2014 - Part I

2. Consider a decision problem with a parameter space Θ having a finite number of values, $\theta_1, \ldots, \theta_l, l < \infty$.

   √(a) Show that a Bayes rule d_B with respect to a prior distribution Λ on Θ having positive probabilities $\lambda_1, \ldots, \lambda_l > 0$ is admissible.

   * (b) The result in part (a) conflicts with other results for continuous parameter spaces where Bayes rules may not be admissible, eg, James-Stein estimation. In the discrete case described above, show that if $\lambda_i = 0$, some $i = 1, \ldots, l$, then the resulting Bayes rule d_B may not be admissible.

   √(c) Suppose that the frequentist risk of d_B in part (b) is finite and constant on those θ_i's having $\lambda_i > 0$. Show that this decision rule is minimax, that is, minimizes the maximum risk, on those θ_i's with $\lambda_i > 0$.

   √(d) Can anything be said about whether or not d_B in part (b) is minimax on θ_i, $i = 1, \ldots, l$? Discuss.

In (e), (f), and (g), consider the following classification problem. Suppose that X is an observation from the density

$$p(x|\theta) = \theta^{-1} I(0 < x < \theta),$$

where $I(.)$ denotes the indicator function and the parameter space is $\Theta = \{1, 2, 3\}$. It is desired to classify X as arising from $p(x|1)$, $p(x|2)$, or $p(x|3)$, under a 0-1 loss function (zero loss for a correct decision, a loss of one for an incorrect decision).

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HW2
#5

   √(e) Find the form of the Bayes rule for this problem.

   √(f) Find the decision rule which minimizes the maximum risk over Θ and the corresponding least favorable prior distribution.

   (g) Find the decision rule which minimizes the maximum risk over $\theta = 1$ and $\theta = 2$ and the corresponding least favorable prior distribution. Is this minimax rule the same as that in (f)? Explain.

admissibility
Bayes risk
Bayes rule
minimax
Form of Bayes rule - classification
Find minimax & least favorable prior

a) $\boxed{\text{Proof}}$ Assume d_B inadmissible.

$\Rightarrow \exists\ d$ s.t. $\forall\ \theta_i \in \Theta$

$$R(\theta_i, d) \le R(\theta_i, d_B)$$

and $\exists\ \theta_j \in \Theta$ s.t.

$$R(\theta_j, d) < R(\theta_j, d_B).$$

$$\Rightarrow \sum_{i=1}^{\ell} \lambda_i R(\theta_i, d) < \sum_{i=1}^{\ell} \lambda_i R(\theta_i, d_B)$$

$$\Rightarrow R(\Lambda, d) < R(\Lambda, d_B)$$

contradicts d_B as Bayes rule

$\Rightarrow d_B$ admissible.

b) $\lambda_i = 0$ for some $i = 1, \dots, \ell \Rightarrow$ Bayes rule may not be admissible

WTS a Bayes rule can be inadmissible

$\boxed{\text{Proof}}$ ~~Let d_B be Bayes rule and a some other rule~~

$$\Rightarrow R(\Lambda, d_B) < R(\Lambda, d)$$

$$\Rightarrow \sum_{i=1}^{\ell} \lambda_i R(\theta_i, d_B) < \sum_{i=1}^{\ell} \lambda_i R(\theta_i, d)$$

Let d_B be such that

$$R(\theta_i, d_B) > R(\theta_i, d) \quad \text{for some } i$$

and

$$R(\theta_j, d_B) \le R(\theta_j, d)$$

suppose d_B inadmissible.

$\Rightarrow \exists\ d$ s.t. $\forall\ \theta_i \in \Theta$

$$R(\theta_i, d) \le R(\theta_i, d_B)$$

and $\exists\ \theta_j \in \Theta$ s.t.

$$R(\theta_j, d) < R(\theta_j, d_B).$$

b) cont'd

Suppose $R(\theta_j, d) < R(\theta_j, d_B)$ for some $j = 1, \dots, \ell$

and $R(\theta_i, d) = R(\theta_i, d_B)$ for all other i

Let $\lambda_j = 0$

$$\Rightarrow \sum_{i=1}^{\ell} \lambda_i R(\theta_i, d) = \sum_{i=1}^{\ell} \lambda_i R(\theta_i, d_B)$$

$$\Rightarrow R(\Lambda, d) = R(\Lambda, d_B)$$

$\Rightarrow$ does not contradict d_B Bayes rule, just shows not unique.

