2) $X_1, \ldots, X_n$ iid with density $f(x) = \theta^{-1} e^{(a-x)/\theta} I(x>a)$; $\theta > 0$

(a) When (a) is known, derive the uniformly most powerful test size α for testing

$H_0: \theta \geq \theta_0$ vs. $\theta < \theta_0$; θ_0 is known constant

Likelihood: $p_\theta(x) = \prod_{i=1}^{n} \theta^{-1} \exp\left(\frac{a-x_i}{\theta}\right) I(x>a) = \theta^{-n} \exp\left\{\sum_{i=1}^{n}\left(\frac{a-x_i}{\theta}\right)\right\} I(x_{(1)} > \overset{\text{known}}{a})$

Let $0 < \theta_0 < \theta_1$

then $\frac{p_{\theta_1}(x)}{p_{\theta_0}(x)} = \frac{\theta_1^{-n} \exp\left\{\sum_{i=1}^{n}(a-x_i)/\theta_1\right\} I(x_{(1)} > a)}{\theta_0^{-n} \exp\left\{\sum_{i=1}^{n}(a-x_i)/\theta_0\right\} I(x_{(1)} > a)} = \left(\frac{\theta_0}{\theta_1}\right)^n \exp\left\{\underbrace{\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)}_{<0} \underbrace{\sum_{i=1}^{n}(a-x_i)}_{\text{If this} > 0 \text{ then fct} \downarrow}\right\} I(x_{(1)}>a)$

So, we can see that this is a nonincreasing $\leftarrow$ opposite H_1
function of $\sum_{i=1}^{n}(a-x_i)$

Thus, our UMP test of size α is

$$\phi(x) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n}(a-x_i) > k \\ 0 & \text{if } \sum_{i=1}^{n}(a-x_i) < k \end{cases} = \begin{cases} 1 & \text{if } \sum_{i=1}^{n}(x_i-a) < k \\ 0 & \text{if } \sum_{i=1}^{n}(x_i-a) > k \end{cases} \quad \text{with } E_{\theta_0}[\phi(x)] = \alpha$$

We know $X_i \sim Exp(\theta)$ on (a, ∞) $\Rightarrow$ $X_i - a \sim Exp(\theta)$ on $(0,\infty)$ $\Rightarrow \sum_{i=1}^{n}(x_i - a) \sim Gamma(n, \theta)$

Thus, $E_{\theta_0}[\phi(x)] = E_{\theta_0}\left[I\left(\sum_{i=1}^{n}(x_i-a) < k\right)\right] = P_{\theta_0}\left(\underbrace{\sum_{i=1}^{n}(x_i-a)}_{Gamma(n,\theta_0)} < k\right) \overset{\text{set}}{=} \alpha$

$\Rightarrow k = \Gamma^{-1}_{n,\theta_0}(\alpha)$

So, the UMP level-α test for $H_0: \theta \geq \theta_0$ vs. $\theta < \theta_0$ is

$$\phi(x) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n}(x_i-a) < \Gamma^{-1}_{n,\theta_0}(\alpha) \\ 0 & \text{if } \sum_{i=1}^{n}(x_i-a) > \Gamma^{-1}_{n,\theta_0}(\alpha) \end{cases} = \begin{cases} 1 & \text{if } \bar{x} = \frac{1}{n}\Gamma^{-1}_{n,\theta_0}(\alpha) + a \\ 0 & \text{ow} \end{cases}$$

(b) When a is known, derive the asymptotic dist. of the MLE of θ.

Likelihood: $\theta^{-n} \exp\left\{\sum_{i=1}^{n}(a-x_i)/\theta\right\} I(X_{(1)} > \overset{\text{known}}{a})$

$\Rightarrow$ log-Likelihood: $-n\log(\theta) + \frac{1}{\theta}\sum_{i=1}^{n}(a-x_i)$ if $X_{(1)} > a$

Then $\frac{d\ell}{d\theta} = \frac{-n}{\theta} - \frac{\sum_{i=1}^{n}(a-x_i)}{\theta^2} \overset{set}{=} 0$

$\Rightarrow n\theta = -\sum_{i=1}^{n}(a-x_i)$

$\Rightarrow \hat{\theta} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \underset{\text{known.}}{\underset{\uparrow}{a}}) = \bar{x} - a$

We know $X_i \sim$ Shifted exponential, shifted by a,

so $X_i - a \sim Exp(\theta)\, I(x>0)$

(Then, $\sum_{i=1}^{n}(x_i - a) \sim Gamma(n, \theta)$ $\leftarrow$ exact, not asymptotic)

We want asymptotic dist.

