Math 362 Assignment 1

Due: Wednesday, September 18

1. *Prove by induction that $30|(19 \cdot 7^{8n} + 11)$ for all integers $n \geq 0$. Follow the example on p. 2.

   Answer

   Let P(n) be the statement "$30|(19 \cdot 7^{8n} + 11)$".

   Basis Step: Since $19 \cdot 7^0 + 11 = 30$, P(0) is true.

   Induction hypothesis: Assume that $30|(19 \cdot 7^{8k} + 11)$ for some integer $k \geq 0$.

   (That is, assume P(k) is true for some integer $k \geq 0$.)

   To prove: P($k + 1$) is true, that is, $30|(19 \cdot 7^{8(k+1)} + 11)$.

   Now,

$$
\begin{aligned}
19 \cdot 7^{8(k+1)} + 11 &= 19 \cdot 7^8 \cdot 7^{8k} + 11 = 19 \cdot 7^8 \cdot 7^{8k} + (7^8 \cdot 11 - 7^8 \cdot 11) + 11 \\
&= (19 \cdot 7^8 \cdot 7^{8k} + 7^8 \cdot 11) - 7^8 \cdot 11 + 11 \\
&= 7^8(19 \cdot 7^{8k} + 11) - 7^8 \cdot 11 + 11 \\
&= 7^8(19 \cdot 7^{8k} + 11) - 63\,412\,800 \\
&= 7^8(19 \cdot 7^{8k} + 11) - 2113\,760 \cdot 30
\end{aligned}
$$

   By the IH, $30|(19 \cdot 7^{8k} + 11)$, hence $30|7^8(19 \cdot 7^{8k} + 11)$, and clearly $30|2113\,760 \cdot 30$. Therefore $30|(19 \cdot 7^{8(k+1)} + 11)$, that is, P($k + 1$) is true.

   By the principle of mathematical induction, $30|(19 \cdot 7^{8n} + 11)$ for all integers $n \geq 0$.

2. Prove the formula for the **sum of a geometric sequence** by mathematical induction: If n is a nonnegative integer, then

$$\sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1}, \quad a \neq 1.$$

Note: Remember this formula!

Answer

Let $\mathrm{P}(n)$ be the statement

$$\sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1}, \quad a \neq 1.$$

Basis step: $\mathrm{P}(0)$ is the statement

$$a^0 = \frac{a-1}{a-1}, \quad a \neq 1,$$

which is obviously true because $a^0 = 1$ and $\frac{a-1}{a-1} = 1$ if $a \neq 0$.

Induction hypothesis: Assume that $\mathrm{P}(k)$ is true for some $k \geq 0$, that is, assume that

$$\sum_{i=0}^{k} a^i = \frac{a^{k+1}-1}{a-1}, \quad a \neq 1. \tag{2.1}$$

To prove: $\mathrm{P}(k+1)$ is true, that is, using (2.1) we must show that

$$\sum_{i=0}^{k+1} a^i = \frac{a^{k+2}-1}{a-1}, \quad a \neq 1.$$

Now,

$$\begin{aligned}
\sum_{i=0}^{k+1} a^i &= \sum_{i=0}^{k} a^i + a^{k+1} \\
&= \frac{a^{k+1}-1}{a-1} + a^{k+1} \qquad \text{(by (2.1))} \\
&= \frac{a^{k+1}-1+a^{k+2}-a^{k+1}}{a-1} \\
&= \frac{a^{k+2}-1}{a-1}
\end{aligned}$$

as required. Thus $\mathrm{P}(k+1)$ is true.

The result follows by the principle of mathematical induction.

3. Use the formula in Question 2 to prove that if n is composite, then $2^n - 1$ is composite.
Note: Remember this result!

Hint: Write $n = pq$ and $2^n = 2^{pq} = (2^p)^q$, and let $a = 2^p$ in the formula.

