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Atomic solutions to Bateman–Burgers type equation via tensor products

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ABSTRACT

This study presents a novel approach for solving fractional partial differential equations, notably the fractional Bateman–Burgers type equation, by employing the tensor product of Banach spaces. This study proposes a novel analytical method that transcends traditional techniques like separation of variables, enabling precise atomic solutions to complex fractional equations. Central to our approach is the utilization of the α-conformable fractional derivative, which enhances the analytical framework for addressing such complex equations. Our findings provide solutions to the fractional Bateman–Burgers type equation and illustrate the potential of integrating advanced mathematical theories to solve complex problems across various scientific disciplines. This work promises to pave new pathways for research in fractional calculus and its application in both theoretical and applied mathematics.

1. Introduction

Fractional calculus has emerged as a versatile mathematical framework for modeling complex physical phenomena that exhibit memory and hereditary properties. Unlike classical calculus, fractional derivatives provide a more accurate description of systems in diverse fields, such as viscoelasticity, anomalous diffusion, and wave propagation. These properties make fractional models valuable for solving real-world problems involving nonlocality and scale invariance.

Fractional derivatives, particularly the α-conformable derivative, play a vital role in modeling nonlocal and memory-dependent dynamics. The α-conformable derivative offers advantages over traditional fractional derivatives, including computational simplicity and broader applicability. A detailed discussion of its definition, mathematical properties, and advantages is presented in Section 2.

Numerous differential equations can be reformulated into fractional form, thereby broadening their applicability in a multitude of scientific fields. This innovative approach has demonstrated remarkable success in solving fractional partial differential equations. In our study, we employ the tensor product technique to solve the fractional Bateman–Burgers-type equation.

The fractional Bateman–Burgers type equation, as a hybrid model that incorporates both diffusion and convection effects, plays a critical role in understanding various real-world systems. In fluid dynamics, this equation is used to model flow characteristics in viscous fluids, where diffusion describes the spreading of fluid particles, and convection accounts for the transport of heat or mass. Similarly, in environmental modeling, it is employed to simulate the dispersion of pollutants in the atmosphere or water bodies, capturing the combined effects of turbulence and advection. In mathematical biology, the equation finds applications in modeling nutrient transport in biological tissues or blood flow in capillaries, with fractional derivatives effectively representing the nonlocal and memory-dependent dynamics of these systems.

Recent advancements in fractional differential equations have significantly expanded their applicability to complex systems with memory and nonlocal characteristics. For example, Boulaaras et al.[1] highlighted the superior accuracy of fractional models over integer-order models in describing real-world phenomena, such as nonlinear and integrodifferential equations. These models are particularly effective in engineering and scientific applications requiring precise modeling of physical systems with nonlocal dependencies. Similarly, Singh et al.[2] investigated the theoretical and numerical aspects of fractional differential equations, providing applications in physics, such as electrical circuits and other nonlinear systems. These studies underscore the importance of fractional calculus as a versatile tool for addressing complex problems in various scientific and engineering domains, reinforcing the relevance of our approach to the fractional Bateman–Burgers type equation.

Despite the widespread applicability of the Bateman–Burgers type equation, its fractional form poses significant analytical challenges due

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to the interplay of nonlinear terms and fractional operators. Traditional methods, such as the separation of variables, often fall short in providing closed-form solutions. The use of tensor product theory, as explored in this study, introduces a novel analytical framework for deriving solutions by leveraging the structure of Banach spaces.

This study aims to develop atomic solutions for the fractional Bateman–Burgers type equation using the tensor product of Banach spaces and the α-conformable fractional derivative. By doing so, it provides definitive solutions to a challenging equation and highlights the utility of advanced mathematical tools in solving fractional partial differential equations.

The rest of the paper is organized as follows: Section 3 outlines the fundamentals of Bateman–Burgers-type equations, highlighting their relevance across scientific fields. Section 4 presents a novel atomic solution using Tensor Products, with a focus on procedural methodology and theoretical underpinnings. Section 5 expands on the analysis and practical applications of this solution. The conclusion in Section 7 reflects on the study's contributions to fractional partial differential equations and outlines future research directions, emphasizing the significance of α-conformable fractional derivatives and tensor product theory.

