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2014, Theory Qual, Section 1, Problem 2

2a) Show that a Bayes rule w.r.t. a prior distn. Λ on Θ having positive probabilities $\lambda_1, \ldots, \lambda_l > 0$ is admissible.

Proof by contradiction.

Take d_B to be a Bayes rule w.r.t. Λ.

Suppose d_B inadmissible $\Rightarrow \exists d' \ni R(\theta, d') \leq R(\theta, d_B) \ \forall \theta$
$<$ for some θ

Have
$$\mathcal{R}(\Lambda, d_B) = E_\Lambda[R(\theta, d_B)] = \sum_{i=1}^{l} \lambda_i R(\theta_i, d_B)$$
$$> \sum_{i=1}^{l} \lambda_i R(\theta_i, d')$$
$$= \mathcal{R}(\Lambda, d')$$

$\Rightarrow \mathcal{R}(\Lambda, d') < \mathcal{R}(\Lambda, d_B) \Rightarrow d_B$ cannot be a Bayes rule

(b/c by Defn. 1.11, in order for d_B to be Bayes, must hold that $\mathcal{R}(\Lambda, d_B) = \inf_{d \in \mathcal{D}} \mathcal{R}(\Lambda, d) =$ Bayes risk)

Since we assumed d_B Bayes w.r.t. Λ.

Thus, d_B must be admissible.

2b) The result in a) conflicts with other results for conts. parameter spaces where Bayes rules may not be admissible, e.g., James Stein estimation. In the discrete case described above, show that if $\lambda_i = 0$ for some $i = 1, \ldots, l$, then the resulting Bayes rule d_B may not be admissible.

⌈ Suppose $\lambda_i = 0$ and d_B is a Bayes rule w.r.t. Λ.

Aim: show d_B may be inadmissible

Suppose $\exists$ a rule $d' \ni R(\theta_i, d') \leq R(\theta_i, d_B) \ \forall \theta_i \in \Theta$
$<$ for some θ_i

Let the strict inequality hold for $\lambda_i = 0$. Let equality hold for $\lambda_i > 0$ with $R(\theta_i, d') = R(\theta_i, d_B)$.

Then, $\sum_i \lambda_i R(\theta_i, d') = \sum_i \lambda_i R(\theta_i, d_B)$

$\Rightarrow \mathcal{R}(\Lambda, d') = \mathcal{R}(\Lambda, d_B) \Rightarrow$ d_B not admissible since it is not unique

(recall, only if unique w/ finite Bayes risk does Bayes $\Rightarrow$ admissible.) ⌋

2c) Suppose that the frequentist risk of d_B in part b) is finite and constant on those θ_i's having $\lambda_i > 0$.

Show that this decision rule is minimax, that is, it minimizes the max risk on those θ_i's w/ $\lambda_i > 0$.

TOPICS 1. Minimax principle : A decision rule d_M is minimax if

$$\inf_{d \in \mathcal{D}} \{ \sup_{\theta \in \Theta} R(\theta, d) \} = \sup_{\theta \in \Theta} R(\theta, d_M)$$

That is, a rule is minimax if it minimizes the worst possible risk $\sup_{\theta \in \Theta} R(\theta, d)$ among all possible randomized rules $d \in \mathcal{D}$.

2. THM 1.12: Suppose that Λ is a prior distn. on Θ $\ni$

$$\mathcal{R}(\Lambda, d_\Lambda) = \int_\Theta R(\theta, d_\Lambda)\, \lambda(\theta)\, d\theta = \sup_\theta R(\theta, d_\Lambda)$$

Then, (i) d_Λ is a minimax

(ii) If d_Λ is unique Bayes w.r.t. Λ, then d_Λ is unique minimax

(iii) Λ is least favorable

Told that the frequentist risk $R(\theta_i, d_B)$ is finite and constant on those θ_i's having $\lambda_i > 0$.

Assume the 1st K θ_i's have $\lambda_i > 0$ and the last $l-K$ θ_i's have $\lambda_i = 0$.

