2009 Theory II #2

2a.i)
$$\begin{bmatrix} Y_{11} \\ \vdots \\ Y_{1,n_1} \\ Y_{21} \\ \vdots \\ Y_{2,n_2} \end{bmatrix} = \begin{bmatrix} 1 & x_{11}-\bar{x}_1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_{1n_1}-\bar{x}_1 & 0 & 0 \\ 0 & 0 & 1 & x_{21}-\bar{x}_2 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & x_{2n_2}-\bar{x}_2 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \gamma_1 \\ \mu_2 \\ \gamma_2 \end{bmatrix} + \underbrace{\varepsilon}_{N(0,\, \sigma^2 I)}$$

2a.ii) From the given, $\begin{bmatrix} 1 & x_{11} \\ \vdots & \vdots \\ 1 & x_{1n_1} \end{bmatrix}$ and $\begin{bmatrix} 1 & x_{21} \\ \vdots & \vdots \\ 1 & x_{2n_2} \end{bmatrix}$ are both full rank and hence $\begin{bmatrix} 1 & x_{11}-\bar{x}_1 \\ \vdots & \vdots \\ 1 & x_{1n_1}-\bar{x}_1 \end{bmatrix}$ and $\begin{bmatrix} 1 & x_{21}-\bar{x}_2 \\ \vdots & \vdots \\ 1 & x_{2n_2}-\bar{x}_2 \end{bmatrix}$ are both full rank from which we conclude that X is full rank. ~~Also Along the Gauss-Markov we obtain that~~

$\cancel{\hat{\beta} = (X'X)^{-1}X'Y}$

~~has minimum variance among the class of linear unbiased estimates.~~ (pg 28 Christensen)

~~is the unique ... has the property~~

Gauss-Markov theorem (corollary pg 43 Seber and Lee): If X has full rank then $a'\hat{\beta}$ is the BLUE of $a'\beta$ for every vector a where $\hat{\beta} = (X'X)^{-1}X'Y$ is the least-squared estimator for β

2b.i) Let $a = (0, 1, 0, -1)$, then $a'\beta = \gamma_1 - \gamma_2$. All linear functions of β are estimable iff X is full rank, which we showed in (2a.ii).

$a'\hat{\beta}$ follows a normal distribution so it remains to find mean, Variance

2b.ii) $E[a'\hat{\beta}] = a'(X'X)^{-1}X'E[Y] = a'(X'X)^{-1}X'X\beta$

$= a'\beta = \gamma_1 - \gamma_2$

$Var[a'\hat{\beta}] = Var[a'(X'X)^{-1}X'Y] = a'(X'X)^{-1}X' \, Var[Y] \, X(X'X)^{-1}a$

$= \sigma^2 a'(X'X)^{-1}X'X(X'X)^{-1}a = \sigma^2 a'(X'X)^{-1}a$

Now $X'X = \begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ x_{11}-\bar{x}_1 & \cdots & x_{1n_1}-\bar{x}_1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 \\ 0 & \cdots & 0 & x_{21}-\bar{x}_2 & \cdots & x_{2n_2}-\bar{x}_2 \end{bmatrix} \begin{bmatrix} 1 & x_{11}-\bar{x}_1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_{1n_1}-\bar{x}_1 & 0 & 0 \\ 0 & 0 & 1 & x_{21}-\bar{x}_2 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & x_{2n_2}-\bar{x}_2 \end{bmatrix}$

$$= \begin{bmatrix} n_1 & 0 & 0 & 0 \\ 0 & \sum_{i=1}^{n_1}(x_{i1}-\bar{x}_1)^2 & 0 & 0 \\ 0 & 0 & n_2 & 0 \\ 0 & 0 & 0 & \sum_{i=1}^{n_2}(x_{i2}-\bar{x}_2)^2 \end{bmatrix}$$

so that

$$Var[a'\hat{\beta}] = \sigma^2 \left\{ \left[\sum_{i=1}^{n_1}(x_{i1}-\bar{x}_1)^2\right]^{-2} + \left[\sum_{i=1}^{n_2}(x_{i2}-\bar{x}_2)^2\right]^{-2} \right\}$$

A 95% CI is given by

$$\left\{ \gamma_1 - \gamma_2 : \; -z_{0.975} < \frac{a'\hat{\beta} - (\gamma_1 - \gamma_2)}{\sqrt{Var[a'\hat{\beta}]}} < z_{0.975} \right\}$$

$$= \left\{ \gamma_1 - \gamma_2 : \; a'\hat{\beta} - \sqrt{Var[a'\hat{\beta}]}\, z_{0.975} < \gamma_1 - \gamma_2 < a'\hat{\beta} + \sqrt{Var[a'\hat{\beta}]}\, z_{0.975} \right\}$$

