Now, $\partial_\xi g(\xi_1,\xi_2)\Big|_{\xi=\xi^*} = \left(\frac{1}{2\xi_2^2}\cdot\frac{1}{\sqrt{\frac{\xi_1}{\xi_2^2}-1}}, \frac{1}{2}(-2)\frac{\xi_1}{\xi_2^3}\frac{1}{\sqrt{\frac{\xi_1}{\xi_2^2}-1}}\right)'\Bigg|_{(\xi_1,\xi_2)'=(\sigma^2+\mu^2,\mu)}$

$$= \left(\frac{1}{2\mu^2}\cdot\frac{1}{\sqrt{\frac{\sigma^2+\mu^2}{\mu^2}-1}}, -\frac{(\sigma^2+\mu^2)}{\mu^3}\cdot\frac{1}{\sqrt{\frac{\sigma^2+\mu^2}{\mu^2}-1}}\right)'$$

$$= \left(\frac{1}{2\mu^2}\cdot\frac{\mu}{\sigma}, -\frac{(\sigma^2+\mu^2)}{\mu^3}\cdot\frac{\mu}{\sigma}\right)' = \left(\frac{1}{2\mu\sigma}, -\frac{(\sigma^2+\mu^2)}{\sigma\mu^2}\right)'$$

Note that $g[(\mu+\sigma^2,\mu)'] = \sqrt{\frac{\mu^2+\sigma^2}{\mu^2}-1} = \sqrt{\frac{\sigma^2}{\mu^2}}$

Thus (*) becomes $\sqrt{n}\left[\frac{C_n}{\bar{Y}} - \frac{\sigma}{\mu}\right] =$

$$\sqrt{n}\left[\sqrt{\frac{\sum Y_i^2}{n\bar{Y}^2}-1} - \frac{\sigma}{\mu}\right] \to_d N\left(0, \left(\frac{1}{2\mu\sigma} \quad \frac{-(\sigma^2+\mu^2)}{\sigma\mu^2}\right)\begin{pmatrix}2\sigma^4+4\mu^2\sigma^2 & 2\mu\sigma^2 \\ 2\mu\sigma^2 & \sigma^2\end{pmatrix}\begin{pmatrix}\frac{1}{2\mu\sigma} \\ -\frac{(\sigma^2+\mu^2)}{\sigma\mu^2}\end{pmatrix}\right)$$

$$\equiv N\left(0, \left(\frac{2\sigma^4}{2\mu\sigma} + \frac{4\mu^2\sigma^2}{2\mu\sigma} - \frac{2\mu\sigma^2(\sigma^2+\mu^2)}{\sigma\mu^2}, \quad \sigma - \frac{\sigma(\sigma^2+\mu^2)}{\mu^2}\right)\begin{pmatrix}\frac{1}{2\mu\sigma} \\ -\frac{(\sigma^2+\mu^2)}{\sigma\mu^2}\end{pmatrix}\right)$$

$$\equiv N\left(0, \frac{\sigma^2}{2\mu^2} + \frac{\sigma^4}{\mu^4}\right)$$

#1.2.12 | 5/5

Question 2: $Y = (Y_0, Y_1, Y_2)'$ follows multinomial distribution with total count m and prob. vector $(\gamma_0, \gamma_1, \gamma_2)'$;

$$\gamma_j = \binom{2}{j}\pi^j(1-\pi)^{2-j}\theta^{-j(2-j)}/f(\pi,\theta)$$

$$f(\pi,\theta) = \sum_{k=0}^{2}\binom{2}{k}\pi^k(1-\pi)^{2-k}\theta^{-k(2-k)}; \quad \lambda = \log\frac{\pi}{1-\pi} \ \& \ \psi = \log\theta$$

