3a) Let $\underbrace{Y_1, \ldots, Y_m}_{R_i=1}^{\overset{Y_o}{\sim}}, \underbrace{Y_{m+1}, \ldots, Y_N}_{R_i=0}^{\overset{Y_m}{\sim}}$ with $X_1, \ldots, X_m, X_{m+1}, \ldots, X_N$

Likelihood: $L(\underline{\beta}, \sigma^2 \mid Y_1 \ldots Y_m X_1 \ldots X_m, \underline{\pi}, \underline{r})$

$$= (2\pi\sigma^2)^{-\frac{m}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{m}(y_i - x_i\beta)^2\right\} \prod_{i=1}^{N} \pi_i^{r_i}(1-\pi_i)^{(1-r_i)}$$

$$f_{\underline{y}_o, \underline{r}}(\underline{y}_o, \underline{r}) = \int_{\underline{y}_m} f_{\underline{y}, \underline{r}}(\underline{y}_o, \underline{y}_m, \underline{r})\, d\underline{y}_m$$

$$= \int_{\underline{y}_m} f_{\underline{y}|\underline{r}}(\underline{y}_o, \underline{y}_m \mid \underline{r}) f_{\underline{r}}(\underline{r})\, d\underline{y}_m$$

$$= \int_{\underline{y}_m} f_{\underline{y}}(\underline{y}_o, \underline{y}_m) f_{\underline{r}}(\underline{r})\, d\underline{y}_m$$

$$= f_{\underline{y}_o}(\underline{y}_o) f_{\underline{r}}(\underline{r}) \int_{\underline{y}_m} f_{\underline{y}_m}(\underline{y}_m)\, d\underline{y}_m$$

$$= \prod_{i=1}^{m} \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) \exp\left\{-\frac{1}{2\sigma^2}(y_i - x_i\beta)\right\} \prod_{j=1}^{N} \pi_j^{r_j}(1-\pi_j)^{1-r_j}$$

b) $$\hat{\beta} = \frac{\sum_{i=1}^{m} y_i x_i}{\sum_{i=1}^{m} x_i^2} \qquad \hat{\sigma}^2 = \frac{\sum_{i=1}^{m}(y - x_i\hat{\beta})^2}{m}$$

c) $$\hat{\beta} = \begin{cases} 0, & m = 0 \\ \dfrac{\sum y_i x_i}{\sum x_i^2}, & m > 0 \end{cases}$$

$$E[\hat{\beta}] = 0 \cdot P(m=0) + E\left[\frac{\sum y_i x_i}{\sum x_i^2}\right] \cdot P(m>0)$$

$$= \frac{1}{\sum x_i^2} \cdot \left(\sum x_i E[y_i]\right) \cdot \left[1 - P(R_1=0)P(R_2=0)\ldots P(R_N=0)\right]$$

$$= \frac{\sum x_i^2}{\sum x_i^2} \beta \cdot \left[1 - (1-\pi_1)\ldots(1-\pi_N)\right]$$

$$= \beta \cdot \left[1 - \prod_{i=1}^{N}(1-\pi_i)\right]$$

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3 c)

$$\hat{\beta} = \frac{\sum_{i=1}^{m} y_i x_i}{\sum_{i=1}^{m} x_i^2} \qquad \hat{\sigma}^2 = \frac{\sum_{i=1}^{m} (y - x_i \hat{\beta})^2}{m}$$

$$Var(\hat{\beta}) = E[\hat{\beta}^2] - E[\hat{\beta}]^2 = E\left[\frac{(\sum x_i y_i)^2}{(\sum x_i^2)^2}\right] \cdot \left[1 - \prod_{i=1}^{N}(1-\pi_i)\right]$$

$$- \beta^2 \left[1 - \prod_{i=1}^{N}(1-\pi_i)\right]^2$$

$$= \frac{1}{(\sum x_i^2)^2} E\left[(\sum x_i y_i)^2\right] \cdot \left[1 - \prod_{i=1}^{N}(1-\pi_i)\right] - \beta^2 \left[1 - \prod_{i=1}^{N}(1-\pi_i)\right]^2$$

$$\parallel$$

$$Var\left(\sum x_i y_i\right) + \left\{E\left(\sum x_i y_i\right)\right\}^2$$

$$= \sum x_i^2 Var(y_i) + \left[\sum x_i E(y_i)\right]^2$$

$$= \sigma^2 \sum x_i^2 + \left[\sum x_i^2 \beta\right]^2$$

$$= \sigma^2 \sum x_i^2 + \beta^2 \left[\sum x_i^2\right]^2$$

$$= \sum x_i^2 \left[\sigma^2 + \beta^2 \sum x_i^2\right]$$

$$\to \frac{\sigma^2 + \beta^2 \sum_{i=1}^{m} x_i^2}{\sum_{i=1}^{m} x_i^2} \left[1 - \prod_{j=1}^{N}(1-\pi_j)\right] - \beta^2 \left[1 - \prod_{j=1}^{N}(1-\pi_j)\right]^2$$

$$E(\hat{\beta}) = \begin{cases} 0 & m = 0 \\ \frac{\sum y_i x_i}{\sum x_i^2} & m > 0 \end{cases}$$

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$$= \prod_{i=1} \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) \exp\left\{-\frac{1}{2\sigma^2}(y_i - x_i\beta)\right\} \prod_{j=1}$$

f) $Var(\tilde{\beta}) = \dfrac{1}{\left[\sum_{i=1}^{N} x_i\right]^2} \sum_{i=1}^{N} \dfrac{1}{\pi_i}\left[\sigma^2 + (1-\pi_i)\left[x_i\beta\right]^2\right]$

g) ?

h) $Var(\tilde{\beta}(g)) = \dfrac{1}{\left[\sum_{i=1}^{N} x_i\right]^2} \sum_{i=1}^{N} \dfrac{1}{\pi_i}\left[\sigma^2 + (1-\pi_i)(x_i\beta - g(x_i))^2\right]$

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