The Big Picture

Naive Set Theory (1800s-1900s)
Ramified Theory of Types (1908)
Simple Type Theory (1920s-1940s)

"Propositions as Terms"

$\Downarrow$

LCF, HOL, etc

Big insight:
Curry-Howard Correspondence (1900s-1969) "Propositions as Types"

$\Downarrow$

NuPRL

Martin-Löf Type Theory (1970s)
Calculus of Constructions (1980s-1990s)

"Dependent Types"

Coq

Step -1: Simple Type Theory ((Church 1940)

Church formulated the λ-calculus as a type theory

Propositions are represented by λ-terms

Proof rules govern syntactic manipulations

$T \triangleq (\lambda x{:}*.x) = (\lambda x{:}*.x)$ $\quad$ $F \triangleq (\lambda x{:}*.T) = (\lambda x{:}*.x)$

$\neg e \triangleq e = F$ $\quad$ $e_1 \wedge e_2 = (\lambda f.\, f\, e_1\, e_2) = (\lambda f.\, f\, T\, T)$

Step 0: Propositions As Types (Howard ~~1964~~ 1980)

Positive Implicational Propositional Logic	λ-Calculus w/o Datatypes
$\dfrac{}{\alpha \vdash \alpha}$	$\dfrac{}{x : \tau \vdash x : \tau}$
$\dfrac{\Gamma, \alpha \vdash \beta}{\Gamma \vdash \alpha \to \beta}$	$\dfrac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda x. e : \tau_1 \to \tau_2}$
$\dfrac{\Gamma \vdash \alpha \to \beta \quad \Gamma \vdash \alpha}{\Gamma \to \beta}$	$\dfrac{\Gamma \vdash e_1 : \tau_1 \to \tau_2 \quad \Gamma \vdash e_2 : \tau_1}{\Gamma \vdash e_1\ e_2 : \tau_2}$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\alpha \vdash \alpha}}{\alpha, \beta \vdash \alpha}}{\alpha \vdash (\beta \to \alpha)}}{\vdash \alpha \to (\beta \to \alpha)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x : \alpha \vdash x : \alpha}}{x : \alpha, y : \beta \vdash x : \alpha}}{x : \alpha \vdash \lambda y. x : \beta \to \alpha}}{\cdot \vdash \lambda x. \lambda y. x : \alpha \to (\beta \to \alpha)}$$

Quantification

Adding Integers (Heyting Arithmetic)

$$\tau ::= 0 \mid S\,\alpha \mid \tau = \tau \mid \ldots$$
$$\mid \tau \to \tau \mid \tau \wedge \tau \mid \forall x.\tau$$

$$\frac{\Gamma, x:\tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda x{:}\tau_1 . e : \tau_1 \to \tau_2} \qquad \frac{\Gamma \vdash e : \tau \quad x \text{ free in } \Gamma}{\Gamma \vdash \lambda x . e : \forall x.\tau}$$

- Numbers in types unrelated to numbers in terms
- Two abstraction forms

Sidebar: Existentials

$$\exists x.\tau \approx (\forall x.\tau \to \beta) \to \beta \qquad \text{vs.} \qquad \exists x.\tau \approx \langle t, \tau[t/x] \rangle$$

???

Dependent Types — Martin-Löf ~~1972~~ ~~1979~~ 1982

Indexed Product and Sum Types

$$\alpha \to \beta \qquad\qquad \forall x{:}\alpha.\ \beta$$

$$\cong \underbrace{\beta * \beta * \cdots * \beta}_{|\alpha|\ \text{times}} \qquad\qquad \cong \beta[a/x] * \beta[b/x] * \cdots \quad (a, b, \ldots ; \alpha)$$

$$\cong \prod_{x:\alpha}.\ \beta \qquad\qquad \cong \prod x{:}\alpha.\ \beta$$

[N.B. $\alpha * \beta \cong \prod_{x:2}.(\text{case } x \text{ of } 1 \to \alpha, 2 \to \beta)$]

$$\alpha * \beta \qquad\qquad \exists x{:}\alpha.\ \beta$$

$$\cong \underbrace{\beta + \beta + \cdots + \beta}_{|\alpha|\ \text{times}} \qquad\qquad \cong \beta[a/x] + \beta[b/x] + \cdots \quad (a, b, \ldots ; \alpha)$$

$$\cong \sum x{:}\alpha.\ \beta \qquad\qquad \cong \sum x{:}\alpha.\ \beta$$

[N.B. $\alpha + \beta = \sum_{x:2}.\ (\text{case } x \text{ of } 1 \to \alpha, 2 \to \beta)$]

