# Math 362 Assignment 6

Due: Tuesday, December 3

- Answer all questions. Each question is worth 5 marks. Full marks will be awarded only for answers that are both mathematically correct and coherently written.
- Please consider the markers and write neatly and legibly! I have instructed the markers to ignore work they cannot read. (And I won't read it, either.)

1. Find **seven(!)** Pythagorean triangles with a side of length 24. Does there exist one with a hypotenuse of length 24?

**Answer**

Factor 24: $24 = 2^3 3$.

For a fundamental Pythagorean triangle, let $a = 24 = 2mn$, where

$(i)$ $m = 4$, $n = 3$. Then $b = m^2 - n^2 = 7$ and $c = m^2 + n^2 = 25$. Hence $\underline{(24, 7, 25)}$ is an FPT with side length 24;

$(ii)$ $m = 12$, $n = 1$. Then $b = 143$ and $c = 145$. Hence $\underline{(24, 143, 145)}$ is an FPT with side length 24.

Next, let $a = 12 = 2mn$, where

$(i)$ $m = 3$, $n = 2$. Then $b = 5$ and $c = 13$. Hence $(12, 5, 13)$ is an FPT with side length 12, so that $\underline{(24, 10, 26)}$ is a Pythagorean triangle with side length 24;

$(ii)$ $m = 6$, $n = 1$. Then $b = 35$ and $n = 37$. Hence $(12, 35, 37)$ is an FPT with side length 12, so that $\underline{(24, 70, 74)}$ is a Pythagorean triangle with side length 24.

Suppose $a = 8 = 2mn$, where $m = 4$ and $n = 1$. Then $m^2-n^2 = 15$ and $m^2+n^2 = 17$. Hence $(8, 15, 17)$ is an FPT, so $\underline{(24, 45, 51)}$ is a Pythagorean triangle with side length 24.

Suppose $a = 4 = 2mn$, where $m = 2$, $n = 1$. Then $b = 3$ and $c = 5$, so $(4, 3, 5)$ is an FPT, which (by multiplying by 8 and 6, respectively) makes $\underline{(32, 24, 40)}$ and $\underline{(24, 18, 30)}$ Pythagorean triangles with a side of length 24.

Suppose next that $b = 3$. Then

$$m^2 - n^2 = (m - n)(m + n) = 1 \cdot 3,$$

hence $m - n = 1$, $m + n = 3$, so that $m = 2$ and $n = 1$. This also gives the Pythagorean triangle $\underline{(32, 24, 40)}$ with side length 24.

Finally, suppose there is a Pythagorean triangle with hypotenuse 24. Then there is an FPT with hypotenuse 1 or 3. Both cases are impossible because neither 1 nor 3 can be written as the sum of the square of two positive integers. Hence no Pythagorean triangle has hypotenuse 24.

2. Find two pairs of relatively prime positive integers $(x, y)$ such that $x^2 + 5929 = y^2$.

**Answer**

Since 5929 is odd, we need to find a fundamental Pythagorean triple $(a, b, c)$ such that $a^2 + 5929 = c^2$.

First find the square root of 5929: $\sqrt{5929} = 77$.

Now suppose $m^2 - n^2 = (m - n)(m + n) = 77$.

Since $77 = 1 \times 77 = 7 \times 11$, we have the two cases $m - n = 1$ and $m + n = 77$, or $m - n = 7$ and $m + n = 11$.

In the former case, $m = 39$ and $n = 38$, so

$$\begin{aligned} a &= 2mn = 2964 \\ b &= 39^2 - 38^2 = 77 \quad \text{(as expected)} \\ c &= 39^2 + 38^2 = 2965. \end{aligned}$$

Hence $\underline{x = 2964 \text{ and } y = 2965}$.

(Test: $2964^2 + 77^2 = 8791\,225 = 5^2 593^2 = (5 \times 593)^2 = 2965^2$.)

Similarly, if $m - n = 7$ and $m + n = 11$, then $m = 9$ and $n = 2$, so

$$a = 36, \ b = 77, \ c = 85.$$

Hence $\underline{x = 36 \text{ and } y = 85}$.

(Test: $36^2 + 77^2 = 7225 = 5^2 17^2 = 85^2$.)

3. A (finite or infinite) sequence of numbers $a, a + d, a + 2d, ...,$ where $d$ is a constant, is said to be "in arithmetic progression". Show that $3n, 4n, 5n$, where $n = 1, 2, ...,$ are the only Pythagorean triples whose terms are in arithmetic progression.

**Answer**

Suppose $x, y, z \in \mathbb{Z}^+$ are in arithmetic progression and satisfy $x^2 + y^2 = z^2$. Then there exist positive integers $a$ and $d$ such that $x = a - d$, $y = a$, $z = a + d$ such that

$$(a - d)^2 + a^2 = (a + d)^2,$$

that is,

$$2a^2 - 2ad + d^2 = a^2 + 2ad + d^2,$$

from which we deduce that

$$a = 4d.$$

Then $x = 3d$, $y = 4d$ and $z = 5d$, $d \in \mathbb{Z}^+$.

4. Show that the equation $x^2 + y^2 + z^2 = 2xyz$ has no nontrivial integer solutions.

**Answer**

Suppose for a contradiction that the equation has nontrivial integer solutions. Then there exists a solution $x = a,\ y = b,\ z = c$, i.e., $a^2 + b^2 + c^2 = 2abc$.

