BASIC PHD WRITTEN EXAMINATION

THEORY, SECTION 1

(9:00 AM–1:00 PM, July 26, 2017)

INSTRUCTIONS:

(a) This is a **CLOSED-BOOK** examination.

(b) The time limit for this examination is four hours.

(c) Answer both questions that follow.

(d) Put the answers to different questions on separate sets of paper.

(e) Put your exam code, **NOT YOUR NAME**, on each page. The same code is used for Section 1 and Section 2 of the PhD Theory Exam. Please keep the code confidential and do not share this information with any students or faculty. Sharing your code with either students or faculty is viewed as a violation of the UNC honor code.

(f) Return the examination with a signed statement of the UNC honor pledge, separately from your answers. The pledge statement is given on the last page of the exam handout.

(g) In the questions to follow, you are required to answer only what is asked, and not to tell all you know about the topics involved.

1. (25 points) Let N be Poisson distributed with parameter $0 < \lambda < \infty$, and let $X_1, X_2, \ldots$ be an i.i.d. sequence of positive random variables, independent of N, with $E \log(X_1) = \mu$, $\text{var}[\log(X_1)] = \sigma^2$, $|\mu| < \infty$, $0 < \sigma^2 < \infty$, and $M(\delta) = EX_1^\delta < \infty$ for some $\delta > 0$. Let $Y = \prod_{i=1}^N X_i$, where $\prod_{i=1}^0$ is defined as 1. Do the following:

   (a) (4 points) Show that $E \log Y = \lambda\mu$ and $\text{var}[\log Y] = \lambda(\sigma^2 + \mu^2)$.

   (b) (5 points) Show that $EY^t = \exp(\lambda[M(t) - 1])$, for all $0 \leq t \leq \delta$.

   (c) (7 points) Show that $Y^{1/\lambda} \to_p e^\mu$, as $\lambda \to \infty$.

   (d) (9 points) Letting $\tau^2 = \lambda(\sigma^2 + \mu^2)$, show that

$$\left(e^{-\lambda\mu} Y\right)^{1/\tau} \to_d e^Z,$$

   as $\lambda \to \infty$, where $Z \sim N(0, 1)$.

2. (25 points) Let F and G be two distinct known cumulative distribution functions on the real line and X be a single observation from the cumulative distribution function $\theta F(x) + (1-\theta)G(x)$, where $\theta \in [0,1]$ is unknown.

   (a) (4 points) Given $0 < \theta_0 < 1$, derive a Uniformly Most Powerful (UMP) test of size α for testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$. You need to specify how the rejection region can be calculated .

   (b) (6 points) Given $0 < \theta_1 < \theta_2 < 1$, derive a UMP test of size α for testing $H_0 : \theta \in [0, \theta_1] \cup [\theta_2, 1]$ versus $H_1 : \theta \in (\theta_1, \theta_2)$.

   (c) (6 points) Show that a UMP test does not exist for testing $H_0 : \theta \in [\theta_1, \theta_2]$ versus $\theta \notin [\theta_1, \theta_2]$.

   (d) (5 points) Obtain a Uniformly Most Powerful Unbiased (UMPU) test of size α for the problem in part (c).

   (e) (4 points) Given $0 < \theta_1 < \theta_2 < 1$, derive the likelihood ratio test statistic for testing $H_0 : \theta \in [\theta_1, \theta_2]$ versus $\theta \notin [\theta_1, \theta_2]$.

Statement of the UNC honor pledge:

> *"In recognition of and in the spirit of the honor code, I certify that I have neither given nor received aid on this examination and that I will report all Honor Code violations observed by me."*

(Signed) ____________________________________
NAME

(Printed) ____________________________________
NAME

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Q 1.

$N \sim Poi(\lambda)$. $X_i > 0$. $E \log X_1 = \mu$, $Var \log X_1 = \sigma^2$. $M(\delta) = E X_1^\delta < \infty$, $\exists \delta > 0$. $Y = \prod_{i=1}^N X_i$

