Framework for hybridization of mixed systems in MFEM

MFEM Community Workshop

Sep 10 – 11, 2025

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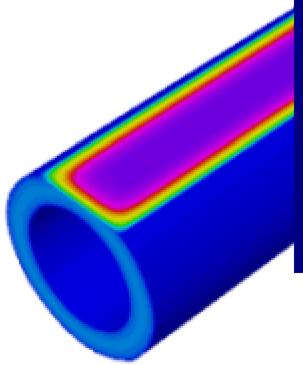
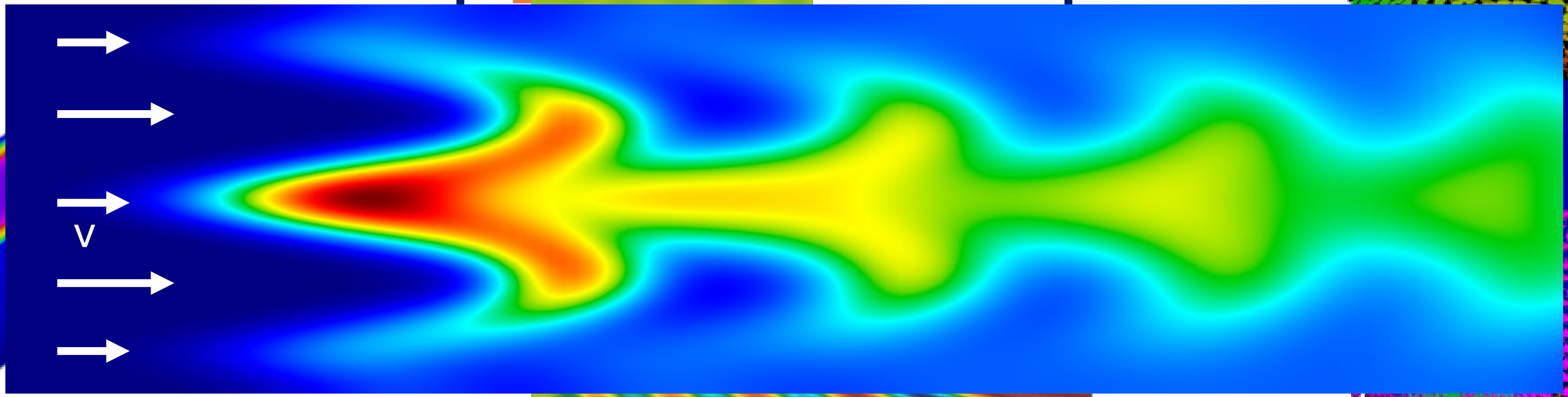
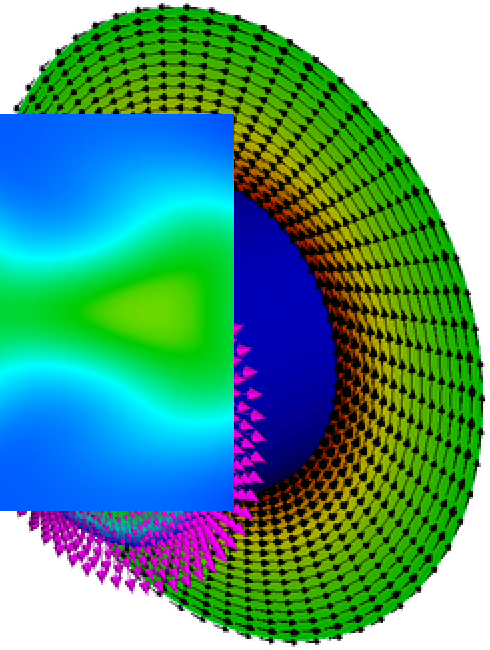
Mixed systems

- (In)definite – Darcy, Heat conduction, Maxwell, …
- Convection-diffusion

$q + \kappa\nabla u = 0, \quad \text{in } \Omega,$







c we … metri

$\begin{bmatrix} A & -B \\ B & D \end{bmatrix}$

Kovasznay flow

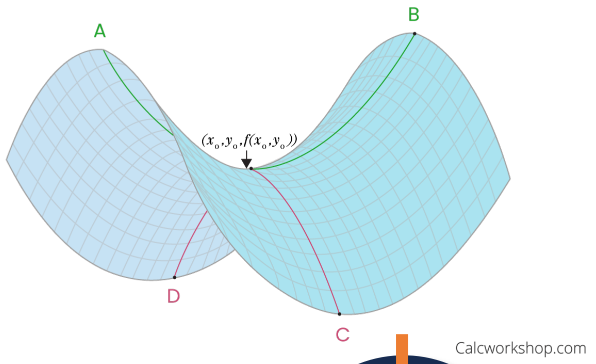
Example 14
(DG diffusion)

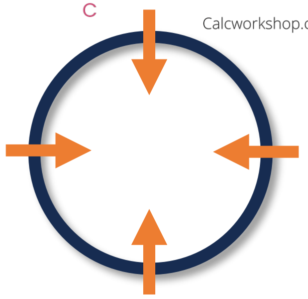
Example 25
(Maxwell problem)

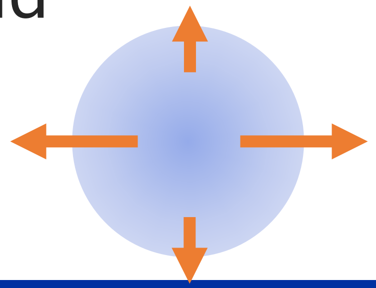
Joule miniapp
(magnetic + thermal diffusion)

Challenges

- **Definiteness** – saddle-point → *primary*?
- **Divergence-free** – potential eq. → *mixed*?
- **Conservation** – local cons. of DG potential
- **Preconditioning** – tight coupling → *primary*?
- **Memory consumption / data motion** – memory bound
- **Anisotropy** – ringing of *CG*, preconditioning → *DG*?