MLTT (cont'd)

$$\frac{\Gamma \vdash \tau_1 \qquad \Gamma, x:\tau_1 \vdash \tau_2}{\Gamma \vdash \Pi x:\tau_1 . \tau_2}$$

$$\frac{\Gamma \vdash \tau_1 \qquad \Gamma, x:\tau_1 \vdash \tau_2}{\Gamma \vdash \Sigma x:\tau_1 . \tau_2}$$

$$\frac{\Gamma, x:\tau_1 \vdash e:\tau_2}{\Gamma \vdash \lambda x.e : \Pi x:\tau_1 . \tau_2}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \qquad \Gamma \vdash e_2 : \tau_2 [e_1/x]}{\Gamma \vdash (\cancel{x:} e_1, e_2) : \Sigma x:\tau_1 . \tau_2}$$

$$\frac{\Gamma \vdash e_1 : \Pi x:\tau_1 . \tau_2 \qquad \Gamma \vdash e_2 : \tau_1}{\Gamma \vdash e_1\ e_2 : \tau_2 [e_1/x]}$$

$$\frac{\Gamma \vdash e_1 : \Sigma x:\tau_1 . \tau_2 \qquad \Gamma, y:\tau_1, z:\tau_2[y/x] \vdash e_2 : \tau_3 [(y,z)/a]}{\Gamma \vdash \text{case } e_1 \text{ of } \{(y,z) \to e_2\} : \tau_3 [e_1/a]}$$

$$\frac{\Gamma \vdash \tau_1 : U_n \qquad \Gamma, x:\tau_1 \vdash \tau_2 : U_n}{\Gamma \vdash \Pi x:\tau_1 . \tau_2 : U_n}$$

$$\frac{\Gamma \vdash \tau_1 : U_n \qquad \Gamma, x:\tau_1 \vdash \tau_2 : U_n}{\Gamma \vdash \Sigma x:\tau_1 . \tau_2 : U_n}$$

(Note: $\frac{\Gamma \vdash \tau_1 : U_n}{\Gamma \vdash \tau_1}$)

MLTT – Equality

$$\frac{\Gamma \vdash \tau : U_n \qquad \Gamma \vdash e_1 : \tau \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 =_\tau e_2 : U_n}$$

$$\frac{\Gamma \vdash e_1 \equiv e_2 : \tau}{\Gamma \vdash \mathrm{refl} : e_1 =_\tau e_2}$$

Extensional

$$\frac{\Gamma \vdash e_1 : e_2 =_\tau e_3}{\Gamma \vdash e_2 \equiv e_3 : \tau}$$

$$\frac{\Gamma \vdash e_1 : e_2 =_{\tau_1} e_3 \qquad \Gamma \vdash e_4 : \tau_2[\mathrm{refl}/z]}{\Gamma \vdash \mathrm{case}\ e_1\ \mathrm{of}\ \{\mathrm{refl} \to e_4\} : \tau_2[e_1/z]}$$

MLTT – Induction

$$\frac{\Gamma \vdash e_1 : \tau_1 \qquad \Gamma \vdash e_2 : \tau_2[e_1/x] \to Wx{:}\tau_1 . \tau_2}{\Gamma \vdash \mathrm{sup}(e_1, e_2) : Wx{:}\tau_1 . \tau_2}$$

$$\frac{\Gamma \vdash e_1 : W_{x:\tau_1} . \tau_2 \qquad \Gamma \vdash x : \tau_1, y : \tau_2 \to W_{x:\tau_1} . \tau_2, z : \Pi v{:}\tau_2 . \tau_3[y(v)/w] \vdash e_2 : \tau_3[\mathrm{sup}(x,y)/w]}{\Gamma \vdash \mathrm{rec}\ e_1 \{x, y, z \to e_2\} : \tau_3[e_1/w]}$$

Trees: $W_{x:\tau_1} . \tau_2$ has $|\tau_1|$ kinds of nodes (constructors), with τ_2 children (recursive occurrences)

$\mathbb{N} = 0 \mid S\, \mathbb{N}$

$\mathbb{N} = W_{x:2} . (\mathrm{case}\ x\ \mathrm{of}\ \{0 \to \bot, 1 \to 1\})$

