BASIC PHD WRITTEN EXAMINATION

THEORY, SECTION 2

(9:00 AM–1:00 PM, July 25, 2019)

INSTRUCTIONS:

(a) This is a **CLOSED-BOOK** examination.

(b) The time limit for this examination is four hours.

(c) Answer both questions that follow.

(d) Put the answers to different questions on separate sets of paper.

(e) Put your exam code, **NOT YOUR NAME**, on each page. The same code is used for Section 1 and Section 2 of the PhD Theory Exam. Please keep the code confidential and do not share this information with any students or faculty. Sharing your code with either students or faculty is viewed as a violation of the UNC honor code.

(f) Return the examination with a signed statement of the UNC honor pledge, separately from your answers. The pledge statement is given on the last page of the exam handout.

(g) In the questions to follow, you are required to answer only what is asked, and not to tell all you know about the topics involved.

1. (25 points) Consider the linear model

$$Y = X\beta + \epsilon, \tag{1}$$

where Y is $n \times 1$, X is an $n \times p$ matrix of fixed covariates with rank $r < p$, β is $p \times 1$, and $\epsilon \sim N_n(0, \Sigma)$, where Σ is a known positive definite matrix.

(a) **Derive** the distribution of

$$U = (Y - X\beta)'\Sigma^{-1}(Y - X\beta),$$

and **derive** the mean and variance of U.

Note: **You are not allowed to simply state the result of a theorem to give your answer. You must *derive* the results.**

(b) Formally derive the set of all possible least squares solutions of β.

Note: **You are not allowed to simply state a result or a formula for your answer. You must *derive* the result.**

(c) Show that $\lambda'\beta$ is estimable if and only if

$$\lambda'(X'\Sigma^{-1}X)^{-}(X'\Sigma^{-1}X) = \lambda',$$

where a "-" denotes generalized inverse.

(d) Assume X has rank p. Show that the BLUE of β is equal to $(X'X)^{-1}X'Y$ if and only if there exists a non-singular $p \times p$ matrix F such that $\Sigma X = XF$.

(e) Assume X has rank p. Let s^2 be defined as

$$s^2 = \frac{Y'(I - M)Y}{n - p}$$

where M denotes the orthogonal projection operator onto the column space of X. Show that

$$E(s^2) \leq \frac{1}{n-p}\sum_{i=1}^{n}\sigma_{ii},$$

where σ_{ii} denotes the ith diagonal element of Σ, $i = 1, \ldots, n$. Can the upper bound on $E(s^2)$ be attained? Justify your answer.

Points: (a) 5; (b) 5; (c) 5; (d) 5; (e) 5.

2. (25 points) Suppose that $Y_1, \cdots, Y_n$ are independent random variables and each Y_i is distributed as exponential with mean $\mu_i = \beta x_i$, where $x_1, \cdots, x_n$ are known positive constants not all equal and $\beta > 0$ is an unknown parameter.

   (a) Find an explicit expression for the maximum-likelihood estimators, $\hat{\beta}$, of β. Also, find the large-sample ($n \to \infty$) distribution of $\sqrt{n}(\hat{\beta} - \beta)$.

   (b) Find a pivotal quantity and use it to construct an exact 95% confidence interval for β.

   (c) Consider the following estimator of β: $\tilde{\beta} = (\sum_{i=1}^n Y_i)/\sum_{i=1}^n x_i$. Show that the finite-sample efficiency of $\tilde{\beta}$ relative to $\hat{\beta}$ is less than 1.

   (d) Now consider a different model for the mean, specifically,

   $$\frac{1}{\mu_i} = \alpha + \gamma x_i$$

   where α and γ are unknown parameters. Find a minimal sufficient statistic for (α, γ).

   (e) By appropriate conditioning, obtain the conditional score function for β (eliminating α). You don't need to simplify it in this part.

Points: (a) 5; (b) 5; (c) 5; (d) 5; (e) 5.

Statement of the UNC honor pledge:

> *"In recognition of and in the spirit of the honor code, I certify that I have neither given nor received aid on this examination and that I will report all Honor Code violations observed by me."*

(Signed) ____________________________________
NAME

(Printed) ____________________________________
NAME

2019 S2 Q1

(a) $U = \|\Sigma^{-1/2}\varepsilon\|^2 \sim \chi^2_n$. $\mathbb{E}U = n$, $\mathrm{Var}\,U = 2n$.