(a) NTS $Y \in$ 2-dim exponential family of form

$$f_Y(y;\eta) = \exp\{Q(y)'\eta - b(\eta) - c(y)\}$$

Solution

$$f_Y(y|\pi,\theta) = \frac{m!}{y_0!\,y_1!\,y_2!}\gamma_0^{y_0}\gamma_1^{y_1}\gamma_2^{y_2}$$

$$\gamma_0 = \binom{2}{0}\pi^0(1-\pi)^2/f(\pi,\theta) = (1-\pi)^2/f(\pi,\theta)$$

$$\gamma_1 = \binom{2}{1}\pi(1-\pi)\theta^{-1}/f(\pi,\theta) = 2\pi(1-\pi)/\theta f(\pi,\theta)$$

$$\gamma_2 = \binom{2}{2}\pi^2(1-\pi)^0\theta^{-2(2-2)}/f(\pi,\theta) = \pi^2/f(\pi,\theta)$$

$$f_Y(y|\pi,\theta) = \frac{m!}{y_0!\, y_1!\, y_2!}\left[\frac{(1-\pi)^2}{f(\pi,\theta)}\right]^{y_0}\left[\frac{2\pi(1-\pi)}{\theta f(\pi,\theta)}\right]^{y_1}\left[\frac{\pi^2}{f(\pi,\theta)}\right]^{y_2}$$

$$\log f_Y(y|\pi,\theta) = \log\left\{\frac{m!}{y_0!\, y_1!\, y_2!}\right\} + y_0\left[2\log(1-\pi) - \log f(\pi,\theta)\right]$$

$$+ y_1\left[\log 2 + \log\pi + \log(1-\pi) - \log\theta - \log f(\pi,\theta)\right]$$

$$+ y_2\left[2\log\pi - \log f(\pi,\theta)\right]$$

where $f(\pi,\theta) = \binom{2}{0}\pi^0(1-\pi)^2 + \binom{2}{1}\pi(1-\pi)\theta^{-1} + \binom{2}{2}\pi^2(1-\pi)^0$

$$= (1-\pi)^2 + 2\pi(1-\pi)\theta^{-1} + \pi^2$$

By substituting

$$\lambda = \log\frac{\pi}{1-\pi} \Rightarrow \pi = \frac{e^\lambda}{1+e^\lambda} \quad \text{and} \quad \psi = \log\theta \Rightarrow \theta = e^\psi$$

$$f(\pi,\theta) = \left[1 - \frac{e^\lambda}{1+e^\lambda}\right]^2 + 2\left[\frac{e^\lambda}{1+e^\lambda}\right]\left[1 - \frac{e^\lambda}{1+e^\lambda}\right]e^{-\psi} + \left[\frac{e^\lambda}{1+e^\lambda}\right]^2$$

$$= \frac{1 + 2e^\lambda e^{-\psi} + e^{2\lambda}}{(1+e^\lambda)^2}$$

Thus,

$$\log f_Y(y|\pi,\theta) = \log\left\{\frac{m!}{y_0!\, y_1!\, y_2!}\right\} + 2y_0\log(1-\pi) - y_0\log f(\pi,\theta) + y_1\log 2$$

$$+ y_1\log\pi + y_1\log(1-\pi) - y_1\log\theta - y_1\log f(\pi,\theta) + 2y_2\log\pi - y_2\log f(\pi,\theta)$$

$$= \log\left\{\frac{m!}{y_0!\, y_1!\, y_2!}\right\} + \{2y_0 + y_1\}\log(1-\pi) - (y_0+y_1+y_2)\log f(\pi,\theta) + \{y_1 + 2y_2\}\log\pi$$

$$+ y_1\log 2 - y_1\log\theta$$

$$= \log\left\{\frac{m!}{y_0!\, y_1!\, y_2!}\right\} + \{m + y_0 - y_2\}\log(1-\pi) + \{m - (y_0 - y_2)\}\log\pi - m\log f(\pi,\theta)$$

$$+ y_1\log 2 - y_1\log\theta \qquad (\text{Since } m = y_0 + y_1 + y_2)$$

$$= \log\left\{\frac{m!}{y_0!\, y_1!\, y_2!}\right\} + m\left\{\log(1-\pi) + \log\pi - \log f(\pi,\theta)\right\} + (y_0 - y_2)\left\{\log(1-\pi) - \log\pi\right\}$$

