Theory Exam Section II 2016

1). $y_1, \ldots, y_n$ positive + indep

$p(y_i|\mu_i) = \frac{1}{\mu_i} \exp\left(-\frac{y_i}{\mu_i}\right) \equiv Exp(\mu_i) \quad \mu_i > 0$

$E[y_i|\mu_i] = \mu_i \quad i = 1, \ldots, n$

$\theta_i = 1/\mu_i$

$\theta_i \sim Gamma(\alpha_i, b_i)$

$\alpha_i = 3$

$\frac{\alpha_i}{b_i} = \frac{3}{b_i} = \exp(-x_i^T B) \Rightarrow b_i = 3\exp(x_i^T B)$

$Var(\theta_i) = \nu \exp(x_i^T B) \quad x_i, B \; p \times 1 \quad B$ unknown

Marginal Mean + Var of y_i:

$$E[y_i] = E[E[y_i|\mu_i]]$$
$$= E[\mu_i]$$
$$= E[1/\theta_i]$$
$$= \int_0^\infty \frac{1}{\Gamma(3)(3\exp(x_i^T B))^3} \frac{1}{\theta_i} \theta_i^{3-1} \exp\left(\frac{-\theta_i}{3\exp(x_i^T B)}\right) d\theta_i$$
$$= \frac{\Gamma(2)}{\Gamma(3)} \frac{1}{(3\exp(x_i^T B))} \underbrace{\int_0^\infty \frac{\theta_i^{2-1} \exp\left(-\theta_i / 3\exp(x_i^T B)\right)}{\Gamma(2)(3\exp(x_i^T B))^2} d\theta}_{\text{pdf of gamma } = 1}$$
$$= \frac{1!}{2!} \frac{1}{(3\exp(x_i^T B))}$$
$$= \boxed{\frac{1}{6\exp(x_i^T \beta)}}$$

$$\begin{aligned}
Var(y_i) &= Var[E(y_i|\mu_i)] + E[Var(y_i|\mu_i)] \\
&= Var(\mu_i) + E[\mu_i^2] \\
&= Var(1/\theta_i) + E[1/\theta_i^2] \\
&= E[1/\theta_i^2] + (E(1/\theta_i))^2 + E[1/\theta_i^2] \\
&= 2E[1/\theta_i^2] + (E(1/\theta_i))^2
\end{aligned}$$

$$E[1/\theta_i^2] = \int_0^\infty \frac{\frac{1}{\theta_i^2}\, \theta^{3-1} \exp(-\theta_i/3\exp(x_i^T B))}{\Gamma(3)(3\exp(x_i^T B))^3}\, d\theta_i$$

$$= \frac{\Gamma(1)}{\Gamma(3)} \frac{1}{(3\exp(x_i^T B))^2} \int_0^\infty \frac{\theta^{1-1}\exp(-\theta_i/3\exp(x_i^T B))}{\Gamma(1)(3\exp(x_i^T B))}\, d\theta_i$$

$$= \frac{1}{2(9)\exp(2x_i^T B)} \quad (1) \rightarrow \text{pdf of Gamma}(1, 3\exp(x^T B))$$

$$= \frac{1}{18\exp(2x_i^T B)}$$

$$\begin{aligned}
Var(y_i) &= \frac{1}{9\exp(2x_i^T B)} + \frac{1}{(6\exp(x_i^T B))^2} \\
&= \frac{2}{18\exp(2x_i^T B)} + \frac{1}{18\exp(2x_i^T B)} \\
&= \boxed{\frac{1}{6\exp(2x_i^T B)}}
\end{aligned}$$

