Hypergeometric

Mingwei Fei

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1 Hypergeometric vs. Binomial

Hypergeometric is drawing objects without replacement, while binomial is drawing objects with replacement. So the binomial distribution has the probability p for each drawn, while the hypergeometric probability mass function:

$$p_X(k) = p(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

Here we already know how many K objects totally, and we would like to draw k out of K. The difference of replacement vs. no replacement is the probability. With replacement, we could get $p = \frac{K}{N}$ for each time, while without replacement, we get the probability by using the designed drawn divided by all possible drawns. That's how we derive the concepts of nuisance parameters as well.

Hypergeometric distribution is derived from binomial, $\binom{a}{b}$ is a binomial coefficients, which means two outcomes, yes or no.

Hypergeometric distribution focuses on the range of k, and the sum of all probabilities of p(k) is 1, so we need to know all possible values for k. $0 \le k \le min(n, K)$.

1.1 Contingency Table- Likelihood function is the Key

Consider a $I \times J$ contingency table of cell counts, where each cell count is denoted by $n_{ij}, i = 1, ..I, j = 1, ..J$, and thus n_{ij} denotes the cell count of ith row and jth column, and $n_{ij} \sim Poisson(\mu_{ij})$ and independent. Further, let $n = \sum_{j=1}^{J}\sum_{i=1}^{I} n_{ij}$ denote the grand total.

(a) Derive the joint distribution of $(n_{11}, n_{12}, ...n_{ij})$ conditional on grand total n. By poisson distribution of each cell counts

$$n = \sum_{i=1}^{I}\sum_{j=1}^{J} n_{ij} \sim \frac{exp(-\mu)\mu^n}{n!}, \qquad \mu = \sum_{i=1}^{I}\sum_{j=1}^{J} \mu_{ij}$$

$$\begin{aligned}
p(n_{11},..n_{ij}|n) &= \frac{\prod_{i=1}^{I}\prod_{j=1}^{J}\frac{exp(-\mu_{ij})\mu_{ij}{}^{n_{ij}}}{n_{ij}!}}{\frac{exp(-\mu)\mu^n}{n!}} \\
&= \binom{n}{n_{11}n_{12}...n_{ij}} \frac{\prod_{i=1}^{I}\prod_{j=1}^{J}\mu_{ij}{}^{n_{ij}}}{\mu^n} \\
&= \binom{n}{n_{11}n_{12}...n_{ij}} \prod_{i=1}^{I}\prod_{j=1}^{J}\left(\frac{\mu_{ij}}{\mu}\right)^{n_{ij}}
\end{aligned}$$

The joint distribution is Multinomial $(n; \pi_{11}, \pi_{12}, ..\pi_{IJ})$, where $\pi_{ij} = \frac{\mu_{ij}}{\sum_{i=1}^{I}\sum_{j=1}^{J}\mu_{ij}}$

(b) Suppose all of the rows margins are assumed fixed. Derive the joint distribution of $(n_{11}, n_{12}, ...n_{ij})$.

$$n_{i+} = \sum_{j=1}^{J} n_{ij}$$

$$n_{i+} \sim Poisson(\sum_{j=1}^{J}\mu_{ij})$$

$$\begin{aligned}
p(n_{11},..n_{ij}|n_{i+}) &= \prod_{i=1}^{I}\prod_{j=1}^{J}\frac{exp(-\mu_{ij})\mu_{ij}{}^{n_{ij}}}{n_{ij}!} \Big/ \prod_{i=1}^{I}\frac{exp(-\mu_i)\mu_i^{n_{i+}}}{n_{i+}!} \\
&= \prod_{i=1}^{I}\binom{n_{i+}}{n_{ij}} \prod_{i=1}^{I}\prod_{j=1}^{J}\left(\frac{\mu_{ij}}{\sum_{j=1}^{J}\mu_{ij}}\right)^{n_{ij}}
\end{aligned}$$

