Analysis Homework 1

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1. (a) $2^{|X|}$

   (b) 2^N – we can assume that the elements of $\mathscr{C}$ are disjoint by replacing C_i with $C_i \setminus \bigcup_{j=1}^{i-1} C_j$ – this is a collection of N disjoint subsets that generates the same algebra. And we can then think of each of the C_i as singletons, since the algebra operations can't separate their elements. And then the algebra generated by $\mathscr{C}$ is just $\mathscr{P}(\bigcup \mathscr{C})$.

   (c) This follows from a similar argument to part (b). The only part that could fail is that one of the disjoint C_i could be empty, but if we throw away all the empty ones, we still get 2^n possible subsets, where n is the number of nonempty C_i.

   (d) Probably not – if $\mathscr{F}$ is a σ-algebra generated by countably many elements, then we could use the same procedure to make them disjoint. But then the σ-algebra operations can't separate their elements so we can once more treat them as singletons. But then we just get $\mathscr{P}(\mathbb{N})$, which is uncountable.

2. It's an elementary fact that inverses preserve all of the algebra operations: $f^{-1}(A^c) = f^{-1}(A)^c$, $f^{-1}(\bigcup A_i) = \bigcup f^{-1}(A_i)$, and that's really all we need.

3. We need to check that $\mathscr{F}$ is a σ-algebra. Clearly $\mathscr{F}$ is closed under complements. And countable unions of countable sets are countable, and any union one of whose terms has countable complement, has countable complement. So $\mathscr{F} \supset \sigma(\mathscr{P})$. And clearly $\sigma(\mathscr{P})$ contains all countable sets, and therefore all sets with countable complements, so $\sigma(\mathscr{P}) \supset \mathscr{F}$.

4. The union of two overlapping such intervals is just another such interval, so we can write elements of $\mathscr{J}$ as $\bigcup_{i=1}^n (a_i, a_i']$ with $a_i < a_i' < a_j$ for $i < j$ (if $a_i' = a_j$, we can also combine to the interval $(a_i, a_j'])$. And the complement of such an element is just $(-\infty, a_0] \cup (a_0', a_1] \cup \cdots \cup (a_{n-1}', a_n] \cup (a_n', \infty)$, is an element of $\mathscr{J}$. So $\mathscr{J}$ is an algebra (it's clearly closed under finite unions since finite unions of finite unions are finite unions), and so clearly $\mathscr{J}$ is the smallest σ-algebra containing $\mathscr{J}_0$.

   $(a, b) = \bigcup_{n\in\mathbb{N}} (a, b - \frac{1}{n}] \in \mathscr{J}$, so $\sigma(\mathscr{J}) \supset \mathscr{B}(\mathbb{R})$, and similarly $(a, b] = \bigcap (a, b + \frac{1}{n}] \in \mathscr{B}(\mathbb{R})$, so $\mathscr{B}(\mathbb{R}) \supset \sigma(\mathscr{J})$.

5. Let $B_n = \bigcap_{k=n}^\infty A_k$. Note that $B_i \subset B_{i+1}$. Let $B = \bigcup B_i$. $\mu(B) = \sum_{j=0}^\infty \mu(B_j \setminus B_{j-1}$, while $\mu(B_i) = \sum_{j=0}^i \mu(B_j \setminus B_{j-1})$, so $\mu(B_i) \to \mu(B)$. So $\mu(\liminf A_n) = \mu(B) = \lim \mu(B_i)$. But $\mu(B_i) \leq \inf k \geq i\mu(A_i)$, since $B_i \subset A_k$ for every $k \geq i$, so we get that $\mu(\liminf A_n) \leq \liminf \mu(A_n)$.

   If $\mu(X) < \infty$, then since $\limsup A_n = (\liminf A_n^c)^c$, $\mu(\limsup A_n) = \mu(X) - \mu(\liminf A_n^c) \geq \mu(X) - \liminf(\mu(X) - \mu(A_n)) = \limsup \mu(A_n)$.

   However, if $\mu(X) = \infty$, we can take $A_n = (n, \infty)$. Then $\limsup A_n = \emptyset$, but $\mu(A_n) = \infty$.

6. Suppose $A \in \mathscr{F}$ has positive measure. Since A is not an atom, we can find $B \subset A$ such that $0 < \mu(B) < \mu(A)$. Now, $\mu(B) + \mu(A \setminus B) = \mu(A)$, so one of them has measure $\leq \frac{1}{2}\mu(A)$. Repeating this procedure, we can get sets of arbitrarily small positive measure.

7. We can translate intervals, so if $\{I_i\}$ is a cover of A, then $\{I_i + x\}$ is a cover of $A+x$ of the same total length. So $m^*(A+x) \leq m^*(A)$ and vice versa.

   If we can cover B by collections of intervals of arbitrarily small total length, then if we take of cover of A by intervals of total length arbitrarily close to $m^*(A)$, and add in a cover of B by intervals of arbitrarily small total length, then we have a cover of $A \cup B$ by intervals of total length arbitrarily close to $m^*(A)$. So $m^*(A \cup B) = m^*(A)$.

   Finally, suppose A is Lebesgue measurable, and B is an arbitrary subset. Then $m^*(B \cap (A+x)) + m^*(B \setminus (A+x)) = m^*((B-x) \cap A) + m^*((B-x) \setminus A) = m^*(B-x) = m^(B)$, so $A+x$ is measurable, and of course $m(A+x) = m(A)$.

8. In the first case the functions that are constant on each C_i are measurable – clearly such functions are measurable, since then f^{-1} of any set is just a union of C_i, and conversely, supposing $f(C_i)$ has two points, $a < b$, then $f^{-1}(-\infty, \frac{a+b}{2})$ has some points of C_i and not others, and such a set does not arise in $\sigma(\mathscr{C})$ (it's easy to see that $\sigma(\mathscr{C})$ is just the collection of finite unions of C_i). In the second case, every function is measurable.

9. This is just $g^{-1}(0)$, where g is the function $\limsup f_n - \liminf f_n$ on the set $(\limsup f_n)^{-1}(\mathbb{R}) \cap (\liminf f_n)^{-1}(\mathbb{R})$.

10. For the first part, just take the union of a cover of A by open intervals of total length less than $\mu(A)+\epsilon$.

    For the second part, we split $\mathbb{R}$ into countably many closed intervals I_n of length 1 that intersect on a set of measure 0. For each I_n, we have an open set O_n containing $I_n \backslash A$ such that $\mu(O_n) \leq 1 - \mu(A \cap I_n) + \frac{\epsilon}{2^n}$, so if $F_n = I_n \backslash O_n$, then $F_n \subset A \cap I_n$, F_n is closed, and $\mu(F_n) \geq \mu(A \cap I_n) - \frac{\epsilon}{2^n}$. Then $\sum \mu(F_i) \geq \mu(A) - \epsilon$, and so some $\sum_{i=1}^n \mu(F_i) \geq \mu(A) - 2\epsilon$, so we can take $F = \bigcup_{i=1}^n F_i$ if we want compactness.

    Then we can take B_i to be the union of all $F \subset A$ closed, and B_e to be the intersection of all $O \supset A$ open.