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On the capacity of a relay network with orthogonal components

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ABSTRACT

In this paper, we consider the relay network in which the source can transmit to the relays and the destination over multiple orthogonal frequency bands. In this relay network, the source–destination pair is supported by a set of cascaded intermediate nodes, the relays. Practically, transmission over orthogonal bands is selected since these intermediate nodes cannot simultaneously receive and forward signals over the same frequency band. For this relay network, we initially derive the capacity of the discrete memoryless model and then extend this result to the additive white Gaussian noise (AWGN) channel. Our numerical results show that the developed encoding scheme can attain a rate higher than the available results in the literature. Further, the benefits of optimally allocating the total available power between the source and the two relays is also numerically investigated.

1. Introduction

Cooperative relaying is a widely used technique for providing a high data rate and improving the reliability of modern wireless networks. Cooperation is done by utilizing another's resources like antennas, power, and bandwidth. The simplest form of a cooperative relay network consists of a source (S) and destination (D) aided by a third node, the relay (R) [1,2], as shown in Fig. 1. Nowadays, the relay is a fundamental components of many wireless networks including the 4G wireless communication. This three-terminal network was originally proposed by Meulen [3] and then comprehensively studied by Cover and El Gamal [1]. Simultaneous to these results, this channel was further investigated in many different scenarios like physically degraded Gaussian relay channel [1,4], fading relay channel [5,6], relay channel with multiple antennas [4,7] and relay channel with orthogonal components [8–10]. Additionally, the capacity of the relay channel is known in a few cases including degraded relay channel [1,11], semi-deterministic relay channel [12], and relay with orthogonal components [8,13]. Moreover, practical coding techniques for relay channel were studied in many different scenarios like [14,15].

The relay channel with orthogonal channel components is motivated by the fact that the relay cannot simultaneously receive and transmit using the same band in practical wireless communication. One solution for this problem is to use a half-duplex relay such that this node does not simultaneously receive and transmit [16,10,17]. Another solution was investigated in [18] in which the source transmits to the relay and destination in a band, and the relay transmits to the destination by using another frequency band. A third solution is obtained by employing a relay with out-of-band reception and in-band transmission [8,9,19]. Practically, consider a multi-standard nodes that can communicate simultaneously over multiple radio interfaces. For instance, recognize a cellular 4G user communicates with the attached base-station is aided by components from other networks like Wi-Fi, Wi-MAX, and Bluetooth. To make a forward step, this paper focuses on investigating of the benefits of cooperation which is enabled by orthogonal radio interfaces.

This paper considers the relay channel in which the source–destination pair is aided by multiple cascaded relays, as depicted in Fig. 2. This channel model was studied in [11] in which the two relays can fully decode the source message and then forward it to its destination. Moreover, the authors in [20] considered the case that these relays can partially decode the source signal and then forward it to its destination. This paper studies the relay network in which the source transmits to the relays and to the destination over different frequency bands. In particular, the first relay can forward its signal to both the second relay and the destination by using orthogonal bands. We first derive the capacity of the general memoryless relay channel by showing that the inner bound and the outer bound of the achievable rate match. The inner bound is obtained by employing superposition block Markov encoding and binning at the source, DF at the relays, and backward decoding at

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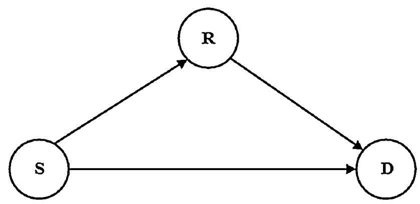

Fig. 1. The basic relay channel model.

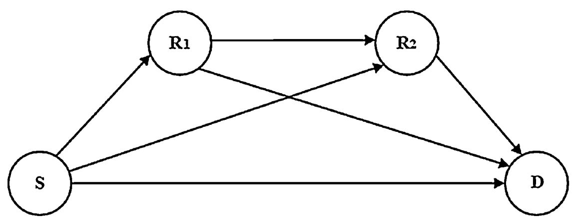

Fig. 2. A relay channel with two relays in cascade.

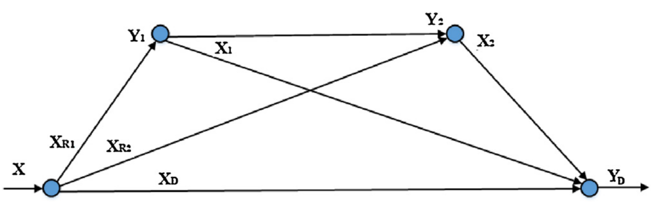

Fig. 3. The general memoryless relay network model with orthogonal components.

the destination. Further, the derived outer bound is based on the "max-flow min-cut". Indeed, the obtained capacity is extended to the additive Gaussian setting. The numerical results show that the developed encoding strategy has a rate higher than the available rates in the literature. Further, the achievable capacity grows as the channel gains from the relays to the destination increase. In addition, more power is allocated to the relay which has the strongest channel gain with the destination.

The remainder of this paper is outlined as follows. Section 2 describes the communication model which we study in this work. The capacity of the relay channel, with two cascaded relays, which employ multi-bands orthogonal frequencies, is derived in Section 3. Next, these capacity results are extended to the Gaussian relay channel in Section 4. Afterwards, numerical examples that validate our theoretical results are presented in Section 5. Finally, we conclude the paper in Section 6.

