Guess: $\ell' = \begin{pmatrix} -3 & 1 & 0 & 3 \\ 0 & 1 & 0 & 0 \end{pmatrix} \longrightarrow \ell' X = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} = \Lambda^T$ ✓

b) ① Find a CSS for $\Lambda^T \beta$?

↳ If we get UMVUE for β, can we apply a linear transformation to get $\Lambda^T\beta$ & have Λ^T(UMVUE) be the UMVUE for $\Lambda^T\beta$?

Try it: ① Find a CSS for β:

$$Y \sim N_4(X\beta, \sigma^2 I_4)$$

$$\Rightarrow f_Y(y) = (2\pi)^{-2}(\sigma^2)^{-2} \exp\left\{-\frac{1}{2}(Y - X\beta)^T(\sigma^2 I_4)^{-1}(Y - X\beta)\right\}$$

$$= \frac{1}{4\pi^2(\sigma^2)^2} \exp\left\{\frac{-1}{2\sigma^2}(y - X\beta)^T(y - X\beta)\right\} \longrightarrow$$

$$\propto \exp\left\{\frac{-1}{2\sigma^2}\left(\underbrace{y^Ty}_{c(y)} - \underbrace{2y^TX}_{\text{css for } \beta}\beta + \underbrace{(X\beta)^TX\beta}_{b(\beta)}\right)\right\}$$

scalars

css for β (since $Y \sim$ exp. fam)

$$CSS_\beta = -2y^TX \Rightarrow CSS_\beta = y^TX = (y_1, y_2, y_3, y_4)\begin{pmatrix} 1 & 3 \\ 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} y_1 + y_2 + y_3 + y_4 \\ 3y_1 + y_2 + y_3 + 2y_4 \end{pmatrix}^T = \begin{pmatrix} \Sigma y_i \\ \Sigma y_i + 2y_1 + y_4 \end{pmatrix}^T$$

$$E[y^TX] = \begin{pmatrix} E[y_1] + E[y_2] + E[y_3] + E[y_4] \\ 3E[y_1] + E[y_2] + E[y_3] + 2E[y_4] \end{pmatrix}^T$$

$$= \begin{pmatrix} \beta_1 + 3\beta_2 + \beta_1 + \beta_2 + \beta_1 + \beta_2 + \beta_1 + 2\beta_2, \\ 3(\beta_1 + 3\beta_2) + \beta_1 + \beta_2 + \beta_1 + \beta_2 + 2(\beta_1 + 2\beta_2) \end{pmatrix}^T$$

$$= \begin{pmatrix} 4\beta_1 + 7\beta_2 \\ 7\beta_1 + 15\beta_2 \end{pmatrix}^T = \left[\begin{pmatrix} 4 & 7 \\ 7 & 15 \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}\right]^T = (\beta_1, \beta_2)\begin{pmatrix} 4 & 7 \\ 7 & 15 \end{pmatrix}$$

Need M s.t. $\begin{pmatrix} 4 & 7 \\ 7 & 15 \end{pmatrix} M = I$, i.e. s.t. $M^{-1} = \begin{pmatrix} 4 & 7 \\ 7 & 15 \end{pmatrix}$.

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$\frac{1}{n}\sum_{i=1} x_{1,i}$, $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ and $J_n^n = J_n J_n'$ (and $n = 4$ in this question).

M

$$\begin{pmatrix} 4 & 7 \\ 7 & 15 \end{pmatrix}^{-1} = \frac{1}{15 \cdot 4 - 7^2}\begin{pmatrix} 15 & -7 \\ -7 & 4 \end{pmatrix} = \frac{1}{60-49}\begin{pmatrix} 15 & -7 \\ -7 & 4 \end{pmatrix} = \begin{pmatrix} 15/11 & -7/11 \\ -7/11 & 4/11 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}}_{A^+}\underbrace{\begin{pmatrix} 15/11 & -7/11 \\ -7/11 & 4/11 \end{pmatrix}}_{M} = \begin{pmatrix} 8/11 & -3/11 \\ 29/11 & -15/11 \end{pmatrix}$$

$$\begin{pmatrix} 8/11 & -3/11 \\ 29/11 & -15/11 \end{pmatrix}\begin{pmatrix} 4 & 7 \\ 7 & 15 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 32-21 & 56-45 \\ 116-105 & 203-225 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 11 & 11 \\ 11 & -22 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$$

$$= \frac{4 \cdot 15 + 7^2}{4 \cdot 15^2} = \frac{109}{4 \cdot 15^2}$$

can get e' s.t. $e'X\beta = \beta_1$ & s.t. $e'X\beta = \beta_2$

works (?) b/c β_1, β_2 are $\boxed{\text{individually estimable}}$, so can find UMVUEs for them.

$\Longrightarrow$ $y^T X M = g(y^T X)$ is an unbiased estimator for θ.

b/c $E[y^T X M] = \beta^T M^{-1} M = \beta^T$. $\Rightarrow$ $y^T X M$ is UMVUE for β

So, UMVUE for $\Lambda^T \beta$ is $\Lambda^T M^T X^T y$ is UMVUE for $\Lambda^T \beta$

* Unbiased ✓
* Min. Variance? ?

Note: $M = M^T$

#2 cont'd

b) We have a CSS for β, X^Ty.
We have just verified that $\Lambda^T M X^T y$ is an unbiased estimator for $\Lambda^T\beta$. So, we know by the Lehmann-Sheffe Thm that the unique UMVUE for $\Lambda^T\beta$ is:

$$E[\Lambda^T M X^T y \mid X^T y] = \Lambda^T M X^T y$$ (the whole thing is a constant).

So this is the unique UMVUE.