1}{\lambda}\left(y_1+\frac{y_2}{n-1}+\frac{y_3}{n-2}-\theta\right)\right\}$$

$$\cdots \frac{1}{\lambda}\exp\left\{-\frac{1}{\lambda}\left(y_1+\sum_{i=2}^{n} y_i/(n-i+1)-\theta\right)\right\}$$

$$= \frac{n}{\lambda}\exp\left\{-\frac{n}{\lambda}(y_1-\theta)\right\}$$

$$\frac{1}{\lambda}\exp\left\{-\frac{1}{\lambda}y_2\right\} \frac{1}{\lambda}\exp\left\{-\frac{1}{\lambda}y_3\right\}$$

$$\cdots \frac{1}{\lambda}\exp\left\{-\frac{1}{\lambda}y_n\right\}, \quad y_1 \geq \theta,\ y_i > 0,\ i=2,\dots,n$$

Thus, we see that

(a) $Y_1, \dots, Y_n$ are independent

(b) $Y_1 = X_{(1)} \sim E(\theta, \lambda/n)$

(c) $Y_i \sim E(0, \lambda)$ for $i = 2, \dots, n$, iid

(d) $\sum_{i=2}^{n} Y_i \sim \text{gamma}(n-1, \frac{1}{\lambda})$

$$\sum_{i=2}^{n} Y_i = \sum_{i=2}^{n} X_{(i)} - (n-1) X_{(1)}$$

$$= \sum_{i=1}^{n} X_{(i)} - n X_{(1)}$$

$$= \sum_{i=1}^{n} X_i - n X_{(1)} \sim \text{gamma}(n-1, \tfrac{1}{\lambda})$$

Since $\sum_{i=2}^{n} Y_i \sim \text{gamma}(n-1, \frac{1}{\lambda})$.

d We wish to test $H_0: \theta \geq \theta_0$ against $H_1: \theta < \theta_0$ where λ is known. Since this family of distributions has the MLR property in $T(\underset{\sim}{X}) = X_{(1)}$, it follows that the UMP level α test is of the form

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & X_{(1)} < k \\ 0 & X_{(1)} > k \end{cases}$$

where

$$\alpha = P(X_{(1)} < k \,/\, \theta = \theta_0)$$

$$= \int_{\theta_0}^{k} \frac{n}{\lambda} \exp\left\{-\frac{n}{\lambda}(u-\theta_0)\right\} du$$

$$= -\exp\left\{-\frac{n}{\lambda}(u-\theta_0)\right\}\Big|_{\theta_0}^{k}$$

$$= 1 - \exp\left\{-\frac{n}{\lambda}(k-\theta_0)\right\}$$

$$\Rightarrow \quad k = \theta_0 - \frac{\lambda}{n}\log(1-\alpha).$$

The power function is given by

$$\pi(\theta) = P(X_{(1)} < k)$$

$$= 1 - \exp\left\{-\frac{n}{\lambda}(k-\theta)\right\}$$

$$= 1 - \exp\left\{-\frac{n}{\lambda}\left(\theta_0 - \frac{\lambda}{n}\log(1-\alpha) - \theta\right)\right\}$$

$$= 1 - (1-\alpha)\exp\left\{-\frac{n}{\lambda}(\theta_0-\theta)\right\}$$

(e) Sufficiency of $\left(X_{(1)}, \sum_{i=1}^{n} X_i - nX_{(1)}\right)$

follows from the factorization criterion

Thus, we wish to factor

$$f(\underset{\sim}{x} \mid \lambda,\theta) = h(\underset{\sim}{x})\, g(T(\underset{\sim}{x}) \mid \lambda,\theta)$$

Where $T(\underset{\sim}{x}) = \left(X_{(1)}, \sum_{i=1}^{n} X_i - nX_{(1)}\right)$, ($g$ depends $\underset{\sim}{x}$ only through $T(\underset{\sim}{x})$.

