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Bios 762, 2014 Section 1

3. Suppose (X, Y) are two RVs w/ joint dist.

$$f(x,y \mid \alpha,\beta) = c(\alpha,\beta)\cdot\exp(-\alpha x-\beta y)\sum_{j=0}^{\infty}\frac{x^j y^j}{(j!)^2} \qquad (1)$$

for $x>0, y>0$.

Also, let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be a random sample from (X,Y) and let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$

a) Show that the joint distn. of (X,Y) in (1) is in the multiparameter $\underbrace{\text{exponential family}}_{i)}$, $\underbrace{\text{identify the rank}}_{ii)}$, $\underbrace{\text{show that } c(\alpha,\beta)=\alpha\beta-1}_{iii)}$, $\underbrace{\text{and find the parameter space of } (\alpha,\beta)}_{iv)}$.

i) A multivariate exponential family has the form,

$$p(x,y \mid \alpha,\beta) = \exp\{Q(x,y)^T T(\alpha,\beta) - b(\alpha,\beta) - c(x,y)\}$$

Here $f(x,y\mid\alpha,\beta) = c(\alpha,\beta)\cdot\exp(-\alpha x-\beta y)\sum_{j=0}^{\infty}\frac{x^j y^j}{(j!)^2}$

$$= \exp\left\{-\alpha x-\beta y + \log(c(\alpha,\beta)) + \log\left(\sum_{j=0}^{\infty}\frac{x^j y^j}{(j!)^2}\right)\right\}$$

$$\Rightarrow Q(x,y) = (-x,-y)$$
$$T(\alpha,\beta) = (\alpha,\beta)$$
$$b(\alpha,\beta) = -\log(c(\alpha,\beta))$$
$$c(x,y) = -\log\left(\sum_{j=0}^{\infty}\frac{x^j y^j}{(j!)^2}\right)$$

$\Rightarrow$ $f(x,y\mid\alpha,\beta)$ is a member of the multiparameter exponential family.

ii) Rank = # L.I. components of $Q(x,y) = 2$ (since $X \perp\!\!\!\perp Y$ b/c random sample).

iii) Want to show $c(\alpha,\beta) = \alpha\beta - 1$.

Know $\int_y\int_x f(x,y\mid\alpha,\beta)\,dx\,dy = 1$

Then, $1 = c(\alpha,\beta)\int_y\int_x e^{-\alpha x-\beta y}\sum_{j=0}^{\infty}\frac{x^j y^j}{(j!)^2}\,dx\,dy$

$$= c(\alpha,\beta)\sum_{j=0}^{\infty}\frac{1}{(j!)^2}\int_0^{\infty} x^{(j+1)-1}e^{-\alpha x}\int_0^{\infty} y^{(j+1)-1}e^{-\beta y}\,dy\,dx$$

cont'd. $\longrightarrow$

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3. a) cont'd.

$$= c(\alpha,\beta)\sum_{j=0}^{\infty}\frac{1}{(j!)^2}\int_0^{\infty}\Gamma(j+1)\cdot\left(\frac{1}{\alpha}\right)^{j+1}\cdot\frac{1}{\Gamma(j+1)\left(\frac{1}{\alpha}\right)^{j+1}}\cdot x^{(j+1)-1}e^{-\alpha x}\int_0^{\infty}\Gamma(j+1)\left(\frac{1}{\beta}\right)^{j+1}\cdot\frac{1}{\Gamma(j+1)\left(\frac{1}{\beta}\right)^{j+1}}y^{(j+1)-1}e^{-\beta x}\,dy\,dx$$

$\underbrace{\cancel{(j!)^2}}_{\parallel}$ (since $\Gamma(n)=(n-1)!$ for $n\in\mathbb{N}$)

$$= c(\alpha,\beta)\sum_{j=0}^{\infty}\frac{1}{\cancel{(j!)^2}}\left[\Gamma(j+1)\right]^2\left(\frac{1}{\alpha\beta}\right)^{j+1}$$

$$= c(\alpha,\beta)\sum_{j=0}^{\infty}\left(\frac{1}{\alpha\beta}\right)^{j+1} = c(\alpha,\beta)\cdot\left(\frac{1}{\alpha\beta}\right)\sum_{j=0}^{\infty}\underbrace{\left(\frac{1}{\alpha\beta}\right)}_{r}^{j} = c(\alpha,\beta)\cdot\frac{1}{(\alpha\beta)}\cdot\frac{1}{(1-1/\alpha\beta)}$$ for $|r|<1$ $\Rightarrow \left|\frac{1}{\alpha\beta}\right|<1$

$$= c(\alpha,\beta)\cdot\frac{1}{(\alpha\beta-1)}.\quad \text{Thus, } 1 = c(\alpha,\beta)\cdot\frac{1}{(\alpha\beta-1)} \Rightarrow \boxed{c(\alpha,\beta)=(\alpha\beta-1)}$$

iv) Find the parameter space of (α,β).