$$+ y_1\log 2 - y_1\log\theta$$

$$= \log\left\{\frac{m!}{y_0!\, y_1!\, y_2!}\right\} + m\log\left\{\frac{e^\lambda}{1+e^\lambda}\cdot\frac{1}{1+e^\lambda}\cdot\frac{(1+e^\lambda)^2}{1+2e^{\lambda-\psi}+e^{2\lambda}}\right\} - (y_0 - y_2)\lambda$$

$$+ y_1\log 2 - y_1\log(e^\psi)$$

$$= \log \left\{ \frac{m!}{y_0!\, y_1!\, y_2!} \right\} + m \log \left\{ \frac{e^{\lambda}}{1+2e^{\lambda-\psi}+e^{2\lambda}} \right\} - (y_0 - y_2)\lambda + y_1 \log 2 - y_1 \psi$$

$\equiv Q(y)'\eta - b(\eta) - c(y)$, a 2 dim exponential family with

$Q(y) = \begin{pmatrix} y_2 - y_0 \\ -y_1 \end{pmatrix}$, $\eta = \begin{pmatrix} \lambda \\ \psi \end{pmatrix}$, $b(\eta) = -m \log \left\{ \frac{e^{\lambda}}{1+2e^{\lambda-\psi}+e^{2\lambda}} \right\}$,

and $c(y) = -\left\{ \log \left(\frac{m!}{y_0!\, y_1!\, y_2!} \right) + y_1 \log 2 \right\}$

(ii) Need to identify $Q(y)$ and η:

(i) Canonical statistics is given as $Q(y) = \begin{pmatrix} y_2 - y_0 \\ -y_1 \end{pmatrix}$

(ii) Canonical parameters are $\eta = \begin{pmatrix} \lambda \\ \psi \end{pmatrix}$.

(iii) Need expression for the log-likelihood fn with canonical parameters as arguments:

Using the exponential form derived in part (i), we obtain the log-likelihood as follows

$$\log f_Y(y|\eta) = \log \left\{ \frac{m!}{y_0!\, y_1!\, y_2!} \right\} + m \log \left\{ \frac{e^{\lambda}}{1+2e^{\lambda-\psi}+e^{2\lambda}} \right\} + (y_2 - y_0)\lambda - y_1\psi + y_1 \log$$

as the desired result.

(b.) Explicit expressions for MLEs of (λ, ψ):

$$\frac{\partial \log f_Y(y|\eta)}{\partial \lambda} = (y_2 - y_0) + m \left\{ 1 - \frac{2e^{\lambda-\psi} + 2e^{2\lambda}}{1+2e^{\lambda-\psi}+e^{2\lambda}} \right\} \overset{\text{set}}{=} 0$$

$$\Rightarrow (y_2 - y_0) + m \left\{ \frac{1+2e^{\lambda-\psi}+e^{2\lambda} - 2e^{\lambda-\psi} - 2e^{2\lambda}}{1+2e^{\lambda-\psi}+e^{2\lambda}} \right\} = 0$$

$$\Rightarrow (y_2 - y_0) + m \left\{ \frac{1 - e^{2\lambda}}{1+2e^{\lambda-\psi}+e^{2\lambda}} \right\} = 0 \quad \text{—— equation (1)}$$

$$\frac{\partial \log f_Y(y|\eta)}{\partial \psi} = -y_1 + \frac{m(1+2e^{\lambda-\psi}+e^{2\lambda})}{e^{\lambda}} \cdot \frac{(-1)e^{\lambda}(-1)2e^{\lambda-\psi}}{(1+2e^{\lambda-\psi}+e^{2\lambda})^2}$$

$$= -y_1 + \frac{m\, 2e^{\lambda-\psi}}{(1+2e^{\lambda-\psi}+e^{2\lambda})} \overset{\text{set}}{=} 0 \quad \text{—— equation (2)}$$

solving equations (1) and (2):

