1. (25 points) Let N be Poisson distributed with parameter $0 < \lambda < \infty$, and let $Z_1, Z_2, \ldots$ be an i.i.d. sequence of $N(0, \sigma^2)$ random variables, independent of N, with $0 < \sigma^2 < \infty$. Let

$$X = 1\{N > 0\} \sum_{j=1}^{N} Z_j,$$

where $1\{A\}$ is the indicator of A. Let $X_1, \ldots, X_n$ be i.i.d. realizations of X, and let $U_i = 1\{X_i = 0\}$, $1 \leq i \leq n$. Do the following:

(a) (4 points) Show that $EU_i = e^{-\lambda}$, $EX_i = 0$, $EX_i^2 = \lambda\sigma^2$, and $EX_i^4 = 3(\lambda + \lambda^2)\sigma^4$.

(b) (5 points) Show that $\hat{T}_n = -\log(n^{-1}\sum_{i=1}^n U_i)$ is almost surely consistent for λ, and that $\hat{W}_n = n^{-1}\sum_{i=1}^n X_i^2/\hat{T}_n$ is almost surely consistent for σ^2, as $n \to \infty$.

(c) (5 points) Show that

$$\sqrt{n}\begin{pmatrix} n^{-1}\sum_{i=1}^n U_i - e^{-\lambda} \\ n^{-1}\sum_{i=1}^n X_i^2 - \lambda\sigma^2 \end{pmatrix} \to_d N(0, \tau_1^2),$$

as $n \to \infty$, and give the form of τ_1^2.

(d) (6 points) Show that

$$\sqrt{n}\begin{pmatrix} \hat{T}_n - \lambda \\ \hat{W}_n - \sigma^2 \end{pmatrix} \to_d N(0, \tau_2^2),$$

as $n \to \infty$, where

$$\tau_2^2 = \begin{pmatrix} e^{\lambda} - 1 & -(e^{\lambda} - \lambda - 1)\sigma^2/\lambda \\ -(e^{\lambda} - \lambda - 1)\sigma^2/\lambda & \left(\frac{e^{\lambda}-1}{\lambda^2} + 2 + \frac{1}{\lambda}\right)\sigma^4 \end{pmatrix}.$$

(e) (5 points) Show that $\hat{W}_n \pm z_{1-\alpha/2}\hat{\rho}_n/\sqrt{n}$, where

$$\hat{\rho}^2 = \left(\frac{e^{\hat{T}_n} - 1}{\hat{T}_n^2} + 2 + \frac{1}{\hat{T}_n}\right)\hat{W}_n^2$$

and z_q is the qth-quantile of a standard normal, is an asymptotically valid $1-\alpha$ level confidence interval for σ^2.

1) $N \sim \text{Poisson}(\lambda)$; $0 < \lambda < \infty$ $\rightarrow f_N = \dfrac{\lambda^N e^{-\lambda}}{N!}$

Let $Z_1, Z_2, \ldots$ iid sequence of $N(0,\sigma^2)$ RV's $\perp\!\!\!\perp N$; $0 < \sigma^2 < \infty$

Let $X = I(N>0) \sum_{j=1}^{N} Z_j$; If $Z_j \sim N(0,\sigma^2)$, then $\sum_{j=1}^{N} Z_j \sim N(0, N\sigma^2)$

$\dfrac{\sum_{j=1}^{N} Z_j}{\sqrt{N\sigma^2}} \sim N(0,1)$ $\Rightarrow \dfrac{\left(\sum_{j=1}^{N} Z_j\right)^2}{N\sigma^2} \sim \chi^2_1$

$\sum_{j=1}^{N} Z_j \sim (N\sigma^2)\chi^2_1$

$X_1, \ldots, X_n$ are iid realizations of X, and $U_i = I(X_i = 0)$; $i \in [1,n]$

(a) Show that $E(U_i) = e^{-\lambda}$, $E(X_i) = 0$, $E(X_i^2) = \lambda\sigma^2$, and $E(X_i^4) = 3(\lambda+\lambda^2)\sigma^4$