(b) Derive the marginal dist of y_i

$$p(y_i) = \int_{\mu_i} \overbrace{p(y_i|\mu_i)\,\pi(\mu_i)}^{P(y_i,\mu_i)}\, d\mu_i$$

$$\Rightarrow p(y_i) = \int_{\theta_i} \underbrace{p(y_i|\theta_i)\,\lambda(\theta_i)}_{p(y_i,\theta_i)}\, d\theta_i$$

$$p(y_i|\theta_i) = \theta_i \exp(-y_i\theta_i)$$

$$\lambda(\theta_i) = \frac{\theta^{3-1}\exp\left(-\theta_i/3\exp(x_i^TB)\right)}{\Gamma(3)\left(3\exp(x_i^TB)\right)^3}$$

$$p(y_i|\theta_i)\lambda(\theta_i) = \frac{\theta_i^{4-1}\exp\left(-\theta_i\left(y_i + 1/3\exp(x_i^TB)\right)\right)}{\Gamma(3)\left(3\exp(x_i^TB)\right)^3}$$

$$\int_0^\infty p(y_i|\theta_i)\lambda(\theta_i)\, d\theta_i$$

$$= \frac{\Gamma(4)}{\Gamma(3)}\left(y_i + \frac{1}{3\exp(x_i^TB)}\right)^4 \frac{1}{\left(3\exp(x_i^TB)\right)^3} \cdot \underbrace{\int_0^\infty \frac{\theta_i^{4-1}\exp\left(-\theta_i\left(y_i + 1/3\exp(x_i^TB)\right)\right)}{\Gamma(4)\left(y_i + \frac{1}{3\exp(x_i^TB)}\right)^4}\, d\theta}_{=1 \text{ (pdf of Gamma}(4,\ y_i + \frac{1}{3}\exp(-x_i^TB)))}$$

$$= \boxed{\frac{4\left(y_i + \frac{1}{3}\exp(-x_i^TB)\right)^4}{27\exp(3x_i^TB)}}$$

(d) μ_i = fixed + unknown parameter

overdispersion: $Var(y_i) = \sigma^2(v_i + \mu_i)$

v_i = var function of GLM in (1)

? $\cancel{v_i = \frac{1}{\mu_i^2}} \Rightarrow v_i = \mu_i^2$

$\mu_i = \exp(x_i^T B)$

(i) Derive the quasilikelihood score eqns for B and a moment estimator of σ^2

Score Eqns:

$$\sum_{i=1}^{n} \frac{\partial \mu_i}{\partial B} \frac{(y_i - \mu_i)}{\sigma^2(v_i + \mu_i)} \qquad \text{Form: } \sum_{i=1}^{n} \frac{\partial \mu_i}{\partial B} \frac{(y_i - \mu_i)}{Var(y_i)} \overset{set}{=} 0$$

$$= \sum_{i=1}^{n} \frac{\partial \mu_i}{\partial B} \frac{(y_i - \mu_i)}{\sigma^2(\mu_i^2 + \mu_i)}$$

$$\frac{\partial \mu_i}{\partial B_j} = \exp(x_i^T B)(x_{ij})$$

(*) $$\frac{\partial \mu_i}{\partial B} = \begin{bmatrix} x_{i1} \exp(x_i^T B) \\ \vdots \\ x_{ip} \exp(x_i^T B) \end{bmatrix} = \underset{\sim}{x}_i \exp(x_i^T B)$$

$$\uparrow \underset{\sim}{x}_i = \begin{bmatrix} x_{i1} \\ \vdots \\ x_{ip} \end{bmatrix}$$

$\mu_i = \exp(x_i^T B)$

$\cancel{\frac{1}{\mu_i^2} = \exp(-2x_i^T B)} \Rightarrow \mu_i^2 = \exp(2x_i^T B$

Score eqns:

$$\boxed{\sum_{i=1}^{n} \frac{\underset{\sim}{x}_i \exp(x_i^T B)(y_i - \exp(x_i^T B))}{\sigma^2(\exp(+2x_i^T B) + \exp(x_i^T B))} \overset{set}{=} 0}$$

Estimator for σ^2:

(Pearson Residual)^2

$$E\left[\sum_{i=1}^{n} \frac{(y_i - \mu_i)^2}{Var(y_i)}\right] \approx n\sigma^2$$

$$\Rightarrow \hat{\sigma}^2 = \frac{1}{n-p} \sum_{i=1}^{n} \frac{(y_i - \mu_i)^2}{\sigma^2(V_i + \mu_i)}$$

$$= \frac{1}{n-p} \sum_{i=1}^{n} \frac{(y_i - \exp(x_i^T B))^2}{\sigma^2(\exp(+2x_i^T B) + \exp(x_i^T B))}$$

(ii) $\hat{B}_p$ = quasilikelihood estimate of B

- Derive the asymptotic cov. matrix for $\hat{B}_p$.

