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2. (25 points) Suppose that Y is a 4×1 vector with $E(Y) = \mu$, $\mu \in E$, where E is the set $E = \{u : u' = (\beta_1 + \beta_2 - \beta_3, \beta_2 + \beta_3, -\beta_2 - \beta_3, -\beta_1 - \beta_2 + \beta_3)\}$, where the β_i are real numbers, $i = 1, 2, 3$ and a $'$ denotes matrix (vector) transposition. Further assume that $\text{Cov}(Y) = \sigma^2 I_{4\times4}$, where σ^2 is unknown.

(a) (5 points) Derive $\hat{\mu}$, the ordinary least squares estimate of μ, by carrying out the appropriate projection.

(b) (4 points) Find the BLUE of $\beta_2 - \beta_3$ or show that it is nonestimable.

(c) (4 points) Consider testing $H_0 : \beta_2 + \beta_3 = 0$ versus $H_1 : \beta_2 + \beta_3 \neq 0$. Let E_0 denote the set E assuming that H_0 is true. Explicitly give the sets E_0 and $E \cap E_0^\perp$, where $E_0^\perp$ denotes the orthogonal complement of E_0.

(d) (6 points) Assuming normality for Y, construct the simplest possible expression for the F statistic for the hypothesis $H_0 : \mu \in E_0$ versus $H_1 : \mu \notin E_0$, where E_0 is specified in part (c), and give the distribution of the F statistic under the null and alternative hypotheses.

(e) (6 points) Assuming normality for Y, construct an exact 95% confidence interval for $\beta_2 + \beta_3$.

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2a) Derive $\hat{\mu}$, the OLS estimate of μ, by carrying out the appropriate projection.

Model: $Y = X\beta + \epsilon$ where $E(\epsilon) = 0$

Then, since $\hat{\beta} = (X'X)^{-1}X'Y \Rightarrow \hat{\mu} = X\hat{\beta} = X(X'X)^{-1}X'Y$

Here $\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}_{3\times 1}$, $X = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$, $Y = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix}_{4\times 1}$

Notice that $\text{Rank}(X) = 2$.

Then $C(X)$ = column space of X = Span of original LI columns of X = $\text{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}\right\}$

Since
$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \xrightarrow{R_1 + R_4 = R_4} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 + R_3 = R_3} \overset{\text{pivots}}{\begin{pmatrix} \textcircled{1} & 1 & -1 \\ 0 & \textcircled{1} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}.$$

Then, let $X^* = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \\ -1 & -1 \end{pmatrix}$ where $C(X^*) = C(X)$.

Then, $\hat{\mu} = MY = X^*(X^{*\prime}X^*)^{-1}X^* Y$

where $X^{*\prime}X^* = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix} \Rightarrow (X^{*\prime}X^*)^{-1} = \frac{1}{8-4}\begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix}$
$= \frac{1}{2}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

$$\Rightarrow M = X^*(X^{*\prime}X^*)^{-1}X^{*\prime} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \hat{\mu} = MY = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix} = \boxed{\frac{1}{2}\begin{pmatrix} Y_1 - Y_4 \\ Y_2 - Y_3 \\ Y_3 - Y_2 \\ Y_4 - Y_1 \end{pmatrix}}$$

2b) Find the BLUE of $\beta_2-\beta_3$ or show that it is nonestimable.

⌈ Write $\beta_2-\beta_3 = \lambda'\beta = (0\ 1\ -1)\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}$ where $\lambda' = (0\ 1\ -1)$

Know that $\beta_2-\beta_3$ is estimable iff $\lambda' = \rho' X$ for some $\rho_{4\times1}$. Let's check.

$$(0\ \ 1\ -1) = (\rho_1, \rho_2, \rho_3, \rho_4)\begin{pmatrix}1 & 1 & -1\\ 0 & 1 & 1\\ 0 & -1 & -1\\ -1 & -1 & 1\end{pmatrix}_{4\times3}$$

$$\Rightarrow \begin{aligned} 0 &= \rho_1-\rho_4 \quad \Rightarrow \rho_1=\rho_4\\ 1 &= \rho_1+\rho_2-\rho_3-\rho_4\\ -1 &= -\rho_1+\rho_2-\rho_3+\rho_4\end{aligned}$$

and
$$\begin{aligned} 1 &= \rho_1+\rho_2-\rho_3-\rho_4\\ + \quad -1 &= -\rho_1+\rho_2-\rho_3+\rho_4\\ \hline 0 &= 0+2\rho_2-2\rho_3\end{aligned}$$
$$\Rightarrow \rho_2=\rho_3$$

$$\overset{\text{sub}}{\Rightarrow} 1 = \rho_1+\rho_3-\rho_3-\rho_1 \Rightarrow 1=0$$ which is a contradiction!

