AMW

1 a) i) $\underline{E[U_i]} = E[1\{X_i = 0\}] = P(X_i = 0) = P\left(1(N>0) \cdot \sum_{j=1}^{N} Z_j = 0\right)$

$$= P(1(N>0) = 0) + \cancel{P\left(\sum_{j=1}^{N} Z_j = 0\right)}^{\,0}$$

b/c a conts. dist. so prob of a point mass is 0.

$$= P(N=0) = \frac{\lambda^0 e^{-\lambda}}{0!} = \boxed{e^{-\lambda}} \checkmark$$

ii) $\underline{E[X_i]} = E\left[1\{N>0\} \cdot \sum_{j=1}^{N} Z_j\right] = E\left[E\left[1\{N>0\} \cdot \sum_{j=1}^{N} Z_j \mid N\right]\right]$

$$= E\left[1\{N>0\} \sum_{j=1}^{N} E[Z_j] \mid N\right] = E[1\{N>0\} \cdot N \cdot 0] = E[0] = \boxed{0} \checkmark$$

iii) $\underline{E[X_i^2]} = Var(X_i) + (\cancel{E[X_i]}^{\,0})^2 = E[Var(X_i \mid N)] + Var[\cancel{E(X_i \mid N)}^{\,0}]$

$$= E\left[Var\left(1\{N>0\} \sum_{j=1}^{N} Z_j \mid N\right)\right] = E\left[\underbrace{1\{N>0\}}_{\text{square of an indicator is itself}} \sum_{j=1}^{N} Var(Z_j \mid N)\right]$$

$$= E[1\{N>0\} \cdot N\sigma^2] = \sigma^2 E[1\{N>0\} \cdot N] = \sigma^2 \sum_{N=1}^{\infty} N \frac{e^{-\lambda} \cdot \lambda^N}{N!}$$

$$= \sigma^2 \sum_{N=1}^{\infty} \frac{e^{-\lambda} \lambda^N}{(N-1)!} = \sigma^2 \sum_{N=1}^{\infty} \frac{e^{-\lambda} \cdot \lambda^{N-1} \cdot \lambda}{(N-1)!} = \lambda \sigma^2 \underbrace{\sum_{u=0}^{\infty} \frac{e^{-\lambda} \lambda^u}{u!}}_{=1} = \lambda \sigma^2$$

let $u = N-1$

iv) $\underline{E[X_i^4]} = Var(X_i^2) + [E(X_i^2)]^2 = Var(X_i^2) + \lambda^2 \sigma^4$

$$= E[Var(X_i^2 \mid N)] + Var[E(X_i^2 \mid N)] + \lambda^2 \sigma^4$$

$$= E\left[Var\left(1\{N>0\}\left(\sum_{j=1}^{N} Z_j \mid N\right)^2\right)\right] + Var\left[E\left(1\{N>0\}\left(\sum_{j=1}^{N} Z_j \mid N\right)^2\right)\right] + \lambda^2 \sigma^4$$

Note:
$Z_j \sim N(0, \sigma^2)$
$\Rightarrow \sum_{j=1}^{N} Z_j \sim N(0, N\sigma^2)$
$\Rightarrow \frac{\sum_{j=1}^{N} Z_j}{\sqrt{N\sigma^2}} \sim N(0,1)$
$\Rightarrow N\sigma^2 \left(\frac{\sum_{j=1}^{N} Z_j}{\sqrt{N\sigma^2}}\right)^2 \sim N\sigma^2 \chi^2_1$

$$= E[1\{N>0\} \cdot N^2 \sigma^4 \cdot 2] + Var[(1\{N>0\} \cdot N\sigma^2 \cdot 1)] + \lambda^2 \sigma^4$$

$$= 2\sigma^4 E[1\{N>0\} N^2] + \underbrace{\sigma^4 Var[1\{N>0\} N]}_{\sigma^4 E[1\{N>0\} N^2] - \underbrace{\sigma^4 [E\{1(N>0) N\}]^2}_{\sigma^4 \cdot \lambda^2}} + \lambda^2 \sigma^4$$

part iii)

$$= 3 \cdot \sigma^4 \sum_{N=1}^{\infty} N^2 \frac{e^{-\lambda} \lambda^N}{N!} + 0 = 2\sigma^4 e^{-\lambda}\left[\sum_{N=0}^{\infty} \frac{N^2 \lambda^N}{N!} - 0\right] + 0$$

