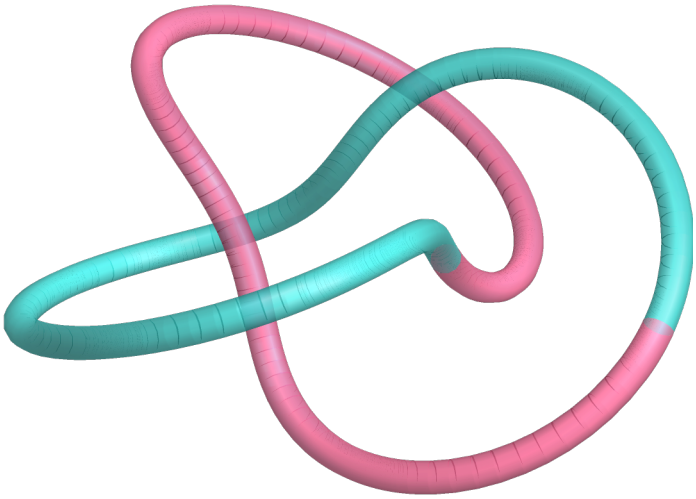
Figure 1

Figure 1a - Unframed knots

In[•]:=
```
BraidOnTorus2[S[2, 3, 1, 0.5, 0.5], False]
```

Out[•]=



In[•]:=
```
SDia[2, 3, 1, 1, 1, 1]
```

Out[•]=

Figure 2a - Framed knots

In[•]:=
```
FramedBraidRep[2, 3, 1, {5, 4}, 0.2]
```

Out[•]=

In[•]:=
```
H = S[2, 3, 1, 0.2, 0.2];
Curves =
  Table[{(1 + H[[j]][[1]]) Cos[h], (1 + H[[j]][[1]]) Sin[h], 3 H[[j]][[2]]}, {j, 1, Dimensions[H][[1]]}];
dk = {2, 2};
Show[Table[Framing[Curves[[j]], dk[[j]], 0.15, ColorBraids[[j]], {0.001 Pi, 2 π 0.999, 500}, 1],
  {j, 1, 2}], ViewPoint → Top]
```

Out[•]=

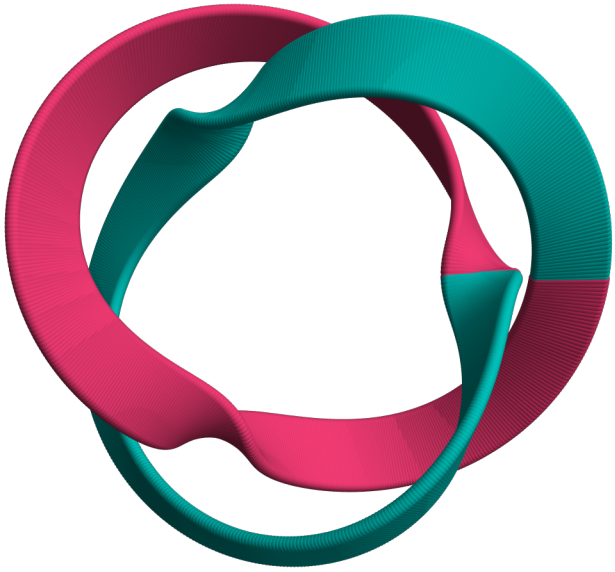
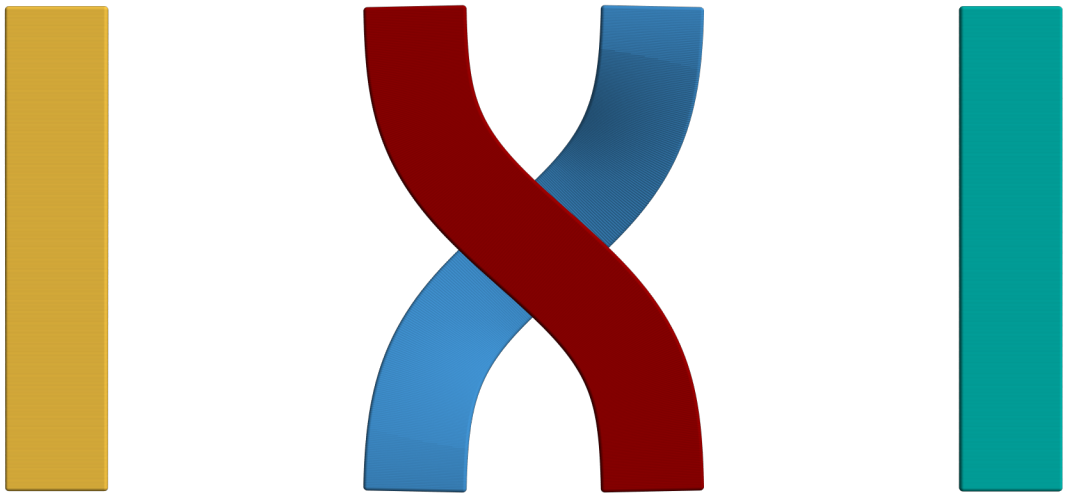


Figure 1c - Braid generators

```
In[•]:= Show[Twist[0, {-4, 0, -2}, Pantone149, 0.4],
          Crossing[{0, 0, 0}, "Left", "Under", PantoneProceBlue, 0.4],
          Crossing[{0, 0, 0}, "Right", "Over", Darker[Red, 0.4], 0.4],
          Twist[0, {4, 0, -2}, Pantone2459, 0.4], Lighting → "Neutral", ImageSize → Large]
```

Out[•]=



```
In[•]:= Show[Twist[0, {-4, 0, -2}, Pantone149, 0.1],
          Crossing[{0, 0, 0}, "Left", "Over", PantoneProceBlue, 0.1],
          Crossing[{0, 0, 0}, "Right", "Under", Darker[Red, 0.4], 0.1],
          Twist[0, {4, 0, -2}, Pantone2459, 0.1]]
```

Out[•]=

```
In[•]:= Show[Twist[0, {-5, 0, -2}, PantoneProceBlue, 0.4], Twist[1, {-3, 0, -2}, Pantone149, 0.4],
          Twist[0, {-1, 0, -2}, Darker[Red, 0.4], 0.4], Lighting → "Neutral", ImageSize → Large]
Out[•]=
```

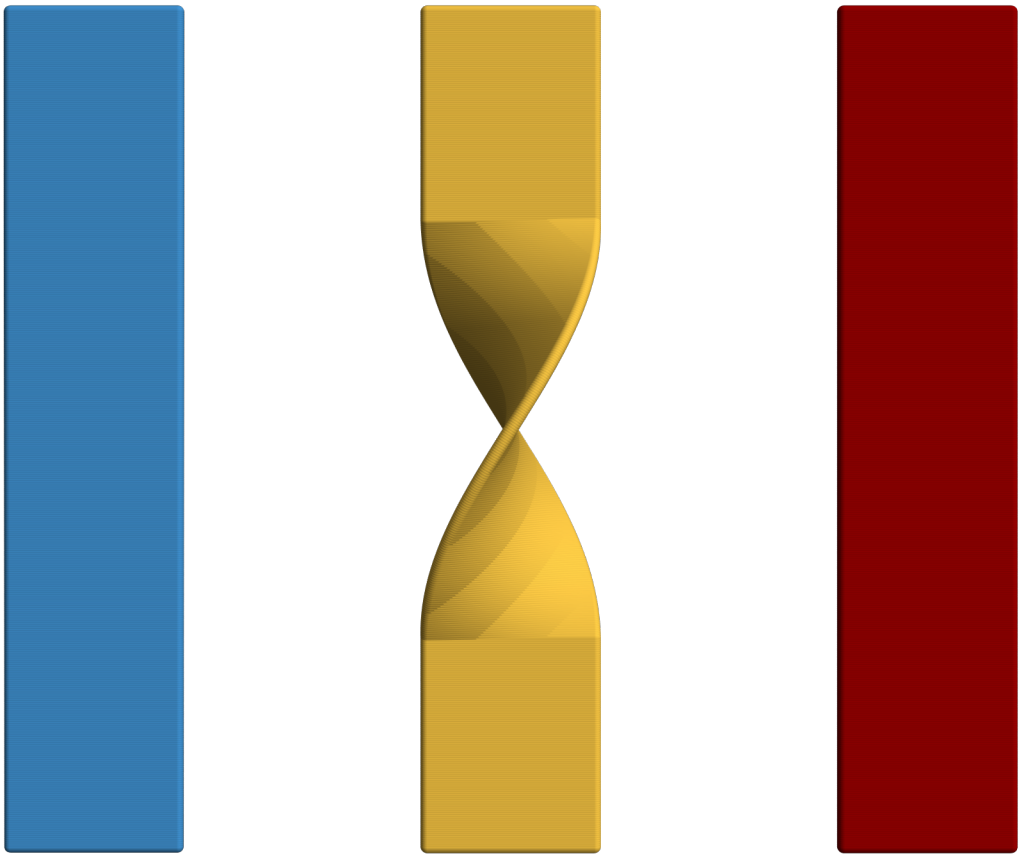
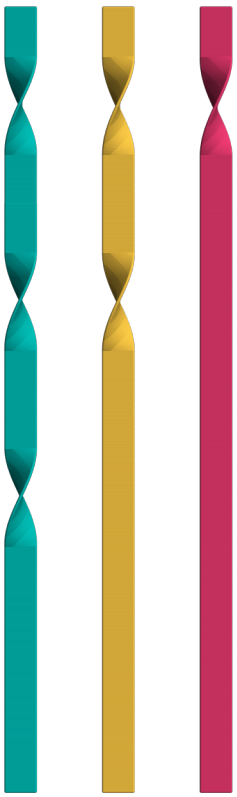


Figure 2 - Sending the message

```
Strand1 = Show[Twist[1, {-3, 0, -2}, Pantone2459, 0.3],
   Crossing[{-2, 0, -4}, "Right", "Under", Pantone2459, 0.3], Crossing[{0, 0, -8},
    "Right", "Over", Pantone2459, 0.3], Twist[2, {1, 0, -14}, Pantone2459, 0.3]];
Strand3 = Show[Crossing[{0, 0, 0}, "Left", "Under", Pantone199, 0.3],
   Crossing[{-2, 0, -4}, "Left", "Over", Pantone199, 0.3], Twist[1, {-3, 0, -10},
    Pantone199, 0.3], Crossing[{-2, 0, -12}, "Right", "Under", Pantone199, 0.3]];
Strand2 = Show[Crossing[{0, 0, 0}, "Right", "Over", Pantone149, 0.3],
   Twist[2, {1, 0, -6}, Pantone149, 0.3], Crossing[{0, 0, -8}, "Left", "Under",
    Pantone149, 0.3], Crossing[{-2, 0, -12}, "Left", "Over", Pantone149, 0.3]];
Show[Strand1, Strand2, Strand3]
```

In[•]:=
```
Strand1 =
  Show[Twist[1, {-3, 0, -2}, Pantone2459, 0.3], Twist[1, {-3, 0, -6}, Pantone2459, 0.3],
   Twist[1, {-3, 0, -10}, Pantone2459, 0.3], Twist[0, {-3, 0, -14}, Pantone2459, 0.3]];
Strand3 = Show[Twist[1, {-1, 0, -2}, Pantone149, 0.3], Twist[1, {-1, 0, -6}, Pantone149, 0.3],
   Twist[0, {-1, 0, -10}, Pantone149, 0.3], Twist[0, {-1, 0, -14}, Pantone149, 0.3]];
Strand2 = Show[Twist[1, {1, 0, -2}, Pantone199, 0.3], Twist[0, {1, 0, -6}, Pantone199, 0.3],
   Twist[0, {1, 0, -10}, Pantone199, 0.3], Twist[0, {1, 0, -14}, Pantone199, 0.3]];
Show[Strand1, Strand2, Strand3]
```

Out[•]=



In[•]:=
```
H = S[3, 2, 2, 0.2, 0.2];
Curves =
  Table[{(1 + H[[j]][[1]]) Cos[h], (1 + H[[j]][[1]]) Sin[h], 2 H[[j]][[2]]}, {j, 1, Dimensions[H][[1]]}];
dk = {3, 2, 1};
Show[Table[FramingZsc[Curves[[j]], dk[[j]], 0.1, ColorBraids[[j]], {0.001 Pi, 2 π 0.999, 250}, 2],
  {j, 1, Dimensions[H][[1]]}], ViewPoint → Top]
```

