HW2 - ISYE 6644

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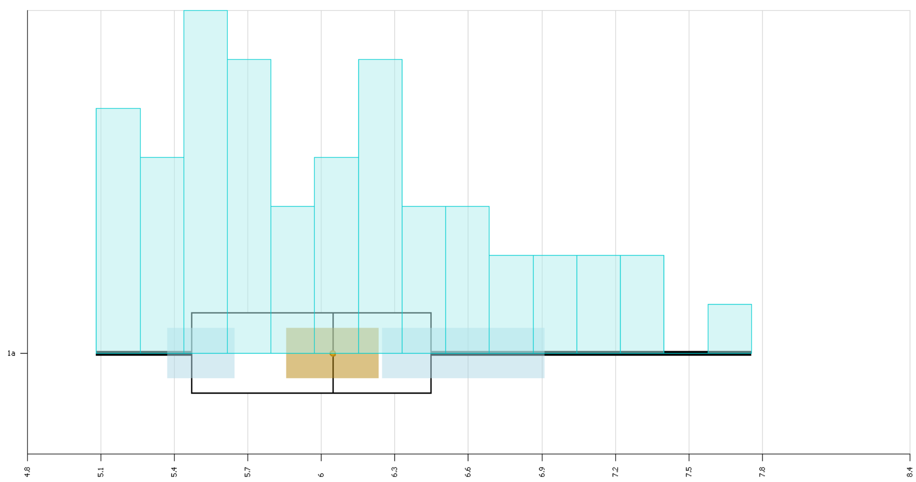
1

The developed Simio model can be found attached in the zip file.

1.1

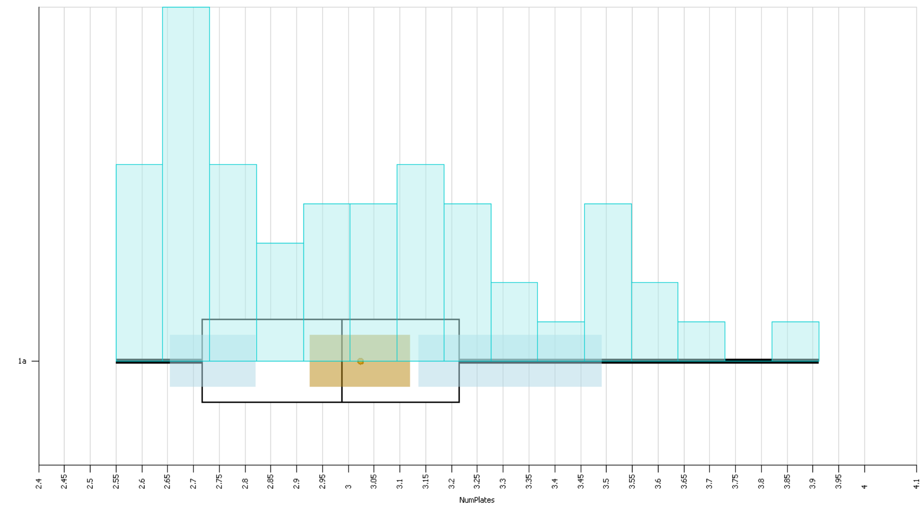
The following are the results for the experiment:
Average time in system for the plates:



The average time in system formula for this scenario is, using Little's Law:

$$W = \frac{3}{.5} = 6$$

Seeing that our average here is 6.05 ± 0.19 minutes, I am quite confident that my simulation was able to capture this relationship effectively, given that the true value is contained within my 95% confidence interval.
Average number of plates in system:

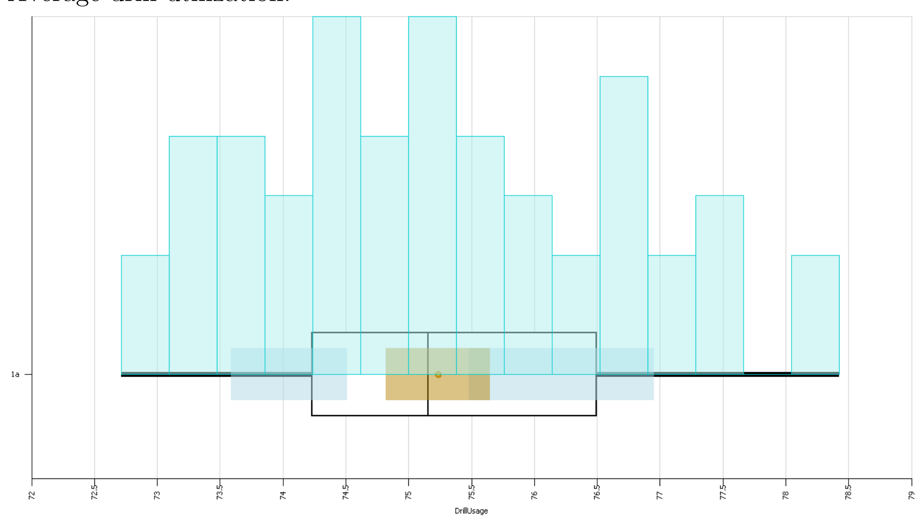


The steady-state average number in system formula for this scenario is given by:

$$L = \frac{0.5}{\frac{1}{1.5} - 0.5} = 3$$

Seeing that our average here is 3.02 ± 0.10, I am quite confident that my simulation was able to capture this relationship effectively, given that the true value is contained within my 95% confidence interval.