(a) $\log Y = \sum_{i=1}^N \log X_i$

$$E(\log Y | N=n) = E\left(\sum_{i=1}^n \log X_i \mid N=n\right)$$
$$= E\left(\sum_{i=1}^n \log X_i\right) \quad (\because)\ X_1, \dots, X_n \perp\!\!\!\perp N$$
$$= \sum_{i=1}^n E \log X_i$$
$$= n\mu$$

$$\Rightarrow E(\log Y) = E[E(\log Y|N)] = E[N\mu] = \mu \cdot EN = \lambda\mu.$$

$$Var(\log Y | N=n) = Var\left(\sum_{i=1}^n \log X_i \mid N=n\right)$$
$$= Var\left(\sum_{i=1}^n \log X_i\right)$$
$$= \sum_{i=1}^n Var(\log X_i)$$
$$= n\sigma^2$$

$$\Rightarrow Var(\log Y) = E[Var(\log Y|N)] + Var(E[\log Y|N])$$
$$= E[N\sigma^2] + Var(N\mu)$$
$$= \lambda\sigma^2 + \mu^2\lambda = \lambda(\sigma^2 + \mu^2)$$

(b) $$E(Y^t | N=n) = E\left(\prod_{i=1}^n X_i^t \mid N=n\right)$$
$$= E\left(\prod_{i=1}^n X_i^t\right)$$
$$= \prod_{i=1}^n E X_i^t$$

Here, Hölder inequality : $\left[\int_{\mathbb{R}} f(x)^p dP(x)\right]^{\frac{1}{p}} \left[\int_{\mathbb{R}} g(x)^q dP(x)\right]^{\frac{1}{q}} \geq \left[\int_{\mathbb{R}} f(x) g(x)\, dP(x)\right]$

$\frac{1}{p} + \frac{1}{q} = 1$. Assume $0 < t < \delta$. Then, $p \leftarrow \frac{\delta}{t}$, $q \leftarrow (1 - \frac{t}{\delta})^{-1}$, $f \leftarrow x^t$, $g \leftarrow 1$

$$\left[\int_{\mathbb{R}} x^\delta dP(x)\right]^{\frac{t}{\delta}} \geq \left[\int_{\mathbb{R}} x^t dP(x)\right]$$

$$\Rightarrow (E X^\delta)^{\frac{1}{\delta}} \geq (EX^t)^{\frac{1}{t}}$$

Thus, $EX_1^\delta < \infty \Rightarrow EX^t < \infty$ for $0 < t < \delta$. When $t = 0, \delta$ also holds.

Then, $E(Y^t | N=n) = \prod_{i=1}^n M(t) = M(t)^n$

then, $EY^t = E[E(Y^t|N)] = E\left(M(t)^N\right) = \sum_{n=0}^\infty M(t)^n \cdot \frac{e^{-\lambda}\lambda^n}{n!} = \sum_{n=0}^\infty e^{-\lambda} \cdot \frac{(M(t)\cdot\lambda)^n}{n!} = e^{-\lambda} \cdot e^{M(t)\cdot\lambda}$

(c) ⟨WTS⟩ $\frac{1}{\lambda}\log Y \to_p \mu$ by Continuous Mapping Thm.

Lemma) $\frac{N}{\lambda} \to_p 1$. as $\lambda \to \infty$

(pf)

Let $\varepsilon > 0$. $P(|\frac{N}{\lambda}-1|>\varepsilon) = P(|N-\lambda| > \lambda\varepsilon) \le \frac{\mathbb{E}|N-\lambda|^2}{(\lambda\varepsilon)^2}$ Chebyshev

$$= \frac{Var(N)}{\lambda^2\varepsilon^2} = \frac{1}{\lambda\varepsilon^2} \xrightarrow[\lambda\to\infty]{} 0.$$

⟨WTS⟩ $\frac{1}{N}\log Y \to_p \mu$ $(\because)$ Slutsky : $\frac{N}{\lambda}\cdot\frac{1}{N}\log Y \to_p 1\cdot\mu = \mu$ will hold if it is true.

⟨WTS⟩ $P(|\frac{1}{N}\log Y - \mu| > \varepsilon) \xrightarrow[\lambda\to\infty]{} 0, \ \forall \varepsilon > 0$.

Let $\varepsilon > 0$.

$$P(|\tfrac{1}{N}\log Y - \mu| > \varepsilon \mid N=n) = P(|\tfrac{1}{n}\sum_{i=1}^{n}\log X_i - \mu| > \varepsilon \mid N=n)$$

$$= P(|\tfrac{1}{n}\sum_{i=1}^{n}\log X_i - \mu| > \varepsilon) = p_n$$

As $var[\log(X_1)] = \sigma^2 < \infty$,

WLLN : $\frac{1}{n}\sum_{i=1}^{n}\log X_i \to_p \mathbb{E}[\log X_1] = \mu$. as $n\to\infty$

$$\Rightarrow P(|\tfrac{1}{n}\sum_{i=1}^{n}\log X_i - \mu| > \varepsilon) \xrightarrow[n\to\infty]{} 0.$$

Thus, $p_n \xrightarrow[n\to\infty]{} 0$.