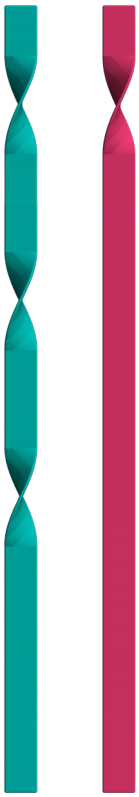
In[•]:=
```
Show[Table[FramingZsc[Curves[[j]], dk[[j]], 0.1, Gray, {0.001 Pi, 2 π 0.999, 250}, 2],
  {j, 1, Dimensions[H][[1]]}], ViewPoint → Top]
```

Out[•]=

Figure 3 - Challenge response

```
Strand1 =
  Show[Twist[1, {-1, 0, -2}, Pantone2459, 0.3], Twist[1, {-1, 0, -6}, Pantone2459, 0.3],
   Twist[1, {-1, 0, -10}, Pantone2459, 0.3], Twist[0, {-1, 0, -14}, Pantone2459, 0.3]];

Strand2 = Show[Twist[1, {1, 0, -2}, Pantone199, 0.3], Twist[0, {1, 0, -6}, Pantone199, 0.3],
   Twist[0, {1, 0, -10}, Pantone199, 0.3], Twist[0, {1, 0, -14}, Pantone199, 0.3]];
Show[Strand1, Strand2]
```

Out[•]=

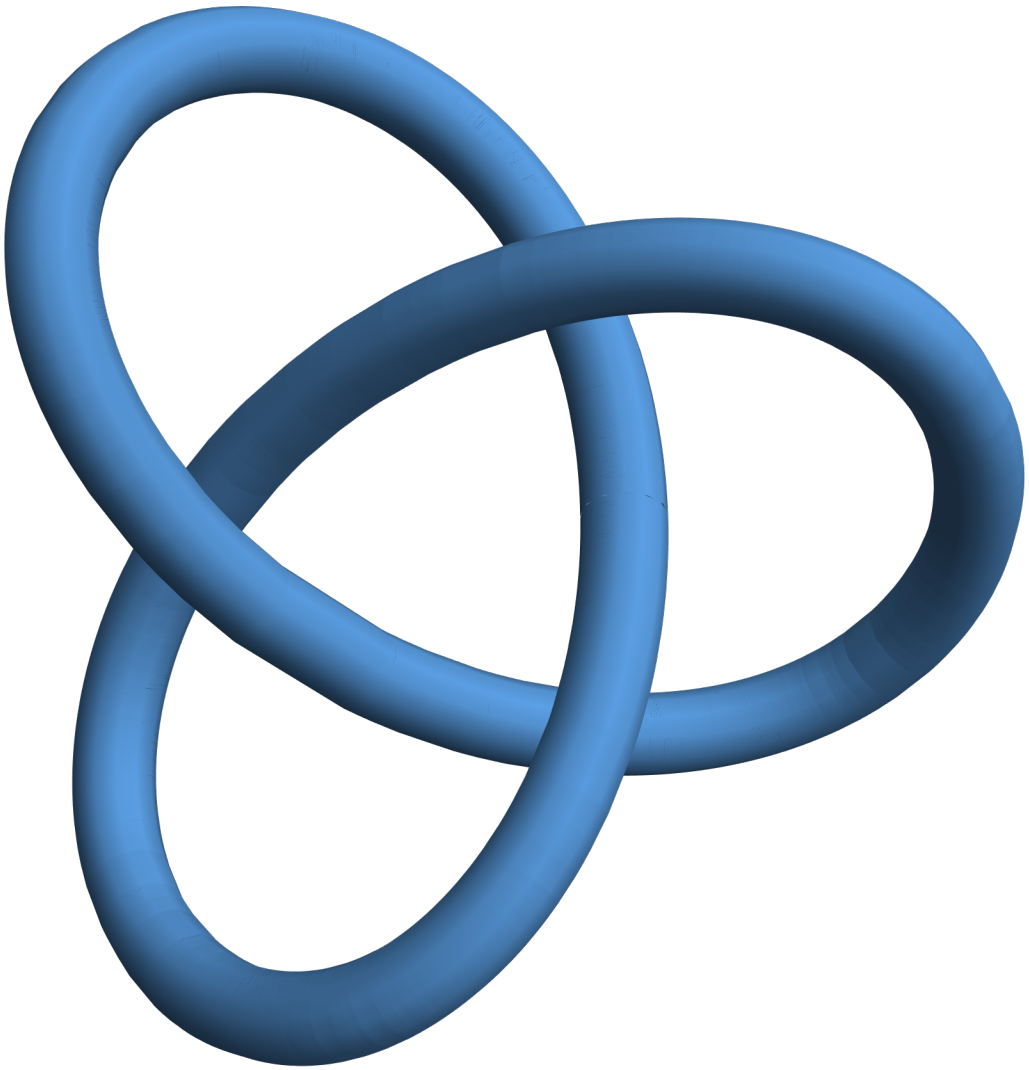


In[•]:=
```
Trefoil[t_, sc_] := sc ({Sin[3 t], 0, 0} + {0, -2 Cos[2 t], 2 Sin[2 t]} + {0, Cos[t], Sin[t]});
```

In[•]:=
```
Strand1 = Show[Crossing[{0, 0, 0}, "Left", "Over", Pantone199, 0.3],
    Twist[1, {-1, 0, -6}, Pantone199, 0.3], Crossing[{0, 0, -8}, "Left", "Over",
     Pantone2459, 0.3], Crossing[{0, 0, -12}, "Right", "Under", Pantone2459, 0.3]];
Strand2 = Show[Crossing[{0, 0, 0}, "Right", "Under", Pantone2459, 0.3],
    Twist[3, {1, 0, -6}, Pantone2459, 0.3], Crossing[{0, 0, -8}, "Right", "Under",
     Pantone199, 0.3], Crossing[{0, 0, -12}, "Left", "Over", Pantone199, 0.3]];

ALiceBraid = Show[Strand1, Strand2]
```

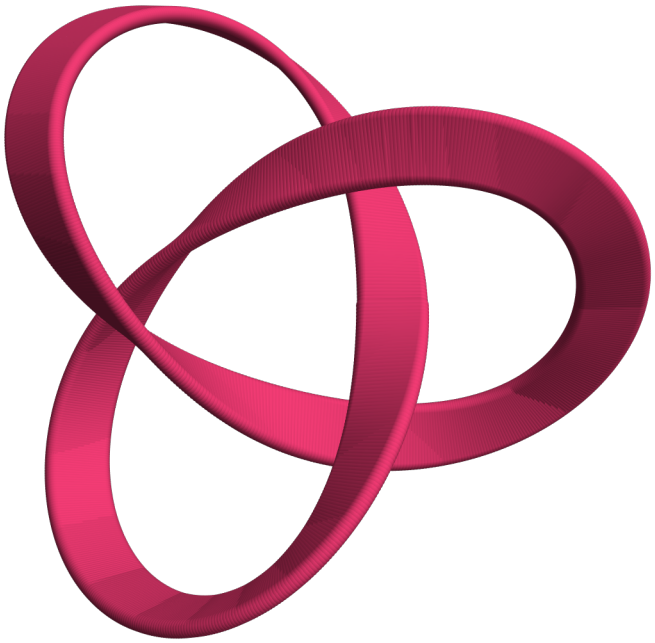
Out[•]=

Out[•]=



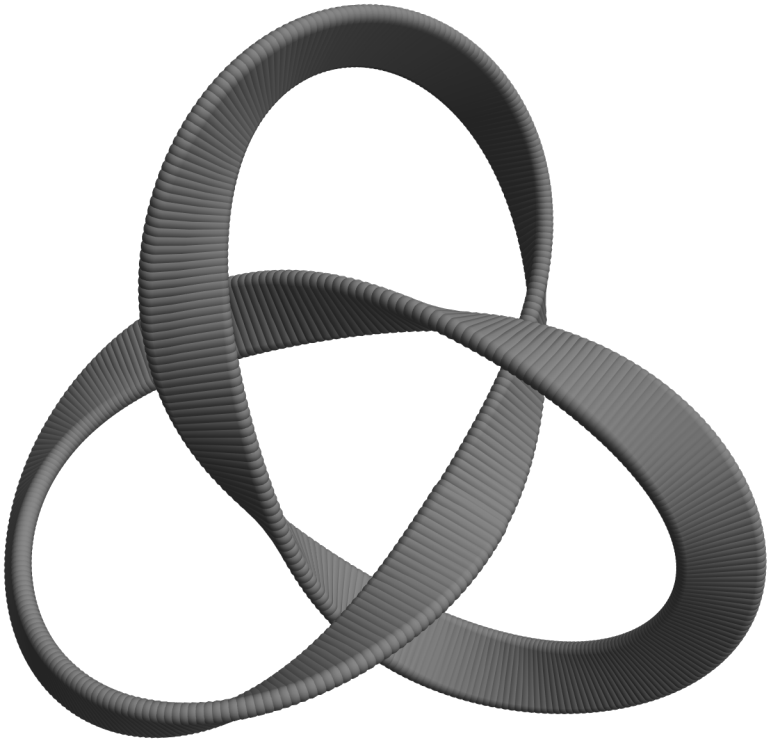
In[•]:=
```
KnotA1 = AdvancedFraming[Trefoil[h, 0.4],
    2, 0.1, Pantone199, {0.001 Pi, π 0.999, 500}, 1, {0, 0, 0}];
KnotA2 = AdvancedFraming[Trefoil[h, 0.4],
    2, 0.1, Pantone199, {π 0.999, 2 π 0.999, 500}, 1, {0, 0, 0}];
Show[KnotA1, KnotA2, ViewPoint → Left]
```

Out[•]=



Out[•]=

$Aborted

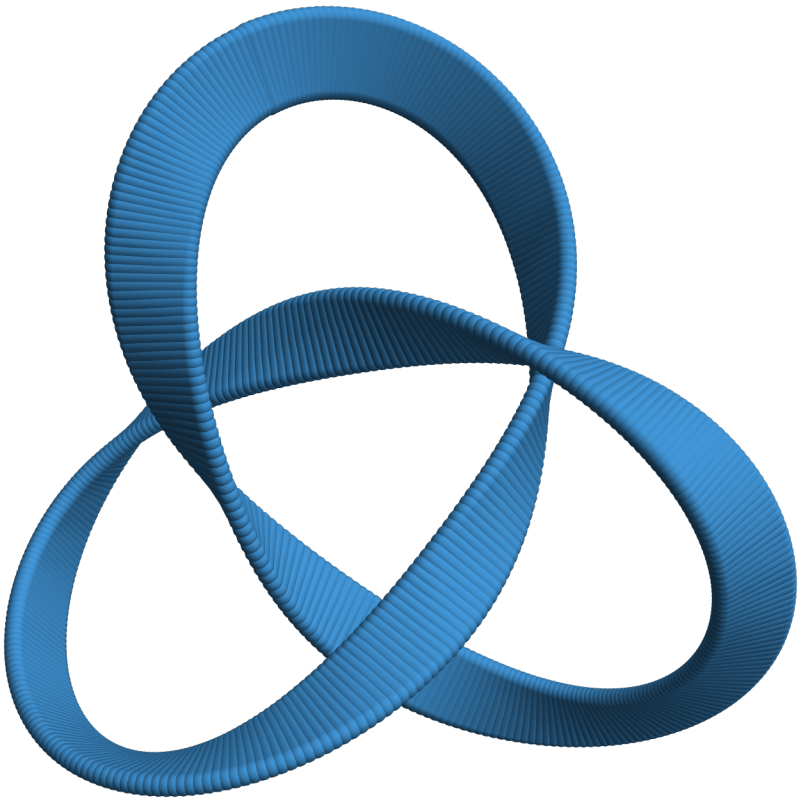


In[•]:=
```
Strand3 = Show[Crossing[{0, 0, 0}, "Left", "Over", Pantone149, 0.3],
    Twist[2, {-1, 0, -6}, Pantone149, 0.3], Crossing[{0, 0, -8}, "Left", "Over",
     Pantone7664, 0.3], Crossing[{0, 0, -12}, "Right", "Under", Pantone7664, 0.3]];
Strand4 = Show[Crossing[{0, 0, 0}, "Right", "Under", Pantone7664, 0.3],
    Twist[2, {1, 0, -6}, Pantone7664, 0.3], Crossing[{0, 0, -8}, "Right", "Under",
     Pantone149, 0.3], Crossing[{0, 0, -12}, "Left", "Over", Pantone149, 0.3]];
BobBraid = Show[Strand3, Strand4]
```

