# 2010 Qualifying Exam Section 1

February 21, 2019

## 1 Question 1

### 1.a

Note that $H_0$ is true iff $\sigma_2^2/\sigma_1^2 = \Delta_0$ iff $1/\sigma_1^2 = \Delta_0/\sigma_2^2$ iff $1/\sigma_1^2 - \Delta_0/\sigma_2^2 = 0$

Let $\eta = (\mu_1, \sigma_1^2, \mu_2, \sigma_2^2)$. The likelihood is given by

$$\begin{aligned}
L(\eta) &= \prod_{i=1}^{n} \sqrt{2\pi}|\Sigma|^{-1/2} \exp\left\{-\frac{1}{2}\left[\frac{1}{\sigma_1^2}(x_i-\mu_1)^2 + \frac{1}{\sigma_2^2}(y_i-\mu_2)^2\right]\right\} \\
&= (2\pi)^{-n}(\sigma_1^2\sigma_2^2)^{-n/2} \exp\left\{-\frac{1}{2}\left[\frac{1}{\sigma_1^2}\sum_{i=1}^{n} x_i^2 - 2n\mu_1\bar{x} + n\mu_1^2\right] + \left[\frac{1}{\sigma_1^2}\sum_{i=1}^{n} y_i^2 - 2n\mu_2\bar{y} + n\mu_2^2\right]\right\} \\
&= (2\pi)^{-n}h(\eta)^{-n/2} \exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma_1^2} - \frac{\Delta_0}{\sigma_2^2}\right)\sum_{i=1}^{n} x_i^2 + \frac{n\mu_1}{\sigma_2^2}\bar{x} - \frac{1}{2\sigma_2^2}\left(\sum_{i=1}^{n} y_i^2 + \Delta_0\sum_{i=1}^{n} x_i^2\right) + \frac{n\mu_2}{\sigma_2^2}\bar{y}\right\}
\end{aligned}$$

The likelihood is written as an exponential family with $U = \sum_{i=1}^n x_i^2$ $\theta = -\frac{1}{\sigma_1^2} - \frac{\Delta_0}{\sigma_2^2}$, $T_1 = \bar{x}$, $\xi_1 = \frac{n\mu_1}{\sigma_1^2}$, $T_2 = \sum_{i=1}^n y_i^2$, $\xi_3 = -\frac{1}{2\sigma_2^2}$, $T_4 = \bar{y}$. This exponential family is full rank. Moreover, the parameter space is $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^- \times \mathbb{R}$, which clearly contains an open set in $\mathbb{R}^4$. Thus, we have a full rank exponential family and we have that the statistics are complete.

Let

$$V = h(U,T) = \frac{\sum_{i=1}^n (x_i-\bar{x})^2}{\sum_{i=1}^n (y_i-\bar{y})^2} = \frac{\sum_{i=1}^n x_i^2 - n\bar{x}^2}{\sum_{i=1}^n y_i^2 - n\bar{y}^2} = \frac{U - T_1}{T_2 - T_3}$$

$V$ is clearly increasing in $U$. We must show it is independent of $T_1, T_2, T_3$. $V$ is independent of $T_1 = \bar{X}$ since the sum of squares is independent of the sample mean for a normal distribution. Arguing rigorously, we note that $U = X'(I - (1/n)\mathbf{1}\mathbf{1}^T)X$. You can show that the matrix is an orthogonal projection operator (it is symmetric and idempotent. $\bar{X}$ is a function of $(1/n)\mathbf{1}\mathbf{1}^T$. Take the product, and you'll see that the product is the 0 matrix. Thus, they are independent.

Moreover, since $\sigma_{12} = 0$, the $x$'s are independent of the $y$'s. Thus, $V$ is increasing in $U$ and independent of $T_1, T_2$, and $T_3$, Under $H_0$, $\sigma_1^2 = \sigma_2^2/\Delta_0$.

We can write $V$ as

$$V = \frac{1}{\Delta_0} \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2/[(n-1)\sigma_2^2/\Delta_0]}{\sum_{i=1}^{n}(y_i - \bar{y})^2/[(n-1)\sigma_2^2]}$$

Under $H_0$, $V$ is a multiple of the ratio of two independent chi-squared random variables divided by their degrees of freedom. Hence, $V \overset{H_0}{\sim} \frac{1}{\Delta_0} F(n-1, n-1)$. We reject $H_0$ if $V < k_1$ or $V > k_2$ such that $\alpha = P(V < k_1) + P(V > k_2)$ and $k_1$ and $k_2$ satisfy $\mathbb{E}(\phi(V)) = \alpha$, $\mathbb{E}((T_2 - T_3)U - T_1 | T = t\phi(V)) = \alpha \mathbb{E}((T_2 - T_3)U - T_1 | T = t)$