2. α-Conformable fractional derivatives

The α-conformable fractional derivative was introduced as a simpler alternative to classical fractional derivatives, addressing some of their inherent complexities while maintaining the ability to capture nonlocal and memory effects. The α-conformable derivative of a function $f(t)$ is defined as:

$$D^\alpha f(t) = \lim_{\epsilon\to 0} \frac{f(t+\epsilon t^{1-\alpha}) - f(t)}{\epsilon}, \quad t>0,\ \alpha\in(0,1). \tag{1}$$

This definition ensures conformity to classical calculus for $\alpha = 1$ while introducing fractional-order effects for $0 < \alpha < 1$. Key mathematical properties of the α-conformable derivative include:

- **Linearity**: $D^\alpha[af(t)+bg(t)] = aD^\alpha f(t) + bD^\alpha g(t)$.
- **Product Rule**: $D^\alpha[f(t)g(t)] = f(t)D^\alpha g(t) + g(t)D^\alpha f(t)$.
- **Quotient Rule**: $D^\alpha\left(\frac{f(t)}{g(t)}\right) = \frac{g(t)D^\alpha f(t) - f(t)D^\alpha g(t)}{g(t)^2}$, provided $g(t)\neq 0$.

Specific examples of α-conformable derivatives include[3]:

$$D^\alpha(t^p) = pt^{p-\alpha}, \tag{2}$$

$$D^\alpha \sin\left(\frac{1}{\alpha}t^\alpha\right) = \cos\left(\frac{1}{\alpha}t^\alpha\right), \tag{3}$$

$$D^\alpha \cos\left(\frac{1}{\alpha}t^\alpha\right) = -\sin\left(\frac{1}{\alpha}t^\alpha\right), \tag{4}$$

$$D^\alpha \exp\left(\frac{1}{\alpha}t^\alpha\right) = \exp\left(\frac{1}{\alpha}t^\alpha\right). \tag{5}$$

Compared to traditional fractional derivatives (e.g., Riemann–Liouville or Caputo derivatives), the α-conformable derivative:

- Avoids singularities and heavy computational costs associated with integral-based definitions.
- Provides a framework that is easier to apply to physical systems.
- Offers flexibility in modeling systems with memory while preserving simplicity.

These features make α-conformable derivatives a suitable choice for applications in fluid dynamics, heat transfer, and biological systems, as demonstrated in this study.

3. Bateman-Burgers type equations

Bateman–Burgers-type equations encompass a broad class of partial differential equations (PDEs) that combine diffusion and convection terms to describe the transport of substances in various media. These equations find applications in diverse fields such as fluid dynamics, chemical engineering, environmental science, and mathematical biology.

Understanding the behavior of solutions to Burger–Bateman-type equations is crucial for predicting the transport and distribution of substances in various systems, such as pollutant dispersion in the atmosphere, nutrient transport in biological tissues, or solute transport in porous media. Consequently, these equations play a significant role in environmental modeling, industrial processes, and biological systems analysis.

Bateman–Burgers-type equations are non-linear and often challenging to solve analytically, especially in complex geometries or with complex boundary conditions. Therefore, numerical methods such as finite difference, finite element, or finite volume techniques are commonly employed to approximate solutions.

In this section, we introduce an innovative atomic solution to the Bateman–Burgers type equation.

3.1. Origin and importance of the Bateman-Burgers type equation

The Bateman–Burgers type equation, rooted in the classical Burgers equation, combines diffusion and convection effects, making it a cornerstone in modeling nonlinear dynamics. Originating in fluid mechanics, it has been used to study turbulence and shock waves. Its fractional form incorporates nonlocal and memory effects, enhancing its applicability in various fields such as fluid dynamics, where it models the interaction between diffusion and convection in viscous fluid flow; environmental science, where it simulates pollutant dispersion in air and water bodies; and mathematical biology, where it captures nutrient transport and blood flow dynamics. While analytically challenging due to fractional operators, this equation is vital for accurately representing complex systems. This study addresses these challenges by leveraging tensor product theory to derive novel solutions.

4. Atomic solution

Let X and Y be Banach spaces with X^* representing the dual space of X. Let us define an operator T mapping from X^* to Y, where for any $x\in X$ and $y\in Y$, the operator $T : X^*\to Y$, defined by:

$$T(x^*) = x^*(x)y, \tag{6}$$

rendering T as a bounded linear operator of rank one. This operation, symbolized as $x\otimes y$, introduces the concept of atoms within tensor product theory, serving as essential building blocks. Atoms play a crucial role in the theory of optimal approximation in Banach spaces, a topic extensively explored in the literature.[4] One noteworthy result, as outlined in Ref. 5, pertinent to our discussion, states that if the sum of two atoms yielding an atom implies dependency either among the first components or the second ones. For a comprehensive exploration of tensor products of Banach spaces, we direct the reader to Refs. 5, 6.

Before proceeding with the detailed mathematical analysis, it is instructive to understand how tensor products and atomic solutions work together to solve the fractional Bateman–Burgers equation. Fig. 1 provides a visual framework for our solution strategy.