Average drill utilization:



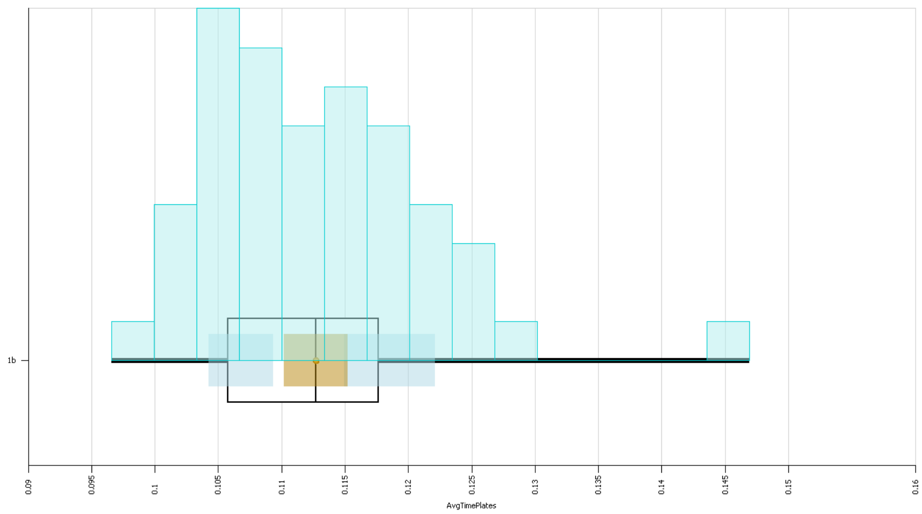
The average drill utilization in steady state for this system p is:

$$p = \frac{0.5}{\frac{1}{1.5}} = 0.75$$

Given that our simulation found a utilization rate of 75.23 ± 0.41, I am quite confident that my simulation was able to capture this relationship effectively, given that the true value is contained within my 95% confidence interval.

1.2

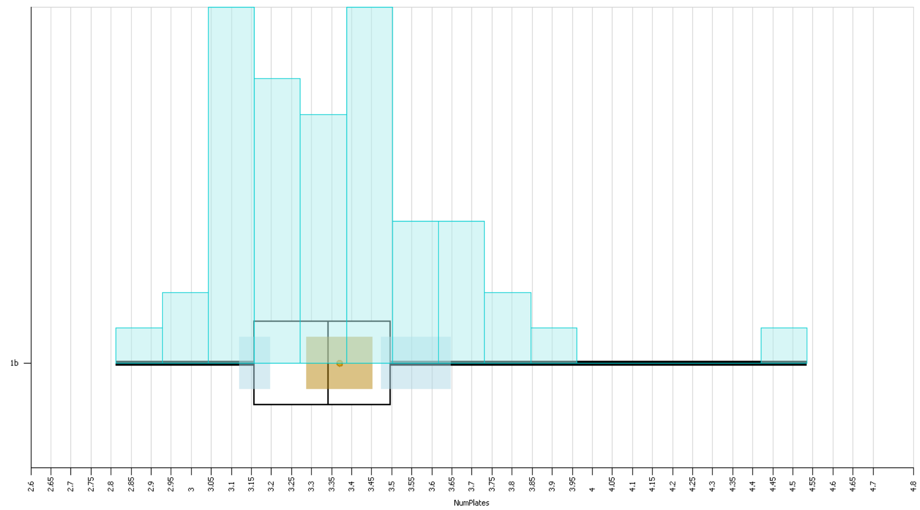
The following are the results for the experiment:
Average time in system for the plates:



The average time in system formula for this scenario is, using an online calculator:

$$W = 6.86$$

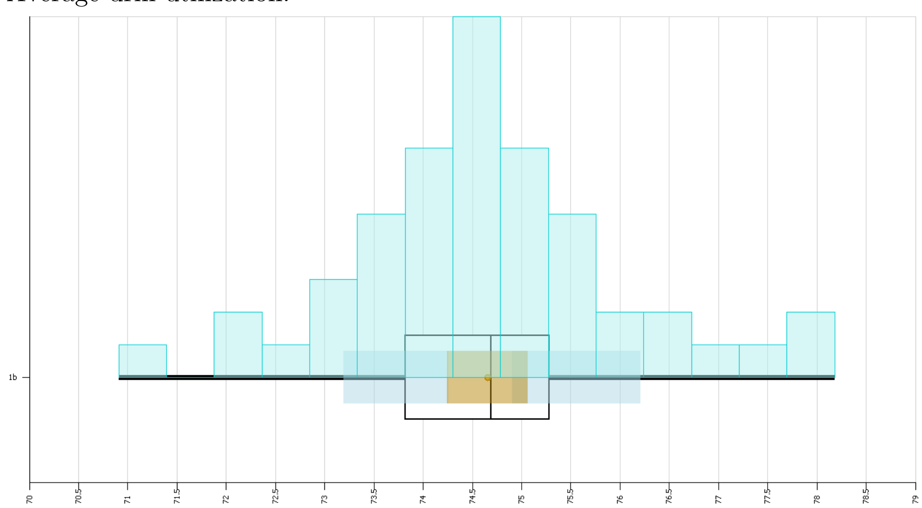
Seeing that our average here is 6.76 ± 0.15 minutes, I am quite confident that my simulation was able to capture this relationship effectively, given that the true value is contained within my 95% confidence interval.
Average number of plates in system:



The steady-state average number in system formula for this scenario is:

$$L = 3.43$$

Seeing that our average here is 3.37 ± 0.08, I am quite confident that my simulation was able to capture this relationship effectively, given that the true value is contained within my 95% confidence interval.
Average drill utilization:



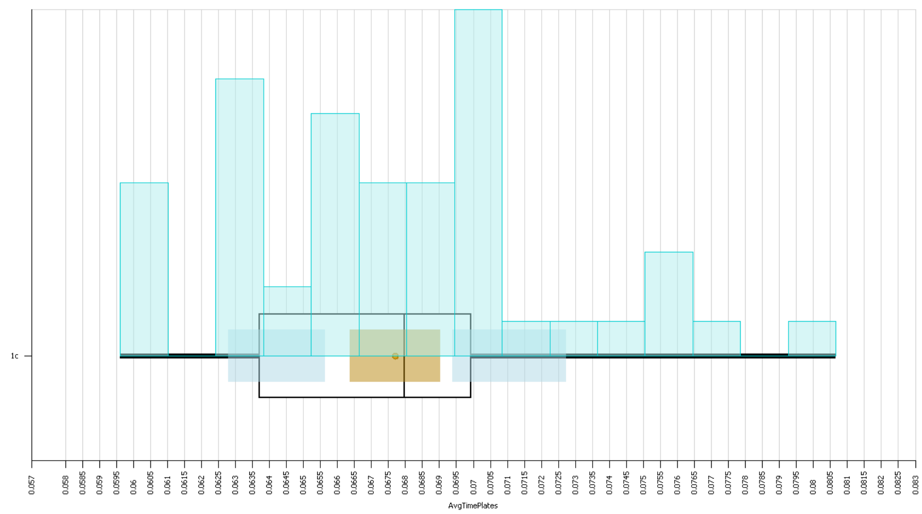
The average drill utilization in steady state for this system p is:

$$p = 0.75$$

Given that our simulation found a utilization rate of 74.66 ± 0.41, I am quite confident that my simulation was able to capture this relationship effectively, given that the true value is contained within my 95% confidence interval.

1.3

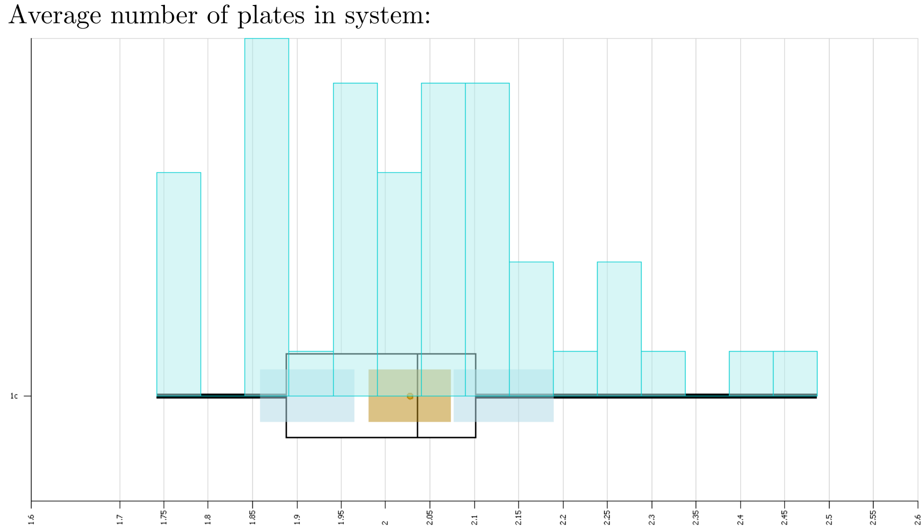
The following are the results for the experiment:
Average time in system for the plates:



The average time in system formula for this scenario is, using Little's Law:

$$W = 2 + \frac{1.5}{0.75} = 4$$

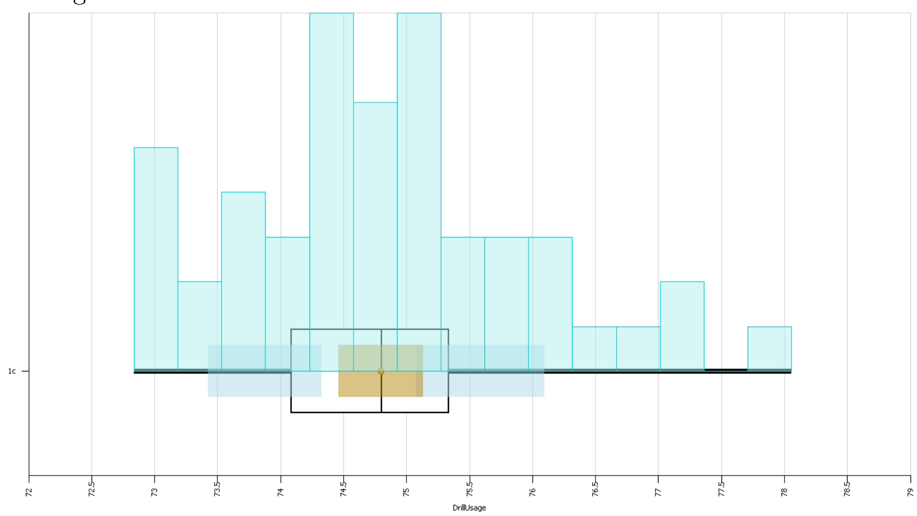
Seeing that our average here is 4.06 ± 0.08 minutes, I am quite confident that my simulation was able to capture this relationship effectively, given that the true value is contained within my 95% confidence interval.
Average number of plates in system:



The steady-state average number in system formula for this scenario is given by:

$$L = 4(0.5) = 2$$

Seeing that our average here is 2.03 ± 0.05, I am quite confident that my simulation was able to capture this relationship effectively, given that the true value is contained within my 95% confidence interval.
Average drill utilization:



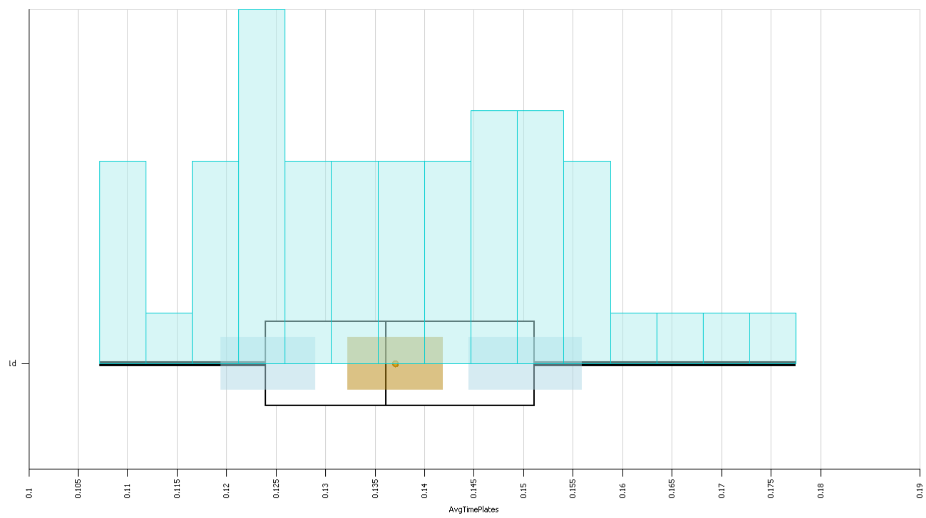
The average drill utilization in steady state for this system p is:

$$p = 0.75$$

Given that our simulation found a utilization rate of 74.80 ± 0.34, I am quite confident that my simulation was able to capture this relationship effectively, given that the true value is contained within my 95% confidence interval.

1.4

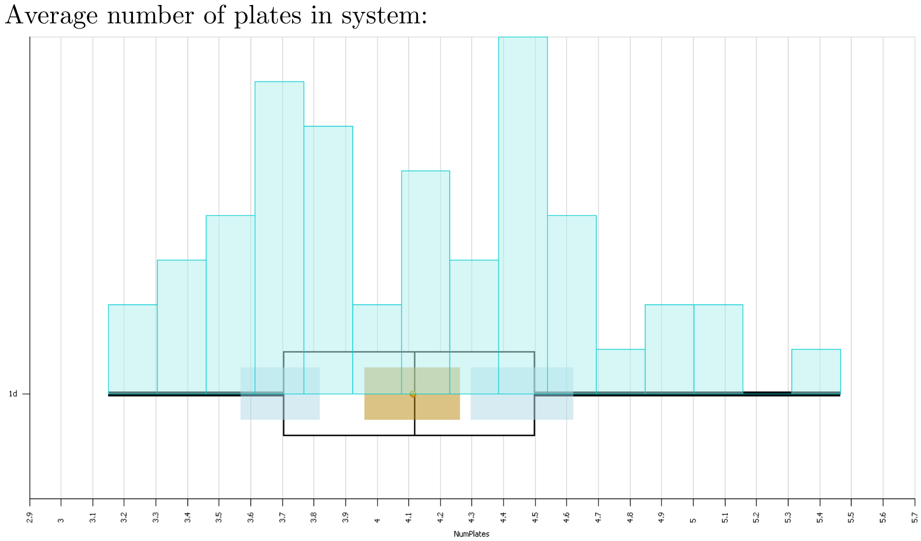
The following are the results for the experiment:
Average time in system for the plates:



The average time in system formula for this scenario is, using Little's Law:

$$W = \frac{.75}{.25}\frac{3}{2}\frac{3}{2} + \frac{3}{2} = 8.25$$

Seeing that our average here is 8.22 ± 0.29 minutes, I am quite confident that my simulation was able to capture this relationship effectively, given that the true value is contained within my 95% confidence interval.
Average number of plates in system:

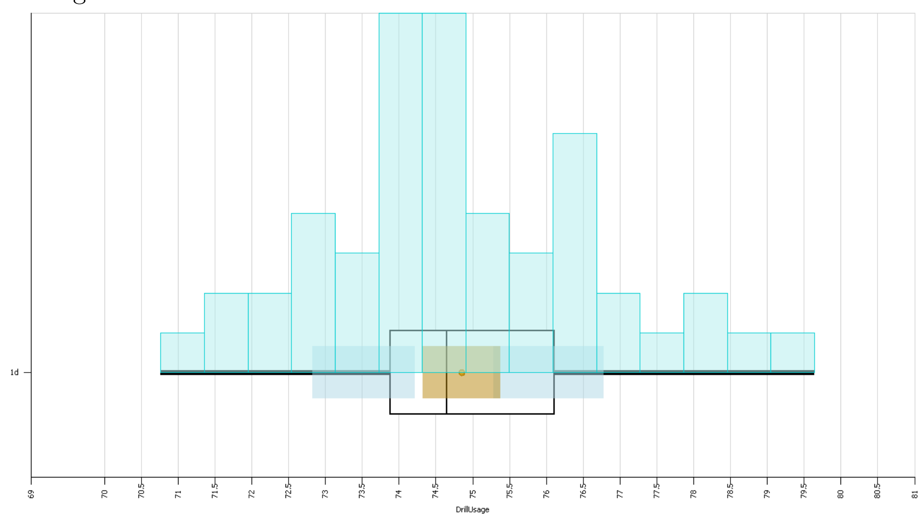


The steady-state average number in system formula for this scenario is given by:

$$L = 8.25\frac{1}{2} = 4.125$$

Seeing that our average here is 4.11 ± 0.15, I am quite confident that my simulation was able to capture this relationship effectively, given that the true value

is contained within my 95% confidence interval.
Average drill utilization:



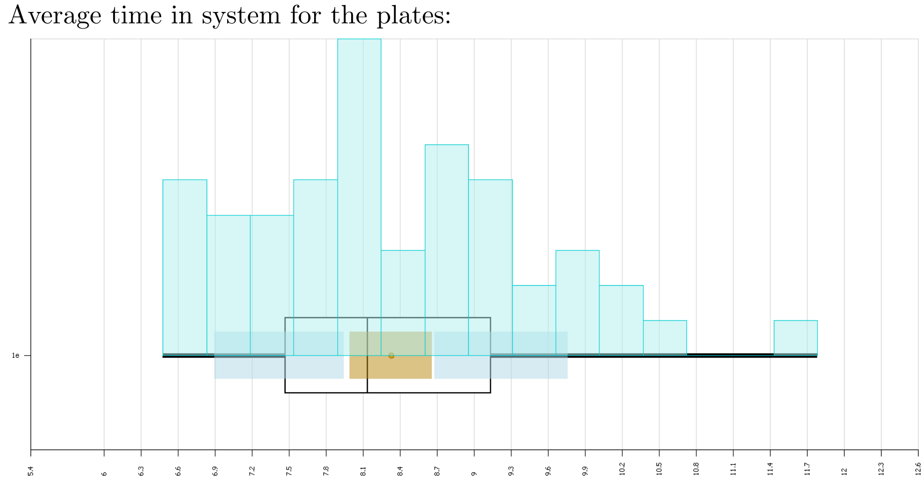
The average drill utilization in steady state for this system p is:

$$p = \frac{0.5}{\frac{1}{1.5}} = 0.75$$

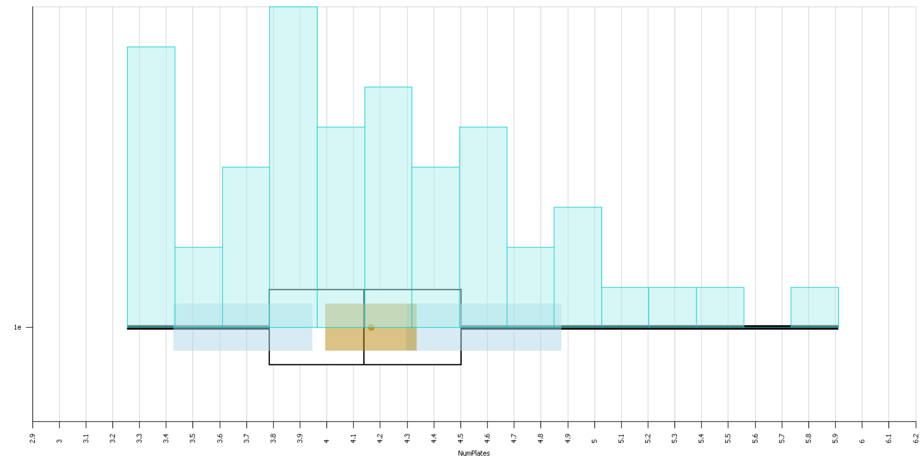
Given that our simulation found a utilization rate of 74.85 ± 0.53, I am quite confident that my simulation was able to capture this relationship effectively, given that the true value is contained within my 95% confidence interval.

1.5

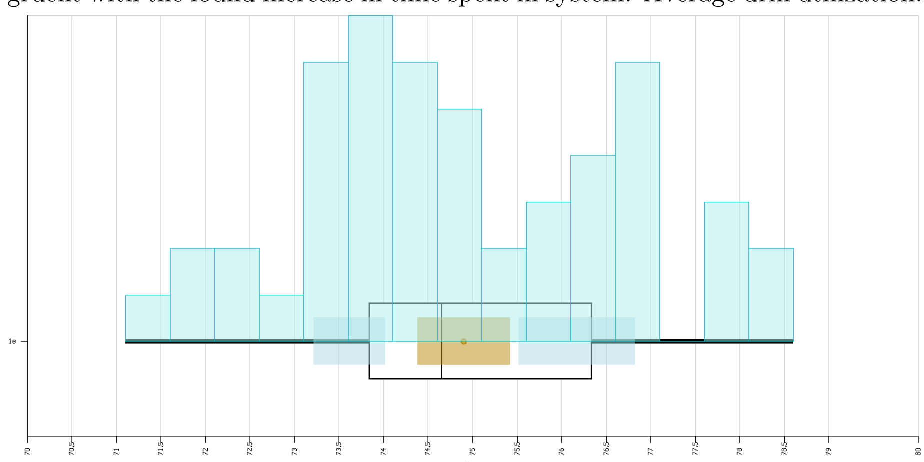
The following are the results for the experiment:
Average time in system for the plates:



Here, our time in system rises from 493.38 second to 499.59 seconds. This makes sense as the expected added time of 3 seconds of travel time would increase the amount of time plates spend in the system. Average number of plates in system:



Here, our average plates in system rises from 4.11 to 4.17. This would be congruent with the found increase in time spent in system. Average drill utilization:



Our average drill utilization seems to stay roughly the same in this new version. This makes sense as the total amount of plates generated by the arrival process didn't change, therefore the drill would be utilized the same amount of time. The added time and plates in system was purely due to changing the connector to a path.