Then, $\mathcal{R}(\Lambda, d_B) = \sum_{i=1}^{l} R(\theta, d_B)\lambda_i = \sum_{i=1}^{K} \lambda_i \underbrace{R(\theta_i, d_B)}_{\text{constant} = C}$ (since $\lambda_i = 0$ for $i > K$)

$= C \underbrace{\sum_{i=1}^{K} \lambda_i}_{=1} = C = \sup_{\theta \in \{\theta_1, \dots, \theta_K\}} R(\theta, d_B)$

By THM 1.12, since $\mathcal{R}(\Lambda, d_K) = \sup_{\theta \in \{\theta_1, \dots, \theta_K\}} R(\theta, d_B) \Rightarrow d_\Lambda$ is a minimax (by (i) of theorem).

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2 d) Can anything be said about whether or not d_B in part b) is minimax on θ_i, $i=1,\dots,\ell$?

This question differs from the last in that here we are trying to evaluate the claim that it's minimax on θ_i for $\lambda_i \geq 0$ notice here can be equal to zero.

It is possible that $R(\theta_i, d_B) > c$ when $\lambda_i = 0$.

Then, $R(\Lambda, d_B) = \sum_{i=1}^{\ell} \lambda_i R(\theta_i, d_B) = \sum_{i=1}^{K} \lambda_i \underbrace{R(\theta_i, d_B)}_{\text{Constant} = c} + \sum_{i=K+1}^{\ell} \underbrace{\lambda_i}_{=0} \underbrace{R(\theta_i, d_B)}_{>c}$

$= c \sum_{i=1}^{K} \lambda_i = c$. However, $\sup_{\theta \in \Theta} R(\theta_i, d_B) > c$ since $R(\theta_i, d_B) > c$ when $\lambda_i = 0$.

Thus, $R(\Lambda, d_B) = c \neq \sup_{\theta \in \Theta} R(\theta_i, d_B) > c$. $\Rightarrow$ cannot claim d_B minimax.

However, if $R(\theta_i, d_B) \leq c$ when $\lambda_i = 0$, then

$R(\Lambda, d_B) = c$ (as above) $= \sup_{\theta \in \Theta} R(\theta_i, d_B) = c$ (since $R(\theta_i, d_B) \leq c$ when $\lambda_i = 0$).

$\Rightarrow$ d_B is minimax.

In summary, if $R(\theta_i, d_B) > c$ when $\lambda_i = 0$, cannot necessarily claim d_B minimax.
if $R(\theta_i, d_B) \leq c$ when $\lambda_i = 0$, can claim d_B minimax.

In e), f), and g), consider the following classification problem. Ann Marie Weideman

Suppose that X is an observation from the density

$$p(x|\theta) = \underbrace{\theta^{-1} I(0 < x < \theta)}_{\text{uniform}}$$

where $I(\cdot)$ denotes the indicator fcn and the parameter space is $\Theta = \{1,2,3\}$.

It is desired to classify X as arising from $p(x|1), p(x|2)$, or $p(x|3)$ under a 0-1 loss function (0 = correct, 1 = incorrect decision).

e) Find the form of the Bayes rule for this problem.

7761

See HWK #2

Problem #5 soln.

Given $p(x|\theta) = \frac{1}{\theta} I(0 < x < \theta)$.

0-1 loss, $L(i, a_k) = \begin{cases} 0 & \text{if } i = k \\ 1 & \text{if } i \neq k \end{cases}$

To derive Bayes rule, need to find posterior expected loss.

Take the prior $\lambda(i) = \lambda_i \ni \lambda_1 + \lambda_2 + \lambda_3 = 1$.