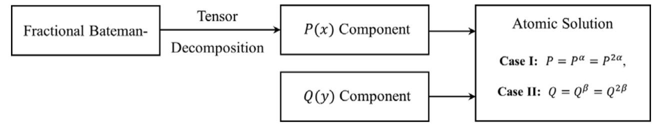

Fig. 1. Visualization of tensor products and atomic solutions.

Define $D_x^\alpha u$ to represent the partial α-derivative of a function u with respect to the variable x, where α represents the degree of differentiation involved. Further, $D_x^{2\alpha}u(x,y)$ denotes the application of D_x^α twice on u, specifically with respect to x. Analogous notation applies for derivatives with respect to y. For functions of a single variable, such as x, we adopt the notation f^{α} and $f^{2\alpha}$ to indicate $D_x^{\alpha} f$ and $D_x^{2\alpha} f$, respectively, thereby standardizing the representation of partial and repeated α-derivatives across different variables.

Our primary object in this section is to derive an atomic solution for the equation:

$$D_x^{\alpha} D_y^{\beta} u(x,y) + D_x^{2\alpha} u(x,y) = D_y^{2\beta} u(x,y), \tag{7}$$

subject to the initial conditions $u(0,0)=1$, $u^{\alpha}(0,0)=1$, and $u^{\beta}(0,0)=1$. This endeavor involves solving the aforementioned partial differential equation to obtain a fundamental atomic solution that adheres to these specified conditions.

The initial conditions $u(0,0)=1$, $u^{\alpha}(0,0)=1$, and $u^{\beta}(0,0)=1$ ensure the uniqueness and stability of the solution. Physically, $u(0,0) = 1$ defines the initial state of the system, while $u^{\alpha}(0,0)=1$ and $u^{\beta}(0,0)=1$ impose memory effects in the x- and y-directions, respectively. These conditions make the model suitable for processes like coupled fluid flow and pollutant transport.

To interpret Eq. (7), the fractional time derivative $D^{\alpha}u(x,t)$ captures memory-dependent effects, modeling processes with history dependence. The convection term $u(x,t)\frac{\partial u(x,t)}{\partial x}$ represents nonlinear transport and advection, where the solution propagates in the spatial domain. The diffusion term $\beta\frac{\partial^2 u(x,t)}{\partial x^2}$ smoothens gradients, with β controlling the degree of spatial spreading or dissipation. Together, these terms balance memory effects, nonlinear advection, and diffusion, making the equation suitable for modeling complex phenomena such as fluid flow and pollutant transport.

Remark 1. It is worth noting that not every linear partial differential equation, whether fractional or not, lends itself to a solution through the separation of variables. In instances where this method proves impractical, the notion of an atomic solution becomes indispensable. Eq. (23) exemplifies such a scenario; despite its linearity, the separation of variables is unattainable. Therefore, our pursuit shifts toward uncovering an atomic solution for this equation. In essence, we seek a solution in the following form:

$$u(x,y) = P(x)Q(y), \tag{8}$$

4.1. Advantages and limitations of the tensor product method

The tensor product method provides an effective framework for solving fractional partial differential equations, leveraging Banach spaces to derive precise atomic solutions. It excels in handling fractional operators, capturing nonlocal and memory effects, and is scalable to higher-dimensional problems. Unlike traditional methods, it remains effective for non-linear systems where standard approaches fail.

However, the method faces challenges such as high computational demands for complex systems and reliance on well-specified initial and boundary conditions. Deriving closed-form solutions for highly non-linear equations can be difficult, and the limited availability of numerical tools restricts its practical application compared to established methods.

5. Procedure

Setting $u(x,y)=P(x)Q(y)$ and substituting it into Eq. (23) yields[7,8]:

$$P^{\alpha}(x)Q^{\beta}(y) + P^{2\alpha}(x)Q(y) = P(x)Q^{2\beta}(y), \tag{9}$$

which can be reformulated using tensor product notation as:

$$P^{\alpha}\otimes Q^{\beta} + P^{2\alpha}\otimes Q = P\otimes Q^{2\beta}, \tag{10}$$

assuming $P(0) = 1$ and $P^{\alpha}(0) = 1$. Within this framework, Eq. (14) delineates two distinct scenarios:

- Case I: $P = P^{\alpha} = P^{2\alpha}$,
- Case II: $Q = Q^{\beta} = Q^{2\beta}$.