2

2.1

Starting from the cdf:

$$f(t) = \alpha\lambda(\lambda t)^{\alpha-1}e^{-(\lambda t)^{\alpha}}, x > 0$$

The pdf can be obtained by the integration of the cdf function relative to t as follows:

$$\begin{aligned} F(t) &= \int_0^t \alpha\lambda(\lambda t)^{\alpha-1}e^{-(\lambda t)^{\alpha}}dt \\ &= -e^{-(\lambda t)^{\alpha}}\int_0^t (\lambda t)^{\alpha-1}dt \\ &= 1 - e^{-(\lambda t)^{\alpha}} \end{aligned} \tag{1}$$

2.2

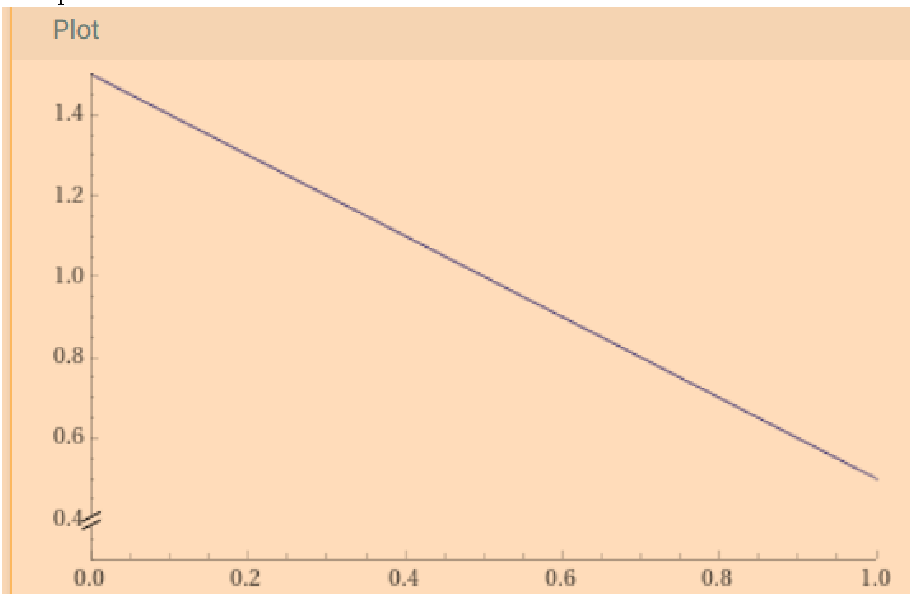
$$\begin{aligned} 1 - e^{-(\lambda t)^{\alpha}} &= g \\ -e^{-(\lambda t)^{\alpha}} &= g - 1 \\ e^{-(\lambda t)^{\alpha}} &= -g + 1 \\ ln(e^{-(\lambda t)^{\alpha}}) &= ln(-g+1) \\ \alpha(ln(e^{-(\lambda t)}) &= ln(-g+1) \\ \alpha\lambda t &= ln(-g+1) \\ t &= \frac{1}{\alpha\lambda}ln(-g+1) \end{aligned}$$

3

3.1

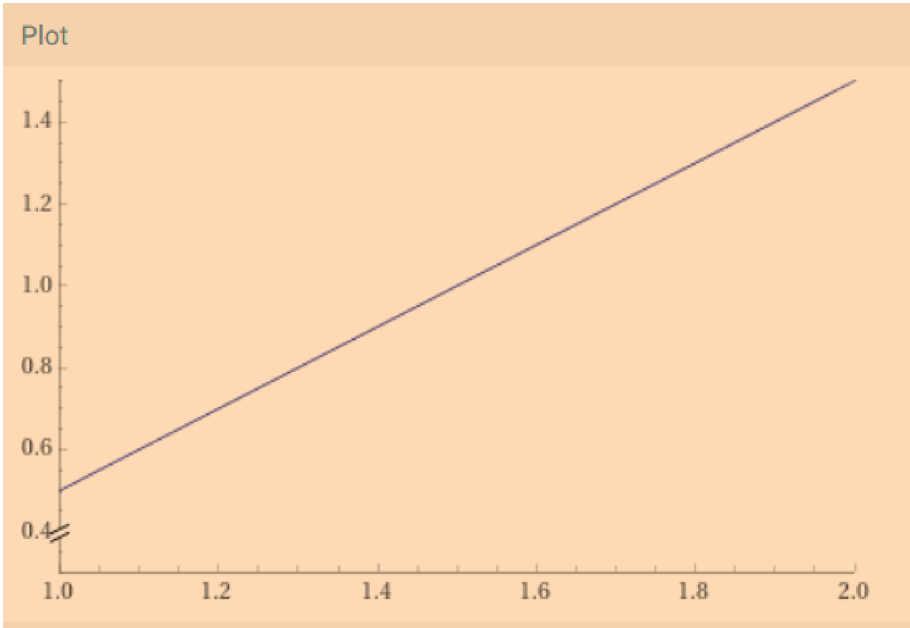
$$\begin{aligned} f_X(x) &= \int_1^2 (y-x)dy \\ &= \int_1^2 ydy - x\int_1^2 1dy \\ &= \frac{y^2}{2}|_1^2 - x\int_1^2 1dy \\ &= \frac{3}{2} - x \end{aligned} \tag{2}$$