By (2), $y_1(1+2e^{\lambda-\psi}+e^{2\lambda}) = 2me^{\lambda-\psi}$

$\Rightarrow y_1 + 2y_1e^{\lambda-\psi} + y_1e^{2\lambda} = 2me^{\lambda-\psi}$

$\Rightarrow y_1(1+e^{2\lambda}) = 2e^{-\psi}(me^{\lambda} - y_1e^{\lambda})$

$\Rightarrow e^{-\psi} = \frac{1}{2}\,\frac{y_1(1+e^{2\lambda})}{e^{\lambda}(m-y_1)} \Rightarrow \psi = \log\left(\frac{2(m-y_1)}{y_1e^{-\lambda}(1+e^{2\lambda})}\right)$ —— (*)

From (1),

$(y_2-y_0)(1+2e^{\lambda-\psi}+e^{2\lambda}) = -m(1-e^{2\lambda})$ —— (* *)

By (*) $2e^{-\psi} = \frac{y_1e^{-\lambda}(1+e^{2\lambda})}{m-y_1} \Rightarrow 2e^{\lambda-\psi} = \frac{y_1(1+e^{2\lambda})}{m-y_1}$

So, (* *) becomes

$(y_2-y_0)\left(1+\frac{y_1(1+e^{2\lambda})}{m-y_1}+e^{2\lambda}\right) = -m(1-e^{2\lambda})$

$\Rightarrow (y_2-y_0)\left[(m-\cancel{y_1}) + \cancel{y_1} + \cancel{y_1e^{2\lambda}} + (m-\cancel{y_1})e^{2\lambda}\right] = -m(1-e^{2\lambda})(m-y_1)$

$\Rightarrow (y_2-y_0)\cancel{m}(1+e^{2\lambda}) = -\cancel{m}(1-e^{2\lambda})(m-y_1)$

$\Rightarrow y_2 + y_2e^{2\lambda} - y_0 - y_0e^{2\lambda} = -(m - me^{2\lambda} - y_1 + y_1e^{2\lambda})$

$\Rightarrow e^{2\lambda}(y_2 - y_0 - m + y_1) = y_1 - m + y_0 - y_2$

$\Rightarrow e^{2\lambda}(y_0+y_1+y_2-2y_0-m) = y_0+y_1+y_2-m-2y_2$

$\Rightarrow e^{2\lambda}(-2y_0) = -2y_2$

$\Rightarrow \hat{\lambda} = \frac{1}{2}\log\left(\frac{y_2}{y_0}\right)$

Also, $\psi = \log\left(\frac{2e^{\lambda}(m-y_1)}{y_1(1+e^{2\lambda})}\right) = \log 2e^{\lambda}(m-y_1) - \log[y_1(1+e^{2\lambda})]$

$= \log e^{\lambda} + \log 2\cdot(y_0+y_1+y_2-y_1) - \log[y_1(1+e^{2\lambda})]$

Since $\hat{\lambda} = \frac{1}{2}\log\left(\frac{y_2}{y_0}\right)$ ✓

$\Rightarrow e^{\lambda} = \left(\frac{y_2}{y_0}\right)^{\frac{1}{2}}$ & $e^{2\lambda} = \left(\frac{y_2}{y_0}\right)$

Thus, we have

$$\frac{1}{2}\log\left(\frac{y_2}{y_0}\right) + \log 2(y_0+y_2) - \log\left[y\left(1+\frac{y_2}{y_0}\right)\right]$$

$$= \frac{1}{2}\log\left(\frac{y_2}{y_0}\right) + \log\left[\frac{2(y_0+y_2)}{y_1\left(\frac{y_0+y_2}{y_0}\right)}\right] = \frac{1}{2}\log\left(\frac{y_2}{y_0}\right) + \log\left(\frac{2y_0}{y_1}\right)$$

Hence, $\hat{\lambda} = \frac{1}{2}\log\left(\frac{y_2}{y_0}\right)$ & $\hat{\psi} = \frac{1}{2}\log\left(\frac{y_2}{y_0}\right) + \log\left(\frac{2y_0}{y_1}\right)$ ✓

(e) Explicit expression for the asymptotic covariance matrix of the MLEs of (λ, ψ):

