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1.a) Derive the distn. of $\overbrace{U = (Y - X\beta)'\Sigma^{-1}(Y - X\beta)}^{i)}$,

and derive $\underbrace{\text{the mean and variance of } U}_{ii)}$.

i) Let $\Sigma^{-1} = \Sigma^{-1/2}\Sigma^{-1/2}$

Then, $U = (Y - X\beta)'\Sigma^{-1/2}\Sigma^{-1/2}(Y - X\beta) = \Big(\underbrace{\Sigma^{-1/2}(Y - X\beta)}_{N(0,I)}\Big)'\Big(\underbrace{\Sigma^{-1/2}(Y - X\beta)}_{N(0,I)}\Big) = \sum_{i=1}^{n} Z_i^2$

where $Z_i := \Sigma_i^{-1/2}(Y_i - X_i\beta) \sim N(0,1)$

$\Rightarrow U = \sum_{i=1}^{n} Z_i^2 \sim \boxed{\chi^2_n}$

ii) $E[U] = E\left[\sum_{i=1}^{n} Z_i^2\right] = \sum_{i=1}^{n} E\underbrace{[Z_i^2]}_{\sim\chi^2_1} = \sum_{i=1}^{n} 1 = \boxed{n}$

$Var[U] = Var\left[\sum_{i=1}^{n} Z_i^2\right] = \sum_{i=1}^{n} Var[Z_i^2] = \sum_{i=1}^{n} 2 = \boxed{2n}$

1 b) Formally derive the set of all possible LS solns. of β.

Consider the transformed model: $\underbrace{Q^{-1}Y}_{Y^*} = \underbrace{Q^{-1}X}_{X^*}\beta + \underbrace{Q^{-1}\epsilon}_{\epsilon^* \sim N(0,I)}$ where $\Sigma = QQ'$

$\Rightarrow Y^* = X^*\beta + \epsilon$, the standard LM.

Let $M^* = X^*(X^{*\prime}X^*)^{-}X^{*\prime}$ be the OPO onto $C(X^*)$,

& $Y^* = M^*Y^* + (I - M^*)Y^*$

Since M^*Y^* is the OPO of Y^* onto $C(X^*)$, then M^*Y^* is the closest vector to Y^* in $C(X^*)$,

So we solve $M^*Y^* = X^*\beta$.

Claim: A general soln. to this system is:

$$\hat{\beta} = \underbrace{(X^{*\prime}X^*)^{-}X^{*\prime}Y}_{i)} + \underbrace{(I - X^{*\prime}(X^*X^{*\prime})^{-}X^*)}_{ii)}z \quad \text{for } z \in \mathbb{R}^p$$

To see this: i) Note that $M^*Y^* \in C(X^*) \Rightarrow X^*b = M^*Y^*$ for some $b \in \mathbb{R}^p$.

Know that $(X^{*\prime}X^*)^{-}X^{*\prime}Y^*$ is a soln. (i.e., $b = (X^{*\prime}X^*)^{-}X^{*\prime}Y^*$)

Since $X^*(X^{*\prime}X^*)^{-}X^{*\prime}Y^* = M^*Y^*$

ii) Next, if β^* is a soln, then the soln. set has the form $\beta^* + w$, where $w \in N(X^*)$.

Know $N(X^*) = C(X^{*\prime})^{\perp} \Rightarrow N(X^*)$ has the same column space as $I - M(X^{*\prime}) = I - X^{*\prime}(X^*X^{*\prime})^{-}X^*$

$\Rightarrow w$ has the form $(I - X^{*\prime}(X^*X^{*\prime})^{-}X^*)z$

Conclude: Thus, the set of all possible LS solns is $LS(\beta) = \{\beta : (X^{*\prime}X^*)^{-}X^{*\prime}Y^* + (I - X^{*\prime}(X^*X^{*\prime})^{-}X^*)z\}$

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1c) Show that $\lambda'\beta$ estimable iff $\lambda'(X'\Sigma^{-1}X)^{-}(X'\Sigma^{-1}X)=\lambda'$, where $^{-}$ denotes a gen-inv.

$\lceil$ ($\Rightarrow$) Assume $\lambda'\beta$ estimable $\Rightarrow$ $\lambda'=\rho'X^*$ for ρ an $n\times 1$ vector of constants.

Let $\Sigma = QQ'$ where Q invertible so Σ P.D.

Then, $\lambda'(X'\Sigma^{-1}X)^{-}(X'\Sigma^{-1}X) = \lambda'(X'(QQ')^{-1}X)^{-}(X'(QQ')^{-1}X)$

$= \lambda'(X'Q'^{-1}Q^{-1}X)^{-}(X'Q'^{-1}Q^{-1}X) = \lambda'(\underbrace{(Q^{-1}X)}_{X^*}'\underbrace{(Q^{-1}X)}_{X^*})^{-}(\underbrace{(Q^{-1}X)}_{X^*}'\underbrace{(Q^{-1}X)}_{X^*}) = \lambda'(X^{*\prime}X^*)^{-}(X^{*\prime}X^*)$

$= \rho'\underbrace{X^*(X^{*\prime}X^*)^{-}(X^{*\prime}}_{M^*}X^*) = \rho' M^* X^* = \rho' X^* = \lambda'$

Thus, $\lambda'\beta$ estimable $\Rightarrow$ $\lambda'(X'\Sigma^{-1}X)^{-}(X'\Sigma^{-1}X) = \lambda'$