c) Recall minimax

$$\sup_\theta R(\theta_i, d_B) < \sup_\theta R(\theta_i, d)$$

Proof

We are given that frequentist risk of d_B is finite & constant on those θ_i's having $\lambda_i > 0$. Let r be this constant risk

$$\Rightarrow R(\Lambda, d_B) = \sum_{i=1}^{\ell} \lambda_i R(\theta_i, d_B) = \sum_{i=1}^{\ell} \lambda_i r = r \sum_{i=1}^{\ell} \lambda_i = r$$

Assume d_B not minimax on those θ_i's with $\lambda_i > 0$.

That is, $\exists\, d^*$ s.t.

$$\sup_\theta R(\theta_i, d^*) < \sup_\theta R(\theta_i, d_B)$$

$$\Rightarrow \sup_\theta R(\theta_i, d^*) < r$$

$$\Rightarrow R(\Lambda, d_*) = \sum_{i=1}^{\ell} \lambda_i R(\theta_i, d^*) < \sum_{i=1}^{\ell} \lambda_i r = r$$

$\Rightarrow$ contradicts d_B as Bayes rule

$\Rightarrow$ d_B is minimax on these θ_i's.

d) For i where $\lambda_i = 0$, it is possible that

$$R(\theta_i, d_B) > R(\theta_i, d) \quad \text{for some } d \in \mathcal{D}$$

and

$$R(\theta_j, d_B) = R(\theta_j, d) \quad \text{for all other } j = 1, \ldots, \ell$$

We are not given that the frequentist risk is finite & constant at θ_i.

$\Rightarrow$ It is possible that

$$\sup_\theta R(\theta_i, d_B) > \sup_\theta R(\theta_i, d)$$

$$\boxed{\text{when } d_B \text{ not admissible} \quad \Rightarrow d_B \text{ not minimax.}}$$

e) $p(x \mid \theta) = \theta^{-1} I(0 < x < \theta)$

0-1 loss $\qquad p(\theta \mid x) \propto p(x \mid \theta)\, \lambda(\theta)$

$\Theta = \{1, 2, 3\}$

Bayes rule minimizes posterior expected loss.

$p(x \mid 1) = I(0 < x < 1)$

$p(x \mid 2) = \frac{1}{2} I(0 < x < 2)$

$p(x \mid 3) = \frac{1}{3} I(0 < x < 3)$

Let λ_i be prior probability for θ_i, $\lambda_1 + \lambda_2 + \lambda_3 = 1$

If $x \geq 2 \Rightarrow d_\wedge(x) = 3$ $\qquad$ Let $\phi_i(x)$ be probability of action i given observation x.

$\phi_3(x) = 1$, $\phi_1(x) = \phi_2(x) = 0$

If $x \geq 1 \Rightarrow$ we need some randomization between 2 & 3.

$\phi_1(x) = 0$

$\lambda_2\, p(x \mid 2) > \lambda_3\, p(x \mid 3)$ $\qquad$ $\phi_2(x) = I(\frac{3}{2}\lambda_2 > \lambda_3) + \sigma_1 I(\frac{3}{2}\lambda_2 = \lambda_3)$

$\lambda_2 \frac{1}{2} > \lambda_3 \frac{1}{3}$

$\frac{3}{2}\lambda_2 > \lambda_3$ $\qquad$ $\phi_3(x) = I(\frac{3}{2}\lambda_2 < \lambda_3) + (1-\sigma_1) I(\frac{3}{2}\lambda_2 = \lambda_3)$

$\boxed{\text{cases}}$ $\quad \frac{3}{2}\lambda_2 > \lambda_3 \Rightarrow \phi_2(x) = 1,\ \phi_3(x) = 0$

$\frac{3}{2}\lambda_2 < \lambda_3 \Rightarrow \phi_2(x) = 0,\ \phi_3(x) = 1$ $\qquad \Rightarrow$ $\phi_2(x) = \sigma_1$, $\phi_3(x) = 1 - \sigma_1$

