2013 Theory I #1

1a.i) $\frac{\partial}{\partial\alpha} \ell_n(\alpha) = \frac{\partial}{\partial\alpha} \sum_{i=1}^{n} \left\{ -(\alpha+2)\log(1+x_i+y_i) + \log\alpha + \log(\alpha+1) \right\}$

$$= \sum_{i=1}^{n} \left\{ -\log(1+x_i+y_i) + \frac{1}{\alpha} + \frac{1}{\alpha+1} \right\} \overset{\text{set}}{=} 0$$

$$\Rightarrow \hat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} \log(1+x_i+y_i) = \frac{1}{\alpha} + \frac{1}{\alpha+1} \equiv g(\alpha)$$

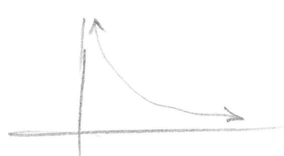
Now $\frac{\partial}{\partial\alpha} g(\alpha) = -\frac{1}{\alpha^2} - \frac{1}{(\alpha+1)^2} < 0$ for all $\alpha > 0$ so that

$g(\cdot)$ is a decreasing function on $(0,\infty)$. Furthermore,

$$\lim_{\alpha\to 0} g(\alpha) = \lim_{\alpha\to 0}\left(\frac{1}{\alpha} + \frac{1}{\alpha+1}\right) = \infty$$

and

$$\lim_{\alpha\to\infty} g(\alpha) = \lim_{\alpha\to\infty}\left(\frac{1}{\alpha} + \frac{1}{\alpha+1}\right) = 0$$

so $g: (0,\infty) \mapsto (0,\infty)$ has form 

and is a bijection from $(0,\infty)$ to $(0,\infty)$. Since $\hat{\mu}_n \in (0,\infty)$,

$g^{-1}(\cdot)$ exists, $\hat{\alpha}_n = g^{-1}(\hat{\mu}_n)$ and is unique.

1a.ii) $\hat{\mu}_n \xrightarrow{a.s.} E\left[\log(1+X_1+Y_1)\right]$ by SLLN. Let

$Z = \log(1+X_1+Y_1)$. Then

Use exponential family property
derivative of potential

$$M_Z(t) = E\left[e^{tZ}\right] = E\left[\exp\left\{t \log(1+X_1+Y_1)\right\}\right]$$

$$= E\left[\exp\left\{\log\left[(1+X_1+Y_1)^t\right]\right\}\right] = E\left[(1+X_1+Y_1)^t\right]$$

$$= \int_0^\infty \int_0^\infty (1+x+y)^t \alpha(\alpha+1)(1+x+y)^{-(\alpha+2)} \, dy \, dx$$

$$= \alpha(\alpha+1) \int_0^\infty \int_0^\infty (1+x+y)^{-(\alpha+2)+t} \, dy \, dx$$

$$= \alpha(\alpha+1) \int_0^\infty \left. \frac{(1+x+y)^{-(\alpha+1)+t}}{-(\alpha+1)+t} \right|_{y=0}^{y=\infty} dx$$

$$= \frac{\alpha(\alpha+1)}{\alpha+1-t} \int_0^\infty (1+x)^{-(\alpha+1)+t} \, dx$$

$$= \frac{\alpha(\alpha+1)}{\alpha+1-t} \left. \frac{(1+x)^{-\alpha+t}}{-\alpha+t} \right|_{x=0}^{x=\infty} = \frac{\alpha(\alpha+1)}{(\alpha+1-t)(\alpha-t)}$$

Notice that

$$\frac{d}{dt}(\alpha+1-t)(\alpha-t) = \frac{\partial}{\partial t}\left[\alpha(\alpha+1) - (\alpha+1)t - \alpha t + t^2\right] = -(2\alpha+1) + 2t$$

Then

$$E\left[\log(1+X_1+Y_1)\right] = \left. \frac{\partial}{\partial t} M_Z(t) \right|_{t=0} = \left. \frac{\partial}{\partial t} \frac{\alpha(\alpha+1)}{(\alpha+1-t)(\alpha-t)} \right|_{t=0}$$

$$= - \left. \frac{\alpha(\alpha+1)\left[-(2\alpha+1)+2t\right]}{(\alpha+1-t)^2(\alpha-t)^2} \right|_{t=0} = \frac{\alpha(\alpha+1)(2\alpha+1)}{(\alpha+1)^2 \alpha^2} = \frac{2\alpha+1}{\alpha(\alpha+1)} = g(\alpha)$$

since

$$\frac{1}{\alpha} + \frac{1}{\alpha+1} = \frac{2\alpha+1}{\alpha(\alpha+1)}$$

Thus

$$\hat{\mu}_n \xrightarrow{a.s.} E\left[\log(1+X_1+Y_1)\right] = g(\alpha)$$

so that by the CMT

$$\hat{\alpha}_n = g^{-1}(\hat{\mu}_n) \xrightarrow{a.s.} (g^{-1} \circ g)(\alpha) = \alpha$$

1a.iii) $p(x, y; \alpha) = \exp\left\{-(\alpha+2)\log(1+x+y) + \log[\alpha(\alpha+1)]\right\}$

$$= \exp\left\{\theta T(x,y) - b(\theta)\right\}$$

where $\theta = -(\alpha+2)$, $T(x,y) = \log(1+x+y)$ and $b(\theta) = -\log[(\theta+1)(\theta+2)]$.