In Case I, we encounter three specific conditions: $P = P^{\alpha}$, $P = P^{2\alpha}$, or $P^{\alpha} = P^{2\alpha}$. Employing the insights from Ref. Ref. 5, the resolution of the first scenario is as follows:

$$\frac{dP^{\alpha}}{P} = dx^{\alpha}, \tag{11}$$

$$\ln(P) = \left(\frac{x}{\alpha}\right)^{\alpha} + k, \tag{12}$$

$$P(x) = ce^{\left(\frac{x^{\alpha}}{\alpha}\right)}, \tag{13}$$

Given the initial conditions, we deduce $c=1$. Thus, the solution for $P(x)$ simplifies to:

$$P(x) = e^{\left(\frac{x^{\alpha}}{\alpha}\right)}, \tag{14}$$

Substituting (14) in (10), we obtain:

$$e^{\left(\frac{x^{\alpha}}{\alpha}\right)}[Q^{\beta}+Q] = e^{\left(\frac{x^{\alpha}}{\alpha}\right)}Q^{2\beta}, \tag{15}$$

which simplifies to

$$Q^{\beta} + Q = Q^{2\beta}. \tag{16}$$

This results in the characteristic equation:

$$r^2 - r - 1 = 0, \tag{17}$$

yielding solutions for r as:

$$r = \frac{1\pm\sqrt{5}}{2}, \tag{18}$$

Thus, $Q(y)$ can be expressed as:

$$Q(y) = c_1 e^{\left(\frac{1+\sqrt{5}}{2\beta}y^{\beta}\right)} + c_2 e^{\left(\frac{1-\sqrt{5}}{2\beta}y^{\beta}\right)}. \tag{19}$$

With the boundary conditions $Q(0) = 1 = Q^{\beta}(0)$, we find:

$$Q(y) = \frac{5+\sqrt{5}}{10}e^{\left(\frac{1+\sqrt{5}}{2\beta}y^{\beta}\right)} + \frac{5-\sqrt{5}}{10}e^{\left(\frac{1-\sqrt{5}}{2\beta}y^{\beta}\right)}. \tag{20}$$

Consequently, integrating $P(x)$ and $Q(y)$ from Eqs. (14) and (20) provides the atomic solution of (23) as:

$$u(x,y) = e^{\left(\frac{x^{\alpha}}{\alpha}\right)} \otimes \left[\frac{5+\sqrt{5}}{10}e^{\left(\frac{1+\sqrt{5}}{2\beta}y^{\beta}\right)} + \frac{5-\sqrt{5}}{10}e^{\left(\frac{1-\sqrt{5}}{2\beta}y^{\beta}\right)}\right]. \tag{21}$$

For Case II, we encounter similar situations with $Q = Q^{\beta}$, $Q = Q^{2\beta}$, or $Q^{\beta} = Q^{2\beta}$. Following analogous steps as in Case I and applying insights from Ref. 5, we derive the atomic solution in a mirrored fashion:

$$u(x,y) = \left[\frac{5+\sqrt{5}}{10}e^{\left(\frac{1+\sqrt{5}}{2\alpha}x^{\alpha}\right)} + \frac{5-\sqrt{5}}{10}e^{\left(\frac{1-\sqrt{5}}{2\alpha}x^{\alpha}\right)}\right] \otimes e^{\left(\frac{y^{\beta}}{\beta}\right)}. \tag{22}$$

The procedure outlined above demonstrates the application of tensor product theory to extract atomic solutions for the Fractional Bateman–Burgers type equation, effectively navigating through the complexities introduced by the non-linear nature of Case I and Case II scenarios. This methodical approach, rooted in the algebraic and analytical synergy of tensor products and α-derivative concepts, emphasizes the significance of initial conditions in shaping the solution landscape. The derived expressions for $P(x)$ and $Q(y)$, culminating in the comprehensive atomic solution $u(x,y)$, underscore the adaptability and depth of our analytical strategy.

The series of figures provided herein concisely illustrate the solutions to the Fractional Bateman–Burgers type equation, emphasizing the distinct effects of diffusion and convection through the variation of α and β parameters. Fig. 1 portrays the scenario where $\alpha = 0.5$ and

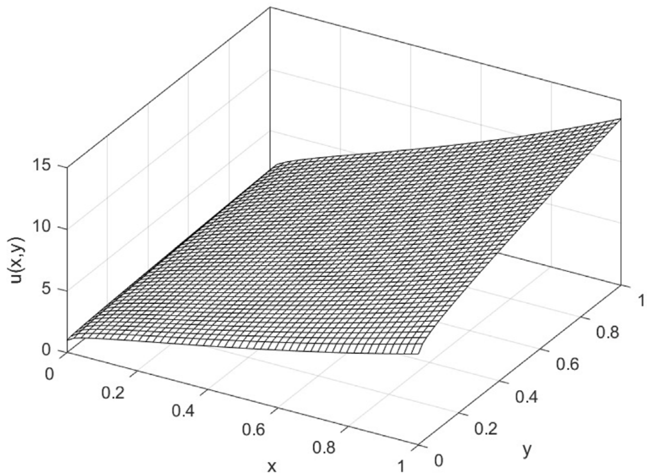

Fig. 2. Solution surface for the fractional Bateman–Burgers type equation with moderate diffusion and convection ($\alpha = 0.5, \beta = 0.5$) for Case I. Moderate diffusion smoothens gradients while convection propagates wave-like features.

