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2a) $$\phi(T) = \begin{cases} 1 & T < c_1(u) \text{ or } T > c_2(u) \\ \gamma_i & T = c_i(u) \quad i=1,2 \\ 0 & \text{else} \end{cases}$$ st $E_{\beta_1=0}[\phi \mid U] = \alpha$ & $E_{\beta_1=0}[U\phi(T) \mid U] = \alpha E(T \mid U)$

where $T = \sum x_i y_i$ $\quad U = \sum y_i$

b) $$E(T \mid U) = E\left[\sum_{i=1}^{n} X_i Y_i \,\Big|\, \sum_{i=1}^{n} Y_i = u\right]$$
$$= \sum_{i=1}^{n} x_i E\left(Y_i \,\Big|\, \sum_{j=1}^{n} Y_j = u\right)$$
$$= \sum_{i=1}^{n} x_i \Pr\left(Y_i = 1 \,\Big|\, \sum_{j=1}^{n} Y_j = u\right)$$

$$\Pr\left(Y_i = 1 \,\Big|\, \sum_{j=1}^{n} Y_j = u\right) = \frac{\Pr\left(Y_i = 1, \sum_{j=1}^{n} Y_j = u\right)}{\Pr\left(\sum_{j=1}^{n} Y_j = u\right)}$$

under null
$$\pi_0 = \Pr(Y_i = 1) \;\forall i = \frac{\exp\{\beta_0\}}{1 + \exp\{\beta_0\}}$$

$$\sum_{j=1}^{n} Y_j \sim Bin(n, \pi_0)$$

$$\Pr\left(Y_i = 1, \sum_{j=1}^{n} Y_j = u\right) = \Pr\left(Y_i = 1, \sum_{j \neq i} Y_j = u-1\right)$$
$$= \Pr(Y_i = 1) \cdot \Pr\left(\sum_{j \neq i} Y_j = u-1\right) \quad \text{by ind.}$$

$\sum_{j \neq i} Y_i \sim Bin(n-1, \pi_0)$

$$= \pi_0 \cdot \binom{n-1}{u-1} \pi_0^{u-1} (1-\pi_0)^{n-u}$$
$$= \binom{n-1}{u-1} \pi_0^{u} (1-\pi_0)^{n-u}$$

$$\Rightarrow \Pr\left(Y_i = 1 \,\Big|\, \sum_{j=1}^{n} Y_j = u\right) = \frac{\binom{n-1}{u-1} \pi_0^{u} (1-\pi_0)^{n-u}}{\binom{n}{u} \pi_0^{u} (1-\pi_0)^{n-u}} = \frac{u}{n}$$

$$\Rightarrow E(T \mid U) = \frac{u}{n} \sum_{i=1}^{n} x_i$$

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b) cont.

$$Var(T|u) = Var\left(\sum_{i=1}^{n} x_i Y_i \,\middle|\, \sum_{i=1}^{n} Y_i = u\right)$$

$$= \sum_{i=1}^{n}\left[x_i^2 Var\left(Y_i \,\middle|\, \sum_{k=1}^{n} Y_k = u\right)\right] + 2\sum_{1\le i\le j\le n} x_i x_j Cov\left(Y_i, Y_j \,\middle|\, \sum_{k=1}^{n} Y_k = u\right)$$

$$Var\left(Y_i \,\middle|\, \sum_{k=1}^{n} Y_k = u\right) = E\left(Y_i^2 \,\middle|\, \sum_{k=1}^{n} Y_k = u\right) - \left[E\left(Y_i \,\middle|\, \sum_{k=1}^{n} Y_k = u\right)\right]^2$$

$$= \frac{u}{n}\left(1 - \frac{u}{n}\right)$$

$$Cov\left(Y_i, Y_j \,\middle|\, \sum_{k=1}^{n} Y_k = u\right) = \underbrace{E\left(Y_i Y_j \,\middle|\, \sum_{k=1}^{n} Y_k = u\right)}_{} - E\left(Y_i \,\middle|\, \sum_{k=1}^{n} Y_k = u\right) \cdot E\left(Y_j \,\middle|\, \sum_{k=1}^{n} Y_k = u\right)$$

$$E\left(Y_i Y_j \,\middle|\, \sum_{k=1}^{n} Y_k = u\right) = Pr\left(Y_i = 1, Y_j = 1 \,\middle|\, \sum_{k=1}^{n} Y_k = u\right)$$

$$= \frac{Pr\left(Y_i = 1, Y_j = 1, \sum_{k=1}^{n} Y_k = u\right)}{Pr\left(\sum_{k=1}^{n} Y_k = u\right)}$$

$$= Pr(Y_i = 1) Pr(Y_j = 1) Pr\left(\sum_{k \ne i,j} Y_k = u - 2\right) / \ldots$$

$$= \pi_0^2 \binom{n-2}{u-2} \pi_0^{u-2} (1-\pi)^{n-u} / \ldots$$

$$= \frac{u}{n} \cdot \frac{u-1}{n-1}$$

$$Cov(\ldots) = \frac{u}{n} \cdot \frac{u-1}{n-1} - \left(\frac{u}{n}\right)^2$$

$$Var(T|U) = \sum_{i=1}^{n} x_i^2 \frac{u}{n}\left[1 - \frac{u}{n}\right] + 2\sum_{1\le i\le j\le n} x_i x_j \left[\frac{u(u-1)}{n(n-1)} - \left(\frac{u}{n}\right)^2\right]$$