2. Channel model and preliminaries

We consider a DF relay network consisting of a source, destination and a two relay as shown in Fig. 3. The discrete memoryless relay channel with orthogonal channel components consists of a finite channel input alphabets $\mathcal{X}$, and a finite channel output alphabet $\mathcal{Y}$. In this scenario, $X \in \mathcal{X}$ is the input signal associated with the source. Further, $Y \in \mathcal{Y}$ represents the output alphabets at the destination. In this channel model, the source has a message $W \in \{1, \ldots, 2^{nR}\}$ to be communicated to the receiver over n channel uses. Further, the following definitions describe the relay channel under consideration.

Definition 1. A $(2^{nR}, n)$ code, $R = R_D + R_{R_1} + R_{R_2}$, for the relay channel with orthogonal components consists of the following:

- A message set $W = (W_D, W_{R_1}, W_{R_2}) \in [1:2^{nR_D}] \times [1:2^{nR_{R_1}}] \times [1:2^{nR_{R_2}}]$
- An encoding function $W = (W_D, W_{R_1}, W_{R_2}) \longrightarrow \mathcal{X}_D^n \times \mathcal{X}_{R_1}^n \times \mathcal{X}_{R_2}^n$
- A set of deterministic relay functions $f_1, f_2, \ldots, f_n$ such that $x_{1i} = f_i(y_{11}, \ldots, y_{1i})$ for $1 \le i \le n$ and $x_{2i} = f_i(y_{21}, \ldots, y_{2i})$ for $1 \le i \le n$.
- For transmitted messages like $W_D = w_D$, $W_{R_1} = w_{R_1}$ and $W_{R_2} = w_{R_2}$, the destination maps y^n to $(w_D, w_{R_1}, w_{R_2}) \in W = W_D \times W_{R_1} \times W_{R_2}$.

Definition 2. A discrete memoryless relay channel with two cascaded relays is said to have orthogonal components if the sender alphabet $\mathcal{X} = \mathcal{X}_D \times \mathcal{X}_{R_1} \times \mathcal{X}_{R_2}$ and the channel transition probability distribution expressed as

$$p(y, y_1, y_2|x, x_1, x_2) = p(y_1|x_{R_1}, x_1)p(y_2|x_{R_2}, x_1, x_2)p(y_D|x_D, x_1, x_2)$$

for all

$$(x_D, x_{R_1}, x_{R_2}, x_1, x_2, y_1, y_2, y_D) \in \mathcal{X}_D \times \mathcal{X}_{R_1} \times \mathcal{X}_{R_2} \times \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{Y}_1 \times \mathcal{Y}_2 \times \mathcal{Y}_D$$

Moreover, a rate R is said to be achievable if, for any $\epsilon > 0$, there exist a sequence of $(2^{nR}, n, P_e)$ code with $P_{e,i}^n$ goes to zero for sufficiently large n. In this definition, the average error probability is defined as

$$P_{e,i} = \sum_{w} \frac{1}{2^{nR}} P[g_i(Y_i) \neq w_i \mid w \quad was \quad sent] \tag{1}$$

where $W = w$ is transmitted messages.

3. Capacity of the relay network

In this section, the capacity of the relay network with orthogonal components is derived. This derivation have two steps in which the achievable rate is derived at first, then the converse, which matches the achievable rate, is also developed. The achievable rate is obtained by using a combination of block Markov encoding, superposition regular encoding [21,22] scheme and binning [22] at the source, and DF at the relays. Further, the converse part is obtained by using "max-flow min-cut".

Theorem 1. *The Capacity of the memoryless relay network with orthogonal components is given by*

$$R = \min\{I(X_D, X_1, X_2, Y_D), I(X_D, Y_D|X_1, X_2) + \min\{I(X_1, X_{R_2}, Y_2|X_2), I(X_{R_1}, Y_1|X_1) + I(X_{R_2}, Y_2|X_1, X_2)\}\}$$

where the maximum is over all joint distribution that factors as

$$P(x_{R_1}, x_{R_2}, x_D, x_1, x_2) = P(x_2)P(x_1|x_2)P(x_{R_1}|x_1)P(x_{R_2}|x_1, x_2)P(x_D|x_1, x_2)$$

Proof. The achievability part is given in Appendix A, in addition, the converse part is provided in Appendix B. □

In brief, by employing codeword splitting and regular encoding scheme, the source divides its signal into three parts. In detail, one part is directly transmitted to the destination, another part is sent through the two relays, and the last part is transmitted only through the relay R_2. In addition, the two relays employ DF in the form of backward decoding.

Remark 1. We note that the channel from both the relay, R_1, and the source to the second relay, R_2, forms a multiple access channel (MAC) [22]. Likewise, the transmission from the source and the two relays to the destination compose another MAC channel. In

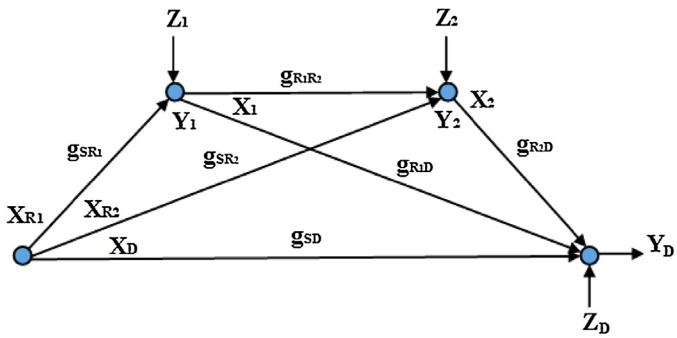

Fig. 4. Gaussian relay network model with orthogonal components.

the model under consideration, at the source and the two relays, the power allocated for each message is allocated to maximize the sum rate for each MAC.

Remark 2. Backward decoding can achieve the capacity of the multiple relay channel but with excess decoding delay. As an alternative solution, nested block may be used to let the relays decode before the destination.

