Math 362 Assignment 2

Due: Wednesday, October 2

- Answer all questions. Each question is worth 5 marks. Full marks will be awarded only for answers that are both mathematically correct and coherently written.
- Please consider the markers and write neatly and legibly! I have instructed the markers to ignore work they cannot read. (And I won't read it, either.)

1. Squares:

   (a) The prime power decomposition (ppd) of an integer $n \geq 2$ is given by $n = p_1^{e_1} p_2^{e_2} ... p_k^{e_k}$. Give a necessary and sufficient condition for n to be a square.

   (b) Which integers n, where $2 \leq n \leq 20$, can be written as the sum of **two** squares? For example, $9 = 0^2 + 3^2$. Express each such integer as such a sum. Also give the ppd of n.

   (c) Show that if $n \equiv 3 \pmod 4$, then n cannot be written as the sum of the squares of two integers. (**Remember this result!**)

   (d) Which integers n, where $2 \leq n \leq 20$, can be written as the sum of **three** squares? Express each such integer as such a sum.

   [Obviously, any integer that can be written as the sum of two squares can be written as the sum of three squares – just add (another) 0^2. You only need to consider the remaining numbers.]

   (e) What do the numbers n, where $2 \leq n \leq 20$, that cannot be expressed as the sum of three squares have in common? (Think of a property that is not shared by any other number from 2 to 20.)

2. (a) Given that n is a positive integer, determine the least residue of $2485234^n \pmod 9$.

   (b) Given that $k \equiv 3 \pmod 5$, determine the least residue of $2k^2 + 19^3 \pmod 5$.

   (c) Given that $k \equiv 3 \pmod 8$, determine the least residue of $5k^{333} + 23^{123} \pmod 8$.

3. Solve the congruences

   (a) $7x \equiv 13 \pmod{15}$

   (b) $8x \equiv 12 \pmod{28}$

   (c) $13x \equiv 352 \pmod{1261}$.

4. A school has between 500 and 800 students who registered to participate in sports days to raise money for charity. On Sports Day 1, they all play tennis. Each tennis team consists of four students, two to play singles matches, and a pair to play doubles matches. So they are grouped into teams of size four, but three students are left over. On Sports Day 2, they all play cricket. Each cricket team consists of 11 students. So

they are grouped into teams of size 11, but nine students are left over. On Sports Day 3, they all play rugby. Each rugby team consists of 15 students. So they are grouped into teams of size 15, and no students are left over.

How many students registered to participate in sports days?

5. The same school as above has between 150 and 200 students who have registered for grade 12 chemistry, grade 12 physics and grade 11 math. The grade 12 chemistry lab can only accommodate 12 students at a time, the physics lab can accommodate 20 students at a time, and the math classroom with computers can accommodate 36 students at a time. When the administrative assistant groups these students in classes of size 12, 20 or 36, seven students are always left over.

   How many students registered to take these three subjects?

6. Use Fermat's Theorem to determine the least residue of

   (a) $3^{311} \pmod{23}$

   (b) $24^{36n+3} + 11 \pmod{19}$, where $n \in \mathbb{N}$.

7. (a) What is the remainder when 2019^{2019} is divided by 22? (No long stories!)

   (b) What is the remainder when $2019^{2019} \cdot 18!$ is divided by 247?