Out[•]=

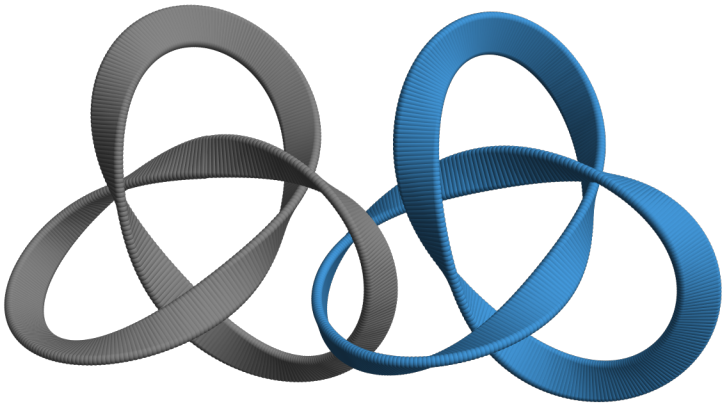
In[•]:=
```
KnotB = AdvancedFraming[Trefoil[h + 0.4 π, 0.4], 4,
  0.1, ColorBraids[4], {0.001 Pi, 2 π 0.999, 500}, 1, {0, 0, 1.8}]
```

Out[•]=



In[•]:=
```
Show[KnotA, KnotB]
```

Out[•]=



Functions

In[ ]:=

```
SetDirectory@NotebookDirectory[];
(*Creates a list of n colors*)
Colors[n_] := With[{partL = Ceiling[Sqrt[n]]},
  DeleteCases[Flatten[Transpose[Partition[Table[Lighter[Darker[Hue[c], .1], .25],
       {c, 0, 1 - 1 / n, 1 / n}], partL, partL, 1, 0]]], 0]]

(*Extra nice Custom colors*)
FunkyT = RGBColor[188 / 255, 224 / 255, 225 / 255];
Pantone2459 = RGBColor[1 / 255, 181 / 255, 174 / 255];
Pantone218 = RGBColor[206 / 255, 102 / 255, 161 / 255];
Pantone199 = RGBColor[227 / 255, 56 / 255, 109 / 255];
Pantone149 = RGBColor[243 / 255, 194 / 255, 66 / 255];
PantoneProceBlue = RGBColor[63 / 255, 143 / 255, 205 / 255];
Pantone7664 = RGBColor[104 / 255, 48 / 255, 120 / 255];

(*Favorite Colors*)
NiceBlue = RGBColor[2 / 5, 178 / 255, 1];
KnottyBlue = RGBColor[1 / 255, 89 / 255, 185 / 255];
MexGreen = RGBColor[0, 104 / 255, 71 / 255];

(*Colouring the braids*)
ColorBraids = {Pantone2459, Pantone199, Pantone149, PantoneProceBlue, Pantone7664};

(* Creation of the Braid representation *)
SDia[s_, r_, l_, a_, b_, sc_] := Module[{A, B, C, DD, DD2, KK},
   (*Braids*)
   A = Table[ParametricPlot3D[{a Cos[(r h + 2 π (j - 1)) / s], b / l Sin[(l (r h + 2 π (j - 1))) / s], sc * h},
       {h, 0, 2 Pi}, PlotStyle → {Opacity[0.6], Lighter[ColorBraids[[j]], 0.1]},
       PlotPoints → 100, ImagePadding → {{Automatic, Automatic}, {None, None}},
       Method → {"ShrinkWrap" → True}] /. Line[pts_, rest___] :→
       {CapForm → None, Specularity[White, 50], Tube[pts, 0.08, rest]}, {j, 1, s}];
   (*Final dots*)
   DD = Graphics3D[Table[{ColorBraids[[j]], Specularity[White, 50], Sphere[Simplify[
         {a Cos[(r h + 2 π (j - 1)) / s], b / l Sin[(l (r h + 2 π (j - 1))) / s], h} /. {h → 0}], 0.1]},
      {j, 1, s}], ImagePadding → {{Automatic, Automatic}, {None, None}},
     Method → {"ShrinkWrap" → True}];
   DD2 = Graphics3D[Table[{ColorBraids[[j]], Specularity[White, 50], Sphere[Simplify[
```

```
        {a Cos[(r h + 2 π (j - 1))/s], b/l Sin[(l (r h + 2 π (j - 1)))/s], sc * h} /. {h → 2 π}], 0.1]},
      {j, 1, s}], ImagePadding → {{Automatic, Automatic}, {None, None}},
     Method → {"ShrinkWrap" → True}];
   KK = Graphics3D[{EdgeForm[], Opacity[0.2], White,
      Cylinder[{{0, 0, -0.1}, {0, 0, sc * 2 π + 0.1}}, 1.2]}, Lighting → "Neutral",
     Boxed → False, ImagePadding → {{Automatic, Automatic}, {None, None}},
     Method → {"ShrinkWrap" → True}];

   (*Top and bottom leminscates*)
   B = ParametricPlot3D[Simplify[{a Cos[ x ], b / l Sin[l x], 0}],
     {x, 0, 2 π}, PlotStyle → Directive[Darker[Gray, 0.8], Thickness[0.02]],
     PlotPoints → 100, ImagePadding → {{Automatic, Automatic}, {None, None}},
     Method → {"ShrinkWrap" → True}];
   C = ParametricPlot3D[Simplify[{a Cos[ x ], b / l Sin[l x], sc * 2 π}],
     {x, 0, 2 π}, PlotStyle → Directive[Darker[Gray, 0.8], Thickness[0.02]],
     PlotPoints → 100, ImagePadding → {{Automatic, Automatic}, {None, None}},
     Method → {"ShrinkWrap" → True}];
   Show[A, B, C, DD, DD2, KK, Boxed → False,
    BoxStyle → {Thick}, Axes → False, Lighting → "Neutral", PlotRange → All,
    ViewPoint → {1.5445548132437892`, -2.864728779434567`, 0.926109847245335`},
    ViewVertical → {0.3340710941969182`, -0.6239446032271305`, 0.7064627634389605`},
    ImagePadding → {{Automatic, Automatic}, {None, None}},
    Method → {"ShrinkWrap" → True}]];

(*Parametric equation of the Braids*)
S[s_, r_, l_, a_, b_] :=
  Table[{a Cos[(r h + 2 π (j - 1))/s], b/l Sin[(l (r h + 2 π (j - 1)))/s], h}, {j, 1, s}];


(*Torus*)

Tor[θ_, ϕ_, R_, r_] :=
  {R Cos[θ], R Sin[θ], 0} + {r Cos[θ] Cos[ϕ], r Cos[ϕ] Sin[θ], r Sin[ϕ]};


(*Closing the Braid on a torus*)
BraidOnTorus2[H_, Torus_] := Module[{A, B, C, DD},
  A = ParametricPlot3D[Tor[θ, ϕ, 1, 0.5], {θ, 0, 2 π},
    {ϕ, 0, 2 π}, Boxed → False, Axes → False, PlotPoints → {70, 70}, Mesh → None,
    PlotStyle → {Opacity[0.5], LightGray, Specularity[White, 500]},
    Lighting → "Neutral", ImagePadding → {{Automatic, Automatic}, {None, None}},
    Method → {"ShrinkWrap" → True}];
  B = Show[Table[
```

```
      ParametricPlot3D[{(1 + H[[j]][[1]]) Cos[h], (1 + H[[j]][[1]]) Sin[h], H[[j]][[2]]}, {h, 0, 2 π},
        PlotStyle → {Opacity[0.6], Lighter[ColorBraids[[j]], 0.2]}, PlotPoints → 100,
        ImagePadding → {{Automatic, Automatic}, {None, None}}, PlotRange → All,
        Axes → False, Boxed → False, Lighting → "Neutral", Method → {"ShrinkWrap" → True}] /.
       Line[pts_, rest___] :→ {CapForm → None, Specularity[White, 100],
         Tube[pts, 0.07, rest]}, {j, 1, Dimensions[H][[1]]}]];
  If[Torus == True, Show[A, B], Show[B]]]



(*N-twist Segment*)
Twist[NumTwist_, Position_, Color_, Scale_] := Module[{PartA, PartB, PartC},
   PartA = Show[ParallelTable[Graphics3D[{Darker[Color, 0.2],
         Tube[{{Position + {-Scale, 0, t}, Position + {Scale, 0, t}}}, 0.03]}],
      {t, 0, 1, 1 / 99}], ViewPoint → Front, Boxed → False, Lighting → "Neutral"];
   PartC = Show[ParallelTable[Graphics3D[{Darker[Color, 0.2],
         Tube[{{Position + {-Scale, 0, t}, Position + {Scale, 0, t}}}, 0.03]}],
      {t, 3, 4, 1 / 99}], ViewPoint → Front, Boxed → False, Lighting → "Neutral"];
   PartB = Show[Table[Graphics3D[{Darker[Color, 0.2], Arrowheads[0.0],
         Arrow[Tube[{{0, 0, 1} + Position + {-Scale Cos[NumTwist π z / 2],
               -Scale Sin[NumTwist π z / 2], z}, {0, 0, 1} + Position +
             {Scale Cos[NumTwist π z / 2], Scale Sin[NumTwist π z / 2], z}}, 0.03]]}],
      {z, 0, 2, 1 / 99}], ViewPoint → Front, Boxed → False, Lighting → "Neutral"];
   Return[Show[PartA, PartB, PartC]]];

(*Crossing *)
Crossing[Position_, Direction_, Cross_, Color_, Scale_] :=
 Module[{Dir, Basis, tangent, normal, binormal, Stuff, Under},
  Dir = If[Direction == "Right", -1, 1];
  Under = If[Cross == "Under", 1, 0];
  Basis = Last[FrenetSerretSystem[{Dir Erf[t], 0, t}, t]] // FullSimplify;
  {tangent, normal, binormal} =
   Map[Arrow[{{Dir Erf[t], 0, t}, {Dir Erf[t], 0, t} + #}] &, Basis];
  Stuff = Show[ParallelTable[
     Graphics3D[{Darker[Color, 0.2], Tube[{{Under {0, 0.3, 0} + Position + {Dir Erf[t],
            Under 0.3 (Cos[π t / 2]), t} - Scale N[Basis[[2]]], Under {0, 0.3, 0} + Position +
           {Dir Erf[t], 0.3 (Under Cos[π t / 2]), t} + Scale N[Basis[[2]]]}}, 0.03]}],
     {t, -2, 2, 4 / (300 - 1)}], ViewPoint → Front, Boxed → False, Lighting →
     "Neutral"];
  Return[Stuff]
 ]


Framing[Curve_, Twists_, Scale_, Color_, Par_, sc_] :=
 Module[{Dir, Basis, tangent, normal, binormal, Stuff},
  Basis = Last[FrenetSerretSystem[Curve, h]] // FullSimplify;
  {tangent, normal, binormal} = Map[Arrow[{Curve, Curve + #}] &, Basis];
```

```
  Stuff =
   Show[ParallelTable[Graphics3D[{Darker[Color, 0.2], Tube[{{Re[(Curve /. {h → sc * h}) +
             Scale (N[Basis⟦2⟧] Cos[Twists h / 2] + Sin[Twists h / 2] N[Basis⟦3⟧])],
          Re[(Curve /. {h → sc * h}) - Scale (N[Basis⟦2⟧] Cos[Twists h / 2] +
               Sin[Twists h / 2] N[Basis⟦3⟧])]}}, 0.03]}, Lighting → "Neutral"],
     {h, Par⟦1⟧, Par⟦2⟧, Abs[Par⟦1⟧ - Par⟦2⟧] / (Par⟦3⟧ - 1)}], ViewPoint →
     Front, Boxed → False]]

VectorFraming[Curve_, Twists_, Scale_, Color_, Par_] :=
 Module[{Dir, Basis, tangent, normal, binormal, Stuff, sc = 1},
  Basis = Last[FrenetSerretSystem[Curve, h]] // FullSimplify;
  {tangent, normal, binormal} = Map[Arrow[{Curve, Curve + #}] &, Basis];
  Stuff = Show[ParallelTable[Graphics3D[
      {Darker[Color, 0.6], Arrowheads[{0.03}], Arrow[Tube[{{Re[(Curve /. {h → sc * h}) -
              0 Scale (N[Basis⟦2⟧] Cos[Twists h / 2] + Sin[Twists h / 2] N[Basis⟦3⟧])],
           Re[(Curve /. {h → sc * h}) - Scale (N[Basis⟦2⟧] Cos[Twists h / 2] +
                Sin[Twists h / 2] N[Basis⟦3⟧])]}}, 0.03]]}, Lighting → "Neutral"],
     {h, Par⟦1⟧, Par⟦2⟧, Abs[Par⟦1⟧ - Par⟦2⟧] / (Par⟦3⟧ - 1)}], ViewPoint →
     Front, Boxed → False]]

(* Creation of the Framed Braid representation *)
FramedBraidRep[s_, r_, l_, dk_, Scale_] :=
  Module[{A, B, Braids, C, DD, DD2, KK, a = 1, b = 1, sc = 1},
   Braids =
    Table[ParametricPlot3D[{a Cos[(r h + 2 π (j - 1)) / s], b/l Sin[(l (r h + 2 π (j - 1))) / s], sc * h},
      {h, 0, 2 Pi}, PlotStyle → {Opacity[0.6], Lighter[ColorBraids⟦j⟧, 0.1]},
      PlotPoints → 100, ImagePadding → {{Automatic, Automatic}, {None, None}},
      Method → {"ShrinkWrap" → True}] /. Line[pts_, rest___] :→
      {CapForm → None, Specularity[White, 50], Tube[pts, 0.08, rest]}, {j, 1, s}];
   A = Show[Table[Framing[S[s, r, l, 1, 1]⟦jj⟧,
       dk⟦jj⟧, Scale, ColorBraids⟦jj⟧, {0, 2 Pi, 200}, 1], {jj, 1, s}]];
   (*Final dots*)
   DD = Graphics3D[Table[{ColorBraids⟦j⟧, Specularity[White, 50], Sphere[Simplify[
         {a Cos[(r h + 2 π (j - 1)) / s], b/l Sin[(l (r h + 2 π (j - 1))) / s], sc * h} /. {h → 0}], 0.1]},
     {j, 1, s}], ImagePadding → {{Automatic, Automatic}, {None, None}},
    Method → {"ShrinkWrap" → True}];
   DD2 = Graphics3D[Table[{ColorBraids⟦j⟧, Specularity[White, 50], Sphere[Simplify[
         {a Cos[(r h + 2 π (j - 1)) / s], b/l Sin[(l (r h + 2 π (j - 1))) / s], sc * h} /. {h → 2 π}], 0.1]},
     {j, 1, s}], ImagePadding → {{Automatic, Automatic}, {None, None}},
```

```
    Method → {"ShrinkWrap" → True}];
  KK = Graphics3D[{EdgeForm[], Opacity[0.2], White,
     Cylinder[{{0, 0, -0.1}, {0, 0, 2 π + 0.1}}, 1.2]}, Lighting → "Neutral",
    Boxed → False, ImagePadding → {{Automatic, Automatic}, {None, None}},
    Method → {"ShrinkWrap" → True}];

  (*Top and bottom leminscates*)
  B = ParametricPlot3D[Simplify[{a Cos[ x ], b / l Sin[l x], 0}],
    {x, 0, 2 π}, PlotStyle → Directive[Darker[Gray, 0.8], Thickness[0.02]],
    PlotPoints → 100, ImagePadding → {{Automatic, Automatic}, {None, None}},
    Method → {"ShrinkWrap" → True}];
  C = ParametricPlot3D[Simplify[{a Cos[ x ], b / l Sin[l x], sc * 2 π}],
    {x, 0, 2 π}, PlotStyle → Directive[Darker[Gray, 0.8], Thickness[0.02]],
    PlotPoints → 100, ImagePadding → {{Automatic, Automatic}, {None, None}},
    Method → {"ShrinkWrap" → True}];
  Show[A, B, C, DD, DD2, KK, Braids, Boxed → False,
   BoxStyle → {Thick}, Axes → False, Lighting → "Neutral", PlotRange → All,
   ViewPoint → {1.5445548132437892`, -2.864728779434567`, 0.926109847245335`},
   ViewVertical → {0.3340710941969182`, -0.6239446032271305`, 0.7064627634389605`},
   ImagePadding → {{Automatic, Automatic}, {None, None}},
   Method → {"ShrinkWrap" → True}]];

(* Creation of the Framed Braid representation *)
FramedBraidRep2[s_, r_, l_, dk_, Scale_] :=
 Module[{A, B, Braids, C, DD, DD2, KK, a = 1, b = 1, sc = 1},
  Braids =
   Table[ParametricPlot3D[{a Cos[(r h + 2 π (j - 1))/s], b/l Sin[(l (r h + 2 π (j - 1)))/s], sc * h},
     {h, 0, 2 Pi}, PlotStyle → {Opacity[0.6], Lighter[ColorBraids[[j]], 0.1]},
     PlotPoints → 100, ImagePadding → {{Automatic, Automatic}, {None, None}},
     Method → {"ShrinkWrap" → True}] /. Line[pts_, rest___] :→
     {CapForm → None, Specularity[White, 50], Tube[pts, 0.08, rest]}, {j, 1, s}];
  A = Show[Table[VectorFraming[S[s, r, l, 1, 1][[jj]],
     dk[[jj]], Scale, ColorBraids[[jj]], {0, 2 Pi, 50}], {jj, 1, s}]];
  (*Final dots*)
  DD = Graphics3D[Table[{ColorBraids[[j]], Specularity[White, 50], Sphere[Simplify[
       {a Cos[(r h + 2 π (j - 1))/s], b/l Sin[(l (r h + 2 π (j - 1)))/s], sc * h} /. {h → 0}], 0.1]},
    {j, 1, s}], ImagePadding → {{Automatic, Automatic}, {None, None}},
   Method → {"ShrinkWrap" → True}];
  DD2 = Graphics3D[Table[{ColorBraids[[j]], Specularity[White, 50], Sphere[Simplify[
```

