2012 Theory II #1

1a) $P(Y_{i1}=0,\ Y_{i2}=0) = 1-\alpha$ ← comprise X_3

reorder

$P(Y_{i1}=1,\ Y_{i2}=0) = \alpha(1-\beta)$ ← comprise X_2

$P(Y_{i1}=1,\ Y_{i2}=1) = \alpha\beta$ ← comprise X_1

Then (X_1, X_2, X_3) follows a multinomial distribution with pmf

$$p(x_1, x_2, x_3) = \frac{n}{x_1!\, x_2!\, x_3!} (\alpha\beta)^{x_1} [\alpha(1-\beta)]^{x_2} (1-\alpha)^{n-x_1-x_2} \qquad x_i = 0, 1, \ldots, n,\ i = 1,2,3 \quad \sum_{i=1}^{3} x_i = n$$

needs more work to show exponential fam.

~~which is a regular exponential fam.~~ see part (1h)

1b) $$\ell(\alpha,\beta) = x_1 \log(\alpha\beta) + x_2 \log[\alpha(1-\beta)] + x_3 \log(1-\alpha) + \log\left(\frac{n!}{x_1!\, x_2!\, x_3!}\right)$$

$$\frac{\partial}{\partial\alpha} \ell(\alpha,\beta) = \frac{x_1}{\alpha} + \frac{x_2}{\alpha} - \frac{x_3}{1-\alpha} \overset{set}{=} 0$$

$$\Rightarrow 0 = (1-\hat\alpha)x_1 + (1-\hat\alpha)x_2 - \hat\alpha x_3 = x_1 + x_2 - n\hat\alpha$$

$$\Rightarrow \hat\alpha = \frac{x_1 + x_2}{n}$$

$$\frac{\partial}{\partial\beta} \ell(\alpha,\beta) = \frac{x_1}{\beta} - \frac{x_2}{1-\beta} \overset{set}{=} 0 \Rightarrow 0 = (1-\hat\beta)x_1 - \hat\beta x_2$$

$$= x_1 - \hat\beta(x_1 + x_2) \Rightarrow \hat\beta = \frac{x_1}{x_1 + x_2}$$

1c) $$\frac{\partial^2}{\partial\alpha^2} \ell(\alpha,\beta) = -\frac{x_1}{\alpha^2} - \frac{x_2}{\alpha^2} - \frac{x_3}{(1-\alpha)^2}$$

$$\frac{\partial^2}{\partial\beta^2} \ell(\alpha,\beta) = -\frac{x_1}{\beta^2} - \frac{x_2}{(1-\beta)^2}$$

$$\frac{\partial^2}{\partial\alpha\,\partial\beta} \ell(\alpha,\beta) = 0$$

$$-E\left[\frac{\partial^2}{\partial\alpha^2} \ell(\alpha,\beta)\right] = \frac{n\alpha\beta}{\alpha^2} + \frac{n\alpha(1-\beta)}{\alpha^2} + \frac{n(1-\alpha)}{(1-\alpha)^2}$$

$$= \frac{n\beta}{\alpha} + \frac{n(1-\beta)}{\alpha} + \frac{n}{1-\alpha} = n\frac{\beta(1-\alpha)+(1-\alpha)(1-\beta)+\alpha}{\alpha(1-\alpha)}$$

$$= n\frac{\cancel{\beta} - \cancel{\alpha\beta} + 1 - \cancel{\beta} - \cancel{\alpha} + \cancel{\alpha\beta} + \cancel{\alpha}}{\alpha(1-\alpha)} = \frac{n}{\alpha(1-\alpha)}$$

$$E\left[-\frac{\partial^2}{\partial\beta^2} \ell(\alpha,\beta)\right] = \frac{n\alpha\beta}{\beta^2} + \frac{n\alpha(1-\beta)}{(1-\beta)^2} = \frac{n\alpha}{\beta} + \frac{n\alpha}{1-\beta}$$

$$= n\frac{\alpha(1-\beta)+\alpha\beta}{\beta(1-\beta)} = \frac{n\alpha}{\beta(1-\beta)}$$

Then the asymptotic covariance matrix is given by

$$\left[\lim_{n\to\infty} \frac{1}{n} I_n(\alpha,\beta)\right]^{-1} = \begin{bmatrix} \alpha(1-\alpha) & 0 \\ 0 & \frac{1}{\alpha}\beta(1-\beta) \end{bmatrix}$$

needs rephrased

Let $\beta^* > 0.5$.