Now, $P(|\frac{1}{N}\log Y - \mu| > \varepsilon) = \mathbb{E}[P(|\frac{1}{N}\log Y - \mu| > \varepsilon \mid N)]$

$$= \sum_{n=0}^{\infty} P(|\tfrac{1}{n}\log Y - \mu| > \varepsilon \mid N=n)\cdot P(N=n)$$

$$= \sum_{n=0}^{\infty} p_n \cdot \frac{e^{-\lambda}\lambda^n}{n!}$$

Now, let $\psi > 0$.

Since $p_n \xrightarrow[n\to\infty]{} 0$, $\exists M$ s.t $n > M \Rightarrow p_n < \psi$

Then, $\sum_{n=0}^{\infty} p_n \cdot \frac{e^{-\lambda}\lambda^n}{n!} = \sum_{n=0}^{M} p_n \cdot \frac{e^{-\lambda}\lambda^n}{n!} + \sum_{n=M+1}^{\infty} p_n \cdot \frac{e^{-\lambda}\lambda^n}{n!}$

$$\le \sum_{n=0}^{M} 1\cdot\frac{e^{-\lambda}\lambda^n}{n!} + \sum_{n=M+1}^{\infty}\psi\cdot\frac{e^{-\lambda}\lambda^n}{n!}$$

$$= e^{-\lambda}\left(\sum_{n=0}^{M}\frac{1}{n!}\lambda^n\right) + \psi\left(1 - e^{-\lambda}\cdot\sum_{n=0}^{M}\frac{1}{n!}\lambda^n\right)$$

$$\xrightarrow[\lambda\to\infty]{} \quad 0 \quad + \psi(1-0) = \psi \quad (\because)\ \frac{Poly(\lambda)}{e^{\lambda}} \xrightarrow[\lambda\to\infty]{} 0$$

It implies that $\overline{\lim_{\lambda\to\infty}} \sum_{n=0}^{\infty} p_n \cdot \frac{e^{-\lambda}\lambda^n}{n!} \le \psi, \ \forall \psi > 0$, thus $0 \le \overline{\lim_{\lambda\to\infty}} \sum_{n=0}^{\infty} p_n \cdot \frac{e^{-\lambda}\lambda^n}{n!} \le 0$

$\therefore \sum_{n=0}^{\infty} p_n \cdot \frac{e^{-\lambda}\lambda^n}{n!} \xrightarrow[\lambda\to\infty]{} 0$.

$\boxed{\text{Chebyshev}}$ "~~Easy~~ solution"

$$P(|\tfrac{1}{\lambda}\log Y - \mu| > \varepsilon)$$

$$= P(|\log Y - \lambda\mu| > \lambda\varepsilon)$$

$$\le \frac{\mathbb{E}|\log Y - \lambda\mu|^2}{(\lambda\varepsilon)^2} = \frac{Var(\log Y)}{\lambda^2\varepsilon^2}$$

$$= \frac{\lambda(\sigma^2+\mu^2)}{\lambda^2\varepsilon^2}$$

$$\xrightarrow[\lambda\to\infty]{} 0$$

(d) ⟨WTS⟩ $\frac{1}{\tau}(-\lambda\mu+\log Y) \xrightarrow{d} N(0,1)$ as $\lambda\to\infty$ by CMT.

⟨WTS⟩ $\log \mathbb{E}\exp\left(t\cdot\frac{1}{\tau}(-\lambda\mu+\log Y)\right) \longrightarrow \frac{1}{2}t^2$ as $\lambda\to\infty$ by 1-1 property of mgf and dist^n

$$\mathbb{E}\left[\exp\left(t\cdot\frac{1}{\tau}(-\lambda\mu+\log Y)\right)\middle| N=n\right] = \mathbb{E}\left[\exp\left(-\frac{\lambda\mu}{\tau}t\right)\cdot\exp\left(\frac{t}{\tau}\sum_{i=1}^{n}\log X_i\right)\middle| N=n\right]$$

$$= \exp\left(-\frac{\lambda\mu}{\tau}t\right)\cdot\mathbb{E}\exp\left(\frac{t}{\tau}\sum_{i=1}^{n}\log X_i\right)$$

$$= \exp\left(-\frac{\lambda\mu}{\tau}t\right)\cdot\left\{\mathbb{E}\exp\left(\frac{t}{\tau}\cdot\log X_1\right)\right\}^n$$

