Algebra Homework 3

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1. Let $K = \mathbb{Q}[\sqrt{2}, \sqrt{3+\sqrt{3}}]$. Let $\beta = \sqrt{3+\sqrt{3}}$. The roots of the minimal polynomial of $\sqrt{2}$, which are $\pm\sqrt{2}$ are in K. β satisfies $(x^2-3)^2 = 3$, and the other roots of that polynomial are $\pm\sqrt{3\pm\sqrt{3}}$. So to see that K is normal, and therefore Galois, it suffices to check that $\sqrt{3-\sqrt{3}} \in K$. It's easy to see that $\sqrt{3} \in K$, and so $\sqrt{6} \in K$, and this element is just $\sqrt{6} - \beta \in K$. So $K/\mathbb{Q}$ is Galois.

   Let σ swap $\{\pm\sqrt{2}\}$, τ swap $\{\pm\sqrt{3+\sqrt{3}}\}$, and ω swap $\{\sqrt{3\pm\sqrt{3}}$ (and also their negatives). Writing $\beta' = \sqrt{3-\sqrt{3}}$, we see that $\langle \omega, \tau \rangle = D_8$, where $\omega\tau = (\beta'\,\beta\,-\beta'\,-\beta)$ is a rotation and $\omega = (\beta'\,\beta)(-\beta'\,-\beta)$ is a reflection. And σ is central so the Galois group is $C_2 \times D_8$.

   Writing $r = \omega\tau$ and $t = \omega$, we find the subgroups of D_8: $\langle 1 \rangle$, $\langle r \rangle$, $\langle r^2 \rangle$, $\langle t \rangle$, $\langle rt \rangle$, $\langle r^2 t \rangle$, $\langle r^3 t \rangle$, $\langle r^2, t \rangle$, and $\langle r, t \rangle$.

   Then, on second thought σ may not be central because $\sqrt{3+\sqrt{3}}\sqrt{3-\sqrt{3}} = \sqrt{2}\sqrt{3}$, but I'm out of time.

2. Shamelessly copying from Wikipedia, if we let $k = \mathbb{F}_p(T, S)$, where T and S are transcendental, and $K = k(t, s)$, where $t^p = T$ and $s^p = S$, then $[K : k] = [K : k(s)][k(s) : k] = p^2$. But for every $\alpha \in K$, $\alpha^p \in k$, so $[k(\alpha) : k] \leq p$, so this extension has no primitive element. So by the primitive element theorem, there must be infinitely many intermediate fields.

3. (a) It's clear how $\mathrm{Aut}_k(K) \hookrightarrow \mathrm{Hom}_k(K, L)$, so we need to show that K/k is normal if and only if there's nothing else in $\mathrm{Hom}_k(K, L)$, that is, K embeds into every k-extension uniquely. Well, if there's some other way to embed K into L, say $K' \subset L$, $K' \cong K$. Then, there's some element $\alpha \in K' \setminus K$, and if f is the minimal polynomial of α, then K contains some root of f since it's isomorphic to K' but doesn't contain α so is not normal. Conversely, if K/k is not normal, then let $\alpha \in K$ such that its minimal polynomial has a root β in $L \setminus K$, where L is some extension. Then $k[\beta] \cong k[\alpha]$ can be extended to an different embedding $K \hookrightarrow L$ which contains β. And this embedding is in $\mathrm{Hom}_k(K, L) \setminus \mathrm{Aut}_k(K)$.

   (b) It's not clear to me how to canonically extend an element of $\mathrm{Gal}(K/k)$ to an element of $\mathrm{Gal}(L/k)$ for $K \subset L$, but if you can do this then it's clear that this gives a functor.

4. k_p is perfect since if f is an irreducible polynomial in $k_p[X]$ with repeated roots then it's a polynomial over some finite purely inseparable extension of k, $K \subset k_p$. So if α is a root of f then $K[\alpha]$ is a finite purely inseparable extension of k, so $\alpha \in k_p$, a contradiction since f is irreducible. Conversely, if $k \subset K \subset k_p$ with K perfect, then k_p is separable over K. We saw last time that any $\alpha \in k_p$ would have minimal polynomial $(X-\alpha)^{p^n}$ over k, where p is the characteristic. So for α to be separable over K, we must have that $\alpha \in K$, so $k_p = K$.

5. By factoring polynomials, we see that every nonconstant polynomial in $k[X]$ has all of its roots in K. But then every element of the algebraic closure of k is algebraic over k and so is in K. So $K = \overline{k}$ is algebraically closed.

6. (a) $n\hat{\mathbb{Z}}$ is closed, so we have that $\mathbb{Z}/n\mathbb{Z} \subset \hat{\mathbb{Z}}/n\hat{\mathbb{Z}}$ is finite and dense, and $\hat{\mathbb{Z}}/n\hat{\mathbb{Z}}$ is Hausdorff, so $\hat{\mathbb{Z}}/n\hat{\mathbb{Z}} = \overline{\mathbb{Z}/n\mathbb{Z}} = \mathbb{Z}/n\mathbb{Z}$.

(b) Since they have finite index, the subgroups given by $n\hat{\mathbb{Z}}$ are both open and closed. And any open subgroup must be given by its residues modulo finitely many n, so by the Chinese Remainder Theorem, these should be the only open subgroups.

7. (a) k^{ab} is clearly Galois since it's the limit of Galois extensions. So then it suffices to show that $\text{Gal}(\bar{k}_s/k^{\text{ab}}) = \overline{G_k'}$, or that k^{ab} is the fixed field of G_k'. And this is true because a normal extension is abelian precisely when it's fixed by the commutator, and every extension is contained in a normal extension.

   (b) $\mathbb{Q}^{\text{ab}}$ is obtained by adjoining all roots of unity, so the Galois group is just $\varprojlim(\mathbb{Z}/n\mathbb{Z})^\times$.

8. (a) Let S be such a basis. Then $\mathbb{C} = \overline{\mathbb{Q}(S)}$. Assume S is infinite for simplicity. Then one can estimate $|\mathbb{Q}(S)| \leq |\mathbb{Q} \times S \times \mathbb{N}| = |S|$. And $|\overline{\mathbb{Q}(S)}| \leq |\mathbb{Q}(S) \times \mathbb{N}| \leq |S|$. So $|\mathbb{C}| = |S|$. (If it's finite, we can see that $|\overline{\mathbb{Q}(S)}| \leq |\overline{\mathbb{Q}(\mathbb{N})}| = |\mathbb{N}| < |\mathbb{C}|$).

   (b) Any permutation of a transcendence basis can be extended to an automorphism over $\mathbb{Q}$, and clearly the permutation group of an uncountable set is uncountable.