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Problem 1: Is it possible to have two divergent sequences (a_n) and (b_n) for which $(a_n + b_n)$ converges? If yes, give an example. If not, prove it.

Yes, it is possible to have two divergent sequences (a_n) and (b_n) for which $(a_n + b_n)$ converges.

Let $(a_n) = n^2$. This sequence diverges to infinity.

Let $(b_n) = -(a_n) = -n^2$. This sequence diverges to $-$ infinity.

Thus both (a_n) and (b_n) are divergent.

However, $(a_n + b_n) = n^2 - n^2 = 0$.

Thus $(a_n + b_n)$ converges to 0.

Problem 2:

(1) Let (s_n) be a sequence of real numbers such that $s_n \to 3$. Use the definition of convergence (without using limit theorems) to prove that $\lim s_n^2 = 9$.

(2) Find a divergent sequence (t_n) with $\lim t_n^2 = 9$.

(1) definition of convergence:

$$\exists L \quad \forall \varepsilon > 0, \exists N \quad \forall n > N : |a_n - a| < \varepsilon$$

We are told $s_n \to 3$. Thus for any $\varepsilon > 0$, there exists N such that for all $n > N$,

$$|s_n - 3| < \varepsilon.$$

Let's call this N as N_1.

Choose $a = 9$.

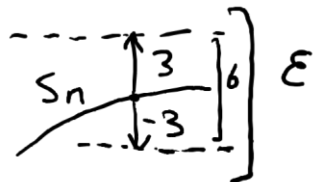
Let $\varepsilon > 0$

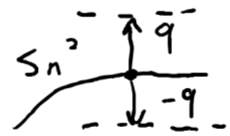
scratch: $|s_n^2 - 9| < \varepsilon$

$$|s_n^2 - 9| = |(s_n - 3)(s_n + 3)| \le |s_n - 3||s_n + 3| < \varepsilon$$

we know $|s_n - 3| < \varepsilon$ for $n \ge N_1$

then let's bind $|s_n + 3| < 6 - \varepsilon$





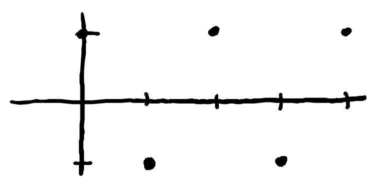
$$|s_n - 3||s_n + 3| < \varepsilon(6 - \varepsilon)$$

Choose $N = \varepsilon(6 - \varepsilon)$

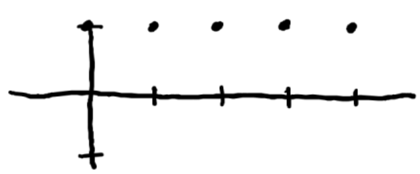
Let $n > N$

Then $|s_n^2 - 9| = |s_n - 3||s_n + 3| < \varepsilon(6 - \varepsilon) < \varepsilon$

(2) We know sequence $s_n = -1^n$ is divergent:

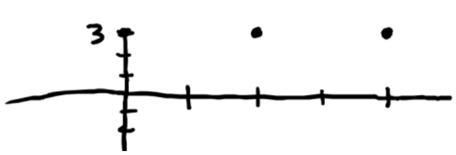


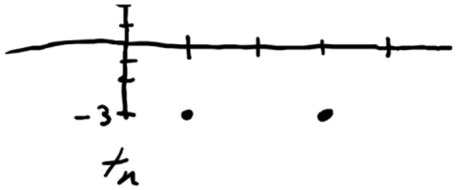
However $s^2_n = (-1^n)^2$ is convergent:

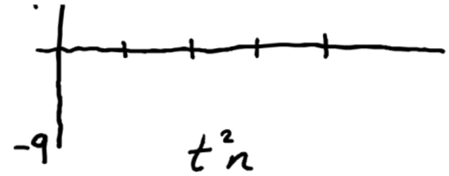


this is convergent as $\lim s_n^2 = 1$.

Thus Let $(t_n) = 3 \cdot (-1)^n$







$$t_n^2 = (3(-1)^n)^2$$

Problem 3: Let $a_1 = 1$, $a_2 = 2$, and $a_n = \frac{1}{2}(a_{n-2} + a_{n-1})$. Does (a_n) converge? Prove your answer. [Hint: is (a_n) Cauchy?]

if cauchy $\rightarrow$ convergent

Cauchy: if $\forall \varepsilon > 0$, $\exists N$ s.t. $|a_m - a_n| < \varepsilon \ \forall m, n > N$

$$a_3 = \frac{1}{2}(a_{3-2} + a_{3-1}) = \frac{1}{2}(1 + 2) = 3/2$$

$$a_4 = \frac{1}{2}(a_2 + a_3) = \frac{1}{2}\left(2 + \frac{3}{2}\right) = 7/4$$

$$a_5 = \frac{1}{2}(a_3 + a_4) = \frac{1}{2}\left(\frac{3}{2} + \frac{7}{4}\right) = \frac{13}{8}$$

$$a_6 = \frac{1}{2}(a_4 + a_5) = \frac{1}{2}\left(7/4 + \frac{13}{8}\right) = \frac{27}{16}$$

