BASIC PHD WRITTEN EXAMINATION

THEORY, SECTION 1

(9:00 AM–1:00 PM, July 27, 2021)

INSTRUCTIONS:

(a) This is a **CLOSED-BOOK** examination.

(b) The time limit for this examination is four hours.

(c) Answer both questions that follow.

(d) Put the answers to different questions on separate sets of paper.

(e) Put your exam code, **NOT YOUR NAME**, on each page. The same code is used for Section 1 and Section 2 of the PhD Theory Exam. Please keep the code confidential and do not share this information with any students or faculty. Sharing your code with either students or faculty is viewed as a violation of the UNC honor code.

(f) Return the examination with a signed statement of the UNC honor pledge, separately from your answers. The pledge statement is given on the last page of the exam handout.

(g) In the questions to follow, you are required to answer only what is asked, and not to tell all you know about the topics involved.

1. (25 points) Assume $T_i, i = 1, ..., n$ are i.i.d., from $Exp(1/\lambda)$, where $\lambda = E[T_i]$ is an unknown parameter and $\lambda > 0$. Instead of observing T_i, we observe whether T_i is within an interval $[0, U_i]$, i.e., $R_i = I(T_i \in [0, U_i])$, where U_i's are i.i.d from the uniform distribution in $[0, 1]$. Thus, the observed data consist of $(R_i, U_i), i = 1, ..., n$.

   (a) (5 points) Write down the observed likelihood function for λ and the score equation for λ.

   (b) (5 points) Give the asymptotic distribution of the maximum likelihood estimator for λ and its asymptotic variance. You can leave integrals in the final solution.

   (c) (6 points) Treat T_i as the missing data. Write down the EM algorithm for calculating the maximum likelihood estimator. You need to give explicit calculation in both E- and M-steps.

   (d) A naive approach to estimate λ is based on the reduced data: we discard the observations whose T_i is larger than U_i, but impute T_i using $U_i/2$, the mid-point of the interval, for those whose T_i is within $[0, U_i]$. The resulting pseudo-likelihood is given as

$$\prod_{i:T_i\in[0,U_i]} \left(\frac{1}{\lambda}\exp\left\{-\frac{U_i}{2\lambda}\right\}\right).$$

      (i) (4 points) Derive the estimator maximizing the pseudo-likelihood function, denoted by $\widetilde{\lambda}$, and find the explicit form of its asymptotic limit.

      (ii) (5 points) Derive the asymptotic distribution of $\widetilde{\lambda}$ after a proper normalization. You may leave expectations in the final expression.

Points: (a) 5; (b) 5; (c) 6; (d) (i) 4; (ii) 5.

2. (25 points) This problem consists of three parts:

   (a) Suppose $X_1, \ldots, X_n$ are i.i.d samples from $N(0, \sigma^2)$, where σ^2 is unknown.

      (i) (4 points) Consider the prior distribution of $\tau = 1/(2\sigma^2)$ to be a Gamma distribution, which has a density function of

$$f(\tau) = \frac{1}{\Gamma(a)b^a}\tau^{a-1}e^{-\tau/b} \text{ with } E(\tau) = ab \text{ and } Var(\tau) = ab^2.$$

      Under the loss function $L(\sigma^2, d) = (d - \sigma^2)^2/\sigma^4$, find the Bayes rule of σ^2.

      (ii) (4 points) Let $Y = n^{-1}\sum_{i=1}^n X_i^2$. Suppose we focus on the rule set $\mathcal{D} = \{\alpha Y + \beta\}$, where α and β are two real numbers. Prove that, under the loss function $L(\sigma^2, d) = (d - \sigma^2)^2$, any rule $Y + \beta$ with $\beta \neq 0$ is inadmissible in $\mathcal{D}$.

   (b) Suppose $X_1, \ldots, X_n$ are i.i.d samples from $N(\mu, 1)$ and the prior distribution of μ is $N(0, 1)$.

      (i) (4 points) Consider the hypothesis test of $H_0 : 0 \leq \mu \leq c$ versus $H_a : \mu > c$, where c is a given positive constant. Compute the posterior probability $\widetilde{p}$ of H_0.

      (ii) (5 points) Note that $\widetilde{p}$ is a function of $X_1, \ldots, X_n$. In the case of $\mu = c$, derive the limiting distribution of $\widetilde{p}$ as $n \to \infty$.

   (c) Suppose $X_1, \ldots, X_n$ are i.i.d samples from $N(\mu, 1)$ and the prior distribution of μ is $P(\mu = 0) = \lambda$ and $P(\mu = 1) = 1 - \lambda$ with $0 < \lambda < 1$. Consider the hypothesis test of $H_0 : \mu = 0$ versus $H_a : \mu = 1$ with the 0–1 loss function.

      (i) (4 points) Derive the Bayes rule for this test.

      (ii) (4 points) Derive the minimax rule for this test.

Points: (a) (i) 4; (ii) 4; (b) (i) 4; (ii) 5; (c) (i) 4; (ii) 4.