From geometric series above, had to assume $|r|<1 \Rightarrow \left|\frac{1}{\alpha\beta}\right|<1 \Rightarrow \alpha\beta>1$ since $\alpha>0,\beta>0$.

Thus, the parameter space is $\boxed{\Theta = \{(\alpha,\beta):\ \alpha > 1/\beta \text{ for } \alpha>0,\beta>0\}}$

3 b) Derive the marginal distribution of X from (1) and show that $E(X)=\frac{\beta}{\alpha\beta-1}$

Need to integrate out y to get marginal pdf of X.

$$f_X(x) = \int_0^{\infty}(\alpha\beta-1)e^{-\alpha x-\beta y}\sum_{j=0}^{\infty}\frac{x^j y^j}{(j!)^2}\,dy = (\alpha\beta-1)e^{-\alpha x}\sum_{j=0}^{\infty}\frac{x^j}{(j!)^2}\int_0^{\infty}e^{-\beta y}y^{(j+1)-1}\,dy$$

$$= (\alpha\beta-1)e^{-\alpha x}\sum_{j=0}^{\infty}\frac{x^j}{(j!)^{\cancel{2}}}\cdot\cancel{\Gamma(j+1)}\cdot\left(\frac{1}{\beta}\right)^{(j+1)} = \frac{(\alpha\beta-1)e^{-\alpha x}}{\beta}\sum_{j=0}^{\infty}\frac{x^j}{\beta^j}$$

$$= \frac{(\alpha\beta-1)}{\beta}e^{-\alpha x}\sum_{j=0}^{\infty}\frac{(x/\beta)^j}{j!} = \frac{(\alpha\beta-1)}{\beta}e^{-\alpha x}\cdot e^{x/\beta} = \boxed{(\alpha-1/\beta)e^{-x(\alpha-1/\beta)} \text{ for } x>0} \Rightarrow X\sim Exp\left(\frac{1}{\alpha-1/\beta}\right)$$

$$\text{Thus, } E(X) = \frac{1}{\alpha-1/\beta} = \boxed{\frac{\beta}{\alpha\beta-1}}$$

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3 c) From 1), Show $E[x^j y^k] = (-1)^{j+k} S^{-1} \dfrac{\partial^{j+k} S}{\partial \alpha^j \partial \beta^k}$ where $S = S(\alpha, \beta) = \dfrac{1}{c(\alpha, \beta)}$

Γ Let $M_{x,y}(s,t) = E[e^{sx+ty}] \Rightarrow \left. \dfrac{\partial^{j+k} M_{x,y}(s,t)}{\partial s^j \partial t^k} \right|_{(s,t)=(0,0)} = E[x^j y^k]$

$$M_{x,y}(s,t) = c(\alpha,\beta) \sum_{j=0}^{\infty} \int_0^{\infty}\int_0^{\infty} \frac{e^{-x(\alpha-s)} x^{(j+1)-1}}{\Gamma(j+1)} \frac{e^{-y(\beta-t)} y^{(j+1)-1}}{\Gamma(j+1)} \, dx\, dy$$

$$= c(\alpha,\beta) \sum_{j=0}^{\infty} \left(\frac{1}{\alpha-s}\right)^{j+1} \left(\frac{1}{\beta-t}\right)^{j+1}$$

$$= c(\alpha,\beta) [(\alpha-s)(\beta-t)]^{-1} \sum_{j=0}^{\infty} \Bigg(\underbrace{\frac{1}{(\alpha-s)(\beta-t)}}_{r \text{ where } |r|<1} \Bigg)^j$$

$$= c(\alpha,\beta) [(\alpha-s)(\beta-t)]^{-1} \left[\frac{1}{1-(\alpha-s)(\beta-t)} \right]$$

$$= c(\alpha,\beta) \left(\frac{1}{(\alpha-s)(\beta-t)-1} \right) = c(\alpha,\beta) [(\alpha-s)(\beta-t)-1]^{-1} = S^{-1} G(s,t)$$

where $S^{-1} = c(\alpha,\beta)$.