In this section, the achievable rate region for the memoryless IC is obtained. The derivation assumes that, for a given user, a noncausal knowledge of the other user's signal is available. Next, these results are extended to the Gaussian cognitive IC. Furthermore, numerical examples are provided to illustrate the value of our theoretical results.

4. Gaussian Relay Network with Orthogonal Components

In this section, the capacity of the relay network with orthogonal components is extended to the Gaussian case. In this model, Y_i, $i \in \{1, 2, D\}$ is the received signal at the relay R_1, the relay R_2 and the destination D, respectively

$$Y_1 = g_{SR_1}X_{R_1} + Z_1 \tag{2a}$$

$$Y_2 = g_{SR_2}X_{R_2} + g_{R_1R_2}X_1 + Z_2 \tag{2b}$$

$$Y_D = g_{SD}X_D + g_{R_1D}X_1 + g_{R_2D}X_2 + Z_D \tag{2c}$$

where X_{R_1}, X_{R_2} and X_D are the transmitted signals by the sender, X_1 is the transmitted signals by the relay R_1, and X_3 is the transmitted signal by the relay R_2. Further, $g_{SR_1}, g_{SR_2}, g_{R_1R_2}, g_{SD}, g_{R_1D}$ and g_{R_2D} are the channel gains of the links as shown in Fig. 4. Z_i, $i \in \{1, 2, D\}$ is assumed to be independent and identically distributed additive white Gaussian noise process with variance N_i, $i \in \{1, 2, D\}$. Moreover, we enforce the power constraints to P_j, $j \in \{S, 1, 2\}$ on the source's and relays' codebooks, respectively. In particular, the following power constraints $E[X_{R_1}^2] \le \alpha_1 P_S$, $E[X_{R_2}^2] \le \alpha_2 P_S$ and $E[X_D^2] \le \alpha_3 P_S$ at the source. In addition, the relay R_1 may forward the signal X_1 to both the relay R_2 and the destination over two different frequency bands. Therefore, this relay allocate part of its power, $\beta_1 P_1$, to forward X_1 to the second relay, R_2, and the remaining part, $\beta_2 P_1$, is used to transmit X_1 to the destination, D. Further, the parameters α_1, α_2, α_3, β_1, and β_2 are the power allocation factors satisfying the following power constraints,

$$\begin{aligned} &\alpha_1 + \alpha_2 + \alpha_3 = 1 \\ &\beta_1 + \beta_2 = 1 \end{aligned} \tag{3}$$

Theorem 2. *The capacity of the AWGN cascade relay channel with orthogonal components subject to the average power constraints P_S, P_1 and P_2 is given as*

$$R = \max_{\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2} \min\left\{ C\left(\frac{A_1}{N_D}\right), C\left(\frac{|g_{SD}|^2 \rho \alpha_3 P_S}{N_D}\right) + \min\left\{ C\left(\frac{A_2}{N_2}\right), C\left(\frac{|g_{SR_1}|^2 \alpha_1 P_S}{N_1}\right) + C\left(\frac{A_3}{N_2}\right)\right\}\right\}$$

where

$$\begin{aligned} A_1 &= |g_{SD}|^2\alpha_3 P_S + |g_{R_1D}|^2\beta_2 P_1 + |g_{R_2D}|^2 P_2 \\ &\quad + 2\rho_2 |g_{SD}||g_{R_1D}|\sqrt{\alpha_3\beta_2 P_S P_1} + 2\rho_3|g_{SD}||g_{R_2D}|\sqrt{\alpha_3 P P_2} \\ &\quad + 2\rho_4 |g_{R_1D}||g_{R_2D}|\sqrt{\beta_2 P_1 P_2}, \\ A_2 &= |g_{SR_2}|^2\alpha_2 P_S + |g_{R_1R_2}|^2\beta_1 P_1 + 2\rho_1|g_{SR_2}||g_{R_1R_2}|\sqrt{\alpha_2 P_S \beta_1 P_1}, \\ A_3 &= |g_{SR_2}|^2(1-\rho_1^2)\alpha_2 P_S. \end{aligned}$$

In this formulation, X_D, X_{R_1} and X_{R_2} are assumed to be independent signals, X_1 is an independent of X_{R_1} but jointly Gaussian with X_D and X_{R_2} such that $E[X_1, X_D] = \rho_2\sqrt{\beta_2 P_1 \alpha_3 P_S}$ and $E[X_1, X_{R_2}] = \rho_1\sqrt{\beta_1 P_1 \alpha_2 P_S}$, respectively. In addition, at the destination, X_2 is assumed to be jointly Gaussian with X_D and X_1 such that $E[X_2, X_D] = \rho_3\sqrt{P_2\alpha_3 P_S}$ and $E[X_1, X_2] = \rho_4\sqrt{\beta_2 P_1 P_2}$.