Know (in general):

$$\sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z$$

$$\sum_{k=0}^{\infty} \frac{k z^k}{k!} = z e^z$$

&

$$\sum_{k=0}^{\infty} \frac{k^2 z^k}{k!} = (z + z^2) e^z$$

$$= 3\sigma^4 \cancel{e^{-\lambda}}\left[(\lambda + \lambda^2) \cancel{e^{+\lambda}}\right] = 3(\lambda + \lambda^2)\sigma^4.$$

1.b) WTS: i) $\hat{T}_n = -\log(\frac{1}{n}\sum_i U_i) \xrightarrow{a.s.} \lambda$

ii) $\hat{W}_n = \frac{\frac{1}{n}\sum_i X_i^2}{\hat{T}_n} \xrightarrow{a.s.} \sigma^2$ as $n \to \infty$

i) In part a), showed $E[U_i] = e^{-\lambda}$.

By SLLN, $\frac{1}{n}\sum_i U_i \xrightarrow{a.s.} e^{-\lambda}$.

Then, by CMT $-\log(\frac{1}{n}\sum_i U_i) \xrightarrow{a.s.} -\log(e^{-\lambda}) = \lambda$ as $n \to \infty$. ✓

ii) Know $E[X_i^2] = \lambda\sigma^2$ from part a)

By SLLN, $\frac{1}{n}\sum_i X_i^2 \xrightarrow{a.s.} E[X_i^2] = \lambda\sigma^2$

Know that, if $X_n \xrightarrow{a.s.} X$ and $Y_n \xrightarrow{a.s.} Y$, then $X_n/Y_n \xrightarrow{a.s.} X/Y$

(Proof: Know that convergence a.s. of $\{X_n\}$ and $\{Y_n\}$ implies their joint convergence a.s., that is, their joint convergence as a vector is $[X_n, Y_n] \xrightarrow{a.s.} [X, Y]$.
Let $g([X,Y]) = X/Y$ which is a continuous function for $Y \neq 0$.
By CMT,

$$\text{a.s.}\lim_{n\to\infty}(X_n/Y_n) = \text{a.s.}\lim_{n\to\infty} g[X_n, Y_n] = g(\text{a.s.}\lim_{n\to\infty}[X_n, Y_n]) = g[X,Y] = X/Y \text{ for } Y \neq 0)$$

Thus, since $\frac{1}{n}\sum_i X_i^2 \xrightarrow{a.s.} \lambda\sigma^2$ & $\hat{T}_n \xrightarrow{a.s.} \lambda$, by the above proof then

$$\frac{\frac{1}{n}\sum_i X_i^2}{\hat{T}_n} \xrightarrow{a.s.} \frac{\lambda\sigma^2}{\lambda} = \sigma^2 \text{ as } n \to \infty. ✓$$

AMW

1.c) Show that $\sqrt{n}\begin{pmatrix} \frac{1}{n}\sum_i U_i - e^{-\lambda} \\ \frac{1}{n}\sum_i X_i^2 - \sigma^2 \end{pmatrix} \xrightarrow{\lambda} N(0, \Sigma_i^2)$

Know from b) that $\frac{1}{n}\sum_i U_i \xrightarrow{a.s.} e^{-\lambda}$ & $\frac{1}{n}\sum_i X_i^2 \xrightarrow{a.s.} \lambda\sigma^2$

