Introduction to Optical knots

Basic definitions

Knot - It is a closed, non-self-intersecting curve embedded in three dimensions.

Braid - A braid in n strands is defined as a set of n non-intersecting smooth lines joining two parallel planes.

Alexander’s Theorem: *“Every oriented knot or link can be represented by a closed braid “*

Lemniscate Knots

This notebook stands as an implementation of the work by Bode et al. [1] on the properties of the lemniscate knots.

In the transverse plane to the braids height, the s strands trace a lemniscate curve - a $(1, l)$ Lissajous Curve parametrized by $h \in [0, 2\pi]$. Therefore, we can write the parametric curve for the j-th strand as

$$S_j^{s,r,l}(h; a, b) = \left(a\,\mathrm{Cos}\left[\tfrac{1}{s}\{\mathrm{rh} + 2\pi(j-1)\}\right],\ \tfrac{b}{l}\,\mathrm{Sin}\left[\tfrac{l}{s}\{\mathrm{rh} + 2\pi(j-1)\}\right],\ h\right)$$

where $a, b \in \mathbb{R}$ are stretching factors. The closure of the braid is parametrized as

$$\left(\mathrm{Cos}[h]\left\{R + a\,\mathrm{Cos}\left[\tfrac{1}{s}\{\mathrm{rh} + 2\pi(j-1)\}\right]\right\},\ \mathrm{Sin}[h]\left\{R + a\,\mathrm{Cos}\left[\tfrac{1}{s}\{\mathrm{rh} + 2\pi(j-1)\}\right]\right\},\ \left\{\tfrac{b}{l}\,\mathrm{Sin}\left[\tfrac{l}{s}\{\mathrm{rh} + 2\pi(j-1)\}\right]\right\}\right)$$

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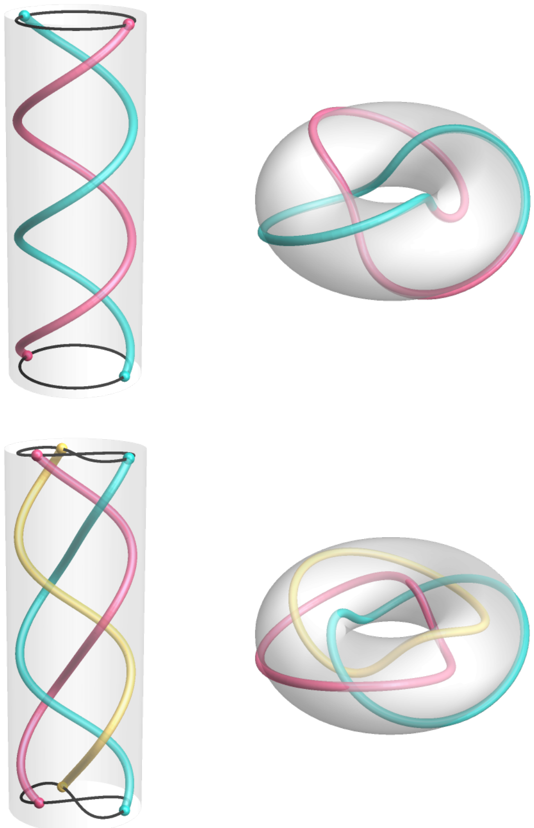
In[310]:=
```
GraphicsGrid[{{SDia[2, 3, 1, 1, 1], BraidOnTorus2[S[2, 3, 1, 0.5, 0.5]]},
  {SDia[3, 3, 2, 1, 1], BraidOnTorus2[S[3, 3, 2, 0.5, 0.5]]}}, Spacings → 50]
```