By (b), $$\frac{\partial \log f_Y}{\partial \lambda} = (y_2 - y_0) + (y_0+y_1+y_2)\left[\frac{1-e^{2\lambda}}{1+2e^{\lambda-\psi}+e^{2\lambda}}\right]$$

$$\Rightarrow \frac{\partial^2 \log f_Y}{\partial \lambda^2} = (y_0+y_1+y_2)\left[\frac{(1+2e^{\lambda-\psi}+e^{2\lambda})(-2e^{2\lambda}) - (1-e^{2\lambda})(2e^{\lambda-\psi}+2e^{2\lambda})}{(1+2e^{\lambda-\psi}+e^{2\lambda})^2}\right]$$

$$= 2m\left[\frac{-2e^{2\lambda} - e^{3\lambda-\psi} - e^{\lambda-\psi}}{(1+2e^{\lambda-\psi}+e^{2\lambda})^2}\right]$$

Also, $$\frac{\partial \log f_Y}{\partial \psi} = -y_1 + (y_0+y_1+y_2)\left\{\frac{2e^{\lambda-\psi}}{1+2e^{\lambda-\psi}+e^{2\lambda}}\right\}$$

and $$\frac{\partial^2 \log f_Y}{\partial \psi^2} = (y_0+y_1+y_2)\left[\frac{(1+2e^{\lambda-\psi}+e^{2\lambda})(-2e^{\lambda-\psi}) - (2e^{\lambda-\psi})(-2e^{\lambda-\psi})}{(1+2e^{\lambda-\psi}+e^{2\lambda})^2}\right]$$

$$= 2m\left[\frac{-e^{\lambda-\psi} - e^{2\lambda}e^{\lambda-\psi}}{(1+2e^{\lambda-\psi}+e^{2\lambda})^2}\right]$$

$$\frac{\partial^2 \log f_Y}{\partial \psi\,\partial \lambda} = \frac{\partial}{\partial \psi}\left[\frac{m(1-e^{2\lambda})}{1+2e^{\lambda-\psi}+e^{2\lambda}}\right]$$

$$= m\left[\frac{(1+2e^{\lambda-\psi}+e^{2\lambda})\cdot 0 - (1-e^{2\lambda})(-2e^{\lambda-\psi})}{(1+2e^{\lambda-\psi}+e^{2\lambda})^2}\right]$$

$$= \frac{2me^{\lambda-\psi}(1-e^{2\lambda})}{(1+2e^{\lambda-\psi}+e^{2\lambda})^2}$$

$$-E\left[\partial^2 \log f_Y(\eta)\right] = \begin{bmatrix} \dfrac{2m(2e^{2\lambda}+e^{3\lambda-\psi}+e^{\lambda-\psi})}{(1+2e^{\lambda-\psi}+e^{2\lambda})^2} & \dfrac{2m\,(e^{\lambda-\psi}(1-e^{2\lambda}))}{(1+2e^{\lambda-\psi}+e^{2\lambda})^2} \\ \dfrac{2m\,(e^{\lambda-\psi}(1-e^{2\lambda}))}{(1+2e^{\lambda-\psi}+e^{2\lambda})^2} & \dfrac{2m\, e^{\lambda-\psi}(1+e^{2\lambda})}{(1+2e^{\lambda-\psi}+e^{2\lambda})^2} \end{bmatrix}$$

which gives the 2×2 $I(\lambda,\psi)$ matrix.

$$I(\lambda,\psi)^{-1} = \frac{1}{\left[(e^{\lambda-\psi}+2e^{2\lambda}+e^{3\lambda-\psi}) * (e^{\lambda-\psi}(1+e^{2\lambda})) - (-e^{\lambda-\psi}(1-e^{2\lambda}) * -e^{\lambda-\psi}(1-e^{2\lambda}))\right]}$$

$$* \frac{(1+2e^{\lambda-\psi}+e^{2\lambda})^2}{2m} \begin{bmatrix} e^{\lambda-\psi}(1+e^{2\lambda}) & e^{\lambda-\psi}(1-e^{2\lambda}) \\ e^{\lambda-\psi}(1-e^{2\lambda}) & e^{\lambda-\psi}+2e^{2\lambda}+e^{3\lambda-\psi} \end{bmatrix}$$