Then, $E_{\theta|x}[L(\theta, a_1)] = \sum_{i=1}^{3} L(\theta, a_1) \cdot P(\theta|x) = \sum_{i=1}^{3} L(\theta_i, a_1) \cdot \frac{\overbrace{P(x|\theta_i) \cdot \lambda_i}^{p(\theta_i, x)}}{p(x)}$

$$= \frac{1}{p(x)} [\underbrace{L(1, a_1)}_{i=k} \cdot P(x|1) \cdot \lambda_1 + \underbrace{L(2, a_1)}_{i \neq k} \cdot P(x|2) \cdot \lambda_2 + \underbrace{L(3, a_1)}_{i \neq k} \cdot P(x|3) \cdot \lambda_3]$$

$$= \frac{1}{p(x)} [0 \cdot P(x|1) \cdot \lambda_1 + 1 \cdot P(x|2) \cdot \lambda_2 + 1 \cdot P(x|3) \cdot \lambda_3]$$

$\Rightarrow E_{\theta|x}[L(\theta, a_1)] = \frac{1}{p(x)} [P(x|2) \cdot \lambda_2 + P(x|3) \cdot \lambda_3]$

Similarly, $E_{\theta|x}[L(\theta, a_2)] = \frac{1}{p(x)} [P(x|1) \cdot \lambda_1 + P(x|3) \cdot \lambda_3]$

$E_{\theta|x}[L(\theta, a_3)] = \frac{1}{p(x)} [P(x|1) \cdot \lambda_1 + P(x|2) \cdot \lambda_2]$

Want to minimize these bad boys.

Let $\phi(i) = p(d(x) = i)$.

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2 e) cont'd.

Case 1: $x \in (0,1)$

Recall that indicator on pdf is $I(0<x<\theta)$, so we will consider 3 cases: $0<x<1$, $1 \le x < 2$, and $2 \le x < 3$

- $$E_{\theta|x}[L(\theta,a_1)] = \frac{1}{p(x)}\Big[\underbrace{\tfrac{1}{2} I(0<x<2)}_{p(x|2)}\cdot\lambda_2 + \underbrace{\tfrac{1}{3} I(0<x<3)}_{p(x|3)}\cdot\lambda_3\Big]$$
$$= \frac{1}{p(x)}\left[\tfrac{1}{2}\lambda_2 + \tfrac{1}{3}\lambda_3\right] \quad \text{(since } x\in(0,1)\text{, both indicators} = 1)$$

- $$E_{\theta|x}[L(\theta,a_2)] = \frac{1}{p(x)}\left[I(0<x<1)\cdot\lambda_1 + \tfrac{1}{3} I(0<x<3)\cdot\lambda_3\right]$$
$$= \frac{1}{p(x)}\left[\lambda_1 + \tfrac{1}{3}\lambda_3\right] \quad \text{(since } x\in(0,1)\text{, both indicators} = 1)$$

- $$E_{\theta|x}[L(\theta,a_3)] = \frac{1}{p(x)}\left[I(0<x<1)\cdot\lambda_1 + \tfrac{1}{2} I(0<x<2)\cdot\lambda_2\right]$$
$$= \frac{1}{p(x)}\left[\lambda_1 + \tfrac{1}{2}\lambda_2\right] \quad \text{(since } x\in(0,1)\text{, both indicators} = 1)$$

Then,

$$\phi_1(x) = 1 \iff \begin{array}{l} \tfrac{1}{2}\lambda_2 + \cancel{\tfrac{1}{3}\lambda_3} < \lambda_1 + \cancel{\tfrac{1}{3}\lambda_3} \\ \cancel{\tfrac{1}{2}\lambda_2} + \tfrac{1}{3}\lambda_3 < \lambda_1 + \cancel{\tfrac{1}{2}\lambda_2} \end{array} \iff \begin{array}{l} \lambda_1 > \tfrac{1}{2}\lambda_2 \\ \lambda_1 > \tfrac{1}{3}\lambda_3 \end{array}$$

$$\phi_1(x) = \gamma_1 = 1 - \phi_2(x) \iff \lambda_1 = \tfrac{1}{2}\lambda_2 \text{ and } \lambda_1 > \tfrac{1}{3}\lambda_3$$

$$\phi_1(x) = \gamma_2 = 1 - \phi_3(x) \iff \lambda_1 = \tfrac{1}{3}\lambda_3 \text{ and } \lambda_1 > \tfrac{1}{2}\lambda_2$$

$$\phi_1(x) = \gamma_3 \iff \lambda_1 = \tfrac{1}{2}\lambda_2 = \tfrac{1}{3}\lambda_3$$