Thus, $\beta_2-\beta_3$ is <u>NOT</u> estimable. ⌋

2c) Consider testing $H_0: \beta_2+\beta_3=0$ vs. $H_1: \beta_2+\beta_3 \neq 0$. Let E_0 denote the set E assuming that H_0 is true. Explicitly give the sets E_0 and $E \cap E_0^\perp$, where $E_0^\perp$ denotes the orthogonal complement of E_0.

⌈ ① First, check that $\beta_2+\beta_3$ is estimable.

$\beta_2+\beta_3 = \lambda'\beta = (0\ 1\ 1)\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}$ where $\lambda' = (0\ 1\ 1)$.

Know that $\beta_2+\beta_3$ is estimable iff $\lambda' = \rho' X$ for sum $\rho_{4\times1}$. Let's check.

$$(0\ \ 1\ \ 1) = (\rho_1\ \rho_2\ \rho_3\ \rho_4)\begin{pmatrix}1 & 1 & -1\\ 0 & 1 & 1\\ 0 & -1 & -1\\ -1 & -1 & 1\end{pmatrix}_{4\times3}$$

$$\Rightarrow \begin{aligned} 0 &= \rho_1-\rho_4 \quad \Rightarrow \rho_1=\rho_4\\ 1 &= \rho_1+\rho_2-\rho_3-\rho_4\\ + \quad 1 &= -\rho_1+\rho_2-\rho_3+\rho_4\\ \hline 2 &= 2\rho_2-2\rho_3\end{aligned} \Rightarrow 1 = \rho_2-\rho_3 \Rightarrow \rho_2 = 1+\rho_3$$

$$\overset{\text{sub}}{\Rightarrow} 1 = \cancel{\rho_1}+1+\cancel{\rho_3}-\cancel{\rho_3}-\cancel{\rho_1} \Rightarrow 1=1$$, which is true. Thus, $\beta_2+\beta_3$ is estimable

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2c) cont'd

② Find the estimation spaces

The estimation space under H_0 is denoted by E_0. Since $H_0: \beta_2+\beta_3=0 \iff \beta_2=-\beta_3$

$$\Rightarrow E_0=\{u: u'=(\beta_1-2\beta_3,\ 0,\ 0,\ 2\beta_3-\beta_1)\}=\{u: u'=(\beta_2-2\beta_3)\cdot(1,0,0,-1)\}$$

$$\text{Thus, } \boxed{E_0=\text{Span}\left\{\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}\right\}} \Rightarrow E_0^{\perp}=\text{Span}\left\{\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\right\} \quad \left(\text{since } (1\ 0\ 0\ {-1})\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}=0\right)$$

$$E=C(X)=\text{Span}\left\{\begin{pmatrix}1\\0\\0\\-1\end{pmatrix},\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}\right\} \Rightarrow \boxed{E\cap E_0^{\perp}=\text{Span}\left\{\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\right\}}$$

$$\text{since } \begin{pmatrix}0\\1\\-1\\0\end{pmatrix}=-\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}+\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}=\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$$

~~d) Assuming normality for Y, construct the simplest possible expression for the F statistic for the hypothesis $H_0: \mu\in E_0$ vs. $H_1: \mu\notin E_0$, where E_0 is specified in part c), and give the distribution of the F statistic under the null & alternative hypotheses.~~

~~Want to test $H_0: \mu\in E_0$ vs. $H_1: \mu\notin E_0$.~~

~~$$F=\frac{\|(M-M_0)Y\|^2/r(M-M_0)}{\|(I-M)Y\|^2/r(I-M)}$$~~ ~~where $M-M_0$ is an OPO onto $\text{span}\left\{\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\right\}$~~

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2d. Assuming normality for Y, construct the simplest possible expression for the F statistic for the hypothesis $H_0: \mu \in E_0$ vs. $H_1: \mu \notin E_0$, where E_0 is specified in part c), and give the distribution of the F statistic under the null and alternative hypotheses.

