1. (25 points) Consider the linear model

$$Y = X\beta + \epsilon, \tag{1}$$

where Y is $n \times 1$, X is an $n \times p$ matrix of fixed covariates with rank $r < p$, β is $p \times 1$, and $\epsilon \sim N_n(0, \Sigma)$, where Σ is a known positive definite matrix.

(a) **Derive** the distribution of

$$U = (Y - X\beta)'\Sigma^{-1}(Y - X\beta),$$

and **derive** the mean and variance of U.

Note: **You are not allowed to simply state the result of a theorem to give your answer. You must *derive* the results.**

(b) Formally derive the set of all possible least squares solutions of β.

Note: **You are not allowed to simply state a result or a formula for your answer. You must *derive* the result.**

(c) Show that $\lambda'\beta$ is estimable if and only if

$$\lambda'(X'\Sigma^{-1}X)^{-}(X'\Sigma^{-1}X) = \lambda',$$

where a "-" denotes generalized inverse.

(d) Assume X has rank p. Show that the BLUE of β is equal to $(X'X)^{-1}X'Y$ if and only if there exists a non-singular $p \times p$ matrix F such that $\Sigma X = XF$.

(e) Assume X has rank p. Let s^2 be defined as

$$s^2 = \frac{Y'(I - M)Y}{n - p}$$

where M denotes the orthogonal projection operator onto the column space of X. Show that

$$E(s^2) \leq \frac{1}{n-p}\sum_{i=1}^{n} \sigma_{ii},$$

where σ_{ii} denotes the ith diagonal element of Σ, $i = 1, \ldots, n$. Can the upper bound on $E(s^2)$ be attained? Justify your answer.

Points: (a) 5; (b) 5; (c) 5; (d) 5; (e) 5.

1. Consider the linear model:

$$Y = X\beta + \varepsilon$$

where $Y_{n\times 1}$, $X_{n\times p}$ rank $r<p$, $\beta_{p\times 1}$ $\varepsilon \sim N(0, \Sigma)$ where Σ known, pos. def.

(a) Derive the distribution of $U = (Y-X\beta)'\Sigma^{-1}(Y-X\beta)$ and derive the mean & variance of U.

as Σ is a known positive definite matrix, it can be decomposed to $\Sigma = \Sigma^{1/2}\Sigma^{1/2}$.

Thus $\Sigma^{-1/2}Y = \Sigma^{-1/2}X\beta + \Sigma^{-1/2}\varepsilon$ $\qquad \Sigma^{-1/2}\varepsilon \sim N(0, I_n)$

$\therefore \Sigma^{-1/2}(Y-X\beta) \sim N(0, I_n)$

let $Z = \Sigma^{-1/2}(Y-X\beta) = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$ then, $z_i \sim N(0,1)$

$U = (Y-X\beta)'\Sigma^{-1}(Y-X\beta) =$

$= (Y-X\beta)'\Sigma^{-1/2}\Sigma^{-1/2}(Y-X\beta)$ $\qquad z_i^2 \sim \chi^2_1$

$= (\Sigma^{-1/2}(Y-X\beta))'\Sigma^{-1/2}(Y-X\beta) = Z'Z = \boxed{\sum_{i=1}^n z_i^2 \sim \chi^2_n}$

$E[U] = n$ $\quad var(U) = 2n$ to derive:

$E[U] = E\left[\sum_{i=1}^n z_i^2\right] = \sum_{i=1}^n E[z_i^2] = \sum_{i=1}^n E[z_i]^2 + Var(z_i) = \sum_{i=1}^n 0^2 + 1 = n$

$var(U) = var\left(\sum_{i=1}^n z_i^2\right) = \sum_{i=1}^n Var(z_i^2) = \sum_{i=1}^n E[z_i^4] - E[z_i^2]^2$

$M_{z_i}(t) = \exp\{\frac{t^2}{2}\}$

$M'(t) = t\exp\{\frac{t^2}{2}\}$

$M''(t) = \exp\{\frac{t^2}{2}\} + t^2\exp\{\frac{t^2}{2}\}$

$M'''(t) = t\exp\{\frac{t^2}{2}\} + 2t\exp\{\frac{t^2}{2}\} + t\exp\{\frac{t^2}{2}\}$

$M^{(4)}(t) = \exp\{\frac{t^2}{2}\} + 2\exp\{\frac{t^2}{2}\} + 2t^2\exp\{\frac{t^2}{2}\}$

$= 3\exp\{\frac{t^2}{2}\} + 2t^2\exp\{\frac{t^2}{2}\}$

$M^{(4)}(t)\big|_0 = 3$

$\therefore var(U) = \sum_{i=1}^n 3 - 1^2 = \sum_{i=1}^n 2 = 2n$

1.(b) Formally derive the set of all possible least squares solutions of β. | 2019 Theory 2