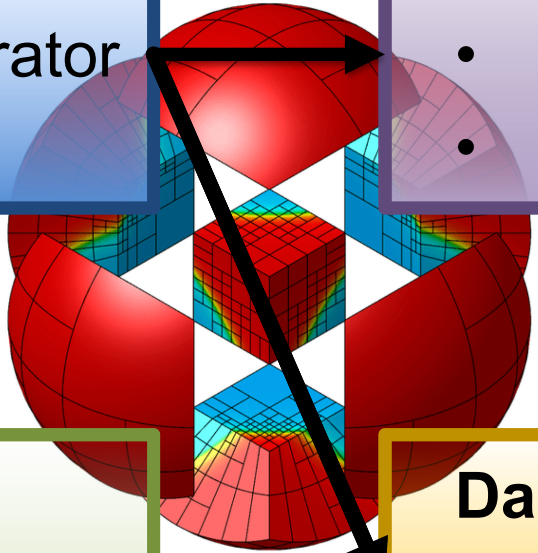


Best of *primary* and *mixed* formulation? *DG*s? And beyond?

Framework for mixed systems

(Par)DarcyForm
- Constructs the block operator
- Elimination of ess. BCs

DarcyHybridization
- Hybridization of total flux
- Static condensation to traces



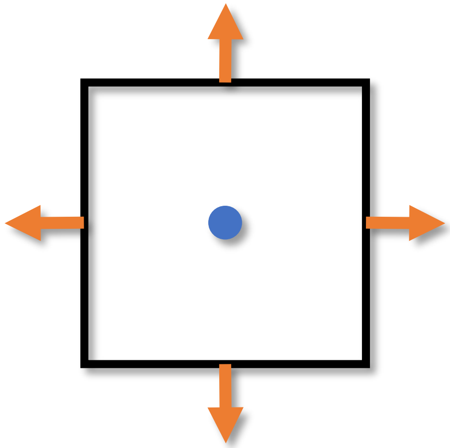
(DarcyOperator)
- Schur complement precond.
- Newton/GMRES/AMG setup

DarcyReduction
- Elimination of flux/potential
- Elimination of ess. BCs

Mixed formulation – RTDG/LDG

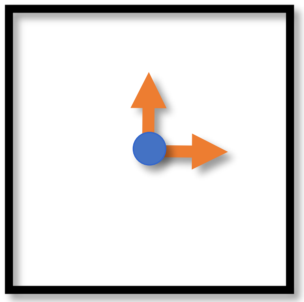
- Finite element method – discrete fxs u_h,q_h test fxs v, w
- (Bi)linear forms – (•,•) volume, ⟨•, •⟩ face
- *Raviart-Thomas + Discontinuous Galerkin (RTDG)*

$$
\begin{aligned}
(\kappa^{-1}\mathrm{q}_h, \mathrm{v})_K - (u_h, \nabla \cdot \mathrm{v})_K &= 0, \\
(\nabla \cdot \mathrm{q}_h, w)_K - (\mathrm{c}u_h, \nabla w)_K + \langle \widehat{\mathrm{c}u_h} \cdot \mathrm{n}, w \rangle_{\partial K} &= (f, w)_K,
\end{aligned}
$$



- *(Local) Discontinuous Galerkin (DG)*

$$
\begin{aligned}
(\kappa^{-1}\mathrm{q}_h, \mathrm{v})_K - (u_h, \nabla \cdot \mathrm{v})_K + \langle \hat{u}_h, \mathrm{v} \cdot \mathrm{n} \rangle_{\partial K} &= 0, \\
(\nabla \cdot \mathrm{q}_h, w)_K - (\mathrm{c}u_h, \nabla w)_K + \langle (\widehat{\mathrm{c}u_h} - \widehat{\mathrm{q}}_h) \cdot \mathrm{n}, w \rangle_{\partial K} &= (f, w)_K,
\end{aligned}
$$



RTDG solution

- **`(Par)DarcyForm`**
  - Constructs B^T operator/matrix
  - Constructs `BlockOperator` (`Mult`, `MultTranspose`)
  - Elimination of essential BCs/DOFs
- Schur complement preconditioner (**`DarcyOperator`**)

$$\begin{bmatrix} A_d^{-1} & \\ & (D + BA_d^{-1}B^T)_{GS}^{-1} \end{bmatrix} \begin{bmatrix} A & -B^T \\ B & D \end{bmatrix}$$

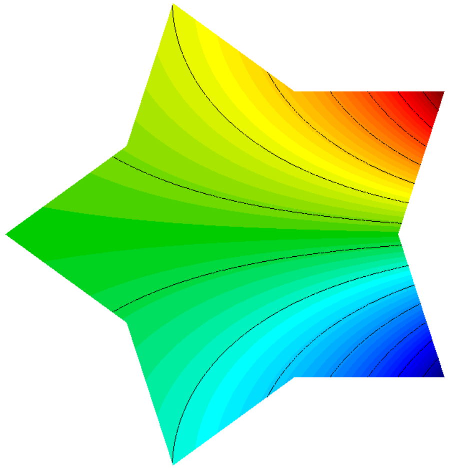
- No convection, steady-state → saddle-point problem (*D* = *0*)
- Anisotropy → direct solver (*SuiteSparse* – UMFPACK)
- (Elim. of potential → **`DarcyReduction`** (~~convection~~, ~~steady~~):

```
void EnablePotentialReduction(const Array<int> &ess_flux_tdof_list)
```
)

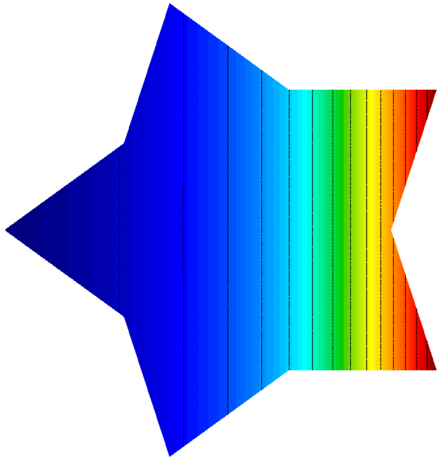
$$(A + B^T D^{-1} B) q_h = B^T f$$

Example 5 - RTDG (ex5.cpp)

```cpp
DarcyForm *darcy = new DarcyForm(R_space, W_space,
                                 false);

BilinearForm *mVarf = darcy->GetFluxMassForm();
MixedBilinearForm *bVarf = darcy->GetFluxDivForm();

mVarf->AddDomainIntegrator(
       new VectorFEMassIntegrator(kcoeff));


ConstantCoefficient cdiv(-1.);
bVarf->AddDomainIntegrator(
       new VectorFEDivergenceIntegrator(cdiv));

if (pa) { darcy->SetAssemblyLevel(
                      AssemblyLevel::PARTIAL); }


darcy->Assemble();
```