- First, $E(U_i) = E(I(X_i=0)) = P(X_i=0) = P(I(N>0)\sum_{j=1}^{N} Z_j = 0) = P(I(N>0)=0) + P(\sum_{j=1}^{N} Z_j = 0)$ ← need to condition on N

due to $Z_j \perp\!\!\!\perp N$

$= P(N \leq 0) + 0 = \dfrac{\lambda^0 e^{-\lambda}}{0!} = \boxed{e^{-\lambda}}$ ✓

- Now, $E(X_i) = E\left[I(N>0)\sum_{j=1}^{N} Z_j\right] = E\left[E\left[I(N>0)\sum_{j=1}^{N} Z_j \mid N\right]\right] = E\left[I(N>0)\,E\left(\sum_{j=1}^{N} Z_j \mid N\right)\right]$

$= E[I(N>0)\cdot 0] = E(0) = \boxed{0}$ ✓

- $E(X_i^2) = E\left[\left(I(N>0)\sum_{j=1}^{N} Z_j\right)^2\right] = E\left[I(N>0)\left(\sum_{j=1}^{N} Z_j\right)^2\right] = E\left[I(N>0)\,E\left(\left(\sum_{j=1}^{N} Z_j\right)^2 \mid N\right)\right]$

← squaring an indicator stays same

$\sum_{0}^{\infty} \dfrac{a^N}{N!} = e^a$

$= E\left[I(N>0)\left\{Var\left(\sum_{j=1}^{N} Z_j \mid N\right) + E\left(\sum_{j=1}^{N} Z_j \mid N\right)^2\right\}\right] = E\left[I(N>0)\{N\sigma^2 + 0\}\right]$

$= \sigma^2 E[I(N>0)N] = \sigma^2 \sum_{N=1}^{\infty} N \dfrac{\lambda^N e^{-\lambda}}{N!} = \sigma^2 \sum_{N=0}^{\infty} \dfrac{\lambda^N e^{-\lambda}}{(N-1)!} = \sigma^2\lambda \sum_{N=0}^{\infty} \dfrac{\lambda^{N-1} e^{-\lambda}}{(N-1)!}$

$= \sigma^2\lambda e^{-\lambda}\left(\sum_{N=0}^{\infty} \dfrac{\lambda^{N-1}}{(N-1)!}\right) = \sigma^2\lambda e^{-\lambda} e^{\lambda} = \boxed{\sigma^2\lambda}$ ✓

- Factorial moments Cancel in Poisson!
- Could use MGF instead

- $E(X_i^4) = E\left(I(N>0)\left(\sum_{j=1}^{N} Z_j\right)^4\right) = E\left(I(N>0)\,E\left(\left(\sum_{j=1}^{N} Z_j\right)^4 \mid N\right)\right)$

$= E\left(I(N>0)\left\{Var\left(\left(\sum_{j=1}^{N} Z_j\right)^2 \mid N\right) + E\left(\left(\sum_{j=1}^{N} Z_j\right)^2 \mid N\right)^2\right\}\right)$

$= E\left(I(N>0)\left\{2(1)(N\sigma^2)^2 + (N\sigma^2)^2(1)\right\}\right)$

$= E\left(I(N>0)\left\{3(N\sigma^2)^2\right\}\right) = 3E\left(I(N>0)N^2\sigma^4\right) = 3\sigma^4 E\left(I(N>0)N^2\right)$

$= 3\sigma^4 \sum_{N=1}^{\infty} N^2 \dfrac{\lambda^N e^{-\lambda}}{N!} = 3\sigma^4 e^{-\lambda} \sum_{N=0}^{\infty} N \dfrac{\lambda^N}{(N-1)!} = 3\sigma^4 e^{-\lambda}\lambda \sum_{N=0}^{\infty} N \dfrac{\lambda^{N-1}}{(N-1)!}$

$= 3\sigma^4 e^{-\lambda}\lambda \sum_{N=0}^{\infty}\left\{\dfrac{(N-1)\lambda^{N-1}}{(N-1)!} + 1\left(\dfrac{\lambda^{N-1}}{(N-1)!}\right)\right\} = 3\sigma^4 e^{-\lambda}\lambda\left\{\lambda e^{\lambda} + e^{\lambda}\right\} = \boxed{3\sigma^4(\lambda^2+\lambda)}$ ✓

(b) Show that $\hat{T}_n = -\log\left(\frac{1}{n}\sum_{i=1}^{n} U_i\right)$ is a.s. consistent for λ and that $\hat{W}_n = \frac{1}{n}\sum_{i=1}^{n} X_i^2 / \hat{T}_n$ is a.s. for σ^2 as $n \to \infty$.