(c) Suppose all of the columns margins are assumed fixed. Derive the joint distribution of $(n_{11}, n_{12}, ...n_{ij})$.

$$n_{+j} = \sum_{i=1}^{I} n_{ij}$$

$$n_{+j} \sim Poisson(\sum_{i=1}^{I}\mu_{ij})$$

$$\begin{aligned}
p(n_{11},..n_{ij}|n_{+j}) &= \prod_{i=1}^{I}\prod_{j=1}^{J}\frac{exp(-\mu_{ij})\mu_{ij}{}^{n_{ij}}}{n_{ij}!} \Big/ \prod_{j=1}^{J}\frac{exp(-\mu_i)\mu_i^{n_{+j}}}{n_{+j}!} \\
&= \prod_{j=1}^{J}\binom{n_{+j}}{n_{ij}} \prod_{i=1}^{I}\prod_{j=1}^{J}\left(\frac{\mu_{ij}}{\sum_{i=1}^{I}\mu_{ij}}\right)^{n_{ij}}
\end{aligned}$$

1.2 Non-central Hypergeometric distribution

Suppose that $I = 2$ and $J = 2$, and both the rows margins and column margins are fixed. Derive the joint distribution of $(n_{11}|n_{1+}, n_{+1}n)$, where $n_{1+} = n_{11} + n_{12}, n_{+1} = n_{11} + n_{21}$.

$$
\begin{aligned}
p(n_{11}|n_{1+}, n_{+1}n) &= \frac{p(n_{11}, n_{1+}, n_{+1}n)}{p(n_{1+}, n_{+1}n)} \\
p(n_{ij}) &= \prod_{i=1}^{2}\prod_{j=1}^{2} \frac{exp(-\mu_{ij})\mu_{ij}^{n_{ij}}}{n_{ij}!} \\
&= \frac{exp(-\mu_{11})\mu_{11}^{n_{11}}}{n_{11}!} \frac{exp(-\mu_{12})\mu_{12}^{n_{12}}}{n_{12}!} \frac{exp(-\mu_{21})\mu_{21}^{n_{21}}}{n_{21}!} \frac{exp(-\mu_{22})\mu_{22}^{n_{22}}}{n_{22}!} \\
n_{12} &= n_{1+} - n_{11}, \qquad n_{21} = n_{+1} - n_{11}, \\
n_{22} &= n - n_{12} - n_{21} - n_{11} = n - n_{1+} - n_{+1} + n_{11} \\
p(n_{11}, n_{1+}, n_{+1}n) &= \frac{exp(-\mu_{11})\mu_{11}^{n_{11}}}{n_{11}!} \frac{exp(-\mu_{12})\mu_{12}^{n_{1+}-n_{11}}}{(n_{1+} - n_{11})!} \frac{exp(-\mu_{21})\mu_{21}^{n_{+1}-n_{11}}}{(n_{+1} - n_{11})!} \frac{exp(-\mu_{22})\mu_{22}^{n-n_{1+}-n_{+1}+n_{11}}}{(n - n_{1+} - n_{+1} + n_{11})!}
\end{aligned}
$$

The Jacobian transformation matrix

$$
J = \begin{pmatrix}
\frac{\partial n_{11}}{\partial n_{11}} & \frac{\partial n_{11}}{\partial n_{1+}} & \frac{\partial n_{11}}{\partial n_{+1}} & \frac{\partial n_{11}}{\partial n} \\
\frac{\partial n_{12}}{\partial n_{11}} & \frac{\partial n_{12}}{\partial n_{1+}} & \frac{\partial n_{21}}{\partial n_{+1}} & \frac{\partial n_{22}}{\partial n} \\
\frac{\partial n_{21}}{\partial n_{11}} & \frac{\partial n_{21}}{\partial n_{1+}} & \frac{\partial n_{21}}{\partial n_{+1}} & \frac{\partial n_{22}}{\partial n} \\
\frac{\partial n_{22}}{\partial n_{11}} & \frac{\partial n_{22}}{\partial n_{1+}} & \frac{\partial n_{22}}{\partial n_{+1}} & \frac{\partial n_{22}}{\partial n}
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
1 & -1 & -1 & 1
\end{pmatrix}
$$

$$\|J\| = 1$$

Then we can get the $p(n_{1+}, n_{+1}, n)$ by summing over n_{11}. We have $n_{11} <= n_{1+}, n_{11} <= n_{+1}$, and $n_{11} >= -n + n_{1+} + n_{+1}$.