Let the score function in part (i) be denoted as follows:

$$S_n(B) = \sum_{i=1}^{n} \frac{\partial \mu_i}{\partial B} \frac{(y_i - \mu_i)}{V(\mu_i)}$$

where $V(\mu_i) = Var(y_i) = \sigma^2 (1/\mu_i^2 + \mu_i)$

Use a Taylor series expansion to expand $S_n(\hat{B}_p)$ about the true value B_0

Since $\hat{B}_p$ solves $S_n(B) \overset{set}{=} 0$

$\Rightarrow S_n(\hat{B}_p) = 0$

$$0 = S_n(\hat{B}_p) = S_n(B_0) + \frac{\partial}{\partial B} S_n(B_0)(\hat{B}_p - B_0) + O_p(\|\hat{B}_p - B_0\|^2)$$

since $\hat{B}_p$ consistent for $B_0 \Rightarrow \hat{B}_p - B_0 \xrightarrow{P} 0$

$\Rightarrow O_p(\|\hat{B}_p - B_0\|^2) = \|\hat{B}_p - B_0\|^2 O_p(1) = o_p(1)$

$$\Rightarrow 0 = \frac{1}{\sqrt{n}} S_n(B_0) + \left(\frac{1}{n}\frac{\partial}{\partial B} S_n(B_0)\right)\sqrt{n}(\hat{B}_p - B_0) + \overbrace{\frac{1}{\sqrt{n}} o_p(1)}^{o_p(1)}$$

$$\Rightarrow \sqrt{n}(\hat{B}_p - B_0) = \left(\frac{1}{\sqrt{n}} S_n(B_0) + o_p(1)\right)\left(-\frac{1}{n}\frac{\partial}{\partial B} S_n(B_0)\right)^{-1}$$

By the CLT,

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} \frac{\partial \mu_i}{\partial B}\frac{(y_i - \mu_i)}{V(\mu_i)} - E\left(\frac{\partial \mu_i}{\partial B}\frac{(y_i - \mu_i)}{V(\mu_i)}\right)\right) \xrightarrow{d}$$

$$N\left(0, E\left[\left(\frac{\partial \mu_i}{\partial B}\frac{(y_i - \mu_i)}{V(\mu_i)}\right)^{\otimes 2}\right]\right)$$

$\longrightarrow$

$$\Rightarrow \underbrace{\sqrt{n}\left(\frac{1}{n} S_n(\beta) - 0\right)}_{\frac{1}{\sqrt{n}} S_n(\beta)} \xrightarrow{d} N\left(0,\ E\left[\left(\frac{\partial \mu_i}{\partial \beta}\frac{(y_i-\mu_i)}{V(\mu_i)}\right)^{\otimes 2}\right]\right)$$

$$E\left[\left(\frac{\partial \mu_i}{\partial \beta}\frac{(y_i-\mu_i)}{V(\mu)}\right)^{\otimes 2}\right] = E\left[\frac{\partial \mu_i}{\partial \beta}\frac{(y_i-\mu_i)}{V(\mu_i)}\frac{(y_i-\mu_i)^T}{V(\mu_i)}\frac{\partial \mu_i^T}{\partial \beta}\right]$$

$$= \frac{\partial \mu_i}{\partial \beta}\frac{E[(y_i-\mu_i)(y_i-\mu_i)^T]}{V(\mu_i)^2}\frac{\partial \mu_i^T}{\partial \beta}$$

$$= \frac{\partial \mu_i}{\partial \beta}\left(\frac{1}{V(\mu_i)}\right)\frac{\partial \mu_i^T}{\partial \beta}$$

By Slutsky's thm,

$$\frac{1}{\sqrt{n}} S_n(\beta_0) + o_p(1) \xrightarrow{d} N\left(\underset{\sim}{0},\ \frac{\partial \mu_i}{\partial \beta}\left(\frac{1}{V(\mu_i)}\right)\frac{\partial \mu_i^T}{\partial \beta}\right)$$