Proof. First, consider the achievable rate at the first relay, R_1,

$$\begin{aligned} I(X_{R_1}; Y_1|X_1) &= h(Y_1|X_1) - h(Y_1|X_1, X_{R_1}) = h(g_{SR_1}X_{R_1} + Z_1) - h(Z_1) \\ &= C\left(\frac{|g_{SR_1}|^2\alpha_1 P_S}{N_1}\right) \end{aligned} \tag{4}$$

Noting that $X_{R_1} \to Y_1 \to X_1$ forms a Markov chain. For instance, given Y_1, X_{R_1} and X_1 are independent. Then, in a similar manner, we have

$$\begin{aligned} I(X_{R_2}; Y_2|X_1, X_2) &= h(Y_2|X_1, X_2) - h(Y_2|X_1, X_2, X_{R_2}) \\ &= h(|g_{SR_2}|X_{R_2} + Z_2|X_1) - h(Z_2) \\ &= C\left(\frac{|g_{SR_2}|^2(1-\rho_1^2)\alpha_2 P_S}{N_2}\right) \end{aligned} \tag{5}$$

We remind that $(X_{R_2}, X_1) \to Y_2 \to X_2$ forms a Markov chain. In this encoding technique, the source always send new information to the relay R_1 through X_{R_1} and to the second relay through part of X_{R_2} with power $(1-\rho_1^2)\alpha_2 P_S$. In addition, the relay's output signal X_1 and the rest of X_{R_2} is used to remove the uncertainty of the second relay, R_2, about the previous message.

Next, consider the multiple access term at the relay, R_2, the sum rate of decoding X_{R_2} and X_1 can be bounded as

$$\begin{aligned} I(X_1, X_{R_2}; Y_2|X_2) &= h(Y_2|X_2) - h(Y_2|X_1, X_{R_2}, X_2) \\ &= h(g_{SR_2}X_{R_2} + g_{R_1R_2}X_1 + Z_2) - h(Z_2) = C\left(\frac{A_2}{N_2}\right) \end{aligned} \tag{6}$$

Last, the signal X_D can be decoded at the destination with rate bounded as

$$\begin{aligned} I(X_D; Y_D|X_1, X_2) &= h(Y_D|X_1, X_2) - h(Y_D|X_D, X_1, X_2) \\ &= h(g_{SD}X_D + Z_D|X_1, X_2) - h(Z_D) \\ &= C\left(\frac{|g_{SD}|^2\,\rho\,\alpha_3 P}{N_D}\right) \end{aligned} \tag{7}$$

where $\rho = 1 - \rho_2^2 - \rho_3^2 - \rho_4^2 + 2\rho_2\rho_3\rho_4$.

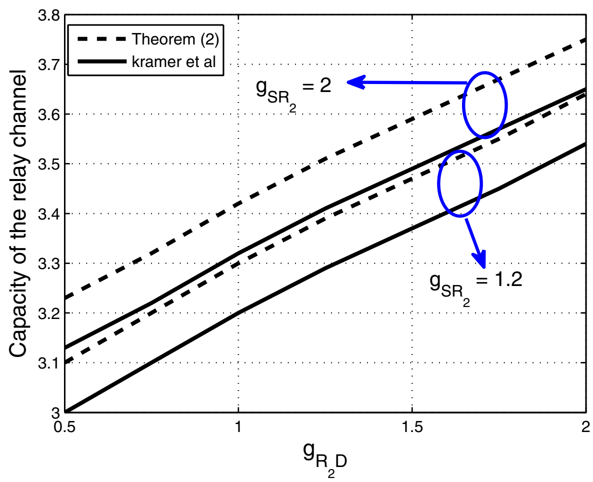

Fig. 5. Comparisons between the established rate in this paper and that derived in [11] vs. g_{R_2D} for different values of g_{SR_2}.

Finally, consider the multiple access term at the destination,

$$I(X_D, X_1, X_2; Y_D) = h(Y_D) - h(Y_D|X_D, X_1, X_2)$$
$$= h(Y_D) - h(Z_D) = C\left(\frac{A_1}{N_D}\right) \quad (8)$$

Which complete the proof of the theorem. □

5. Numerical examples

This section presents numerical results for a degraded faded relay channel with orthogonal components. Unless otherwise specified, the channel gains are initially set as follows: $g_{SD} = 1$, $g_{SR_1} = 2$, $g_{SR_2} = 1.6$, $g_{R_1R_2} = 3$, $g_{R_1D} = 1.5$ and $g_{R_2D} = 2$. Further, the average power at the source, P_S, is limited to 20, whereas the two relays are allocated equal power such as P_1 and P_2 are set to 10. In addition the following noise variances N_1, N_2, and N_D are normalized to 1. The results in Section 5.1 are obtained with equal power allocation between the two relays, i.e., $P_1 = P_2$, whereas the impact of allocating different powers to these relays i.e., $P1 \neq P2$ are investigated in Section 5.2. We need to note that the capacity of the point-to-point channel (without relays) is $0.5 \log\left(1 + \frac{20+10+10}{1}\right) = 2.6788$ bit per second per channel use (bpc). Thus, unless they can increase the capacity of this channel model, the relays can not be used. Finally, we note that the capacity is given in bits per second/channel use (bps).

5.1. Equal power allocation between the relays

Fig. 5 compares between the achievable rate, which is developed in Theorem 2, and the available achievable rate in the literature. It clearly shows that our established encoding technique can achieve a rate higher than that was obtained by the authors in [11]. In addition, It shows that our developed achievable rate is an increasing function of g_{R_2D}. Furthermore, the capacity also grows as the channel gain g_{SR_2} increases.

5.2. Impact of power allocation between the two relays

Now, we turn to the problem of optimizing the allocated power to each relay. Simply, the total power allocated to the two relays is assumed to be constant, i.e., $P_T = 20 = P_1 + P_2$ but the allocated power to each relay is varied such that the capacity is maximized. Fig. 6 shows that different power allocation can significantly increase the capacity for low values of the channel gain g_{R_2D}, and marginally improve the values of g_{R_2D} that is close to g_{R_1D}. In detail, as shown in Fig. 7, for low values of g_{R_2D}, more power is allocated to the relay, R_1, since $g_{R_1D} > g_{R_2D}$ and thus the capacity significantly increases. Furthermore, as g_{R_2D} increases, then more power is allocated to the second relay, R_2. Moreover, in the case that $g_{R_1D} = g_{R_2D} = 2$ and $g_{SR1} = g_{DR_2} = 2$, then equal power is allocated to both relays.