At this point we want to work with the transformed model:

$$Q^{-1}Y = Q^{-1}X\beta + Q^{-1}\varepsilon \qquad \text{where } QQ' = \Sigma.$$

let $Y^* = Q^{-1}Y$, $X^* = Q^{-1}X$, $\varepsilon^* = Q^{-1}\varepsilon \sim N(0, I_n)$

which leads us to the standard linear model:

$$Y^* = X^*\beta + \varepsilon^*.$$

need g-inv as X is not full rank.

let M^* be the OPO onto $C(X^*)$ $M^* = X^*(X^{*\prime}X^*)^- X^{*\prime}$

$$Y^* = M^*Y^* + (I - M^*)Y^*$$

M^*Y^* is the closest to Y^* in $C(X^*)$ $\therefore$ we use this to solve the system.

$$M^*Y^* = X^*\beta$$

$\therefore (X^{*\prime}X^*)^{-1}X^*Y^* = \beta^*$ is a solution for β

if β^* is a solution for β, the solution set is of the form

$\beta^* + w \qquad w \in N(X^*) \equiv C(X^{*\prime})^\perp \qquad \therefore N(X^*) \equiv I - X^{*\prime}(X^*X^{*\prime})^- X^*$

$\therefore w = (I - X^{*\prime}(X^*X^{*\prime})^- X^*)z \qquad z \in \mathbb{R}^n$

$\therefore$ The solution set for all LS solutions of β is:

$$\{\beta : (X^{*\prime}X^*)^- X^*Y^* + (I - X^{*\prime}(X^*X^{*\prime})^- X^*)z\}$$

1.(c) Show that $\lambda'\beta$ is estimable $\Leftrightarrow$

$$\lambda'(X'\Sigma^{-1}X)^{-}(X'\Sigma^{-1}X) = \lambda'$$

($\Rightarrow$)

$\lambda'\beta$ is estimable $\Leftrightarrow \lambda \in C(X^{*\prime})$

$\therefore \lambda = X^{*\prime}P$ for some $P \in \mathbb{R}^n$

$= X'Q^{-1}P$

$$(X'\Sigma^{-1}X)(X'\Sigma^{-1}X)^{-}\lambda = (X'\Sigma^{-1}X)(X'\Sigma^{-1}X)^{-}X'Q^{-1}P$$

$$= ((Q^{-1}X)'(Q^{-1}X))((Q^{-1}X)'(Q^{-1}X))^{-}(Q^{-1}X)'P$$

$$= (Q^{-1}X)'M_{X^*}P$$

$$= (M_{X^*}(Q^{-1}X))'P$$

$$= (Q^{-1}X)'P = X'Q^{-1}P = \lambda$$

$\Rightarrow \lambda' = \lambda'(X'\Sigma^{-1}X)(X'\Sigma^{-1}X)^{-}$

($\Leftarrow$) Know $\lambda'(X'\Sigma^{-1}X)^{-}(X'\Sigma^{-1}X) = \lambda'$ WTS $\lambda'\beta$ is estimable

NTS $\lambda \in C(X^{*\prime})$

$$\lambda = (X'\Sigma^{-1}X)(X'\Sigma^{-1}X)^{-}\lambda$$

$$= (X^{*\prime}X^*)(X^{*\prime}X^*)^{-}\lambda$$

$$= (X^{*\prime})((X^*)(X^{*\prime}X^*)^{-}\lambda)$$

$= X^{*\prime}P$ for $P = X^*(X^{*\prime}X^*)^{-}\lambda$

$\therefore \lambda \in C(X^{*\prime})$ $\therefore \lambda'\beta$ is estimable.

1(d) Assume X has rank p. Show that the BLUE of β is equal to $(X'X)^{-1}X'Y \iff \exists$ a nonsingular matrix F s.t. $\Sigma X = XF$

we again will use the transformed model

$$Y^* = X^*\beta + \varepsilon^*$$

under this model, the BLUE is $(X^{*\prime}X^*)^{-1}X^{*\prime}Y^*$

$$(X^{*\prime}X^*)^{-1}X^{*\prime}Y = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}Y = \hat{\beta}$$

