2009 Theory I #3

3a) $E[B_{n+1}] = E\left[\sum_{k=1}^{B_n} A_{nk}\right] = E\left(E\left[\sum_{k=1}^{B_n} A_{nk} \,\middle|\, B_n\right]\right)$

$= E\left(B_n E[A]\right) = E[B_n]\,E[A]$

$Var[B_{n+1}] = Var\left[\sum_{k=1}^{B_n} A_{nk}\right] = E\left(Var\left[\sum_{k=1}^{B_n} A_{nk} \,\middle|\, B_n\right]\right) + Var\left(E\left[\sum_{k=1}^{B_n} A_{nk} \,\middle|\, B_n\right]\right)$

$= E\left(B_n Var[A]\right) + Var\left(B_n E[A]\right)$

$= E[B_n]\,Var[A] + Var[B_n]\,(EA)^2$

3b) $EB_n = \mu EB_{n-1} = \dots = \mu^n$

See slide 760 for variance

3c) $P\left(|B_n - EB_n| > \varepsilon\right) \le \dfrac{Var[B_n]}{\varepsilon^2} \to 0$ as $n \to \infty$

Now $P\left(|B_n| > \varepsilon\right) - P\left(|B_n - EB_n| > \varepsilon\right) \to 0$ so that

as $n \to \infty$

by Slutsky's we obtain

$P\left(|B_n| > \varepsilon\right) \to 0$

$|B_n| - |B_n - EB_n| \xrightarrow{a.s.} 0$

3d) $E[z^{B_n}] = E\left(E[z^{B_n} \mid B_0 = 1]\right) \overset{\text{assumpt.}}{=} E[z^{B_n} \mid B_0 = 1]$

$= \sum_{j=0}^{\infty} P(B_n = j \mid B_0 = 1) z^j = \sum_{j=0}^{\infty} P_{1j}^{(n)} z^j$

$\phi_n(0) = \sum_{j=0}^{\infty} P_{1j}^{(n)} 0^j = P_{10} = P(B_n = 0 \mid B_0 = 1)$

3e) $\sum_{j=0} P\left(\sum_{i=1}^{k} A_{ni} = j \mid B_n = k\right) z^j = E\left[z^{\sum_{i=1}^{k} A_{ni}}\right]$

$\overset{\text{iid}}{=} \prod_{i=1}^{k} E[z^{A_{ni}}] = [\phi(z)]^k$

3f) $\phi_n(z) = E[z^{B_n}] = E\left[z^{\sum_{k=1}^{B_{n-1}} A_{nk}}\right] = E\left(E\left[z^{\sum_{k=1}^{B_{n-1}} A_{nk}} \mid B_{n-1}\right]\right)$

$= E\left[(\phi(z))^{B_{n-1}}\right] = \phi_{n-1}(\phi(z))$