Want to test $H_0: \mu \in E_0$ vs. $H_1: \mu \notin E_0$

$$F = \frac{\| M_{MP} Y \|^2 / r(M_{MP})}{\| (I-M) Y \|^2 / r(I-M)} \sim F(r(M_{MP}), r(I-M), \gamma)$$

where $Y = X\hat{\beta}$, $\Lambda' = P'X$, $M = X(X'X)^- X'$, $M_{MP} = (MP)(P'MP)^-(PM)'$
$= (MP)(P'MP)^- P'M$

and the non-centrality parameter is $\gamma = \dfrac{\| M_{MP} X\beta \|^2}{2\sigma^2}$

Thus, using the relationship $\Lambda' = P'X$, can find projection operator P.

Since $(0\ 1\ 1) = (\rho_1\ \rho_2\ \rho_3\ \rho_4) \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix}_{4\times 3}$

where $\rho_1 = \rho_4$

$1 = \rho_1 + \rho_2 - \rho_3 - \rho_4$

$1 = -\rho_1 + \rho_2 - \rho_3 + \rho_4$

$2 = 2\rho_2 - 2\rho_3 \implies 1 = \rho_2 - \rho_3 \implies \rho_2 = 1 + \rho_3$

One possibility for rho is $\rho = (1211)$ since if let $\rho_1 = \rho_4 = 1$ and $\rho_2 = 2 \implies \rho_3 = 1$

Then, $M_{MP} = (MP)(P'MP)^- P'M = \left[\frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}_{4\times4} \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} \right] \left[\frac{1}{2} (1\ 2\ 1\ 1) \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} \right]^- (1211) \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}_{4\times4}$

$= \frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \cdot 2 \left[(0\ 1\ {-1}\ 0) \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} \right]^{-1} \frac{1}{2} (0\ 1\ {-1}\ 0) = \frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} (0\ 1\ {-1}\ 0) = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\implies M_{MP} Y = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 \\ Y_2 - Y_3 \\ Y_3 - Y_2 \\ 0 \end{pmatrix} \implies \| M_{MP} Y \|^2 = \frac{1}{4} (0\ (Y_2 - Y_3)\ (Y_3 - Y_2)\ 0) \begin{pmatrix} 0 \\ (Y_2 - Y_3) \\ (Y_3 - Y_2) \\ 0 \end{pmatrix}$

$= \frac{1}{4} [(Y_2 - Y_3)^2 + \underbrace{(Y_3 - Y_2)^2}_{(Y_2 - Y_3)^2}] = \frac{1}{2} (Y_2 - Y_3)^2$

where Rank$(M_{MP}) = 1$ (since can row reduce to get one pivot).

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2d) cont'd

And
$$I - M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1/2 & 0 & 0 & -1/2 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ -1/2 & 0 & 0 & 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix} = 1/2 \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow (I-M)Y = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} Y_1 + Y_4 \\ Y_2 + Y_3 \\ Y_2 + Y_3 \\ Y_1 + Y_4 \end{pmatrix}$$

Then, $\|(I-M)Y\|^2$

$$= \frac{1}{4}\left[(Y_1+Y_4)^2 + (Y_2+Y_3)^2 + (Y_2+Y_3)^2 + (Y_1+Y_4)^2\right]$$
$$= \frac{1}{\cancel{4}_{2}}\left[\cancel{2}(Y_1+Y_4)^2 + \cancel{2}(Y_2+Y_3)^2\right]$$
$$= \frac{1}{2}(Y_1+Y_4)^2 + \frac{1}{2}(Y_2+Y_3)^2$$

where Rank$(I-M) = 2$ (if we row reduce $I-M$, get two pivots)

Thus,
$$\boxed{F = \frac{1/2\,(Y_2 - Y_3)^2}{\left[\frac{1}{2}(Y_1+Y_4)^2 + \frac{1}{2}(Y_2+Y_3)^2\right]/2} = \frac{2\,(Y_2-Y_3)^2}{\left[(Y_1+Y_4)^2 + (Y_2+Y_3)^2\right]} \overset{H_0}{\sim} F(1,2)}$$