Since $2abc$ is even, one or all of $a,\ b$ and $c$ are even (the sum of an odd number of odd numbers is odd). Assume without loss of generality that $a$ is even, say $a = 2r$, where $r \in \mathbb{Z}^+$. Then

$$4r^2 + b^2 + c^2 = 4rbc,$$

from which we deduce that $b^2 + c^2 \equiv 0 \pmod 4$. Now, if $b$ and $c$ are both odd, then $b^2 \equiv c^2 \equiv 1 \pmod 4$, hence $b^2 + c^2 \equiv 2 \pmod 4$, which is not the case. Therefore $b$ and $c$ are both even. Let $e \geq 1$ be the largest exponent such that $2^e|a,\ 2^e|b,\ 2^e|c$ and let $a = 2^e r,\ b = 2^e s,\ c = 2^e t$. Then at least one of $a, b, c$ is odd, and

$$2^{2e}r^2 + 2^{2e}s^2 + 2^{2e}t^2 = 2 \cdot 2^{3e}rst,$$

that is,

$$r^2 + s^2 + t^2 = 2 \cdot 2^e rst.$$

But now we have $r^2 + s^2 + t^2 \equiv 0 \pmod 4$, and since the square of any integer is congruent to either 0 or 1 (mod 4), the only possibility is that $r^2 \equiv s^2 \equiv t^2 \equiv 0 \pmod 4$, that is, all of $r, s$ and $t$ are even, a contradiction. We conclude the equation has no nontrivial integer solutions.

5. Find all possible ways of writing $9945 = 3^2 \times 5 \times 13 \times 17$ as the sum of two squares. (No marks for using trial and error!)

**Answer**

Since 9945 contains three primes congruent to 1 (mod 4) to an odd power in its prime power decomposition, there are $2^{3-1} = 4$ ways of writing 9945 as the sum of two squares. we first write 5, 13 and 17 as the sum of two squares:

$$5 = 1^2 + 2^2, \quad 13 = 2^2 + 3^2, \quad 17 = 4^2 + 1^2$$

are the <u>only</u> possibilities (up to the signs and order of the summands). Now use Lemma 18.1 and first apply it to $65 = 5 \times 13$:

$$5 \times 13 = (1^2 + 2^2)(2^2 + 3^2) = (2 + 6)^2 + (3 - 4)^2 = 8^2 + 1^2$$

and

$$5 \times 13 = (1^2 + 2^2)(3^2 + 2^2) = (3 + 4)^2 + (2 - 6)^2 = 7^2 + 4^2.$$

Next, apply Lemma 18.1 to $65 \times 17 = 1105$:

$$\begin{aligned} 65 \times 17 &= (8^2 + 1^2)(4^2 + 1) = 33^2 + 4^2 \\ &= (8^2 + 1^2)(1 + 4^2) = 12^2 + 31^2 \end{aligned}$$

and

$$\begin{aligned}65 \times 17 &= (7^2+4^2)(4^2+1) = 32^2+9^2\\ &= (7^2+4^2)(1+4^2) = 23^2+24^2.\end{aligned}$$

Therefore

$$\begin{aligned}9945 &= (3\cdot 33)^2+(3\cdot 4)^2 = 99^2+12^2\\ &= (3\cdot 12)^2+(3\cdot 31)^2 = 36^2+93^2\\ &= (3\cdot 32)^2+(3\cdot 9)^2 = 96^2+27^2\\ &= (3\cdot 23)^2+(3\cdot 24)^2 = 69^2+72^2.\end{aligned}$$

(Test: $99^2+12^2 = 9945 = 36^2+93^2 = 96^2+27 = 69^2+72^2$)

6. Prove that

   (a) a positive integer $n$ is representable as the difference of two squares if and only if $n$ is the product of two numbers that are either both even or both odd;

   (b) a positive even integer $n$ can be written as the difference of two squares if and only if $n \equiv 0 \pmod 4$.

**Answer**

(a) Suppose $n = a^2-b^2$. Then $n = (a-b)(a+b)$. If $a-b$ is even, then $a \equiv b \pmod 2$, hence $a+b \equiv 0 \pmod 2$ and so $a-b$ and $a+b$ are both even. If $a-b$ is odd, then $a \equiv b+1 \pmod 2$, hence $a+b \equiv 1 \pmod 2$ and so $a-b$ and $a+b$ are both odd. Moreover, $n$ is the product of $a+b$ and $a-b$.

Conversely, suppose $n = ab$, where $a \equiv b \pmod 2$. Assume first that $a$ and $b$ are both even, say $a = 2r$ and $b = 2s$. Then

$$(rs+1)^2-(rs-1)^2 = 4rs = ab = n.$$

Assume next that $a$ and $b$ are both odd. Then $n = ab$ is also odd; say $n = 2r+1$. Now we have $(r+1)^2-(r)^2 = 2r+1 = n$.

(b) Suppose $n$ is even and $n = a^2-b^2$. As shown in (a), $a \equiv b \equiv 0 \pmod 2$ and hence $a^2 \equiv b^2 \equiv 0 \pmod 4$, from which it follows that $n = a^2-b^2 \equiv 0 \pmod 4$.

Conversely, if $n \equiv 0 \pmod 4$, then $n$ can be written as the product of two even numbers, and again the result follows from (a).

7. Find all primes $p$ for which $29p + 1$ is a perfect square.

**Answer**

Suppose $29p+1$ is a perfect square. Then there exists an integer $a$ such that $29p+1 = a^2$, that is, $29p = a^2 - 1 = (a-1)(a+1)$. By the Unique Factorization Theorem, $29p$ and $(a-1)(a+1)$ have the same prime power decomposition, hence we either have $29 = a - 1$ and $p = a + 1$, or $p = a - 1$ and $29 = a + 1$.

In the former case, $a = 30$ and $p = 31$, and in the latter case, $a = 28$ and $p = 27$. Since 27 is not prime, we deduce that $p = 31$ (and $a = 30$).