Out[310]=



The braid can represented by the family of complex polynomials $p_h^{s,r,l}(u)$ with $u \in \mathbb{C}$, that have roots $Z_j^{s,r,l}(h)$ given in the intersection of the parametrized braid with the horizontal plan so

$$p_h^{s,r,l}(u) = \prod_{j=1}^{s} \left[u - Z_j^{s,r,l}(h) \right]$$

where we had parametrized $Z_j^{s,r,l}(h) = a \operatorname{Cos}\left[\frac{1}{s}\{\mathrm{rh} + 2\pi(j-1)\}\right] + i\,\frac{b}{l} \operatorname{Sin}\left[\frac{l}{s}\{\mathrm{rh} + 2\pi(j-1)\}\right]$. The semiholomorphic map $f(u, v, \overline{v})$ with knotted zero line is fond by the replacements, in $p_h^{s,r,l}(u)$, $\exp(i\,h) \to v$ and $\exp(-i\,h) \to \overline{v}$. The mapping to the complex plane can be obtained by considering the substitutions

$$u \to \frac{\rho^2-1}{\rho^2+1}, \qquad v \to \frac{2\,\rho\,\exp[i\phi]}{\rho^2+1}, \qquad \overline{v} \to \frac{2\,\rho\,\exp[-i\phi]}{\rho^2+1}$$

From here, we can obtained the complex polynomial in the complex plane from the numerator of $p_h^{s,r,l}(u, v, \overline{v}) \to p_h^{s,r,l}(\rho, \phi)$.

In[315]:=
```
Expand[Numerator[Pol[2, 3, 1, 1, 1]]] (*Trefoil*)
```

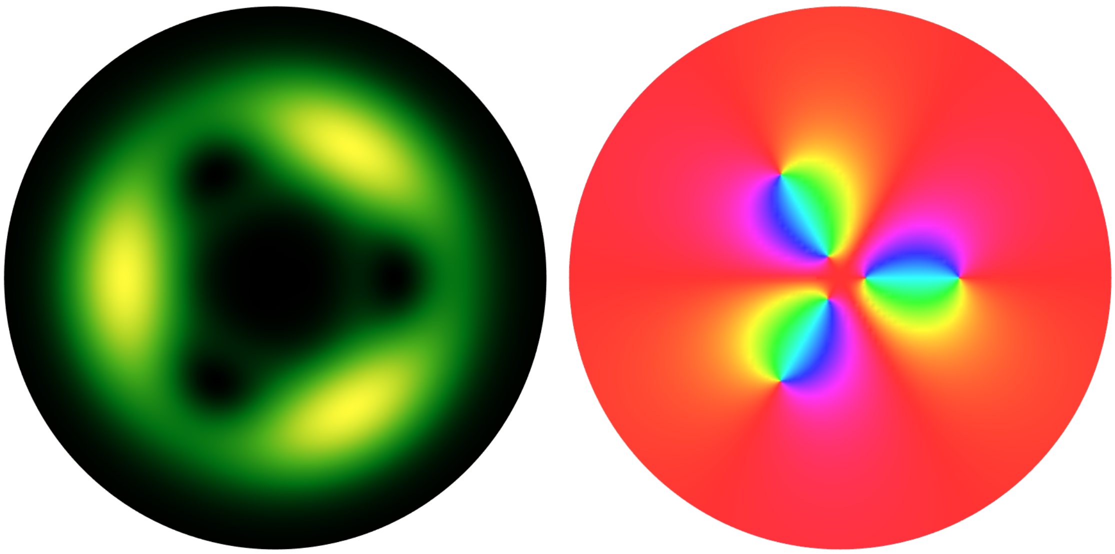
Out[315]= $1 - \rho^2 - 8\, e^{3\,i\,\phi}\, \rho^3 - \rho^4 + \rho^6$

The knotted field can be constructed as the product of the later polynomial and a Gaussian envelope

$$\boldsymbol{E}\,(\boldsymbol{r}) = \mathrm{Exp}\left[-\frac{r^2}{w_0{}^2}\right]\mathrm{Numerator}\left[p_h{}^{s,r,l}(r,\,\phi)\right]$$

In[325]:=
```
IntPhase[Expand[Numerator[Pol[2, 3, 1, 1, 1]]]]
```

Out[325]=



The evolution of the electric field as it propagates can be obtained by means of the Fresnel Propagator [2, 3]. (Still has to be implemented properly)

The singular points of the field at any plane z can be obtained by calculating the intersections of the curves when Re[$\boldsymbol{E}(\boldsymbol{r})$] = Im[$\boldsymbol{E}(\boldsymbol{r})$] = 0