Thus $(X_1, Y_1), (X_2, Y_2), \dots$ are iid exponential r.v.'s and consequently the regularity properties for convergence and asymptotic efficiency of MLEs are met. From previous calculations,

$$\frac{\partial^2}{\partial \alpha^2} \log\left[p(x,y;\alpha)\right] = -n\left[\frac{1}{\alpha^2} + \frac{1}{(\alpha+1)^2}\right] = -n\,\frac{\alpha^2 + (\alpha+1)^2}{\alpha^2(\alpha+1)^2}$$

Thus by MLE theory we obtain

$$\sqrt{n}\left(\hat{\alpha}_n - \alpha\right) \xrightarrow{L} N\left(0, \frac{\alpha^2(\alpha+1)^2}{\alpha^2 + (\alpha+1)^2}\right)$$

1b) $p(x;\alpha) = \int_0^\infty p(x,y;\alpha)\,dy = \int_0^\infty \alpha(\alpha+1)(1+x+y)^{-(\alpha+2)}\,dy$

$$= \alpha(\alpha+1)\left.\frac{(1+x+y)^{-(\alpha+1)}}{-(\alpha+1)}\right|_{y=0}^{y=\infty} = \alpha(1+x)^{-(\alpha+1)}, \quad x>0, \quad \alpha>0$$

Then

$$p(y\mid x;\alpha) = \frac{p(x,y;\alpha)}{p(x;\alpha)} = \frac{\alpha(\alpha+1)(1+x+y)^{-(\alpha+2)}}{\alpha(1+x)^{-(\alpha+1)}}$$

$$= \frac{\alpha+1}{1+x}\left(\frac{1+x+y}{1+x}\right)^{-(\alpha+2)} = \frac{\alpha+1}{1+x}\left(1+\frac{y}{1+x}\right)^{-(\alpha+2)}, \quad y>0, \quad \alpha>0$$

1c.i) Define $\ell_n^*(\alpha)$ to be the log-likelihood of the conditional density of $(y_1, \dots, y_n)$. Then

$$\ell_n^*(\alpha) = \sum_{i=1}^n \left\{-(\alpha+2)\log\left(1+\frac{y_i}{1+x_i}\right) + \log(\alpha+1) - \log(x_i+1)\right\}$$

Next,

$$\frac{\partial}{\partial\alpha}\ell_n^*(\alpha) = \sum_{i=1}^n \left\{-\log\left(1+\frac{y_i}{1+x_i}\right) + \frac{1}{\alpha+1}\right\} \overset{\text{set}}{=} 0$$

$$\Rightarrow \underbrace{\frac{1}{\hat\alpha+1}}^{g(\hat\alpha)\equiv} = \frac{1}{n}\sum_{i=1}^n \log\left(1+\frac{y_i}{1+x_i}\right) \equiv \tilde\mu_n$$

$$\Rightarrow \hat\alpha = \frac{1-\tilde\mu_n}{\tilde\mu_n} \equiv g^{-1}(\tilde\mu_n)$$

We notice that the graph of g^{-1} is of the form

Thus we define

$$\tilde{\alpha}_n = \begin{cases} \frac{1-\tilde{\mu}_n}{\tilde{\mu}_n}, & 0 < \tilde{\mu}_n < 1 \\ 0, & \tilde{\mu}_n \geq 1 \end{cases}$$

argument for this at the end of the problem (pg 6)

It is clear that any other estimator does not maximize the likelihood and thus $\tilde{\alpha}_n$ is unique (we have constructed an estimator that maximizes the likelihood so no other estimator will maximize)

1c.ii) Let $U_i = \frac{Y_i}{1+x_i}$. Then $U_i \in (0, \infty)$ and $Y_i = U_i(1+x_i)$

so that

$$f_{U_i}(u_i) = f_{Y_i}\left(u_i(1+x_i)\right)\left|\frac{d\, u_i(1+x_i)}{du_i}\right|$$

$$= \frac{\alpha+1}{1+x_i}\left(1 + \frac{u_i(1+x_i)}{1+x_i}\right)^{-(\alpha+2)}(1+x_i) = (\alpha+1)\left(1+u_i\right)^{-(\alpha+2)}, \quad u_i > 0, \ \alpha > 0$$

for $i = 1, 2, \ldots$. Thus $U_1, U_2, \ldots$ are iid and consequently

$$Z_i \equiv \log(1 + U_i), \quad i = 1, 2, \ldots$$

are also iid. Now

$$M_{Z_1}(t) = E\left[(1+U_1)^t\right] = \int_0^\infty (1+u)^t \overset{(\alpha+1)}{\downarrow} (1+u)^{-(\alpha+2)}\, du$$

$$= (\alpha+1)\int_0^\infty (1+u)^{-(\alpha+2)+t}\, du = (\alpha+1)\left.\frac{(1+u)^{-(\alpha+1)+t}}{-(\alpha+1)+t}\right|_{u=0}^{u=\infty} = \frac{\alpha+1}{\alpha+1-t}$$