$G(s,t) = [(\alpha-s)(\beta-t)-1]^{-1} = S(\alpha-s, \beta-t)$

Now note that since we can interchange α w/ $-s$ and β w/ $-t$, then we have

$$\left. \frac{\partial^{j+k} M_{x,y}(s,t)}{\partial s^j \partial t^k} \right|_{(s,t)=(0,0)} = \left. \frac{\partial^{j+k} S^{-1} G(s,t)}{\partial s^j \partial t^k} \right|_{(s,t)=(0,0)} \overset{\text{by chain rule}}{=} \left. (-1)^{j+k} S^{-1} \frac{\partial S(\alpha-s, \beta-t)}{\partial \alpha^j \partial \beta^k} \right|_{(s,t)=(0,0)}$$

$$= (-1)^{j+k} S^{-1} \frac{\partial^{j+k}}{\partial \alpha^j \partial \beta^k} S(\alpha,\beta) = (-1)^{j+k} S^{-1} \frac{\partial^{j+k} S}{\partial \alpha^j \partial \beta^k} \checkmark \quad \lrcorner$$

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3 d) Show that the conditional distribution of $Y|X=x$ depends on β but is free of α, and derive the asymptotic distn of $\bar{Y}|\bar{X}=\bar{x}$, properly normalized.

i) Know
$$f(Y|X=x) = \frac{f(X=x, Y=y)}{f(X=x)} = \frac{c(\alpha,\beta)\exp(-\alpha x - \beta y)\sum_{j=0}^{\infty}\frac{x^j y^j}{(j!)^2}}{(\alpha - \frac{1}{\beta})e^{-x(\alpha - 1/\beta)}}$$

$c(\alpha,\beta) = (\alpha\beta - 1)$, $\quad (\alpha - \frac{1}{\beta}) = \frac{(\alpha\beta - 1)}{\beta}$, $\quad \exp(-\alpha x - \beta y) = \cancel{e^{-\alpha x}} \cdot e^{-\beta y}$

$$= \frac{\beta \exp(-\alpha x - \beta y)\sum_{j=0}^{\infty}\frac{x^j y^j}{(j!)^2}}{\cancel{e^{-\alpha x}} \cdot e^{x/\beta}} = \frac{\beta e^{-\beta y}\sum_{j=0}^{\infty}\frac{x^j y^j}{(j!)^2}}{e^{x/\beta}} \quad \perp\!\!\!\perp \alpha, \quad y>0,\ \beta>0$$

ii) By CLT, know
$$\sqrt{n}\left(\bar{Y}|\bar{X}=\bar{x} - E[Y|X=x]\right) \xrightarrow{d} N(0, Var[Y|X=x])$$

Then,
$$E[Y|X=x] = \beta\sum_{j=0}^{\infty}\frac{x^j}{(j!)^2}e^{-x/\beta}\int_0^{\infty} e^{-\beta y}\cdot y^{(j+2)-1}\,dy$$

$$= \beta\sum_{j=0}^{\infty}\frac{x^j}{(j!)^2}\cdot e^{-x/\beta}\cdot\Gamma(j+2)\cdot\left(\frac{1}{\beta}\right)^{j+2} = \beta^{-1}e^{-x/\beta}\sum_{j=0}^{\infty}\frac{x^j}{(j!)^{\cancel{2}}}\beta^{-j}(j+1)!$$

$(j+1)! = \cancel{j!}\cdot(j+1)$

$$= \beta^{-1}e^{-x/\beta}\sum_{j=0}^{\infty}\frac{x^j}{j!}\beta^{-j}(j+1) = \beta^{-1}e^{-x/\beta}\left[\sum_{j=0}^{\infty}\frac{j x^j}{j!\,\beta^j} + \sum_{j=0}^{\infty}\frac{x^j}{j!\,\beta^j}\right]$$

$$= \beta^{-1}e^{-x/\beta}\left[\sum_{j=0}^{\infty}\frac{j(x/\beta)^j}{j!} + \sum_{j=0}^{\infty}\frac{(x/\beta)^j}{j!}\right]$$

$$= \beta^{-1}e^{-x/\beta}\left[x/\beta\, e^{x/\beta} + e^{x/\beta}\right] = \beta^{-1}\cancel{e^{-x/\beta}}\cdot\cancel{e^{x/\beta}}[x/\beta + 1] = \beta^{-1}[x/\beta + 1]$$

Replace x w/ $n\bar{x}$ to get $E[Y|X=x] = \beta^{-1}[n\bar{x}/\beta + 1]$

Cont'd next pg. $\longrightarrow$

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3d. ii) cont'd.