Potential



Flux

Local Discontinuous Galerkin (LDG)

$$(\kappa^{-1}\mathrm{q}_h, \mathrm{v})_K - (u_h, \nabla\cdot\mathrm{v})_K + \langle \hat{u}_h, \mathrm{v}\cdot\mathrm{n}\rangle_{\partial K} = 0, \quad \forall \mathrm{v} \in (\mathcal{P}^p(K))^d,$$

$$-(\mathrm{c}u_h + \mathrm{q}_h, \nabla w)_K + \langle(\widehat{\mathrm{c}u_h} + \widehat{\mathrm{q}}_h)\cdot\mathrm{n}, w\rangle_{\partial K} = (f, w)_K, \quad \forall w \in \mathcal{P}^p(K).$$

- Traces definition → local stabilization

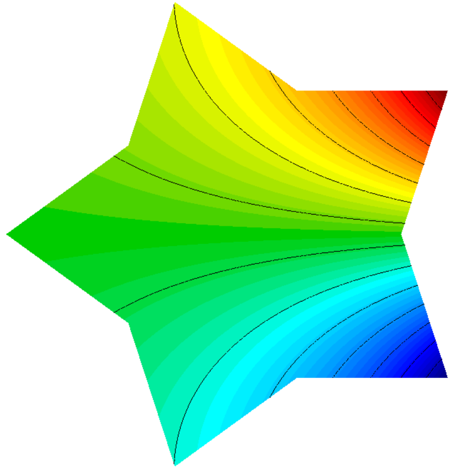
$$\widehat{\mathrm{q}}_h = \{\{\mathrm{q}_h\}\} + \mathrm{C}_{11}[\![u_h\mathrm{n}]\!] + \mathrm{C}_{12}[\![\mathrm{q}_h\cdot\mathrm{n}]\!],$$
$$\lambda_h = \hat{u}_h = \{\{u_h\}\} - \mathrm{C}_{12}\cdot[\![u_h\mathrm{n}]\!] + \mathrm{C}_{22}[\![\mathrm{q}_h\cdot\mathrm{n}]\!],$$

- LDG: C_{22}=0 (flux elimination → **DarcyReduction** $(D + BA^{-1}B^T)u_h = f$)
- Centered scheme: C_{12}=0, $C_{11}=\kappa h^{-1}/2$

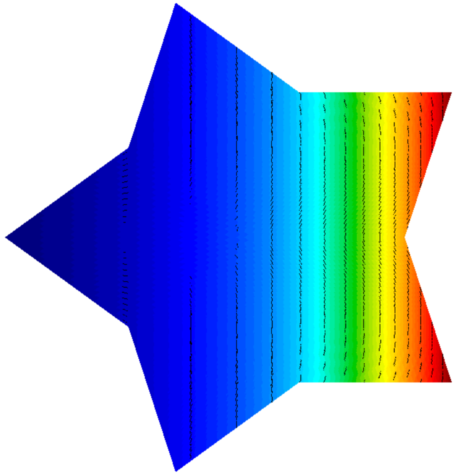
$$(\kappa^{-1}q_h, v) - (u_h, \nabla\cdot v) + \langle\{\{u_h\}\}, [\![v\cdot n]\!]\rangle = 0,$$
$$(\nabla\cdot q_h, w) - \langle[\![q_h\cdot n]\!], \{\{w\}\}\rangle + \underbrace{\langle\tfrac{\kappa h^{-1}}{2}[\![u_h]\!], [\![v]\!]\rangle}_{\approx\text{DG diffusion}} = (f, w)$$

Example 5 – LDG (ex5-hdg.cpp)

```
DarcyForm *darcy = new DarcyForm(R_space, W_space);

BilinearForm *mVarf = darcy->GetFluxMassForm();
MixedBilinearForm *bVarf = darcy->GetFluxDivForm();
BilinearForm *mtVarf = GetPotentialMassForm();

mVarf->AddDomainIntegrator(new
              VectorMassIntegrator(kcoeff));
bVarf->AddDomainIntegrator(new
              VectorDivergenceIntegrator());
bVarf->AddInteriorFaceIntegrator(new
              TransposeIntegrator(new
                DGNormalTraceIntegrator(-1.)));
mtVarf->AddInteriorFaceIntegrator(new
              HDGDiffusionIntegrator(ikcoeff));

darcy->Assemble();
```



Potential

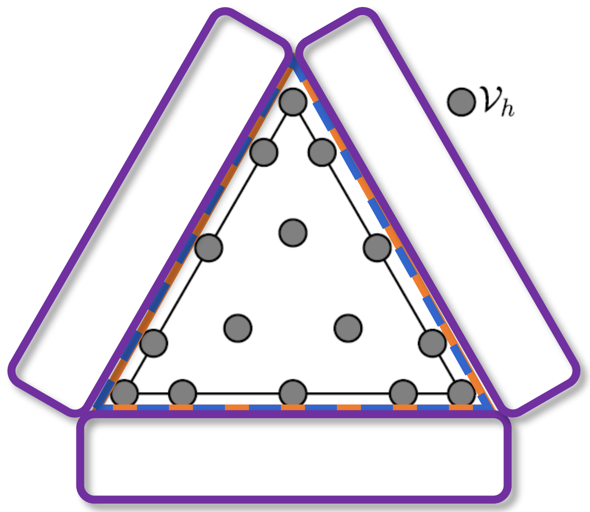


Flux

Hybridization

- Lagrange multipliers $\lambda_h \approx \hat{u}_h$
- Weak continuity of total flux

$$\langle [\![(\widehat{c u}_h + \widehat{\mathrm{q}}_h) \cdot \mathrm{n}]\!], \mu \rangle_{\mathcal{E}_h} = 0 \qquad , \qquad \forall \mu$$



Credit: G. Giorgiani et al. / CPC 254 (2020) 107375

- *N.C. Nguyen, J. Peraire & B. Cockburn (2009), JCP, 228, 3232–3254.*
- Reduction to trace DOFs of λ_h (**`DarcyHybridization`**)
- `darcy-hdg-dev` - [PR #4350]