Then, by $\boxed{\text{multivariate CLT}}$, $\sqrt{n}\begin{pmatrix} \frac{1}{n}\sum_i U_i - e^{-\lambda} \\ \frac{1}{n}\sum_i X_i^2 - \sigma^2 \end{pmatrix} \xrightarrow{d} N(0, \Sigma)$

↑ Not fucking Delta Method! Lesson learned!

where $\Sigma = \begin{pmatrix} Var(U_i) & Cov(U_i, X_i^2) \\ & Var(X_i^2) \end{pmatrix}$

where $\underline{Var(U_i)} = Var(1\{X_i = 0\}) = E[\underbrace{1\{X_i=0\}^2}_{=1\{X_i=0\}}] - (E[1\{X_i=0\}])^2$

$= P(X_i=0) - [P(X_i=0)]^2 = \frac{e^{-\lambda}\lambda^0}{0!} - \left(\frac{e^{-\lambda}\lambda^0}{0!}\right)^2 = e^{-\lambda} - e^{-2\lambda} = e^{-\lambda}(1-e^{-\lambda})$

$\underline{Cov(U_i, X_i^2)} = E[U_i \cdot X_i^2] - \underbrace{E[U_i]}_{e^{-\lambda}\text{ part a)}} \cdot \underbrace{E[X_i^2]}_{\lambda\sigma^2\text{ part a)}} = E[1\{X_i=0\}\cdot X_i^2] - \lambda e^{-\lambda}\sigma^2$

$= E\left[E[1\{X_i=0\}X_i^2 \mid X_i=0]\right] - \lambda e^{-\lambda}\sigma^2 = E[1\{X_i=0\}\cancel{\underbrace{E[X_i^2 \mid X_i=0]}_{0}}] - \lambda e^{-\lambda}\sigma^2 = -\lambda e^{-\lambda}\sigma^2$

$Var(X_i^2) = \underbrace{E[X_i^4]}_{3(\lambda+\lambda^2)\sigma^4\text{ part a)}} - (\underbrace{E[X_i^2]}_{\lambda\sigma^2\text{ part a)}})^2 = 3(\lambda+\lambda^2)\sigma^4 - \lambda^2\sigma^4 = 3\lambda\sigma^4 + \underline{3\lambda^2\sigma^4} - \underline{\lambda^2\sigma^4}$

$= 3\lambda\sigma^4 + 2\lambda^2\sigma^4$

$= (3+2\lambda)\lambda\sigma^4$

$\Rightarrow \Sigma = \begin{pmatrix} e^{-\lambda}(1-e^{-\lambda}) & -\lambda e^{-\lambda}\sigma^2 \\ & (3+2\lambda)\lambda\sigma^4 \end{pmatrix}$

AMW

1 d) Show that $\sqrt{n}\begin{pmatrix}\hat{T}_n - \lambda \\ \hat{W}_n - \sigma^2\end{pmatrix} \xrightarrow{d} N(0, \tau_2^2)$

as $n \to \infty$ where $\tau_2^2 = \begin{pmatrix} e^\lambda - 1 & -(e^\lambda-\lambda-1)\sigma^2/\lambda \\ -(e^\lambda-\lambda-1)\sigma^2/\lambda & \left(\frac{e^\lambda-1}{\lambda^2}+2+\frac{1}{\lambda}\right)\sigma^4\end{pmatrix}$

From b), $\hat{T}_n = -\log\left(\frac{1}{n}\sum_i U_i\right) \xrightarrow{a.s.} \lambda$

$\hat{W}_n = \frac{\frac{1}{n}\sum_i X_i^2}{\hat{T}_n}$ where $\frac{1}{n}\sum_i X_i^2 \xrightarrow{a.s.} \lambda\sigma^2$

Since $\sqrt{n}\begin{pmatrix}\frac{1}{n}\sum_i U_i - e^{-\lambda} \\ \frac{1}{n}\sum_i X_i^2 - \sigma^2\end{pmatrix} \xrightarrow{d} N(0, \tau_1^2)$ from part e)