Also, $Var[Y|X=x] = \underbrace{E[Y^2|X=x]}_{\text{Need to find}} - \underbrace{E[Y|X=x]}_{\text{from last part}}{}^2$

$$E[Y^2|X=x] = \beta \sum_{j=0}^{\infty} \frac{x^j}{(j!)^2} e^{-x/\beta} \int_0^{\infty} e^{-\beta y} \cdot y^{(j+3)-1}\, dy$$

$$= \beta \sum_{j=0}^{\infty} \frac{x^j}{(j!)^2} \cdot e^{-x/\beta} \cdot \Gamma(j+3) \cdot \left(\frac{1}{\beta}\right)^{j+3} = \beta^{-2} e^{-x/\beta} \sum_{j=0}^{\infty} \frac{x^j}{(j!)^{\cancel{2}}} \cdot \beta^{-j} (j+2)(j+1) \cdot \cancel{j!}$$

$$= \beta^{-2} e^{-x/\beta} \left[\sum_{j=0}^{\infty} \frac{(j^2+3j+2)(x/\beta)^j}{j!} \right] = \beta^{-2} e^{-x/\beta} \left[\sum_{j=0}^{\infty} \frac{j^2 (x/\beta)^j}{j!} + 3 \sum_{j=0}^{\infty} \frac{j (x/\beta)^j}{j!} + 2 \sum_{j=0}^{\infty} \frac{(x/\beta)^j}{j!} \right]$$

$$= \beta^{-2} e^{-x/\beta} \left[(x/\beta + x^2/\beta^2) e^{x/\beta} + 3 (x/\beta) e^{x/\beta} + 2 e^{x/\beta} \right]$$

$$= \beta^{-2} e^{-x/\beta} \left[x^2/\beta^2\, e^{x/\beta} + 4\, x/\beta\, e^{x/\beta} + 2 e^{x/\beta} \right]$$

Replace x with $n\bar{x}$ in the above to get,

$$E[Y^2|X=x] = \beta^2 e^{-n\bar{x}/\beta} \left[\frac{n^2 \bar{x}^2}{\beta^2} e^{n\bar{x}/\beta} + 4 \frac{n\bar{x}}{\beta} e^{n\bar{x}/\beta} + 2 e^{n\bar{x}/\beta} \right]$$

Then, $$Var[Y|X=x] = \beta^2 e^{-n\bar{x}/\beta} \left[\frac{n^2 \bar{x}^2}{\beta^2} e^{n\bar{x}/\beta} + 4 \frac{n\bar{x}}{\beta} e^{n\bar{x}/\beta} + 2 e^{n\bar{x}/\beta} \right] - \beta^{-2} \left[\frac{n^2 \bar{x}^2}{\beta^2} + 1 \right]$$

Thus, $\sqrt{n} \left(\bar{Y} | \bar{X} = x - E[Y|X=x] \right) \xrightarrow{d} N(0, Var[Y|X=x])$.

for $E[Y|X=x]$ and $Var[Y|X=x]$ as derived above. ⌋

3. e) Based on a sample of size n, derive a UMPU size α^* test for $H_0: \beta = 2$ vs. $H_1: \beta > 2$ and obtain an explicit expression for the critical value of the test.

⌈ First, write as a multiparameter exponential family.