Let $Z_1=\log X_1$.

$$= \quad '' \quad \cdot\left\{M_{Z_1}\left(\frac{t}{\tau}\right)\right\}^n$$

Then, $\mathbb{E}\left[\exp\left(t\cdot\frac{1}{\tau}(-\lambda\mu+\log Y)\right)\right] = \sum_{n=0}^{\infty}\exp\left(-\frac{\lambda\mu}{\tau}t\right)\cdot\left\{M_{Z_1}\left(\frac{t}{\tau}\right)\right\}^n\cdot\frac{e^{-\lambda}\lambda^n}{n!}$

$$= \exp\left(-\frac{\lambda\mu}{\tau}t-\lambda\right)\cdot\exp\left(\lambda M_{Z_1}\left(\frac{t}{\tau}\right)\right)$$

Since $\lim_{x\to\infty}\frac{(\log x)^k}{x^\delta}=0$, $\forall k\in\mathbb{N}$, $\exists M$ s.t $x>M \Rightarrow \frac{(\log x)^k}{x^\delta}<1$.

$$\Rightarrow \mathbb{E}Z_1^k = \mathbb{E}Z_1^k I(X_1\le M)+\mathbb{E}Z_1^k I(X_1>M)$$

$$\le (\log M)^k+\mathbb{E}X_1^\delta < \infty, \quad \forall k\in\mathbb{N}.$$

Thus, $M_{Z_1}\left(\frac{t}{\tau}\right) = 1+m_1\frac{t}{\tau}+\frac{m_2}{2!}\left(\frac{t}{\tau}\right)^2+\frac{m_3}{3!}\left(\frac{t}{\tau}\right)^3+\cdots$ $\quad\left(m_k := \mathbb{E}Z_1^k = \mathbb{E}\left((\log X_1)^k\right)\right)$

$$= 1+\mu\frac{t}{\sqrt{\lambda(\mu^2+\sigma^2)}}+\frac{\sigma^2+\mu^2}{2!}\frac{t^2}{\lambda(\mu^2+\sigma^2)}+o(\lambda^{-1})$$

$$\therefore\ \mathbb{E}\left[\exp\left(t\cdot\frac{1}{\tau}(-\lambda\mu+\log Y)\right)\right] = \exp\left(\cancel{-\frac{\lambda\mu}{\sqrt{\lambda}\sqrt{\mu^2+\sigma^2}}t}\cancel{-\lambda}+\cancel{\lambda}+\cancel{\lambda\mu\frac{t}{\sqrt{\lambda(\mu^2+\sigma^2)}}}+\frac{1}{2}t^2+o(1)\right)$$

$$= \exp\left(\frac{1}{2}t^2+o(1)\right)$$

$$= \exp\left(\frac{1}{2}t^2\right)+o(1)$$

$$\therefore \lim_{\lambda\to\infty} \quad '' \quad = \exp\left(\frac{1}{2}t^2\right) \text{ : mgf of } N(0,1).$$

Q2.

(a) Given the dominating measure $F+G$,

$$p_X(x;\theta) = \frac{d}{d(F+G)}\left(\theta F(x) + (1-\theta)G(x)\right) = \theta f(x) + (1-\theta) g(x), \quad f(x) = \frac{dF}{d(F+G)}, \quad g(x) = \frac{dG}{d(F+G)}$$

$$= \theta\left(f(x) - g(x)\right) + g(x)$$

Then, for $\theta_1 < \theta_2$,

$$\frac{p_X(x;\theta_2)}{p_X(x;\theta_1)} = \frac{\theta_2(f(x)-g(x)) + g(x)}{\theta_1(f(x)-g(x)) + g(x)} = \frac{\theta_2\left(\frac{f(x)}{g(x)} - 1\right) + 1}{\theta_1\left(\frac{f(x)}{g(x)} - 1\right) + 1} \quad : \text{MLR in } \frac{f(x)}{g(x)}$$

$\therefore$ UMP α test for $H_0: \theta \le \theta_0$ vs $H_1: \theta > \theta_0$ is

$$\phi(x) = \begin{cases} 1, & \frac{f(x)}{g(x)} < c \\ \gamma, & = c \\ 0, & > c \end{cases}$$

where $P_{\theta_0}\left(\frac{f(X)}{g(X)} < c\right) + \gamma \cdot P_{\theta_0}\left(\frac{f(X)}{g(X)} = c\right) = \alpha$, $\exists c$ & $\exists \gamma \in [0,1]$.