$$\phi_2(x) = 1 \iff \begin{array}{l} \lambda_1 + \cancel{\tfrac{1}{3}\lambda_3} < \tfrac{1}{2}\lambda_2 + \cancel{\tfrac{1}{3}\lambda_3} \\ \cancel{\lambda_1} + \tfrac{1}{3}\lambda_3 < \cancel{\lambda_1} + \tfrac{1}{2}\lambda_2 \end{array} \iff \begin{array}{l} \tfrac{1}{2}\lambda_2 > \lambda_1 \\ \tfrac{1}{2}\lambda_2 > \tfrac{1}{3}\lambda_3 \end{array}$$

$$\phi_2(x) = \gamma_4 \iff \tfrac{1}{2}\lambda_2 = \tfrac{1}{3}\lambda_3 \text{ and } \tfrac{1}{2}\lambda_2 > \lambda_1$$

$$\phi_2(x) = (1-\gamma_1) \iff \lambda_1 = \tfrac{1}{2}\lambda_2 \text{ and } \underbrace{1/2\,\lambda_2}_{\lambda_1} > \tfrac{1}{3}\lambda_3$$

$$\phi_2(x) = \gamma_5 \iff \lambda_1 = \tfrac{1}{2}\lambda_2 = \tfrac{1}{3}\lambda_3$$

$$\phi_3(x) = 1 \iff \begin{array}{l} \lambda_1 + \cancel{\tfrac{1}{2}\lambda_2} < \cancel{\tfrac{1}{2}\lambda_2} + \tfrac{1}{3}\lambda_3 \\ \cancel{\lambda_1} + \tfrac{1}{2}\lambda_2 < \cancel{\lambda_1} + \tfrac{1}{3}\lambda_3 \end{array} \iff \begin{array}{l} \tfrac{1}{3}\lambda_3 > \lambda_1 \\ \tfrac{1}{3}\lambda_3 > \tfrac{1}{2}\lambda_2 \end{array}$$

$$\phi_3(x) = (1-\gamma_4) \iff \tfrac{1}{2}\lambda_2 = \tfrac{1}{3}\lambda_3 \text{ and } \tfrac{1}{2}\lambda_2 > \lambda_1$$

$$\phi_3(x) = (1-\gamma_2) \iff \lambda_1 = \tfrac{1}{3}\lambda_3 \text{ and } \underbrace{\tfrac{1}{3}\lambda_3}_{\lambda_1} > \tfrac{1}{2}\lambda_2$$

$$\phi_3(x) = (1-\gamma_3-\gamma_5) \iff \lambda_1 = \tfrac{1}{2}\lambda_2 = \tfrac{1}{3}\lambda_3$$

cont'd →

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2e) cont'd.

Combining the above pieces

In total, the Bayes rule is:

$$\phi_1(x) = I(\lambda_1 > \tfrac{1}{2}\lambda_2) I(\lambda_1 > \tfrac{1}{3}\lambda_3) + \gamma_1 I(\lambda_1 = \tfrac{1}{2}\lambda_2) I(\lambda_1 > \tfrac{1}{3}\lambda_3) + \gamma_2 I(\lambda_1 = \tfrac{1}{3}\lambda_3) I(\lambda_1 > \tfrac{1}{2}\lambda_2) + \gamma_3 I(\lambda_1 = \tfrac{1}{2}\lambda_2 = \tfrac{1}{3}\lambda_3)$$

$$\phi_2(x) = I(\tfrac{1}{2}\lambda_2 > \lambda_1) I(\tfrac{1}{2}\lambda_2 > \tfrac{1}{3}\lambda_3) + \gamma_4 I(\tfrac{1}{2}\lambda_2 = \tfrac{1}{3}\lambda_3) I(\tfrac{1}{2}\lambda_2 > \lambda_1) + (1-\gamma_1) I(\lambda_1 = \tfrac{1}{2}\lambda_2) I(\underbrace{\tfrac{1}{2}\lambda_2}_{\lambda_1} > \tfrac{1}{3}\lambda_3) + \gamma_5 I(\lambda_1 = \tfrac{1}{2}\lambda_2 = \tfrac{1}{3}\lambda_3)$$

