Chi-square distribution

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1 Normal Distribution

Generally, the normal distribution encountered with σ^2 known, so the sufficient statistics for μ is $\sum_{i=1}^n y_i$. However, if σ^2 is unknown, the sufficient statistics for μ depends on σ^2, and we need to pay attention to that.

The second we need to pay attention is that, the normal distribution has different μ_i for each patient. And we usually write in the matrix form of the likelihood function.

The projection operator generally applies to normal distribution, so we will usually write in matrix form for multivariate normal distribution. Because the likelihood function could be considered as MVN, as we are estimating β, σ^2 using all the y_i simultaneously.

2 Chi-square distribution

If $Z_1, ..., Z_k$ are independent, standard normal random variables, then the sum of their squares,

$$Q = Z_i^2 \sim \chi^2(k)$$
$$p(k) = \frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1} exp(-\frac{x}{2})$$

3 Non-Central Chi-square distribution

The non-central chi-square distribution: Let $(X_1, X_2, \ldots, X_i, \ldots, X_k)$ be k independent, normally distributed random variables with means μ_i and unit variances. Then the random variable

$$Q = \sum_{i=1}^k X_i^2 \sim \chi^2(k, \lambda), \qquad \lambda = \sum_{i=1}^k \mu_i^2$$

where the degrees of freedom is k.

The sample mean of n i.i.d. chi-squared variables of degree k is distributed according to a gamma distribution with shape α and scale θ parameters:

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim Gamma(\alpha = nk/2, \theta = 2/n)$$

3.1 Lemma

Let $Q_i \sim \chi^2_{k_i}(\lambda_i)$ for $i = 1, \ldots, n$, be independent. Then, $Q = \sum_{i=1}^{n} Q_i$ is a noncentral $\chi^2_k(\lambda)$, where $k = \sum_{i=1}^{n} k_i$ and $\lambda = \sum_{i=1}^{n} \lambda_i$.

Proof:

The distribution transformation use moment generating function.

3.1.1 Moment Generating Function

We can get MGF from $E[x^2 t]$

$$\begin{aligned}
M_i(t) = E[x^2 t] &= \frac{1}{\sqrt{2\pi}} \int exp(x^2 t) exp\left(-\frac{(x-\mu)^2}{2}\right) dx \\
&= \frac{1}{\sqrt{2\pi}} \int exp\left((t - \frac{1}{2})x^2 + \mu x - \frac{\mu^2}{2}\right) dx \\
&= \frac{1}{\sqrt{2\pi}} \int exp\left(-\frac{1}{2}(1-2t)\{x^2 - \frac{2\mu x}{(1-2t)} + \frac{\mu^2}{(1-2t)^2}\} + \frac{\mu^2}{2(1-2t)} - \frac{\mu^2}{2}\right) dx \\
&= \frac{1}{\sqrt{(1-2t)}} \int \frac{(1-2t)}{\sqrt{2\pi}} exp\left(-\frac{(x - \frac{\mu}{1-2t})^2}{2(1-2t)^{-1}}\right) dx \left[exp\left(\frac{\mu^2 t}{1-2t}\right)\right] \\
&= \frac{1}{\sqrt{(1-2t)}} exp\left(\frac{\mu^2 t}{1-2t}\right), \qquad \lambda = \mu^2 \\
&= \frac{1}{\sqrt{(1-2t)}} exp\left(\frac{\lambda t}{1-2t}\right)
\end{aligned}$$

Then the MGF for $Q_i \sim \chi^2_{k_i}(\lambda_i)$

$$\begin{aligned}
M(t) = E[\sum_{i=1}^{k} x_i^2 t] &= \prod_{i=1}^{k} M_i(t) \\
&= \left(\frac{1}{\sqrt{(1-2t)}}\right)^k exp\left(\frac{\sum_{i=1}^{k} \lambda_i t}{1-2t}\right) \\
&= \left(\frac{1}{\sqrt{(1-2t)}}\right)^k exp\left(\frac{\lambda t}{1-2t}\right) \\
&= (1-2t)^{-k/2} exp\left(\frac{\lambda t}{1-2t}\right), \qquad \text{i.i.d}
\end{aligned}$$

The general case of a linear combination of independent $\chi^2_{k_i}(\lambda_i)$

$$Q = \sum_{i=1}^{k} a_i Q_i$$

We also can prove using MGF.