(b) $MY = X\hat\beta \Rightarrow \hat\beta = \hat\beta_0 + \eta$, where $\hat\beta_0 = (X^TX)^-X^TY$: LSE & $\eta \in C(X^T)^\perp$

$\therefore \hat\beta = (X^TX)^-X^TY + (I - X^T(XX^T)^-X)r$

(c) $(\Rightarrow)$ $\lambda^T = p^TX$, $\exists p$

$$p^TX(X^T\Sigma^{-1}X)^-X^T\Sigma^{-1}X$$

$$= p^T\Sigma^{1/2}\Big[\underbrace{(\Sigma^{-1/2}X)(X^T\Sigma^{-1}X)^-X^T\Sigma^{-1/2}}_{\text{o.p.o onto } C(\Sigma^{-1/2}X)}\Big]\Sigma^{-1/2}X = p^T\Sigma^{1/2}\cdot\Sigma^{-1/2}X = p^TX = \lambda^T$$

$(\Leftarrow)$ $\lambda^T = \underbrace{\lambda^T(X^T\Sigma^{-1}X)^-(X^T\Sigma^{-1})}_{p^T}X \Rightarrow \lambda^T\beta = p^TX\beta$: estimable.

(d) $\Sigma^{-1/2}Y = \Sigma^{-1/2}X\beta + \Sigma^{-1/2}\varepsilon$

$$\hat\beta^{BLUE} = \hat\beta^{LSE} = (X_*^TX_*)^{-1}X_*^TY_*$$

$$= (X^T\Sigma^{-1}X)^{-1}X^T\Sigma^{-1}Y \text{ by } \underline{\text{Gauss-Markov Thm}}.$$

If $\hat\beta^{BLUE} = (X^TX)^{-1}X^TY \Leftrightarrow (X^T\Sigma^{-1}X)^{-1}X^T\Sigma^{-1} = (X^TX)^{-1}X^T$

$\Rightarrow (X^T\Sigma^{-1}X)^{-1}X^T = (X^TX)^{-1}X^T\Sigma$

$\Rightarrow X(X^T\Sigma^{-1}X)^{-1} = \Sigma X(X^TX)^{-1}$

$\Rightarrow X\underbrace{(X^T\Sigma^{-1}X)^{-1}X^TX}_{F} = \Sigma X$

If $XF = \Sigma X \Leftarrow$ $\Sigma^{-1}X = XF^{-1} \Rightarrow X^T\Sigma^{-1} = (F^{-1})^TX^T \Rightarrow X^T\Sigma^{-1}X = (F^{-1})^TX^TX$

$\Rightarrow (X^T\Sigma^{-1}X)^{-1}X^T\Sigma^{-1} = ((F^{-1})^TX^TX)^{-1}\cdot(F^{-1})^TX^T$

$= (X^TX)^{-1}F^T\cdot(F^T)^{-1}X^T = (X^TX)^{-1}X^T$ //

(e) $\mathbb{E}s^2 = \frac{1}{n-p}\left((X\beta)^T(I-M)(X\beta) + tr((I-M)\mathrm{Var}\,Y)\right)$

$= \frac{1}{n-p}\cdot tr((I-M)\Sigma) = \frac{1}{n-p}tr(\Sigma) - \frac{1}{n-p}tr(M\Sigma)$

$$e_i = \begin{array}{c} \\ \\ \\ \leftarrow i\text{th} \\ \\ \\ \end{array}\!\!\!\!\!\!\!\!\!\!\!\!\! \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$tr(M\Sigma) = tr(\Sigma^{1/2}M\Sigma^{1/2}) = \sum_{i=1}^{p} e_i^T\Sigma^{1/2}M\Sigma^{1/2}e_i = \sum_{i=1}^{p}\|M\Sigma^{1/2}e_i\|^2 \ge 0$

$\therefore \mathbb{E}s^2 \le \frac{1}{n-p}tr(\Sigma) = \frac{1}{n-p}\sum_{i=1}^{n}\sigma_{ii}$. "$=$" iff $M\Sigma^{1/2}e_i = 0, \forall i \Rightarrow M\Sigma^{1/2} = 0 \Rightarrow M = 0$: Impossible!

2019 S2 Q2

(a) $l_n(\beta) = \sum_{i=1}^{n} \log\left(\frac{1}{\mu_i} e^{-y_i/\mu_i}\right) = \sum_{i=1}^{n} -\log \mu_i - y_i/\mu_i = \sum_{i=1}^{n} -\log \beta x_i - y_i/\beta x_i$

$= -n\log\beta - \sum_i \log x_i - \frac{1}{\beta}\sum_i \frac{y_i}{x_i}$

$\dot{l}_n(\beta) = -n\beta^{-1} + \beta^{-2}\sum_i \frac{y_i}{x_i} \overset{set}{=} 0 \Rightarrow \hat{\beta} = \frac{1}{n}\sum_i \frac{y_i}{x_i}$

$\ddot{l}_n(\beta) = n\beta^{-2} - 2\beta^{-3}\sum_i \frac{y_i}{x_i} \Rightarrow I_n(\beta) = \mathbb{E}\left[-\ddot{l}_n(\beta)\right] = -n\beta^{-2} + 2\beta^{-3}\sum_i \frac{\beta x_i}{x_i}$

$= n\beta^{-2}$

$\Rightarrow I(\beta) = \lim_{n\to\infty} \frac{1}{n} I_n(\beta) = \beta^{-2}$.