$$\phi_3(x) = I(\tfrac{1}{3}\lambda_3 > \lambda_1) I(\tfrac{1}{3}\lambda_3 > \tfrac{1}{2}\lambda_2) + (1-\gamma_4) I(\tfrac{1}{2}\lambda_2 = \tfrac{1}{3}\lambda_3) I(\tfrac{1}{2}\lambda_2 > \lambda_1) + (1-\gamma_2) I(\lambda_1 = \tfrac{1}{3}\lambda_3) I(\underbrace{\tfrac{1}{3}\lambda_3}_{\lambda_1} > \tfrac{1}{2}\lambda_2) + (1-\gamma_3-\gamma_5) I(\lambda_1 = \tfrac{1}{2}\lambda_2 = \tfrac{1}{3}\lambda_3)$$

<u>Case 2: $x \in [1,2)$</u> (Similar to above approach - will skip some steps)

- $E_{\theta|x}[L(\theta|a_1)] = \frac{1}{p(x)}[\tfrac{1}{2}\lambda_2 + \tfrac{1}{3}\lambda_3]$
- $E_{\theta|x}[L(\theta|a_2)] = \frac{1}{p(x)}[\tfrac{1}{3}\lambda_3]$
- $E_{\theta|x}[L(\theta|a_3)] = \frac{1}{p(x)}[\tfrac{1}{2}\lambda_2]$

In total, the Bayes rule is:

$\phi_1(x) = 0$ (b/c the sum of $\tfrac{1}{2}\lambda_1 + \tfrac{1}{3}\lambda_3$ can never be greater than the individual terms)

$$\phi_2(x) = I(\tfrac{1}{3}\lambda_3 < \tfrac{1}{2}\lambda_2) + \gamma_6 I(\tfrac{1}{3}\lambda_3 = \tfrac{1}{2}\lambda_2)$$

$$\phi_3(x) = I(\tfrac{1}{2}\lambda_2 < \tfrac{1}{3}\lambda_3) + (1-\gamma_6) I(\tfrac{1}{3}\lambda_3 = \tfrac{1}{2}\lambda_2)$$

Case 3: $x \in [2,3)$ ← not equal to 3 since indicator ends with $<\theta$

- $E_{\theta|x}[L(\theta|a_1)] = \frac{1}{p(x)}[\tfrac{1}{3}\lambda_3]$
- $E_{\theta|x}[L(\theta|a_2)] = \frac{1}{p(x)}[\tfrac{1}{3}\lambda_3]$
- $E_{\theta|x}[L(\theta|a_3)] = \frac{1}{p(x)}[0] = 0$

$\Rightarrow$ $\boxed{\phi_3(x) = 1 \Rightarrow \phi_1(x) = \phi_2(x) = 0.}$

Ann Marie Weideman

2f). Find the decision rule which minimizes the max risk over Θ and the corresponding least favorable prior distn.

To find the minimax rule, we want to find the Bayes rule with constant risk.

By slide 176, $R(\theta_i, \phi) = \sum_{j=1}^{3} L(\theta_i, a_j)\, E_{\theta_i}[\phi_j(x)] = 1 - E_{\theta_i}[\phi_i(x)]$

$R(\theta_i, \phi)$ constant $\Rightarrow$ $E_{\theta_i}[\phi_i(x)]$ also constant.

Notice that $E_{\theta_3}[\phi_3(x)] \geq P(2 \leq x < 3) > 0$

<u>why is this true?</u>

Because
$$E_{\theta_3}\Big[I(0<x<1)\, \overbrace{\phi_3(x \mid 0<x<1)}^{=\text{ a messy expression}} + I(1 \leq x < 2) \cdot \overbrace{\phi_3(x \mid 1 \leq x < 2)}^{=\text{ a messy expression}} + I(2 \leq x < 3) \cdot \underbrace{\phi_3(x \mid 2 \leq x < 3)}_{=1} \Big]$$

$$= P(0<x<1) \cdot \phi_3(x \mid 0<x<1) + P(1 \leq x < 2) \cdot \phi_3(x \mid 1 \leq x < 2) + P(2 \leq x < 3) \cdot 1$$

$$\rightarrow E_{\theta_3}[\phi_3(x)] \geq P(2 \leq x < 3) > 0$$

Now, we aim to solve for λ_1, λ_2, and λ_3 so we can find the decision rule which minimizes the max risk over Θ.