The plot for this function is shown below:



$$
\begin{aligned}
f_Y(y) &= \int_0^1 (y-x)dx \\
&= -\int_0^1 xdx + y\int_0^1 1dx \\
&= -\frac{1}{2} + y
\end{aligned} \tag{3}
$$

The plot for this function is shown below:



3.2

$$f(x,y) \neq (\frac{3}{2} - x)(-\frac{1}{2} + y)$$

Due to the above, X and Y are not independent.

3.3

I will be referring ton $F_X(x)$ as $F_X(a)$ for clarity and differentiation from the already existing x.

$$\begin{aligned} F_X(a) &= \int_0^a (\frac{3}{2} - x)dx \\ &= (\frac{3x}{2} - \frac{x^2}{2})|_0^a \\ &= \frac{3a}{2} - \frac{a^2}{2} - \frac{3(0)}{2} - \frac{0^2}{2} \\ &= -\frac{1}{2}(a-3)a \end{aligned} \tag{4}$$

I will be referring ton $F_Y(y)$ as $F_Y(b)$ for clarity and differentiation from the already existing y.

$$\begin{aligned}
F_Y(b) &= \int_1^b (-\frac{1}{2} + y)dy \\
&= (\frac{y^2}{2} - \frac{y}{2})|_1^b \\
&= \frac{b^2}{2} - \frac{b}{2} - \frac{1}{2} - \frac{1^2}{2} \\
&= \frac{1}{2}(b-1)b
\end{aligned} \tag{5}$$

3.4

$$\begin{aligned}
E(X) &= \int_0^1 (\frac{3x}{2} - x^2)dx \\
&= (\frac{3x^2}{4} - \frac{x^3}{3})|_0^1 \\
&= \frac{3}{4} - \frac{1}{3} \\
&= \frac{5}{12}
\end{aligned} \tag{6}$$

$$\begin{aligned}
E(X^2) &= \int_0^1 (\frac{3x^2}{2} - x^3)dx \\
&= (\frac{x^3}{2} - \frac{x^4}{4})|_0^1 \\
&= \frac{1}{2} - \frac{1}{4} \\
&= \frac{1}{4}
\end{aligned} \tag{7}$$

$$\begin{aligned}
Var(X) &= E(X^2) - (E(X))^2 \\
&= \frac{1}{4} - (\frac{5}{12})^2 \\
&= \frac{11}{144}
\end{aligned} \tag{8}$$

$$\begin{aligned}
E(Y) &= \int_1^2 (y^2 - \frac{y}{2})dy \\
&= (\frac{y^3}{3} - \frac{y^2}{4})|_1^2 \\
&= \frac{8}{3} - 1 - \frac{1}{3} - \frac{1}{4} \\
&= \frac{19}{12}
\end{aligned} \tag{9}$$

$$
\begin{aligned}
E(Y^2) &= \int_1^2 (y^3 - \frac{y^2}{2}) dy \\
&= (\frac{y^4}{4} - \frac{y^3}{6})|_1^2 \\
&= 4 - \frac{8}{6} - \frac{1}{4} - \frac{1}{6} \\
&= \frac{31}{12}
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
Var(Y) &= E(Y^2) - (E(Y))^2 \\
&= \frac{31}{12} - (\frac{19}{12})^2 \\
&= \frac{11}{144}
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
E(XY) &= \int_0^1 \int_1^2 xy(y-x) dy dx \\
&= \int_0^1 (\frac{xy^3}{3} - x^2 \frac{y^2}{2})|_1^2 dx \\
&= \int_0^1 (\frac{8x}{3} - x^2 \frac{4}{2} - \frac{x}{3} - x^2 \frac{1}{2}) dx \\
&= \int_0^1 (\frac{7x}{3} - \frac{3x^2}{2}) dx \\
&= (\frac{7x^2}{6} - \frac{3x^3}{6})|_0^1 \\
&= \frac{7}{6} - \frac{3}{6} \\
&= \frac{2}{3}
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
Cov(X,Y) &= E(XY) - E(X)E(Y) \\
&= \frac{2}{3} - \frac{5}{12} \frac{19}{12} \\
&= \frac{1}{144}
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
Cor(X,Y) &= \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} \\
&= \frac{\frac{1}{144}}{\sqrt{(\frac{11}{144})^2}} \\
&= \frac{1}{11}
\end{aligned} \tag{14}
$$

4

$$q = \frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}$$

$$f(x_1,x_2) = \frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2}} e^{\frac{-q}{2}}$$

$$f(x_1,x_2) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{\frac{(x_1-\mu_1)^2}{\sigma_1^2}} \frac{1}{\sqrt{2\pi}\sigma_2} e^{\frac{(x_2-\mu_2)^2}{\sigma_2^2}}$$

$$f(x_1) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{\frac{(x_1-\mu_1)^2}{\sigma_1^2}}$$

$$f(x_2) = \frac{1}{\sqrt{2\pi}\sigma_2} e^{\frac{(x_2-\mu_2)^2}{\sigma_2^2}}$$

$$f(x_1,x_2) = f(x_1)f(x_2)$$

Due to this, we can conclude that X_1 and X_2 are independent when $p_{12} = 0$.