$$= \frac{(1+2e^{\lambda-\psi}+e^{2\lambda})^2}{4m\left[2e^{4\lambda-2\psi}+e^{5\lambda-\psi}+e^{3\lambda-\psi}\right]} \begin{bmatrix} e^{\lambda-\psi}(1+e^{2\lambda}) & e^{\lambda-\psi}(1-e^{2\lambda}) \\ e^{\lambda-\psi}(1-e^{2\lambda}) & e^{\lambda-\psi}+2e^{2\lambda}+e^{3\lambda-\psi} \end{bmatrix}$$

is the covariance matrix of (λ,ψ).

(iii) Estimate of matrix: Derived by plugging in $\hat{\psi}$ & $\hat{\lambda}$ in matrix

$$\frac{(1+2e^{\hat{\lambda}-\hat{\psi}}+e^{2\hat{\lambda}})^2}{4m\left[2e^{4\hat{\lambda}-2\hat{\psi}}+e^{5\hat{\lambda}-\hat{\psi}}+e^{3\hat{\lambda}-\hat{\psi}}\right]} \begin{bmatrix} e^{\hat{\lambda}-\hat{\psi}}(1+e^{2\hat{\lambda}}) & e^{\hat{\lambda}-\hat{\psi}}(1-e^{2\hat{\lambda}}) \\ e^{\hat{\lambda}-\hat{\psi}}(1-e^{2\hat{\lambda}}) & e^{\hat{\lambda}-\hat{\psi}}+2e^{2\hat{\lambda}}+e^{3\hat{\lambda}-\hat{\psi}} \end{bmatrix}$$

where $\hat{\lambda} = \frac{1}{2}\log\left(\frac{y_2}{y_0}\right) \Rightarrow e^{\hat{\lambda}} = \left(\frac{y_2}{y_0}\right)^{1/2}$

$$\hat{\psi} = \frac{1}{2}\log\left(\frac{y_2}{y_0}\right) + \log\left(\frac{2y_0}{y_1}\right) \Rightarrow e^{\hat{\psi}} = \left(\frac{y_2}{y_0}\right)^{1/2}\cdot\left(\frac{2y_0}{y_1}\right)$$

$$\hat{\lambda}-\hat{\psi} = \log\left(\frac{y_2}{y_0}\right)^{1/2} - \log\left(\frac{y_2}{y_0}\right)^{\frac{1}{2}} + \log\left(\frac{y_1}{2y_0}\right) = \log\left(\frac{y_1}{2y_0}\right)$$

So $e^{\hat{\lambda}-\hat{\psi}} = \dfrac{y_1}{2y_0}$

Further simplification:

$$\mathrm{Cov}(\lambda,\psi) = \frac{\left(1+\cancel{2}\left(\cancel{\frac{y_2}{y_0}}\right)^{1/2}\left(\cancel{\frac{y_0}{y_2}}\right)^{1/2}\left(\frac{y_1}{\cancel{2}y_0}\right)+\left(\frac{y_2}{y_0}\right)\right)^2}{4(y_0+y_1+y_2)(y_2/y_0)} \begin{bmatrix} 1+\left(\frac{y_2}{y_0}\right) & 1-\left(\frac{y_2}{y_0}\right) \\ 1-\left(\frac{y_2}{y_0}\right) & 1+2\left(\frac{y_2}{\cancel{y_0}}\right)^{1/2}\left(\frac{y_2}{\cancel{y_0}}\right)^{1/2}\left(\frac{2\cancel{y_0}}{y_1}\right)+\left(\frac{y_2}{y_0}\right) \end{bmatrix}$$

$$= \frac{(\cancel{y_0+y_1+y_2})/y_2}{4(\cancel{y_0+y_1+y_2})} \begin{bmatrix} 1+y_2/y_0 & 1-y_2/y_0 \\ 1-y_2/y_0 & 1+4\,y_2/y_1+y_2/y_0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} \frac{1}{y_2}\left(1+y_2/y_0\right) & 1/y_2\left(1-y_2/y_0\right) \\ 1/y_2\left(1-y_2/y_0\right) & 1/y^2\left(1+4\,y_2/y_1+y_2/y_0\right) \end{bmatrix}$$