### 1.b

The likelihood can be written as

$$L(\eta) = (2\pi)^{-n}(\sigma_1^2\sigma_2^2)^{-n/2}\exp\left\{-\frac{1}{2\sigma_1^2}\sum_{i=1}^{n}(x_i-\mu_1)^2 - \frac{1}{2\sigma_2^2}\sum_{i=1}^{n}(y_i-\mu_2)^2\right\}$$

The likelihood only depends on $\mu_1$ through $\sum_{i=1}^{n}(x_i-\mu_1)^2$ and is strictly decreasing in this term. Note that we can write

$$\sum_{i=1}^{n}(x_i-\mu_1)^2 = \sum_{i=1}^{n}(x_i-\bar{x}+\bar{x}-\mu_1)^2 = \sum_{i=1}^{n}(x_i-\bar{x})^2 + n(\bar{x}-\mu_1)^2$$

which is minimized when $\mu_1 = \bar{x}$ since $(\bar{x}-\mu_1)^2 \geq 0$ with equality holding if and only if $\mu_1 = \bar{x}$. Thus, we have $\hat{\mu}_1 = \bar{x}$. Similarly, we have $\hat{\mu}_2 = \bar{y}$.

The log-likelihood is given by

$$\ell(\mu_1,\mu_2,\sigma_2^1,\sigma_2^2) = -n\log 2\pi - \frac{n}{2}[\log\left(\sigma_1^2\right)+\log\left(\sigma_2^2\right)] - \frac{1}{2\sigma_1^2}\sum_{i=1}^{n}(x_i-\mu_1)^2 - \frac{1}{2\sigma_2^2}\sum_{i=1}^{n}(y_i-\bar{y})^2$$

We have

$$\frac{\partial\ell}{\partial\sigma_1^2} = -\frac{n}{2\sigma_1^2} + \frac{1}{(\sigma_1^2)^2}\sum_{i=1}^{n}(x_i-\mu_1)^2$$

setting this derivative equal to 0 and substituting $\hat{\mu_1} = \bar{x}$ for $\mu_1$, we have

$$\hat{\sigma}_1^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2$$

Similarly, we have

$$\hat{\sigma}_2^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i-\bar{y})^2$$

The likelihood evaluated at the MLEs is

$$\sup_{\eta\in H} L(\eta) = (2\pi)^{-n}(\hat{\sigma_1}^2\hat{\sigma_2}^2)^{-n/2}\exp\left\{-\frac{1}{2\hat{\sigma_1}^2}n\hat{\sigma_1}^2 - \frac{1}{2\hat{\sigma}_2^2}n\hat{\sigma}_2^2\right\}$$

$$= (2\pi)^{-n}(\hat{\sigma}_1^2\hat{\sigma}_2^2)^{-n/2}e^{-n}$$

Under $H_0$, we have that $\sigma_1^2 = \Delta_0 \sigma_2^2$. Let $\sigma_2^2 = \sigma^2$ so that under $H_0$, $\sigma_1^2 = \Delta_0 \sigma^2$. Let $\tilde{L}$ denote the likelihood under $H_0$. Then

$$\begin{aligned}\tilde{L}(\mu_1, \mu_2, \sigma^2) &= L(\mu_1, \mu_2, \sigma^2, \sigma^2) \\ &= (2\pi)^{-n} (\Delta_0)^{-n/2} (\sigma^2)^{-n} \exp\left\{ -\frac{1}{2\sigma^2} \left[ \frac{1}{\Delta_0} \sum_{i=1}^n (x_i - \mu_1)^2 + \sum_{i=1}^n (y_i - \mu_2)^2 \right] \right\}\end{aligned}$$

$\tilde{L}$ only depends on $\mu_1$ through the sum of squares $\sum_{i=1}^n (x_i - \mu_1)^2$ and is decreasing in this quantity since $\Delta_0 > 0$. Thus, for the same reasons as the unrestricted MLE, we have $\tilde{\mu}_1 = \bar{x}$ and analogously, $\tilde{\mu}_2 = \bar{y}$. The restricted log likelihood is

$$\ell(\mu_1, \mu_2, \sigma^2) \propto -n \log \sigma^2 - \frac{1}{2\sigma^2} \left[ \frac{1}{\Delta_0} \sum_{i=1}^n (x_i - \mu_1)^2 + \sum_{i=1}^n (y_i - \mu_2)^2 \right]$$

Thus,

$$\frac{\partial \ell}{\partial \sigma^2} = -\frac{n}{\sigma^2} + \frac{1}{2(\sigma^2)^2} \left[ \frac{1}{\Delta_0} \sum_{i=1}^n (x_i - \mu_1)^2 + \sum_{i=1}^n (y_i - \mu_2)^2 \right]$$

and hence $\tilde{\sigma}^2 = \dfrac{1}{2n} \left[ \frac{1}{\Delta_0} \sum_{i=1}^n (x_i - \bar{x})^2 + \sum_{i=1}^n (y_i - \bar{y})^2 \right]$