Then

$$E[Z_1] = \left.\frac{d}{dt}\frac{\alpha+1}{\alpha+1-t}\right|_{t=0} = \left.\frac{\alpha+1}{(\alpha+1-t)^2}\right|_{t=0} = \frac{1}{\alpha+1}$$

Then by the SLLN,

$$\tilde{\mu}_n \xrightarrow{a.s.} \frac{1}{\alpha+1}$$

and by the CMT

$$\frac{1-\tilde{\mu}_n}{\tilde{\mu}_n} \xrightarrow{a.s} \frac{1-\frac{1}{\alpha+1}}{\frac{1}{\alpha+1}} = \frac{\frac{\alpha}{\alpha+1}}{\frac{1}{\alpha+1}} = \alpha$$

It remains to argue that $\tilde{\alpha}_n \xrightarrow{a.s.} \alpha$. But

$$P\left(\lim_{n\to\infty} \tilde{\alpha}_n = \alpha\right) = 1 - P\left(\lim_{n\to\infty} \tilde{\alpha}_n \neq \alpha\right)$$

$$\geq 1 - P\left(\left\{\tilde{\mu}_n \not\to \frac{1}{\alpha+1}\right\}\right) = 1 - 0 = 1$$

Equivalently:

$$P\left(\lim_{n\to\infty} \tilde{\alpha}_n = \alpha\right) \geq P\left(\lim_{n\to\infty} \tilde{\mu}_n = \frac{1}{\alpha+1}\right) = 1$$

Argument for the form of $\tilde{\alpha}_n$: Let $\Theta = (0,\infty)$

When $\tilde{\mu}_n \leq 1$ then $\exists\ \alpha \in \bar{\Theta}$ that maximizes $\ell_n^*(\alpha)$. Suppose that $\tilde{\mu}_n > 1$. Then

$$\frac{d}{d\alpha}\ell_n^*(\alpha) = n\left[-\tilde{\mu}_n + \frac{1}{\alpha+1}\right] < 0 \text{ for every } \alpha \in \bar{\Theta}.$$

Thus $\ell_n^*(\alpha)$ is a strictly decreasing in α for all $\alpha \in \bar{\Theta}$ so that $\alpha = 0$ maximizes $\ell_n^*(\alpha)$

1c. iii) $f(u_1;\alpha) = \exp\{-(\alpha+2)\log(1+u_1) + \log(\alpha+1)\}$

$$= \exp\{\theta(\alpha)T(u_1) - \eta(\alpha)\}$$

so $U_1, U_2, \ldots$ are iid exponential-family r.v.'s. Thus the regularity conditions for asymptotic efficiency are met and

$$\sqrt{n}\left(g^{-1}(\tilde{\mu}_n) - \alpha\right) \xrightarrow{L} N\left(0, (\alpha+1)^2\right)$$

recall $g^{-1}(\tilde{\mu}_n)$ is RLE at each n

since

$$\frac{\partial^2}{\partial\alpha^2} \ell_n^*(\alpha) = \frac{\partial}{\partial\alpha}\sum_{i=1}^{n}\left\{-\log(1+u_i) + \frac{1}{\alpha+1}\right\} = -\frac{n}{(\alpha+1)^2}$$

Now let $h(x) = \begin{cases} x, & x>0 \\ 0, & \text{else} \end{cases}$

Notice that $h(\alpha)$ is differentiable at every α and $\dot{h}(\alpha) = 1$.

Thus by delta method,

$$\sqrt{n}\left(\tilde{\alpha}_n - \alpha\right) = \sqrt{n}\left((h \circ g^{-1})(\tilde{\mu}_n) - h(\alpha)\right) \xrightarrow{L} N\left(0, 1^2(\alpha+1)^2\right)$$

$$\stackrel{d}{=} N\left(0, (\alpha+1)^2\right)$$

Another sol'n is to use the cgf to obtain $Var[U_1] = (\alpha+1)^{-2}$ then by CLT $\sqrt{n}\left(\tilde{\mu}_n - \frac{1}{\alpha+1}\right) \xrightarrow{L} N\left(0, (\alpha+1)^{-2}\right)$. Then choose $h(x) = \begin{cases} (1-x)/x, & a>0 \\ 0, & \text{else} \end{cases}$

$\dot{h}(x) = -x^{-2}$ so that $\sqrt{n}(\tilde{\alpha}_n - \alpha) = \sqrt{n}\left(h(\tilde{\mu}_n) - h(1/(\alpha+1))\right) \xrightarrow{L} N\left(0, (\alpha+1)^4(\alpha+1)^{-2}\right)$

1d) The asymptotic rel. eff. is $\frac{\sigma_1^2}{\sigma_2^2} = \frac{\alpha^2(\alpha+1)^2}{\alpha^2+(\alpha+1)^2} \Big/ (\alpha+1)^2 = \frac{\alpha^2}{\alpha^2+(\alpha+1)^2} < 1$

so that $\hat{\alpha}_n$ is a more efficient estimator