[WIP] Hybridization of mixed systems (HRT, HDG) [darcy-hdg-dev] #4350

Open najlkin wants to merge 497 commits into `master` from `darcy-hdg-dev`

Hybridized Raviart-Thomas (HRT)

- Lagrange mulitplier $\lambda_h \approx \hat{u}_h$ (`EnableHybridization()`)

$$
\begin{aligned}
(\kappa^{-1}\mathrm{q}_h, \mathrm{v})_{\mathcal{T}_h} - (u_h, \nabla\cdot\mathrm{v})_{\mathcal{T}_h} + \langle\lambda_h, \mathrm{v}\cdot\mathrm{n}\rangle_{\partial\mathcal{T}_h} &= 0 &&, \quad \forall \mathrm{v}\in \mathrm{V}_h^p,\\
(\nabla\cdot\mathrm{q}_h, w)_{\mathcal{T}_h} &= (f, w)_{\mathcal{T}_h} &&, \quad \forall w \in W_h^p,\\
\langle [\![\widehat{\mathrm{q}}_h\cdot\mathrm{n}]\!], \mu\rangle_{\mathcal{E}_h} = 0 & &&, \quad \forall \mu \in M_h^p(0).
\end{aligned}
$$

- Reduction of the system: (`FormLinearSystem()`)

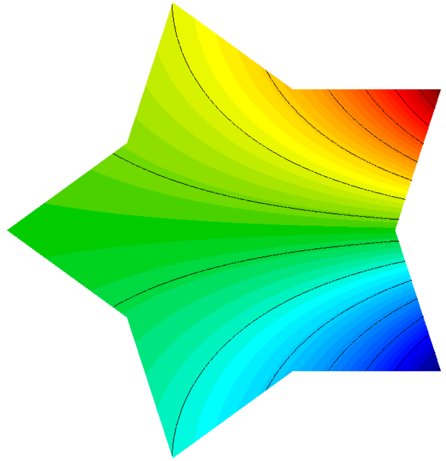
$$
\begin{bmatrix} A & -B^T & C^T \\ B & 0 & 0 \\ C & 0 & 0 \end{bmatrix}
\begin{bmatrix} \mathrm{Q} \\ U \\ \Lambda \end{bmatrix}
=
\begin{bmatrix} 0 \\ F \\ 0 \end{bmatrix}.
\Rightarrow
\begin{aligned}
\mathbb{K} &= -[C \;\; 0]\begin{bmatrix} A & -B^T \\ B & 0 \end{bmatrix}^{-1}\begin{bmatrix} C^T \\ 0 \end{bmatrix}, \\
\mathbb{F} &= -[C \;\; 0]\begin{bmatrix} A & -B^T \\ B & 0 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ F \end{bmatrix}.
\end{aligned}
\Rightarrow
\boxed{\mathbb{K}\,\Lambda = \mathbb{F},}
$$

- Recovery of the solution: (`RecoverFEMSolution()`)

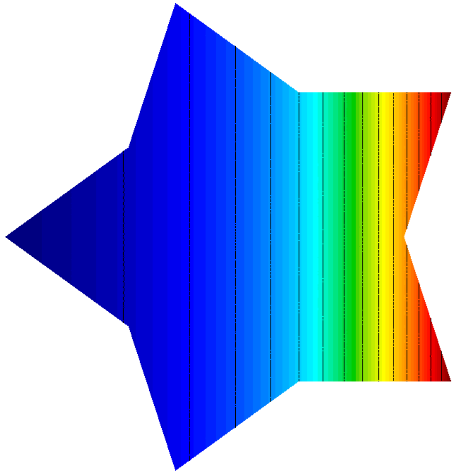
$$
\begin{bmatrix} \mathrm{Q} \\ U \end{bmatrix} = \begin{bmatrix} A & -B^T \\ B & 0 \end{bmatrix}^{-1}\left(\begin{bmatrix} 0 \\ F \end{bmatrix} - \begin{bmatrix} C^T \\ 0 \end{bmatrix}\Lambda\right),
$$

Example 5 – HRT (ex5.cpp / ex5-hdg.cpp)

```
…

if (hybridization)
{
   trace_coll = new DG_Interface_FECollection(
                     order, dim);
   trace_space = new FiniteElementSpace(
                     mesh, trace_coll);
   darcy->EnableHybridization(trace_space,
                     new NormalTraceJumpIntegrator(),
                     ess_flux_tdofs_list);
}
darcy->Assemble();
```



Potential



Flux

Hybridizable Discontinuous Galerkin (HDG)

- Lagrange mulitplier $\lambda_h \approx \hat{u}_h$

$$(\kappa^{-1}\mathrm{q}_h, \mathrm{v})_{\mathcal{T}_h} - (u_h, \nabla\cdot\mathrm{v})_{\mathcal{T}_h} + \langle \lambda_h, \mathrm{v}\cdot\mathrm{n}\rangle_{\partial\mathcal{T}_h} = 0 \quad , \quad \forall \mathrm{v} \in \mathrm{V}_h^p,$$

$$-(\mathrm{c}u_h + \mathrm{q}_h, \nabla w)_{\mathcal{T}_h} + \langle(\widehat{\mathrm{c}u_h} + \widehat{\mathrm{q}}_h)\cdot\mathrm{n}, w\rangle_{\partial\mathcal{T}_h} = (f, w)_{\mathcal{T}_h} \quad , \quad \forall w \in W_h^p,$$

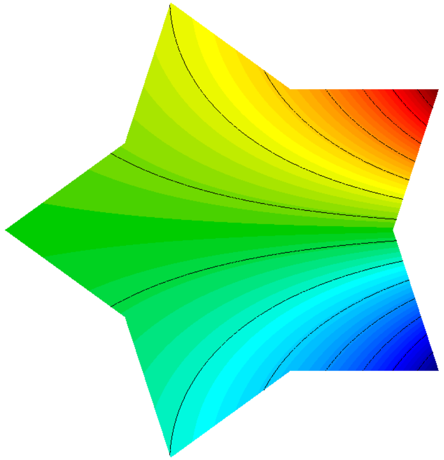
$$\langle [\![(\widehat{\mathrm{c}u_h} + \widehat{\mathrm{q}}_h)\cdot\mathrm{n}]\!], \mu\rangle_{\mathcal{E}_h} = 0 \quad , \quad \forall \mu \in M_h^p(0).$$

