2010 Theory I #3

(a)

$$e^x - 1 - x = \left\{1 + x + \frac{x^2}{2} + o(x^2)\right\} - 1 - x = \frac{x^2}{2} + o(x^2)$$

$\bar{X}_n \in O_p(n^{-1/2})$, $\bar{X}_n \in o_p(1)$

(b) (i) $e^{\bar{X}_n} - 1 - \bar{X}_n = \left\{1 + \bar{X}_n + O_p(n^{-1/2})\right\} - 1 - \bar{X}_n = O_p(n^{-1/2}) = o_p(1)$

(ii) $e^{\bar{X}_n} - \bar{X}_n - 1 = \left\{1 + \bar{X}_n + o_p(\bar{X}_n)\right\} - 1 - \bar{X}_n = o_p(\bar{X}_n)$

So by Slutsky thm. / CMT

$$\frac{e^{\bar{X}_n} - \bar{X}_n - 1}{\bar{X}_n} = \frac{o_p(\bar{X}_n)}{\bar{X}_n} = o_p(1)$$

(iii) $e^{\bar{X}_n} - \bar{X}_n - 1 = \left\{1 + \bar{X}_n + \frac{\bar{X}_n^2}{2} + o_p(\bar{X}_n^2)\right\} - \bar{X}_1 - 1$

$$= \frac{\bar{X}_n^2}{2} + o_p(\bar{X}_n^2)$$

So by Slutsky's Thm/ CMT $\quad \frac{e^{\bar{X}_n} - \bar{X}_n - 1}{\bar{X}_n^2} = \frac{\bar{X}_n^2/2 + o_p(\bar{X}_n^2)}{\bar{X}_n^2} = \frac{1}{2} + o_p(1)$

(c) $\frac{2n}{S_n^2}\left(e^{\bar{X}_n} - 1 - \bar{X}_n\right) = \frac{n\bar{X}_n^2}{\sigma^2} \cdot \frac{\sigma^2}{S_n^2} \cdot \frac{2(e^{\bar{X}_n} - 1 - \bar{X}_n)}{\bar{X}_n^2}$

By CLT

$$\sqrt{n}\,\bar{X}_n \xrightarrow{L} N(0, \sigma^2) \overset{CMT}{\Longrightarrow} \frac{\sqrt{n}\,\bar{X}_n}{\sigma} \xrightarrow{L} N(0,1) \overset{CMT}{\Longrightarrow} \frac{n\bar{X}_n^2}{\sigma^2} \xrightarrow{L} \chi_1^2$$

$\tilde{S}_n^2 \equiv \frac{1}{n}\sum(X_i - \bar{X}_n)^2 = \frac{1}{n}\sum X_i^2 - \bar{X}_n^2.$

Since $EX_i^2 = \sigma^2 \overset{SLLN}{\Rightarrow} \frac{1}{n}\sum X_i^2 \xrightarrow{P} \sigma^2$

Since $E\bar{X}_n = 0 \overset{SLLN}{\Rightarrow} \bar{X}_n \xrightarrow{P} 0 \overset{CMT}{\Rightarrow} \bar{X}_n^2 \xrightarrow{P} 0$

So by Slutsky's / CMT $\tilde{S}_n^2 \to \sigma^2$. Finally, since $\frac{n-1}{n} \to 1$

by Slutsky's/CMT we obtain $S_n^2 = \frac{n-1}{n}\tilde{S}_n^2 \xrightarrow{P} 1 \cdot \sigma^2 = \sigma^2$

From part (b) we obtain $\frac{2(e^{\bar{X}_n} - 1 - \bar{X}_n)}{\bar{X}_n^2} \xrightarrow{P} 1$

~~Thus~~ Notice that convergence in ~~law~~ prob. $\Rightarrow$ convergence in law, Thus

By Slutsky's/CMT

$$\frac{2n}{S_n^2}(e^{\bar{X}_n} - 1 - \bar{X}_n) = \frac{n\bar{X}_n^2}{\sigma^2} \cdot \frac{\sigma^2}{S_n^2} \cdot \frac{2(e^{\bar{X}_n} - 1 - \bar{X}_n)}{\bar{X}_n^2}$$

$$\xrightarrow{L} \chi_1^2 \cdot 1 \cdot 1 \overset{d}{=} \chi_1^2$$

(d)

$$\frac{2\sqrt{n}}{S_n} \cdot \frac{e^{\bar{X}_n} - 1 - \bar{X}_n}{\bar{X}_n} = \frac{\sqrt{n}\,\bar{X}_n}{\sigma} \cdot \frac{\sigma}{S_n} \cdot \frac{2(e^{\bar{X}_n} - 1 - X_n)}{\bar{X}_n^2}$$

$$\xrightarrow{L} N(0,1) \cdot 1 \cdot 1 \overset{d}{=} N(0,1)$$

by same arguments as (c)

(e)

SLLN, $\bar{X}_n \xrightarrow{P} 0 \overset{CMT}{\Longrightarrow} \tan \bar{X}_n \xrightarrow{P} \tan(0) = 0$

~~Then by (c) and Slutsky's / CMT~~

~~$\frac{2n}{S_n^2}(e^{\bar{X}_n} - 1 - \bar{X}_n) \tan \bar{X}_n \xrightarrow{P}$~~

~~From (c) we saw that $S_n^2 \xrightarrow{P} \sigma^2$ and from (a) we saw that~~ Now from (c),

$$\frac{2n}{S_n^2}(e^{\bar{X}_n} - 1 - \bar{X}_n) \in O_p(1)$$

so that

$$\frac{2n}{S_n^2}(e^{\bar{X}_n} - 1 - \bar{X}_n) \tan \bar{X}_n = O_p(1)\, o_p(1) = o_p(1 \cdot 1) = o_p(1)$$

(f) From (e) $\tan \bar{X}_n \xrightarrow{P} 0$ and by SLLN, $\bar{X}_n \xrightarrow{P} 0$.

~~Then by CMT~~ Let $g(x) = \begin{cases} \frac{\tan(x)}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$. Then

~~by CMT $g(\bar{X}_n) \to 1$~~ Now $\frac{\tan(0)}{0} = \frac{0}{0}$ so by

L'Hospital's rule ~~$\frac{\tan(0)}{0} \overset{L.H.}{=} \sec^2(0)$~~

$$\lim_{x \to 0} \frac{\tan(x)}{x} \overset{L.H.}{=} \lim_{x \to 0} \frac{\sec^2(x)}{1} = \frac{1}{1} = 1$$

so by CMT

$$\frac{\tan \bar{X}_n}{\bar{X}_n} \xrightarrow{P} 1 \quad \text{and hence} \quad \frac{2\sqrt{n}}{S_n}\left(\frac{e^{\bar{X}_n} - 1 - \bar{X}_n}{\bar{X}_n}\right) \tan \bar{X}_n \xrightarrow{L} N(0,1) \cdot 1$$