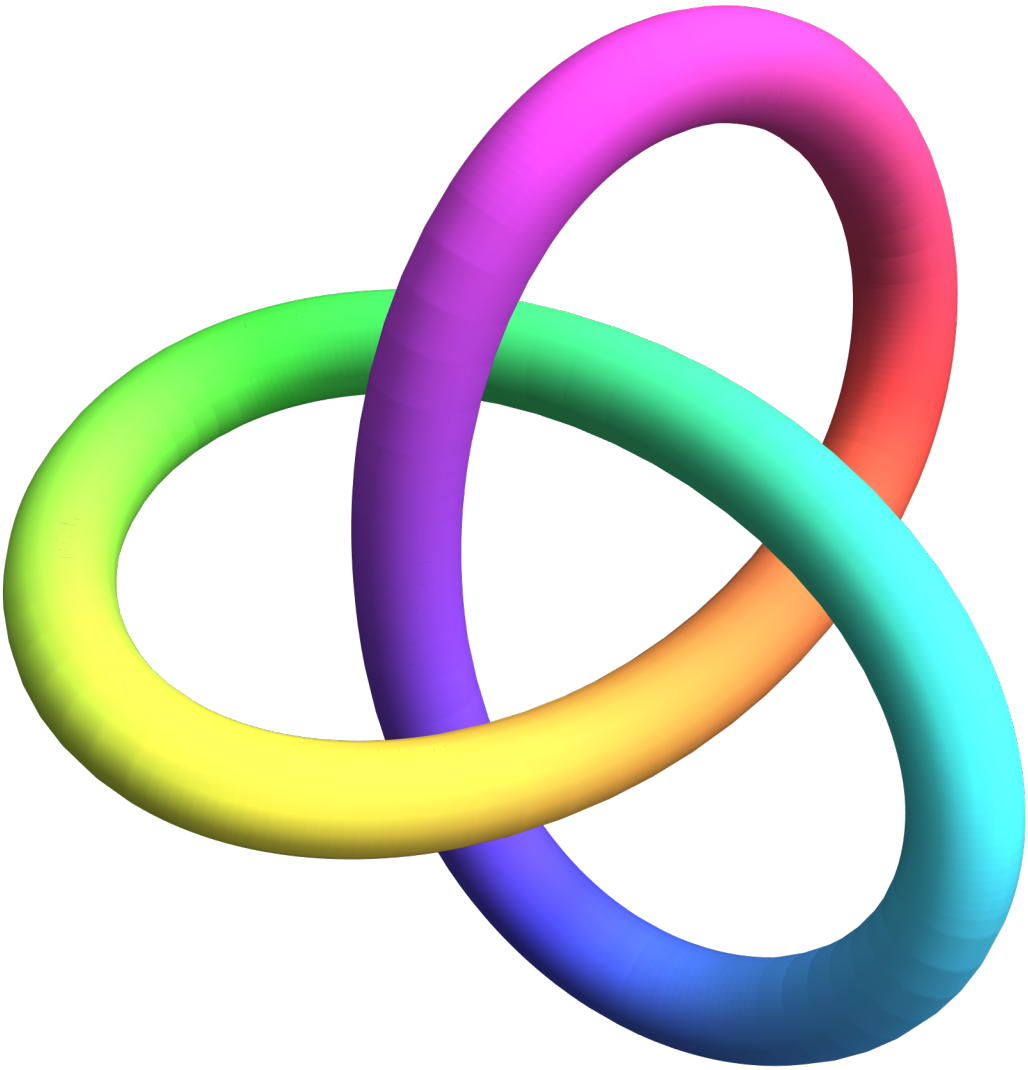
```
        {a Cos[(r h + 2 π (j - 1))/s], b/l Sin[(l (r h + 2 π (j - 1)))/s], sc * h} /. {h → 2 π}], 0.1]},
      {j, 1, s}], ImagePadding → {{Automatic, Automatic}, {None, None}},
     Method → {"ShrinkWrap" → True}];
   KK = Graphics3D[{EdgeForm[], Opacity[0.2], White,
      Cylinder[{{0, 0, -0.1}, {0, 0, 2 π + 0.1}}, 1.2]}, Lighting → "Neutral",
     Boxed → False, ImagePadding → {{Automatic, Automatic}, {None, None}},
     Method → {"ShrinkWrap" → True}];

   (*Top and bottom leminscates*)
   B = ParametricPlot3D[Simplify[{a Cos[ x ], b / l Sin[l x], 0}],
     {x, 0, 2 π}, PlotStyle → Directive[Darker[Gray, 0.8], Thickness[0.02]],
     PlotPoints → 100, ImagePadding → {{Automatic, Automatic}, {None, None}},
     Method → {"ShrinkWrap" → True}];
   C = ParametricPlot3D[Simplify[{a Cos[ x ], b / l Sin[l x], sc * 2 π}],
     {x, 0, 2 π}, PlotStyle → Directive[Darker[Gray, 0.8], Thickness[0.02]],
     PlotPoints → 100, ImagePadding → {{Automatic, Automatic}, {None, None}},
     Method → {"ShrinkWrap" → True}];
   Show[A, B, C, DD, DD2, KK, Braids, Boxed → False,
    BoxStyle → {Thick}, Axes → False, Lighting → "Neutral", PlotRange → All,
    ViewPoint → {1.5445548132437892`, -2.864728779434567`, 0.926109847245335`},
    ViewVertical → {0.3340710941969182`, -0.6239446032271305`, 0.7064627634389605`},
    ImagePadding → {{Automatic, Automatic}, {None, None}},
    Method → {"ShrinkWrap" → True}]];


AdvancedFraming[Curve_, Twists_, Scale_, Color_, Par_, sc_, X_] :=
 Module[{Dir, Basis, tangent, normal, binormal, Stuff},
  Basis = Last[FrenetSerretSystem[Curve, h]] // FullSimplify;
  {tangent, normal, binormal} = Map[Arrow[{Curve, Curve + #}] &, Basis];
  Stuff = Show[
    ParallelTable[Graphics3D[{Darker[Color, 0.2], Tube[{{X + Re[(Curve /. {h → sc * h}) +
              Scale (N[Basis[[2]]] Cos[Twists h / 2] + Sin[Twists h / 2] N[Basis[[3]]])],
          X + Re[(Curve /. {h → sc * h}) - Scale (N[Basis[[2]]] Cos[Twists h / 2] +
                Sin[Twists h / 2] N[Basis[[3]]])]}}, 0.03]}, Lighting → "Neutral"],
     {h, Par[[1]], Par[[2]], Abs[Par[[1]] - Par[[2]] / (Par[[3]] - 1)}], ViewPoint →
     Front, Boxed → False]]


FramingZsc[Curve_, Twists_, Scale_, Color_, Par_, Zscale_] :=
  Module[{Dir, Basis, tangent, normal, binormal, Stuff},
   Basis = Last[FrenetSerretSystem[Curve, h]] // FullSimplify;
   {tangent, normal, binormal} = Map[Arrow[{Curve, Curve + #}] &, Basis];
   Stuff = Show[ParallelTable[
      Graphics3D[{Darker[Color, 0.2], Tube[{{Re[Curve + Zscale {0, 0, Curve[[3]]} +
```

```
            Scale (N[Basis[[2]]] Cos[Twists h / 2] + Sin[Twists h / 2] N[Basis[[3]]]) ],
          Re[Curve + Zscale {0, 0, Curve[[3]]} - Scale (N[Basis[[2]]] Cos[Twists h / 2] +
               Sin[Twists h / 2] N[Basis[[3]]])]}}, 0.03]}, Lighting → "Neutral"],
    {h, Par[[1]], Par[[2]], Abs[Par[[1]] - Par[[2]]] / (Par[[3]] - 1)}], ViewPoint →
    Front, Boxed → False]];
```

