ISyE 6644 – Spring 2023

Homework #5 — Due Friday, April 7

Follow the instructions on the syllabus to submit a single archive containing all codes and a single PDF document containing your discussion and findings. Name the archive as `LastName-FirstName.zip`.

1. Random Number Generators (RNGs). **[This problem will not be graded, but you should attempt to answer all questions.]**
   (a) Visit the site `www.mathworks.com/help/techdoc/ref/randstream.html` and check the RNGs incorporated in Matlab.
   - What is the default RNG in Matlab?
   - What is the keyword for the Mersenne Twister generator in Matlab?
   - The combined generator we discussed in class was proposed by Pierre L'Ecuyer. The name of this generator is "mrg32k3a". What is the period of this generator?

   (b) Try to find C and Java codes for mrg32k3a. Display the web links.
   (c) What RNG is implemented by default in the most recent release of Python?
   (d) What RNG is implemented by default in the most recent release of the R project? Which other RNGs are available?
   (e) Visit the site `https://support.office.com/en-us/article/rand-function-4cbfa695-8869-4788-8d90-021ea9f5be73` and read about the RNG employed in Microsoft Excel.

2. Consider the following discrete distribution with $p_i = \Pr(X = i)$:

i	1	2	3	4	5	6	7	8	9	10
p_i	.05	.09	.12	.08	.03	.16	.07	.22	.04	.14

   The next two tasks can be performed easily in Matlab, R or Python.
   (a) Use the cutpoint method (Algorithm CMSET) from the posted slides to determine $m = 10$ cutpoints. The last (auxiliary) cutpoint will be equal to 10.
   (b) Using these cutpoints as input to a computer program based on Algorithm CM, randomly generate 10,000 samples from the distribution $\{p_i\}$. Evaluate the correctness of your algorithm(s) by plotting the actual c.d.f. $q_i \equiv \Pr\{X \le i\}$ versus its estimate

$$\hat{q}_i = \frac{\text{no. of samples} \le i}{10{,}000}, \quad i = 1, \ldots, 10.$$

   Ideally, the P-P plot of the pairs $(q_i, \hat{q}_i)$ should be close to the 45-degree line in the unit square $[0, 1]^2$.

3. The logistic distribution has density function

$$f(x) = \frac{(1/\beta)e^{-(x-\gamma)/\beta}}{[1 + e^{-(x-\gamma)/\beta}]^2}, \quad x \in \mathbb{R},$$

where $\gamma \in \mathbb{R}$ and $\beta > 0$ are parameters. Show that γ is a location parameter and β is a scale parameter. Then give a formula for generating realizations from this distribution.

4. In each of the following cases, give a formula that uses *exactly one uniform random number* for generating realizations of the random variable X:
   (a) $X = \min\{U_1, U_2\}$, where U_1 and U_2 are i.i.d. uniform$(0, 1)$.
   (b) $X = \max\{U_1, U_2\}$, where U_1 and U_2 are i.i.d. uniform$(0, 1)$.
   (c) $X = \max\{X_1, X_2\}$, where X_1 and X_2 are i.i.d. exponential with parameter λ.

5. Derive algorithms based on the inverse-transform and composition methods for generating realizations from the density
$$f(x) = \frac{3x^2}{2}, \quad -1 \le x \le 1.$$

6. Use the inverse transform method to develop a generator for the random variable X with density function
$$f(x) = \begin{cases} e^{2x}, & x < 0 \\ e^{-2x}, & x \ge 0. \end{cases}$$

7. Consider the beta density $f(x) = 20x(1-x)^3$, $0 \le x \le 1$. Since the c.d.f. cannot be inverted easily, we will use the acceptance-rejection method to generate realizations from this distribution.
   (a) Find the maximum value of $f(x)$ in $[0, 1]$, say m. To make your life easier, you can try the equivalent problem of maximizing the (natural) log function $\ln[f(x)]$.
   (b) Use the majorizing function $t(x) = m$, $0 \le x \le 1$ to develop an acceptance-rejection (A-R) algorithm for generating realizations from $f(x)$. First derive the constant
$$c = \int_0^1 t(x)\,dx,$$
   the auxiliary density $h(x) = t(x)/c$ on $[0, 1]$, and the acceptance function $g(x) = f(x)/t(x)$. Then list the entire A-R algorithm, including the step for generating realizations from $h(x)$.
   (c) How many iterations of the A-R algorithm are expected until a realization from $f(x)$ is delivered?

8. Construct an algorithm based on the ratio-of-uniforms method to generate observations from the beta density in Problem 7. What is the mean number of uniform$(0, 1)$ observations required to generate a single observation from $f(\cdot)$?