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2c) $$\left.\begin{pmatrix} \frac{d}{d\beta_0} l \\ \frac{d}{d\beta_1} l \end{pmatrix}\right|_{\beta_1=0,\tilde{\beta}_0} = \begin{pmatrix} \sum y_i - n\tilde{\pi}_0 \\ \sum x_i(y_i - \tilde{\pi}_0) \end{pmatrix} \xrightarrow{} =0$$

$$\tilde{\pi}_0 \quad \frac{e^{\tilde{\beta}_0}}{1+e^{\tilde{\beta}_0}}$$

where $\tilde{\beta}_0$ is mle under null

$$\left.\begin{pmatrix} \frac{d^2}{d\beta_0^2} l & \frac{d^2}{d\beta_0 d\beta_1} l \\ \frac{d}{d\beta_0 d\beta_1} l & \frac{d^2}{d\beta_1^2} l \end{pmatrix}\right|_{\beta_1=0,\tilde{\beta}_0} = \begin{pmatrix} n\tilde{\pi}_0(1-\tilde{\pi}_0) & \tilde{\pi}_0(1-\tilde{\pi}_0)\sum x_i \\ \tilde{\pi}_0(1-\tilde{\pi}_0)\sum x_i & \tilde{\pi}_0(1-\tilde{\pi}_0)\sum x_i^2 \end{pmatrix}$$

$$\tilde{\beta}_0 = \log\left(\frac{\bar{y}}{1-\bar{y}}\right)$$

$$\Rightarrow \tilde{\pi}_0 = \frac{\frac{\bar{y}}{1-\bar{y}}}{1+\frac{\bar{y}}{1-\bar{y}}} = \frac{\frac{\bar{y}}{\cancel{1-\bar{y}}}}{\left(\frac{1-\cancel{\bar{y}}+\cancel{\bar{y}}}{\cancel{1-\bar{y}}}\right)} = \bar{y}$$

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d) $Y_1, \ldots, Y_n \overset{ind}{\sim} Bern(\pi_i)$

$$\pi_i = \frac{\exp\{\underset{\sim}{x}_i'\underset{\sim}{\beta}\}}{1+\exp\{\underset{\sim}{x}_i'\underset{\sim}{\beta}\}} \qquad \underset{\sim}{\beta} \in \mathbb{R}^p$$

$$\pm \underset{\sim}{\lambda}'\underset{\sim}{\beta}\left(\sum_{i=1}^n x_{i1}y_i + \ldots + \sum_{i=1}^n x_{ip}y_i\right)$$

Test

$H_0: \underset{\sim}{\lambda}'\underset{\sim}{\beta} = \theta_0 \qquad \underset{\sim}{\lambda} = (\lambda_1, \ldots, \lambda_p)$

$H_1: \underset{\sim}{\lambda}'\underset{\sim}{\beta} \neq \theta_0$

$$f_{\underset{\sim}{x}}(x \mid \underset{\sim}{\beta}) = \prod_{i=1}^n \frac{\exp\{y_i \underset{\sim}{x}_i'\underset{\sim}{\beta}\}}{1+\exp\{\underset{\sim}{x}_i'\underset{\sim}{\beta}\}}$$

$$= \frac{\exp\left\{\sum_{i=1}^n y_i \underset{\sim}{x}_i'\underset{\sim}{\beta}\right\}}{\prod_{i=1}^n \left[1+\exp\{\underset{\sim}{x}_i'\underset{\sim}{\beta}\}\right]}$$

$$= \exp\left\{\beta_1 \sum_{i=1}^n x_{i1}y_i + \ldots + \beta_p \sum_{i=1}^n x_{ip}y_i - \sum_{i=1}^n \ln\left(1+\exp\{\underset{\sim}{x}_i'\underset{\sim}{\beta}\}\right)\right\}$$

$$= \exp\left\{\beta_1 \sum_{i=1}^n x_{i1}y_i + \ldots + \beta_p \sum_{i=1}^n x_{ip}y_i + (\lambda_1\beta_1 + \ldots + \lambda_p\beta_p)\left(\sum_{i=1}^n x_{i1}y_i + \ldots + \sum_{i=1}^n x_{ip}y_i\right) - (\lambda_1\beta_1 + \ldots + \lambda_p\beta_p)\left(\sum_{i=1}^n x_{i1}y_i + \ldots + \sum_{i=1}^n x_{ip}y_i\right) - C(\underset{\sim}{\beta})\right\}$$

$$C(\underset{\sim}{\beta}) = \sum_{i=1}^n \ln(\cdots)$$

$$T(\underset{\sim}{x}) = \sum_{i=1}^n x_{i1}y_i + \ldots + \sum_{i=1}^n x_{ip}y_i$$

$$= \exp\left\{(\underset{\sim}{\lambda}'\underset{\sim}{\beta})T(\underset{\sim}{x}) + \beta_1\left(\sum x_{i1}y_i - \lambda_1 T(\underset{\sim}{x})\right) + \ldots + \beta_p\left(\sum x_{ip}y_i - \lambda_p T\right) - C(\underset{\sim}{\beta})\right\}$$