(b) ϕ: arbitrary test of size α

$$\sup_{\theta \in [0,\theta_1] \cup [\theta_2, 1]} \underbrace{\theta \cdot \left(\int \phi dF - \int \phi dG\right) + \int \phi dG}_{\text{linear in } \theta} \le \alpha \Rightarrow \int \phi dF \le \alpha, \int \phi dG \le \alpha \text{ hold}$$

$\Rightarrow$ power at $\theta_* \in (\theta_1, \theta_2)$:

$$E_{\theta_*}\phi(X) = \theta_* \left(\int \phi dF - \int \phi dG\right) + \int \phi dG \le \max\left\{\int \phi dF, \int \phi dG\right\} \le \alpha \text{ hold}$$

$\therefore \psi(x) \equiv \alpha$ is a UMP test of size α

(c) Assume ψ: UMP test $\boxed{\text{of size } \alpha}$ for $H_0: \theta \in [\theta_1, \theta_2]$ vs $H_1: \theta \notin [\theta_1, \theta_2]$.

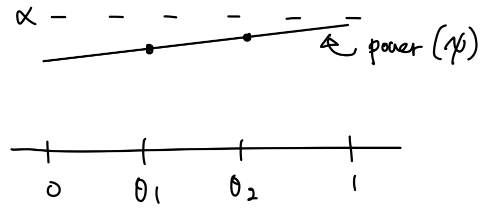
$$\sup_{\theta \in [\theta_1,\theta_2]} \theta \cdot \underbrace{\left(\int \psi dF - \int \psi dG\right)}_{a_1} + \underbrace{\int \psi dG}_{a_2} \le \alpha \Leftrightarrow \begin{array}{l} a_1\theta_1 + a_2 \le \alpha \\ a_1\theta_2 + a_2 \le \alpha. \end{array}$$

For any arbitrary ϕ s.t. $\sup_{\theta \in [\theta_1,\theta_2]} \theta \cdot \overbrace{\left(\int \phi dF - \int \phi dG\right)}^{b_1} + \overbrace{\int \phi dG}^{b_2} \le \alpha$,

$$b_1\theta + b_2 \le a_1\theta + a_2, \ \forall \theta \notin [\theta_1, \theta_2]$$

We know that the power at $\theta = \theta_1, \theta_2$ fully determines a test's power function, which is linear in θ.

Thus, ψ : UMP $\Leftrightarrow$ the line corresponds to the power ft^n of ψ is always upper than other line " " " " " " arbitrary test ϕ.



If both $\mathbb{E}_{\theta=0}\psi$, $\mathbb{E}_{\theta=1}\psi = 0$, then the power ft^n is always 0

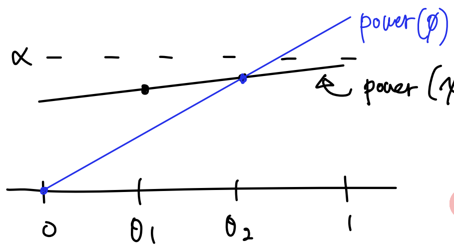
$\Rightarrow$ <u>NOT</u> UMP.

Thus, wlog, $\mathbb{E}_{\theta=0}\psi \neq 0$

Now, consider a test s.t $\mathbb{E}_{\theta=0}\phi = 0$ and $\mathbb{E}_{\theta_2}\phi = \mathbb{E}_{\theta_2}\psi \leq \alpha$.

Then, $\mathbb{E}_{\theta_1}\phi = \frac{\theta_1}{\theta_2}\mathbb{E}_{\theta_2}\phi < \alpha$ holds. $\Rightarrow$ ϕ : size α test.

However, it is obvious that $\mathbb{E}_{\theta=1}\phi > \mathbb{E}_{\theta=1}\psi$ because two line intersects at $\theta=\theta_2$, but power of ϕ has a larger slope.



$\therefore$ ψ : <u>NOT</u> UMP.

Final step is to construct such test ϕ s.t $\mathbb{E}_{\theta=0}\phi = 0$, $\mathbb{E}_{\theta_2}\phi = \mathbb{E}_{\theta_2}\psi \leq \alpha$.

