2. **Recursion**

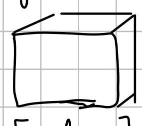
$$\beta_t(i) = \sum_{j=1}^{N} a_{ij}\, b_j(o_{t+1})\, \beta_{t+1}(j), \quad 1 \le i \le N, 1 \le t < T$$

$$\beta_t(i) = \sum a_{ij}\, b_j(o_{t+1})\, \beta_{t+1}(j) =$$

$$\sum_N \exp\left(\log\left(a_{ij} \cdot b_j(o_{t+1})\, \beta_{t+1}(j)\right)\right) =$$

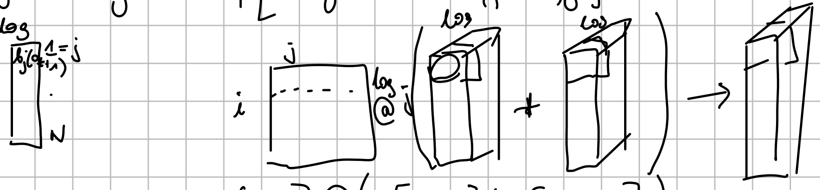
$$= \sum_{j=1} \exp\left(\log(a_{ij}) + \log(b_j(o_{t+1})) + \log(\beta_{t+1}(j))\right)$$

$$a_{ij} \to A_{N \times N}$$



$[n, t, 3]$

logdow@ $\to$ $out_{\{ij\}} = \text{logsumexp}\left[\log A_{ik} + \log B_{kj}\right]$

$\log A \cdot$ 

$[i,j]\,@\left([N,1,n] + [\quad]\right)$

$[n, N]^T \Rightarrow [N, n]$

$\downarrow$ unsqueeze (1)

$[N, 1, n]$

$\log\left(b_j^1(o_{t+1}) + \log(\beta_{t+1}^1(j)\right)$

$N \times n$

$$\xi_t(i,j) = \frac{\alpha_t(i)\, a_{ij}\, b_j(o_{t+1})\, \beta_{t+1}(j)}{\sum_{k=1}^{N} \alpha_t(k)\, \beta_t(k)}$$

$\xi \sim [n\ t\ N, N]$

$\log \alpha \sim [n, t, N]$

$\log \beta \sim [n, t, N]$

$$\log(\xi) = \log(\alpha_t(i)) + \log(a_{ij}) + \log(b_j(o_{t+1})) + \log(\beta_{t+1}(j)) - \log\left(\sum_{j=1}^{N} \alpha_t(j)\, \beta_t(j)\right)$$

$$\underset{[n,t,N]}{\log \alpha} + \underset{[n,t,N]}{\log \beta} \to [n,t,N] \xrightarrow[N]{\text{log sum exp}} [n, t, 1]$$

$\log \beta \to [n, t, N]$

$[n, 1{:}t, N]$

$\log \alpha \to [n, 0{:}(t-1), N]$

$\log b \quad [n, 1{:}t, N]$

$\log a \quad [N, N]$

$[n, \overset{t-1}{t}, \underbrace{N, N}]$

$$\overset{i}{\log \alpha} + \overset{ij}{\log a} + \overset{j}{\log b} + \overset{j}{\log \beta}$$

$$\underset{3\ 3\ 2\ 1}{[n, 0{:}t{-}1, N, 1]} + \underset{1\ 1\ \ 2\ 2}{[1, 1, N, N]} + \left([n, 1{:}t, 1, N] + [n, 1{:}t, 1, N]\right) \to [n, t-1, N, N]$$

$$- [n, t-1, 1, 1]$$

$\downarrow$

$[n, t-1, N, N]$

$$\hat{a}_{ij} = \frac{\sum_{t=1}^{T-1} \xi_t(i,j)}{\sum_{t=1}^{T-1} \sum_{k=1}^{N} \xi_t(i,k)}$$

$$\text{log sum exp}(\log \xi\, [n, \overset{\text{po tym}}{t-1}, N, N])$$

$$- \log(\text{sum}(\text{sum}(\exp(\log \xi\, [n, t-1, N, N])))) \to \underset{[n,N,N]}{\log \tilde{a}_{ij}} \xrightarrow{\text{mean po ousp}} \underset{[N,N]}{\log \tilde{a}}$$

2 po tym

1. po tym

$[n, N, 1]$