2016 Day 1 Problem 3

a) $f_{S,X_1,\dots,X_{S+1}}(s, x_1,\dots,x_{s+1}) = f_S(s)\, f_{X_1,\dots,X_{S+1}|S}(x_1,\dots,x_{s+1})$ $\to \prod_{i=1}^{s+1} f_{X_i|S}(x_i) \to N(\mu,1)$

$= \binom{n}{s} p^s (1-p)^{n-s} (2\pi)^{-(s+1)/2} \exp\left\{-\frac{1}{2}\sum_{i=1}^{s+1}(x_i-\mu)^2\right\}$

$= \binom{n}{s}(2\pi)^{-(s+1)/2} \exp\left\{-\frac{1}{2}\sum_{i=1}^{s+1} x_i^2\right\} \cdot$

$\exp\left\{s\log p + (n-s)\log(1-p) + \mu\sum_{i=1}^{s+1} x_i - \frac{1}{2}(s+1)\mu^2\right\}$

$\hookrightarrow \exp\left\{n\log(1-p) + s\log\frac{p}{1-p} + \mu\sum_{i=1}^{s+1} x_i - \underbrace{\frac{1}{2}(s+1)\mu^2}_{\text{redundant with } s}\right\}$

$\mu=0 \Rightarrow f = \binom{n}{s}(2\pi)^{-(s+1)/2}\exp\left\{-\frac{1}{2}\sum_{i=1}^{s+1} x_i^2\right\} \cdot \quad] \, h(x)$

$\exp\{\underbrace{n\log(1-p)}_{b(\theta)} + \underbrace{s\log\frac{p}{1-p}}_{T(x)\cdot q(\theta)}\}$

$\Rightarrow s$ is CSS for $\log\frac{p}{1-p}$ aka for p

$\mu \neq 0 \Rightarrow$ CSS for (p,μ) is $(S, \sum_{i=1}^{s+1} x_i) \Rightarrow$ full rank $=2$

b) $l_n(p,\mu) = \log\left[\binom{n}{s}(2\pi)^{-\frac{(s+1)}{2}}\right] - \frac{1}{2}\sum_{i=1}^{s+1} x_i^2 + s\log p$
$+ (n-s)\log(1-p) + \mu\sum_{i=1}^{s+1} x_i - \frac{1}{2}(s+1)\mu^2$

$\frac{\partial l_n}{\partial \mu} = \sum_{i=1}^{s+1} x_i - (s+1)\mu := 0 \Rightarrow \hat{\mu} = \frac{1}{s+1}\sum_{i=1}^{s+1} x_i$

$\frac{\partial l_n}{\partial p} = \frac{1}{p}s - (n-s)\frac{1}{1-p} = \frac{1}{p(1-p)}\left[s(1-p) - (n-s)p\right]$
$= \frac{1}{p(1-p)}[s - np] := 0 \Rightarrow \hat{p} = \frac{s}{n}$

c) $\sqrt{n}\left(\begin{pmatrix}\hat{p}\\ \hat{\mu}\end{pmatrix} - \begin{pmatrix}p\\ \mu\end{pmatrix}\right) \xrightarrow{d} N(0, I(p,\mu)^{-1})$

$\frac{\partial^2 l_n}{\partial \mu^2} = -(s+1) \qquad -E[\cdot] = E(s)+1 = np+1$

$\frac{\partial^2 l_n}{\partial\mu\partial p} = 0 \qquad -E[\cdot] = 0$

$\frac{\partial^2 l_n}{\partial p^2} = -\frac{1}{p^2}s - (n-s)\frac{1}{(1-p)^2} \qquad -E[\cdot] = \frac{1}{p^2}E(s) + \frac{1}{(1-p)^2}[n - E(s)]$
$= \frac{n}{p} + \frac{n}{1-p} = \frac{n}{p(1-p)}[1-p+p] = \frac{n}{p(1-p)}$

$I_n(\mu,p) = \begin{bmatrix} np+1 & 0 \\ 0 & \frac{n}{p(1-p)}\end{bmatrix} \qquad I_n(\mu,p)^{-1} = \begin{bmatrix}\frac{1}{np+1} & 0 \\ 0 & \frac{p(1-p)}{n}\end{bmatrix}$

$I(\mu,p) = \lim_{n\to\infty}\frac{1}{n}I_n(\mu,p) = \lim_{n\to\infty}\begin{bmatrix} p+\frac{1}{n} & 0 \\ 0 & \frac{1}{p(1-p)}\end{bmatrix} = \begin{bmatrix} p & 0 \\ 0 & \frac{1}{p(1-p)}\end{bmatrix}$

$\sqrt{n}\left(\begin{pmatrix}\hat{p}\\ \hat{\mu}\end{pmatrix} - \begin{pmatrix}p\\ \mu\end{pmatrix}\right) \xrightarrow{d} N\left(0, \begin{bmatrix} p(1-p) & 0 \\ 0 & 1/p\end{bmatrix}\right)$

d) unbiased test $\phi \Leftrightarrow \forall \theta \in \Theta_1, \beta_\phi(\theta) \geq \alpha$
and $\forall \theta \in \Theta_0, \beta_\phi(\theta) \leq \alpha$

That is, $\beta_\phi(p, \mu) \geq \alpha \quad \forall \mu > 0, \forall p$

$\beta_\phi(p, \mu) \leq \alpha \quad \forall \mu \leq 0 \ \forall p$

so $\beta(p, 0) \leq \alpha \ \forall p$

exponential family $\Rightarrow$ power function is continuous

and $\beta(p, \mu) \geq \alpha \ \forall \mu > 0 \ \forall p$ and continuity implies that $\beta(p, 0) \geq \alpha \ \forall p$ so

$\alpha \leq \beta(p, 0) \leq \alpha \Rightarrow \beta(p, 0) = \alpha \ \forall p$

e) UMPU test of $H_0: \mu \leq 0$ vs $H_1: \mu > 0$

$$\phi(x) = \begin{cases} 1 & U > c'(t) \\ \gamma(t) & U = c'(t) \\ 0 & U < c'(t) \end{cases} \quad \text{where } E_{\mu=0}[\phi(U) \mid T = t] = \alpha$$

$U = \sum_{i=1}^{s+1} X_i, \quad T = S$

$X_i \mid S = s \sim N(\mu, 1) \Rightarrow U \mid T = s \sim N((s+1)\mu, s+1)$

equivalently, let $\bar{X} = \frac{1}{s+1} \sum_{i=1}^{s+1} X_i \quad \bar{X} \mid T = s \sim N(\mu, \frac{1}{s+1})$

$Z \sim N(0, 1)$

$$\phi(x) = \begin{cases} 1 & \bar{X} > c(t) \\ 0 & \bar{X} < c(t) \end{cases}$$ at boundary

$\alpha = E_{\mu=0}[\bar{X} > c(t) \mid S = s] = P_{\mu=0}\left(\frac{1}{\sqrt{s+1}} Z > c(t)\right)$

$= P(Z > c(t)\sqrt{s+1}) \Rightarrow 1 - \alpha = P(Z \leq c(t)\sqrt{s+1})$

$\Rightarrow c(t)\sqrt{s+1} = z_{1-\alpha} \Rightarrow c(t) = z_{1-\alpha}/\sqrt{s+1}$

Thus, the UMPU test of H_0 vs. H_1 is

$\phi(x) = I\left(\bar{X} > \frac{1}{\sqrt{s+1}} z_{1-\alpha}\right)$