BASIC PHD WRITTEN EXAMINATION IN BIOSTATISTICS

THEORY, SECTION 1

(9:00 AM- 1:00 PM
Monday, July 28, 2014)

INSTRUCTIONS:

(a) This is a **CLOSED-BOOK** examination.

(b) The time limit for this Examination is four hours.

(c) Answer any TWO (2) (BUT ONLY TWO) of the THREE (3) questions that follow.

(d) Put the answers to different questions on separate sets of paper.

(e) Put your exam code, **NOT YOUR NAME**, on each page. The same code will be used for Section 1 and Section 2 of the PhD Theory Exam. Please keep the code confidential and do not share this information with any students or faculty. Sharing your code with either students or faculty is viewed as a violation of the UNC honor code.

(f) Return the examination with a signed statement of the UNC honor pledge, separately from your answers. The pledge statement is given on the last page of the exam handout.

(g) In the questions to follow, you are required to answer only what is asked, and not to tell all you know about the topics involved.

$\frac{1}{\alpha}$

1. We consider two groups of independent observations: $X_1, ..., X_n$ are i.i.d from $Unif(0, \alpha)$ and $Y_1, ..., Y_n$ are i.i.d from $Unif(0, \beta)$, where both α and β are unknown parameter assumed to be positive. For comparison, we are interested in inference on $\theta = \beta/\alpha$.

$\frac{1}{\beta}$

① 找到 CSS $T(x)$

② $E[g(T(x))] = \theta$

   (a) Derive the UMVUEs for α and β and calculate their respective variances.

   (b) Calculate the MLEs for α and β, denoted as $\hat{\alpha}$ and $\hat{\beta}$ respectively. Derive the asymptotic distributions for $\hat{\alpha}$ and $\hat{\beta}$ are some normalization.

   (c) The MLE for θ is then $\hat{\theta} = \hat{\beta}/\hat{\alpha}$. Derive the asymptotic distribution of $\hat{\theta}$ after normalization. Construct an asymptotic 95% confidence interval for θ based on the observations.

   (d) We wish to test the hypothesis $H_0 : \alpha = \beta$ versus $H_a : \alpha \neq \beta$. What is the likelihood ratio test static. Derive the exact distribution of this test statistic.

   (e) Note $E[X_k] = \alpha/2$ and $E[Y_k] = \beta/2$. Thus, a simple estimator for θ is $\bar{Y}_n/\bar{X}_n$. Derive the asymptotic distribution of this estimator after normalization. What is the asymptotic relative efficiency of this estimator with respect to $\hat{\theta}$, $2\bar{Y}_n/\hat{\alpha}$ and $\hat{\beta}/(2\bar{X}_n)$?

(a) $f(\alpha) = \left(\frac{1}{\alpha}\right)^n I_{(X_{(1)} > 0)} I_{(X_{(n)} < \alpha)}$ According to Factor Zation theorem

$T(x) = X_{(n)}$ $T(Y) = Y_{(n)}$ $\because f(X_{(n)}) = n \frac{1}{\alpha} I_{(0<X_{(n)}<\alpha)} \left(\frac{X}{\alpha}\right)^{n-1} = \frac{n X^{n-1}}{\alpha^n} I_{(0<X_{(n)}<\alpha)}$

$\because F(x) = \frac{X}{\alpha}$ $\therefore E(X_{(n)}) = \frac{n}{\alpha^n} \int_0^\alpha X^n \, dX = \frac{n}{\alpha^n} \frac{\alpha^{n+1}}{n+1} = \frac{n\alpha}{n+1}$ $\therefore g(T(x)) = \frac{n+1}{n} X_{(n)}$

$\therefore$ UMVUE for α is $\frac{n+1}{n} X_{(n)}$ and β is $\frac{n+1}{n} Y_{(n)}$

$Var\left(\frac{n+1}{n} X_{(n)}\right) = \left(\frac{n+1}{n}\right)^2 E(X_{(n)}^2) - E^2(X_{(n)})$

(b) $l(\alpha) = -n\log\alpha$ as $\alpha \uparrow$ $l(\alpha) \downarrow$ since $\alpha > X_{(n)}$ $\therefore \hat{\alpha} = X_{(n)}$

$\hat{\beta} = Y_{(n)}$

2. Consider a decision problem with a parameter space Θ having a finite number of values, $\theta_1, \ldots, \theta_l, l < \infty$.