Answer

If n is composite, we can write $n = pq$, where $2 \leq p \leq q < n$. Hence $2^n = 2^{pq} = (2^p)^q$. Let $a = 2^p$. Then $a \neq 1$ because $p \geq 2$, and

$$\frac{a^q - 1}{a - 1} = \sum_{i=0}^{q-1} a^i$$

(Question 2). Hence

$$a^q - 1 = (a - 1)(1 + a + \cdots a^{q-2} + a^{q-1}).$$

Substituting $a = 2^p$ in the left-hand side, we get

$$(2^p)^q - 1 = (a - 1)(1 + a + \cdots a^{q-2} + a^{q-1}).$$

Now $a - 1 = 2^p - 1 \geq 2$ because $p \geq 2$, and $(1 + a + \cdots a^{q-2} + a^{q-1}) \geq 2$ because $a, q > 1$. Thus $2^n - 1$ is composite.

4. Let n be composite and let p be the smallest prime factor of n. Prove that if $p > n^{1/3}$, then n/p is prime.

Answer

Since n is composite and p is a prime factor of n, $n/p > 1$. Suppose, to the contrary, that n/p is not prime. Since $n/p > 1$, n/p is composite. Therefore there exist integers r and s such that $1 < r, s < n/p$ and $n/p = rs$, that is, $n = prs$. Then $r|n$ and $s|n$. Since p is the smallest prime factor of n, $r \geq p$ and $s \geq p$. Therefore

$$n = prs \geq p^3 > (n^{1/3})^3 = n,$$

which is impossible. Therefore n/p is prime.

5. Use the Euclidean algorithm and other results to explain why the linear Diophantine equation

$$273x + 401y = 162$$

has a solution. Find all solutions (x, y) such that $x, y \geq -4$. (Work the Euclidean algorithm backwards to get the first solution.)

Answer

First find $(273, 401)$. Since

$$\begin{aligned}
401 &= 273 \cdot 1 + 128 \\
273 &= 128 \cdot 2 + 17 \\
128 &= 17 \cdot 7 + 9 \\
17 &= 9 \cdot 1 + 8 \\
9 &= 8 \cdot 1 + 1,
\end{aligned}$$

$(273, 401) = 1$. Since $1|162$, the equation has a solution.

Working backwards, we get

$$\begin{aligned}
1 = 9 - 8 &= 9 - (17 - 9) = 9 \cdot 2 - 17 \\
&= (128 - 17 \cdot 7) \cdot 2 - 17 = 128 \cdot 2 - 17 \cdot 15 \\
&= 128 \cdot 2 - (273 - 128 \cdot 2) \cdot 15 = 128 \cdot 32 - 273 \cdot 15 \\
&= (401 - 273) \cdot 32 - 273 \cdot 15 = 401 \cdot 32 - 273 \cdot 47.
\end{aligned}$$

Therefore

$$401 \cdot 32 - 273 \cdot 47 = 1.$$

Multiply both sides by 162:

$$401 \cdot 5184 - 273 \cdot 7614 = 162.$$

Therefore $(x, y) = (-7614, 5184)$ is one solution. By Theorem 3.1 all solutions are given by

$$\left.\begin{array}{r} x = -7614 + 401t \\ y = 5184 - 273t \end{array}\right\}, t \in \mathbb{Z}.$$

Now suppose $-7614 + 401t \geq -4$. Then

$$t \geq \frac{-4 + 7614}{401} \approx 18.978,$$

i.e. $t \geq 19$.

On the other hand, if $5184 - 273t \geq -4$, then

$$t \leq \frac{5184 + 4}{273} \approx 19.004.$$

That is, $t \leq 19$, and so there is only one such solution: when $t = 19$, we get $x = -7614 + 401 \cdot 19 = 5$ and $y = 5184 - 273 \cdot 19 = -3$. The only solution is

$$\boxed{(x, y) = (5, -3)}.$$

Test: $273 \cdot 5 + 401(-3) = 162$.

6. Find infinitely many integers x, y, z that satisfy the equation $10x + 25y + 19z = 0$. (You don't have to find them all.)

Answer

(Any infinite set of correct solutions is sufficient.)