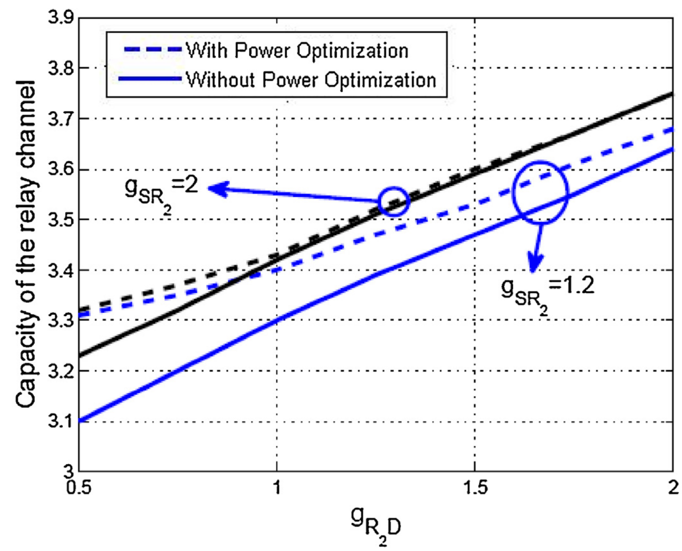

Fig. 6. Achievable rate vs. g_{R_2D} with and without power optimization.

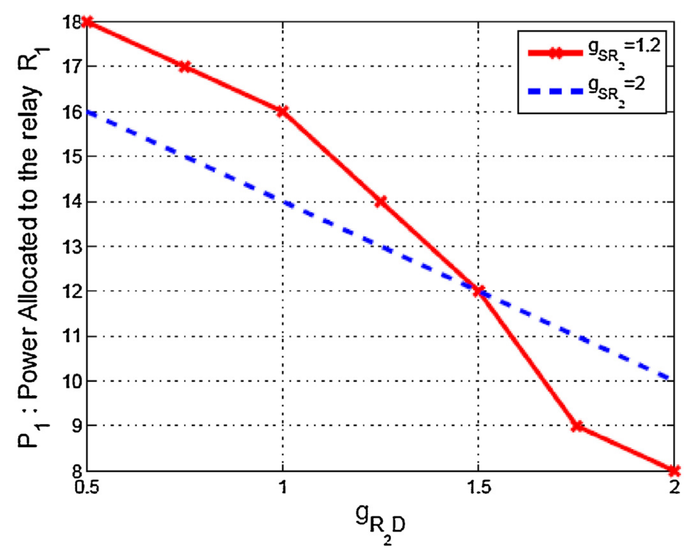

Fig. 7. Power allocated to the first relay versus the channel gain g_{R_2D} for different values of g_{SR_2}.

Remark 3.

- In the case of $g_{R_2D} = 0$, the total power is allocated to the first relay. Thus, the model is returned to that studied in [8] in which the source–destination is aided by a relay.
- In the case of $g_{SR_2} = 0$, then based on the channel gain g_{R_2D}, the relay, $R1$, may allocate part of its power to forward its signal to the destination via the second relay, R_2.

6. Concluding remarks

In this paper, the authors consider the relay network which is aided by multiple cascaded relays. In this relay network, orthogonal frequency bands are used to transmit from the source to the relays and to the destination. The capacity of this network is developed in the case of discrete memoryless channel. Then, the derivation is extended to the case of additive noise channel. Numerical results are drawn to understand the benefits of optimizing the power for each relay such that the achievable capacity is maximized.

Future research should propose practical coding techniques such that information can be sent reliably over multi-standard relays. Further, some standards and rules should be drawn to manage the transmission over multi-standard devices.

Appendix A. Achievability part

Suppose the source uses superposition block Markov encoding, and binning scheme. Both relays, R_1, and, R_2, can employ DF scheme to help the source–destination pair. Specifically, regular encoding/Willems(tm) backward decoding is employed at the both relays. Furthermore, the channels from the source to the relays R_1, R_2, and D are assumed to be orthogonal. The source divides its signal into B equally sized blocks. Then, the transmission to the two relays and the destination is performed over $B+2$ blocks. In particular, for a given block index t, where $t \in [1, B]$, the source divides its signal w into tuple $(w_D, w_{R_1}, w_{R_2}) \in [1, 2^{nR_D}] \times [1, 2^{nR_{R_1}}] \times [1, 2^{nR_{R_2}}]$ such that w_D is transmitted directly to the destination, w_{R_1} is transmitted to the destination via the two relays, and w_{R_2} is sent to its destination via only relay R_2. We note that the average rate of the triple $\left(R_D \frac{B}{B+2}, R_{R_1} \frac{B}{B+2}, R_{R_2} \frac{B}{B+2}\right)$ approaches $\left(R_D, R_{R_1}, R_{R_2}\right)$ as $B \longrightarrow \infty$.

We use random codes and fix a joint probability distribution

$$P(x_{R_1}, x_{R_2}, x_D, x_1, x_2) = P(x_2)P(x_1|x_2)P(x_{R_1}|x_1)P(x_{R_2}|x_1, x_2)P(x_D|x_1, x_2)$$

Further, A_ϵ^n denote the strongly jointly ϵ-typical set of length n.