And, under H_1 $F \sim F(1,2,\gamma)$ where the non-centrality parameter
$$\gamma = \frac{\|M_{MP} X\beta\|^2}{2\sigma^2}$$

where
$$M_{MP}X\beta = \underbrace{\underset{4\times4}{\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}} \underset{4\times3}{\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix}}}_{4\times3} \underset{4\times1}{\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}} = \frac{1}{2}\underbrace{\underset{4\times3}{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{pmatrix}} \underset{3\times1}{\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}}}_{4\times1} = \frac{1}{2}\begin{pmatrix} 0 \\ 2\beta_2 + 2\beta_3 \\ -2\beta_2 - 2\beta_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \beta_2 + \beta_3 \\ -\beta_2 - \beta_3 \\ 0 \end{pmatrix}$$

$$= (\beta_2 + \beta_3)\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \Rightarrow \|M_{MP}X\beta\|^2 = 2(\beta_2+\beta_3)^2$$

$$\Rightarrow \gamma = \frac{\cancel{2}(\beta_2+\beta_3)^2}{\cancel{2}\sigma^2} = \boxed{\frac{(\beta_2+\beta_3)^2}{\sigma^2}}$$

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2e) Assuming normality for Y, construct an exact 95% CI for $\beta_2+\beta_3$.

General Formula: $(1-\alpha)\times 100\%\ CR(\Lambda'\beta)$

$$= \left\{ \beta : \frac{(\Lambda'\hat{\beta}-\Lambda'\beta)'(\Lambda'(X'X)^{-}\Lambda)^{-}(\Lambda'\hat{\beta}-\Lambda'\beta)/r(\Lambda)}{MSE} \leq F(1-\alpha, r(\Lambda), r(I-M)) \right\}$$

where $F(1-\alpha, r(\Lambda), r(I-M))$ is the upper $(1-\alpha)\times 100$th percentile of a central F distribution.

Here $MSE = \| (1-M)Y\|^2 / r(I-M)$

Find $\Lambda'\hat{\beta}$: $\lambda'\hat{\beta} = \rho'\underbrace{X\hat{\beta}}_{\hat{\mu}} = \rho'\hat{\mu}$. (REMEMBER: "Lambda prime = rho prime X)

$$\text{Thus, } \lambda'\hat{\beta} = \underset{\uparrow \atop \rho'}{(1\ 2\ 1\ 1)} \cdot \underbrace{\frac{1}{2}\begin{pmatrix} Y_1-Y_4 \\ Y_2-Y_3 \\ Y_3-Y_2 \\ Y_4-Y_1 \end{pmatrix}}_{\hat{\mu}} = \frac{1}{2}\left(\cancel{(Y_1-Y_4)} + \underbrace{2(Y_2-Y_3)-(Y_2-Y_3)}_{Y_2-Y_3} - \cancel{(Y_1-Y_4)} \right) = \frac{1}{2}(Y_2-Y_3)$$

Find $\Lambda'\beta$: $\lambda'\beta = (0\ 1\ 1)\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \beta_2+\beta_3$

Find $(\Lambda'(X'X)^{-}\Lambda)^{-}$: Know from previous part that $(X^{*\prime}X^{*})^{-} = \frac{1}{2}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

$\parallel$

$(\Lambda^{*\prime}(X^{*\prime}X^{*})^{-}\Lambda^{*})^{-}$

Then, $\Lambda^{*} = (1\ 1)$. $\Rightarrow \Lambda^{*\prime}(X^{*\prime}X^{*})^{-}\Lambda^{*} = (1\ 1)\cdot\frac{1}{2}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$= \frac{1}{2}(1\ 0)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} \Rightarrow (\Lambda^{*\prime}(X^{*\prime}X^{*})\Lambda^{*})^{-} = 2$$

Simplify Numerator: $(\Lambda'\hat{\beta}-\Lambda'\beta)'(\Lambda^{*}(X^{*\prime}X^{*})^{-}\Lambda^{*})^{-}(\Lambda'\hat{\beta}-\Lambda'\beta)/r(\Lambda)$

$$2\left(\tfrac{1}{2}(Y_2-Y_3)-(\beta_2+\beta_3)\right)^2/1$$

Simplify Denominator: Know from previous part that $MSE = [\frac{1}{2}(Y_1+Y_4)^2 + \frac{1}{2}(Y_2+Y_3)^2]/2 = \frac{1}{4}(Y_1+Y_4)^2 + \frac{1}{4}(Y_2+Y_3)^2$

Write final CI:

$$\boxed{95\%\ CR(\beta_2+\beta_3) = \left\{ \beta : \frac{8\left(\frac{1}{2}(Y_2-Y_3)-(\beta_2+\beta_3)\right)^2}{[(Y_1+Y_4)^2+(Y_2+Y_3^2)]} \leq F(0.95, 1, 2) \right\}}$$

(upper 95th percentile of a central $F(1,2)$ distn.)