1d) By the Neyman-Peason lemma, a UMP test of $H_0: \beta = \beta^*$ vs. $H_1: \beta \neq \beta^*$ is given by

$$\phi(X_1, X_2, X_3) = \begin{cases} 1, & p_1(x) > k p_0(x) \\ \gamma, & p_1(x) = p_0(x) \\ 0, & \text{else} \end{cases} \iff \begin{cases} 1, & \frac{p_1(x)}{p_0(x)} > k \\ \gamma, & \frac{p_1(x)}{p_0(x)} = k \\ 0, & \text{else} \end{cases}$$

where k, γ satisfy

$E_{\beta^*}[\phi(X_1, X_2, X_3)] = \delta$ ← type I error

Now

$$\frac{p_1(x)}{p_0(x)} = \frac{(\alpha\beta^*)^{x_1}[\alpha(1-\beta^*)]^{x_2}(1-\alpha)^{x_3}}{(\alpha\cdot\frac{1}{2})^{x_1}[\alpha(1-\frac{1}{2})]^{x_2}(1-\alpha)^{x_3}} = 2^{x_1+x_2}(\beta^*)^{x_1}(1-\beta^*)^{x_2}$$

Since the rejection region depends on β^* we conclude that no UMP test exists

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1e) We derive the MLE for the restricted (H_0) case. Substituting α for β we have

$$L(\alpha) \equiv L(\alpha,\alpha) = \frac{n}{x_1!\,x_2!\,x_3!}\alpha^{2x_1}[\alpha(1-\alpha)]^{x_2}(1-\alpha)^{x_3}$$

$$\ell(\alpha,\alpha) = 2x_1\log\alpha + x_2\log[\alpha(1-\alpha)] + x_3\log(1-\alpha)$$

$$\frac{\partial}{\partial\alpha}\ell(\alpha,\alpha) = \frac{2x_1}{\alpha} + \frac{x_2(1-2\alpha)}{\alpha(1-\alpha)} - \frac{x_3}{1-\alpha} \overset{\text{set}}{=} 0$$

$$\Rightarrow 0 = 2x_1(1-\tilde{\alpha}) + x_2(1-2\tilde{\alpha}) - x_3\tilde{\alpha}$$
$$= 2x_1 - 2x_1\tilde{\alpha} + x_2 - 2x_2\tilde{\alpha} - x_3\tilde{\alpha}$$

$$= 2x_1 + x_2 - (2x_1 + 2x_2 + x_3)\tilde{\alpha}$$

$$\Rightarrow \tilde{\alpha} = \frac{2x_1 + x_2}{2x_1 + 2x_2 + x_3}$$

$$LRT_n = 2\ell(\hat{\alpha}, \hat{\beta}) - 2\ell(\tilde{\alpha}, \tilde{\alpha})$$

$$= x_1 \log\left(\frac{\hat{\alpha}\hat{\beta}}{\tilde{\alpha}}\right) + x_2 \log\left(\frac{\hat{\alpha}(1-\hat{\beta})}{\tilde{\alpha}(1-\tilde{\alpha})}\right) + x_3 \log\left(\frac{1-\hat{\alpha}}{1-\tilde{\alpha}}\right)$$

$$= x_1 \log\left(\frac{\frac{x_1+x_2}{n} \cdot \frac{x_1}{x_1+x_2}}{\frac{2x_1+x_2}{2x_1+2x_2+x_3}}\right) + x_2 \log\left(\qquad\right) + x_3 \log(\cdot)$$

$$\overset{H_0}{\sim} \chi^2_1$$

Doesn't simplify further

1f) $$s_n(\alpha, \beta) = \begin{bmatrix} \frac{x_1}{\alpha} + \frac{x_2}{\alpha} - \frac{x_3}{1-\alpha} \\ \frac{x_1}{\beta} - \frac{x_2}{\beta} \end{bmatrix}, \quad I_n^{(\alpha,\beta)} = \begin{bmatrix} \frac{n}{\alpha(1-\alpha)} & 0 \\ 0 & \frac{n\alpha}{\beta(1-\beta)} \end{bmatrix}$$