$0 = \mathrm{sup}(0, \lambda x . \mathrm{case}\ x\ \mathrm{of}\ \{\})$

$S n = \mathrm{sup}(1, \lambda x . n)$

$\boxed{\text{Note: fails intensionally}}$

Calculus of Constructions - simplify & add Prop

$$\frac{\Gamma \vdash}{\Gamma \vdash \mathrm{Prop} : \mathrm{Type}_1} \qquad \frac{\Gamma \vdash}{\Gamma \vdash \mathrm{Type}_i : \mathrm{Type}_{i+1}}$$

$$\frac{\Gamma \vdash \quad x : \tau \in \Gamma}{\Gamma \vdash x : \tau} \qquad \frac{\Gamma \vdash \tau : s \quad x \notin \Gamma \quad s \in \{\mathrm{Prop}, \mathrm{Type}_i\}}{\Gamma, x : \tau \vdash}$$

$$\frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda x. e : \Pi x : \tau_1 . \tau_2} \qquad \frac{\Gamma \vdash e_1 : \Pi x : \tau_1 . \tau_2 \quad \Gamma \vdash e_2 : \tau_1}{\Gamma \vdash e_1\ e_2 : \tau_2 [e_2 / x]}$$

$$\frac{\Gamma, x : \tau_1 \vdash \tau_2 : \mathrm{Prop}}{\Gamma \vdash \Pi x : \tau_1 . \tau_2 : \mathrm{Prop}} \qquad \frac{\Gamma \vdash x : \tau_1 \vdash \tau_2 : \mathrm{Type}_i \quad \Gamma \vdash \tau_1 : \mathrm{Type}_i}{\Gamma \vdash \Pi x : \tau_1 . \tau_2 : \mathrm{Type}_i}$$

$$\frac{\Gamma \vdash e : \tau_1 \quad \Gamma \vdash \tau_2 : s \quad \tau_1 \preceq \tau_2}{\Gamma \vdash e : \tau_2}$$

$\exists x : A, B \triangleq \forall C : \mathrm{Prop}, (\forall x : A, B \to C) \to C$

$\bot \triangleq \forall C : \mathrm{Prop}, C$

$x = y \triangleq \forall P : A \to \mathrm{Prop}. P x \to P y$

Calculus of Constructions – Conversion Rules

$$\frac{\Gamma \vdash e_1 \mapsto e_2 \qquad \Gamma \vdash e_2 \mapsto e_3}{\Gamma \vdash e_1 \mapsto e_3}$$

$$\frac{\Gamma, x : \tau \vdash e_1 \mapsto e_2}{\Gamma \vdash \Pi x : \tau\ e_1 \mapsto \Pi x : \tau\ e_2}$$

$$\frac{\Gamma, x : \tau \vdash e_1 \mapsto e_2 \qquad \Gamma, x : \tau \vdash e_1 : s}{\Gamma \vdash \lambda x : \tau . e_1 \mapsto \lambda x : \tau . e_2}$$

$$\frac{\Gamma, x : \tau_1 \vdash e : s \qquad \Gamma \vdash \tau_1 \mapsto \tau_2}{\Gamma \vdash \Pi x : \tau_1 . e \mapsto \Pi x : \tau_2 . e}$$

$$\frac{\Gamma, x : \tau \vdash e : s \qquad \Gamma \vdash \tau_1 \mapsto \tau_2}{\Gamma \vdash \lambda x : \tau_1 . e \mapsto \lambda x : \tau_2 . e}$$

$$\frac{\Gamma \vdash e_1\ e_2 : s \qquad \Gamma \vdash e_1 \mapsto e_3}{\Gamma \vdash e_1\ e_2 \mapsto e_3\ e_2}$$

$$\frac{\Gamma \vdash e_1\ e_2 : s \qquad \Gamma \vdash e_2 \mapsto e_3}{\Gamma \vdash e_1\ e_2 \mapsto e_1\ e_3}$$

$$\frac{\Gamma, x : \tau_1 \vdash e_1 : s \qquad \Gamma \vdash e_2 : \tau_1}{\Gamma \vdash (\lambda x : \tau_1 . e_1)\ e_2 \mapsto e_1[e_2/x]}$$

Calculus of Constructions – Inductives

Axiomatize inductives

$$\mathbb{N} ::= \mid z : \mathbb{N} \qquad \mathbb{N} : \text{Type} \quad z : \mathbb{N} \quad S : \mathbb{N} \to \mathbb{N}$$
$$\mid S : \mathbb{N} \to \mathbb{N}$$

$$\text{ind}\,\mathbb{N} : \forall P : \mathbb{N} \to \text{Prop}.\ P\,z \to (\forall x : \mathbb{N},\ P\,x \to P\,(S\,x)) \to \forall x : \mathbb{N},\ P\,x$$

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