- *N.C. Nguyen, J. Peraire & B. Cockburn (2009), JCP, 228, 3232–3254.*
- Stabilization: $\widehat{\mathrm{c}u_h} + \widehat{\mathrm{q}}_h = \mathrm{c}\widehat{u}_h + \mathrm{q}_h + \tau(u_h - \widehat{u}_h)\mathrm{n},$ → Centered/upwinded
- Reduction of the system:

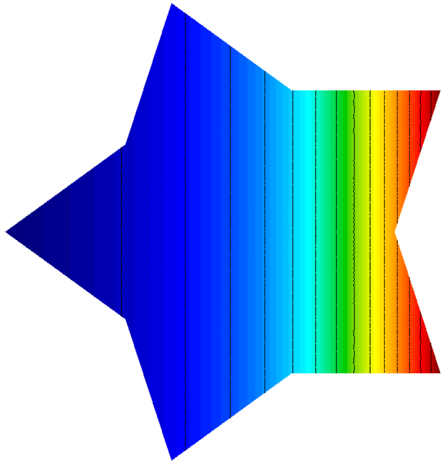
$$\begin{bmatrix} A & -B^T & C^T \\ B & D & E \\ C & G & H \end{bmatrix} \begin{bmatrix} Q \\ U \\ \Lambda \end{bmatrix} = \begin{bmatrix} R \\ F \\ L \end{bmatrix}. \Rightarrow \quad \mathbb{K} = -[C \quad G] \begin{bmatrix} A & -B^T \\ B & D \end{bmatrix}^{-1} \begin{bmatrix} C^T \\ E \end{bmatrix} + H, \quad \mathbb{F} = L - [C \quad G] \begin{bmatrix} A & -B^T \\ B & D \end{bmatrix}^{-1} \begin{bmatrix} R \\ F \end{bmatrix}. \Rightarrow \boxed{\mathbb{K}\,\Lambda = \mathbb{F},}$$

Example 5 – HDG (ex5-hdg.cpp)

```
…
BilinearForm *mtVarf = GetPotentialMassForm();
…
mtVarf->AddInteriorFaceIntegrator(new
              HDGDiffusionIntegrator(ikcoeff));

if (hybridization)
{
   trace_coll = new DG_Interface_FECollection(
                    order, dim);
   trace_space = new FiniteElementSpace(
                    mesh, trace_coll);
   darcy->EnableHybridization(trace_space,
                    new NormalTraceJumpIntegrator(),
                    ess_flux_tdofs_list);
}

darcy->Assemble();
```



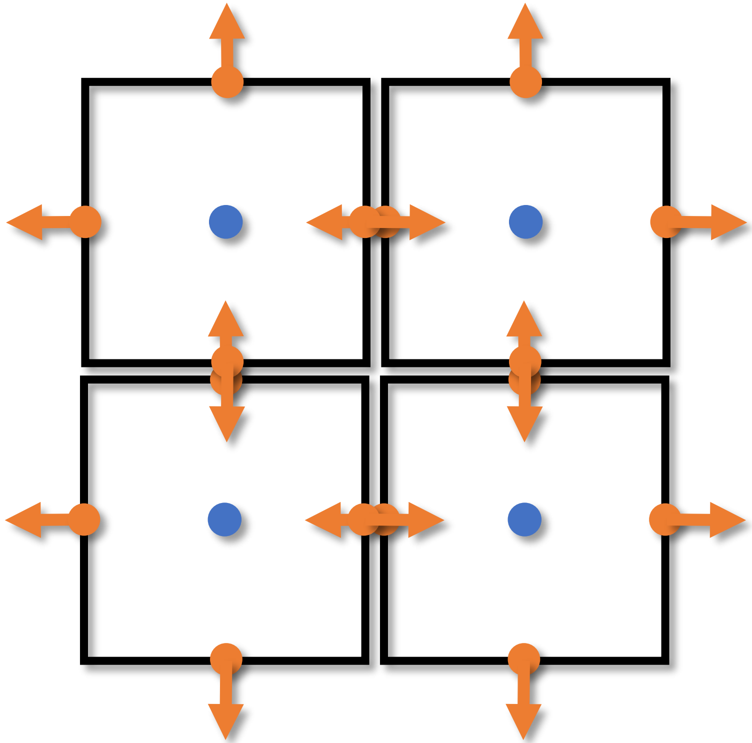
Potential



Flux

H/LBRT – broken RT

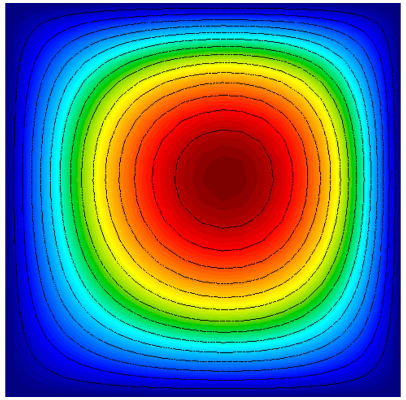
- **Broken Raviart-Thomas** (`BrokenRT_FECollection`)
- *H. Egger, J. Schöberl (2010), IMA J. Numer. Anal., 30, 1206–1234.*
- Local compatibility → *No stabilization*
- Convection-diffusion
- LDG-style system (LBRT)



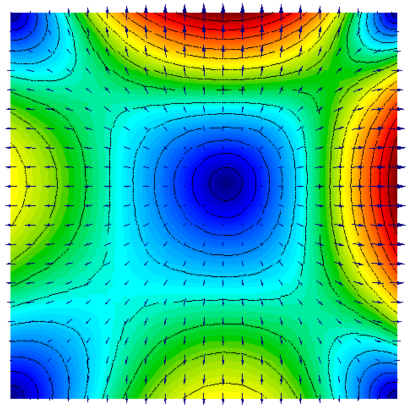
Convection-diffusion (ex5-nguyen.cpp)

- Problem 2 (-p 2) – steady advection-diffusion

```
…
BilinearForm *Mt = GetPotentialMassForm();
…

Mt->AddDomainIntegrator(
     new ConservativeConvectionIntegrator(ccoeff));

if (upwinded) {
    Mt->AddInteriorFaceIntegrator(
      new HDGConvectionUpwindedIntegrator(ccoeff));
} else {
    Mt->AddInteriorFaceIntegrator(
      new HDGConvectionCenteredIntegrator(ccoeff));
   }
}
```