($\Rightarrow$) If $\hat{\beta} = \tilde{\beta} = (X'X)^{-1}X'Y$ then

$$(X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}Y = (X'X)^{-1}X'Y$$

$$(X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}Y - (X'X)^{-1}X'Y = 0$$

$$[(X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1} - (X'X)^{-1}X']Y = 0$$

$$(X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1} - (X'X)^{-1}X' = 0$$

$$(X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1} = (X'X)^{-1}X'$$

$$\Sigma^{-1}X(X'\Sigma^{-1}X)^{-1} = X(X'X)^{-1}$$

$$X(X'\Sigma^{-1}X)^{-1} = \Sigma X(X'X)^{-1}$$

$$X(X'\Sigma^{-1}X)^{-1}(X'X) = \Sigma X$$

$\Rightarrow \Sigma X = XF$ where $F = (X'\Sigma^{-1}X)^{-1}(X'X)$ non-singular

($\Leftarrow$) $\exists$ a non-singular F s.t. $\Sigma X = XF$

$\Sigma X = XF \iff X = \Sigma^{-1}XF \iff XF^{-1} = \Sigma^{-1}X$

$$\therefore \hat{\beta} = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}Y = ((\Sigma^{-1}X)'X)^{-1}(\Sigma^{-1}X)'Y$$

$$= ((XF^{-1})'X)^{-1}(XF^{-1})'Y$$

$$= (F^{-1\prime}X'X)^{-1}F^{-1\prime}X'Y$$

$$= (X'X)^{-1}F'F^{-1\prime}X'Y$$

$$= (X'X)^{-1}X'Y$$

1.(e) Assume X has rank p. Let s^2 be defined as

$$s^2 = \frac{Y'(I-M)Y}{n-p}$$

(i)
Show that $E(s^2) \le \frac{1}{n-p}\sum_{i=1}^{n}\sigma_{ii}$ where σ_{ii} denotes the i^{th} diagonal element of Σ.

(ii)
Can the upper bound of $E[s^2]$ be attained?

$$E[s^2] = E\left[\frac{Y'(I-M)Y}{n-p}\right] = \frac{1}{n-p}E[Y'(I-M)Y] \le \frac{1}{n-p}\sum\sigma_{ii}$$

$$\equiv E[Y'(I-M)Y] \le \sum\sigma_{ii}$$

$$\begin{aligned}
Y'(I-M)Y &= (X\beta+\varepsilon)'(I-M)(X\beta+\varepsilon) \\
&= \varepsilon'(I-M)\varepsilon \\
&= \varepsilon^{*\prime}Q'(I-M)Q\varepsilon^* \\
&= \varepsilon^{*\prime}UDU'\varepsilon^* \\
&= (U'\varepsilon^*)'D(U'\varepsilon^*) \\
&= A'DA \\
&= \sum_{i=1}^{n} a_i^2 d_i
\end{aligned}$$

$\varepsilon^* = Q^{-1}\varepsilon \Rightarrow \varepsilon = Q\varepsilon^*$

by EVD: $Q'(I-M)Q = UDU'$

let $U'\varepsilon^* = A \sim N(0,I)$
$a_i \sim N(0,1)$

$$\therefore E[Y'(I-M)Y] = E\left[\sum_{i=1}^{n}a_i^2 d_i\right] = \sum_{i=1}^{n} d_i E[a_i^2] = \sum_{i=1}^{n} d_i$$

$$\begin{aligned}
\sum_{i=1}^{n} d_i = tr(D) = tr(UDU') &= tr(Q'(I-M)Q) \\
&= tr((I-M)Q'Q) \\
&= tr((I-M)\Sigma) \\
&= tr(\Sigma - M\Sigma) \\
&= tr(\Sigma) - tr(M\Sigma) \\
&= tr(\Sigma) - tr(M^2\Sigma) \\
&= tr(\Sigma) - tr(M\Sigma M') \\
&< tr(\Sigma)
\end{aligned}$$

Σ is positive definite
$\Rightarrow M\Sigma M'$ is pos definite
$\Rightarrow \lambda_i$ eigenvalue of $M\Sigma M'$
$\lambda_i \ge 0$
$\Rightarrow tr(M\Sigma M) = \sum\lambda_i \ge 0$

The upper bound could only be attained if $\lambda_i = 0 \ \forall\, i=1,\dots,n$ which will never occur in a positive definite matrix.
Thus, the upper bound will never be attained

1.con't $Y|X \sim$ log-normal let $N(0,\sigma^2) \equiv Z$

$\gamma X \sim N(0,\gamma^2)$ $\phi_{\gamma X}(t) = \exp\left\{-\frac{\gamma^2 t^2}{2}\right\}$ $\phi_Z(t) = \exp\left\{-\frac{\sigma^2 t^2}{2}\right\}$

$$\phi_{\gamma X+Z}(t) = \phi_{\gamma X}(t)\,\phi_Z(t) = \exp\left\{-\frac{(\gamma^2+\sigma^2)t}{2}\right\}$$