Want to get in the form $p(\theta, \xi) = c(\theta, \xi)\exp\left(\theta u(x) + \sum_{i=1}^{K} \xi_i T_i(x)\right)$

$\uparrow$ nuisance

Here, $f(\underset{\sim}{x}, \underset{\sim}{y} | \alpha, \beta) = [c(\alpha,\beta)]^n \exp(-\alpha \sum x_i - \beta \sum y_i) \prod_{i=1}^{n} \left(\sum_{j=0}^{\infty} \frac{x_i^j y_i^j}{(j!)^2} \right)$

where $\theta = \beta \leftarrow$ parameter of interest b/c it's in the null

$u = -\sum_i y_i$

$T_1 = -\sum_i x_i$

$\xi_1 = \alpha \leftarrow$ nuisance parameter b/c it's not in the null

Thus, the UMPU size α^* test is of the form,

$$\phi(x) = \begin{cases} 1 & \text{if } -\sum_i y_i > -c(t) \\ 0 & \text{if } -\sum_i y_i < -c(t) \end{cases} = \begin{cases} 1 & \text{if } \sum_i y_i < c(t) \\ 0 & \text{if } \sum_i y_i > c(t) \end{cases}$$

where $E_{\beta=2}[\sum_i y_i < c(t) \mid \sum_i x_i] = \alpha^*$

$\Rightarrow \alpha^* = 1 \cdot P_{\beta=2}(\sum_i y_i < c(t) \mid T = t) + \cancel{0 \cdot P_{\beta=2}(\sum_i y_i > c(t) \mid T=t)}^{\,0}$

$= \int_0^{c(t)} f(\sum_i y_i \mid \sum x_i, \beta = 2)\, d\sum_i y_i$

Use the above to solve for $c(t)$. ⌋

3f) Based on a sample of size n, derive an exact 95% CI for β.

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A 95% CI for β can be found by inverting the two-sided test of $H_0: \beta = 2$ vs. $H_1: \beta \neq 2$.

Thus, a 95% CI for β is the set of all β's in the interval

$$1 - \phi^* = \begin{cases} 1, & \text{if } c_1(t) < \sum_i Y_i < c_2(t) \\ 0, & \text{else} \end{cases}$$

where $0.05 = E_\beta[1 - \phi^* \mid T = t]$ &

$$E_\beta[\textstyle\sum_i Y_i (1 - \phi^*) \mid T = t] = 0.05\, E_\beta[\textstyle\sum_i Y_i \mid T = t]$$

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3g) Derive the score test for testing $H_0: \beta = 2$ and obtain its asymptotic distribution.

General score test: $SC_n = \frac{\partial l_n}{\partial \xi}(\xi)^T I_n(\xi)^{-1} \frac{\partial l_n}{\partial \xi}(\xi) \Big|_{\xi = \hat{\xi}}$

Specific to this problem: $SC_n = \frac{\partial l_n}{\partial \beta}(\beta)^T I_n(\beta)^{-1} \frac{\partial l_n}{\partial \beta}(\beta) \Big|_{\beta = \tilde{\beta}}$

$$\mathcal{L}(\alpha, \beta \mid X, Y) = (\alpha\beta - 1)^n e^{-\alpha \sum_i x_i - \beta \sum_i y_i} \prod_{i=1}^{n} \sum_{j=0}^{\infty} \frac{x^i y^j}{(j!)^2}$$

$$l(\alpha, \beta \mid X, Y) = n \log(\alpha\beta - 1) - \alpha \sum_i x_i - \beta \sum_i y_i + \log\left(\sum_{i=1}^{n} \sum_{j=0}^{\infty} \frac{x^i y^j}{(j!)^2}\right)$$

$$\propto n \log(\alpha\beta - 1) - \alpha \sum_i x_i - \beta \sum_i y_i$$

$$\Rightarrow \frac{\partial l}{\partial \alpha} = \frac{n\beta}{(\alpha\beta - 1)} - \sum_i x_i \qquad \frac{\partial l}{\partial \beta} = \frac{n\alpha}{(\alpha\beta - 1)} - \sum_i y_i$$

$$\Rightarrow \frac{\partial^2 l}{\partial \alpha^2} = \frac{-n\beta^2}{(\alpha\beta - 1)^2} \qquad \frac{\partial^2 l}{\partial \beta^2} = \frac{-n\alpha^2}{(\alpha\beta - 1)^2}$$

$$\frac{\partial^2 l}{\partial \alpha \partial \beta} = \frac{\partial}{\partial \alpha}\left(\frac{n\alpha}{(\alpha\beta - 1)} - \sum_i y_i\right) = \frac{-n\alpha\beta}{(\alpha\beta - 1)^2} + \frac{n}{(\alpha\beta - 1)}$$

$$= \frac{-n\cancel{\alpha\beta} + n\cancel{\alpha\beta} - n}{(\alpha\beta - 1)^2} = \frac{-n}{(\alpha\beta - 1)^2}$$

$$\Rightarrow I_n(\alpha, \beta) = \begin{pmatrix} \frac{n\beta^2}{(\alpha\beta - 1)^2} & \frac{n}{(\alpha\beta - 1)^2} \\ \frac{n}{(\alpha\beta - 1)^2} & \frac{n\alpha^2}{(\alpha\beta - 1)^2} \end{pmatrix}$$

