2014 D1

3) a) rank 2??

$\Theta = \{\alpha, \beta : \underline{\alpha\beta > 1}, \alpha > 0, \beta > 0\}$

b) $X \sim Exp(\alpha - \frac{1}{\beta})$

correction

c) Show $E(X^j Y^k) = (-1)^{j+k} S^{-1} \frac{d}{d\alpha^k d\beta^j} S$

let $t = (t_1, t_2)^T$

$$M_{(X,Y)}(t) = E\left(e^{t_1 X + t_2 Y}\right) = \int_0^\infty \int_0^\infty e^{t_1 x} e^{t_2 y} \left[(\alpha\beta - 1) e^{-\alpha x} e^{-\beta y} \sum_{j=0}^\infty \frac{(xy)^j}{(j!)^2}\right] dy\, dx$$

$$= (\alpha\beta - 1) \int_0^\infty e^{-x(\alpha - t_1)} \sum_{j=0}^\infty \frac{x^j}{(j!)^2} \left[\int_0^\infty y^j e^{-y(\beta - t_2)} dy\right] dx$$

$\to 1$ b/c $G(j+1, \beta - t_2)$

$$= (\alpha\beta - 1) \int_0^\infty e^{-x(\alpha - t_1)} \sum_{j=0}^\infty \frac{x^j}{(j!)^2} \frac{\Gamma(j+1)}{(\beta - t_2)^{j+1}} \left[\int_0^\infty \frac{(\beta - t_2)^{j+1}}{\Gamma(j+1)} y^{(j+1)-1} e^{-y(\beta - t_2)} dy\right] dx$$

$$= (\alpha\beta - 1) \int_0^\infty e^{-x(\alpha - t_1)} \frac{1}{(\beta - t_2)} \underbrace{\sum_{j=0}^\infty \frac{(x/(\beta - t_2))^j}{j!}}_{e^{x/(\beta - t_2)}} dx$$

$$= \frac{(\alpha\beta - 1)}{\beta - t_2} \int_0^\infty e^{-x\left(\alpha - t_1 - \frac{1}{\beta - t_2}\right)} dx$$

$$= \frac{(\alpha\beta - 1)}{(\beta - t_2)} \left[\frac{-1}{\alpha - t_1 - \frac{1}{\beta - t_2}} e^{-x\left(\alpha - t_1 - \frac{1}{\beta - t_2}\right)}\right]_0^\infty = \frac{(\alpha\beta - 1)}{(\beta - t_2)} \left(\frac{1}{\frac{(\alpha - t_1)(\beta - t_2) - 1}{(\beta - t_2)}}\right) = \frac{(\alpha\beta - 1)}{(\alpha - t_1)(\beta - t_2) - 1}$$

$$E(XY) = \frac{d}{dt_1\, dt_2} M_{(X,Y)}(t) \Big|_{t_1 = t_2 = 0}$$

$$= S^{-1} \frac{d}{dt_1 dt_2} \frac{1}{(\alpha - t_1)(\beta - t_2) - 1} \Big|_{t_1 = t_2 = 0}$$

$$= S^{-1} \frac{d}{dt_2} \left[\frac{(\beta - t_2)}{\{(\alpha - t_1)(\beta - t_2) - 1\}^2}\right]_{t=0} = S^{-1} \left[\frac{-[(\alpha - t_1)(\beta - t_2) - 1]^{\cancel{2}} + (\beta - t_2)\, 2\, \cancel{[(\alpha - t_1)(\beta - t_2) - 1]}\, (\alpha - t_1)}{[(\alpha - t_1)(\beta - t_2) - 1]^{\cancel{4}\, 3}}\right]_{t=0}$$

$$= S^{-1} \left[\frac{-(\alpha\beta - 1) + 2\alpha\beta}{(\alpha\beta - 1)^3}\right] = S^{-1} \left[\frac{\alpha\beta + 1}{(\alpha\beta - 1)^3}\right] = (-1)^{1+1} S^{-1} \frac{d}{d\alpha\, d\beta} S$$

$$\frac{d}{d\alpha} \frac{1}{(\alpha\beta - 1)} = \frac{-\beta}{(\alpha\beta - 1)^2}$$

$$\frac{d}{d\alpha\, d\beta} \frac{1}{(\alpha\beta - 1)} = \frac{-(\alpha\beta - 1)^2 + \beta\, 2(\alpha\beta - 1)\alpha}{(\alpha\beta - 1)^4}$$

$$= \frac{-(\alpha\beta - 1) + 2\alpha\beta}{(\alpha\beta - 1)^3} = \frac{\alpha\beta + 1}{(\alpha\beta - 1)^2}$$

2014 P1 3 e setup

Use $f(\vec{y}|\vec{x}) = \prod_{i=1}^{n} f(y_i|x_i)$

i) use NPL to test $H_0: \beta=2$ v $H_1: \beta=\beta_1$ $\beta_1 > 2$

$\Rightarrow \phi(y) = \begin{cases} 1 & f(\vec{y}|\vec{x},\beta_1) > k f(\vec{y}|\vec{x},2) \\ 0 & f(\vec{y}|\vec{x},\beta_1) < k f(\vec{y}|\vec{x},2) \end{cases}$ st $E_2[\phi(y)] = \alpha$

Show it's RR doesn't depend on H_1 (ie β_1)

$\Rightarrow$ it's a UMP test for $H_0: \beta=2$ v $H_1: \beta > 2$