Framed Links

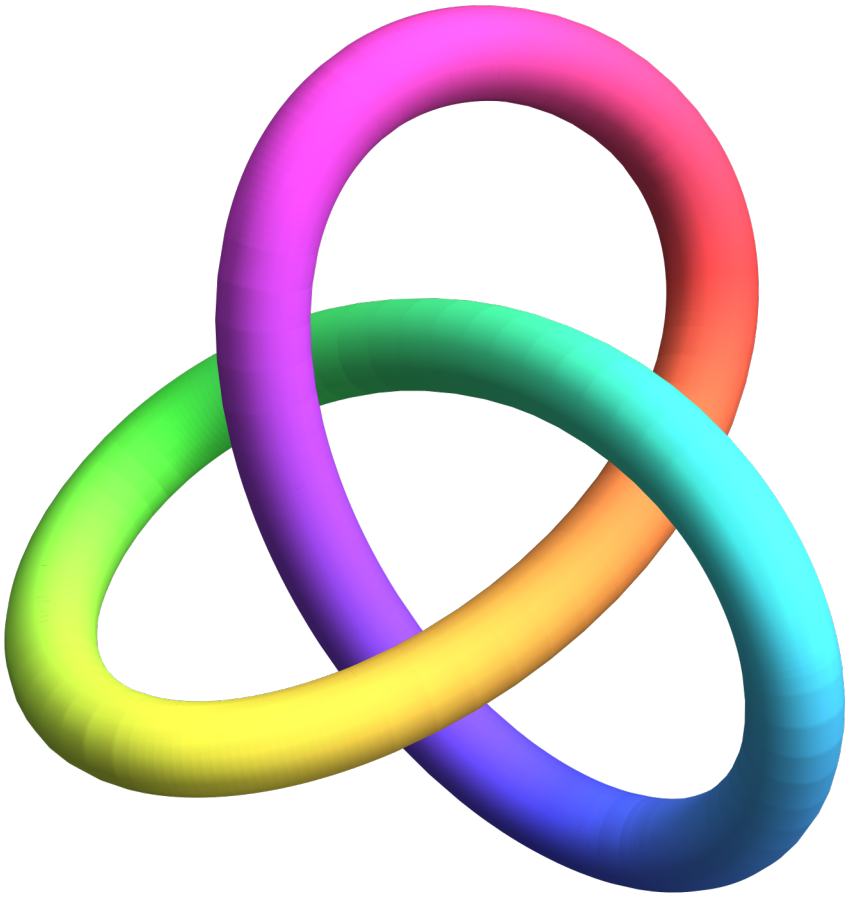
In[•]:=
```
PolRec3D[G_] :=
 Show[{ParametricPlot3D[Chop[Re[G]], {t, Pi / 2, 2 Pi + Pi / 2}, ColorFunction → Function[
       {x, y, z, t}, Lighter[Hue[t], 0.3]], ColorFunctionScaling → True, PlotRange → All,
     ImageSize → Large, Boxed → False, Axes → False, Method → {"TubePoints" → 20},
     PlotPoints → 120, MaxRecursion → 2, ViewPoint → {Pi, Pi / 2, 2}, AxesLabel → Automatic] /.
    Line[pts_, rest___] :→ Tube[pts, 0.08, rest]}];
Show[{PolRec3D[0.25 {Sin[3 t], 0, 0} + 0.25 {0, -2 Cos[2 t], 2 Sin[2 t]} +
    0.25 {0, Cos[t], Sin[t]}]}, ViewPoint → Right]
```

Out[•]=



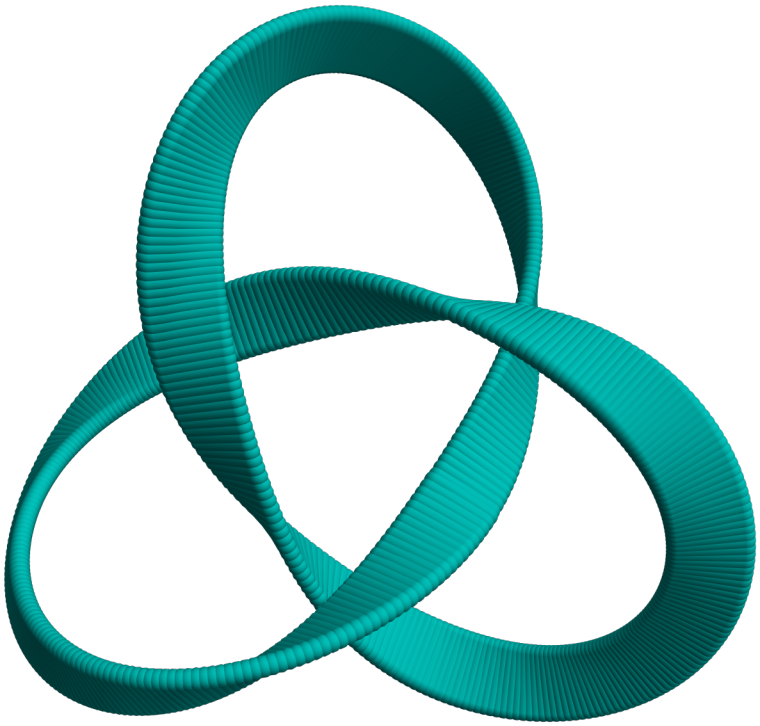
```
In[•]:= ParametricPlot3D[Trefoil[t], {t, Pi / 2, 2 Pi + Pi / 2},
  ColorFunction → Function[{x, y, z, t}, Lighter[Hue[t], 0.3]], ColorFunctionScaling → True,
  PlotRange → All, ImageSize → Large, Boxed → False, Axes → False,
  Method → {"TubePoints" → 20}, PlotPoints → 120, MaxRecursion → 2, ViewPoint → {Pi, Pi / 2, 2},
  AxesLabel → Automatic] /. Line[pts_, rest___] :→ Tube[pts, 0.08, rest]
```

Out[•]=



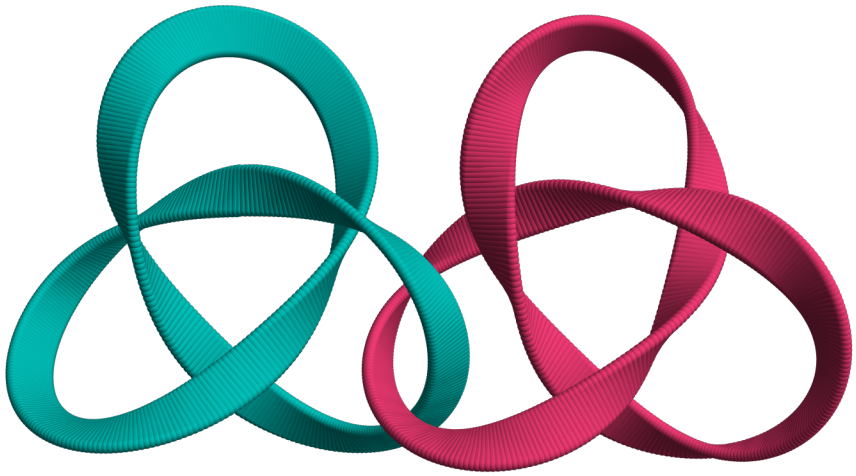
In[•]:= **KnotA**

Out[•]=



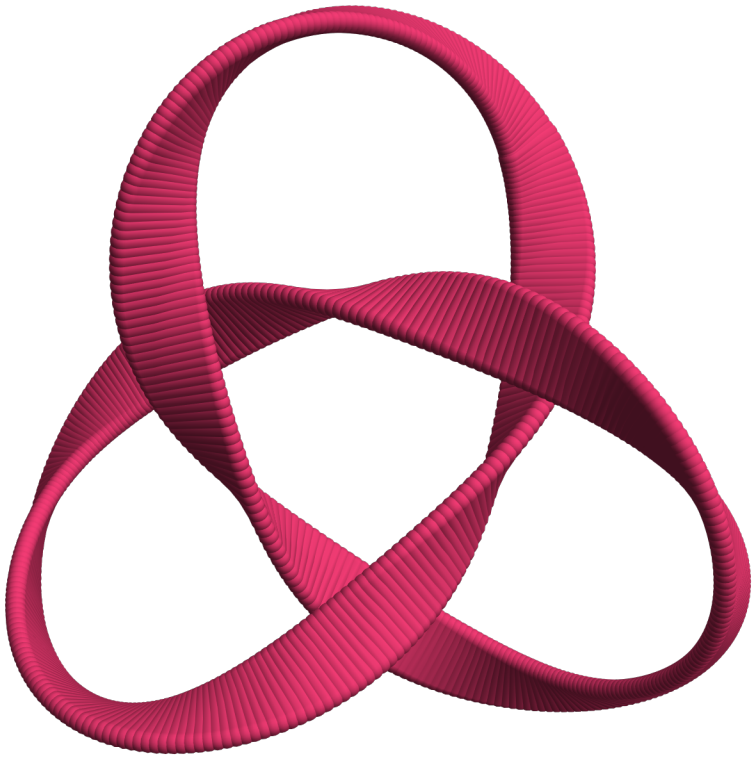
```
KnotA = AdvancedFraming[Trefoil[h, 0.4], 4,
   0.1, ColorBraids[1], {0.001 Pi, 2 π 0.999, 500}, 1, {0, 0, 0}];

Show[KnotA, KnotB]
```

Out[•]=



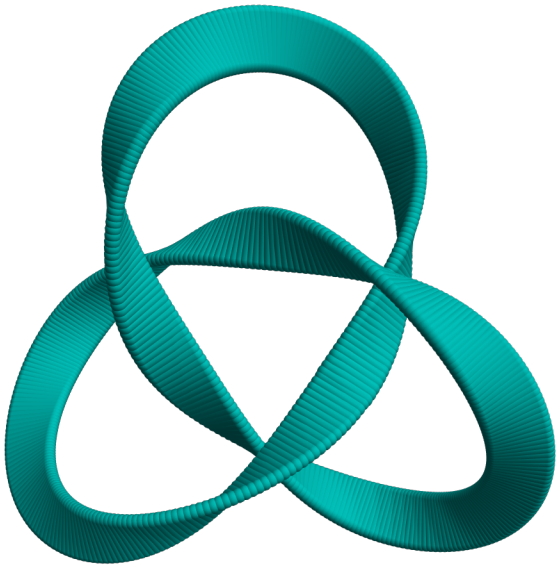
In[•]:= **KnotB**

Out[•]=



In[•]:= **KnotA**

Out[•]=



```
Framing[Curve_, Twists_, Scale_, Color_, Par_, sc_] :=
 Module[{Dir, Basis, tangent, normal, binormal, Stuff},
  Basis = Last[FrenetSerretSystem[Curve, h]] // FullSimplify;
  {tangent, normal, binormal} = Map[Arrow[{Curve, Curve + #}] &, Basis];
  Stuff =
   Show[ParallelTable[Graphics3D[{Darker[Color, 0.2], Tube[{{Re[(Curve /. {h → sc * h}) +
              Scale (N[Basis[[2]]] Cos[Twists h / 2] + Sin[Twists h / 2] N[Basis[[3]]])],
           Re[(Curve /. {h → sc * h}) - Scale (N[Basis[[2]]] Cos[Twists h / 2] +
               Sin[Twists h / 2] N[Basis[[3]]])]}}, 0.03]}, Lighting → "Neutral"],
     {h, Par[[1]], Par[[2]], Abs[Par[[1]] - Par[[2]]] / (Par[[3]] - 1)}], ViewPoint →
     Front, Boxed → False]]
```