$$
\begin{aligned}
p(n_{11}, n_{1+}, n_{+1}n) &= \frac{exp(-\mu_{11})\mu_{11}^{n_{11}}}{n_{11}!} \frac{exp(-\mu_{12})\mu_{12}^{n_{1+}-n_{11}}}{(n_{1+} - n_{11})!} \frac{exp(-\mu_{21})\mu_{21}^{n_{+1}-n_{11}}}{(n_{+1} - n_{11})!} \frac{exp(-\mu_{22})\mu_{22}^{n-n_{1+}-n_{+1}+n_{11}}}{(n - n_{1+} - n_{+1} + n_{11})!} \\
&= \frac{exp(-\sum_{i=1}^{2}\sum_{j=1}^{2}\mu_{ij}) \left(\frac{\mu_{11}\mu_{22}}{\mu_{12}\mu_{21}}\right)^{n_{11}} \left(\frac{\mu_{12}}{\mu_{22}}\right)^{n_{1+}} \left(\frac{\mu_{21}}{\mu_{22}}\right)^{n_{+1}} \mu_{22}^{n}}{n_{11}!(n_{1+} - n_{11})!(n_{+1} - n_{11})!(n - n_{1+} - n_{+1} + n_{11})!} \\
p(n_{1+}, n_{+1}n) &= \sum_{\max(0, -n+n_{1+}+n_{+1})}^{\min(n_{1+}, n_{+1})} \frac{exp(-\sum_{i=1}^{2}\sum_{j=1}^{2}\mu_{ij}) \left(\frac{\mu_{11}\mu_{22}}{\mu_{12}\mu_{21}}\right)^{n_{11}} \left(\frac{\mu_{12}}{\mu_{22}}\right)^{n_{1+}} \left(\frac{\mu_{21}}{\mu_{22}}\right)^{n_{+1}} \mu_{22}^{n}}{n_{11}!(n_{1+} - n_{11})!(n_{+1} - n_{11})!(n - n_{1+} - n_{+1} + n_{11})!}
\end{aligned}
$$

So we can have

$$
\begin{aligned}
p(n_{11}|n_{1+}, n_{+1}n) &= \frac{p(n_{11}, n_{1+}, n_{+1}n)}{p(n_{1+}, n_{+1}n)} \\
&= \frac{exp(-\sum_{i=1}^{2}\sum_{j=1}^{2}\mu_{ij})\left(\frac{\mu_{11}\mu_{22}}{\mu_{12}\mu_{21}}\right)^{n_{11}}\left(\frac{\mu_{12}}{\mu_{22}}\right)^{n_{1+}}\left(\frac{\mu_{21}}{\mu_{22}}\right)^{n_{+1}}\mu_{22}^{n}}{n_{11}!(n_{1+}-n_{11})!(n_{+1}-n_{11})!(n-n_{1+}-n_{+1}+n_{11})!} \\
&\Big/ \sum_{x\in\max(0,-n+n_{1+}+n_{+1})}^{\min(n_{1+},n_{+1})} \frac{exp(-\sum_{i=1}^{2}\sum_{j=1}^{2}\mu_{ij})\left(\frac{\mu_{11}\mu_{22}}{\mu_{12}\mu_{21}}\right)^{x}\left(\frac{\mu_{12}}{\mu_{22}}\right)^{n_{1+}}\left(\frac{\mu_{21}}{\mu_{22}}\right)^{n_{+1}}\mu_{22}^{n}}{x!(n_{1+}-x)!(n_{+1}-x)!(n-n_{1+}-n_{+1}+x)!}
\end{aligned}
$$