Instead: $\left[\frac{\partial}{\partial \beta}\left(\frac{\partial \mu_i}{\partial \beta}\frac{1}{V(\mu_i)}\right)(y_i-\mu_i) + \frac{\partial \mu_i}{\partial \beta}\left(\frac{1}{V(\mu_i)}\right)\frac{\partial \mu_i}{\partial \beta}\right]$

$$\frac{1}{n}\frac{\partial}{\partial \beta} S_n(\beta) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\partial^2 \mu_i}{\partial \beta^2}\frac{(y_i-\mu_i)}{V(\mu_i)} + \frac{\partial \mu_i}{\partial \beta}\frac{\partial}{\partial \beta}\left(\frac{(y_i-\mu_i)}{V(\mu_i)}\right)\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\frac{\partial^2 \mu_i}{\partial \beta^2}\frac{(y_i-\mu_i)}{V(\mu_i)} + \frac{1}{n}\sum_{i=1}^{n}\frac{\partial \mu_i}{\partial \beta}\left(\frac{V(\mu_i)(\partial \mu_i/\partial \beta) - (y_i-\mu_i)\,\partial V(\mu_i)/\partial \beta}{V(\mu_i)^2}\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\frac{\partial^2 \mu_i}{\partial \beta^2}\frac{(y_i-\mu_i)}{V(\mu_i)} - \frac{1}{n}\sum_{i=1}^{n}\frac{\partial \mu_i}{\partial \beta}\left(\frac{\partial \mu_i^T}{\partial \beta}\right)\frac{1}{V(\mu_i)} + \frac{1}{n}\sum_{i=1}^{n}\frac{\partial V(\mu_i)}{\partial \beta}\frac{(y_i-\mu_i)}{V(\mu_i)^2}$$

By the strong Law of Large #s,

$$\frac{1}{n}\sum_{i=1}^{n} q_i \xrightarrow{a.s.} E[q_i]$$

Since $E[y_i-\mu_i] = 0$

$$\Rightarrow \text{above} \xrightarrow{a.s.} 0 - E\left[\frac{\partial \mu_i}{\partial \beta}\frac{1}{V(\mu_i)}\frac{\partial \mu_i^T}{\partial \beta}\right] + 0$$

Consequently, by the continuous mapping thm,

$$\left[-\frac{1}{n}\frac{\partial}{\partial B} S_n(B_0)\right]^{-1} \xrightarrow{a.s} \left[\frac{\partial \mu_i}{\partial B}\frac{1}{V(\mu_i)}\frac{\partial \mu_i}{\partial B}\right]^{-1}$$

By Slutsky's thm,

$$\left(\frac{1}{\sqrt{n}} S_n(B) + o_p(1)\right)\left[-\frac{1}{n}\frac{\partial}{\partial B} S_n(B_0)\right] \xrightarrow{d}$$

$$N\left(\underset{\sim}{0}, \left[\frac{\partial \mu_i}{\partial B}\frac{1}{V(\mu_i)}\frac{\partial \mu_i^T}{\partial B}\right]^{-1}\left[\frac{\partial \mu_i}{\partial B}\frac{1}{V(\mu_i)}\frac{\partial \mu_i^T}{\partial B}\right]\left[\frac{\partial \mu_i}{\partial B}\frac{1}{V(\mu_i)}\frac{\partial \mu_i^T}{\partial B}\right]^{-1}\right)$$

$$\stackrel{d}{\equiv} N\left(0, \left[\frac{\partial \mu_i}{\partial B}\frac{1}{V(\mu_i)}\frac{\partial \mu_i^T}{\partial B}\right]^{-1}\right)$$

$\Rightarrow$ Asymptotic Cov. Matrix of $\hat{B}_P$ is

$$\left(\frac{\partial \mu_i}{\partial B}\frac{1}{V(\mu_i)}\frac{\partial \mu_i^T}{\partial B}\bigg|_{B=B_0}\right)^{-1}$$

Estimate of Asymp. Cov Matrix:

$$\left[\frac{\partial \mu_i}{\partial B}\frac{1}{V(\mu_i)}\frac{\partial \mu_i^T}{\partial B}\right]^{-1}\Bigg|_{B=\hat{B}_P}.$$