Thus,

$$\begin{aligned}\sup_{\eta \in H_0} L(\mu_1, \sigma_1^2, \mu_2, \sigma_2^2) &= (2\pi)^{-n} \Delta_0^{-n/2} (\tilde{\sigma}^2)^{-n} \exp\left\{ -\frac{1}{2\tilde{\sigma}^2} 2n\tilde{\sigma}^2 \right\} \\ &= (2\pi)^{-n} \Delta_0^{-n/2} (\tilde{\sigma}^2)^{-n} e^{-n}\end{aligned}$$

Hence, the likelihood ratio statistic is

$$\begin{aligned}\Lambda &= \frac{\Delta_0^{-n/2} (\tilde{\sigma}^2)^{-n}}{(\hat{\sigma}_1^2 \hat{\sigma}_2^2)^{-n/2}} \\ &= \left( \frac{\hat{\sigma}_1 \hat{\sigma}_2}{\Delta_0^{1/2} \tilde{\sigma}^2} \right)^n \\ &= \frac{\frac{1}{n} \left[ \sqrt{\sum_{i=1}^n (x_i - \bar{x})^2} \sqrt{\sum_{i=1}^n (y_i - \bar{y})^2} \right]}{\Delta_0^{1/2} \frac{1}{2n} \left[ \frac{1}{\Delta_0} \sum_{i=1}^n (x_i - \bar{x})^2 + \sum_{i=1}^n (y_i - \bar{y})^2 \right]} \\ &= \frac{2 \left[ \sqrt{\sum_{i=1}^n (x_i - \bar{x})^2} \sqrt{\sum_{i=1}^n (y_i - \bar{y})^2} \right]}{\Delta_0^{1/2} \left[ \frac{1}{\Delta_0} \sum_{i=1}^n (x_i - \bar{x})^2 + \sum_{i=1}^n (y_i - \bar{y})^2 \right]}\end{aligned}$$

## 1.c

Let $z_i = (x_i, y_i)^T$ and let $\mu = (\mu_1, \mu_2)^T$. Then the likelihood can be written as

$$L(\mu, \Sigma) = (2\pi \det\{\Sigma\})^{-n/2} \exp\left\{ -\frac{1}{2} \sum_{i=1}^{n} (z_i - \mu)^T \Sigma^{-1} (z_i - \mu) \right\}$$

Thus, the log-likelihood is

$$\begin{aligned} \ell(\mu, \Sigma) &= -\frac{n}{2} \log |\Sigma| - \frac{1}{2} \sum_{i=1}^{n} (z_i - \mu)^T \Sigma^{-1} (z_i - \mu) && (1.1) \\ &= -\frac{n}{2} \log |\Sigma| - \frac{1}{2} \operatorname{tr}\left( \Sigma^{-1} \sum_{i=1}^{n} (z_i - \mu)(z_i - \mu)^T \right) && (1.2) \end{aligned}$$

From (1.1), note that we can write the second term as

$$\sum_{i=1}^{n} (z_i - \bar{z})^T \Sigma^{-1} (z_i - \bar{z}) + n(\bar{z} - \mu)^T \Sigma^{-1} (\bar{z} - \mu)$$

Since $\Sigma$ is positive definite, so is $\Sigma^{-1}$. Thus since the likelihood is decreasing in the term above, and the term above is minimized when $\mu = \bar{z}$, we have $\hat{\mu} = \bar{z}$

Now, we have

$$\frac{\partial \ell}{\partial \Sigma^{-1}} = -\frac{n}{2}\Sigma - \frac{1}{2} \sum_{i=1}^{n} (z_i - \mu)(z_i - \mu)^T$$

This derivative is 0 iff $\hat{\Sigma} = \frac{1}{n} \sum_{i=1}^{n} (z_i - \bar{z})(z_i - \bar{z})^T$

Thus, the likelihood evaluated at the MLE is

$$\begin{aligned} L(\hat{\mu}, \hat{\Sigma}) &= (2\pi)^{-n/2} |\hat{\Sigma}|^{-n/2} - \frac{1}{2} \operatorname{tr}\left( \hat{\Sigma}^{-1} n \hat{\Sigma} \right) \\ &= (2\pi)^{-n/2} |\hat{\Sigma}|^{-n/2} e^{-n^2/2} \end{aligned}$$

Note that $H_0$ is true iff $\sigma_{12} = 0$. By part (b), we already know the MLEs under the restricted case and we can see that the LRT rejects $H_0$ when

$$\begin{aligned} &\frac{|\tilde{\Sigma}|^{-n/2}}{|\hat{\Sigma}^{-n/2}|} < k_1 \\ \iff\ &\frac{\hat{\sigma_1}^2 \hat{\sigma}_2^2 - \hat{\sigma_{12}}^2}{\hat{\sigma}_1^2 \hat{\sigma}_2^2} < k_2 \\ \iff\ &(1 - \rho^2) < k_2 \\ \iff\ &\rho^2 > k_3 \\ \iff\ &|\rho| > k \end{aligned}$$