Since $E_{\theta_3}[\phi_3(x)] > 0 \Rightarrow E_{\theta_1}[\phi_1(x)] > 0$ on $0 \leq x < 1$ (b/c we know it's equal to 0 on $1 \leq x < 2$ and $2 \leq x < 3$).

Since we have to equate risks $\ni E_{\theta_1}[\phi_1(x)] = E_{\theta_2}[\phi_2(x)] = E_{\theta_3}[\phi_3(x)]$

Notice that, when we equate risks, due to the other indicators being zero, we only have constant risk when $\lambda_1 = \frac{1}{2}\lambda_2 = \frac{1}{3}\lambda_3$

Since $\lambda_1 + \lambda_2 + \lambda_3 = 1 \underset{\uparrow \text{sub}}{\Rightarrow} \lambda_1 + 2\lambda_1 + 3\lambda_1 = 1 \Rightarrow \lambda_1 = 1/6$

$\Rightarrow \lambda_2 = 2(1/6) = 1/3$ and $\lambda_3 = 3(1/6) = 1/2$.

Thus, the least favorable prior is $(\lambda_1, \lambda_2, \lambda_3) = (1/6, 1/3, 1/2)$

cont'd. →

Equating risks, we get

$E_{\theta_1}[I(\lambda_1 = 1/2\,\lambda_2 = 1/3\,\lambda_3)] = P(\lambda_1 = 1/2\,\lambda_2 = 1/3\,\lambda_3 \mid \theta_1 \text{ true}) = 1$

$E_{\theta_2}[I(\lambda_1 = 1/2\,\lambda_2 = 1/3\,\lambda_3)] = P(\lambda_1 = 1/2\,\lambda_2 = 1/3\,\lambda_3 \mid \theta_2 \text{ true}) = 1$

$E_{\theta_2}[I(1/3\,\lambda_3 = 1/2\,\lambda_2)] = P(1/3\,\lambda_3 = 1/2\,\lambda_2 \mid \theta_2 \text{ true}) = 1$

$$\gamma_3 \cdot 1 \cdot \underbrace{P(0<x<1 \mid 0<x<1)}_{=1 \text{ since } X \sim Unif(0,1)} = \gamma_3 \cdot 1 \cdot \underbrace{P(0<x<1 \mid 0<x<2)}_{=1/2 \text{ since } X \sim Unif(0,2)} + \gamma_6 \cdot 1 \cdot \underbrace{P(1 \leq x < 2 \mid 0<x<2)}_{=1/2 \text{ since } X \sim Unif(0,2)}$$

$$= (1-\gamma_3-\gamma_5) \cdot 1 \cdot \underbrace{P(0<x<1 \mid 0<x<3)}_{=1/3 \text{ since } X \sim Unif(0,3)} + (1-\gamma_6) \cdot 1 \cdot \underbrace{P(2 \leq x < 3 \mid 0<x<3)}_{=1/3 \text{ since } X \sim Unif(0,3)}$$

Ann Marie
Weideman

2 f) cont'd

$\Rightarrow \quad \gamma_3 = \frac{1}{2}\gamma_5 + \frac{1}{2}\gamma_6 = \frac{1}{3}(1-\gamma_3-\gamma_5) + \frac{1}{3}(1-\gamma_6)$

$\Rightarrow \quad 2\gamma_3 = \gamma_5 + \gamma_6$ and $3\gamma_5 + 3\gamma_6 = 2 - 2\gamma_3 - 2\gamma_5 + 2 - 2\gamma_6$

$\Rightarrow \quad 2\gamma_3 = 4 - 5\gamma_5 - 5\gamma_6$

Thus, $\overbrace{2\gamma_3 = \gamma_5 + \gamma_6}^{i)}$ and $2\gamma_3 = 4 - 5\gamma_3 - 5\gamma_6$