2). Y 4x1 vector

$E(Y) = \mu \quad \mu \in E$

$$E = \left\{ \mu : \mu = \begin{bmatrix} B_1 + B_2 - B_3 \\ B_2 + B_3 \\ -B_2 - B_3 \\ -B_1 - B_2 + B_3 \end{bmatrix} \right\}$$

$Cov(Y) = \sigma^2 I_{4\times4} \quad \sigma^2$ unknown

(a) Derive $\hat{\mu}$, the LSE of μ

$$E(Y) = XB = \begin{bmatrix} B_1 + B_2 - B_3 \\ B_2 + B_3 \\ -B_2 - B_3 \\ -B_1 - B_2 + B_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

$\mu = \Lambda' B = XB = IXB \quad (P' = I)$

Clearly, $\Lambda \in C(X')$ since $\Lambda' = IX \Rightarrow \lambda = X'$

$\Rightarrow (\Lambda'\hat{B}$ estimable ✓)

LSE of $\Lambda' B = P'MY = MY$

since $P' = I$

$M = X(X'X)^{-}X'$

$= X^*(X^{*\prime}X^*)^{-1}X^{*\prime}$

where X^* = linearly indep columns of X.

$C(X) = \text{rowspace } X'$

$$X' = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix} \xrightarrow[-R_1+R_2]{R_1+R_3} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \xrightarrow{-R_2+R_3}$$

$$= \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rows 1 & 2 of X' linearly indep

$\Rightarrow$ Columns 1 & 2 of X linearly indep.

$$X^* = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \\ -1 & -1 \end{bmatrix}$$

$$X^{*\prime} X^* = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}$$

$$(X^{*\prime} X^*)^{-1} = \frac{1}{8-4} \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$X^*(X^{*\prime} X^*)^{-1} X^{*\prime}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \\ -1 & -1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\Lambda' B = MY = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} Y_1 - Y_4 \\ Y_2 - Y_3 \\ -Y_2 + Y_3 \\ -Y_1 + Y_4 \end{bmatrix}$$

(b) Find the BLUE of $\beta_2 - \beta_3$ or show that it is non-estimable.

Estimable if $\lambda'\beta = \rho' E(Y) = \rho' X \beta$

$\Rightarrow \lambda' = \rho' X$

$\Rightarrow \lambda = X'\rho$

Need $\lambda \in C(X') \Leftrightarrow \lambda \in$ rowspace X

$$X = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \xrightarrow[R_2+R_3]{R_1+R_4} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

If $\lambda \in \text{Span}\left\{ \underset{v_1}{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}} \right\}$ then $\lambda'\beta$ estimable

$$\begin{array}{ccc} c_1 & c_2 & \lambda \end{array} \rightarrow c_1 v_1 + c_2 v_2 = \lambda$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{array}\right] \xrightarrow[R_1+R_3]{-R_1+R_2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{array}\right] \xrightarrow{-R_2+R_3} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{array}\right]$$

We have no soln to this system of eqns.
$(0c_1 + 0c_2 \neq -2$ ever$)$

Consequently, $\beta_2 - \beta_3$ not estimable

(c) $H_0: \beta_2 + \beta_3 = 0$ vs. $H_1: \beta_2 + \beta_3 \neq 0$

E_0 = set E assuming H_0 true

Explicitly give sets E_0 & $E \cap E_0^{\perp}$ where

$E_0^{\perp}$ = orthogonal complement of E_0

$$E = \left\{ \mu : \mu = \begin{bmatrix} \beta_1 + \beta_2 - \beta_3 \\ \beta_2 + \beta_3 \\ -\beta_2 + \beta_3 \\ -\beta_1 - \beta_2 + \beta_3 \end{bmatrix} \right\}$$

$$\beta_2 + \beta_3 = 0 \Rightarrow -\beta_2 - \beta_3 = 0$$

$$\beta_1 + \beta_2 - \beta_3 = \beta_1 + \underbrace{\beta_2 + \beta_3}_{0} - 2\beta_3 = \beta_1 - 2\beta_3$$

$$-\beta_1 - \beta_2 + \beta_3 = -(\beta_1 + \beta_2 - \beta_3) = -\beta_1 + 2\beta_3$$

