2012 Theory I #1

1a) $$EU = E\left(E\left[I_{N>0}\sum_{i=1}^{N} X_i \,\middle|\, N\right]\right) = \sum_{k=1}^{\infty} k\lambda\, P(N=k)$$

$$= \sum_{k=1}^{\infty} k\lambda \frac{\mu^k e^{-\mu}}{k!} = \lambda \sum_{k=1}^{\infty} \frac{\mu^k e^{-\mu}}{(k-1)!} = \mu\lambda \sum_{k=1}^{\infty} \frac{\mu^{k-1} e^{-\mu}}{(k-1)!} = \mu\lambda$$

$$Var[U] = E\left(Var\left[I_{N>0}\sum_{i=1}^{N} X_i \,\middle|\, N\right]\right) + Var\left(E\left[I_{N>0}\sum_{i=1}^{N} X_i \,\middle|\, N\right]\right)$$

$$= \sum_{k=1}^{\infty} k\lambda \frac{\mu^k e^{-\mu}}{k!} + Var[N\lambda] = \mu\lambda + \mu\lambda^2 = \mu\lambda(1+\lambda)$$

1b.i) $N_k \sim pois(k)$, $X_1^{(k)}, X_2^{(k)}, \ldots \overset{iid}{\sim} pois(h/k)$

$$U_k \equiv I_{N_k>0}\sum_{i=1}^{N} X_i^{(k)}$$

$$EU_k = E\left(E\left[I_{N_k>0}\sum_{i=1}^{N} X_i^{(k)} \,\middle|\, N_k\right]\right) = E\left[N_k \cdot \frac{h}{k}\right] = h$$

$$Var[U_k] = E\left(Var\left[I_{N_k>0}\sum_{i=1}^{N} X_i^{(k)} \,\middle|\, N\right]\right) + Var\left(E\left[I_{N_k>0}\sum_{i=1}^{N} X_i^{(k)} \,\middle|\, N\right]\right)$$

$$= E\left[N_k \frac{h}{k}\right] + Var\left[N_k \frac{h}{k}\right] = h + \frac{h^2}{k^2}k = h + \frac{h^2}{k} = \frac{h(k+h)}{k}$$

Thus,

$$\lim_{k\to\infty} EU_k = h, \quad \lim_{k\to\infty} Var[U_k] = h$$

1b.ii) $P(X_i^{(k)} \neq D_i^{(k)}) = 1 - P(X_i^{(k)} = D_i^{(k)}) = 1 - [P(X_i^{(k)} = 0) + P(X_i^{(k)} = 1)]$

$$= 1 - \frac{\lambda_k^0 e^{-\lambda_k}}{0!} - \frac{\lambda_k^1 e^{-\lambda_k}}{1!} = 1 - e^{-\lambda_k} - \lambda_k e^{-\lambda_k}$$

$$\boxed{e^{-\lambda_k} = \sum_{p=0}^{\infty} \frac{(-\lambda_k)^p}{p!} = 1 - \lambda_k + \frac{\lambda_k^2}{2} - \frac{\lambda_k^3}{6} + \dots}$$

$$= 1 - [1 - \lambda_k + o(\lambda_k)] - \lambda_k[1 - \lambda_k + o(\lambda_k)] = \lambda_k^2 + o(\lambda_k) + o(\lambda_k^2)$$

$$= \lambda_k^2 + o(\lambda_k) = \lambda_k^2 (1 + o(\lambda_k^{-1}))$$

This term doesn't match

Proof by contr.

1b.iii) We notice that if it were possible, LHS > RHS $\Rightarrow$ LHS $= I(U \neq T) = 1$. Now $\{U \neq T\} = \{N > 0\} \cap \{\bigcup_{i=1}^{N} X_i \neq D_i\}$. Let Z be the r.v. denoting the value of the RHS. But for any $\omega \in \{N > 0\} \cap \{\bigcup_{i=1}^{N} X_i \neq D_i\}$, $Z(\omega) \geq 1$ so a contradiction is reached.

Next,

$$P(|U-T| > \varepsilon) \leq P(U \neq T) \leq P(Z \geq 1) = 1 - P(Z = 0)$$

* Use iterated expectation; cleaner

Now,

$$P(Z=0) = P(N=0) + E\left[P(Z=0 \mid N=n\geq 1)\right]$$

$$\boxed{\begin{aligned} P(Z=0 \mid N=n\geq 1) &= P(X_1 = D_1, \ldots, X_n = D_n \mid N=n\geq 1) \\ &= \left[P(X_1 = D_1)\right]^N = \left[1 - P(X_1 \neq D_1)\right]^N = \left[1 - \lambda_k^2 + o(\lambda_k)\right]^N \end{aligned}}$$

$$= P(N=0) + E\left[\left(1 - \lambda_k^2 + o(\lambda_k)\right)^N \mid N \geq 1\right]$$

$$= e^{-\mu_K} + \sum_{x=1}^{\infty}\left[1 - \lambda_k^2 + o(\lambda_k)\right]^x \frac{\mu_K^x e^{-\mu_K}}{x!}$$

$$= \sum_{x=0}^{\infty}\left[1 - \lambda_k^2 + o(\lambda_k)\right]^x \frac{\mu_k^x e^{-\mu_K}}{x!} = e^{-\mu_K} \exp\left\{\left[1 - \lambda_k^2 + o(\lambda_k)\right]\mu_K\right\}$$