Which we can rewrite

$$
\begin{aligned}
p(n_{11}|n_{1+}, n_{+1}n) = &\binom{n_{1+}}{n_{11}}\binom{n-n_{1+}}{n_{+1}-n_{11}}\left(\frac{\pi_{11}\pi_{22}}{\pi_{12}\pi_{21}}\right)^{n_{11}} \\
&\Big/ \sum_{x\in\max(0,-n+n_{1+}+n_{+1})}^{\min(n_{1+},n_{+1})} \binom{n_{1+}}{x}\binom{n-n_{1+}}{n_{+1}-x}\left(\frac{\pi_{11}\pi_{22}}{\pi_{12}\pi_{21}}\right)^{x}
\end{aligned}
$$

Dervie the hypergeometric distribution:

For a fixed sample size n, the joint distribution of the cell counts in the 2×2 table is given by

$$
\begin{aligned}
p &= \frac{n!}{n_{11}!n_{12}!n_{21}!n_{22}!}\pi_{11}^{n_{11}}\pi_{12}^{n_{12}}\pi_{21}^{n_{21}}\pi_{22}^{n_{22}} \\
\psi &= \frac{\pi_{11}\pi_{22}}{\pi_{12}!\pi_{21}!}
\end{aligned}
$$

Let ψ be the parameter of interest, and π_{21}, π_{12} are the nuisance parameters. By looking at the sufficient statistics of π_{12}, π_{21}, which is $n_{12} = n_{1+} - n_{11}, n_{21} = n_{+1} - n_{11}$. We have a distribution of n_{11} which is the parameter of interest.

There are two ways to get the distribution of conditional probability, one is directly use the conditional probability definition, while the other is to use the conditional log-likelihood formula. Which way should we go will depend on the situation.

If it is easier to get the log-likelihood, then go with the log-likelihood function. But for hypergeometric distribution, it is easier to just use definition as the binomial coefficient is not easy to deal with in log form.

Method 2: Use multinomial distribution definition. When we fixed n_{1+}, n_{+1} which is equal to fix n_{12}, n_{21}

$$
\begin{aligned}
p(n_{11}, n_{1.}, n_{.1}|n) &= \frac{n!}{n_{11}!n_{12}!n_{21}!n_{22}!}\psi^{n_{11}}\pi_{12}^{n_{1+}}\pi_{21}^{n_{+1}}\pi_{22}^{n-n_{1+}-n_{+1}} \\
p(n_{11}|n_{1.}, n_{.1}, n) &= \frac{p(n_{11}, n_{1.}, n_{.1}|n)}{p(n_{1.}, n_{.1}|n)} \\
&= \frac{n!n_{1.}!(n-n_{1.})!}{n_{1.}!(n-n_{1.})!n_{11}!n_{12}!n_{21}!n_{22}!} \\
&= \binom{n}{n_{1.}}\binom{n_{1.}}{n_{11}}\binom{n-n_{1.}}{n_{.1}-n_{11}}
\end{aligned}
$$

The marginal distribution of $p(n_{1.}, n_{.1}|n)$

$$
p(n_{1.}, n_{.1}|n) = \sum_{N_{11}\in\max(0,-n+n_{1+}+n_{+1})}^{\min(n_{1+},n_{+1})} \frac{n!}{n_{11}!n_{12}!n_{21}!n_{22}!}\psi^{n_{11}}\pi_{12}^{n_{1+}}\pi_{21}^{n_{+1}}\pi_{22}^{n-n_{1+}-n_{+1}}
$$