$\therefore \log(Y) \sim N(0,\gamma^2+\sigma^2)$ let $\log(Y) = A$

$$f_A(a) = \frac{1}{\sqrt{2\pi(\gamma^2+\sigma^2)}} \exp\left\{-\frac{A^2}{2(\gamma^2+\sigma^2)}\right\}$$

we want to find the density of $\exp^A$: let $Y = e^A \Rightarrow A = \log(Y)$ $|J| = \left|1/y\right| = 1/y$

$$f_{Y|X}(y) = f_A(a)\cdot\left|\frac{1}{y}\right| = \frac{1}{\sqrt{2\pi(\gamma^2+\sigma^2)}} \exp\left\{-\frac{\log(y)^2}{2(\gamma^2+\sigma^2)}\right\}\left|\frac{1}{y}\right|$$

$Y|X \sim N(\gamma X, \sigma^2)$

$$f_{Y|X} = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(y-\gamma x)^2}{2\sigma^2}\right\}$$

$$f_{X,Y}(x,y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(y-\gamma x)^2}{2\sigma^2}\right\}\frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}$$

$$= \frac{1}{2\pi\sigma} \exp\left\{-\frac{(y-\gamma x)^2 + x^2\sigma^2}{2\sigma^2}\right\}$$

$E[Y] = E[E[Y|X]] = E[\gamma X] = 0$?

$E[XY] = \gamma E[Y]$

1.(c) con't

$E[YX] = \cdots = E[X\exp(\gamma X)]E[\exp(Z)]$

$= \exp\left(\frac{\gamma^2}{2}\right)\gamma \exp(E(Z)) = \exp\left(\frac{\gamma^2}{2}\right)\gamma$

1.(c) Now suppose that we know the distribution of $Y|X \sim$ lognormal s.t. $\log Y = \gamma X + N(0,\sigma^2) \Rightarrow \log Y|X \sim N(\gamma X, \sigma^2+\gamma^2)$

let $T = \log Y \Rightarrow f_{T,X}(t,x) = f_{T|X}(t) f_X(x)$

$$f_{T,X}(t,x) = \frac{1}{\sqrt{2\pi(\sigma^2+\gamma^2)}} \exp\left\{\frac{-(t-\gamma x)^2}{2(\sigma^2+\gamma^2)}\right\} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}$$

$$= \frac{1}{2\pi\sqrt{\sigma^2+\gamma^2}} \exp\left\{-\frac{(t-\gamma x)^2}{2(\sigma^2+\gamma^2)} - \frac{x^2}{2}\right\}$$

$$l(\gamma,\sigma^2) \propto -\frac{n}{2}\log(\sigma^2+\gamma^2) - \frac{\Sigma(t_i-\gamma x_i)^2}{2(\sigma^2+\gamma^2)} = -\frac{n}{2}\log(\sigma^2+\gamma^2) - \frac{\Sigma(t_i^2 - 2\gamma t_i x_i + \gamma^2 x_i^2)}{2(\sigma^2+\gamma^2)}$$

$$= -\frac{n}{2}\log(\sigma^2+\gamma^2) - \frac{\Sigma t_i^2 - 2\gamma \Sigma t_i x_i + \gamma^2 \Sigma x_i^2}{2(\sigma^2+\gamma^2)}$$

$$= -\frac{n}{2}\log(\sigma^2+\gamma^2) - \frac{\Sigma(t_i^2)}{2(\sigma^2+\gamma^2)} + \frac{\gamma\Sigma t_i x_i}{\sigma^2+\gamma^2} - \frac{\gamma^2\Sigma x_i^2}{2(\sigma^2+\gamma^2)}$$

$$\frac{\partial l(\gamma,\sigma^2)}{\partial\gamma} = -\frac{n}{2}\frac{2\gamma}{\sigma^2+\gamma^2} + \frac{\Sigma(t_i^2)\gamma}{(\sigma^2+\gamma^2)^2} + \frac{\overset{(\sigma^2-\gamma^2)\Sigma t_i x_i}{(\sigma^2+\gamma^2)\Sigma t_i x_i - \gamma\Sigma t_i x_i(2\gamma)}}{(\sigma^2+\gamma^2)^2} -$$

1. Now suppose we know $Y|X$ is from a log-normal family [2019 Theory 1]

s.t. $\log Y = \gamma X + N(0,\sigma^2)$

1c) obtain the MLE estimators for α^* & β^* given in (1) & derive their asymptotic distribution

we know $X \sim N(0,1)$ $\therefore$ $\gamma X \sim N(0,\gamma^2)$

$$\gamma X + N(0,\sigma^2) = N(0,\gamma^2) + N(0,\sigma^2) \quad \text{assuming independence}$$
$$= N(0, \gamma^2 + \sigma^2)$$