By CLT, $\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}(x_i-a) - E(x_i-a)\right) \xrightarrow[d]{} N(0, Var(x_i - a))$

$\Rightarrow \sqrt{n}\left(\hat{\theta} - \theta\right) \xrightarrow[d]{} N(0, \theta^2)$

Need to state this for either method

Note: When a is known, this distribution is a member of the exponential family, and hence regularity conditions hold, so we know $\sqrt{n}(\hat{\theta}-\theta) \xrightarrow[d]{} N(0, I(\theta)^{-1})$

$I(\theta) = -\frac{1}{n}E\left[\frac{d^2 \log f_\theta(x)}{d\theta}\right]$; $\log f_\theta(x) \propto -n\log(\theta) - \frac{\sum_{i=1}^{n}(x_i-a)}{\theta}$; $\frac{d\ell}{d\theta} = -\frac{n}{\theta} + \frac{\sum(x_i-a)}{\theta^2} \overset{set}{=} 0$

$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n}(x_i-a)$; $\frac{d^2\ell}{d\theta^2} = \frac{n}{\theta^2} - \frac{2\theta\sum(x_i-a)}{\theta^2}$

$-\frac{1}{n}E\left[\frac{n}{\theta^2} - \frac{2\theta\sum(x_i-a)}{\theta^4}\right] = \frac{-1}{\theta^2} + \frac{2n\theta(\theta)}{n\theta^4} = \frac{-1}{\theta^2} + \frac{2}{\theta^2} = \frac{1}{\theta^2} \rightarrow I(\theta)^{-1} = \theta^2$

(c) Now $a=\theta$; $f(x) = \theta^{-1} e^{(\theta-x)/\theta} I(x>\theta)$

{ So X has an exponential dist. shifted by θ
$\Rightarrow X-\theta \sim Exp(\theta)$ and $\frac{X-\theta}{\theta} \sim Exp(1)$ }

$\rightarrow$ Prove $\frac{\overline{X}}{\theta}$ and $\frac{X_{(1)}}{\theta}$ are pivotal quantities.

Likelihood: $\theta^{-n} \exp\{\sum_{i=1}^{n}(\theta-x_i)/\theta\} I(x_{(1)}>\theta)$

We will show the CDFs of the 2 proposed pivots are parameter free:

Note: $X-\theta \sim Exp(\theta)$ and $\frac{X-\theta}{\theta} \sim Exp(1)$

Note that $\sum_{i=1}^{n}(X_i-\theta) \sim Gamma(n,\theta)$

$\sum_{i=1}^{n}\left(\frac{X_i-\theta}{\theta}\right) \sim Gamma(n,1) \Rightarrow \frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i-\theta}{\theta}\right) \sim Gamma(n,\frac{1}{n})$

$$P\left(\frac{\overline{X}}{\theta} \le t\right) = P\left(\frac{\frac{1}{n}\sum_{i=1}^{n}X_i}{\theta} \le t\right) = P\left(\frac{\frac{1}{n}\sum_{i=1}^{n}X_i}{\theta} - \frac{\theta}{\theta} \le t-1\right) = P\Bigg(\underbrace{\frac{\frac{1}{n}\sum_{i=1}^{n}(X_i-\theta)}{\theta}}_{Gamma(n,\frac{1}{n})\ \perp\!\!\!\perp\ \theta} \le t-1\Bigg)$$

OR $P\left(\frac{X_i}{\theta} \le t\right) = P\Bigg(\underbrace{\frac{X_i-\theta}{\theta}}_{Exp(1)} \le t-1\Bigg) = 1-\exp(t-1)$, which is clearly $\perp\!\!\!\perp\ \theta$

So $\frac{X_i}{\theta}$ is a pivotal quantity, and hence $\frac{\overline{X}}{\theta}$ is a pivotal quantity

We know $P\left(\frac{X_{(1)}}{\theta} \le t\right) = 1-P\left(\frac{X_{(1)}}{\theta} \ge t\right) = 1-P\left(\frac{X_1}{\theta},\ldots,\frac{X_n}{\theta} \ge t\right) = 1-P\left(\frac{X_1}{\theta} \ge t\right)^n$

$$= 1-\left[1-P\left(\frac{X_1}{\theta} \le t\right)\right]^n = 1-\Bigg[1-P\Bigg(\underbrace{\frac{X_1-\theta}{\theta}}_{Exp(1)} \le t-1\Bigg)\Bigg]^n$$

$$= 1-\left[1-(1-e^{-(t-1)})\right]^n = 1-\left[e^{-(t-1)}\right]^n = 1-e^{-n(t-1)}$$

which is also parameter free, and hence $\frac{X_{(1)}}{\theta}$ is a pivot

(d) Obtain two confidence intervals w/ $1-\alpha$ level for θ based on two pivots in (c):

In (c), we found $P\left(\frac{\bar{X}}{\theta} \le t\right) = 1-e^{-(t-1)}$

So, $\frac{\bar{X}}{\theta}$

$$\Rightarrow \frac{\bar{X}}{\theta} - 1 \sim \text{Gamma}\left(n, \tfrac{1}{n}\right)$$