$\Rightarrow \quad 2\gamma_3 = \frac{2}{3} - \cancel{\gamma_6} + \cancel{\gamma_6} \quad \Rightarrow \quad \gamma_3 = \frac{1}{3}$

sub into (i)

$\gamma_5 = \frac{2}{3} - \gamma_6$

Thus, the minimax rule is any rule $\ni (\lambda_1, \lambda_2, \lambda_3) = (\frac{1}{6}, \frac{1}{3}, \frac{1}{2})$

and $\gamma_3 = \frac{1}{3}$, $\gamma_5 = \frac{2}{3} - \gamma_6 \ni E_{\theta_1}[\phi_1(x)] = E_{\theta_2}[\phi_2(x)] = E_{\theta_3}[\phi_3(x)]$.

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2g) Find the decision rule which minimizes the maximum risk over $\theta = 1$ and $\theta = 2$ and the corresponding least favorable prior distn.

Is this minimax rule the same as in f)? Explain.

$\ulcorner$ To derive the Bayes rule, need to again find the posterior expected loss.

$$E_{\theta|x}[L(\theta,a_1)] = \frac{1}{p(x)}[p(x|2)\cdot\lambda_2]$$

$$E_{\theta|x}[L(\theta,a_2)] = \frac{1}{p(x)}[p(x|1)\cdot\lambda_1]$$

Let $\phi_i = p(d(x)=i)$.

<u>Case 1 : $0<x<1$</u> : $E_{\theta|x}[L(\theta,a_1)] = \frac{1}{p(x)}[\frac{1}{2}I(0<x<2)\cdot\lambda_2] = \frac{1}{p(x)}[\frac{1}{2}\lambda_2]$

$E_{\theta|x}[L(\theta,a_2)] = \frac{1}{p(x)}[I(0<x<1)\cdot\lambda_1] = \frac{1}{p(x)}[\lambda_1]$

$$\Rightarrow \left.\begin{array}{l} \phi_1(x) = 1 \iff \frac{1}{2}\lambda_2 < \lambda_1 \\ \phi_1(x) = \gamma_1 \iff \frac{1}{2}\lambda_2 = \lambda_1 \\ \phi_2(x) = 1 \iff \lambda_1 < \frac{1}{2}\lambda_2 \\ \phi_2(x) = 1-\gamma_1 \iff \frac{1}{2}\lambda_2 = \lambda_1 \end{array}\right\} \Rightarrow \begin{array}{l} \phi_1(x) = I(\frac{1}{2}\lambda_2<\lambda_1) + \gamma_1 I(\frac{1}{2}\lambda_2 = \lambda_1) \\ \phi_2(x) = I(\lambda_1 < \frac{1}{2}\lambda_2) + (1-\gamma_1)I(\frac{1}{2}\lambda_2 = \lambda_1) \end{array}$$

<u>Case 2 : $1 \le x < 2$</u> : $E_{\theta|x}[L(\theta,a_1)] = \frac{1}{p(x)}[\frac{1}{2}I(0<x<2)\cdot\lambda_2] = \frac{1}{p(x)}[\frac{1}{2}\lambda_2]$

$E_{\theta|x}[L(\theta,a_2)] = \frac{1}{p(x)}[I(0<x<1)\cdot\lambda_1] = 0$

$\Rightarrow \phi_2(x) = 1 \Rightarrow \phi_1(x) = 0$

Thus, $\phi_1(x) = I(0<x<1)[I(\frac{1}{2}\lambda_2<\lambda_1) + \gamma_1 I(\frac{1}{2}\lambda_2=\lambda_1)]$

$\phi_2(x) = I(0<x<1)[I(\lambda_1<\frac{1}{2}\lambda_2) + (1-\gamma_1)I(\frac{1}{2}\lambda_2=\lambda_1)] + I(1\le x<2)$

To find minimax rule, want to find Bayes rule w/ constant risk

Note that $E_{\theta_1}[\phi_1(x)] = E_{\theta_2}[\phi_2(x)]$ only if $\lambda_1 = \frac{\lambda_2}{2}$.

