Algebra Homework 1

Mendel Feygelson

October 17, 2011

1. (a) We take N to be the subgroup consisting of upper triangular matrices with 1s on the diagonal. We note that when multiplying two matrices in $A \in B$, their diagonal entries multiply termwise, so a diagonal entry of A^{-1} is the inverse of the corresponding diagonal entry of A, and so if $X \in N$, we see that the diagonal entries of $A^{-1}XA$ are 1, so N is normal. Clearly $N \cap T = 1$, and if we have a matrix $(a_{ij}) \in B$, then multiplying the matrix in T whose diagonal is given by (a_{ii}) by the matrix in N whose entries are given by $(\frac{a_{ij}}{a_{ii}})$ (on the right) gives us back A, so $B = TN$.

   (b) T is abelian, so we need to show that N is solvable. We consider the subspace of R^n where the last entry is zero. N acts on this space like the corresponding group in dimension $n-1$. And the kernel of this action, the subgroup where all of the non-diagnal entries outisde the last column are zero, is isomorphic to the additive group of $n-1$, so is abelian. So by induction, we have that N is solvable.

2. (a) The commutator of S_4 contains every element of the form $\sigma\sigma^\tau$, with σ and τ in S_4. In particular, since the 2-cycles form a single conjugacy class, it contains every element that can be written as the product of two 2-cycles, and these generate A_4, so $S_4' = A_4$. But A_4 is not abelian, so S_4 is not metabelian. However, A_4 is solvable as we'll see later, and S_4/A_4 has order 2, so S_4 is solvable.

   (b) We've seen that every group of order p^2 is abelian, so every group of order p^4 is metabelian (we use the fact that a p-group has a normal subgroup corresponding to every power of p dividing the order of the group, so in particular a group of order p^4 has a normal subgroup of order p^2). So all p-groups of order less than 24 are metabelian. We've also seen that groups of order pq are metabelian. Thus the only orders that remain to check are 12, 18, and 20. But these all have order p^2q and all groups of order p^2q are metabelian as a consequence of the solution to 3(a).

3. (a) If $p^2 < q$, then the Sylow q-subgroup is normal and solvable, and we know that groups of order p^2 are solvable, so G is solvable.

   Suppose $p^2 > q$, and suppose the Sylow p-subgroup is not normal. That means that there are q of them, and that $q \equiv 1 \mod p$. Now if the Sylow q-subgroup is not normal, then there must be at least p of them. If there are exactly p, then $p \equiv 1 \mod q$, which can't happen since $q > p$. So there must be p^2 q-subgroups. Distinct q-groups can only intersect at the identity, so that's $p^2q - p^2$ elements of order q. But that leaves only p^2 elements not of order q, so there can only be one Sylow p-subgroup, which is normal. So either the Sylow p-subgroup or Sylow q-subgroup is normal, so by taking the quotient we see that G is solvable.

   (b) The only orders remaining to check are $24 = 2^3 \cdot 3$, $30 = 2 \cdot 3 \cdot 5$, $32 = 2^5$, $36 = 2^2 \cdot 3^2$, $40 = 2^3 \cdot 5$, $42 = 2 \cdot 3 \cdot 7$, $48 = 2^4 \cdot 3$, $54 = 2 \cdot 3^3$, and $56 = 2^3 \cdot 7$. p-groups are always solvable, so groups of order 32 aren't a concern, and a group of order 54 has a normal Sylow subgroup of order 27, so is solvable.

   In the case of 30, there can be either one or six Sylow 5-subgroups. If there are six, that accounts for 24 elements of order 5, so there cannot be ten Sylow 3-subgroups.

In the case of 40, there can only be one Sylow 5-subgroup. In the case of 42, there can only be one Sylow 7-subgroup. In the case of 56, there must be eight Sylow 7-subgroups, yielding 48 elements of order 7, and only enough room for one Sylow 2-subgroup.

That leaves 24, 36, and 48. I'm going to work out the technique to do these some other time.

