Math 362 Assignment 1

Due: Wednesday, September 18

- Answer all questions. Each question is worth 5 marks. Full marks will be awarded only for answers that are both mathematically correct and coherently written.
- Please consider the markers and write neatly and legibly! I have instructed the markers to ignore work they cannot read. (And I won't read it, either.)
- No help given with questions marked *.

1. *Prove by induction that $30 | (19 \cdot 7^{8n} + 11)$ for all integers $n \geq 0$. Follow the example on p. 2.

   Hint: $19 \cdot 7^8 \cdot 7^{8k} + 11 = 19 \cdot 7^8 \cdot 7^{8k} + (7^8 \cdot 11 - 7^8 \cdot 11) + 11$

2. Prove the formula for the **sum of a geometric sequence** by mathematical induction:

   If n is a nonnegative integer, then
   $$\sum_{i=0}^{n} a^i = \frac{a^{n+1} - 1}{a - 1}, \quad a \neq 1.$$

   Note: Remember this formula!

3. Use the formula in Question 2 to prove that if n is composite, then $2^n - 1$ is composite.
   Note: Remember this result!

   Hint: Write $n = pq$ and $2^n = 2^{pq} = (2^p)^q$, and let $a = 2^p$ in the formula.

4. Let n be composite and let p be the smallest prime factor of n. Prove that if $p > n^{1/3}$, then n/p is prime.

5. Use the Euclidean algorithm and other results to explain why the linear Diophantine equation
   $$273x + 401y = 162$$
   has a solution. Find all solutions (x, y) such that $x, y \geq -4$. (Work the Euclidean algorithm backwards to get the first solution.)

6. Find infinitely many integers x, y, z that satisfy the equation $10x + 25y + 19z = 0$. (You don't have to find them all.)

7. An integer n is *square* if $n = r^2$ for some integer r, and *triangular* if $n = 1+2+3+\cdots+s$ for some integer n. The smallest integer that is both square and triangular is 1.

   (a) Find the next two numbers that are both triangular and square.

   (b) *Prove that n is triangular if and only if $8n + 1$ is square.

Example for Question 1.

Prove by induction that $3|(5^{2n} - 4^n)$ for all nonnegative integers n.

Answer

Let P(n) be the statement "$3|(5^{2n} - 4^n)$".

Basis Step: Since $5^{2\cdot 0} - 4^0 = 0$ and $3|0$, P(0) is true.

Induction hypothesis: Assume that $3|(5^{2k} - 4^k)$ for some integer $k \geq 0$. (That is, assume P(k) is true for some integer $k \geq 0$.)

Now,

$$\begin{aligned} 5^{2(k+1)} - 4^{k+1} &= 25 \cdot 5^{2k} - 4 \cdot 4^k \\ &= 4 \cdot 5^{2k} + 21 \cdot 5^{2k} - 4 \cdot 4^k \\ &= 4(5^{2k} - 4^k) + 21 \cdot 5^{2k}. \end{aligned}$$

But $3|(5^{2k} - 4^k)$ by the IH, and $3|21$, hence $3|(4(5^{2k} - 4^k) + 21 \cdot 5^{2k})$ by Lemma 1.2. Hence $3|(5^{2(k+1)} - 4^{k+1})$. (That is, P($k+1$) is true.)

By the principle of mathematical induction, $3|(5^{2n} - 4^n)$ for all integers $n \geq 0$.