The source generates the following signals:

- Generate $2^{n(L_1+L_2)}$ independent identically distributed n-sequences x_2^n, each drawn according to $p(x_2^n) = \prod_{t=1}^n P(x_{2,t})$. Index them as $x_2^n(i)$, $i \in [1:2^{nR}]$.
- For each $x_2^n(i)$, generate 2^{nL} conditional independent n-sequences x_1^n, drawn according to $p(x_1^n|x_2^n(i)) = \prod_{t=1}^n P(x_{1,t}|x_{2,t}(i))$. Index them as $x_1^n(i,j)$, $j \in [1:2^{nL}]$.
- For each $x_1^n(i,j)$, generate $2^{nR_{R_1}}$ conditional independent n-sequences $x_{R_1}^n$, drawn according to $p(x_{R_1}^n|x_1^n(i,j)) = \prod_{t=1}^n p(x_{R_1,t}|x_{1,t}(i,j))$. Index them as $x_{R_1}^n(i,j,k)$, $k \in [1:2^{nR_{R_1}}]$.
- For each $x_2^n(i)$ and $x_1^n(i,j)$, generate $2^{nR_{R_2}}$ conditional independent n-sequences $x_{R_2}^n$ drawn according to $p(x_{R_2}^n|x_2^n(i), x_1^n(i,j)) = \prod_{t=1}^n p(x_{R_2,t}|x_{2,t}(i), x_{1,t}(i,j))$. Index them as $x_{R_2}^n(i,v)$, $v \in [1:2^{nR_{R_2}}]$.
- For each $x_2^n(i)$ and $x_2^n(i,j)$, generate 2^{nR_D} conditional independent n-sequences x_D^n, drawn according to $p(x_D^n|x_2^n(i), x_2^n(i,j)) = \prod_{t=1}^n p(x_{D,t}|x_{2,t}(i), x_{1,t}(i,j))$. Index them as $x_{R_D}^n(i,m)$, $m \in [1:2^{nR_D}]$.

Encoding: We now illustrate the encoding process at the source and the two relays. In particular, the encoding is performed in $B+2$ blocks, the coding strategy is shown in Table 1 and it can be explained as follows:

- *Source terminal*: During block index b, $b \in [1, B]$, the message w_b is split into B equally sized blocks $w_{R_1,b}$,$w_{R_2,b}$, $w_{D,b}$. Then, by using superposition encoding, the sender transmits $x_{R_1,b}^n(w_{R_1,b}, w_{R_1,b-1}, w_{R_1,b-2})$, $x_{R_2,b}^n(w_{R_2,b}, w_{R_2,b-1}, w_{R_2,b-2})$ and $x_{D,b}^n(w_{D,b}, w_{D,b-1}, w_{D,b-2})$ where $w_{R_{1,b-1}}$, $w_{R_{2,b-1}}$, and $w_{R_{D,b-1}}$ are the messages transmitted in block $b-1$.
- *Relay terminal 1*: After completing the transmission of block b. The relay R_1 can see $y_{1,b}^n$, then, this relay tries to find $\hat{w}_{R_1,b}$ such that

$$\begin{aligned}&(x_{R_1,b}^n(\hat{w}_{R_1,b-2}, \hat{w}_{R_1,b-1}, \hat{w}_{R_1,b}),\\ &\quad x_{1,b}^n(\hat{w}_{R_1,b-2}, \hat{w}_{R_1,b-1}), y_{1,b}^n) \in A_{\epsilon_1}^n(X_{R_1}, X_1, Y_1)\end{aligned}$$

where $A_{\epsilon_1}^n$ denote the strongly jointly ϵ-typical set of length n. Further, $\hat{w}_{R_1,b-2}$ and $\hat{w}_{R_1,b-1}$ are the relay terminal(tm)s estimate of $w_{R_1,b-2}$, and $w_{R_1,b-1}$ respectively. Then, the relay, R_1, transmits $x_{1,b+1}^n(\hat{w}_{R_1,b-1}, \hat{w}_{R_1,b})$ in block $b+1$.
- *Relay terminal 2*: After the transmission of block b is completed, relay R_2 has seen $y_{2,b}^n$. The relay tries to find $\hat{w}_{R_1,b-1}$ and $\hat{w}_{R_2,b-1}$ such that

$$\begin{aligned}&(x_{R_2,b-1}^n(w_{R_2,b-3}, w_{R_2,b-2}, w_{R_2,b-1}),\\ &\quad x_{1,b-1}^n(w_{R_1 1,b-3}, w_{R_1 1,b-2}), y_{2,b-1}^n) \in A_\epsilon^n(X_1, X_{R_1}, Y_2)\end{aligned}$$

and

$$(x_{2,b}^n(\hat{w}_{R_1,b}, \hat{w}_{R_2,b}), y_{2,b}^n) \in A_\epsilon^n(X_2, Y_2)$$

Then, this relay transmits transmit $x_2^n(\hat{w}_{R_1,b}, \hat{w}_{R_2,b})$ in block $b+2$.