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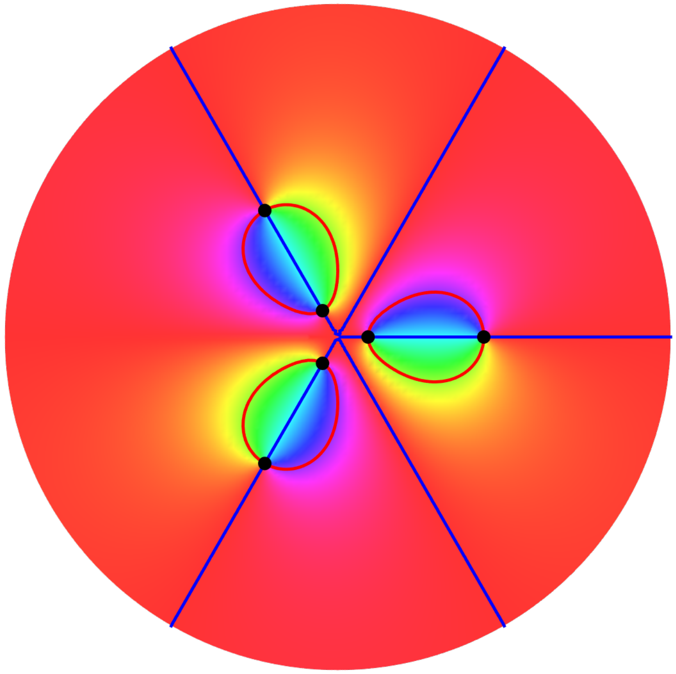
In[373]:= `Intersections[Expand[Numerator[Pol[2, 3, 1, 1, 1]]]]`

Out[373]=



Out[337]= `$Aborted`

Examples

Trefoil knot

Cinquefoil

Borromean rings

Hopf Link

References

1. Bode, Benjamin, Mark R. Dennis, David Foster, and Robert P. King. *“Knotted fields and explicit fibrations for lemniscate knots.”* Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 473, no. 2202 (2017): 20160829.

2. Goodman, Joseph W.. *Introduction to Fourier optics.* United States: W. H. Freeman, 2005.

3. Breckinridge, J., D. Voelz, and J. B. Breckinridge. *"Computational Fourier Optics: A MATLAB Tutorial."* (2011).

Functions

Preamble

In[295]:=

```
SetDirectory@NotebookDirectory[];
(*Creates a list of n colors*)
Colors[n_] := With[{partL = Ceiling[Sqrt[n]]},
  DeleteCases[Flatten[Transpose[Partition[Table[Lighter[Darker[Hue[c], .1], .25],
       {c, 0, 1 - 1 / n, 1 / n}], partL, partL, 1, 0]]], 0]]

(*Extra nice Custom colors*)
FunkyT = RGBColor[188 / 255, 224 / 255, 225 / 255];
Pantone2459 = RGBColor[1 / 255, 181 / 255, 174 / 255];
Pantone218 = RGBColor[206 / 255, 102 / 255, 161 / 255];
Pantone199 = RGBColor[227 / 255, 56 / 255, 109 / 255];
Pantone149 = RGBColor[243 / 255, 194 / 255, 66 / 255];
PantoneProceBlue = RGBColor[63 / 255, 143 / 255, 205 / 255];
Pantone7664 = RGBColor[104 / 255, 48 / 255, 120 / 255];

(*Colouring the braids*)
ColorBraids = {Pantone2459, Pantone199, Pantone149, PantoneProceBlue, Pantone7664};
```