We don't change n_{11} in this formula in order to construct the hypergeometric coefficients in the conditional probability. Most of the terms could be canceled and left n_{11}

$$
\begin{aligned}
p(n_{11}|n_{1.}, n_{.1}, n) &= \frac{p(n_{11}, n_{1.}, n_{.1}|n)}{p(n_{1.}, n_{.1}|n)} \\
&= \frac{n!}{n_{11}!n_{12}!n_{21}!n_{22}!}\psi^{n_{11}}\pi_{12}^{n_{1+}}\pi_{21}^{n_{+1}}\pi_{22}^{n-n_{1+}-n_{+1}} \\
&\Bigg/ \sum_{x\in\max(0,-n+n_{1+}+n_{+1})}^{\min(n_{1+},n_{+1})} \frac{n!}{n_{11}!n_{12}!n_{21}!n_{22}!}\psi^{x}\pi_{12}^{n_{1+}}\pi_{21}^{n_{+1}}\pi_{22}^{n-n_{1+}-n_{+1}} \\
&= \binom{n_{1.}}{n_{11}}\binom{n-n_{1.}}{n_{.1}-n_{11}}\psi^{n_{11}} \Bigg/ P_0(\psi) \\
P_0(\psi) &= \sum_{x\in\max(0,-n+n_{1+}+n_{+1})}^{\min(n_{1+},n_{+1})} \binom{n_{1.}}{x}\binom{n-n_{1.}}{n_{.1}-x}\psi^{x}
\end{aligned}
$$

The n_{1+}, n_{+1} could be considered as the K from above PDF, that we already know the total row and column, and see what the probability is in each draw.

1.3 HG distribution exponential family

Although the hypergeometric distribution looks ugly, the characteristics are the same as other distribution. Here need to be aware that, the ψ is the random variable, while n_{11} is y.

1.3.1 Exponential family

$$P(n_{11}|n_{1+}, n_{+1}, n, \psi) = exp\{n_{11} log\psi - logP_0(\psi) + const\}$$
$$\theta = log\psi, \qquad \phi = 1, \qquad b(\theta) = logP_0(\psi) = logP_0(exp(\theta))$$

1.3.2 M(t), K(t)

$$\begin{aligned}
M(t) = E[exp(ty)] &= \int_{\max(0,-n+n_{1+}+n_{+1})}^{\min(n_{1+},n_{+1})} exp(ty)exp\{ylog\psi - logP_0(\psi) + c\}dy \\
&= \int_{\max(0,-n+n_{1+}+n_{+1})}^{\min(n_{1+},n_{+1})} exp\{y(\theta+t) - logP_0(exp(\theta)) + c\}dy \\
&= \int_{\max(0,-n+n_{1+}+n_{+1})}^{\min(n_{1+},n_{+1})} \\
& exp\{y(\theta+t) - logP_0(exp(\theta+t)) + logP_0(exp(\theta+t)) - logP_0(exp(\theta)) + c\}dy \\
M(t) &= exp\{\phi b(\theta + t/\phi) - b(\theta)\} = exp(b(\theta+t) - b(\theta)) \\
&= P_0(exp(\theta+t))/P_0(exp(\theta)) \\
M(t) &= P_0(exp(t)exp(\theta))/P_0(exp(\theta)) = P_0(exp(t)\psi)/P_0(\psi)
\end{aligned}$$

The cumulant moment generating function

$$K(t) = logM(t) = logP_0(exp(t)\psi) - logP_0(\psi)$$

1.3.3 μ, σ^2

$$\mu = \partial_t K(t) = \left.\frac{P_0(exp(t)\psi)'}{P_0(exp(t)\psi)}\right|_{t=0} = \frac{P_1(\psi)}{P_0(\psi)}$$
$$\sigma^2 = \left.\frac{\partial \partial_t K(t)}{\partial t}\right|_{t=0} = \frac{P_2(\psi)}{P_0(\psi)} - \mu_i^2$$
$$P_j(\psi) = \int_{x \in \max(0,-n+n_{1+}+n_{+1})}^{\min(n_{1+},n_{+1})} \binom{n_{1+}}{x}\binom{n-n_{1+}}{n_{+1}-x}\psi^x x^j$$

1.3.4 Conditional MLE

The conditional maximum likelihood estiamte (CMLE) of ψ is not calculated directly from the conditional distribution of ψ. While we get it from $p(n_{11}|n_{1+}, n_{+1}, n, \psi)$. We should be able to get the MLE from the log-likelihood conditional.

