2015 D1

3 b) $f(y_i \mid obs_i) = \dfrac{f(y_i, obs_i)}{f(obs_i)} = \dfrac{f(y_{complete_i})}{f(obs_i)} = \dfrac{f(y_i, X_i) f(R_i)}{f(obs_i)}$

$$= \frac{f(y_i, X_i) \cancel{f(r_i)}}{[f(y_i, X_i)]^{R_i} \; f(X_i)^{1-R_i} \cancel{f(r_i)}}$$

$$= \frac{f(y_i, X_i)^{1-R_i}}{f(X_i)^{1-R_i}} = \left(\frac{f(y_i, X_i)}{f(X_i)}\right)^{1-R_i} = f(y_i \mid X_i)^{1-R_i}$$

$$y_i \mid X_i \sim N\left(\mu^* = \mu_y + \frac{\sigma_{12}}{\sigma_{11}}(x_i - \mu_x),\ \theta^* = \sigma_{22} - \frac{\sigma_{12}^2}{\sigma_{11}}\right)$$

$$f(y_i \mid obs_i) = (2\pi\theta^*)^{\frac{-(1-R_i)}{2}} \exp\left\{-\frac{1-R_i}{2\theta^*}(y_i - \mu^*)^2\right\}$$

$$E(y_i \mid obs_i) = (2\pi\theta^*)^{\frac{-(1-R_i)}{2}} \int_{-\infty}^{\infty} y_i \exp\left\{\frac{-1}{2(\theta^*/(1-R_i))}(y_i - \mu^*)^2\right\} dy_i$$

$$= (2\pi\theta^*)^{\frac{-(1-R_i)}{2}} \left(2\pi \frac{\theta^*}{1-R_i}\right)^{\frac{1}{2}} \underbrace{\int_{-\infty}^{\infty} y_i \left(2\pi \frac{\theta^*}{1-R_i}\right)^{-1/2} \exp\left\{\frac{-1}{2\theta^*/(1-R_i)}(y_i - \mu^*)^2\right\} dy_i}_{E(Z_i) \quad Z_i \sim N(\mu^*, \theta^*/(1-R_i))}$$

$$= (2\pi\theta^*)^{\frac{-(1-R_i)}{2}} (2\pi\theta^*)^{\frac{1}{2}} (1-R_i)^{-\frac{1}{2}} \mu^*$$

$$= (2\pi\theta^*)^{\frac{R_i}{2}} (1-R_i)^{-1/2}$$

d)

$$Y = \alpha + \beta X + \varepsilon$$

$$E[Y|X] = \alpha + \beta X$$

$$Y_i | X_i \sim N\left(\mu_Y + \sigma_{12}\sigma_{11}^{-1}(X - \mu_X),\ \sigma_{22} - \sigma_{12}\sigma_{11}^{-1}\sigma_{21}\right)$$

$$\alpha + \beta X \overset{\text{set}}{=} \mu_Y + \sigma_{12}\sigma_{11}^{-1}(X - \mu_X)$$

$$= \underbrace{(\mu_Y - \mu_X\sigma_{12}\sigma_{11}^{-1})}_{\alpha} + \underbrace{\sigma_{12}\sigma_{11}^{-1}}_{\beta} X$$

① $E(Y) = \mu_Y$

$E(\alpha + \beta X + \varepsilon) = \alpha + \beta\mu_X$

② $Var(Y) = \sigma_{22}$

$Var(\alpha + \beta X + \varepsilon) = \beta^2 Var(X) + Var(\varepsilon) = \beta^2\sigma_{11} + Var(\varepsilon)$

③ $Cov(X, Y) = \sigma_{12}$

$Cov(X, \alpha + \beta X + \varepsilon) = \beta Cov(X) = \beta\sigma_{11}$

③ $\sigma_{12} = \beta\sigma_{11} \Rightarrow \beta = \frac{\sigma_{12}}{\sigma_{11}}$

$\Rightarrow$

① $\mu_Y = \alpha + \beta\mu_X \Rightarrow \alpha = \mu_Y - \beta\mu_X = \mu_Y - \frac{\sigma_{12}}{\sigma_{11}}\mu_X$

② $\sigma_{22} = \beta^2\sigma_{11} + Var(\varepsilon) \Rightarrow Var(\varepsilon) = \sigma_{22} - \beta^2\sigma_{11} = \sigma_{22} - \left(\frac{\sigma_{12}}{\sigma_{11}}\right)^2\sigma_{11}$

$= \sigma_{22} - \frac{\sigma_{12}^2}{\sigma_{11}}$

Scanned by CamScanner