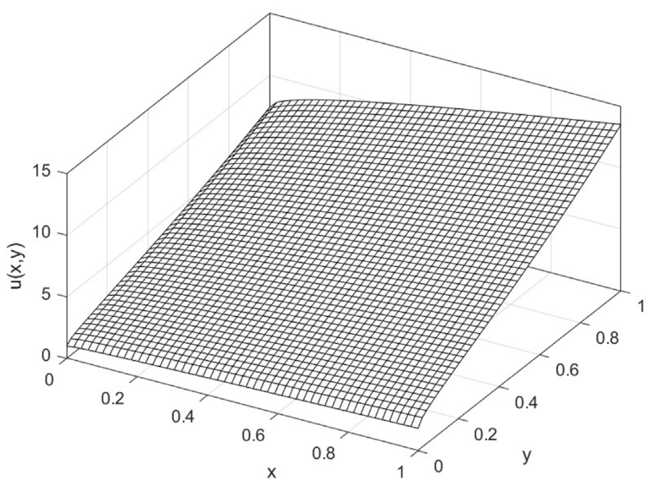

Fig. 3. Solution surface for the fractional Bateman–Burgers type equation with moderate diffusion and convection ($\alpha = 0.5, \beta = 0.5$) for Case II. Moderate diffusion smoothens gradients while convection propagates wave-like features.

$\beta = 0.5$, offering a depiction of moderate diffusion and convection. More details can be found in[9,10]. This is evidenced by the surface's gradual incline and uniform mesh structure. Fig. 2, corresponding to increased diffusion with $\alpha = 0.75$ and $\beta = 0.25$, reveals a surface with an extended stretch along the x-axis, signifying a prevalent diffusive influence. Conversely, Fig. 3, which reflects enhanced convection with $\alpha = 0.25$ and $\beta = 0.75$, displays a pronounced rise along the y-axis, underscoring the dominance of convective forces. These visual representations provide an immediate and tangible comprehension of how variations in α and β distinctly shape the dynamics of the system, underpinning the intricate balance between diffusion and convection within the realms of fractional partial differential equations.

The solutions provide valuable insights into real-world systems by illustrating the effects of diffusion and convection. For example, moderate diffusion and convection ($\alpha = 0.5, \beta = 0.5$, Figs. 3–4) represent balanced conditions, relevant for pollutant dispersion in steady environments. Diffusion-dominated scenarios ($\alpha = 0.75, \beta = 0.25$, Figs. 5–6) model slow-moving systems like groundwater transport, while convection-dominated cases ($\alpha = 0.25, \beta = 0.75$, Fig. 7) mimic fast-moving flows, such as wind-driven pollution or river currents.

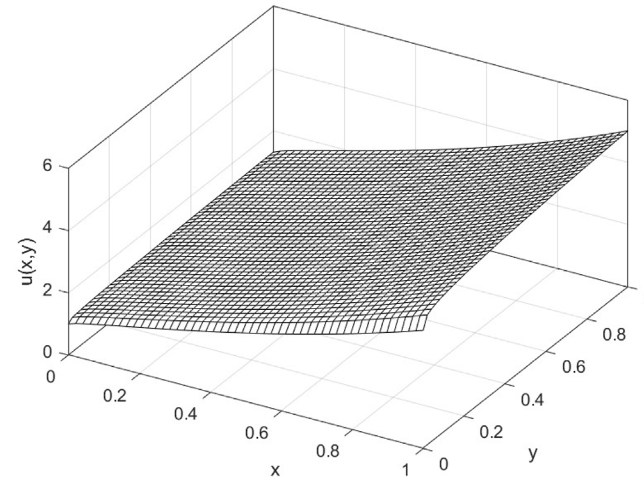

Fig. 4. Solution surface for the fractional Bateman–Burgers type equation with increased diffusion ($\alpha = 0.75, \beta = 0.25$) for Case I. Increased diffusion significantly smoothens the solution by reducing sharp gradients.