Under $H_0: \beta = 2$: $l(\alpha, \beta = 2 \mid X, Y) \propto n \log(2\alpha - 1) - \alpha \sum_i x_i$

$$\Rightarrow \frac{\partial l}{\partial \alpha} = \frac{2n}{2\alpha - 1} - \sum_i x_i \overset{set}{=} 0 \Rightarrow \frac{2n}{2\alpha - 1} = \sum_i x_i \Rightarrow \frac{2\alpha - 1}{2} = \frac{1}{\bar{x}} \Rightarrow 2\alpha = 2\frac{1}{\bar{x}} + 1$$

$$\Rightarrow \tilde{\alpha} = \frac{1}{\bar{x}} + \frac{1}{2}$$

$$\Rightarrow \frac{\partial l}{\partial \alpha} = 0 \text{ and } \frac{\partial l}{\partial \beta} = \frac{n\tilde{\alpha}}{(2\tilde{\alpha} - 1)} - \sum_i y_i$$

Then, $$SC_n = \left(0 \quad \frac{n\tilde{\alpha}}{(2\tilde{\alpha} - 1)} - \sum_i y_i\right) \begin{pmatrix} \frac{4n}{(2\tilde{\alpha} - 1)^2} & \frac{n}{(2\tilde{\alpha} - 1)^2} \\ \frac{n}{(2\tilde{\alpha} - 1)^2} & \frac{n\tilde{\alpha}^2}{(2\tilde{\alpha} - 1)^2} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \frac{n\tilde{\alpha}}{(2\tilde{\alpha} - 1)} - \sum_i y_i \end{pmatrix}$$

$$= \frac{1}{\left[\frac{4n^2\tilde{\alpha}^2}{(2\tilde{\alpha} - 1)^4} - \frac{n^2}{(2\tilde{\alpha} - 1)^4}\right]} \left(0 \quad \frac{n\tilde{\alpha}}{(2\tilde{\alpha} - 1)} - \sum_i y_i\right) \begin{pmatrix} \frac{n\tilde{\alpha}^2}{(2\tilde{\alpha} - 1)^2} & -\frac{n}{(2\tilde{\alpha} - 1)^2} \\ -\frac{n}{(2\tilde{\alpha} - 1)^2} & \frac{4n}{(2\tilde{\alpha} - 1)^2} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{n\tilde{\alpha}}{(2\tilde{\alpha} - 1)} - \sum_i y_i \end{pmatrix}$$

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3g. cont'd

$$= \frac{1}{\frac{n^2(4\tilde{\alpha}^2-1)}{(2\tilde{\alpha}-1)^4}} \left(\frac{-n}{(2\tilde{\alpha}-1)^2}\left(\frac{n\tilde{\alpha}}{(2\tilde{\alpha}-1)} - \sum_i y_i\right) \quad \frac{4n}{(2\tilde{\alpha}-1)^2}\left(\frac{n\tilde{\alpha}}{(2\tilde{\alpha}-1)} - \sum_i y_i\right)\right) \begin{pmatrix} 0 \\ \frac{n\tilde{\alpha}}{(2\tilde{\alpha}-1)} - \sum_i y_i \end{pmatrix}$$

$$= \frac{(2\tilde{\alpha}-1)^{\cancel{4}\,2}}{n^{\cancel{2}}(4\tilde{\alpha}^2-1)} \left[\frac{4\cancel{n}}{\cancel{(2\tilde{\alpha}-1)^2}} \left(\frac{n\tilde{\alpha}}{(2\tilde{\alpha}-1)} - \sum_i y_i\right)^2 \right] = \boxed{\frac{4(2\tilde{\alpha}-1)^2}{(4n\tilde{\alpha}^2-n)} \left(\frac{n\tilde{\alpha}}{2\tilde{\alpha}-1} - \sum_i y_i\right)^2}$$

where $\tilde{\alpha} = \frac{1}{\bar{x}} + \frac{1}{2}$

$$SC_n \xrightarrow{H_0} \chi^2_1, \text{ as } n \to \infty$$

3h)