5

$$\begin{aligned}
Cor(X,Y) &= \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} \\
&= \frac{Cov(X,aX+b)}{\sqrt{Var(X)Var(aX+b)}} \\
&= \frac{Cov(X,aX)+Cov(X,b)}{\sqrt{Var(X)Var(aX+b)}} \\
&= \frac{aCov(X,X)}{\sqrt{a^2Var(X)Var(X+b)}} \\
&= \frac{aVar(X)}{\sqrt{a^2Var(X)Var(X+b)}} \\
&= \frac{aVar(X)}{\sqrt{a^2Var(X)(Var(X)+Var(b)+2Cov(X,b)}} \\
&= \frac{aVar(X)}{\sqrt{a^2Var(X)Var(X)}} \\
&= \frac{aVar(X)}{\sqrt{a^2}Var(X)} \\
&= \frac{a}{|a|}
\end{aligned} \tag{15}$$

So, if $a < 0$, the covariance will be -1, since the denominator is negative, and positive if $a > 0$, since the denominator is positive.

6

6.1

$$E[S_n] = \frac{0+1}{2} = 0.5$$
$$Var(S_n) = \frac{(0-1)^2}{2} = \frac{1}{12}$$

The mean is 0.5 and the variance is $\frac{1}{12}$.

6.2

As N approaches ∞, the CLT states that a uniform (0, 1) distribution of Z_n, the standardized form of S_n will approach a normal distribution with the following mean and variance:

$$E[S_n] = \frac{n}{2}$$
$$Var(S_n) = \frac{n}{12}$$

$$\begin{aligned} E[Z_n] &= E[\frac{S_n - E[S_n]}{\sqrt{Var(S_n)}}] \\ &= \frac{1}{\sqrt{\frac{n}{12}}}(E[S_n - \frac{n}{2}]) \\ &= \frac{1}{\sqrt{\frac{n}{12}}}(E[S_n] - \frac{n}{2}]) \\ &= \frac{1}{\sqrt{\frac{n}{12}}}(\frac{n}{2} - \frac{n}{2}]) \\ &= 0 \end{aligned} \tag{16}$$

$$\begin{aligned} Var(z_n) &= Var(\frac{S_n - \frac{n}{2}}{\sqrt{\frac{n}{12}}}) \\ &= \frac{1}{\frac{n}{12}} Var(S_n) \\ &= \frac{1}{\frac{n}{12}} * \frac{n}{12} \\ &= \frac{n}{n} \\ &= 1 \end{aligned} \tag{17}$$

Therefore, the limiting distribution will be a normal distribution N(0, 1).

6.3

$$Z_{12}\ norm(0,1)$$
$$p(z_{12} > 2) = 1 - p(z_{12} \leq 2)$$
$$p(z_{12} \leq 2) = 0.9772$$
$$p(z_{12} > 2) = 1 - 0.9772 = 0.0228$$

7

7.1

$$\bar{X}_{10} = 6.38$$
$$S_{10} = 1.47$$

The 95% confidence interval for μ is:

$$6.38 \pm 2.262(\frac{1.47}{\sqrt{10}}) = 6.38 \pm 1.05$$
$$or$$
$$[5.33, 7.43]$$

7.2

$$H_0 : \mu = 6$$
$$H_1 : \mu \neq 6$$

$$\begin{aligned} t &= \frac{\bar{X}_{10} - \mu}{\frac{S_{10}}{\sqrt{n}}} \\ &= \frac{6.38 - 6}{\frac{1.47}{\sqrt{10}}} \\ &= 0.8176 \end{aligned} \tag{18}$$

Given that the t value given is less than the critical t statistic found at 0.95 of 2.262, we maintain H_0.

7.3

$$\alpha = 0.05$$

$$\begin{aligned} t &= \frac{\bar{X}_{10} - \mu}{\frac{S_{10}}{\sqrt{n}}} \\ &= \frac{6.38 - 7}{\frac{1.47}{\sqrt{10}}} \\ &= -1.3338 \end{aligned} \tag{19}$$

For the upper confidence interval:

$$6.38 + 1.833\frac{1.47}{\sqrt{10}} = 7.2321$$

$$H_0 : \mu \leq 7$$

$$H_1 : \mu > 7$$

Given that the absolute value of the t value given is less than the critical t statistic found at 0.95 of 1.833, we maintain H_0.

7.4

$$\alpha = 0.05$$

$$\begin{aligned} t &= \frac{\bar{X}_{10} - \mu}{\frac{S_{10}}{\sqrt{n}}} \\ &= \frac{6.38 - 5.25}{\frac{1.47}{\sqrt{10}}} \\ &= 2.4309 \end{aligned} \tag{20}$$

For the lower confidence interval:

$$6.38 - 1.833\frac{1.47}{\sqrt{10}} = 5.5279$$

$$H_0 : \mu < 5.25$$

$$H_1 : \mu \geq 5.25$$

Given that the t value given is greater than the critical t statistic found at 0.95 of 1.833, we reject H_0.