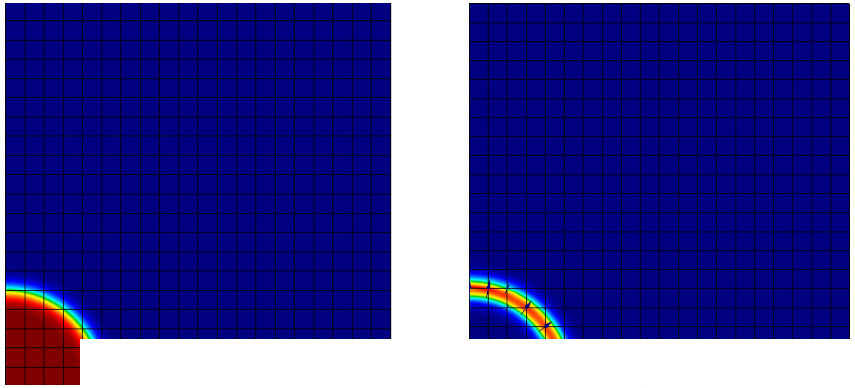


Potential



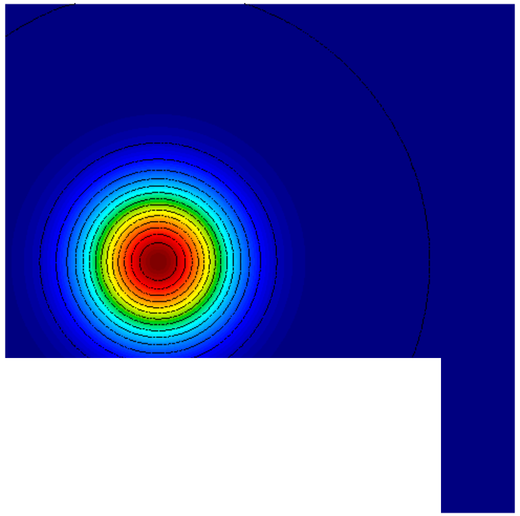
Flux

Example 5 – Nguyen (ex5-nguyen.cpp)



Potential *Flux*

Problem 3 – steady advection

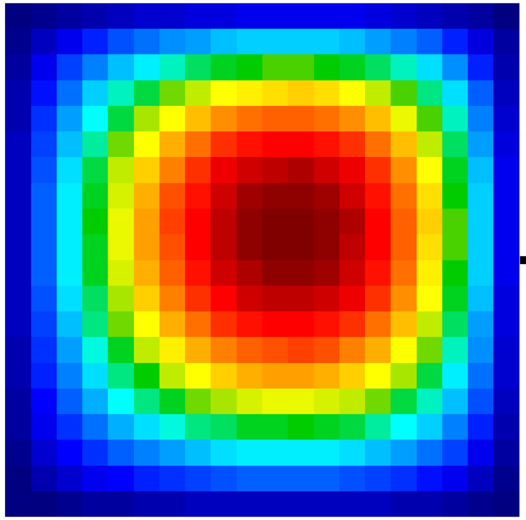


Problem 4 – non-steady advection(-diffusion)

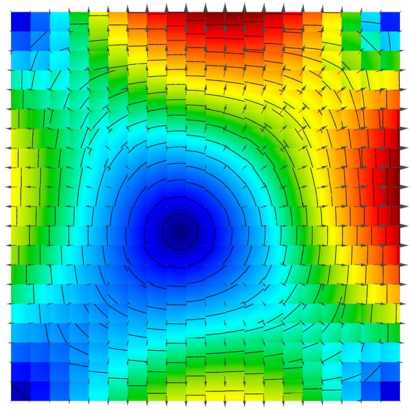
Problem 5 – Kovasznay flow

Superconvergent reconstruction

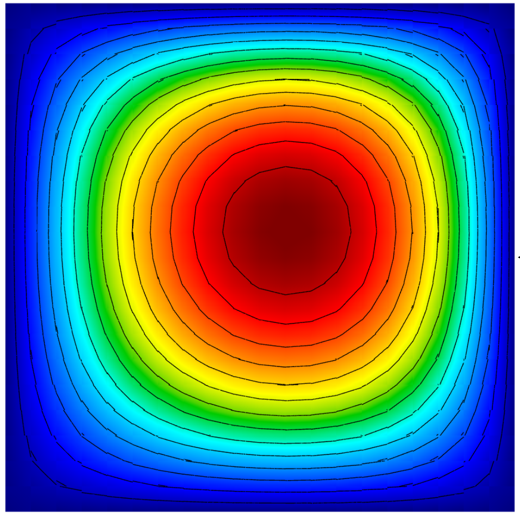
- DG diff. flux (+ DG conv. flux) + HDG trace
  → **RT total flux**
- `DarcyForm::ReconstructTotalFlux()`
  - Auto FE space construction
  - Auto velocity recognition from conv. integs.
- RT total flux (+ DG potential) →
  superconvergent diff. flux + potential
- `DarcyForm::ReconstructFluxAndPot()`
  - Auto FE space construction
  - Auto (non-)steady case treatment



Potential (Q0)



Tot. flux



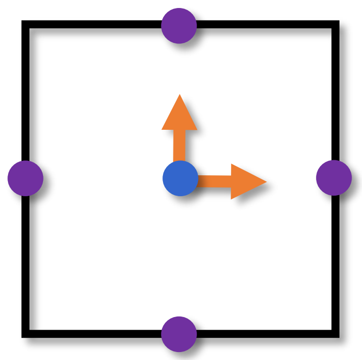
Reconstructed pot. (Q1)

Non-linear convection

- Non-linear flux ***F**(u)*

$$\begin{aligned} \boldsymbol{q} + \kappa \nabla u &= 0, && \text{in} \quad \Omega, \\ \nabla \cdot (\boldsymbol{q} + \boldsymbol{F}(u)) &= f, && \text{in} \quad \Omega, \end{aligned}$$

- `HyperbolicFormIntegrator` + `NumericalFlux`
  - `RusanovFlux`
  - `ComponentwiseUpwindFlux`
  - (`HDGFlux` – HDG-I / HDG-II schemes)