(DEF: $\hat{T}_n \xrightarrow[as]{} \lambda$ iff $P\left(\sup_{m \geq n} |\hat{T}_m - \lambda| > \varepsilon\right) \to 0$)

First, let's focus on $\frac{1}{n}\sum_{i=1}^{n} U_i$

By SLLN, $\frac{1}{n}\sum_{i=1}^{n} U_i \xrightarrow[as]{} E(U_i) = e^{-\lambda}$, as shown in (a).

Then, let $g(a) = -\log(a)$, which is continuous function $\forall a > 0$, so by continuous mapping theorem,

$$\hat{T}_n = -\log\left(\frac{1}{n}\sum_{i=1}^{n} U_i\right) \xrightarrow[as]{} -\log(e^{-\lambda}) = \lambda \checkmark$$

Now, let's focus on $\frac{1}{n}\sum_{i=1}^{n} X_i^2$.

Again, by SLLN, $\frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow[as]{} E(X_i^2) = \lambda\sigma^2$, from (a).

So, we have $\begin{pmatrix} \hat{T}_n \\ \frac{1}{n}\sum_{i=1}^{n} X_i^2 \end{pmatrix} \xrightarrow[as]{} \begin{pmatrix} \lambda \\ \lambda\sigma^2 \end{pmatrix}$

← note, we could not use this for convergence in dist.

Unless: $X_n \to_d X$, $Y_n \to_d Y$ and X or Y are constant, then $\begin{pmatrix} X_n \\ Y_n \end{pmatrix} \xrightarrow[d]{} \begin{pmatrix} X \\ Y \end{pmatrix}$

Now, let $h(a,b) = b/a$, Then again by CMT,

* Note $P(\hat{T}_n = 0)$ here, so $h(a,b)$ is continuous!

↳ $P(\hat{T}_n = 0) = P\left(-\log\left(\frac{1}{n}\sum_{i=1}^{n} U_i\right) = 0\right)$

$= P\left(\sum_{i=1}^{n} U_i = n\right)$

$= P\left(\sum_{i=1}^{n} I(X_i = 0) = n\right) = 0$

b/c all X_i's not 0!

$$\hat{W}_n = \frac{\frac{1}{n}\sum_{i=1}^{n} X_i^2}{\hat{T}_n} \xrightarrow[as]{} \frac{\lambda\sigma^2}{\lambda} = \sigma^2 \checkmark$$

(c) Show that $\sqrt{n}\begin{pmatrix} \frac{1}{n}\sum_{i=1}^{n} U_i - e^{-\lambda} \\ \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \lambda\sigma^2 \end{pmatrix} \xrightarrow{d} N(0, \Upsilon_1^2)$ and give Υ_1^2

<u>Note</u>: We know the second moments are bdd, so we can use CLT!

We know by CLT that $\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} U_i - E(U_i)\right) \xrightarrow{d} N(0, Var(U_i))$

$\hookrightarrow E(U_i) = e^{-\lambda}$ and
$$Var(U_i) = E\left((I(X_i=0))^2\right) - E\left(I(X_i=0)\right)^2$$
$$= E\left(I(X_i=0)\right) - E\left(I(X_i=0)\right)^2$$
$$= E(U_i) - E(U_i)^2 = e^{-\lambda} - e^{-2\lambda}$$

Thus, $\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} U_i - e^{-\lambda}\right) \xrightarrow{d} N(0, e^{-\lambda} - e^{-2\lambda})$

Also by CLT, we know $\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - E(X_i^2)\right) \xrightarrow{d} N(0, Var(X_i^2))$