$\therefore$ $\log Y \sim N(0, \gamma^2+\sigma^2) \Rightarrow Y \sim \exp(N(0, \gamma^2+\sigma^2))$

$$E[\log Y] = 0 \Rightarrow E[\exp(\log Y)] = \exp(0)$$
$$E[Y] = 1$$

let $Z \sim N(0, \sigma^2)$

$\therefore$ $A = \gamma X + Z$ $\quad E[AX] = E[(\gamma X + Z)X] = E[\gamma X^2 + ZX] = \gamma E[X^2] + ZE[X] = \gamma$

but in this case we have $\log Y = \gamma X + Z \rightarrow Y = \exp\{\gamma X + Z\}$

$$E[YX] = E[\exp(\gamma X + Z)X] = E[X \exp(\gamma X)\exp(Z)] = E[X\exp(\gamma X)]E[\exp(Z)]$$

$X \sim N(0,1)$ $\therefore$

$$E[X\exp(\gamma X)] = \int_{-\infty}^{\infty} X \exp(\gamma X) \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{X^2}{2}\right\} dx$$

$$= \int_{-\infty}^{\infty} \frac{X}{\sqrt{2\pi}} \exp\left\{-\frac{X^2 - 2\gamma X}{2}\right\} dx \qquad X^2 - 2\gamma X + \gamma^2 = (X-\gamma)^2$$

$$= \int_{-\infty}^{\infty} \frac{X}{\sqrt{2\pi}} \exp\left\{-\frac{(X^2 - 2\gamma X + \gamma^2 - \gamma^2)}{2}\right\} dx$$

$$= \int_{-\infty}^{\infty} \frac{X}{\sqrt{2\pi}} \exp\left\{-\frac{(X-\gamma)^2}{2} + \frac{\gamma^2}{2}\right\} dx$$

$$= \int_{-\infty}^{\infty} \frac{X}{\sqrt{2\pi}} \exp\left\{-\frac{(X-\gamma)^2}{2}\right\} \exp\left(\frac{\gamma^2}{2}\right) dx$$

$$= \exp\left(\frac{\gamma^2}{2}\right) \underbrace{\int_{-\infty}^{\infty} \frac{X}{\sqrt{2\pi}} \exp\left\{\frac{(X-\gamma)^2}{2}\right\} dx}_{E[N(\gamma,1)]}$$

$$= \exp\left(\frac{\gamma^2}{2}\right) \gamma$$

2019 Theory 2

2. (25 points) Suppose that $Y_1, \cdots, Y_n$ are independent random variables and each Y_i is distributed as exponential with mean $\mu_i = \beta x_i$, where $x_1, \cdots, x_n$ are known positive constants not all equal and $\beta > 0$ is an unknown parameter.

   √(a) Find an explicit expression for the maximum-likelihood estimators, $\hat{\beta}$, of β. Also, find the large-sample ($n \to \infty$) distribution of $\sqrt{n}(\hat{\beta} - \beta)$.

   √(b) Find a pivotal quantity and use it to construct an exact 95% confidence interval for β.

   √(c) Consider the following estimator of β: $\tilde{\beta} = (\sum_{i=1}^n Y_i)/\sum_{i=1}^n x_i$. Show that the finite-sample efficiency of $\tilde{\beta}$ relative to $\hat{\beta}$ is less than 1. ~ MSE

   √(d) Now consider a different model for the mean, specifically,

$$\frac{1}{\mu_i} = \alpha + \gamma x_i$$

   where α and γ are unknown parameters. Find a minimal sufficient statistic for (α, γ).

   (e) By appropriate conditioning, obtain the conditional score function for β (eliminating α). You don't need to simplify it in this part. ς γ?

Points: (a) 5; (b) 5; (c) 5; (d) 5; (e) 5.

2019 Theory 2

2. Suppose $Y_1, \dots, Y_n \overset{iid}{\sim} \exp(\mu_i)$ $\mu_i = \beta x_i$ where $x_1, \dots, x_n$ are known >0 $\neq 0$

$\beta > 0$ an unknown parameter

$$f(y_i) = \frac{1}{\mu_i} \exp\left\{-\frac{y_i}{\mu_i}\right\} = \frac{1}{\beta x_i} \exp\left\{-\frac{y_i}{\beta x_i}\right\}$$

(a) Find the explicit expression for the MLE, $\hat{\beta}$, of β. Also find the large sample ($n \to \infty$) distribution of $\sqrt{n}(\hat{\beta} - \beta)$.