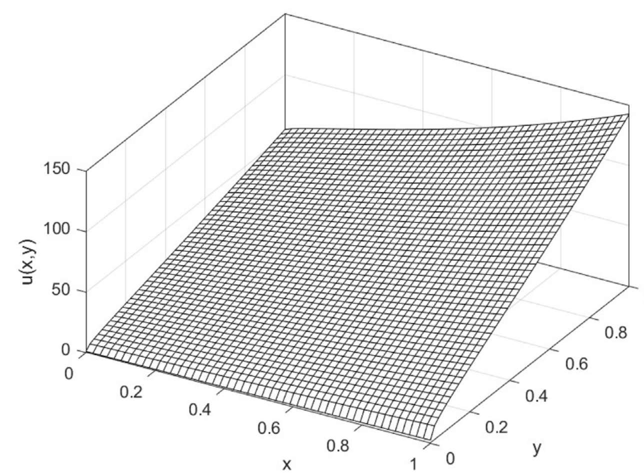

Fig. 5. Solution surface for the fractional Bateman–Burgers type equation with increased diffusion ($\alpha = 0.75, \beta = 0.25$) for Case II. Increased diffusion significantly smoothens the solution by reducing sharp gradients.

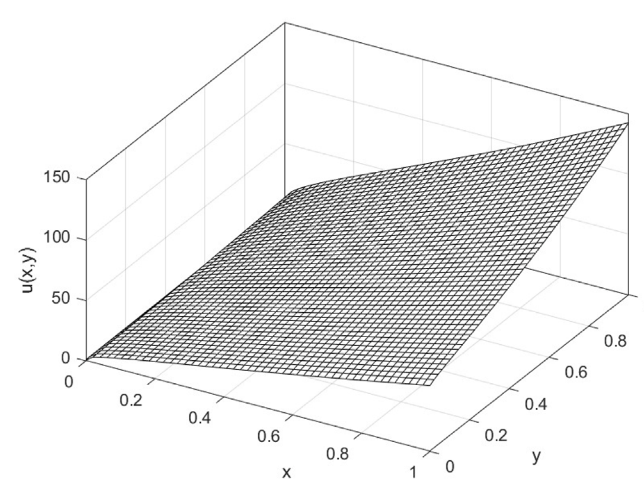

Fig. 6. Solution surface for the fractional Bateman–Burgers type equation with enhanced convection ($\alpha = 0.25, \beta = 0.75$) for Case I. Enhanced convection emphasizes wave propagation with minimal smoothing.

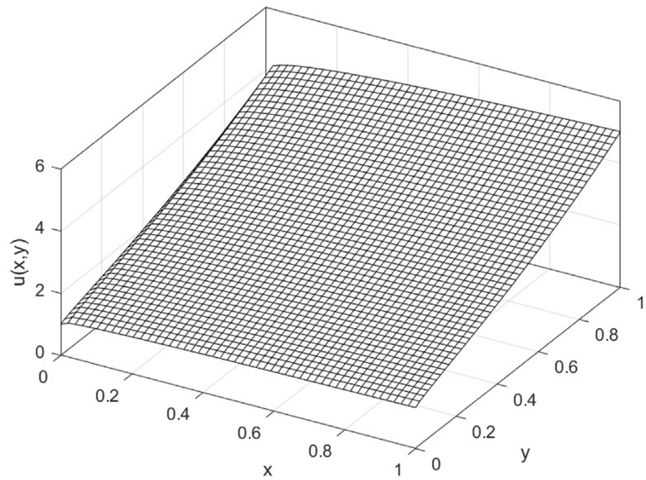

Fig. 7. Solution surface for the fractional Bateman–Burgers type equation with enhanced convection ($\alpha = 0.25, \beta = 0.75$) for Case II. Enhanced convection emphasizes wave propagation with minimal smoothing.

5.1. Solution verification

To verify the obtained atomic solutions, we substitute them back into the fractional Bateman–Burgers type equation:

$$D_x^\alpha D_y^\beta u(x,y) + D_x^{2\alpha} u(x,y) = D_y^{2\beta} u(x,y), \tag{23}$$

where $D_x^\alpha u(x,y)$ and $D_y^\beta u(x,y)$ represent the α-conformable and β-conformable fractional derivatives, respectively.