$\hat{\psi}_c$ is the solution to

$$
\begin{aligned}
n_{11} &= P_1(\hat{\psi}_c)/P_0(\hat{\psi}_c) = \mu \\
logP(n_{11}|n_{1+}, n_{+1}, n, \psi) &= n_{11}log\psi - logP_0(\psi) + c \\
\frac{\partial logP}{\partial \psi} &= \frac{n_{11}}{\psi} - \frac{P_0(\psi)'}{P_0(\psi)} = 0 \\
n_{11} &= P_1(\hat{\psi}_c)/P_0(\hat{\psi}_c)
\end{aligned}
$$

The variance of $\hat{\psi}_c$ can be approximated by the inverse of the Fisher information matrix $I_n(\hat{\psi}_c)$, which is given

$$
I_n(\hat{\psi}_c) = E\{[\partial_\psi logP(n_{11}|n_{1+}, n_{+1}, n, \hat{\psi}_c)]^2\} = \frac{Var(n_{11}|n_{1+}, n_{+1}, n, \hat{\psi}_c)}{\hat{\psi}_c^2}
$$

(e) Let π_{ij} denote the cell probability and assume n is fixed. Consider testing $H_0 : \pi_{ij} = \pi_{i+}\pi_{+j}, i = 1, ..I, j = 1, ..J$. Derive the MLE of π_{ij} under H_0.

The H_0 could be written as

$$
H_0 : \pi_{ij} = \pi_{i+}\pi_{+j}
$$

The multinomial distribution of π_{ij}

$$
p(\pi_{ij}) = \binom{n}{n_{11}n_{12}n_{21}n_{22}} \pi_{ij}^{n_{ij}}, \sum_{i=1}^{I}\sum_{j=1}^{J} \pi_{ij} = 1
$$

The log-likelihood function

$$
logp(\pi_{ij}) = log\binom{n}{n_{11}n_{12}n_{21}n_{22}} + n_{ij}log\pi_{ij}, \sum_{i=1}^{I}\sum_{j=1}^{J} \pi_{ij} = 1
$$

Under H_0, the log-likelihood

$$
logp(\pi_{ij}) = log\binom{n}{n_{11}n_{12}n_{21}n_{22}} + n_{ij}log\pi_{i+}\pi_{+j}, \sum_{i=1}^{I} \pi_{i+} = 1, \sum_{j=1}^{J} \pi_{+j} = 1
$$

By Lagrangian multiplier theorem,

$$
\begin{aligned}
ln(\pi_{ij}) &= nlog\binom{n}{n_{11}n_{12}n_{21}n_{22}} + \sum_{i=1}^{I}\sum_{j=1}^{J} n_{ij}log\pi_{i+}\pi_{+j} + \lambda(\sum_{i=1}^{I}\sum_{j=1}^{J} \pi_{ij} - 1), \\
&= nlog\binom{n}{n_{11}n_{12}n_{21}n_{22}} + \sum_{i=1}^{I}\sum_{j=1}^{J} n_{ij}log\pi_{i+} + \sum_{j=1}^{J}\sum_{i=1}^{I} n_{ij}log\pi_{+j} - \lambda(\sum_{i=1}^{I} \pi_{i+} - 1)
\end{aligned}
$$

Take first derivative of log-likelihood

$$\begin{aligned}
\frac{\partial ln}{\partial \pi_{i+}} &= \frac{\sum_{j=1}^{J} n_{ij}}{\pi_{i+}} + \lambda = 0 \\
\hat{\pi}_{i+} &= \frac{\sum_{j=1}^{J} n_{ij}}{\lambda} \\
\sum_{i=1}^{I} \pi_{i+} &= 1, \qquad \lambda = \sum_{j=1}^{J}\sum_{i=1}^{I} n_{ij} \\
\hat{\pi}_{i+} &= \frac{n_{i+}}{n}
\end{aligned}$$