2b.iii) $$F = \frac{(a'\hat{\beta})' \left[a'(X'X)^{-1} a \right]^{-1} (a'\beta) / 1}{\|Y - X\hat{\beta}\| / (n-p)}$$

$$= \frac{(a'\beta)^2}{a'(X'X)^{-1} a \, S^2} \overset{H_0}{\sim} F_{1, n-p} \overset{d}{=} F_{1, n_1+n_2-4}$$

3c.i) New model given by

$$Y = \begin{bmatrix} x_{11} - \bar{x}_1 & 0 \\ \vdots & \vdots \\ x_{1n_1} - \bar{x}_1 & 0 \\ 0 & x_{21} - \bar{x}_2 \\ \vdots & \vdots \\ 0 & x_{2n_2} - \bar{x}_2 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} + \varepsilon$$

Let $a = (1, -1)$. Then $E[a'\hat{\beta}] = a'(X'X)^{-1}X' E[Y] = a'\beta = \gamma_1 - \gamma_2$

Since $X'X = \begin{bmatrix} nS_1^2 & 0 \\ 0 & nS_2^2 \end{bmatrix}$ where $S_1^2 \equiv \frac{1}{n_1} \sum (x_{1i} - \bar{x}_1)^2$, $S_2^2 \equiv \frac{1}{n_2} \sum (x_{2i} - \bar{x}_2)^2$

and therefore $a'(X'X)^{-1}a$ is the same as before so that the result is unchanged from (2b.ii). In (2c.iii) the term $\|Y-X\hat{\beta}\|$ will in general change and the $n-p$ term will certainly change, from n_1+n_2-4 to n_1+n_2-2.

$$Y'Y - Y'X\beta - \beta'X'Y + \beta'X'X\beta$$

2c.iii) Want to minimize $\underbrace{(Y-X\beta)'(Y-X\beta)}_{L(\beta,\lambda)\equiv} + \lambda a'\beta$

$$\begin{cases} \dfrac{\partial}{\partial\beta} L(\beta,\lambda) = -2X'Y + 2X'X\beta + \lambda a \overset{set}{=} 0 \\[2ex] \dfrac{\partial}{\partial\lambda} L(\beta,\lambda) = a'\beta \overset{set}{=} 0 \end{cases}$$

$$\Rightarrow \hat{\beta}_H = \underbrace{(X'X)^{-1}X'Y}_{\hat{\beta}} - \frac{\hat{\lambda}}{2}(X'X)^{-1}a \Rightarrow 0 = a'\hat{\beta}_H = a'\hat{\beta} - \frac{\hat{\lambda}}{2}a'(X'X)^{-1}a$$

$$\Rightarrow \hat{\lambda} = 2\frac{a'\hat{\beta}}{a'(X'X)^{-1}a} \quad \Rightarrow \hat{\beta}_H = \hat{\beta} - \frac{a'\hat{\beta}}{a'(X'X)^{-1}a}(X'X)^{-1}a$$

Now, $(X'X)^{-1} = \begin{bmatrix} \frac{1}{n_1} & & & \\ & \frac{1}{n_1 S_1^2} & & \\ & & \frac{1}{n_2} & \\ & & & \frac{1}{n_2 S_2^2} \end{bmatrix}$, $\quad X'Y = \begin{bmatrix} n_1\bar{y}_1 \\ \sum(x_{1i}-\bar{x}_1)y_{1i} \\ n_2\bar{y}_2 \\ \sum(x_{2i}-\bar{x}_2)y_{2i} \end{bmatrix}$

$$\hat{\beta} = \begin{bmatrix} \bar{y}_1 \\ \frac{1}{n_1 S_1^2}\sum(x_{1i}-\bar{x}_1)y_{1i} \\ \bar{y}_2 \\ \frac{1}{n_2 S_2^2}\sum(x_{2i}-\bar{x}_2)y_{2i} \end{bmatrix}, \quad \text{i.e.} \begin{cases} \mu_1 = \frac{1}{n_1}\sum y_{1i} \\ \mu_2 = \frac{1}{n_2}\sum y_{2i} \end{cases}$$

