2010 Theory II #2

unfinished!

2a) $Y = X\beta + UJ + \varepsilon$

$U \sim N(\alpha J,\ k\sigma^2 JJ')$

$\varepsilon \sim N(0,\ \sigma^2 I)$

Since $U \perp\!\!\!\perp \varepsilon$ we have

$$Var[Y] = Var[U] + Var[\varepsilon] = k\sigma^2 JJ' + \sigma^2 I = \sigma^2(I + kJJ')$$

so that

$$Y \sim N(0, \sigma^2\Sigma) \quad \text{where } \Sigma = I + kJJ'$$

Let $0 \neq v \in \mathbb{R}^n$. Then

$$v'\Sigma v = \underbrace{\sigma^2}_{>0}\Big[\underbrace{v'v}_{>0} + \underbrace{k(\Sigma v_i)^2}_{\geq 0}\Big] > 0$$

2c) We can express the model as

$$Y = W\theta + \gamma = (1, X)\begin{pmatrix}\alpha \\ \beta\end{pmatrix} + \underset{\sim}{\gamma}$$

$N(0, \sigma^2\Sigma)$

Then

$$L_n(\theta, \Sigma) = (2\pi)^{-n/2}|\sigma^2\Sigma|^{-1/2}\exp\left\{-\frac{1}{2\sigma^2}(Y - W\theta)'\Sigma^{-1}(Y - W\theta)\right\}$$

$$= (2\pi)^{-n/2}|\sigma^2\Sigma|^{-1/2}\exp\left\{-\frac{1}{2\sigma^2}\left[Y'\Sigma^{-1}Y - 2Y'\Sigma^{-1}W\theta + \theta'W'\Sigma^{-1}W\theta\right]\right\}$$

so that

$$\frac{\partial}{\partial\theta} l_n(\theta, \Sigma) = \sigma^2 W'\Sigma^{-1}Y - \sigma^2 W'\Sigma^{-1}W\theta \overset{set}{=} 0$$

W is pd

$$\Rightarrow \hat\theta = (W'\Sigma^{-1}W)^{-1}W'\Sigma^{-1}Y$$

since $\text{rank}(W'\Sigma^{-1}W) = \text{rank}(W'\Sigma^{-1/2}\Sigma^{-1/2}W) = \text{rank}(W'\Sigma^{-1/2}) = \text{rank}(W) = p+1$

$\text{rank}(A'A) = \text{rank}(A)$

$\text{rank}(AB) = \text{rank}(A)$ if B is nonsingular

Next,

$$\cancel{l_n(\theta, \Sigma) = -\frac{n}{2}\log(2\pi) + \frac{1}{2}\log(|\Sigma^{-1}|) - \frac{1}{2}tr\{\Sigma^{-1}(Y - W\theta)(Y - W\theta)'\}}$$

so that

$$\cancel{\frac{\partial}{\partial\Sigma^{-1}} l_n(\theta, \Sigma) = \frac{1}{2}\frac{1}{|\Sigma^{-1}|}|\Sigma^{-1}|\Sigma - \frac{1}{2}(Y - W\theta)(Y - W\theta)'}$$

$$\cancel{= \frac{1}{2}\Sigma - \frac{1}{2}(Y - W\theta)(Y - W\theta)' \overset{set}{=} 0}$$

$$\cancel{\Rightarrow \hat\Sigma = (Y - W\hat\theta)(Y - W\hat\theta)'}$$

don't need

Then $\hat\theta$ is normally distributed with

$$E\hat\theta = (W'\Sigma^{-1}W)^{-1}W'\Sigma^{-1}EY = (W'\Sigma^{-1}W)^{-1}W'\Sigma^{-1}W\theta = \theta$$

$$Var[\hat\theta] = (W'\Sigma^{-1}W)^{-1}W'\Sigma^{-1}Var[Y]\Sigma^{-1}W(W'\Sigma^{-1}W)$$

$$= (W'\Sigma^{-1}W)^{-1}W'\Sigma^{-1}\sigma^2\Sigma\Sigma^{-1}W(W'\Sigma^{-1}W) = (W'\Sigma^{-1}W)^{-1}$$

2d) $$\frac{\partial}{\partial\theta} \sum_{i=1}^{n} \left[Y_i - (\alpha + X_{i1}\beta_1 + \dots + X_{ip}\beta_p)\right]^2 = \frac{\partial}{\partial\theta}(Y - W\theta)'(Y - W\theta)$$

$$= \frac{\partial}{\partial\theta}\left\{Y'Y - 2\theta'W'Y + \theta'W'W\theta\right\} = -2W'Y + 2W'W\theta \overset{set}{=} 0$$

$$\Rightarrow \tilde{\theta} = (W'W)^{-1}W'Y$$

Wish to determine if $v'\left(Var[\hat{\theta}] - Var[\tilde{\theta}]\right)v < 0 \quad \forall\, v \neq 0$.

Recall Gauss-Markov thm: let $\hat{\beta}$ be the least-squares estimate of β. then $v'\hat{\beta}$ is an unbiased estimate of $v'\beta$. Let $d'Y$ be any other unbiased estimate of $v'\beta$. Then

$$Var[d'Y] \geq Var[v'\hat{\beta}] \text{ with equality only if}$$

$$(I - P)d = 0 \Leftrightarrow d = Pd = Pv$$

part of the thm

Now we write this theorem in terms of the transformed model. We have

$$\underbrace{\Sigma^{-1/2}Y}_{Z} = \underbrace{\Sigma^{-1/2}W}_{B}\theta + \varepsilon$$

$$Z = B\theta + \varepsilon$$

Then for $d'Y$ an unbiased estimate of $v'\theta$

$$Var[d'Y] \geq Var[v'\hat{\theta}] \text{ with equality only if}$$

$$d'Y = v'(W'\Sigma^{-1}W)^{-1}W'\Sigma^{-1}Y = v'\hat{\theta}$$

Notice that $E\tilde{\theta} = \theta$ so Gauss-Markov applies

Thus letting $d = v'(W'W)^{-1}W'$

$$v'\left(Var[\hat{\theta}] - Var[\tilde{\theta}]\right)v = Var[v'\hat{\theta}] - Var[v'\tilde{\theta}] = Var[v'\hat{\theta}] - Var[d'Y] \leq 0$$

with equality only if

$$v'(W'\Sigma^{-1}W)^{-1}W'\Sigma^{-1} = v'(W'W)^{-1}W'$$

Updating (Sherman-Morrison) formula: suppose A is invertible and u, v are vectors s.t. $1 + v'A^{-1}u \neq 0$. Then

$$(A + uv')^{-1} = A^{-1} - \frac{A^{-1}uv'A^{-1}}{1 + v'A^{-1}u}$$

$$LHS = v'\left\{(W'W)^{-1} - \frac{\rho(W'W)^{-1}W'JJ'W(W'W)^{-1}}{1+\rho J'W(W'W)^{-1}W'J}\right\}W'[I + \rho JJ']$$

$$= v'(W'W)^{-1}W - \rho\frac{v'(W'W)^{-1}W'JJ'W(W'W)^{-1}W'[I+\rho JJ']}{1+\rho J'W(W'W)^{-1}W'J}$$

$$+ v'(W'W)^{-1}W'[I+\rho JJ']$$

$$= v'(W'W)^{-1}W - \frac{\rho}{1+\rho n}v'(W'W)^{-1}WJJ'[I+\rho JJ'] + v'(W'W)^{-1}W'[I+\rho JJ']$$