This is done by letting $\leftarrow$ Showing the existence of such ϕ is Really hard ㅠㅠ

Instead, use the fact that $\sup_{\theta\in[\theta_1,\theta_2]} \mathbb{E}_\theta\psi \leq \mathbb{E}_{\theta_3}\psi$, $\forall \theta_3 \notin [\theta_1, \theta_2]$

: property of UMP. ("power $\geq$ size")

It is only true when $\mathbb{E}_{\theta_1}\psi = \mathbb{E}_{\theta_2}\psi$. (zero slope ..)

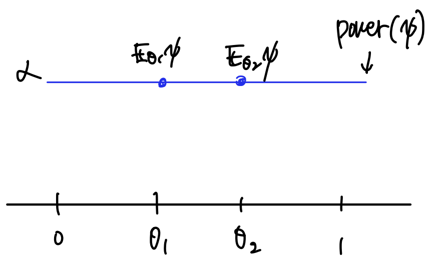
(d) ψ: UMPU test of size α

$\Leftrightarrow$ ① ψ: unbiased. (i.e) $\sup_{\theta\in[\theta_1,\theta_2]} \mathbb{E}_\theta \psi \le \mathbb{E}_{\theta_3}\psi$, $\forall \theta_3 \notin [\theta_1,\theta_2]$

② If ϕ: unbiased, then $\mathbb{E}_{\theta_3}\psi \ge \mathbb{E}_{\theta_3}\phi$, $\forall \theta_3 \notin [\theta_1,\theta_2]$

③ $\sup_{\theta\in[\theta_1,\theta_2]} \mathbb{E}_\theta \psi \le \alpha$

ψ $\boxed{\text{'s power ft}^n}$ is determined by the value of $\mathbb{E}_{\theta_1}\psi$ and $\mathbb{E}_{\theta_2}\psi$. (linear in θ)



Similarly, for any test ϕ, its power ft^n is determined by the value of $\mathbb{E}_{\theta_1}\phi$, $\mathbb{E}_{\theta_2}\phi$.

Note that if ϕ's power ft^n's slope is NOT zero, than $\mathbb{E}_{\theta=0}\phi < \mathbb{E}_{\theta_1}\phi$ (slope >0)

or

$\mathbb{E}_{\theta=1}\phi < \mathbb{E}_{\theta_2}\phi$ (slope <0) occurs

$\Rightarrow$ NOT unbiased.

$\therefore$ ϕ: unbiased $\Leftrightarrow$ slope $=0$ $\Leftrightarrow$ $\mathbb{E}_\theta\phi = \mathbb{E}_{\theta_1}\phi$, $\forall\theta\in[0,1]$

Thus, by setting $\psi(x)=\alpha$, it is a UMPU size α test.

(e) Assume $f(x), g(x)$: density of F, G dist^n

Then X's density is $\theta f(x) + (1-\theta) g(x) =: L(\theta)$

1. Without restriction

$$\sup_{\theta\in[0,1]} L(\theta) = \max\{f(x), g(x)\}$$

2. Under H_0

$$\sup_{\theta\in[\theta_1,\theta_2]} L(\theta) = \begin{cases} \theta_2\cdot f(x) + (1-\theta_2)\cdot g(x) & \text{if } f(x)\ge g(x) \\ \theta_1\cdot f(x) + (1-\theta_1)\cdot g(x) & \text{if } f(x) < g(x)\end{cases}$$

Thus, $$\Lambda = \frac{\sup_{\theta\in[\theta_1,\theta_2]} L(\theta)}{\sup_{\theta\in[0,1]} L(\theta)} = \begin{cases} \dfrac{\theta_2 f + (1-\theta_2) g}{f(x)}, & f(x)\ge g(x) \\ \dfrac{\theta_1 f + (1-\theta_1) g}{g}, & f(x) < g(x)\end{cases}$$

Rejection Region: $\Lambda < c$ $\Leftrightarrow$ i) $f(x)\ge g(x)$: $\theta_2 + (1-\theta_2)\frac{g}{f} < c$ $\Leftrightarrow$ $\frac{g}{f} < k_1$

(Note that $\Lambda\le 1$ always) ii) $f(x) < g(x)$: $\theta_1\frac{f}{g} + (1-\theta_1) < c$ $\Leftrightarrow$ $\frac{f}{g} < k_2$

$\therefore$ Reject H_0 if $k_3 < \frac{g}{f} < k_1$, $\exists k_3 \le 1 \le k_1$.