Here : $E_{\theta_1}[\phi_1(x)] = E[I(0<x<1)[I(\frac{1}{2}\lambda_2<\lambda_1) + \gamma_1 I(\frac{1}{2}\lambda_2=\lambda_1)]]$

$= \underbrace{P(0<x<1)}_{=1 \text{ since } X\sim Unif(0,1)}[P(\frac{1}{2}\lambda_2<\lambda_1) + \gamma_1 P(\frac{1}{2}\lambda_2=\lambda_1)]$

Since $\frac{1}{2}\lambda_2 = \lambda_1 \Rightarrow P(\frac{1}{2}\lambda_2<\lambda_1) = 0$ & $P(\frac{1}{2}\lambda_2 = \lambda_1) = 1 \Rightarrow E_{\theta_1}[\phi_1(x)] = \gamma_1$

$E_{\theta_2}[\phi_2(x)] = E\{I(0<x<1)[I(\lambda_1<\frac{1}{2}\lambda_2) + (1-\gamma_1)I(\frac{1}{2}\lambda_2=\lambda_1)] + I(1\le x<2)\}$

$= \underbrace{P(0<x<1)}_{=\frac{1}{2} \text{ since } X\sim Unif(0,2)}[P(\lambda_1<\frac{1}{2}\lambda_2) + (1-\gamma_1)P(\frac{1}{2}\lambda_2=\lambda_1)] + \underbrace{P(1\le x<2)}_{=\frac{1}{2} \text{ since } X\sim Unif(0,2)}$

Since $\frac{1}{2}\lambda_2 = \lambda_1 \Rightarrow P(\lambda_1 < \frac{1}{2}\lambda_2) = 0$ & $P(\frac{1}{2}\lambda_2 = \lambda_1) = 1$

$\Rightarrow E_{\theta_2}[\phi_2(x)] = \frac{1}{2}(1-\gamma_1) + \frac{1}{2}$

cont'd next pg. $\longrightarrow$

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2g) cont'd

Then, equating risks we get: $\gamma_1 = \frac{1}{2}(1-\gamma_1) + \frac{1}{2}$

$\Rightarrow \gamma_1 + \frac{1}{2}\gamma_1 = \frac{1}{2} + \frac{1}{2}$

$\Rightarrow \frac{3}{2}\gamma_1 = 1 \Rightarrow \gamma_1 = \frac{2}{3}$

Also, $\lambda_1 = \frac{1}{2}\lambda_2$ & $\lambda_1 + \lambda_2 = 1$

$\Rightarrow \lambda_1 + \frac{1}{2}\lambda_2 = 1 \Rightarrow \frac{3}{2}\lambda_2 = 1 \Rightarrow \lambda_2 = \frac{2}{3}$

$\Rightarrow \lambda_1 = \frac{1}{2}(\frac{2}{3}) = \frac{1}{3}$

Thus, the minimax rule is any rule $\ni (\lambda_1, \lambda_2) = (\frac{1}{3}, \frac{2}{3})$ and $\gamma_1 = \frac{2}{3} \ni E_{\theta_1}[\phi_1(x)] = E_{\theta_2}[\phi_2(x)]$.

No, it is not equivalent to minimax rule in f).

- Part f) minimizes the risk over a parameter space of three parameters, $\Theta = \{\theta_1, \theta_2, \theta_3\}$, resulting in a minimax rule of $(\lambda_1, \lambda_2, \lambda_3) = (\frac{1}{6}, \frac{1}{3}, \frac{1}{2})$ and two constraints of $\gamma_3 = \frac{1}{3}$ and $\gamma_5 = \frac{2}{3} - \gamma_6$
- However, part g) minimizes the risk over a parameter space involving two parameters, $\Theta = \{\theta_1, \theta_2\}$, resulting in a minimax rule of $(\lambda_1, \lambda_2) = (\frac{1}{3}, \frac{2}{3})$ and one constraint of $\gamma_1 = \frac{2}{3}$. $\lrcorner$