$$E_0 = \left\{ \mu : \mu = \begin{bmatrix} \beta_1 - 2\beta_3 \\ 0 \\ 0 \\ -\beta_1 + 2\beta_3 \end{bmatrix} \right\}$$

$$\equiv \left\{ \mu : \mu = (\beta_1 - 2\beta_3) \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\}$$

$$= \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\}$$

$E \cap E_0^{\perp}$ = span of vectors in E that are orthogonal to $E_0 = \text{Span}\{[1\ 0\ 0\ {-1}]^T\}$

$$E = C(X) = \left\{ (\beta_1 + \beta_2 - \beta_3)\begin{bmatrix}1\\0\\0\\-1\end{bmatrix} + (\beta_2 + \beta_3)\begin{bmatrix}0\\1\\-1\\0\end{bmatrix} \right\}$$

$$= \text{Span}\left\{ \begin{bmatrix}1\\0\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\\0\end{bmatrix} \right\}$$

Note: $\begin{bmatrix}1 & 0 & 0 & -1\end{bmatrix}\begin{bmatrix}0\\1\\-1\\0\end{bmatrix} = 0$

$$\Rightarrow E \cap E_0^{\perp} = \text{Span}\left\{ \begin{bmatrix}0\\1\\-1\\0\end{bmatrix} \right\}$$ ✓

(d) $Y \sim$ Normal

Construct the simplest possible expression for the F statistic for the hypothesis

$H_0: \mu \in E_0$ vs. $H_1: \mu \notin E_0$

(E_0 specified in (c))

- Give the dist of the F statistic under null & alternative hypotheses.

$$F = \frac{Y'(M-M_0)Y / r(M-M_0)}{Y'(I-M)Y / r(M)}$$

$M_0 = X_0(X_0^T X_0)^{-1} X_0^T$

By thm, $X_0 = M - M_{mp}$

Note: Want to accept $H_0: Y \in C(X_0)$ if

Error $Y \in C(X) \cap C(X_0)^\perp$ is as small as possible

$\Rightarrow Y'(M-M_0)Y / r(M-M_0)$ smaller than total

error $Y'(I-M)Y/r(M)$

$M - M_0 =$ orthog proj. operator onto $E \cap E_0^\perp$

$$= \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \Bigg(\underbrace{\begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}}_{2} \Bigg)^{-1} \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M = X(X'X)^{-}X^{T}$$

Find linearly indep columns of $X \Rightarrow E$

$$= X^{*}(X^{*\prime}X^{*})^{-1}X^{*T}$$

$$X^{*} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$X^{*\prime}X^{*} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$(X^{*\prime}X^{*})^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$X^{*}(X^{*\prime}X^{*})^{-1}X^{*\prime}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

$$Y'(M-M_0)Y = ((M-M_0)Y)^{T}((M-M_0)Y)$$

$$(M-M_0)Y = \frac{1}{2}\begin{bmatrix} 0 \\ Y_2 - Y_3 \\ -Y_2 + Y_3 \\ 0 \end{bmatrix}$$

$$= \frac{1}{4}\left((Y_2 - Y_3)^2 + (-1)^2(Y_2 - Y_3)^2\right)$$

$$= \frac{1}{2}(Y_2 - Y_3)^2$$

$$(I-M)Y = \frac{1}{2}\begin{bmatrix} y_1 - y_4 \\ y_2 - y_3 \\ -(y_2-y_3) \\ -(y_1-y_4) \end{bmatrix}$$

$$Y'(I-M)Y = \frac{1}{4}\left[(y_1-y_4)^2 + (y_2-y_3)^2 + (y_2-y_3)^2 + (y_1-y_4)^2\right]$$
$$= \frac{1}{2}\left[(y_1-y_4)^2 + (y_2-y_3)^2\right]$$

$$r(M-M_0) = r(E \cap E_0^{\perp}) = 1$$
$$r(M) = r(E) = 2$$

$$F = \frac{\cancel{\frac{1}{2}}[y_2-y_3]^2}{\frac{1}{2}\left[(y_1-y_4)^2+(y_2-y_3)^2\right]\left(\frac{1}{2}\right)}$$
$$= \frac{2(y_2-y_3)^2}{(y_1-y_4)^2+(y_2-y_3)^2}$$