$$L(\beta) = \prod_{i=1}^n \frac{1}{\beta x_i} \exp\left\{-\frac{y_i}{\beta x_i}\right\} = \frac{1}{\beta^n} \prod_{i=1}^n \left(\frac{1}{x_i}\right) \exp\left\{-\frac{1}{\beta} \sum_{i=1}^n \frac{y_i}{x_i}\right\}$$

$$\ell(\beta) = -n \log \beta - \sum_{i=1}^n \log x_i - \frac{1}{\beta} \sum_{i=1}^n \frac{y_i}{x_i}$$

$$\frac{\partial \ell(\beta)}{\partial \beta} = -\frac{n}{\beta} + \frac{1}{\beta^2} \sum_{i=1}^n \frac{y_i}{x_i} \overset{set}{=} 0$$

$$\frac{\partial^2 \ell(\beta)}{\partial \beta^2} = \frac{n}{\beta^2} - \frac{2}{\beta^3} \sum_{i=1}^n \frac{y_i}{x_i} = \frac{n\beta - 2\sum y_i / x_i}{\beta^3} \overset{?}{<} 0$$

$\therefore$ maximum

$$\frac{n}{\beta} = \frac{1}{\beta^2} \sum_{i=1}^n \frac{y_i}{x_i} \Rightarrow \boxed{\hat{\beta} = \frac{1}{n} \sum_{i=1}^n \frac{y_i}{x_i}}$$

$$\frac{\partial^2 \ell(\beta)}{\partial \beta^2} = \frac{n}{\beta^2} - \frac{2}{\beta^3} \sum_{i=1}^n \frac{y_i}{x_i}$$

$$E\left[-\frac{\partial^2 \ell(\beta)}{\partial \beta^2}\right] = \frac{2}{\beta^3} \sum_{i=1}^n \frac{E[y_i]}{x_i} - \frac{n}{\beta^2}$$

$$= \frac{2}{\beta^3} \sum_{i=1}^n \frac{\beta x_i}{x_i} - \frac{n}{\beta^2}$$

$$= \frac{2n\beta}{\beta^3} - \frac{n}{\beta^2}$$

$$= \frac{2n}{\beta^2} - \frac{n}{\beta^2} = \frac{n}{\beta^2}$$

$$I_n(\beta) = \frac{n}{\beta^2} \Rightarrow I_1(\beta) = \frac{1}{n} I_n(\beta) = \frac{1}{\beta^2}$$

$\therefore$ by properties of MLE, $\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{d} N(0, \beta^2)$

2 (b) Find a pivotal quantity & use it to construct an exact 95% CI for β. | 2019 Theory 2

$Y_i \sim exp(\beta x_i)$ ∴ as a member of a location-scale family, the following is true

$$\frac{y_i}{x_i} \sim exp(\beta) \Rightarrow \frac{y_i}{\beta x_i} \sim exp(1)$$

let $z_i = \frac{y_i}{\beta x_i} \sim exp(1) \Rightarrow \sum_{i=1}^{n} z_i \sim Gamma(n, 1)$ by distributional relationships.

$$\sum_{i=1}^{n} z_i = \sum_{i=1}^{n} \frac{y_i}{\beta x_i} = \frac{1}{\beta} \sum_{i=1}^{n} \frac{y_i}{x_i} = \frac{n\hat{\beta}}{\beta} \sim Gamma(n, 1)$$

∴ $\frac{n\hat{\beta}}{\beta}$ is a pivotal quantity we can base the CI on.

a 95% CI for β is:

$$.95 = P\left(\Gamma(.025, n, 1) < \frac{n\hat{\beta}}{\beta} < \Gamma(.975, n, 1)\right)$$

$$= P\left(\frac{n\hat{\beta}}{\Gamma(.975, n, 1)} < \beta < \frac{n\hat{\beta}}{\Gamma(.025, n, 1)}\right)$$

where $\Gamma(\alpha, n, 1)$ is the α^{th} quantile of the $\Gamma(n, 1)$ distribution.

2(c) consider the following estimator of β:

$\tilde{\beta} = \frac{\sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} x_i}$ Show that the finite sample efficiency of $\tilde{\beta}$ relative to $\hat{\beta}$ is less than 1.

we need to first find $var(\hat{\beta})$ & $var(\tilde{\beta})$ in finite samples.