3.1.2 Linear Combination of Chi-Square Distribution

The linear combination of chi-square distribution Y_j. Let us denote by $X \sim \Gamma(r, \lambda)$ the fact that the r.v. X has a Gamma distribution with shape parameter r and rate parameter λ

$$f_X(x) = \frac{\lambda^x}{\Gamma(r)} exp(-\lambda x) x^{r-1}, \qquad (r, \lambda > 0, x > 0)$$

Then we have, for $j = 1, ..p$,

$$Y_j \sim \Gamma(\frac{k_j}{2}, \frac{1}{2}) \rightarrow Z_j = w_j Y_j \sim \Gamma(\frac{k_j}{2}, \frac{1}{2w_j})$$

The MGF for linear combinations $Z_j = w_j Y_j$

$$\begin{aligned} M(t) &= E[exp(Y_j t)] = (1-2t)^{-k/2} exp\left(\frac{\lambda t}{1-2t}\right) \\ M_{Z_j}(t) &= E[exp(w_j Y_j t)] = E[exp(Y_j(w_j t))] \\ &= (1-2w_j t)^{-1/2} exp\left(\frac{\lambda w_j t}{1-2w_j t}\right) \end{aligned}$$

$$\begin{aligned} M_Y(t) &= E[exp(Yt)] = E[exp(t[w_1Y_1 + w_2Y_2 + w_3Y_3 + ..w_nY_n])] \\ &= E[exp(w_1tY_1)]E[exp(w_2tY_2)]...E[exp(w_ntY_n)] \\ &= M_{X_1}(w_1t)M_{X_2}(w_2t)M_{X_3}(w_3t)..M_{X_n}(w_nt) \\ &= \prod_{i=1}^{n} M_{X_i}(w_it) \end{aligned}$$

The third equation comes from the properties of exponents, as wells as from the expectation of the product of functions of independent random variables.

I need to pay attention that, only under independent and identical situation, we can write

$$M_Y(t) = M_X(t)^n$$

Other than that, we can not further simplify that. So back to the non-central chi-square distribution, we have the MGF of Y

$$\begin{aligned} M_Y(t) &= \prod_{i=1}^{n} M_{X_i}(w_i t) \\ &= \prod_{i=1}^{n} (1 - 2w_j t)^{-1/2} exp\left(\frac{\lambda w_j t}{1 - 2w_j t}\right) \end{aligned}$$

Then we can see that the shape parameter is $\frac{1}{2w_i}$. If we want to have a non-central chi-square distribution for Y, then all w_j need to be the same.

4 Quadratic Forms

4.1 Independent

If $y \sim N(0, \sigma^2 I)$, M is a symmetric idempotent matrix of order n, and L is a $k \times n$ matrix, then Ly and $y'My$ are independently distribution if $LM = 0$.

Proof: The proof will use the orthogonal matrix Q, which is $Q'Q = I$, we could add this term wherever we want to.

Define the matrix Q so that

$$Q^T M Q = \Lambda = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

Let r denote the dimension of the identity matrix which is equal to the rank of M. Thus $r = trM$.

Let $v = Q'y$ and partition v as follows

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ .. \\ v_r \\ v_{r+1} \\ .. \\ v_n \end{bmatrix}$$

The number of elements of v_1 is r, while v_2 contains $n - r$ elements. Clearly v_1 and v_2 are independent of each other since they are independent standard normals. What we will show now is that $y'My$ depends only on v_1 and Ly depends only on v_2. Given that the v_i are independent, $y'My$ and Ly will be independent.

$$
\begin{aligned}
y'My &= v'Q'MQv \\
&= v' \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} v \\
&= v_1^T v_1
\end{aligned}
$$

Now consider the product of L and Q which we denote C. Partition C as (C_1, C_2). C_1 has k rows and r columns. C_2 has k rows and $n - r$ columns. Now consider the following product

$$
\begin{aligned}
C(Q'MQ) &= LQQ'MQ, \qquad & C = LQ \\
&= LMQ = 0, \qquad & LM = 0
\end{aligned}
$$

Now consider the product of C and matrix $Q'MQ$

$$
\begin{aligned}
C(Q'MQ) &= (C_1, C_2) \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \\
&= 0
\end{aligned}
$$

This implies that $C_1 = 0$, then implies that $LQ = C = (0, C_2)$

Now consider Ly. It can be written

$$
Ly = LQQ'y = Cv = C_2 v_2
$$

Now note that Ly depends only on v_2, and $y'My$ depends only on v_1. But since v_1 and v_2 are independent, so are Ly and $y'My$.