#2 | 10/10

Question #2

$$Y = (Y_0, Y_1, Y_2) \sim \text{Multinomial}(\gamma_0, \gamma_1, \gamma_2)^T$$

$$\gamma_j = \binom{2}{j} \pi^j (1-\pi)^{2-j} \theta^{-j(2-j)} \Big/ f(\pi,\theta) \qquad j = 0, 1, 2$$

where $f(\pi,\theta) = \sum_{k=0}^{2} \binom{2}{k} \pi^k (1-\pi)^{2-k} \theta^{-k(2-k)}$

$$\lambda = \log\frac{\pi}{1-\pi} \qquad \psi = \log\theta$$

(d). $p(\underset{\sim}{y} \mid \pi, \theta) = \dfrac{m!}{y_0!\, y_1!\, y_2!} \gamma_0^{y_0} \gamma_1^{y_1} \gamma_2^{y_2}$

by HW 1

$$l_{\underset{\sim}{y}}(y \mid \lambda, \psi) = (y_2 - y_0)\lambda - y_1\psi + m\log\frac{e^\lambda}{1+2e^{\lambda-\psi}+e^{2\lambda}} + y_1\log 2 + \log\left(\frac{m!}{y_0!\, y_1!\, y_2!}\right)$$

So $y_2 - y_0$ is a sufficient statistic for λ, assuming $\psi - \psi_0$ is known

Let $t = y_2 - y_0$

$$P(y_2 - y_0 = t) = \sum_{y_2 - y_0 = t} \frac{m!}{y_0!\, y_1!\, y_2!} 2^{y_1} \left(\frac{e^\lambda}{1+2e^{\lambda-\psi}+e^{2\lambda}}\right)^m e^{\lambda(y_2-y_0)} e^{-y_1\psi}$$

$$P(\underset{\sim}{y}, y_2 - y_0 = t) = \frac{m!}{y_0!\, y_1!\, y_2!} 2^{y_1} \left(\frac{e^\lambda}{1+2e^{\lambda-\psi}+e^{2\lambda}}\right)^m e^{\lambda(y_2-y_0)} e^{-y_1\psi}$$

$$\Rightarrow P(y_1 \mid y_2 - y_0 = t) = \frac{P(y_2 - y_0 = t, \underset{\sim}{y})}{P(y_2 - y_0 = t)}$$

$$= \frac{\frac{1}{y_0!\, y_1!\, y_2!} 2^{y_1} e^{-y_1\psi}}{\sum_{y_2' - y_0' = t} 2^{y_1'} e^{-y_1'\psi} \cdot \frac{1}{y_0'!\, y_1'!\, y_2'!}}$$

(e) $H_0: \theta = 1 \iff H_0: \psi = 0$

$y_0 = 3$. $y_1 = 0$ $y_2 = 2$ $\Rightarrow t = y_2 - y_0 = -1$, $m = 5$

possible cases for $y_2 - y_0 = -1$

y_1	y_0	y_2	
0	3	2	✓
1			✗
2	2	1	✓
3			✗
4	1	0	✓
5			✗

under H_0

$$P(y_1 \mid y_2 - y_0 = -1) = \frac{2^{y_1} \frac{1}{y_0! y_1! y_2!}}{\sum_{y_2' - y_0' = -1} 2^{y_1'} \frac{1}{y_0'! y_1'! y_2'!}} = \begin{cases} \frac{1}{21} & y_1 = 0 \\ \frac{12}{21} & y_1 = 2 \\ \frac{8}{21} & y_1 = 4 \end{cases}$$

reject H_0. if $y_1 \geq k$. k is a constant

One sided exact p-value $= \frac{1}{21} = 0.0476$