To obtain a $1-\alpha$ confidence interval for θ, we must have

$$P\left(a \le \frac{\bar{X}}{\theta} - 1 \le b\right) = 1-\alpha$$

where a and b are quantiles of $\text{Gamma}(n, \frac{1}{n})$

So, we need a to be the $\frac{\alpha}{2}$ percentile of $\Gamma(n, \frac{1}{n}) \Rightarrow a = \Gamma^{-1}_{n,\frac{1}{n}}(\alpha/2)$

b to be $(1-\frac{\alpha}{2})$ percentile of $\Gamma(n, \frac{1}{n}) \Rightarrow b = \Gamma^{-1}_{n,\frac{1}{n}}(1-\frac{\alpha}{2})$

So, a $1-\alpha$ CI for θ based on pivot $\frac{\bar{X}}{\theta}$ is:

$$a \le \frac{\bar{X}}{\theta} - 1 \le b \Leftrightarrow \frac{\bar{X}}{b+1} \le \theta \le \frac{\bar{X}}{a+1}$$

$$\Leftrightarrow \boxed{\frac{\bar{X}}{1+\Gamma^{-1}_{n,\frac{1}{n}}(1-\frac{\alpha}{2})} \le \theta \le \frac{\bar{X}}{1+\Gamma^{-1}_{n,\frac{1}{n}}(\alpha/2)}}$$

Now, in (c) we also found $P\left(\frac{X_{(1)}}{\theta} \le t\right) = 1-e^{-n(t-1)}$

So, to get a $1-\alpha$ CI for θ, we will find a and b s.t. $P\left(a \le \frac{X_{(1)}}{\theta} \le b\right) = 1-\alpha$

We need $1-e^{-n(a-1)} = \frac{\alpha}{2}$ and $1-e^{-n(b-1)} = 1-\frac{\alpha}{2}$

$$e^{-n(a-1)} = 1-\frac{\alpha}{2} \qquad -n(b-1) = \log\left(\frac{\alpha}{2}\right)$$

$$a-1 = -\frac{1}{n}\log\left(1-\frac{\alpha}{2}\right) \qquad b = 1-\frac{1}{n}\log\left(\frac{\alpha}{2}\right)$$

$$a = 1-\frac{1}{n}\log\left(1-\frac{\alpha}{2}\right)$$

So, a $1-\alpha$ CI for θ based on pivot $\frac{X_{(1)}}{\theta}$ is: $\boxed{\frac{X_{(1)}}{1-\frac{1}{n}\log(\frac{\alpha}{2})} \le \theta \le \frac{X_{(1)}}{1-\frac{1}{n}\log(1-\frac{\alpha}{2})}}$

(e) When n is sufficiently large, which CI has shorter length?

The interval based on $\frac{X_{(1)}}{\theta}$ is $\frac{X_{(1)}}{1-\frac{1}{n}\log(\frac{\alpha}{2})} \leq \theta \leq \frac{X_{(1)}}{1-\frac{1}{n}\log(1-\frac{\alpha}{2})}$

As $n \to \infty$ $\quad \frac{1}{n}\log(\frac{\alpha}{2}) \to 0$ and $\frac{1}{n}\log(1-\frac{\alpha}{2}) \to 0$

and so the length of this interval is

$$\frac{X_{(1)}}{1-0} - \frac{X_{(1)}}{1-0} = 0$$

The interval based on $\frac{\overline{X}}{\theta}$ is $\frac{\overline{X}}{1+T^{-1}_{n,\frac{1}{n}}(1-\frac{\alpha}{2})} \leq \theta \leq \frac{\overline{X}}{1+T^{-1}_{n,\frac{1}{n}}(\alpha/2)}$

As $n \to \infty$, $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \longrightarrow E(X_i)$

$$E(X_i) = \int_{\theta}^{\infty} \frac{X_i}{\theta} \exp\left(-\frac{X_i}{\theta}+1\right) dx$$

$$= \exp(1) \int_0^{\infty} \frac{X_i}{\theta} \exp\left(\frac{-X_i}{\theta}\right) dx = \theta e^1 \int u e^u \, du = \theta e \left[u e^u - \int e^u du\right]$$

$u = \frac{-X_i}{\theta} \quad du = \frac{-1}{\theta} dx$

$c = u \quad v = e^u$

$dc = du \quad dv = e^u du$

$$= \theta e\left[u e^u - e^u\right] = \theta e\left[\frac{-X_i}{\theta} e^{-X_i/\theta} - e^{-X_i/\theta}\right]_{\theta}^{\infty}$$

$$= \theta e\left[+e^{-1} + e^{-1}\right] = \theta e\left[+2e^{-1}\right] = +2\theta$$

So, the length of the CI is $\quad 2\theta\left[\frac{1}{1+T_{n,\frac{1}{n}}(1-\frac{\alpha}{2})} - \frac{1}{1+T_{n,\frac{1}{n}}(\alpha/2)}\right]$

For a given n, this has length 0 if $\alpha = 1$,
but that would give a 0% CI! Haha!
However, as $n \to \infty$, Gamma$(n, \frac{1}{n}) \to N(1, \frac{1}{n}) \to N(1, 0$
So would have length 0?

So, CI based on $\frac{X_{(1)}}{\theta}$ is shorter as $n \to \infty$?