$\frac{3}{2}\lambda_2 = \lambda_3 \Rightarrow$ randomization needed

e) more details

Bayes rule minimizes posterior expected loss

$\cancel{E_{\theta|x}[L(\theta, a_1}$

$$E_{\theta|x}[L(\theta,1)] = \frac{\lambda_2}{2} + \frac{\lambda_3}{3}$$

$$E_{\theta|x}[L(\theta,2)] = \lambda_1 + \frac{\lambda_3}{3}$$

$$E_{\theta|x}[L(\theta,3)] = \lambda_1 + \frac{\lambda_2}{2}$$

consider

$0 < x < 1$

If $\frac{\lambda_2}{2} + \frac{\lambda_3}{3} < \lambda_1 + \frac{\lambda_3}{3}$

$\Rightarrow \phi_1(x) = 1$

&

$\frac{\lambda_2}{2} + \frac{\lambda_3}{3} < \lambda_1 + \frac{\lambda_2}{2}$

$$\Rightarrow \left.\begin{array}{l} \lambda_1 > \frac{\lambda_2}{2} \\ \& \\ \lambda_1 > \frac{\lambda_3}{3} \end{array}\right\} \text{equivalent to previous page.}$$

e) cont'd

$\boxed{\text{If } \quad 0 < x < 1}$

$$\lambda_1 \, p(x \mid 1) > \lambda_2 \, p(x \mid 2)$$

$$\lambda_1 > \tfrac{1}{2} \lambda_2$$

$\boxed{\text{cases}}$

$\lambda_1 > \frac{1}{2}\lambda_2$ & $\lambda_1 > \frac{1}{3}\lambda_3$ $\Rightarrow$ $\phi_1(x) = 1$
$\phi_2(x) = \phi_3(x) = 0$

$\frac{1}{2}\lambda_2 > \lambda_2$ & $\frac{1}{2}\lambda_2 > \frac{1}{3}\lambda_3$ $\Rightarrow$ $\phi_2(x) = 1$
$\phi_1(x) = \phi_3(x) = 0$

$\frac{1}{3}\lambda_3 > \lambda_1$ & $\frac{1}{3}\lambda_3 > \frac{1}{2}\lambda_2$ $\Rightarrow$ $\phi_3(x) = 1$
$\phi_1(x) = \phi_2(x) = 0$

$\lambda_1 = \frac{1}{2}\lambda_2$ & $\lambda_1 > \frac{1}{3}\lambda_3$ $\Rightarrow$ $\phi_3(x) = 0$
$\phi_1(x) = \gamma_2$
$\phi_2(x) = 1 - \gamma_2$

$\lambda_1 = \frac{1}{3}\lambda_3$ & $\lambda_1 > \frac{1}{2}\lambda_2$ $\Rightarrow$ $\phi_2(x) = 0$
$\phi_1(x) = \gamma_3$
$\phi_3(x) = 1 - \gamma_3$

$\frac{1}{2}\lambda_2 = \frac{1}{3}\lambda_3$ & $\frac{1}{2}\lambda_2 > \lambda_1$ $\Rightarrow$ $\phi_1(x) = 0$
$\phi_2(x) = \gamma_4$
$\phi_3(x) = 1 - \gamma_4$

$\lambda_1 = \frac{1}{2}\lambda_2 = \frac{1}{3}\lambda_3$ $\Rightarrow$ $\phi_1(x) = \gamma_5$
$\phi_2(x) = \gamma_6$
$\phi_3(x) = 1 - \gamma_5 - \gamma_6$

$$\phi_1(x) = I(\lambda_1 > \tfrac{1}{2}\lambda_2)\, I(\lambda_1 > \tfrac{1}{3}\lambda_3) + \gamma_2\, I(\lambda_1 = \tfrac{1}{2}\lambda_2)\, I(\lambda_1 > \tfrac{1}{3}\lambda_3) + \gamma_3\, I(\tfrac{1}{3}\lambda_3 = \lambda_1)\, I(\lambda_1 > \tfrac{1}{2}\lambda_2) + \gamma_5\, I(\lambda_1 = \tfrac{1}{2}\lambda_2)\, I(\lambda_1 = \tfrac{1}{3}\lambda_3)$$