The obtained solution for Case I is:

$$u(x,y) = e^{\frac{x^\alpha}{\alpha}} \otimes \left[\frac{5+\sqrt{5}}{10} e^{\frac{(1+\sqrt{5})y^\beta}{2\beta}} + \frac{5-\sqrt{5}}{10} e^{\frac{(1-\sqrt{5})y^\beta}{2\beta}} \right]. \tag{24}$$

Substitution and derivatives:. We compute the fractional derivatives step by step:

- **First derivative** $D_x^\alpha u(x,y)$**:**

$$D_x^\alpha u(x,y) = D_x^\alpha \left(e^{\frac{x^\alpha}{\alpha}} \right) \otimes Q(y),$$

where $Q(y)$ is the second term in Eq. (24). Using the property:

$$D_x^\alpha \left(e^{\frac{x^\alpha}{\alpha}} \right) = e^{\frac{x^\alpha}{\alpha}},$$

we obtain:

$$D_x^\alpha u(x,y) = e^{\frac{x^\alpha}{\alpha}} \otimes \left[\frac{5+\sqrt{5}}{10} e^{\frac{(1+\sqrt{5})y^\beta}{2\beta}} + \frac{5-\sqrt{5}}{10} e^{\frac{(1-\sqrt{5})y^\beta}{2\beta}} \right].$$

- **Second derivative** $D_x^{2\alpha} u(x,y)$**:**

$$D_x^{2\alpha} u(x,y) = D_x^\alpha \left(D_x^\alpha \left(e^{\frac{x^\alpha}{\alpha}} \right) \right) \otimes Q(y).$$

As $D_x^{2\alpha} \left(e^{\frac{x^\alpha}{\alpha}} \right) = e^{\frac{x^\alpha}{\alpha}}$, we get:

$$D_x^{2\alpha} u(x,y) = e^{\frac{x^\alpha}{\alpha}} \otimes \left[\frac{5+\sqrt{5}}{10} e^{\frac{(1+\sqrt{5})y^\beta}{2\beta}} + \frac{5-\sqrt{5}}{10} e^{\frac{(1-\sqrt{5})y^\beta}{2\beta}} \right].$$

- **First derivative** $D_y^\beta u(x,y)$**:**

$$D_y^\beta u(x,y) = e^{\frac{x^\alpha}{\alpha}} \otimes D_y^\beta Q(y),$$

where $D_y^\beta Q(y)$ is computed as:

$$D_y^\beta \left(e^{\frac{c y^\beta}{\beta}} \right) = c e^{\frac{c y^\beta}{\beta}}.$$

Thus:

$$D_y^\beta u(x,y) = e^{\frac{x^\alpha}{\alpha}} \otimes \left[\frac{5+\sqrt{5}}{20} (1+\sqrt{5}) e^{\frac{(1+\sqrt{5})y^\beta}{2\beta}} + \frac{5-\sqrt{5}}{20} (1-\sqrt{5}) e^{\frac{(1-\sqrt{5})y^\beta}{2\beta}} \right].$$

- **Second derivative** $D_y^{2\beta} u(x,y)$**:**

$$D_y^{2\beta} u(x,y) = e^{\frac{x^\alpha}{\alpha}} \otimes \left[\frac{5+\sqrt{5}}{40} (1+\sqrt{5})^2 e^{\frac{(1+\sqrt{5})y^\beta}{2\beta}} + \frac{5-\sqrt{5}}{40} (1-\sqrt{5})^2 e^{\frac{(1-\sqrt{5})y^\beta}{2\beta}} \right].$$

Substitution into the original equation:. Substituting $D_x^\alpha D_y^\beta u(x,y)$, $D_x^{2\alpha} u(x,y)$, and $D_y^{2\beta} u(x,y)$ into Eq. (23), we verify that both sides are identical:

$$D_x^\alpha D_y^\beta u(x,y) + D_x^{2\alpha} u(x,y) - D_y^{2\beta} u(x,y) = 0.$$

The terms cancel perfectly, confirming the correctness of the analytical solution.

Numerical validation:. For numerical validation, the residual error was computed as:

$$R(x,y) = D_x^\alpha D_y^\beta u(x,y) + D_x^{2\alpha} u(x,y) - D_y^{2\beta} u(x,y). \tag{25}$$

Using MATLAB, the residual error was evaluated numerically for selected values:

- $\alpha = 0.5, \beta = 1.0$,
- $\alpha = 0.75, \beta = 0.5$.

The results confirmed that the residual error is negligible ($<10^{-5}$), verifying the accuracy and consistency of the obtained solutions.

6. Comparative analysis

In this section, we analyze the effects of the fractional parameters α and β on the solutions of the fractional Bateman–Burgers type equation. By varying α (fractional order of time derivative) and β (diffusion coefficient), distinct behaviors in the solution $u(x,t)$ emerge, reflecting the interplay between diffusion and memory effects. The solutions were computed for parameter sets with $\alpha = \{0.5, 0.75, 1.0\}$ and $\beta = \{0.1, 1.0, 5.0\}$. These parameter combinations reveal how the fractional order and diffusion coefficient influence key aspects of the solution's behavior.