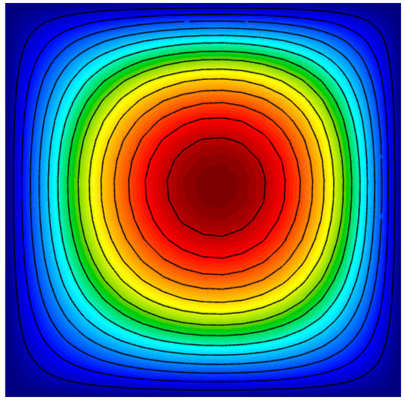
- *N.C. Nguyen, J. Peraire & B. Cockburn (2009), JCP, 228, 8841–8855.*
- `DarcyHybridization` → `Operator`
- Global+Local solver (LBFGS/LBB/Newton)

$$-[C \quad G]\begin{bmatrix} A & -B^T \\ B & D \end{bmatrix}^{-1} \begin{bmatrix} C^T \\ E \end{bmatrix} \Lambda + H\Lambda$$

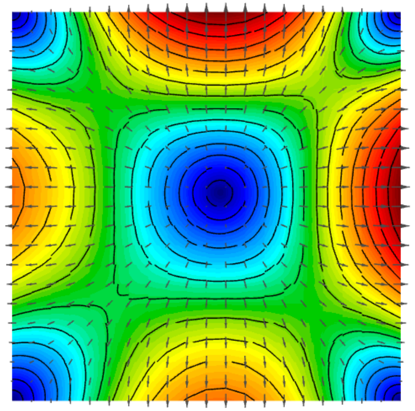
Burgers + diffusion (ex5-nguyen.cpp)

- Problem 6 (-p 6) – steady Burgers-diffusion

```
NonlinearForm *Mtnl = darcy->GetPotentialMassNonlinearForm();
…
FluxFun = new BurgersFlux(ccoef.GetVDim());

switch (hdg_scheme)
{
case 1: FluxSolver = new HDGFlux(*FluxFun,
             HDGFlux::HDGScheme::HDG_1); break;
case 2: FluxSolver = new HDGFlux(*FluxFun,
             HDGFlux::HDGScheme::HDG_2); break;
case 3: FluxSolver = new RusanovFlux(*FluxFun); break;
case 4: FluxSolver = new ComponentwiseUpwindFlux(*FluxFun);
break;
}

Mtnl->AddDomainIntegrator(
             new HyperbolicFormIntegrator(*FluxSolver, 0, -1.));

Mtnl->AddInteriorFaceIntegrator(
             new HyperbolicFormIntegrator(*FluxSolver, 0, -1.));
```



Potential



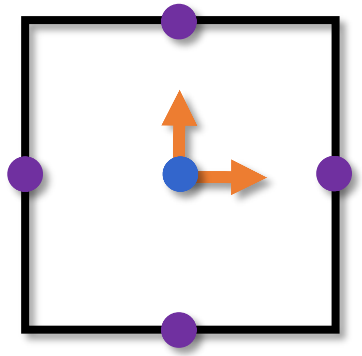
Flux

Non-linear diffusion

- Non-linear conductivity *κ(u)*

$$\begin{aligned} \mathbf{q} + \kappa(u)\nabla u &= 0, \\ \nabla \cdot \mathbf{q} &= 0 \end{aligned}$$

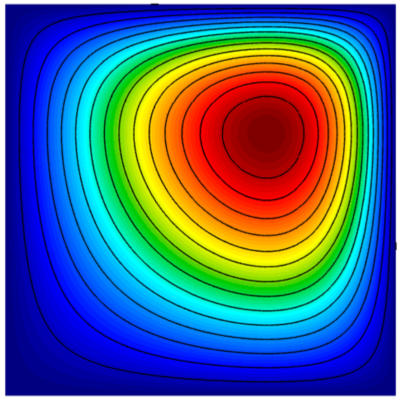
- `MixedConductionNLFIntegrator` + `MixedFluxFunction`
  - `LinearDiffusionFlux`
  - `FunctionDiffusionFlux`
- `BlockNonlinearForm + BlockOperator`
- Global+Local solver (LBFGS/LBB/Newton)



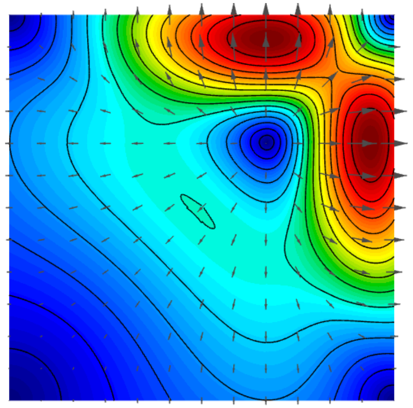
Non-linear diffusion (ex5-nguyen.cpp)

- Problem 8 (-p 8) – lin. conductivity (*κ(u)* = *k*+*u*)

```
BlockNonlinearForm *Mnl = darcy->GetBlockNonlinearForm();
…
auto ikappa = [=](const Vector &x, real_t u)
                   { return 1./(k+u); };
auto dikappa = [=](const Vector &x, real_t u)
                   { return -1./((k+u)*(k+u)); };

 HeatFluxFun = new FunctionDiffusionFlux(dim, ikappa, dikappa);

 Mnl->AddDomainIntegrator(
          new MixedConductionNLFIntegrator(*HeatFluxFun));

 if (upwinded) {
     Mnl->AddInteriorFaceIntegrator(
          new MixedConductionNLFIntegrator(*HeatFluxFun, ccoef));
 } else {
     Mnl->AddInteriorFaceIntegrator(
          new MixedConductionNLFIntegrator(*HeatFluxFun));
 }
```