Long story: Note that $10x + 25y = 5(2x + 5y)$. Let $w = 2x + 5y$ and solve the equation $5w + 19z = 0$. One solution is $(w, z) = (19, -5)$, hence all solutions are

$$\left.\begin{array}{l} w = 19 + 19t \\ z = -5 - 5t \end{array}\right\}, t \in \mathbb{Z}. \tag{1}$$

Now solve $2x + 5y = 19 + 19t$; for example, $x = 2 + 2t$ and $y = 3 + 3t$ is a solution. All solutions are

$$\left.\begin{array}{l} x = 2 + 2t + 5r \\ y = 3 + 3t - 2r \end{array}\right\}, r \in \mathbb{Z}. \tag{2}$$

Combining (1) and (2), we get

$$\left.\begin{array}{l} x = 2 + 2t + 5r \\ y = 3 + 3t - 2r \\ z = -5 - 5t, \end{array}\right\}, r, t \in \mathbb{Z},$$

which gives infinitely many solutions.

[Test: Take (for example) $t = r = 0$. Then $10(2) + 25(3) + 19(-5) = 0$ as required. Or, $t = r = 1$. Then $10(9) + 25(4) + 19(-10) = 0$.]

Short story: Let $z = 0$ (or $x = 0$ or $y = 0$) and solve the equation $10x + 25y = 0$, i.e., $2x + 5y = 0$. One solution is $(x, y) = (5, -2)$, hence

$$\left.\begin{array}{l} x = 5 + 5t \\ y = -2 - 2t \\ z = 0 \end{array}\right\}, t \in \mathbb{Z}$$

gives infinitely many solutions of the original equation.

7. An integer n is *square* if $n = r^2$ for some integer r, and *triangular* if $n = 1+2+3+\cdots+s$ for some integer n. The smallest integer that is both square and triangular is 1.

   (a) Find the next two numbers that are both triangular and square.

   (b) *Prove that n is triangular if and only if $8n+1$ is square.

Answer

(a) From Math 122,

$$1+2+3+\cdots+s = \frac{s(s+1)}{2}.$$

Therefore we need a number $n = r^2 = \frac{s(s+1)}{2}$, where r and s are integers. With trial and error (and thinking of values of s such that s or $s+1$ is an odd square), substituting different values of s and checking whether we get a square, we see that for $s = 8$,

$$\frac{s(s+1)}{2} = \boxed{36} = 6^2,$$

and for $s = 49$,

$$\frac{49(49+1)}{2} = \boxed{1225} = 35^2.$$

(b) Suppose n is triangular. Then $n = \frac{s(s+1)}{2}$, that is, $s^2 + s - 2n = 0$. Using the quadratic formula to solve for s (and remembering that $s > 0$), we get

$$s = \frac{-1+\sqrt{8n+1}}{2}.$$

Now, s is an integer if and only if $\sqrt{8n+1}$ is an odd integer, which happens if and only if $8n+1$ is a square. [If it is a square, it is the square of an odd integer, because $8n+1$ is odd for all n.]

Example for Question 1.

Prove by induction that $3|(5^{2n} - 4^n)$ for all nonnegative integers n.

Answer

Let $\mathrm{P}(n)$ be the statement "$3|(5^{2n} - 4^n)$".

Basis Step: Since $5^{2\cdot 0} - 4^0 = 0$ and $3|0$, $\mathrm{P}(0)$ is true.

Induction hypothesis: Assume that $3|(5^{2k} - 4^k)$ for some integer $k \geq 0$. (That is, assume $\mathrm{P}(k)$ is true for some integer $k \geq 0$.)

Now,

$$\begin{aligned} 5^{2(k+1)} - 4^{k+1} &= 25 \cdot 5^{2k} - 4 \cdot 4^k \\ &= 4 \cdot 5^{2k} + 21 \cdot 5^{2k} - 4 \cdot 4^k \\ &= 4(5^{2k} - 4^k) + 21 \cdot 5^{2k}. \end{aligned}$$

But $3|(5^{2k} - 4^k)$ by the IH, and $3|21$, hence $3|(4(5^{2k} - 4^k) + 21 \cdot 5^{2k})$ by Lemma 1.2. Hence $3|(5^{2(k+1)} - 4^{k+1})$. (That is, $\mathrm{P}(k+1)$ is true.)

By the principle of mathematical induction, $3|(5^{2n} - 4^n)$ for all integers $n \geq 0$.