   (a) Show that a Bayes rule d_B with respect to a prior distribution Λ on Θ having positive probabilities $\lambda_1, \ldots, \lambda_l > 0$ is admissible.

   (b) The result in part (a) conflicts with other results for continuous parameter spaces where Bayes rules may not be admissible, eg, James-Stein estimation. In the discrete case described above, show that if $\lambda_i = 0$, some $i = 1, \ldots, l$, then the resulting Bayes rule d_B may not be admissible.

   (c) Suppose that the frequentist risk of d_B in part (b) is finite and constant on those θ_i's having $\lambda_i > 0$. Show that this decision rule is minimax, that is, minimizes the maximum risk, on those θ_i's with $\lambda_i > 0$.

   (d) Can anything be said about whether or not d_B in part (b) is minimax on $\theta_i, i = 1, \ldots, l$? Discuss.

   In (e), (f), and (g), consider the following classification problem. Suppose that X is an observation from the density

$$p(x|\theta) = \theta^{-1} I(0 < x < \theta),$$

   where $I(.)$ denotes the indicator function and the parameter space is $\Theta = \{1, 2, 3\}$. It is desired to classify X as arising from $p(x|1)$, $p(x|2)$, or $p(x|3)$, under a 0-1 loss function (zero loss for a correct decision, a loss of one for an incorrect decision).

   (e) Find the form of the Bayes rule for this problem.

   (f) Find the decision rule which minimizes the maximum risk over Θ and the corresponding least favorable prior distribution.

   (g) Find the decision rule which minimizes the maximum risk over $\theta = 1$ and $\theta = 2$ and the corresponding least favorable prior distribution. Is this minimax rule the same as that in (f)? Explain.

3. Suppose that (X, Y) are two random variables with joint distribution

$$f(x, y|\alpha, \beta) = c(\alpha, \beta) \exp(-\alpha x - \beta y) \sum_{j=0}^{\infty} \frac{x^j y^j}{(j!)^2} \tag{1}$$

for $x > 0, y > 0$. Also, let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be a random sample from (X, Y), and let $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$ and $\bar{Y} = \frac{1}{n} \sum_{i=1}^{n} Y_i$.

(a) Show that the joint distribution of (X, Y) in (1) is in the multiparameter exponential family and identify the rank, show that $c(\alpha, \beta) = \alpha\beta - 1$, and find the parameter space of (α, β).

(b) Derive the marginal distribution of X from (1) and show that $E(X) = \frac{\beta}{\alpha\beta - 1}$.

(c) From (1), show that $E(X^j Y^k) = (-1)^{j+k} S^{-1} \frac{\partial^{j+k} S}{\partial \alpha^j \beta^k}$, where $S \equiv S(\alpha, \beta) = 1/c(\alpha, \beta)$.

(d) Show that the conditional distribution of $Y|X = x$ depends on β but is free of α, and derive the asymptotic distribution of $\bar{Y}|\bar{X} = \bar{x}$, properly normalized.

(e) Based on a sample of size n, derive a UMPU size α^* test for $H_0 : \beta = 2$ against $H_1 : \beta > 2$ and obtain an explicit expression for the critical value of the test.

(f) Based on a sample of size n, derive an exact 95% confidence interval for β.

(g) Derive the score test for testing $H_0 : \beta = 2$ and obtain its asymptotic distribution.

(h) Based on a sample of size n, under squared error loss, derive the generalized Bayes estimator of $\theta = \frac{\alpha}{\beta}$ assuming the joint prior $\pi(\alpha, \beta) \propto \alpha^{-1}\beta^{-1}$ and determine whether the generalized Bayes estimator is admissible.

Statement of the UNC honor pledge:

"In recognition of and in the spirit of the honor code, I certify that I have neither given nor received aid on this examination and that I will report all Honor Code violations observed by me."

(Signed) ____

NAME

(Printed) ____

NAME