Clearly

$$f(\underset{\sim}{x} \mid \lambda,\theta) = \lambda^{-n}\exp\left\{-\frac{1}{\lambda}\sum(X_i-\theta)\right\} I(X_{(1)} \geq \theta)$$

$$= \lambda^{-n}\exp\left\{-\frac{1}{\lambda}\sum(X_i - X_{(1)} + X_{(1)} - \theta)\right\} I(X_{(1)} \geq \theta)$$

$$= \lambda^{-n} \exp\left\{ -\frac{1}{\lambda} \sum_{i=1}^{n} (X_i - X_{(1)}) - \frac{1}{\lambda} \sum_{i=1}^{n} (X_{(1)} - \theta) \right\} I(X_{(1)} \geq \theta)$$

Let $h(\underset{\sim}{X}) = 1$

$$g(T(\underset{\sim}{x}) | \lambda, \theta)$$

$$= \lambda^{-n} \exp\left\{ -\frac{1}{\lambda} \sum_{i=1}^{n} (X_i - X_{(1)}) - \frac{1}{\lambda} \sum_{i=1}^{n} (X_{(1)} - \theta) \right\} I(X_{(1)} \geq \theta)$$

We see that $g(T(\underset{\sim}{x}) | \lambda, \theta)$ depends on $\underset{\sim}{X}$ only through $T(\underset{\sim}{X})$, and thus

$$T(\underset{\sim}{X}) = \left(X_{(1)}, \sum_{i=1}^{n} X_i - n X_{(1)} \right) \equiv (T_1, T_2)$$

a joint sufficient statistic for (θ, λ).

To prove completeness, we must show that

$$E_{(\theta,\lambda)}\left[f(T_1, T_2) \right] = 0 \quad \Rightarrow \quad f(T_1, T_2) \equiv 0$$

with probability 1 for all (θ, λ).

Suppose that

$$E_{(\theta,\lambda)}\left[f(T_1, T_2) \right] = 0 \text{ for all } (\theta, \lambda).$$

Then if

$$g(t_1, \lambda) = E_\lambda [f(t_1, T_2)], \qquad (*)$$

We must show that for any fixed λ,

$$\int_\theta^\infty g(t_1, \lambda)\, e^{-\frac{n t_1}{\lambda}}\, dt_1 = 0 \quad \text{for all } \theta.$$

Now split $g(t_1, \lambda)$ into it's positive and negative parts: $g(t_1, \lambda) = g^+(t_1, \lambda) - g^-(t_1, \lambda)$

Where $g^+ = \max(g, 0)$, $g^- = -\min(g, 0)$

So that both g^+, g^- are non-negative.

Thus
$$\int_\theta^\infty g(t_1, \lambda)\, e^{-\frac{n t_1}{\lambda}}\, dt_1 = 0$$

$$\Leftrightarrow \int_\theta^\infty g^+(t_1, \lambda)\, e^{-\frac{n t_1}{\lambda}}\, dt_1 = \int_\theta^\infty g^-(t_1, \lambda)\, e^{-\frac{n t_1}{\lambda}}\, dt_1$$

$\Leftrightarrow$ $g(t_1, \lambda) = 0$ except on a set N_λ of t_1 values which has Lebesgue measure 0 and which may depend on λ. Then, by Fubini's theorem, for almost all t_1, we have

$$g(t_1, \lambda) = 0 \quad \text{a.e. in } \lambda.$$

Since the densities of $T_2 = \sum_{i=1}^{n} X_i - n X_{(1)}$ constitute an exponential family, $g(t_1, \lambda)$ by (*) is a continuous function of λ for any fixed t_1. It follows that for almost all t_1, $g(t_1, \lambda) = 0$ for <u>all</u> λ. Applying completeness of T_2 to (*), we see that for almost all t_1, $f(t_1, t_2) = 0$ a.e. in t_2.

Thus, finally, $f(t_1, t_2) = 0$ a.e. w.r.t. Lebesgue measure in the (t_1, t_2) plane.

$(X_{(1)}, \sum X_i - n X_{(1)})$ is indeed a joint minimal sufficient statistic since it is complete, a complete sufficient statistic is <u>always</u> minimal.