Under H_0: $F \sim F(1,2)$

Under H_A: $F \sim F(1,2,\delta)$

$$\delta = \frac{E(Y)'(M-M_0)E(Y)}{2\sigma^2} \quad \text{(noncentrality parameter)}$$
$$= \frac{\beta' X'(M-M_0)X\beta}{2\sigma^2}$$

* δ = Expectation of the numerator form $Y'(M-M_0)Y$ divided by $2\sigma^2$

$\longrightarrow$

$$(M-M_0)XB = \frac{1}{2}\begin{bmatrix}0&0&0&0\\0&1&-1&0\\0&-1&1&0\\0&0&0&0\end{bmatrix} \begin{array}{c} XB \\ \begin{bmatrix} B_1+B_2-B_3\\ B_2+B_3 \\ -(B_2+B_3) \\ -(B_1+B_2-B_3)\end{bmatrix}\end{array}$$

$$= \frac{1}{2}\begin{bmatrix} 0 \\ 2(B_2+B_3) \\ -2(B_2+B_3) \\ 0 \end{bmatrix}$$

$$B'X'(M-M_0)XB = \frac{1}{4}(4)(B_2+B_3)^2(2)$$
$$= 2(B_2+B_3)^2$$

$$\Rightarrow \delta = \frac{(B_2+B_3)^2}{\sigma^2}$$

(e) Assuming normality for Y, construct an exact 95% CI for $\beta_2 + \beta_3$.

F statistic:

$$\frac{(\lambda'\hat{\beta} - \lambda'\beta_0)^T \rho' M \rho (\lambda'\hat{\beta} - \lambda'\beta_0)}{\hat{\sigma}^2 / n-p}$$

$$\lambda'\beta = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

Since $\lambda \in \text{Span}\left(\left\{ \overset{v_1}{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}, \overset{v_2}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}} \right\} \right)$

$\lambda'\beta$ is estimable because

$\lambda = 0 v_1 + 1 v_2$ ✓

$$\lambda' = \rho' X$$
$$= \begin{bmatrix} \rho_1 & \rho_2 & \rho_3 & \rho_4 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \rho_1 - \rho_4, & \rho_1 + \rho_2 - \rho_3 - \rho_4, & -\rho_1 + \rho_2 - \rho_3 + \rho_4 \end{bmatrix}$$

$$\begin{array}{c} \quad \rho_1 \quad \rho_2 \quad \rho_3 \quad \rho_4 \\ = \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 \end{array}\right] \end{array} \xrightarrow[R_1+R_3]{-R_1+R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{-R_2+R_3}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Need $\rho_1 - \rho_4 = 0$

$\rho_2 - \rho_3 = 1$

Possible $\rho = [1 \quad 1 \quad 0 \quad 1]$

$$\lambda'\hat{\beta} = \rho' M Y$$

From part ⓐ, $MY = \frac{1}{2}\begin{bmatrix} Y_1 - Y_4 \\ Y_2 - Y_3 \\ -Y_2 + Y_3 \\ -Y_1 + Y_4 \end{bmatrix}$

$$\rho' M Y = \frac{1}{2}\left[\cancel{Y_1} - \cancel{Y_4} + Y_2 - Y_3 - \cancel{Y_1} + \cancel{Y_4}\right]$$
$$= \frac{1}{2}\left[Y_2 - Y_3\right]$$

Note: $M_0 = M - M_{MP}$

$$\Rightarrow M - M_0 = M - (M - M_{MP}) = M_{MP}$$
$$= (MP)\left((MP)'(MP)\right)^{-1}(MP)'$$
$$= MP(P'MP)^{-}P'M$$

$$Y' M_{MP} Y = \underbrace{Y'MP}_{(\widehat{\lambda'\beta})'}(P'MP)^{-}\underbrace{P'MY}_{\widehat{\lambda'\beta}}$$
$$= (\widehat{\lambda'\beta})'(P'MP)^{-}(\widehat{\lambda'\beta})$$