Then, by Delta method, have

$$\sqrt{n}\left(g\left(\tfrac{1}{n}\textstyle\sum_i U_i, \tfrac{1}{n}\sum_i X_i^2\right) - g(e^{-\lambda}, \lambda\sigma^2)\right) \xrightarrow{d} N(0, \nabla g\ \tau_1^2\ \nabla g')$$

Where $g(a,b) = \begin{pmatrix}-\log(a) \\ -b/\log(a)\end{pmatrix} \Rightarrow \nabla g(a,b) = \begin{pmatrix}\frac{\partial}{\partial a}(-\log(a)) & \frac{\partial}{\partial b}(-\log(a)) \\ \frac{\partial}{\partial a}\left(-\frac{b}{\log(a)}\right) & \frac{\partial}{\partial b}\underbrace{\left(\frac{-b}{\log(a)}\right)}_{=-b[\log(a)]^{-1}}\end{pmatrix} = \begin{pmatrix}-1/a & 0 \\ \frac{b}{a[\log(a)]^2} & -\frac{1}{\log(a)}\end{pmatrix}$

sub $a = e^{-\lambda}$, $b = \lambda\sigma^2$

$$= \begin{pmatrix}-e^{\lambda} & 0 \\ \frac{\lambda\sigma^2}{\underbrace{e^{-\lambda}[\log(e^{-\lambda})]^2}_{\lambda^2}} & -\frac{1}{\underbrace{\log(e^{-\lambda})}_{-\lambda}}\end{pmatrix} = \begin{pmatrix}-e^{\lambda} & 0 \\ \frac{\sigma^2 e^{\lambda}}{\lambda} & 1/\lambda\end{pmatrix}$$

Then, $\nabla g' \tau_1^2 \nabla g = \begin{pmatrix}-e^{\lambda} & 0 \\ \frac{\sigma^2 e^{\lambda}}{\lambda} & 1/\lambda\end{pmatrix}\begin{pmatrix}e^{-\lambda}(1-e^{-\lambda}) & -\lambda e^{-\lambda}\sigma^2 \\ -\lambda e^{-\lambda}\sigma^2 & (3+2\lambda)\lambda\sigma^4\end{pmatrix}\begin{pmatrix}-e^{\lambda} & \frac{\sigma^2 e^{\lambda}}{\lambda} \\ 0 & 1/\lambda\end{pmatrix}$

$$= \begin{pmatrix} e^{-\lambda} - 1 & \lambda\sigma^2 \\ \left(\frac{\sigma^2(1-e^{-\lambda})}{\lambda} - e^{-\lambda}\sigma^2\right) & \left(-\sigma^4 + (3+2\lambda)\sigma^4\right)\end{pmatrix}\begin{pmatrix}-e^{\lambda} & \frac{\sigma^2 e^{\lambda}}{\lambda} \\ 0 & 1/\lambda\end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda} - 1 & \left[\frac{(e^{-\lambda}-1)\sigma^2 e^{\lambda}}{\lambda} + \sigma^2\right] \\ \left[-\frac{\sigma^2(1-e^{-\lambda})e^{\lambda}}{\lambda} + \sigma^2\right] & \left[\frac{\sigma^4 e^{\lambda}(1-e^{-\lambda})}{\lambda^2} - \frac{\sigma^4}{\lambda}\right] - \frac{\sigma^4}{\lambda} + \frac{(3+2\lambda)\sigma^4}{\lambda}\end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda} - 1 & \frac{-\sigma^2(e^{\lambda}-1)+\lambda\sigma^2}{\lambda} \\ \frac{-\sigma^2(e^{\lambda}-1)+\lambda\sigma^2}{\lambda} & \frac{\sigma^4(e^{\lambda}-1)-\lambda\sigma^4}{\lambda^2} - \frac{\lambda\sigma^4}{\lambda^2} + \frac{(3+2\lambda)\lambda\sigma^4}{\lambda^2}\end{pmatrix} = \begin{pmatrix} e^{\lambda} - 1 & \frac{-\sigma^2(e^{\lambda}-1-\lambda)}{\lambda} \\ \frac{-\sigma^2(e^{\lambda}-1-\lambda)}{\lambda} & \left(\frac{e^{\lambda}-1}{\lambda^2}+\frac{1}{\lambda}+2\right)\sigma^4\end{pmatrix}$$