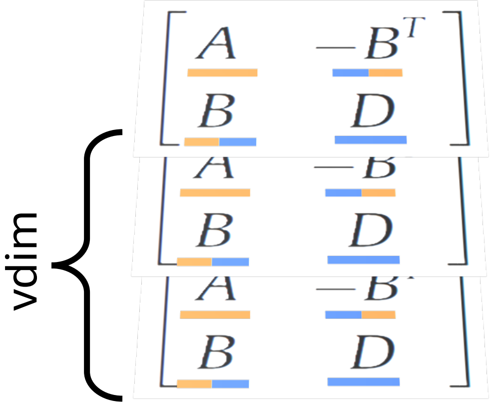
Potential

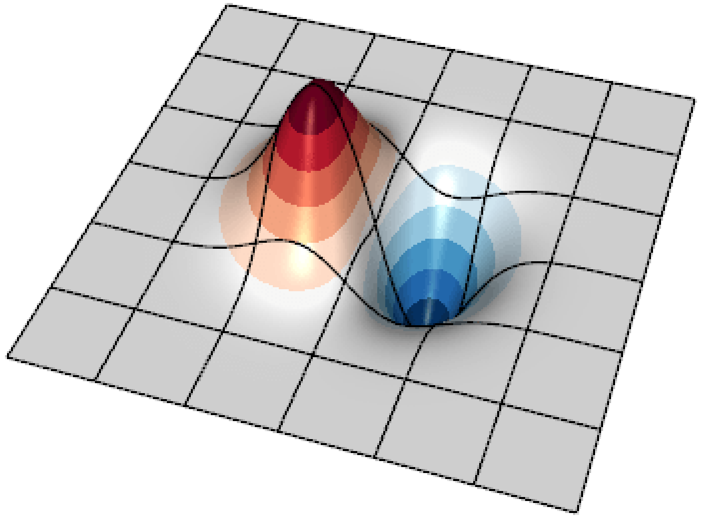


Flux

Systems of equations [WIP]

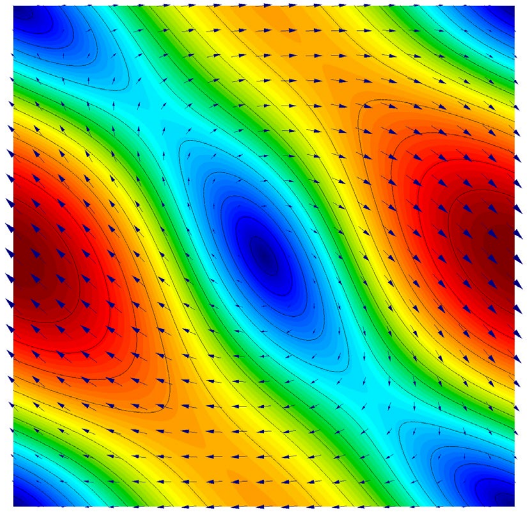
- Vector dimension
- `VectorBlockDiagonalIntegrator`
- *Implicit* Euler equations with diffusion (`ex18-hdg.cpp`) – `HyperbolicFormIntegrator` + `EulerFlux` + `DarcyOperator`
- Status:
  - Mixed ✓
  - Reduced (linear) ✓
  - Hybridized [WIP]



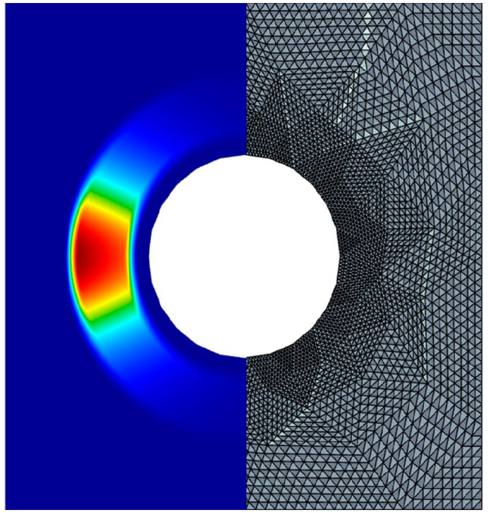


Anisotropic diffusion (`ex5-aniso.cpp`)

- Problems:
  - Stationary/asymptotic diffusion
  - MFEM text random conv-diffusion
  - Diffusion ring arc/Gauss/sine
  - Boundary layer
  - Steady peak/varying angle
  - Sovinec
  - Umansky



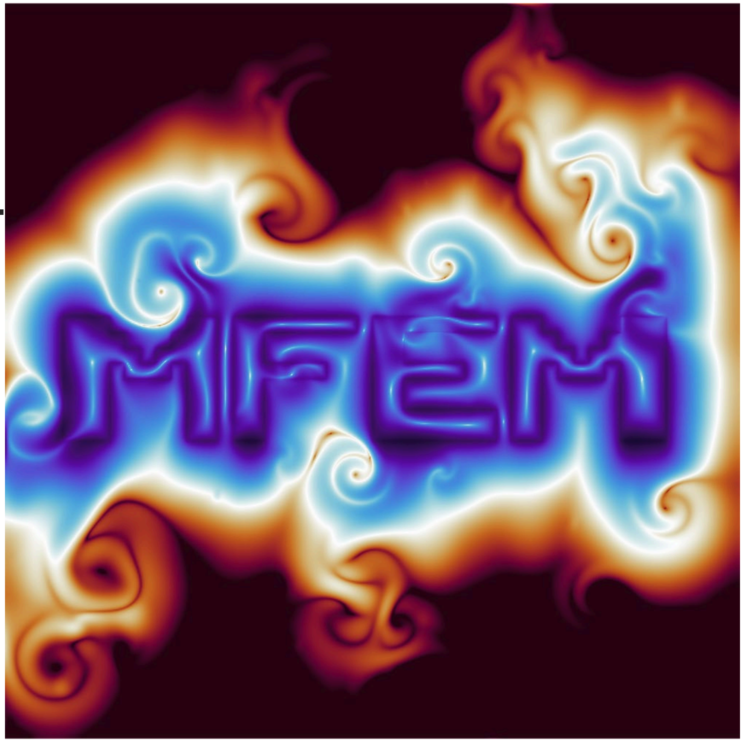
Steady diffusion (`-p 1`)



Diffusion ring (`-p 3`)

Conclusions

- Framework for mixed systems – `(Par)DarcyForm`
- Potential/flux reduction (`DarcyReduction`)
- Total flux hybridization (`DarcyHybridization`) – reduced system, preconditioning, convergence, stabilization, …
- Superconv. reconstruction – H(div) total flux
- Non-linear convection / diffusion
- Systems of equations [WIP]
- TODOs – miniapps, GPUs, Maxwell, …



Single-step anisotropic diffusion-convection simulation

Thank you for your attention

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