Math 362 Assignment 5

Due: Friday, November 22

- Answer all questions. Each question is worth 5 marks. Full marks will be awarded only for answers that are both mathematically correct and coherently written.
- Please consider the markers and write neatly and legibly! I have instructed the markers to ignore work they cannot read. (And I won't read it, either.)

1. Suppose p is prime, $a \in \Phi(p)$ and $a^{(p-1)/2} \equiv -1 \pmod{p}$. Does it follow that a is a primitive root of p? If so, prove it. If not, give a counterexample.
2. Prove that for any integer $n \geq 3$, the integer 2^n has no primitive roots. Proceed as follows:
   (a) Prove by induction on n that if a is an odd integer, then
   $$a^{2^{n-2}} \equiv 1 \pmod{2^n}$$
   for all $n \geq 3$.
   (b) Deduce the required result.
3. Use Legendre symbols to determine whether the congruence $x^2 \equiv 91 \pmod{103}$ has solutions.
4. Use Legendre symbols to determine whether 7 is a primitive root of the prime 4583.
5. Find (i) the period and (ii) the length of the non-periodical part of the
   (a) decimal expansion of $\frac{45}{252}$
   (b) base 6 expansion of $\frac{45}{252}$
   without finding the expansion itself.
6. Find the decimal expansion of $\frac{5}{28}$.
7. Let $p = 24k-1$, $k \in \mathbb{N}$, be a prime number. Explain why neither 2 nor 3 is a primitive root of p.
8. [4] (Bonus question; no help given) Find an integer n such that the last ten digits of 7^n are 0000000001.