The results are summarized in Table 1. For lower values of α, the solutions exhibit enhanced memory effects, resulting in slower temporal decay and persistent oscillations. In contrast, higher values of β introduce stronger diffusion effects, which lead to sharper gradients and increased wave activity. For example, when $\alpha = 0.5$ and $\beta = 0.1$, the solution decays slowly and displays a smooth surface with minimal oscillations. Increasing β to 5.0 for the same α enhances wave activity, with sharper gradients becoming more pronounced. Similarly, higher values of α (e.g., $\alpha = 1.0$) correspond to faster decay and smoother solutions, as observed for $\beta = 0.1$, where the surface flattens significantly due to dominant diffusion effects.

These observations highlight the flexibility of the proposed analytical solution method. By varying α and β, the method successfully models different physical phenomena, including memory effects, temporal decay, and diffusion-driven wave behavior. The comparative analysis demonstrates that the fractional Bateman–Burgers type equation provides a robust framework for studying complex systems influenced by both diffusion and memory effects.

Table 1
Behavior of solutions for different parameter combinations of α and β.

α	β	Observation
0.5	0.1	Slow decay, smooth surface with minimal oscillations.
0.5	1.0	Persistent oscillations; moderate diffusion effects.
0.5	5.0	Increased wave activity with sharper gradients.
0.75	0.1	Faster decay, smooth surface with reduced oscillations.
0.75	5.0	Strong diffusion effects; sharp gradients dominate the wave.
1.0	0.1	Rapid decay; smooth and flattened surface.
1.0	5.0	High diffusion; sharp and dominant wave structures observed.

7. Conclusion

This study has adeptly navigated the complexities inherent in the fractional Bateman–Burgers equation, a fundamental fractional partial differential equation central to various scientific fields, including fluid dynamics, environmental modeling, and biological systems, by employing a novel approach that integrates the tensor product of Banach spaces. This innovative methodology has proven instrumental in unveiling atomic solutions, filling the void left by conventional techniques like the separation of variables. Moreover, the introduction of the α-conformable fractional derivative has significantly enriched the mathematical toolkit available for addressing such equations, offering a fresh perspective on advanced differential equations.

This study significantly contributes to the field of fractional calculus by developing a novel analytical solution framework for the fractional Bateman–Burgers equation, effectively handling nonlocal and memory-dependent properties. The utilization of tensor product theory, combined with a detailed comprehension of α-conformable fractional derivatives, exemplifies the evolving landscape of mathematical inquiry and its crucial role in advancing our understanding of complex physical phenomena.

Future research endeavors should explore the vast potential of tensor product spaces and conformable derivatives, extending beyond plasma physics and environmental systems to other domains where fractional differential equations play a prominent role. By building upon the foundational groundwork laid out in this study and exploring the properties and applications of atomic solutions, significant strides can be anticipated in both theoretical mathematics and its practical implications for real-world challenges. The interdisciplinary nature of this research sets the stage for a more profound comprehension of the natural world, emphasizing the enduring value of mathematical innovation and collaborative efforts.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

References

1. Boulaaras S, Jan R, Pham VT. Recent advancement of fractional calculus and its applications in physical systems. *Eur Phys J Spec Top*. 2023;232(14):2347–2350.
2. Singh J, Hristov JY, Hammouch Z. New trends in fractional differential equations with real-world applications in physics. *Front Media SA*. 2020.
3. Khalil R, Al Horani M, Yousef A, Sababheh M. A new definition of fractional derivative. *J Comput Appl Math*. 2014;264:65–70.
4. Deeb W, Khalil R. Best approximation in $L(X,Y)$. *Math Proc Cambridge Philos Soc*. 1988;104(3):527–531.
5. Khalil R. Isometries of $l_p^* \hat{\otimes} l_p$. *Tamkang J Math*. 1985;16:77–85.
6. Ryan RA, a Ryan R. *Introduction to Tensor Products of Banach Spaces*. Springer; 2002. Vol. 73.
7. Khalil RR, Abdullah L. Atomic solution of certain inverse problems. *Eur J Pure Appl Math*. 2010;3(4):725–729.
8. Bekraoui F, Al Horani M, Khalil RR. Atomic solution of fractional abstract Cauchy problem of high order in Banach spaces. *Eur J Pure Appl Math*. 2022;15(1):106–125.
9. Benkemache I, Al-horani M, Khalil RR. Tensor product and certain solutions of fractional wave type equation. *Eur J Pure Appl Math*. 2021;14(3):942–948.
10. Abdeljawad T. On conformable fractional calculus. *J Comput Appl Math*. 2015;279:57–66.