$\hookrightarrow E(X_i^2) = \lambda\sigma^2$ and
$$Var(X_i^2) = E(X_i^4) - E(X_i^2)^2$$
$$= 3(\lambda+\lambda^2)\sigma^4 - (\lambda\sigma^2)^2$$
$$= 3\sigma^4(\lambda+\lambda^2) - \lambda^2\sigma^4$$
$$= 3\sigma^4\lambda^2 - \lambda^2\sigma^4 + 3\sigma^4\lambda$$
$$= 2\sigma^4\lambda^2 + 3\sigma^4\lambda = \sigma^4\lambda(2\lambda+3)$$

Thus, $\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \lambda\sigma^2\right) \xrightarrow{d} N(0, \sigma^4\lambda(2\lambda+3))$

By properties of MVN, we have

$$\sqrt{n}\begin{pmatrix} \frac{1}{n}\sum_{i=1}^{n} U_i - e^{-\lambda} \\ \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \lambda\sigma^2 \end{pmatrix} \xrightarrow{d} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Var(U_i) & Cov(U_i, X_i^2) \\ Cov(X_i^2, U_i) & Var(X_i^2) \end{pmatrix}\right)$$

$\hookrightarrow Cov(U_i, X_i^2) = Cov\left(I(X_i=0), X_i^2\right) = E\left[I(X_i=0)X_i^2\right] - E\left(I(X_i=0)\right)E(X_i^2)$

$$= \underset{\substack{\uparrow \\ \text{either } I(X_i=0)=0 \text{ or } X_i=0}}{0} - E(U_i)E(X_i^2) = -(e^{-\lambda}\cdot\lambda\sigma^2) = -\lambda\sigma^2 e^{-\lambda}$$

Thus, $\sqrt{n}\begin{pmatrix} \frac{1}{n}\sum_{i=1}^{n} U_i - e^{-\lambda} \\ \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \lambda\sigma^2 \end{pmatrix} \xrightarrow{d} N(0, \Upsilon_1^2)$, where $\Upsilon_1^2 = \begin{pmatrix} e^{-\lambda} - e^{-2\lambda} & -\lambda\sigma^2 e^{-\lambda} \\ -\lambda\sigma^2 e^{-\lambda} & \sigma^4\lambda(2\lambda+3) \end{pmatrix}$

(d) Show that $\sqrt{n}\begin{pmatrix}\hat{T}_n-\lambda\\ \hat{W}_n-\sigma^2\end{pmatrix}\xrightarrow{d} N(0,\tau_2^2)$; $\tau_2^2 = \begin{pmatrix} e^\lambda-1 & -(e^\lambda-\lambda-1)\sigma^2/\lambda \\ -(e^\lambda-\lambda-1)\sigma^2/\lambda & \left(\frac{e^\lambda-1}{\lambda^2}+2+\frac{1}{\lambda}\right)\sigma^4\end{pmatrix}$

From (c), we have $\sqrt{n}\begin{pmatrix}\frac{1}{n}\sum_{i=1}^n U_i - e^{-\lambda}\\ \frac{1}{n}\sum_{i=1}^n X_i^2-\lambda\sigma^2\end{pmatrix}\xrightarrow{d} N(0,\tau_1^2)$

Now, let $g\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}-\log(a)\\ b/-\log(a)\end{pmatrix}$, then $\nabla g\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}-\frac{1}{a} & 0\\ \frac{b(1/a)}{(\log(a))^2} & \frac{-1}{\log(a)}\end{pmatrix}$

So, $\nabla g\begin{pmatrix}e^{-\lambda}\\ \lambda\sigma^2\end{pmatrix} = \begin{pmatrix}-1/e^{-\lambda} & 0\\ \frac{\lambda\sigma^2/e^{-\lambda}}{\lambda^2} & 1/\lambda\end{pmatrix} = \begin{pmatrix}-e^\lambda & 0\\ \frac{\sigma^2 e^\lambda}{\lambda} & \frac{1}{\lambda}\end{pmatrix}$