($\Leftarrow$) In first direction, had $\lambda'(X'\Sigma^{-1}X)^{-}(X'\Sigma^{-1}X) = \lambda'(X^{*\prime}X^*)^{-}(X^{*\prime}X^*)$

Then, since it is assumed that $\lambda'(X'\Sigma^{-1}X)^{-}(X'\Sigma^{-1}X)=\lambda'$, then substituting the above expression, we get:

$\underbrace{\lambda'(X^{*\prime}X^*)^{-}(X^{*\prime}}_{\rho'}X^*) = \lambda'$ $\Rightarrow$ $\lambda' = \rho'X^*$ $\Rightarrow$ $\lambda'\beta$ is estimable. $\rfloor$

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1 d) Assume X has rank p. Show that the BLUE of β is equal to $(X'X)^{-1}X'Y$

$\iff \exists$ a non-singular $p \times p$ matrix $F \ni \Sigma X = XF$.

$\ulcorner$ Let $\tilde{\beta} = (X'X)^{-1}X'Y$ 　　 For $\Sigma = QQ'$ for Q invertible and Σ P.D.

$$\hat{\beta} = (X^{*\prime}X^*)^{-1}X^{*\prime}Y^* = ((Q^{-1}X)'(Q^{-1}X))^{-1}(Q^{-1}X)'(Q^{-1}Y) = (X'\underbrace{Q'^{-1}Q^{-1}}_{(QQ')^{-1}}X)^{-1}(X'\underbrace{Q'^{-1}Q^{-1}}_{(QQ')^{-1}}Y)$$

$$= (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}Y$$

$(\Rightarrow)$ Assume $\hat{\beta} = \tilde{\beta}$

$$\Rightarrow (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}Y = (X'X)^{-1}X'Y$$

$$\Rightarrow (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}Y - (X'X)^{-1}X'Y = 0$$

$$\Rightarrow [(X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1} - (X'X)^{-1}X']Y = 0$$

$$\Rightarrow (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1} - (X'X)^{-1}X' = 0 \Rightarrow (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1} = (X'X)^{-1}X'$$

$$\Rightarrow \{(X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}\}' = \{(X'X)^{-1}X'\}'$$

$$\Rightarrow \Sigma^{-1}X(X'\Sigma^{-1}X)^{-1} = X(X'X)^{-1}$$

$$\Rightarrow \Sigma\Sigma^{-1}X(X'\Sigma^{-1}X)^{-1} = \Sigma X(X'X)^{-1}$$

$$\Rightarrow X(X'\Sigma^{-1}X)^{-1} = \Sigma X(X'X)^{-1} \Rightarrow X\underbrace{(X'\Sigma^{-1}X)^{-1}(X'X)}_{F} = \Sigma X$$

$\Rightarrow XF = \Sigma X$ for a non-singular $p \times p$ matrix F.

$(\Leftarrow)$ Assume $\exists$ a non-singular $p \times p$ matrix $F \ni \Sigma X = XF$.

Note that $\Sigma X = XF \Rightarrow X = \Sigma^{-1}XF \Rightarrow XF^{-1} = \Sigma^{-1}X \Rightarrow (XF^{-1})' = X'\Sigma^{-1}$

Then, $\beta = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}Y = [(XF^{-1})'X]^{-1}F'^{-1}X'Y = [F'^{-1}X'X]^{-1}F'^{-1}X'Y$

$$= (X'X)^{-1}\underbrace{F'F'^{-1}}_{I}X'Y = (X'X)^{-1}X'Y$$

Thus, the BLUE of β is equal to $(X'X)^{-1}X'Y$. $\lrcorner$

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1.e) Assume X has rank p. Let s^2 be defined as $s^2 = \frac{Y'(I-M)Y}{n-p}$

where M denotes the OPO onto $C(X)$.

Show that $E(s^2) \leq \frac{1}{n-p} \sum_{i=1}^{n} \sigma_{ii}$ i)

where σ_{ii} denotes the ith diagonal element of Σ, $i=1,\dots,n$. Can the upper bound on

ii) → $E(s^2)$ be attained? Justify.

i) $E(s^2) = E\left[\frac{Y'(I-M)Y}{n-p}\right] = \frac{1}{n-p} E[Y'(I-M)Y]$.

In general, $E[Y'AY] = \mu' A \mu + tr[\Sigma A]$, then applying to the above, we get;

$$E(s^2) = \frac{1}{n-p}\left\{\mu'(I-M)\mu + tr(\Sigma(I-M))\right\}$$

$0 \leftarrow$ since $I-M$ is an OPO onto $C(X')^{\perp} = N(X)$.

$$= \frac{1}{n-p}\left\{\cancel{(X\beta)'(I-M)(X\beta)} + tr(\Sigma) - tr(\Sigma M)\right\}$$

$$= \frac{1}{n-p}\left\{tr(\Sigma) - \underbrace{tr(\Sigma M)}\right\} \leq \frac{1}{n-p} tr(\Sigma) = \frac{1}{n-p}\sum_{i=1}^{n}\sigma_{ii}$$

$= tr(\Sigma M^2)$

$= tr(M \Sigma M)$

Since Σ P.D. $\Rightarrow$ $M\Sigma M$ P.D.

$\Rightarrow tr(M\Sigma M) \geq 0$

where σ_{ii} denotes the ith diag element of Σ, for $i=1,\dots,n$.

ii) The upper bound on $E(s^2)$ can be attained if ΣM has eigenvalues that sum to zero.