(f) We want to derive a <u>joint</u> $(1-\alpha)\times 100\%$ confidence region for (θ, λ)

Now since $(X_{(1)}, \sum X_i - nX_{(1)})$ are independent, we have

$$P(a_1 < X_{(1)} < b_1,\ a_2 < \sum X_i - nX_{(1)} < b_2)$$
$$= P(a_1 < X_{(1)} < b_1)\, P(a_2 < \sum X_i - nX_{(1)} < b_2)$$
$$= (1-2\delta)(1-2\varepsilon),$$

where $a_1 = \delta$ percentile of $E(\theta, \lambda/n)$

$b_1 = 1-\delta$ percentile of $E(\theta, \lambda/n)$

$a_2 = \varepsilon$ percentile of $\text{gamma}(n-1, \frac{1}{\lambda})$

$b_2 = 1-\varepsilon$ percentile of $\text{gamma}(n-1, \frac{1}{\lambda})$

Thus, if we want a $(1-\alpha)\times 100\%$ confidence region for (θ, λ), we need to choose (δ, ε) so that $(1-2\delta)(1-2\varepsilon) = 1-\alpha$.

(g) We want the joint asymptotic distribution of $(X_{(1)}, \Sigma X_i - nX_{(1)})$. Since $X_{(1)}$ and $\Sigma X_i - nX_{(1)}$ are independent for all n, they are also asymptotically independent.

To find the asymptotic distribution of $X_{(1)}$, we look for sequences $\{a_n\}$, $\{b_n\}$, $b_n > 0$ such that

$$P\left(\frac{X_{(1)} - a_n}{b_n} \le t\right) \to \text{a non-degenerate cdf}$$

as $n \to \infty$.

$$P\left(\frac{X_{(1)} - a_n}{b_n} \le t\right) = P(X_{(1)} \le b_n t + a_n)$$

$$= F_{X_{(1)}}[b_n t + a_n]$$

$X_{(1)} \sim E(\theta, \lambda/n)$, so that

$$F_{X_{(1)}}(x) = \int_\theta^x \frac{n}{\lambda} e^{-\frac{n}{\lambda}(u-\theta)}\, du$$

$$= -e^{-\frac{n}{\lambda}(u-\theta)}\Big|_\theta^x = 1 - e^{-\frac{n}{\lambda}(x-\theta)}$$

$F_{X_{(1)}}[b_n t + a_n]$

$$= 1 - e^{-\frac{n}{\lambda}[b_n t + a_n - \theta]}$$

Let $b_n = \frac{\lambda}{n}$, $a_n = \theta$

then

$$F_{X_{(1)}}[b_n t + a_n] = 1 - e^{-\frac{n}{\lambda}[\frac{\lambda}{n} t + \theta - \theta]}$$

$$= 1 - e^{-t} \to 1 - e^{-t} \quad \text{as } n \to \infty$$

which is the cdf of a standard exponential distribution.

thus

$$Z_{1,n} = \frac{X_{(1)} - \theta}{\lambda/n} \to \text{exponential } (1)$$

The asymptotic density of $Z_{1,n}$ is $f(z_1) = e^{-z_1}$

for $z_1 > 0$.

Now we want the asymptotic distribution of

$$\sum X_i - nX_{(1)} \sim \text{gamma}(n-1, \tfrac{1}{\lambda})$$

$$\sum X_i - nX_{(1)} = \sum_{i=1}^{n-1} Y_i \text{ , where } Y_i \sim E(0, \lambda),$$

iid.

Thus, by the CLT

$$\frac{\sum X_i - nX_{(1)}}{n-1} = \frac{1}{n-1}\sum_{i=1}^{n-1} Y_i \rightarrow N\left(\lambda, \frac{\lambda^2}{n-1}\right)$$

Thus

$$Z_{2,n} = \sqrt{n-1}\,\frac{\left[\left(\sum X_i - nX_{(1)}\right) - \lambda\right]}{\lambda} \xrightarrow{d} N(0,1)$$

Thus the asymptotic joint density of $(Z_{1,n}, Z_{2,n})$ is

$$f(z_1, z_2) = \begin{cases} e^{-z_1}\,(2\pi)^{-1/2} e^{-z_2^2/2} & z_1 > 0,\ -\infty < z_2 < \infty \\ 0 & \text{otherwise} \end{cases}$$