4. There is of course the direct product $\mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. To get a different semidirect product, we need a nontrivial homomorphism $\mathbb{Z}/p\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/p^2\mathbb{Z}) = (\mathbb{Z}/p^2\mathbb{Z})^\times$. $(\mathbb{Z}/p^2\mathbb{Z})^\times$ is a cyclic group of order $p(p-1)$, so is isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p-1\mathbb{Z}$. So we can get a semidirect product by mapping $\mathbb{Z}/p\mathbb{Z}$ to the subgroup of $\mathrm{Aut}(\mathbb{Z}/p^2\mathbb{Z})$ isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

5. First, if $q = 2$, then $b = b^{-1}$, and this is just the usual presentation of the dihedral group D_{2p}, and similarly if $p = 2$, this is just the usual presentation of D_{2q} (the relation $ab^{-1}a = b$ follows from the relation $aba = b^{-1}$). Suppose p and q are both odd. We have $a^{-1} = bab = b^{-1}ab^{-1}$, so $a^{-2} = b^{-1}a^2b$, and if r is the inverse of 2 modulo p, then taking this relation to the power r, we see that $a^b = a^{-1}$. But then $1 = abab = b^2 = b^q$, so $b = 1$, so $1 = a^2 = a^p$, so $a = 1$ and the group is trivial.

6. f_* takes a map $h : G \to A$ to $f \circ h : G \to B$, and f^* takes a map $h : B \to G$ to $h \circ f : A \to G$. These are homomorphisms since f is a homomorphism. f_* is injective since f is injective (f has trivial kernel, so if $f_*(h)$ is trivial, then $\mathrm{im}\, h \subset \ker f = 1$, so g is trivial). $\mathrm{im}(g_*(f_*(h)) \subset \mathrm{im}(g \circ f) = 1$, so $g_* \circ f_* = 1$. And if $g_*(h) = 1$ then $\mathrm{im}\, h \subset \ker g = \mathrm{im}\, f$, and so since f injects A into B, it induces an isomorphism $A \cong \mathrm{im}\, f$, so this gives us a map $\tilde{h} : G \to \mathrm{im}\, f \cong A$ such that $f \circ \tilde{h} = h$. So the first diagram is exact.

   Similarly, g^* is injective because g is surjective – if $g^*(h) = 1$, then h must be identically 1 on C. And $f^*(g^*(h)) = h \circ g \circ f \equiv 1$ since $g \circ f = 1$. And if we have $h : B \to G$ such that $h \circ f = 1$, so h is trivial on the image of f, which is the kernel of g, then h factors through g, so $h = \tilde{h} \circ g$ for $\tilde{h} : C \to G$, so h is in the image of g^*.

   Finally, consider the exact sequence $1 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 1$, where the first map is multiplication by 2. We have the identity map $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$, but that doesn't factor through $\mathbb{Z}$ since every map $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$ is trivial, so the first sequence need not be surjective. Also, the identity map $\mathbb{Z} \to \mathbb{Z}$ doesn't factor through multiplication by 2, so we see that the second sequence also isn't surjective.

7. An element of $\mathrm{Hom}(G, H_1) \times_{\mathrm{Hom}(G,K)} \mathrm{Hom}(G, H_2)$ is given by a pair of maps $G \to H_1$ and $G \to H_2$ that give the same map to K. But then by the universal property of pullback, this gives us a unique map $G \to H_1 \times_K H_2$, and conversely, any map $G \to H_1 \times_K H_2$ gives us maps $G \to H_1$ and $G \to H_2$ which give the same map to K. And it's pretty clear that this identification is natural.

   Similarly, an element of $\mathrm{Hom}(H_1, G) \times_{\mathrm{Hom}(K,G)} \mathrm{Hom}(H_2, G)$ is given by two maps $H_1 \to G$ and $H_2 \to G$ which agree on K, and this is precisely what maps $H_1 *_K H_2 \to G$ are.

8. It suffices to check that the group defined by these generators and relations satisfies the universal property of coproduct. Call this group E. We have a map $S_1 \to E$, which lifts to a map $F(S_1) \to E$, and since this map is zero on R_1, it factors through G, so we have a map $G \to E$, and similarly we have a map $H \to E$. Now suppose we have maps f and g from G and H respectively to a group X. These give us maps $S_1 \to G \xrightarrow{f} X$ and $S_2 \to H \xrightarrow{g} X$, and so we have maps from the free groups. And these maps are zero on R_1 and R_2, so this gives us a map $E \to X$, which is uniquely determined since f and g determine where the generators go.