Lemniscate Knots calculations

In[386]:=

```
(* Creation of the Braid representation *)
SDia[s_, r_, l_, a_, b_] := Module[{A, B, C, DD, DD2, KK},
   (*Braids*)
   A = Table[ParametricPlot3D[{a Cos[(r h + 2 π (j - 1))/s], b/l Sin[(l (r h + 2 π (j - 1)))/s], h},
        {h, 0, 2 Pi}, PlotStyle → {Opacity[0.6], Lighter[ColorBraids[[j]], 0.1]},
        PlotPoints → 100, ImagePadding → {{Automatic, Automatic}, {None, None}},
        Method → {"ShrinkWrap" → True}] /. Line[pts_, rest___] ⧴
        {CapForm → None, Specularity[White, 50], Tube[pts, 0.08, rest]}, {j, 1, s}];
   (*Final dots*)
   DD = Graphics3D[Table[{ColorBraids[[j]], Specularity[White, 50], Sphere[Simplify[
```

```
        {a Cos[(r h + 2 π (j - 1))/s], b/l Sin[(l (r h + 2 π (j - 1)))/s], h} /. {h → 0}], 0.1]},
     {j, 1, s}], ImagePadding → {{Automatic, Automatic}, {None, None}},
    Method → {"ShrinkWrap" → True}];
  DD2 = Graphics3D[Table[{ColorBraids[[j]], Specularity[White, 50], Sphere[Simplify[
        {a Cos[(r h + 2 π (j - 1))/s], b/l Sin[(l (r h + 2 π (j - 1)))/s], h} /. {h → 2 π}], 0.1]},
     {j, 1, s}], ImagePadding → {{Automatic, Automatic}, {None, None}},
    Method → {"ShrinkWrap" → True}];
  KK = Graphics3D[{EdgeForm[], Opacity[0.2], White,
     Cylinder[{{0, 0, -0.1}, {0, 0, 2 π + 0.1}}, 1.2]}, Lighting → "Neutral",
    Boxed → False, ImagePadding → {{Automatic, Automatic}, {None, None}},
    Method → {"ShrinkWrap" → True}];

  (*Top and bottom leminscates*)
  B = ParametricPlot3D[Simplify[{a Cos[ x ], b / l Sin[l x], 0}],
    {x, 0, 2 π}, PlotStyle → Directive[Darker[Gray, 0.8], Thickness[0.02]],
    PlotPoints → 100, ImagePadding → {{Automatic, Automatic}, {None, None}},
    Method → {"ShrinkWrap" → True}];
  C = ParametricPlot3D[Simplify[{a Cos[ x ], b / l Sin[l x], 2 π}],
    {x, 0, 2 π}, PlotStyle → Directive[Darker[Gray, 0.8], Thickness[0.02]],
    PlotPoints → 100, ImagePadding → {{Automatic, Automatic}, {None, None}},
    Method → {"ShrinkWrap" → True}];
  Show[A, B, C, DD, DD2, KK, Boxed → False,
   BoxStyle → {Thick}, Axes → False, Lighting → "Neutral", PlotRange → All,
   ViewPoint → {1.5445548132437892`, -2.864728779434567`, 0.926109847245335`},
   ViewVertical → {0.3340710941969182`, -0.6239446032271305`, 0.7064627634389605`},
   ImagePadding → {{Automatic, Automatic}, {None, None}},
   Method → {"ShrinkWrap" → True}]];

(*Parametric equation of the Braids*)
S[s_, r_, l_, a_, b_] :=
  Table[{a Cos[(r h + 2 π (j - 1))/s], b/l Sin[(l (r h + 2 π (j - 1)))/s], h}, {j, 1, s}];



(*Torus*)

Tor[θ_, ϕ_, R_, r_] :=
  {R Cos[θ], R Sin[θ], 0} + {r Cos[θ] Cos[ϕ], r Cos[ϕ] Sin[θ], r Sin[ϕ]};



(*Closing the Braid on a torus*)
BraidOnTorus2[H_] := Module[{A, B, C, DD},
```

```
A = ParametricPlot3D[Tor[θ, ϕ, 1, 0.5], {θ, 0, 2 π}, {ϕ, 0, 2 π},
   Boxed → False, Axes → False, PlotPoints → {70, 70}, Mesh → None,
   PlotStyle → {Opacity[0.5], LightGray, Specularity[White, 500]},
   Lighting → "Neutral", ImagePadding → {{Automatic, Automatic}, {None, None}},
   Method → {"ShrinkWrap" → True}];
B = Show[Table[ParametricPlot3D[{(1 + H[[j]][[1]]) Cos[h], (1 + H[[j]][[1]]) Sin[h], H[[j]][[2]]},
      {h, 0, 2 π}, PlotStyle → {Opacity[0.6], Lighter[ColorBraids[[j]], 0.2]},
      PlotPoints → 100, ImagePadding → {{Automatic, Automatic}, {None, None}},
      Method → {"ShrinkWrap" → True}] /. Line[pts_, rest___] :→ {CapForm → None,
       Specularity[White, 100], Tube[pts, 0.05, rest]}, {j, 1, Dimensions[H][[1]]}]];
Show[A, B]]
```