$\therefore \sqrt{n}(\hat{\beta}-\beta) \to_d N(0, \beta^2)$

(b) $Y_i \sim Exp(\beta x_i) \overset{d}{=} Ga(1, \beta x_i) \Rightarrow Y_i/x_i \overset{iid}{\sim} Ga(1, \beta)$

$\Rightarrow \sum_i \frac{Y_i}{x_i} \sim Ga(n, \beta) \Rightarrow \frac{2}{\beta}\sum_i \frac{Y_i}{x_i} \sim Ga(n, 2) \overset{d}{=} \chi^2(2n)$

$\therefore P\left(\chi^2_{\frac{\alpha}{2}}(2n) \le \frac{2}{\beta}\sum_i \frac{Y_i}{x_i} \le \chi^2_{1-\frac{\alpha}{2}}(2n)\right) = 1-\alpha \Rightarrow \beta \in \left[\frac{2\sum \frac{Y_i}{x_i}}{\chi^2_{1-\frac{\alpha}{2}}(2n)}, \frac{2\sum \frac{Y_i}{x_i}}{\chi^2_{\frac{\alpha}{2}}(2n)}\right]$

(c) $Var(\tilde{\beta}) = \sum_i Var(Y_i)/(\sum x_i)^2 = \sum \beta^2 x_i^2/(\sum x_i)^2 = \beta^2 \cdot \frac{\sum x_i^2}{(\sum x_i)^2}$

$Var(\hat{\beta}) = \frac{1}{n^2}\sum_i \frac{1}{x_i^2} Var(Y_i) = \frac{1}{n^2}\sum_i \frac{1}{x_i^2}\beta^2 x_i^2 = \frac{\beta^2}{n}$

<u>Cauchy-Schwarz inequality</u> : $n\sum x_i^2 \underset{\neq}{>} (\sum x_i)^2 \Rightarrow var(\tilde{\beta}) > var(\hat{\beta})$.

$x_1 \sim x_n$ NOT all equal

$\Rightarrow$ Finite sample efficiency of $\tilde{\beta}$ relative to $\hat{\beta}$ $= \frac{1/var(\tilde{\beta})}{1/var(\hat{\beta})} = \frac{var(\hat{\beta})}{var(\tilde{\beta})} < 1$

(d) $l_n(\alpha, \gamma) = \sum_i -\log\mu_i - \frac{Y_i}{\mu_i} = \sum_i \log(\alpha+\gamma x_i) - Y_i(\alpha+\gamma x_i)$

$= \alpha\cdot(\sum_i -Y_i) + \gamma\cdot(\sum_i -x_iY_i) + \sum_i \log(\alpha+\gamma x_i)$: exp'l family.

$\Rightarrow (\overset{\alpha}{\sum_i Y_i}, \overset{\gamma}{\sum_i x_iY_i})$: MSS.

(e) Compute $P(y_1, \cdots, y_n \mid \sum_i y_i = t)$ to eliminate α's effect.

$$p(y_1,\dots,y_n,\textstyle\sum_i y_i=t) = \exp\left(\alpha(-t)+\gamma\left(\sum_i -x_i y_i\right)+\sum_i \log(\alpha+\gamma x_i)\right) I\left(\sum y_i = t\right)$$

$$p(\textstyle\sum_i y_i=t) = \int p(y_1,\dots,y_n,\sum_i y_i=t)\, dy_1\cdots dy_n$$

$$= \exp\left(\alpha(-t)+\sum_i \log(\alpha+\gamma x_i)\right)\cdot \underbrace{\int\cdots\int \exp\left(\gamma\left(\sum_i -x_i y_i\right)\right)\cdot I\left(\sum y_i = t\right) dy_1\cdots dy_n}_{\text{Let } G(t)}$$

$\therefore\ p(y_1,\dots,y_n \mid \sum_i y_i = t)$

$$= \frac{p(y_1,\dots,y_n,\sum_i y_i=t)}{p(\sum_i y_i = t)}$$

$$= \frac{\exp\left(\alpha(-t)+\gamma\left(\sum_i -x_i y_i\right)+\sum_i \log(\alpha+\gamma x_i)\right)\cdot I\left(\sum y_i=t\right)}{\exp\left(\alpha(-t)+\sum_i \log(\alpha+\gamma x_i)\right)\cdot G(t,\gamma)}$$

$$= \frac{\exp\left(\gamma\sum_i -x_i y_i\right)}{G(t)}\cdot I\left(\sum y_i = t\right) \,_{\prime\prime}$$

$$\Rightarrow\ l_n(\gamma\mid\alpha) = \gamma\sum_i -x_i y_i - \ln G(t,\gamma)$$

$$\Rightarrow\ S_n(\gamma\mid\alpha) = \dot{l}_n(\gamma\mid\alpha) = -\sum_i x_i Y_i - \frac{\partial_\gamma G(t,\gamma)}{G(t,\gamma)} \,_{\prime\prime}$$