Decoding: The decoding process at both the relay and the destination is obtained as

- The relay R_1 can reliably decode $w_{R_1,b}$ after receiving block b for sufficiently large n and if its past message estimate $(\hat{w}_{R_1,b-2}, \hat{w}_{R_1,b-1})$ of $(w_{R_1,b-2}, w_{R_1,b-1})$ and

$$R_{R_1} < I(X_{R_1}; Y_1|X_1) \tag{9}$$

- The relay R_2 can correctly decode $w_{R_1,b-1}$ by using $y_{2,b-1}$ and $y_{2,b}$ if the past estimates $(\hat{w}_{R_1,b-2}, \hat{w}_{R_1,b-3})$ are correct. Similarly, $w_{R_2,b}$ can be reliably decoded by using $y_{2,b-1}$ and $y_{2,b}$ if the past message estimate $\hat{w}_{R_2,b-1}$ is correct. These estimates can be done reliably as long as n is sufficiently large and

$$R_{R_2} \le \min(L_{R_2}, R_{R_1} + L_{12}) \tag{10}$$

where

$$L_{R_2} = I(X_1, X_{R_2}; Y_2|X_2)$$
$$L_{12} = I(X_{R_2}; Y_2|X_1, X_2)$$

- Using backward decoding, the destination knows $w_{R_1,b-4}, w_{R_1,b-3}$ and $w_{D,b-3}$ and searches for a unique $w_{R_1,b-2}, w_{R_2,b-2}$, and $w_{D,b-2}$ such that

$$\begin{aligned}&\left(x_1^n(\hat{w}_{R_1,b-2}, \hat{w}_{R_1,b-3}), x_2^n(\hat{w}_{R_1,b-2}, \hat{w}_{R_2,b-2}),\right.\\ &\quad \left. x_D^n(\hat{w}_{D,b-2}), y_{D,b-2}^n\right) \in A_\epsilon^n(X_1, X_2, X_D, Y_D)\end{aligned}$$

Table 1
Common signal m_{13} transmitted over three consecutive blocks.

	Block 1	Block 2	Block 3
Primary transmits	$x_1(1, 1, m_{13[1]})$	$x_1(1, m_{13[1]}, m_{13[2]})$	$x_1(m_{13[1]}, m_{13[2]}, m_{13[3]})$
Cognitive transmits	$x_2(1, 1)$	$x_2(1, m_{13[1]})$	$x_2(m_{13[1]}, m_{13[2]})$
Relay transmits	$x_3(1)$	$x_3(1)$	$x_3(m_{13[1]})$

Then, the destination can decode reliably as long as n is large and

$$R \le \min\left(I(X_D, X_1, X_2; Y_D), I(X_D; Y_D|, X_1, X_2) + R_{R_2}\right) \tag{11}$$

Appendix B. Converse

The best known upper bound on the capacity of the relay channel is the max-flow min-cut upper bound (also referred to as the cut-set bound). This upper bound is tight in all the cases in which the capacity of the relay channel is known. Although this bound is not believed to be the capacity of the relay channel in general, there is not a single example proving suboptimality of this upper bound. We present an upper bound on capacity of the discrete memoryless cascade relay channel with orthogonal components using the cut-set bound. We will see that this upper bound meets the lower bound which leads to the capacity of the channel.

Proof. (converse part) In order to prove this upper bound we divide the terms under inequality into multiple access term and broadcast term. First, consider the broadcast terms. Therefore, starting from Fanos inequality, the achievable rate the first relay, R_1 may be given as

$$\begin{aligned}
nR_{R_1} &= H(W_{R_1}) \\
&= I(W_{R_1}; Y_1^n) + H(W_{R_1}|Y_1^n) \\
&\le^a I(W_{R_1}; Y_1^n) + n\epsilon_1 \\
&= \sum_{i=1}^{n} I(W_{R_1}; Y_{1,i}|Y_{1,i-1}) + n\epsilon_1 \\
&=^b I(W_{R_1}; Y_{1,i}|Y_{1,i-1}, X_{1,i}) + n\epsilon_1 \\
&= \sum_{i=1}^{n} H(Y_{1,i}|Y_{1,i-1}, X_{1,i}) + n\epsilon_1 \\
&\quad -H(Y_{1,i}|Y_{1,i-1}, X_{1,i}, W_{R_1}, X_{R_1,i}) + n\epsilon_1 \\
&\le^c H(Y_{1,i}|X_{1,i}) - H(Y_{1,i}|X_{1,i}, X_{R_1,i}) + n\epsilon_1 \\
&= \sum_{i=1}^{n} I(X_{R_1,i}; Y_{1,i}|X_{1,i}) + n\epsilon_1
\end{aligned} \tag{12}$$

Therefore, we have

$$R_{R_1} \le I(X_{R_1}; Y_1|X_1) = I_1 \tag{13}$$

where:

a: follow from Fano's inequality,

b: follows from the fact that $X_{1,i}$, is a function of $Y_{1,i-1}$,

c: follows from the fact that conditioning reduces entropy and from Markov chain.

□

In a similar manner, we may get

$$R_{R_2} \le I(X_{R_2}; Y_2|X_1, X_2) = I_2 \tag{14}$$

Then, we may consider the achievable sum rate at the second relay, R_2, thus, we have

$$\begin{aligned}
n(R_{R_1} + R_{R_2}) &= H(W_{R_1}, W_{R_2}) \\
&= I(W_{R_1}, W_{R_2}; Y_2^n) + H(W_{R_1}, W_{R_2}|Y_2^n) \\
&= I(W_{R_1}, W_{R_2}; Y_2^n) + H(W_{R_1}, W_{R_2}|Y_2^n) \\
&\le^a I(W_{R_1}, W_{R_2}; Y_2^n) + n\epsilon_2 \\
&= \sum_{i=1}^{n} I(W_{R_1,i}, W_{R_2,i}; Y_{2,i}|Y_{2,i-1}) + n\epsilon_2 \\
&= \sum_{i=1}^{n} (H(Y_{2,i}|Y_{2,i-1}) - H(Y_{2,i}|Y_{2,i-1}, W_{R_1}, W_{R_2}) + n\epsilon_2 \\
&=^b \sum_{i=1}^{n} H(Y_{2,i}|Y_{2,i-1}, X_{2,i}) - H(Y_{2,i}|Y_{2,i-1}, X_{R_2,i}, X_{1,i}, X_{2,i}, W_{R_1}, W_{R_2}) + n\epsilon_2 \\
&\le^c \sum_{i=1}^{n} H(Y_{2,i}|X_{2,i}) - H(Y_{2,i}|Y_{2,i-1}, X_{R_2,i}, X_{1,i}, X_{2,i}, W_{R_1}, W_{R_2}) + n\epsilon_2 \\
&=^d \sum_{i=1}^{n} \left(H(Y_{2,i}|X_{2,i}) - H(Y_{2,i}|X_{R_2,i}, X_{1,i}, X_{2,i})\right) + n\epsilon_2 \\
&= \sum_{i=1}^{n} I(X_{1,i}, X_{R_2,i}; Y_{2,i}|X_{2i}) + n\epsilon_2
\end{aligned} \tag{15}$$