$$Z[j\_, a\_, b\_, r\_, s\_] := \frac{a}{2}\left(vv^{r/s}\, Exp[i\, 2\, \pi\, (j-1) / s] + vp^{r/s}\, Exp[i\, 2\, \pi\, (j-1) / s]\right) + \frac{b}{2}\left(vv^{r/s}\, Exp[i\, 2\, \pi\, (j-1) / s] - vp^{r/s}\, Exp[i\, 2\, \pi\, (j-1) / s]\right);$$

```
Pol[s_, r_, l_, a_, b_] := Block[{p, vv, vp, u},
  p = Product[u - Z[j, a, b, r, s], {j, 1, s}];
```

$$Return\left[FullSimplify\left[p\ /.\ \left\{u \to \frac{\rho^2 - 1}{\rho^2 + 1},\ vv \to \frac{2\,\rho\, Exp[i\,\phi]}{\rho^2 + 1},\ vp \to \frac{2\,\rho\, Exp[-i\,\phi]}{\rho^2 + 1}\right\}\right]\right]\right];$$

```
KnotBeam[H_, x_, y_, w_, k_] := Block[
  {r = Sqrt[x^2 + y^2] / k, ϕ = ArcTan[x, y]}, Exp[-r^2 / w^2] Expand[H /. {ρ → r}]];


IntPhase[H_] := Module[{E, FF, A, B},
  FF = KnotBeam[H, x, y, 1.7, 0.4];
  A = DensityPlot[Abs[FF]^2, {x, -2, 2}, {y, -2, 2}, Exclusions → None,
    PlotPoints → 100, RegionFunction → Function[{x, y}, Sqrt[x^2 + y^2] ≤ 2],
    PlotRange → All, MaxRecursion → 1, ColorFunction → ColorData["AvocadoColors"],
    PlotRange → All, ColorFunctionScaling → True, Frame → None];
  B = DensityPlot[Arg[FF], {x, -2, 2}, {y, -2, 2}, Exclusions → None, PlotPoints → 100,
    RegionFunction → Function[{x, y}, Sqrt[x^2 + y^2] ≤ 2], MaxRecursion → 2,
    PlotRange → All, ColorFunction → (Lighter[Hue[Rescale[#, {0, 2 π}, {0, 1}]], 0.2] &),
    PlotRange → All, ColorFunctionScaling → False, Frame → None];
  Return[GraphicsRow[{A, B}, Spacings → 0]]]

Intersections[H_] := Module[{RealPart, ImagPart, Contour1, FF, A, Intections},
  RealPart = Re[KnotBeam[H, x, y, 1.7, 0.4]];
  ImagPart = Im[KnotBeam[H, x, y, 1.7, 0.4]];
  Contour1 = ContourPlot[{RealPart == 0, ImagPart == 0},
    {x, -2, 2}, {y, -2, 2}, ContourStyle → {Red, Blue},
    RegionFunction → Function[{x, y}, Sqrt[x^2 + y^2] ≤ 2], Frame → None];
```

```
Intections = plot // Normal // Cases[#, Line[_], Infinity] & // Subsets[#, {2}] & //
    Map@Apply@RegionIntersection // DeleteCases[_EmptyRegion];
FF = KnotBeam[H, x, y, 1.7, 0.4];
A = DensityPlot[Arg[FF], {x, -2, 2}, {y, -2, 2}, Exclusions → None, PlotPoints → 100,
   RegionFunction → Function[{x, y}, Sqrt[x^2 + y^2] ≤ 2], MaxRecursion → 2,
   PlotRange → All, ColorFunction → (Lighter[Hue[Rescale[#, {0, 2 π}, {0, 1}]], 0.2] &),
   PlotRange → All, ColorFunctionScaling → False, Frame → None];
Show[A, Contour1, Graphics[{Black, PointSize@Large, Intections}]]
```

Out[386]= Null^4