Then, by multivariate Delta method,

$$\sqrt{n}\begin{pmatrix}\hat{T}_n - \overbrace{(-\log(e^{-\lambda}))}^{\lambda}\\ \hat{W}_n - \underbrace{(\lambda\sigma^2/-\log(e^{-\lambda}))}_{\sigma^2}\end{pmatrix}\xrightarrow{d} N\Big(0, \underbrace{\nabla g\begin{pmatrix}e^{-\lambda}\\ \lambda\sigma^2\end{pmatrix}\tau_1^2\,\nabla g'\begin{pmatrix}e^{-\lambda}\\ \lambda\sigma^2\end{pmatrix}}_{\tau_2^2}\Big)$$

$$(\nabla g)'\tau_1^2(\nabla g) = \begin{pmatrix}-e^\lambda & 0\\ \frac{\sigma^2e^\lambda}{\lambda} & \frac{1}{\lambda}\end{pmatrix}\begin{pmatrix}e^{-\lambda}-e^{-2\lambda} & -\lambda\sigma^2e^{-\lambda}\\ -\lambda\sigma^2e^{-\lambda} & \sigma^4\lambda(2\lambda+3)\end{pmatrix}\nabla g'$$

$$= \begin{pmatrix}-1+e^{-\lambda} & \lambda\sigma^2\\ \frac{\sigma^2}{\lambda}-\frac{\sigma^2e^{-\lambda}}{\lambda}-\sigma^2e^{-\lambda} & -\sigma^4+\sigma^4(2\lambda+3)\end{pmatrix}\begin{pmatrix}-e^\lambda & \frac{\sigma^2e^\lambda}{\lambda}\\ 0 & \frac{1}{\lambda}\end{pmatrix}$$

$$= \begin{pmatrix}e^\lambda-1 & \frac{-\sigma^2e^\lambda+\sigma^2+\lambda\sigma^2}{\lambda}\\ \frac{-\sigma^2e^\lambda+\sigma^2+\lambda\sigma^2}{\lambda} & \frac{\sigma^4e^\lambda}{\lambda^2}-\frac{\sigma^4}{\lambda^2}\underbrace{-\frac{\sigma^4}{\lambda}-\frac{\sigma^4}{\lambda}+\frac{\sigma^4(2\lambda+3)}{\lambda}}_{-\frac{2\sigma^4}{\lambda}+\frac{2\lambda\sigma^4}{\lambda}+\frac{3\sigma^4}{\lambda}=\frac{\sigma^4}{\lambda}+2\sigma^4}\end{pmatrix}$$

$$\Rightarrow \tau_2^2 = \begin{pmatrix}e^\lambda-1 & -(e^\lambda-\lambda-1)\sigma^2/\lambda\\ -(e^\lambda-\lambda-1)\sigma^2/\lambda & \left(\frac{e^\lambda-1}{\lambda}+2+\frac{1}{\lambda}\right)\sigma^4\end{pmatrix}$$ ✓

(e) Show that $\hat{W}_n \pm Z_{1-\alpha/2}\, \hat{\rho}_n / \sqrt{n}$ where $\hat{\rho}^2 = \left(\frac{e^{\hat{T}_n} - 1}{\hat{T}_n} + 2 + \frac{1}{\hat{T}_n} \right) \hat{W}_n^2$ and Z_q is the q^{th}- quantile of std Normal is an asymptotically valid $1-\alpha$ level CI for σ^2

Again, from (c), $\sqrt{n} \begin{pmatrix} \frac{1}{n}\sum U_i - e^{-\lambda} \\ \frac{1}{n}\sum X_i^2 - \lambda\sigma^2 \end{pmatrix} \xrightarrow{d} N(0, T_1^2)$

Now, let $g(a,b) = \frac{b}{-\log(a)}$, so $\nabla g(a,b) = \left(\frac{b/a}{\log(a)^2}, \frac{-1}{\log(a)} \right)$

so $\nabla g(e^{-\lambda}, \lambda\sigma^2) = \left(\frac{\lambda\sigma^2 e^{\lambda}}{\lambda^2}, \frac{-1}{-\lambda} \right) = \left(\frac{\sigma^2 e^{\lambda}}{\lambda}, \frac{1}{\lambda} \right)$