First Protocol implementation

In[•]:= **listProduct[x_List] := Times @@ x**

Alice

Input parameters

```
In[•]:= dk = {3, 2, 1}; (* Number of twist per strand*)
        np = {2, 3, 5}; (* Primer number assignation*)
        α = 2;          (* Parameter selection *)
        M = Total[dk];
```

Computing the parameters that Alice will send to Bob

```
In[•]:= AliceMessage =
          {Total[dk], α, N[listProduct[Table[np[[ii]]^(α^(dk[[ii]]-M)), {ii, 1, Dimensions[dk][[1]]}]]]}
Out[•]=
        {6, 2, 1.22826}
```

Bob

```
In[•]:= (* Calculus of the invariant*)
        NN = AliceMessage[[3]]^(AliceMessage[[2]]^AliceMessage[[1]])
        (* Prime Factorization*)
        PF = FactorInteger[IntegerPart[NN]]
Out[•]=
        518400.
Out[•]=
        {{2, 8}, {3, 4}, {5, 2}}

In[•]:=
```

Challenge-response protocol

1.- Alice's knot generation

```
In[•]:= dk = {3, 1}; (* Number of twist per strand*)
        np = {2, 3}; (* Primer number assignation*)
        α = 2;        (* Parameter selection *)
        M = Total[dk];

        (*Computation of the message for bob*)
        AliceMessage =
         {Total[dk], α, N[listProduct[Table[np[[ii]]^(α^(dk[[ii]]-M)), {ii, 1, Dimensions[dk][[1]]}]]]}
Out[•]=
        {4, 2, 1.62239}
```

2.- Bob calculations

```
In[•]:= (* Calculus of the invariant*)
        NN = AliceMessage[[3]]^(AliceMessage[[2]]^AliceMessage[[1]])

        (* Prime Factorization*)
        PF = FactorInteger[IntegerPart[NN]]

        (*From here, we need α to find d_k*)
        db = Table[1 / Log[PF[[ii, 2]], AliceMessage[[2]]], {ii, 1, Dimensions[PF][[1]]}]
Out[•]=
        2304.
Out[•]=
        {{2, 8}, {3, 2}}
Out[•]=
        {3, 1}
```

3.- Bob's knot generation

```
In[•]:= dkb = {4, 2}; (* Number of twist per strand (????)*)
        npb = {5, 7}; (* Primer number assignation*)
        M2 = Total[dkb];
```

3.1 Computation of β'

```
In[•]:= βp = N[listProduct[Table[npb[[ii]]^(α^(dkb[[ii]]-M2)), {ii, 1, Dimensions[dkb][[1]]}]]];

        BobKnot = {M2, AliceMessage[[2]], βp}
Out[•]=
        {6, 2, 1.68873}

In[•]:= 1.6887335383685038`^2^6
Out[•]=
        3.66364×10^14

In[•]:= (* Calculus of the invariant*)
        NNB = N[DecimalForm[BobKnot[[3]]^(BobKnot[[2]]^BobKnot[[1]])]]
Out[•]//DecimalForm=
        366363525390626.

In[•]:= EulerPhi[1550625]
Out[•]=
        826000

Out[•]=
        826000
```

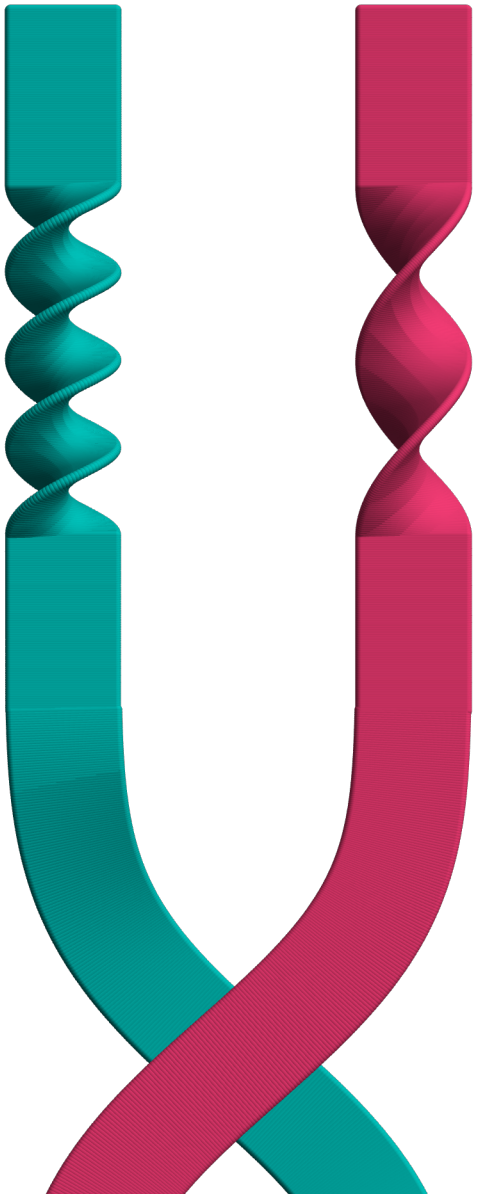
3.2 Computation of β"

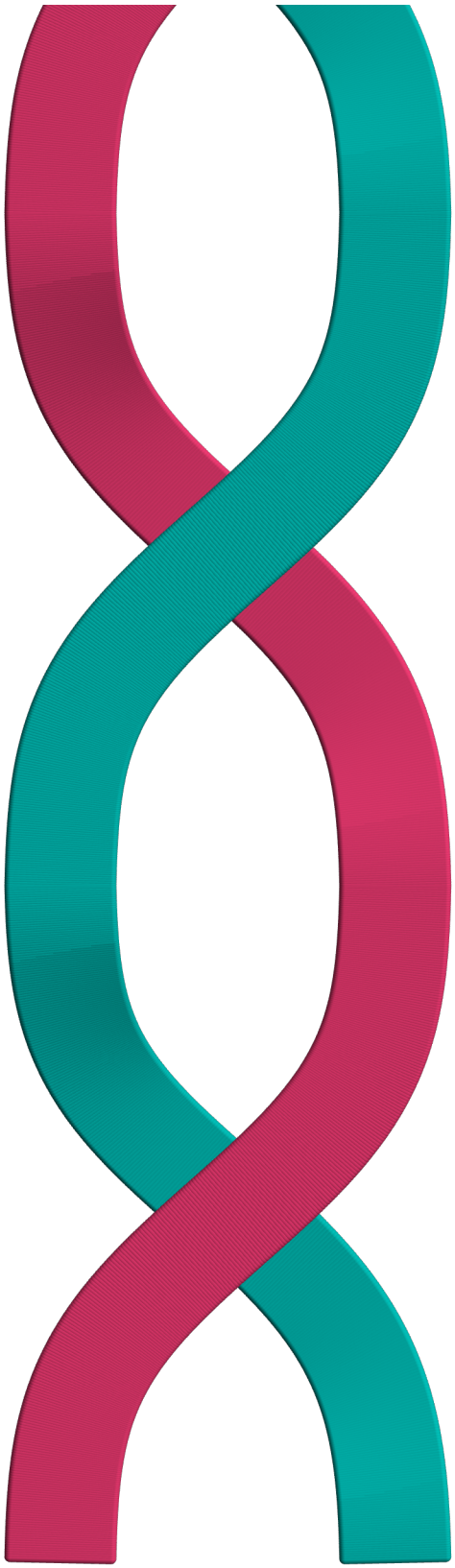
```
In[•]:= dkAB = Join[dk, dkb];
        npkAB = Join[np, npb];
        MAB = Total[dkAB];
        βAB = N[listProduct[Table[npkAB[[ii]]^(α^(dkAB[[ii]]-MAB)), {ii, 1, Dimensions[dkAB][[1]]}]]]
Out[•]=
        1.01146

In[•]:= (* Calculus of the invariant*)
        NNAB = 1.011464918860659`^2^10
Out[•]=
        117394.

In[•]:= NNAB * NN
Out[•]=
        2.70475×10^8
```

asdsad

In[ ]:=
```
Strand1 = Show[Twist[4, {-3, 0, -2}, Pantone2459, 0.3],
    Crossing[{-2, 0, -4}, "Right", "Under", Pantone2459, 0.3],
    Crossing[{-2, 0, -8}, "Left", "Over", Pantone2459, 0.3],
    Crossing[{-2, 0, -12}, "Right", "Under", Pantone2459, 0.3]];
Strand2 = Show[Twist[2, {-1, 0, -2}, Pantone199, 0.3],
    Crossing[{-2, 0, -4}, "Left", "Over", Pantone199, 0.3],
    Crossing[{-2, 0, -8}, "Right", "Under", Pantone199, 0.3],
    Crossing[{-2, 0, -12}, "Left", "Over", Pantone199, 0.3]];
Show[Strand1, Strand2]
```