$$var(\hat{\beta}) = var\left(\frac{1}{n}\sum_{i=1}^{n}\frac{y_i}{x_i}\right) = \frac{1}{n^2}\sum_{i=1}^{n}\frac{var(y_i)}{x_i^2} = \frac{1}{n^2}\sum_{i=1}^{n}\frac{x_i^2\beta^2}{x_i^2} = \frac{n\beta^2}{n^2} = \frac{\beta^2}{n}$$

$$var(\tilde{\beta}) = var\left(\frac{\sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} x_i}\right) = \frac{1}{(\sum_{i=1}^{n} x_i)^2}\sum_{i=1}^{n} var(y_i) = \frac{1}{(\sum_{i=1}^{n} x_i)^2}\sum_{i=1}^{n}\beta^2 x_i^2 = \beta^2\frac{\sum_{i=1}^{n}(x_i^2)}{(\sum_{i=1}^{n} x_i)^2}$$

relative efficiency: $\frac{var(\tilde{\beta})}{var(\hat{\beta})} = \frac{\beta^2 \frac{\sum_i x_i^2}{(\sum_i x_i)^2}}{\beta^2/n}$

$$= \frac{\sum_i x_i^2}{(\sum_i x_i)^2} \Big/ \frac{1}{n} = \frac{n\sum_{i=1}^{n} x_i^2}{(\sum_{i=1}^{n} x_i)^2} < 1$$

as $x_i \geq 0$ where strict inequality holds for some x_i.

2(d) Now consider a different model for the mean:

$$\frac{1}{\mu_i} = \alpha + \gamma x_i \quad \text{where } \alpha, \gamma \text{ are unknown parameters}$$

Find a minimal sufficient statistic for (α, γ)

$$f(y_i) = (\alpha + \gamma x_i) \exp\{-y_i(\alpha + \gamma x_i)\}$$

$$l(\alpha, \gamma) = \sum_{i=1}^{n} \log(\alpha + \gamma x_i) - \alpha \sum_{i=1}^{n} y_i - \gamma \sum_{i=1}^{n} y_i x_i$$

$\therefore$ by properties of the exponential family

$\left(\sum_{i=1}^{n} y_i, \sum_{i=1}^{n} x_i y_i\right)$ is a CSS of (α, γ).

Is $\left(\sum_{i=1}^{n} y_i, \sum_{i=1}^{n} x_i y_i\right)$ minimal? <u>YES</u>

both are necessary & not redundant as if $\sum_{i=1}^{n} y_i$ is known, $\sum_{i=1}^{n} x_i y_i$ is not directly known & vice versa (even though x_is are known, if the summation is known the individual values you need to multiply $x_i y_i$ in the summation are not).

Thus $\left(\sum_{i=1}^{n} y_i, \sum_{i=1}^{n} x_i y_i\right)$ is the minimal sufficient statistic for (α, γ).

2(e) By appropriate conditioning, obtain the score function for β (eliminating α)

From (d) we know Σy_i is sufficient for α & $\Sigma x_i y_i$ is sufficient for γ.

$$p(y_i|\xi) = (\alpha+\gamma x_i)\exp\{-y_i(\alpha+\gamma x_i)\} = \exp\{\log(\alpha+\gamma x_i) - y_i(\alpha+\gamma x_i)\}$$

$$\ell n(\xi) = \sum_{i=1}^{n} \log(\alpha+\gamma x_i) - y_i(\alpha+\gamma x_i)$$
$$= \sum_{i=1}^{n} \log(\alpha+\gamma x_i) - \alpha\sum_{i=1}^{n} y_i + \gamma\sum_{i=1}^{n} x_i y_i$$

$$L(\xi|Y) = \prod_{i=1}^{n} \exp\{\log(\alpha+\gamma x_i) - y_i(\alpha+\gamma x_i)\}$$
$$= \exp\{\Sigma\log(\alpha+\gamma x_i) - \alpha\Sigma y_i - \gamma\Sigma x_i y_i\}$$

since Σy_i is sufficient for α, the nuisance parameter, we condition using $S = \{Y' : \Sigma y_i' = \Sigma y_i\}$

$$\partial_\gamma \ell n(\xi) = \sum_{i=1}^{n} \frac{x_i}{\alpha+\gamma x_i} - y_i x_i$$

$$P_c(\xi|y_1,\dots,y_n) = \frac{\exp\{\Sigma\log(\alpha+\gamma x_i) - \alpha\Sigma y_i - \gamma\Sigma x_i y_i\}}{\sum_{y'\in S}\exp\{\Sigma\log(\alpha+\gamma x_i) - \alpha\Sigma y_i' - \gamma\Sigma x_i y_i'\}}$$

$$= \frac{\exp\{-\gamma\Sigma x_i y_i\}}{\sum_{y'\in S}\exp\{-\gamma\Sigma x_i y_i'\}}$$

$$\therefore \ell_c(\xi|y_1,\dots,y_n) = -\gamma\Sigma x_i y_i - \log\left(\sum_{y'\in S}\exp\{-\gamma\Sigma x_i y_i'\}\right)$$

$$\frac{\partial \ell_c(\xi)}{\partial\gamma} = -\Sigma x_i y_i + \frac{\sum_{y'\in S}(\Sigma x_i y_i')\exp\{-\gamma\Sigma x_i y_i'\}}{\sum_{y'\in S}\exp\{-\gamma\Sigma x_i y_i'\}} = \partial_\gamma \ell_c(\xi)$$

$$U_\gamma(\xi) = \partial_\gamma \ell n(\xi) - E[\partial_\gamma \ell_c(\xi)]$$

where $\partial_\gamma \ell n(\xi)$ and $\partial_\gamma \ell_c(\xi)$ are defined as above.