Then

$$\hat{\beta}_H = \begin{bmatrix} \bar{y}_1 \\ \dfrac{n_1^{-1}\sum (x_{1i}-\bar{x}_1)\, y_{1i}}{s_1^2} \\ \bar{y}_2 \\ \dfrac{n_2^{-1}\sum (x_{2i}-\bar{x}_2)\, y_{2i}}{s_2^2} \end{bmatrix} - \frac{\dfrac{n_1^{-1}\sum (x_{1i}-\bar{x})\, y_{1i}}{s_1^2} - \dfrac{n_2^{-1}\sum (x_{2i}-\bar{x}_2)\, y_{2i}}{s_2^2}}{\dfrac{1}{n_1 s_1^2} + \dfrac{1}{n_2 s_2^2}} \begin{bmatrix} 0 \\ \dfrac{1}{n_1 s_1^2} \\ 0 \\ \dfrac{1}{n_2 s_2^2} \end{bmatrix}$$

not needed

2d.i) Recall the updating formula:

> Let A be a $p \times p$ full rank symmetric matrix and let a and b be $q \times p$ rank q matrices. Then, provided that the inverses exist,
>
> $$(A + a'b)^{-1} = A^{-1} - A^{-1}a'(I + bA^{-1}a')^{-1}bA^{-1}$$

Then choosing $A = X'X$, $a = -x_i'$, $b = x_i$ (and assuming X is full rank) we have

$$(X_{(k)}' X_{(k)})^{-1} = (X'X - x_k' x_k)^{-1} = (X'X)^{-1} + \frac{(X'X)^{-1} x_k x_k' (X'X)^{-1}}{1 - x_i'(X'X)^{-1}x_i}$$

$$= (X'X)^{-1} + \frac{(X'X)^{-1} x_k x_k' (X'X)^{-1}}{1 - P_{ii}}$$

where $P = X(X'X)^{-1}X'$

Then

$$\hat{\beta}_{(k)} = (X_{(k)}' X_{(k)})^{-1} X_{(k)}' Y_{(k)} = (X_{(k)}' X_{(k)})^{-1} [X'Y - y_k \underline{x}_k]$$

$$= \left\{ (X'X)^{-1} + \frac{(X'X)^{-1} x_k x_k' (X'X)^{-1}}{1 - P_{kk}} \right\} [X'Y - y_k x_k]$$

$$= \hat{\beta} - y_k (X'X)^{-1} x_k + \frac{(X'X)^{-1} x_k x_k' \hat{\beta}}{1 - P_{kk}} - y_k \frac{(X'X)^{-1} x_k x_k' (X'X)^{-1} x_k}{1 - P_{ii}}$$

$$= \hat{\beta} - y_k \frac{(1 - P_{ii})(X'X)^{-1} x_k + (X'X)^{-1} x_k P_{kk}}{1 - P_{kk}} + \frac{(X'X)^{-1} x_k x_k' \hat{\beta}}{1 - P_{kk}}$$

$$= \hat{\beta} - \frac{y_k (X'X)^{-1} x_k \cancel{(X'X)^{-1} x_k x_k' \hat{\beta}} - (X'X)^{-1} x_k x_k' \hat{\beta}}{1 - P_{kk}}$$

$$= \hat{\beta} - \frac{(X'X)^{-1} x_k (y_k - x_k' \hat{\beta})}{1 - P_{kk}}$$

$$= \hat{\beta} - \frac{(X'X)^{-1} x_k \hat{\varepsilon}_i}{1 - P_{kk}}$$

20.ii) $Y_k \perp\!\!\!\perp Y_{(k)} \Rightarrow Y_k \perp\!\!\!\perp \hat{\beta}_{(k)}$. Then D_k is normally distributed with mean

$$EY_k - x_i' E\hat{\beta}_{(k)} = x_i'\beta - x_i'\beta = 0$$

$$Var[D_k] = Var[Y_k] + x_k' \, Var\left[\hat{\beta}_{(k)}\right] x_k$$

$$\cancel{= \sigma^2 + 2x_k' \sigma^2 (X_{(k)}' X_{(k)})^{-1} x_k = \sigma^2\left[1 + x_k' (X_{(k)}' X_{(k)})^{-1} x_k\right]}$$

$$= \sigma^2 + x_k' \sigma^2 (X_{(k)}' X_{(k)})^{-1} x_k = \sigma^2\left[1 + x_k' (X_{(k)}' X_{(k)})^{-1} x_k\right]$$

2d.iii) Let $\hat{\sigma}^2_{(k)} = \frac{1}{n_1 + n_2 - 1 - 4} \|Y - X\hat{\beta}_{(k)}\|$. Then

$$\frac{D_k}{\widehat{se}(D_k)} = \frac{D_k}{\sigma_{(k)} \sqrt{1 + x_k' (X_{(k)}' X_{(k)})^{-1} x_k}} \sim t_{n_1 + n_2 - 4}$$