$$\phi_2(x) = I(\tfrac{1}{2}\lambda_2 > \lambda_1)\, I(\tfrac{1}{2}\lambda_2 > \tfrac{1}{3}\lambda_3) + (1-\gamma_2)\, I(\lambda_1 = \tfrac{1}{2}\lambda_2)\, I(\lambda_1 > \tfrac{1}{3}\lambda_3) + \gamma_4\, I(\tfrac{1}{2}\lambda_2 = \tfrac{1}{3}\lambda_3)\, I(\tfrac{1}{2}\lambda_2 > \lambda_1) + \gamma_6\, I(\lambda_1 = \tfrac{1}{2}\lambda_2)\, I(\lambda_1 = \tfrac{1}{3}\lambda_3)$$

$$\phi_3(x) = I(\tfrac{1}{3}\lambda_3 > \lambda_1)\, I(\tfrac{1}{3}\lambda_3 > \tfrac{1}{2}\lambda_2) + (1-\gamma_3)\, I(\lambda_1 = \tfrac{1}{3}\lambda_3)\, I(\lambda_1 > \tfrac{1}{2}\lambda_2) + (1-\gamma_4)\, I(\tfrac{1}{2}\lambda_2 = \tfrac{1}{3}\lambda_3)\, I(\tfrac{1}{2}\lambda_2 > \lambda_1) + (1-\gamma_5-\gamma_6)\, I(\lambda_1 = \tfrac{1}{2}\lambda_2)\, I(\lambda_1 = \tfrac{1}{3}\lambda_3)$$

f) $R(\theta_i, \phi] = 1 - E_{\theta_i}[\phi_i] = \text{constant}$ — want constant risk for Bayes rule to get minimax

Now

$$E_{\theta_3}[\phi_3(x)] \geq E_{\theta_3}[I(2 \leq x \leq 3)]$$
$$= P(2 \leq x \leq 3) \geq 0$$

$\Rightarrow$ $R(\theta_3, \phi)$ constant

For
$$E_{\theta_1}[\phi_1(x)] = E_{\theta_2}[\phi_2(x)] = E_{\theta_3}[\phi_3(x)]$$
it is best to look at a case that occurs in all 3 rules

$$\Rightarrow \lambda_1 = \frac{\lambda_2}{2} = \frac{\lambda_3}{3}$$

$$\Rightarrow \lambda_1 = \frac{\lambda_2}{2}$$

$$\lambda_1 = \frac{\lambda_3}{3}$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$

$$\Rightarrow \lambda_1 + 2\lambda_1 + 3\lambda_1 = 1$$

$$\Rightarrow 6\lambda_1 = 1 \Rightarrow \boxed{\lambda_1 = \frac{1}{6}}$$

$$\Rightarrow \boxed{\lambda_2 = \frac{1}{3}}$$

$$\boxed{\lambda_3 = \frac{1}{2}}$$

Least favorable prior

$$\Rightarrow \phi_1(x) = \gamma_5 I(0 < x < 1)$$

$$\phi_2(x) = \gamma_1 I(1 \leq x < 2) + \gamma_6 I(0 < x < 1)$$

$$\phi_3(x) = I(2 \leq x < 3) + (1-\gamma_1) I(1 \leq x < 2) + (1-\gamma_5-\gamma_6) I(0 < x < 1)$$

Minimax rule.

g)

want

$$E_{\theta_1}[\phi_1(x)] = E_{\theta_2}[\phi_2(x)]$$

look for cases that occur in ~~at~~ both

$\lambda_1 = \frac{\lambda_2}{2}$ let $\lambda_3 = 0$

$\Rightarrow \lambda_1 + \lambda_2 = 1$

$\Rightarrow \lambda_1 + 2\lambda_1 = 1$

$\Rightarrow 3\lambda_1 = 1 \Rightarrow \lambda_1 = \frac{1}{3}$

$\lambda_2 = 2/3$

$$\Rightarrow \boxed{\begin{aligned} \phi_1(x) &= [\gamma_2]\, I(0 < x < 1) \\ \phi_2(x) &= I(1 \le x < 2) + (1-\gamma_2)\, I(0 < x < 1) \\ \phi_3(x) &= I(2 \le x < 3) \end{aligned}}$$

$\boxed{\lambda_1 = \frac{1}{3},\ \lambda_2 = \frac{2}{3},\ \lambda_3 = 0}$ Least favorable prior

not the same as (f).