$$= \exp\left\{-\frac{h^2}{k} + o(1)\right\} \to 1 \text{ as } k \to \infty$$

Thus

$$P(|U - T| > \varepsilon) \leq 1 - P(Z=0) \to 0 \text{ as } k \to \infty$$

$$I\left[T - \sum_{i=1}^{k} D_i > 0\right] = I\left[\bigcup_{i \in (k,N)} \{X_i \neq 0\}\right]$$

$$\Rightarrow P\left(T - \sum D_i > 0\right) \leq E[N-k]\,P(\lambda_i \neq 0)$$

1b.iv) $E\left[T - \sum_{i=1}^{k} D_i\right] = 0$

$$P\left(\left|T - \sum_{i=1}^{k} D_i\right| > \varepsilon\right) \leq \frac{Var\left[T - \sum_{i=1}^{k} D_i\right]}{\varepsilon^2}$$

$$= \frac{1}{\varepsilon^2} Var\left[\sum_{i=1}^{|N-k|} D_i\right] \leq \frac{1}{\varepsilon^2} Var\left[\sum_{i=1}^{\max(N,k)} D_i\right]$$

$$= \frac{1}{\varepsilon^2}\left\{ E\left(Var\left[\sum_{i=1}^{|N-k|} D_i \,\middle|\, N \right]\right) + Var\left(E\left[\sum_{i=1}^{|N-k|} D_i \,\middle|\, N \right]\right)\right\}$$

$$= \frac{1}{\varepsilon^2}\left\{ E\left(|N-k|\, Var[D_1] \right) + Var\left(|N-k|\, ED_1 \right)\right\}$$

$$\boxed{\begin{aligned} &ED_1 = P(X_1 = 1) = \lambda_k e^{-\lambda_k}, \quad Var[D_1] = ED_1^2 - (ED_1)^2 \\ &= ED_1 - (ED_1)^2 = ED_1(1 - ED_1) = \lambda_k e^{-\lambda_k}(1 - \lambda_k e^{-\lambda_k}) \end{aligned}}$$

$$= \frac{1}{\varepsilon^2}\left\{ Var[D_1]\, E\big[|N-k|\big] + (ED_1)^2 Var\big[|N-k|\big] \right\}$$

$$\le \frac{1}{\varepsilon^2}\left\{ Var[D_1]\, E\big[|N| + |k|\big] + (ED_1)^2 \left(E\big[(N-k)^2\big] - \left(E\big[|N-k|\big]\right)^2 \right) \right\}$$

$$\le \frac{1}{\varepsilon^2}\left\{ Var[D_1]\, E\big[|N| + |k|\big] + (ED_1)^2 E\big[(N-k)^2\big] \right\}$$

$$= \frac{1}{\varepsilon^2}\left\{ 2Var[D_1]\, EN + (ED_1)^2 \left(EN^2 - (EN)^2 \right) \right\}$$

$$= \frac{1}{\varepsilon^2}\left\{ 2\lambda_k e^{-\lambda_k}(1 - \lambda_k e^{-\lambda_k})\mu_k + \lambda_k^2 e^{-2\lambda_k}\mu_k \right\}$$ ← Treated D_i as Bernoulli

$$= \frac{1}{\varepsilon^2}\left\{ 2h e^{-\lambda_k}(1 - \lambda_k e^{-\lambda_k}) + \frac{h^2}{k} e^{-\lambda_k} \right\} \longrightarrow 0$$

1b.v) $D_i \sim bern(\lambda_k e^{-\lambda_k})$ so that $Y_k \equiv \sum_{i=1}^{k} D_i \sim bin(k, \lambda_k e^{-\lambda_k})$

Then

$$M_{Y_k}(t) = \left[1 - \lambda_k e^{-\lambda_k} + \lambda_k e^{-\lambda_k} e^t\right]^k$$

$$= \left[1 + \frac{1}{k} h\left(-e^{-h/k} + e^{-h/k} e^t\right)\right]^k$$

$$\rightarrow \exp\left\{ \lim_{k \to \infty} h\left(-e^{-h/k} + e^{-h/k} e^t\right)\right\}$$

$$= \exp\left\{ h(e^t - 1)\right\}$$

1c.i) $EU = E\left(E[U|N]\right) = E[Nk] = h$

$$Var[U] = E\left(Var[U|N]\right) + Var\left(E[U|N]\right)$$

$$= E[Nk] + Var[Nk] = h + k^2 \cdot \frac{h}{k} = h + hk = h(1+k)$$

Then take $h \to \infty$

1c.ii) $P(U=0) = E\left[P(U=0|N)\right] = E\left[\left(P(X_1=0)\right)^N\right]$

$$= E\left[e^{-N\lambda_k}\right] = \sum_{x=0}^{\infty} e^{-x\lambda_k} \cdot \frac{\mu_k^x e^{-\mu_k}}{x!} = \sum_{x=0}^{\infty} \frac{(e^{-\lambda_k}\mu_k)^x e^{-\mu_k}}{x!}$$

$$= e^{-\mu_k} \exp\{e^{-\lambda_k}\mu_k\} = e^{-h/k} \exp\left\{e^{-k} \cdot \frac{h}{k}\right\} \rightarrow 1$$