Out[ ]=

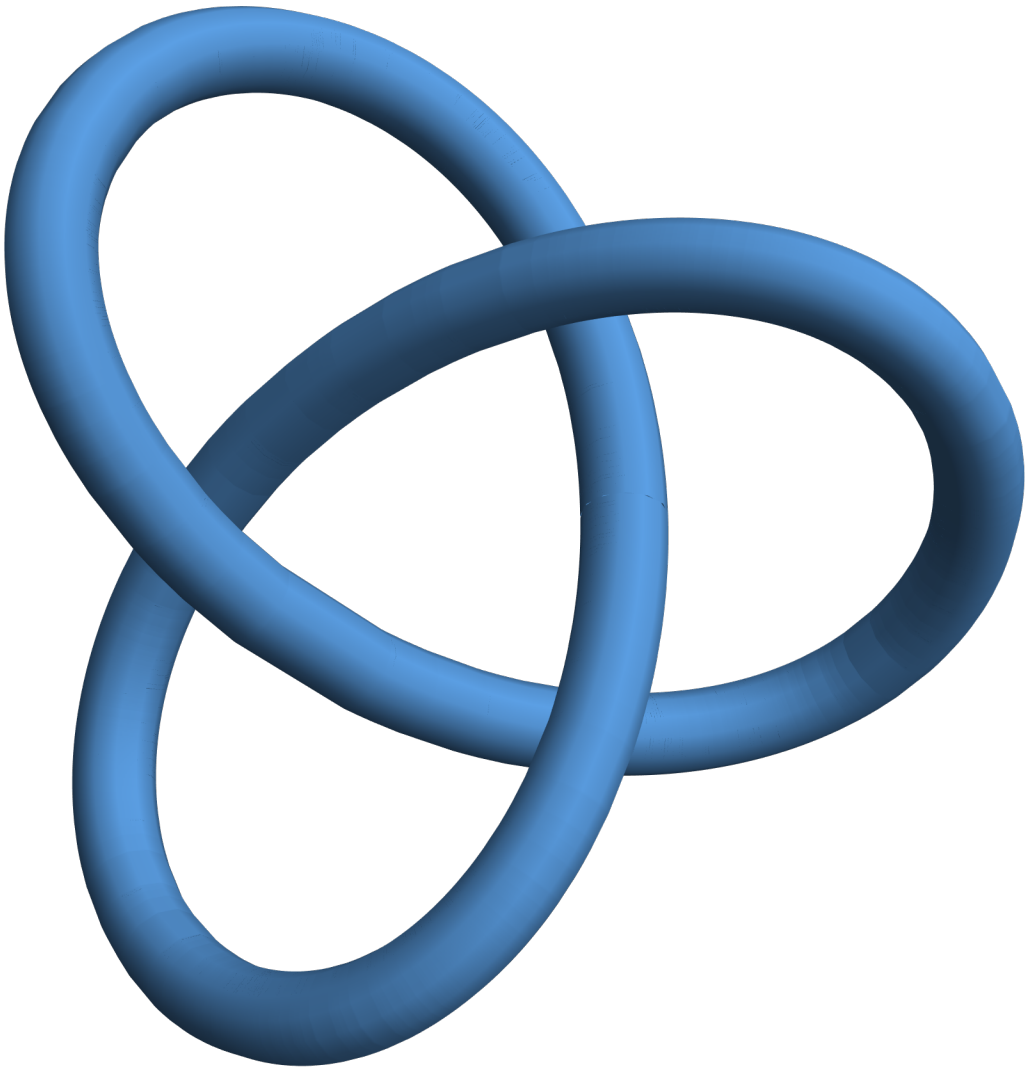




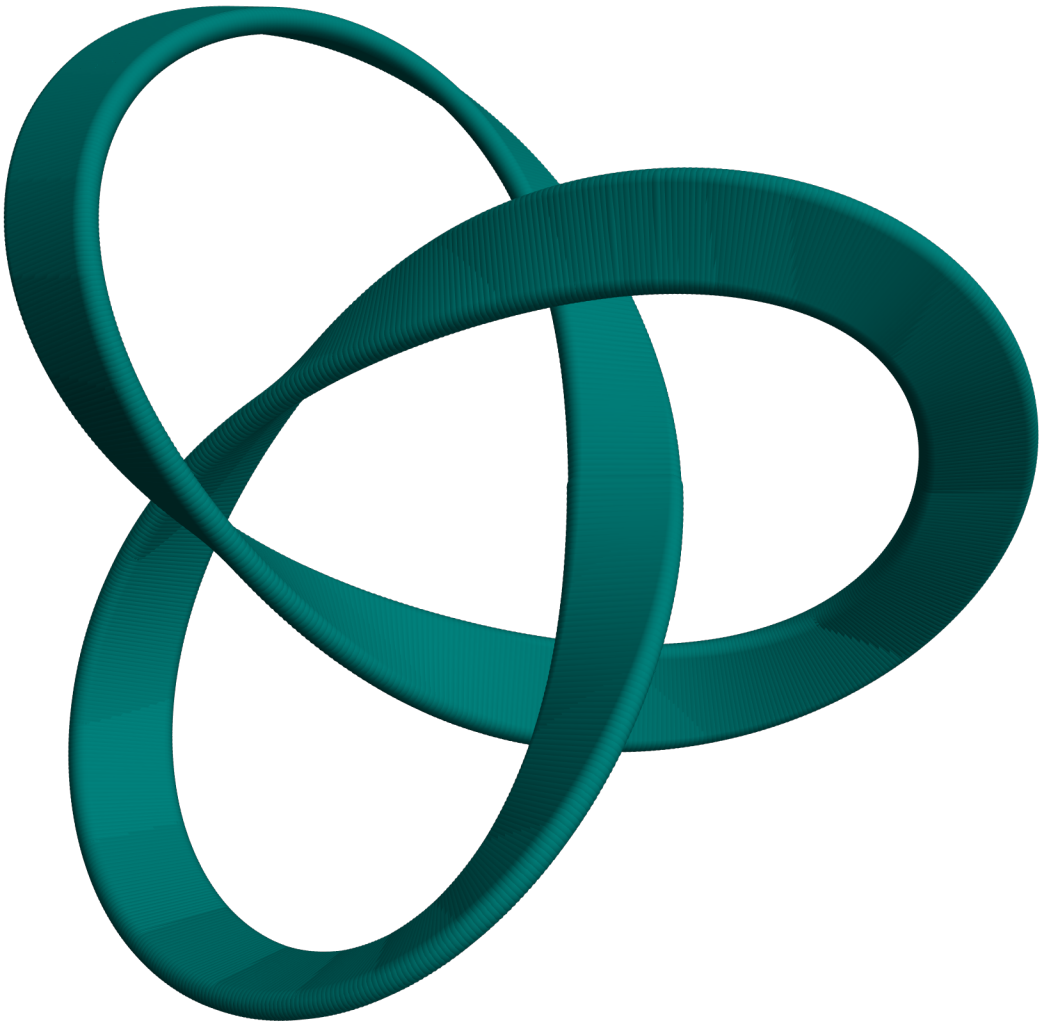
In[•]:=

```
Trefoil[t_, sc_] := sc ({Sin[3 t], 0, 0} + {0, -2 Cos[2 t], 2 Sin[2 t]} + {0, Cos[t], Sin[t]});
ParametricPlot3D[Trefoil[h, 0.4], {h, 0, 2 π},
  PlotStyle → Darker@NiceBlue, PlotPoints → 25, PlotRange → All, Lighting → "Neutral",
  ViewPoint → Left, Boxed → False, Axes → False, ImageSize → Large] /.
 Line[pts_, rest___] :→ {CapForm → None, Tube[pts, 0.1, rest]}

KnotA1 = AdvancedFraming[Trefoil[h, 0.4], 2,
   0.1, Darker@Pantone2459, {0.001 Pi, π 0.999, 500}, 1, {0, 0, 0}];
KnotA2 = AdvancedFraming[Trefoil[h, 0.4], 2,
   0.1, Darker@Pantone2459, {π 0.999, 2 π 0.999, 500}, 1, {0, 0, 0}];
Show[KnotA1, KnotA2, ViewPoint → Left]
```