Then

$$R_n = [s_n(\tilde{\alpha}, \tilde{\alpha})]' [I_n(\tilde{\alpha}, \tilde{\alpha})]^{-1} s_n(\tilde{\alpha}, \tilde{\alpha})$$

$$\boxed{[I_n(\alpha, \beta)]^{-1} = \begin{bmatrix} \frac{\alpha(1-\alpha)}{n} & 0 \\ 0 & \frac{\beta(1-\beta)}{n\alpha} \end{bmatrix}}$$

$$\overset{H_0}{\sim} \chi^2_1$$

1g) Let $h(\alpha,\beta) = \alpha - \beta$. ~~...~~ Let $H(\alpha,\beta) = \frac{\partial}{\partial(\alpha,\beta)} h(\alpha,\beta) = [1 \ \ -1]$

Then

$$W_n = (h(\alpha,\beta) - 0)\left[H(\hat\alpha,\hat\beta)\left[I_n(\hat\alpha,\hat\beta)\right]^{-1}\left[H(\hat\alpha,\hat\beta)\right]^{-1}\right]^{-1}(h(\alpha,\beta) - 0)$$

$$= \frac{(\hat\alpha - \hat\beta)^2}{\frac{\hat\alpha(1-\hat\alpha)}{n} + \frac{\hat\beta(1-\hat\beta)}{n\alpha}} = \frac{n\alpha(\hat\alpha - \hat\beta)^2}{\hat\alpha(1-\hat\alpha) + \hat\beta(1-\hat\beta)}$$

1h) $p(x_1, x_2, x_3; \alpha, \beta) = \frac{n}{x_1!\,x_2!\,x_3!}(\alpha\beta)^{x_1}[\alpha(1-\beta)]^{x_2}(1-\alpha)^{x_3}$

$$= \alpha^{x_1+x_2}(1-\alpha)^{x_3}\beta^{x_1}(1-\beta)^{x_2}\frac{n}{x_1!\,x_2!\,x_3!}$$

$$= \exp\Big\{(x_1+x_2)\log\alpha + x_3\log(1-\alpha)$$
$$+ x_2\log\beta + x_2\log(1-\beta) + \log\left(\frac{n}{x_1!\,x_2!\,x_3!}\right)\Big\}$$

$$= \exp\Big\{(x_1+x_2)\log\left(\frac{\alpha}{1-\alpha}\right) + n\log(1-\alpha)$$
$$+ x_1\log\beta + x_2\log(1-\beta) + \log(\cdot)\Big\}$$

$$= \exp\{(x_1+x_2)\theta_1 - n\log(1+\theta_1) + x_1\log\theta_2 + x_2\log(1-e^{\theta_2}) + \log(\cdot)\}$$

still curved exponential but suggests $X_1 + X_2$ sufficient for α

$$p(x_1, x_2, x_3 \mid x_1 + x_2) = \frac{p(x_1, x_2, x_3)}{p(x_1 + x_2)}$$

$\text{bin}(n, \alpha\beta + \alpha(1-\beta)) \overset{?}{=} \text{bin}(n, \alpha)$

$$= \frac{\dfrac{n!}{x_1!\, x_2!\, x_3!}\, \alpha^{x_1 + x_2} (1-\alpha)^{x_3} \beta^{x_1} (1-\beta)^{x_2}}{\dfrac{n!}{(x_1+x_2)!\, x_3!}\, \alpha^{x_1 + x_2} (1-\alpha)^{x_3}}$$

$$= \frac{(x_1 + x_2)!}{x_1!\, x_2!}\, \beta^{x_1} (1-\beta)^{x_2} \sim \text{bin}(x_1 + x_2, \beta)$$

Thus the conditional MLE is $\frac{x_1}{x_1 + x_2}$. Intuitive since the unconditional MLE did not depend on $\hat{\alpha}$