Then, by Delta method,

$$\sqrt{n} \left(\hat{W}_n - \frac{\lambda\sigma^2}{\lambda} \right) \xrightarrow{d} N\left(0, \nabla g' T_1^2 \nabla g\right)$$

$$\nabla g' T_1^2 \nabla g = \left(\frac{\sigma^2 e^{\lambda}}{\lambda}, \frac{1}{\lambda} \right)_{1\times 2} \begin{pmatrix} e^{-\lambda} - e^{-2\lambda} & -\lambda\sigma^2 e^{-\lambda} \\ -\lambda\sigma^2 e^{-\lambda} & \sigma^4 \lambda(2\lambda+3) \end{pmatrix}_{2\times 2} \begin{pmatrix} \sigma^2 e^{\lambda}/\lambda \\ 1/\lambda \end{pmatrix}_{2\times 1}$$

$$= \left(\frac{\sigma^2 e^{\lambda}(e^{-\lambda} - e^{-2\lambda})}{\lambda} - \sigma^2 e^{-\lambda}, -\sigma^4 + e^{\lambda}\sigma^4(2\lambda+3) \right) \begin{pmatrix} \sigma^2/\lambda \\ 1/\lambda \end{pmatrix}$$

$$= \left(\frac{e^{\lambda} - 1}{\lambda} + 2 + \frac{1}{\lambda} \right) \sigma^4 \quad \text{(as can be found in (d) as well)}$$

So, $\sqrt{n}\, \frac{\hat{W}_n - \sigma^2}{\sqrt{V}} \xrightarrow{d} N(0,1)$, Since we don't know λ, we need an estimate for λ, and in (b) we showed $\hat{T}_n \xrightarrow{as} \lambda$ as $n \to \infty$, so we use $\hat{T}_n$ as an estimate for λ.

We also showed $\hat{W}_n \xrightarrow{as} \sigma^2$

So a CI for σ^2 takes form

$\hat{W}_n \pm Z_{1-\frac{\alpha}{2}} \frac{\text{Std Error}(\hat{W}_n)}{\sqrt{n}}$

So, an estimate for ρ_n is:

$$\hat{\rho}_n^2 = \left(\frac{e^{\hat{T}_n} - 1}{\hat{T}_n} + 2 + \frac{1}{\hat{T}_n} \right) \hat{W}_n^2$$

And thus a $1-\alpha$ CI for σ^2 is $\hat{W}_n \pm Z_{1-\frac{\alpha}{2}} \frac{\hat{\rho}_n}{\sqrt{n}}$ ✓

<u>WTS</u>: $\frac{\sqrt{n}\,(\hat{W}_n - \sigma^2)}{\hat{\rho}} \xrightarrow[d]{} N(0,1)$

want to standardize

(show $\hat{\rho}^2 \xrightarrow[p]{}$ this, and then by slutsky's can just divide by this!)

We know $\sqrt{n}\,(\hat{W}_n - \sigma^2) \xrightarrow[d]{} N\left(0, \underbrace{\left(\frac{e^{\lambda}-1}{\lambda} + 2 + \frac{1}{\lambda}\right)\sigma^4}\right)$

We know $\hat{T}_n \xrightarrow[as]{} \lambda$ and $\hat{W}_n \xrightarrow[as]{} \sigma^2$

So, $\begin{pmatrix} \hat{T}_n \\ \hat{W}_n \end{pmatrix} \xrightarrow[as]{} \begin{pmatrix} \lambda \\ \sigma^2 \end{pmatrix}$ letting $g(\lambda, \sigma^2) = \left(\frac{e^{\lambda}-1}{\lambda} + 2 + \frac{1}{\lambda}\right)\sigma^4$

and by CMT, $\hat{\rho}^2 \xrightarrow[as.]{} \left(\frac{e^{\lambda}-1}{\lambda} + 2 + \frac{1}{\lambda}\right)\sigma^4$

So, by Slutsky's theorem,

$$\frac{\sqrt{n}\,(\hat{W}_n - \sigma^2)}{\hat{\rho}} \to N(0,1)$$

And so CI is $\hat{W}_n \pm Z_{1-\alpha/2} \frac{\hat{\rho}_n}{\sqrt{n}}$