Given $\theta = \frac{\alpha}{\beta}$. Know $p(\alpha,\beta \mid \underset{\sim}{x}, \underset{\sim}{y}) \propto p(\underset{\sim}{x}, \underset{\sim}{y} \mid \alpha,\beta)\cdot \pi(\alpha,\beta)$

$$\text{Bayes rule} = \underset{d(\underset{\sim}{x},\underset{\sim}{y})}{\operatorname{argmin}} \left[\int_{\Theta} L(\theta, d(\underset{\sim}{x},\underset{\sim}{y}))\, p(\theta \mid \underset{\sim}{x},\underset{\sim}{y})\, d\theta \right]$$

Given squared error loss, $L(\theta, a) = (\theta - a)^2$.

Then, the posterior expected loss is $G(a) = \int_{\Theta} (\theta-a)^2 p(\theta \mid \underset{\sim}{x},\underset{\sim}{y})\, d\theta$

Minimizing $G(a)$ w.r.t. a, get $\frac{dG(a)}{da} = \int_{\Theta} -2(\theta-a)\, p(\theta \mid \underset{\sim}{x},\underset{\sim}{y})\, d\theta = 0$

$$\Rightarrow \int_{\Theta} \theta\, p(\theta \mid \underset{\sim}{x},\underset{\sim}{y})\, d\theta = a \underbrace{\int p(\theta \mid \underset{\sim}{x},\underset{\sim}{y})\, d\theta}_{=1} \Rightarrow a = \int_{\Theta} \theta\, p(\theta \mid \underset{\sim}{x},\underset{\sim}{y})\, d\theta$$

Now, need to find $p(\theta \mid \underset{\sim}{x},\underset{\sim}{y})$.

Know $p(\alpha,\beta \mid \underset{\sim}{x},\underset{\sim}{y}) \propto p(\underset{\sim}{x},\underset{\sim}{y} \mid \alpha,\beta)\cdot \pi(\alpha,\beta)$

$$= \underbrace{(\alpha\beta-1)^n e^{-\alpha\sum_i x_i - \beta\sum_i y_i} \prod_{i=0}^{n} \sum_{j=0}^{\infty} \frac{x_i^j y_i^j}{(j!)^2}}_{p(\underset{\sim}{x},\underset{\sim}{y} \mid \alpha,\beta)} \cdot \frac{1}{\alpha\beta}$$

Convolution:

$$\text{Let } \left.\begin{aligned} \theta &= \frac{\alpha}{\beta} \\ \gamma &= \beta \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \alpha &= \beta\theta = \gamma\theta \\ \beta &= \gamma \end{aligned}\right\} \Rightarrow |J| = \begin{vmatrix} \frac{\partial\alpha}{\partial\theta} & \frac{\partial\alpha}{\partial\gamma} \\ \frac{\partial\beta}{\partial\theta} & \frac{\partial\beta}{\partial\gamma} \end{vmatrix} = \begin{vmatrix} \gamma & \theta \\ 0 & 1 \end{vmatrix} = |\gamma| = \gamma$$

since $\gamma > 0$

Then, $p(\alpha,\beta \mid \underset{\sim}{x},\underset{\sim}{y}) \propto p(\gamma\theta, \gamma \mid \underset{\sim}{x},\underset{\sim}{y})|\gamma|$

$$= (\gamma^2\theta - 1)^n e^{-\gamma\theta\sum_i x_i - \gamma\sum_i y_i} \prod_{i=1}^{n} \sum_{j=0}^{\infty} \frac{x_i^j y_i^j}{(j!)^2} \cdot \frac{1}{\gamma^{\cancel{2}}\theta} \cdot \underbrace{\cancel{\gamma}}_{\text{jacobian}}$$

$$\propto \frac{(\gamma^2\theta-1)^n}{\gamma\theta} e^{-\gamma\theta\sum_i x_i - \gamma\sum_i y_i}, \quad \begin{aligned} 0 < \theta < \infty \\ 0 < \gamma < \infty \end{aligned}, \; x > 0,\; y > 0$$

Integrate out γ to get marginal of θ $\rightarrow$ No clue where to go from here. Not a recognizable distn.