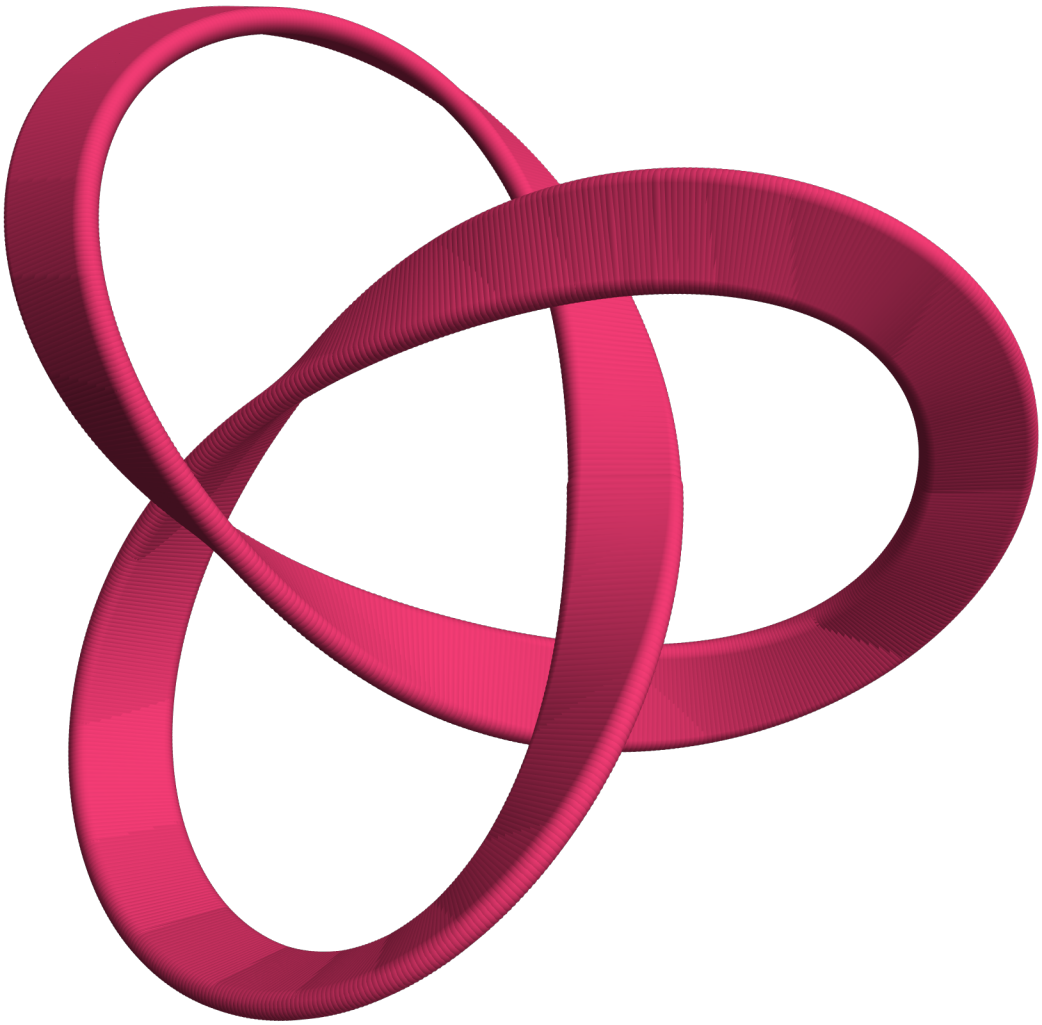
Out[•]=



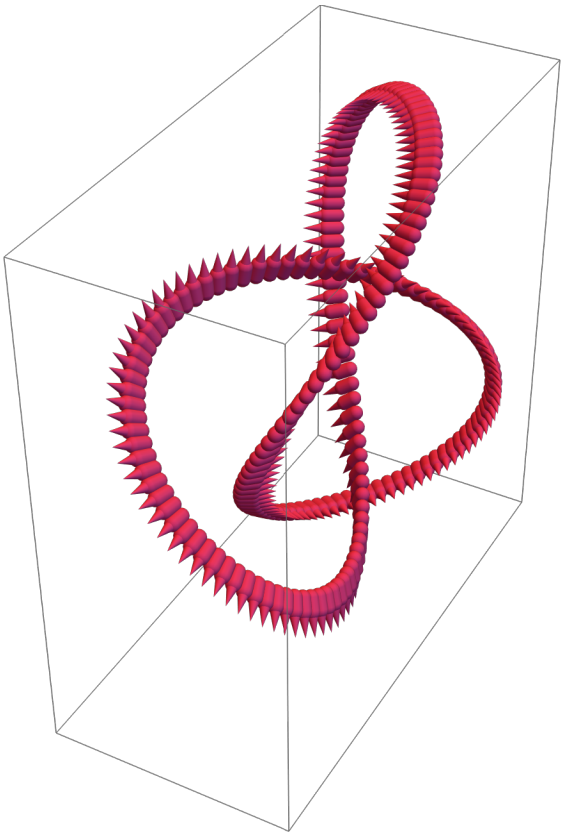
Out[ ]=



Out[•]=

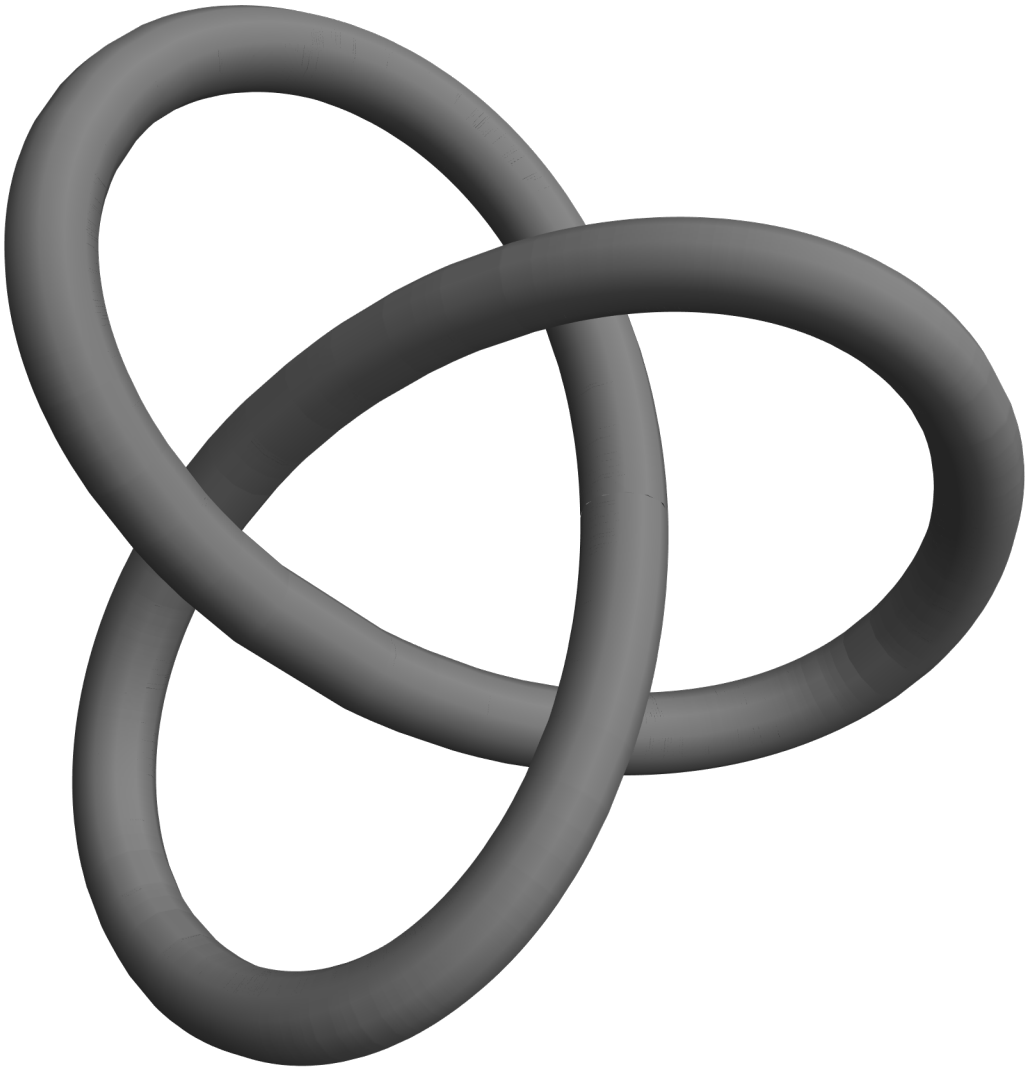


Out[ ]=



In[•]:=
```
Trefoil[t_, sc_] := sc ({Sin[3 t], 0, 0} + {0, -2 Cos[2 t], 2 Sin[2 t]} + {0, Cos[t], Sin[t]});
ParametricPlot3D[Trefoil[h, 0.4], {h, 0, 2 π},
  PlotStyle → Darker[Gray, 0.2], PlotPoints → 25, PlotRange → All, Lighting → "Neutral",
  ViewPoint → Left, Boxed → False, Axes → False, ImageSize → Large] /.
 Line[pts_, rest___] :↦ {CapForm → None, Tube[pts, 0.1, rest]}
```

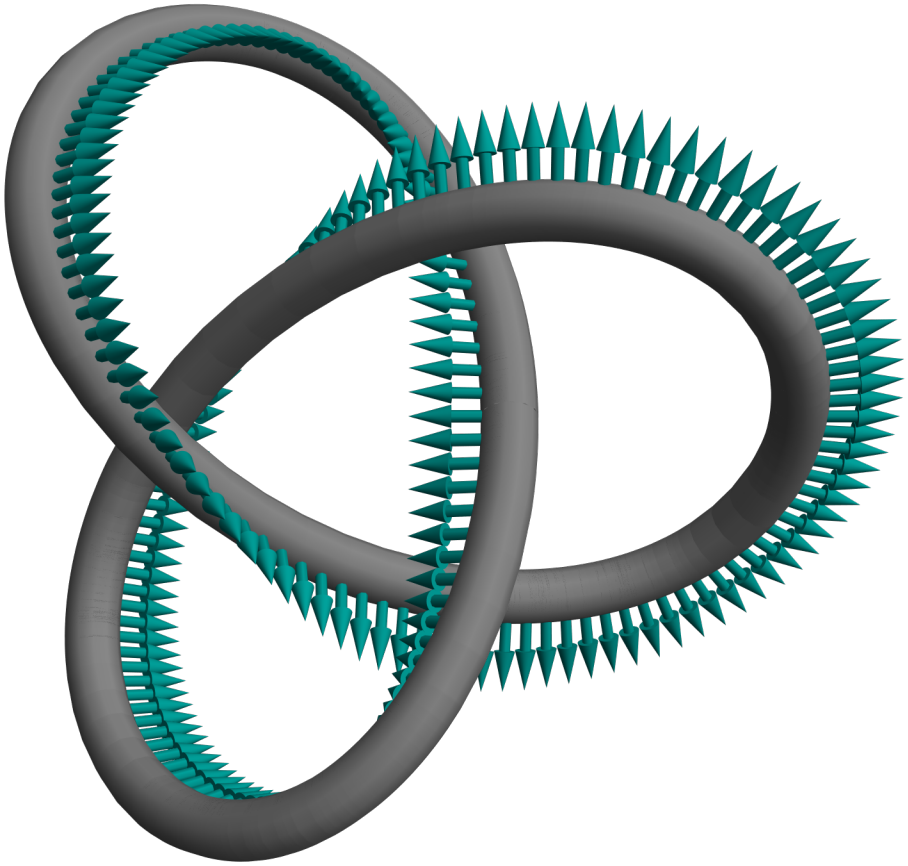
Out[•]=



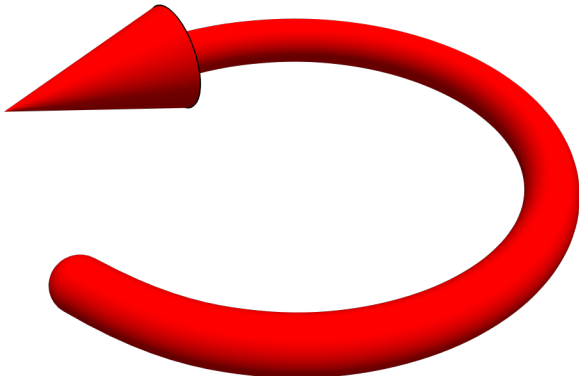
In[•]:=

```
Knot[t_] := {Sin[3 t], 0, 0} + {0, -2 Cos[2 t], 2 Sin[2 t]} + {0, Cos[t], Sin[t]};
basis = Last[FrenetSerretSystem[Knot[t], t]] // FullSimplify;
fram = Graphics3D[
   Table[{Arrowheads[0.03], Darker@Pantone2459, Arrow[Tube[{0.5 Knot[t], 0.5 Knot[t] +
         0.35 (Cos[t] basis[[2]] + Sin[t] basis[[3]])}, 0.02]]}, {t, 0, 2 π, 0.01 π}]];
framed = Show[ParametricPlot3D[Trefoil[h, 0.5], {h, 0, 2 π},
    PlotStyle → Darker[Gray, 0.2], PlotPoints → 100, PlotRange → All, Lighting → "Neutral",
    ViewPoint → Left, Boxed → False, Axes → False, ImageSize → Large] /.
   Line[pts_, rest___] :→ {CapForm → None, Tube[pts, 0.1, rest]},
  fram, Lighting → "Neutral", ViewPoint → Left]
```

Out[•]=

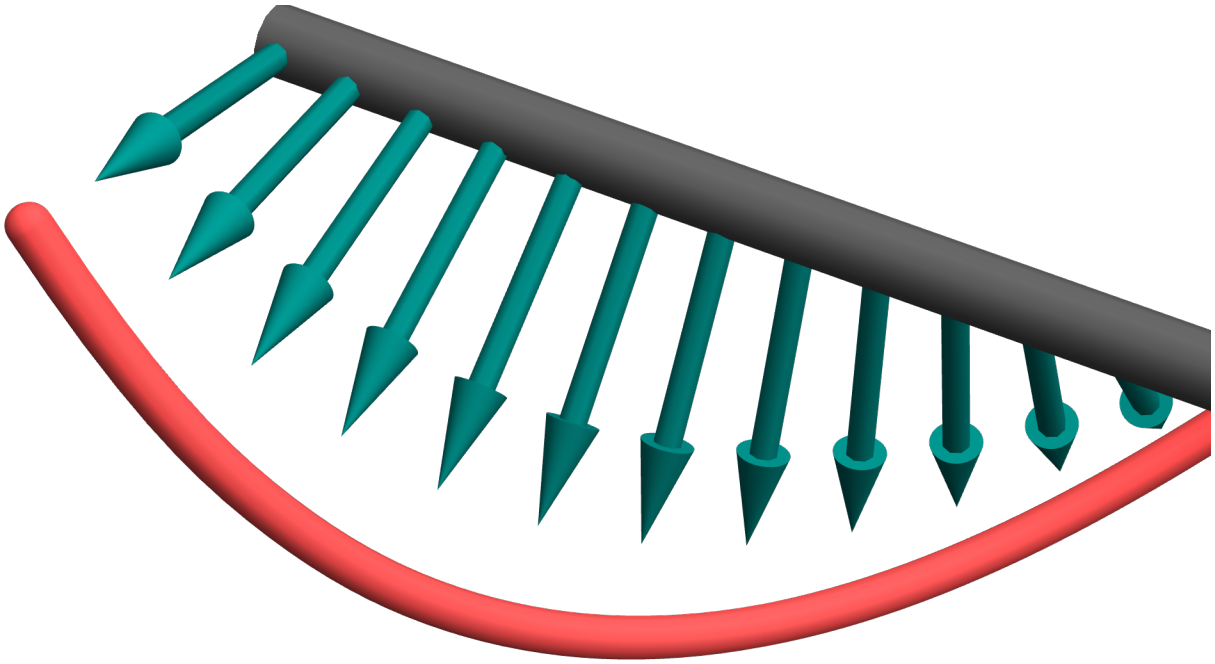


Out[•]=



In[•]:=
```
arrowtube = With[{ra = .06, arhd = .25,
     arrowtip = Graphics3D[{EdgeForm[], Cone[{{0, 0, 0}, {0.35, 0, 0}}, 0.08]}]},
    Graphics3D[{Lighter[Red, 0.3], Arrowheads[{{arhd, 1, arrowtip}}], Arrow[
       Tube[BSplineCurve[Table[{t, 1.3 Cos[t], 1.3 Sin[t]}, {t, 0, 1.75 π, Pi / 10}]], ra]]},
     Boxed → False, ViewPoint → {1, 1, 1}, ImageSize → {300, Automatic},
     Method → {"ShrinkWrap" → True}]];
Show[
 Graphics3D[{CapForm["Square"], Darker@Gray, Tube[{{0, 0, 0}, {3 π, 0, 0}}, 0.1]},
  ViewPoint → Above, Boxed → False],
 Graphics3D[Table[{CapForm["Square"],
    Darker@Pantone2459, , Arrow[Tube[{{t, 0, 0}, {t, Cos[t], Sin[t]}}, 0.04]]},
   {t, 0, 2 π, 0.07 π}], ViewPoint → Above, Boxed → False],
 arrowtube,
 PlotRange → {{0, 2 π}, {-1.4, 1.4}, {-1.4, 1.4}}, Lighting → "Neutral"
]
```

Out[•]=



In[•]:=
```
Knot[t_] := {Sin[3 t], 0, 0} + {0, -2 Cos[2 t], 2 Sin[2 t]} + {0, Cos[t], Sin[t]};
basis = Last[FrenetSerretSystem[Knot[t], t]] // FullSimplify;
fram = Graphics3D[Table[{Arrowheads[0.035], Pantone7664, Arrow[
       Tube[{0.5 Knot[t], 0.5 Knot[t] + 0.3 basis[[3]]}, 0.015]]}, {t, 0.6 π, 0.9 π, 0.02 π}]];
fram2 = Graphics3D[Table[{Arrowheads[0.035], Darker@Pantone199, Arrow[
       Tube[{0.5 Knot[t], 0.5 Knot[t] + 0.3 basis[[2]]}, 0.015]]}, {t, 0.6 π, 0.9 π, 0.02 π}]];
framed =
 Show[ParametricPlot3D[Trefoil[h, 0.5], {h, 0.6 π, 0.9 π}, PlotStyle → Darker[Gray, 0.2],
    PlotPoints → 100, PlotRange → All, Lighting → "Neutral", ViewPoint → Left,
    Boxed → False, Axes → False, ImageSize → Large] /.
   Line[pts_, rest___] :→ {CapForm["Square"], Tube[pts, 0.07, rest]},
  fram, fram2, Lighting → "Neutral", ViewPoint → Left]
```

Out[•]=