Therefore,

$$F = \frac{(\widehat{\lambda'\beta})'(P'MP)^{-}(\widehat{\lambda'\beta}) / \overbrace{r(\Lambda)}^{=1}}{Y'(I-M)Y / \underbrace{r(M)}_{=2}}$$

Consequently, a 95% CI for $\lambda'\beta$ is:

$$\frac{(\overbrace{\widehat{\lambda'\beta}}^{P'MY} - \lambda'\beta)'(P'MP)^{-}(\overbrace{\widehat{\lambda'\beta}}^{P'MY} - \lambda'\beta)}{\frac{1}{4}\left[(Y_1 - Y_4)^2 + (Y_2 - Y_3)^2\right]} \leq F(1, 2, (1-\alpha))$$

... (calculate $P'MP$, plug in $P'MY = \frac{1}{2}(Y_2 - Y_3)$ ...)

3). $y_i = B^T x_i + e_i$

$e_i \sim N(0, \Sigma_R)$

$w^T y_i = w^T B^T x_i + w^T e_i = B_w^T x_i + \varepsilon_i$

$w = q \times 1, \quad w^T w = 1$ (fixed)

MLE of B_w?

$\varepsilon_i = w^T e_i \sim N(w^T \underset{\sim}{0}, w^T \Sigma_R w)$

$\overset{d}{=} N(0, \sigma^2)$

where $\sigma^2 = w^T \Sigma_R w$

since Σ_R positive definite, $w^T \Sigma_R w > 0$ ✓

$\underbrace{w^T y_i}_{y_{wi}} \sim N(B_w^T x_i, \sigma^2)$

$$L(y_w) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2\sigma^2}(y_{wi} - B_w^T x_i)^2\right)$$

$$\Rightarrow \ell(y_w) \propto -\frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_{wi} - B_w^T x_i)^2$$

$$\frac{\partial}{\partial B_w}\ell(y_w) = -\frac{1}{\cancel{2}\sigma^2}\sum_{i=1}^{n}(\cancel{2})(y_{wi} - \overbrace{B_w^T x_i}^{x_i^T B_w})(-x_i^T) \overset{set}{=} 0$$

$$\Rightarrow \sum_{i=1}^{n} y_{wi} \underset{\sim}{x}_i^T - B_w \sum_{i=1}^{n} \underset{\sim}{x}_i^T \underset{\sim}{x}_i \overset{set}{=} 0$$

$$\Rightarrow \quad B_w = \frac{\sum_{i=1}^{n} y_{wi} \underset{\sim}{x}_i}{\sum_{i=1}^{n} \underset{\sim}{x}_i^T \underset{\sim}{x}_i}$$

$$\Rightarrow \quad \boxed{\hat{B}_w = \frac{\sum_{i=1}^{n} w^T y_i \underset{\sim}{x}_i}{\sum_{i=1}^{n}\sum_{j=1}^{p} x_{ij}^2}}$$

Dist of $\hat{B}_w$:

$$w^T y_i \sim N(x_i^T B_w, \sigma^2)$$

$$w^T y_i \underset{\sim}{x}_i \sim N(x_i^T B_w x_i, \sigma^2 \underset{\sim}{x}_i \underset{\sim}{x}_i^T)$$

$$\Rightarrow \frac{\sum_{i=1}^n w^T y_i \underset{\sim}{x}_i}{\sum_{i=1}^n \sum_{j=1}^p x_{ij}^2} \sim N\left(\frac{\sum_{i=1}^n B_w x_i^T x_i}{\sum_{i=1}^n x_i^T x_i}, \frac{\sigma^2 x_i x_i^T}{(\sum_{i=1}^n x_i^T x_i)^2}\right)$$

$$\overset{d}{=} N\left(B_w, \frac{\sigma^2 x_i \underset{\sim}{x}_i^T}{(\sum_{i=1}^n \underset{\sim}{x}_i^T \underset{\sim}{x}_i)^2}\right)$$

(b) $CB = B_0 \Rightarrow CB\underbrace{w^Tw}_{1} = B_0\underbrace{w^Tw}_{1} \Rightarrow CBw = B_0w$

$\Rightarrow CB_w = b_0$

Similarly,

$CB_w \neq b_0 \Rightarrow CBw \neq B_0w \Rightarrow CB \neq B_0$

(c)