Therefore, from (13), we have

$$R_{R_1} + R_{R_2} \le I(X_1, X_{R_2}; Y_2|X_2) = I_3 \tag{16}$$

where:

a: follows from Fano's inequality

b: follows from the fact that $X_{2,i}$ is function of $Y_{2,i-1}$ and $X_{R_2,i}$. Further, $X_{1,i}$ and $X_{2,i}$ are function of $W_{R_1,i}$ $W_{R_2,i}$, respectively,

c: follows from the fact that conditioning reduces entropy,

d: follows form the fact that $(Y_{2,i-1}, W_{R_2 i}) \longrightarrow (X_{R_2,i}, X_{1,i}, X_{2,i}) \longrightarrow Y_{2,i}$ form a Markov chain.

Indeed, in a similar manner, we may have $R_{R_1} + R_{R_2} \le I(X_{R_1}; Y_1|X_1) + I(X_{R_2}; Y_2|X_1, X_2)$. Thus, the achievable sum rate at relay, R_2, may be bounded as

$$R_{R_1} + R_{R_2} \le \min(I_1 + I_2, I_3) \tag{17}$$

Next, the achievable rate over the direct link may be bounded as

$$\begin{aligned}
nR_D &= H(W_D) \\
&= I(W_D; Y^n) + H(W_D|Y^n) \\
&\le^a I(W_D; Y^n) + n\epsilon_3 \\
&= \sum_{i=1}^{n} I(W_D; Y_i|Y_{i-1}) + n\epsilon_3 \\
&= \sum_{i=1}^{n} H(Y_i|Y_{i-1}) - H(Y_i|W_D, Y_{i-1}) + n\epsilon_3 \\
&= \sum_{i=1}^{n} H(Y_i|X_{1i}, X_{2i}) \\
&\quad -H(Y_i|W_{Di}, X_{Di}, X_{1i}, X_{2i}) + n\epsilon_3 \\
&= \sum_{i=1}^{n} I(X_{Di}; Y_i|X_{1i}, X_{2i}) + n\epsilon_3
\end{aligned} \tag{18}$$

So, we have

$$R_D \le I(X_D; Y_D|X_1, X_2) = I_4 \tag{19}$$

Finally, consider the multiple access term under inequality,

$$\begin{aligned}
nR &= H(W) \\
&= I(W;Y_D^n) + H(W|Y_D^n) \\
&\leq^a I(W;Y_D^n) + n\epsilon_4 \\
&= \sum_{i=1}^{n} I(W;Y_{D,i}|Y_{D,i-1}) + n\epsilon_4 \\
&= \sum_{i=1}^{n} (H(Y_{D,i}|Y_{D,i-1}) - H(Y_i|W, Y_{D,i-1})) + n\epsilon_4 \\
&\leq^b \sum_{i=1}^{n} (H(Y_{D,i}) - H(Y_{D,i}|W, Y_{D,i-1})) + n\epsilon_4 \\
&\leq^c \sum_{i=1}^{n} (H(Y_{D,i}) \\
&\quad - H(Y_{D,i}|X_{D_i}, X_{1_i}, X_{2_i}, W, Y_{D,i-1})) + n\epsilon_4 \\
&=^d \sum_{i=1}^{n} (H(Y_{D,i}) \\
&\quad - H(Y_{D,i}|X_{D_i}, X_{1_i}, X_{2_i})) + n\epsilon_4 \\
&= \sum_{i=1}^{n} I(X_{D_i}, X_{1_i}, X_{2_i}; Y_{D,i}) + n\epsilon_4
\end{aligned} \tag{20}$$

Thus, we have

$$R \leq I(X_D, X_1, X_2; Y_D) = I_5 \tag{21}$$

where:

a: follows from Fano's inequality

b: follows from the fact that conditioning reduces entropy.

c: follows from the fact that Y_{i-1} is a function of X_{D_i}, X_{1_i}, and, X_{2_i}

d: follows from the fact that $(W, Y_{i-1}) \longrightarrow (X_{D_i}, X_{1_i}, X_{2_i}) \longrightarrow Y_i$ form a Markov chain.

Similarly, the following inequalities may be proved,

$$R = I(X_D; Y_D|X_1, X_2) + I(X_1, X_{R_2}; Y_2|X_2) = I_4 + I_3$$

$$R = I(X_D; Y_D|X_1, X_2)$$

$$+ I(X_{R_1}; Y_1|X_1) + I(X_{R_2}; Y_2|X_1, X_2) = I_4 + I_1 + I_2$$

Thus, the over all achievable rate may be given by

$$R = \min\left\{I_5, I_4 + \min(I_1 + I_2, I_3)\right\}$$

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