2(e) By appropriate conditioning, obtain the score function for β (eliminating α). $\boxed{\text{2019 Theory 2}}$

From (d) we know by properties of exponential family that $\Sigma x_i y_i$ is sufficient for γ and $\sum_{i=1}^n y_i$ is sufficient for α. Thus we need to derive the distribution of $\underbrace{\sum_{i=1}^n x_i y_i \mid \sum_{i=1}^n y_i}$

$$y_i \sim \exp\{\mu_i\} \Rightarrow \cancel{\sum_{i=1}^n y_i \sim Gamma(n, \mu_i) = Gamma\left(n, \tfrac{1}{\alpha+\gamma x_i}\right)} \rightarrow \frac{(\alpha+\gamma x_i)^n}{\Gamma(n)}$$

we need the joint distribution of $(\Sigma y_i, \Sigma x_i y_i)$

$$l = \log(p(Y|\xi)) - \log(p(s|\xi))$$

$$l = \left(\sum_{i=1}^n \log(\alpha+\gamma x_i) - \alpha \Sigma y_i - \gamma \Sigma y_i x_i\right) -$$

$$f(y_i) = (\alpha+\gamma x_i)\exp\{-y_i(\alpha+\gamma x_i)\} \Rightarrow f_Y(y_i) = \prod_{i=1}^n (\alpha+\gamma x_i)\exp\{-y_i(\alpha+\gamma x_i)\}$$

$$f_Y(y) = \left(\prod_{i=1}^n \alpha+\gamma x_i\right)\exp\left\{-\alpha\sum_{i=1}^n y_i - \gamma\sum_{i=1}^n x_i y_i\right\}$$

want to find the distribution of Σy_i.

let $Z_1 = y_1$, $Z_2 = y_1 + y_2$, $\ldots$, $Z_n = \sum_{i=1}^n y_n$

$$\frac{dy'}{dz} = \left(\frac{dz'}{dy}\right)^{-1} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ & & & \ddots & \vdots \\ 0 & & & & 1 \end{pmatrix}^{-1} \Rightarrow \left|\frac{dy}{dz}\right|' = \left|\frac{dz'}{dy}\right|^{-1} = 1$$

$$f_Z(z) = f_Y(y)\left|\frac{dy'}{dz}\right|$$

$$= \left(\prod_{i=1}^n \alpha+\gamma x_i\right)\exp\left\{-\alpha\sum_{i=1}^n y_i - \gamma\sum_{i=1}^n x_i y_i\right\}$$

$$y_i \sim \exp\left(\frac{1}{\alpha+\gamma x_i}\right) \Rightarrow y_i(\alpha+\gamma x_i) \sim \exp(1)$$

$$\Rightarrow \sum_{i=1}^n y_i(\alpha+\gamma x_i) \sim Gamma(n,1)$$

$$\alpha\Sigma y_i + \gamma\Sigma x_i y_i \sim Gamma(n,1)$$

$$y_i \sim \exp\left(\frac{1}{\alpha+\gamma x_i}\right) \quad x_i y_i \sim \exp\left(\frac{x_i}{\alpha+\gamma x_i}\right)$$

$$\phi_{y_i}(t) = \cancel{\frac{1/\alpha+\gamma x_i}{\frac{1}{\alpha+\gamma x_i}+it}} = \cancel{\frac{1/\alpha+\gamma x_i}{\frac{1+it(\alpha+\gamma x_i)}{\alpha+\gamma x_i}}} = \cancel{\frac{1}{1+it(\alpha+\gamma x_i)}} = \frac{\alpha+\gamma x_i}{\alpha+\gamma x_i+it}$$

$$\phi_{\Sigma y_i}(t) = \cancel{\prod_{j=1}^n \frac{1}{1+it(\alpha+\gamma x_j)}} = \cancel{\frac{1}{\prod_{j=1}^n(1+it(\alpha+\gamma x_j))}} \qquad \phi_{\Sigma y_i}(t) = \prod_{j=1}^n \frac{\alpha+\gamma x_j}{\alpha+\gamma x_j - it}$$