Similarly, we have $\hat{\pi}_{+j} = \frac{n_{+j}}{n}$, the MLE of π_{ij} under H_0 is

$$\hat{\pi}_{ij} = \hat{\pi}_{i+}\hat{\pi}_{+j} = \frac{n_{i+}n_{+j}}{n^2}$$

(f) Derive the likelihood ratio test for the hypothesis in part (e) and derive its asymptotic distribution under H_0. From part (e), we have the parameter estimates under H_0. While under alternative hypothesis, we have $\mu_{ij} = n_{ij}$.

$$\begin{aligned}
LRT_n &= 2(LR(\pi_{H_1}) - LR(\pi_{H_0})) = 2\left(\sum_{i=1}^{I}\sum_{j=1}^{J} n_{ij} log \pi_{ij} - \sum_{i=1}^{I}\sum_{j=1}^{J} n_{ij} log \pi_{i+}\pi_{+j}\right) \\
&= 2\left(\sum_{i=1}^{I}\sum_{j=1}^{J} n_{ij} log \frac{\pi_{ij}}{\pi_{i+}\pi_{+j}}\right) \\
&= 2\left(\sum_{i=1}^{I}\sum_{j=1}^{J} n_{ij} log \frac{n_{ij}n}{n_{i+}n_{+j}}\right) \sim \chi^2_{(I-1)(J-1)}
\end{aligned}$$

Note that the full model has $(IJ - 1)$ parameters, and the null hypothesis has $(I-1)+(J-1)$ parameters.

$$\begin{aligned}
df &= I \times J - 1 - (I-1) - (J-1) \\
&= (I-1)(J-1)
\end{aligned}$$

(g) Suppose that π_{11}, π_{12} are parameters of interest and the rest of the parameters are treated as nuisance. Derive the conditional likelihood of (π_{11}, π_{12}) and the conditional MLE's of (π_{11}, π_{12}). If not specified, we treat as general contingency table that total n is fixed. If only π_{11}, π_{12} are parameters of interest and the rest of the parameters are treated as nuisance, then we will set the rest of the parameters as one parameter, and get its distribution, which is to find the sufficient statistics for rest of the parameters. Write the Multinomial distribution in exponential family distribution.

We can find marginal distribution by summing over along all possible values of (n_{11}, n_{12}). Note that $n_{11} \leq \min n_{1+} - n_{12}, n_{+1}$ for a given value of n_{12}. Similarly, $n_{12} \leq \min n_{1+} - n_{11}, n_{+1}$ for a given value of n_{11}.
Additionally,

$$\begin{aligned} n &\geq n_{1+} + n_{+1} + n_{+2} - n_{11} - n_{12} \\ n_{11} + n_{12} &\geq \max 0, n_{+1} + n_{1+} + n_{+2} \end{aligned}$$

Let

$$\begin{aligned} S(n_{11}, n_{12}) = \{&(n_{11}, n_{12}) : n_{11} + n_{12} \geq \max 0, n_{+1} + n_{1+} + n_{+2}, \\ &n_{11} \leq \min (n_{1+} - n_{12}, n_{+1}), n_{12} \leq \min (n_{1+} - n_{11}, n_{+1})\} \end{aligned}$$

The conditional distribution

$$\begin{aligned} p(n_{11}, n_{12} | n_{13}, ...n_{IJ}, n) &= \frac{p(n_{ij}}{p(S_n)} \\ &= \frac{\frac{1}{n_{11}! n_{12}!} \pi_{11}^{n_{11}} \pi_{12}^{n_{12}}}{\sum_{(x,y \in S_n)} \frac{1}{x! y!} \pi_{11}^{x} \pi_{12}^{y}} \end{aligned}$$

And $\hat{\pi}_{11}, \hat{\pi}_{12}$ are the CMLE that maximize $p(n_{11}, n_{12} | n_{13}, ...n_{IJ}, n)$.