$$\frac{\sigma^4 e^{\lambda} - \sigma^4 - \lambda\sigma^4 - \lambda\sigma^4 + 3\lambda\sigma^4 + 2\lambda^2\sigma^4}{\lambda^2}$$

$$\frac{\sigma^4 e^{\lambda} - \sigma^4 + \lambda\sigma^4 + 2\lambda^2\sigma^4}{\lambda^2}$$

$$= \frac{\sigma^4(e^{\lambda}-1)}{\lambda^2} + \frac{\sigma^4}{\lambda} + 2\sigma^4$$

$$= \left(\frac{e^{\lambda}-1}{\lambda^2} + \frac{1}{\lambda} + 2\right)\sigma^4$$

✓

AMW

1.e) Show that $\hat{W}_n \pm Z_{1-\alpha/2}\,\hat{\rho}_n/\sqrt{n}$ where $\hat{\rho}^2 = \left(\frac{e^{\hat{T}_n}-1}{\hat{T}_n^2} + 2 + \frac{1}{\hat{T}_n}\right)\hat{W}_n^2$

and Z_q is the q^{th} quantile of a standard normal is an asymptotically valid $1-\alpha$ level CI for σ^2.

[Know $\hat{T}_n \xrightarrow{a.s.} \lambda$ and $\hat{W}_n \xrightarrow{a.s.} \sigma^2$ by part b).

$\overset{\text{by CMT}}{\Rightarrow} \quad (\hat{W}_n)^2 \xrightarrow{a.s.} (\sigma^2)^2 = \sigma^4$

Using the results from d), we know that (marginally),

$$\sqrt{n}\,(\hat{W}_n - \sigma^2) \xrightarrow{d} N\left(0,\ \left(\frac{e^{\lambda}-1}{\lambda^2} + 2 + \frac{1}{\lambda}\right)\sigma^4\right)$$

Thus, $Var(\hat{W}_n) = \frac{1}{\sqrt{n}}\left(\frac{e^{\hat{T}_n}-1}{\hat{T}_n^2} + 2 + \frac{1}{\hat{T}_n}\right)\hat{W}_n^2$

since $\sqrt{n}\,Var(\hat{W}_n) \xrightarrow{a.s.} \left(\frac{e^{\lambda}-1}{\lambda^2} + 2 + \frac{1}{\lambda}\right)\sigma^4$

Thus, $(1-\alpha)\times 100\%\ CI(\sigma^2) = \hat{W}_n \pm \underbrace{Z_{1-\alpha/2}\,\hat{\rho}/\sqrt{n}}$ where $\hat{\rho}^2 = \left(\frac{e^{\hat{T}_n}-1}{\hat{T}_n^2} + 2 + \frac{1}{\hat{T}_n}\right)\hat{W}_n^2$

$\hat{W}_n \downarrow a.s.\ \sigma^2$

$\downarrow$ a.s. (by CMT)

$Z_{1-\alpha/2}\left(\frac{e^{\lambda}-1}{\lambda^2} + 2 + \frac{1}{\lambda}\right)$

$\Rightarrow \quad (1-\alpha)\times 100\%\ CI(\sigma^2) \xrightarrow{a.s.} \sigma^2 + Z_{1-\alpha/2}\left(\frac{e^{\lambda}-1}{\lambda^2} + 2 + \frac{1}{\lambda}\right)$ by a final application of CMT]