1		2		$\frac{3}{2}$		$\frac{7}{4}$		$\frac{13}{8}$		$\frac{27}{16}$	
	$+\frac{1}{1}$		$-\frac{1}{2}$		$\frac{1}{4}$		$-\frac{1}{8}$		$\frac{1}{16}$		$-\frac{1}{32}$

from this we can see that the difference between the n and $n+1$ term is: $\left(\frac{-1}{2}\right)^{n-1}$

e.g. $n=1,\ n+1=2 \rightarrow \left(\frac{-1}{2}\right)^0 = 1$

$n=2,\ n+1=3 \rightarrow \left(\frac{-1}{2}\right)^1 = -\frac{1}{2}$

$n=3,\ n+1=4 \rightarrow \left(\frac{-1}{2}\right)^2 = \frac{1}{4}$

the magnitude of difference between terms is decreasing.

$$|a_2 - a_1| > |a_3 - a_2| > |a_4 - a_3| > \dots > |a_{n+1} - a_n|$$

Thus let $\varepsilon > 0$. Choose $N > 0$ such that $\frac{1}{2^N} < \varepsilon/2$. Then for any $m,n > N$,

$$|a_n - a_m| = \varepsilon.$$

We have shown a_n is Cauchy, and thus it follows that a_n converges.

Problem 4: Suppose $x_n \to \infty$ and $y_n \to 2$. Prove that $x_n + y_n \to \infty$. (Note that (x_n) diverges, so this is not covered by our limit theorems).

assume for contradiction that there exists some a s.t.:

$$x_n + y_n \to a.$$

Let $\varepsilon = \frac{1}{2}$.

It follows that there exists some N for which, for all $n > N$, we have $|(x_n + y_n) - a| < \frac{1}{2}$

$$-\frac{1}{2} < (x_n + y_n) - a < \frac{1}{2}$$

$$-\frac{1}{2} - x_n < y_n - a < \frac{1}{2} - x_n$$

$$\frac{1}{2} + x_n > -y_n + a > -\frac{1}{2} + x_n$$

$$1 + x_n > \underbrace{-y_n + a + \tfrac{1}{2} > x_n}$$

Here lies our contradiction. We supposed $x_n \to \infty$, yet we have it bounded by $-y_n + a + \frac{1}{2}$, a finite number. Thus $x_n + y_n$ diverges.

Problem 5: Let f be a continuous function on $\mathbb{R}$. Show that g defined by $g(x) = \max\{f(x), 1\}$ is also continuous.

To show $g(x)$ is continuous, Let's consider the 2 cases:

Case 1: $\max\{f(x), 1\} = f(x)$

It follows that $g(x) = f(x)$ and since f is continuous, g is also continuous.

Case 2: $\max\{f(x), 1\} = 1$

Then $g(x) = 1$. Since $g(x) = 1$ is a constant function, it is also continuous. Thus we have shown that in either case, g is continuous.

Problem 6: Use the definition of the derivative to find $f'(x)$ for $f(x) = \frac{1}{x^2+4}$ on $\mathbb{R}$.

$$\lim \frac{f(x) - f(c)}{}$$

$$x \to c \qquad x - c$$

$$\lim_{x \to c} \frac{\frac{1}{x^2+4} - \frac{1}{c^2+4}}{x - c} \to \frac{\frac{1}{(x^2+4)(c^2+4)}(c^2+4) - \frac{1}{(c^2+4)(x^2+4)}(x^2+4)}{x - c}$$

$$\lim_{x \to c} \frac{\frac{(c^2+4) - (x^2+4)}{(x^2+4)(c^2+4)}}{x - c} \to \frac{\frac{c^2 - x^2}{x^2c^2+4x^2+4c^2+16}}{\frac{x-c}{1}}$$

$$\lim_{x \to c} \frac{(c-x)(c+x)}{x^2c^2+4x^2+4c^2+16} \cdot \frac{1}{x-c}$$

$$\lim_{x \to c} \frac{-1\cancel{(x-c)}(c+x)}{\cancel{(x-c)}(x^2c^2+4x^2+4c^2+16)}$$

$$\lim_{x \to c} \frac{-c - x}{(x^2c^2+4x^2+4c^2+16)}$$

$$= \frac{-c - c}{(c^2c^2+4c^2+4c^2+16)}$$

$$f'(c) = \frac{-2c}{c^4+8c^2+16}$$

$$f'(c) = \frac{-2c}{(c^2+4)^2} \to \boxed{f'(x) = \frac{-2x}{(x^2+4)^2}}$$

Problem 7: Let $f : [-1,1] \to \mathbb{R}$, and $f(x) = \begin{cases} 0, & x \neq 0, \\ 1, & x = 0. \end{cases}$

Prove, using the definition of the integral, that

$$\int_{-1}^{1} f(x)\, dx = 0.$$

Definition 8.10: A bounded function $f : [a,b] \to \mathbb{R}$ is integrable if $L(f) = U(f)$. When this happens, we denote $\int_a^b f$ and $\int_a^b f(x)\,dx$ to be this common value.

That is, $\int_a^b f(x)\,dx = L(f) = U(f)$

given any partition p on $[-1,1]$ that includes $x = 0$, the subinterval $[x_{i-1}, x_i]$ will be one of the following cases:

case 1: $0 \notin [x_{i-1}, x_i]$. Then $m_i = M_i = 0$, where m_i and M_i are min/max values of $f(x)$ on $[x_{i-1}, x_i]$

case 2: $0 \in [x_{i-1}, x_i]$. Then $m_i = 0$ and $M_i = 1$

Lower sum:

$$L(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = 0$$

$$L(f) = \sup_P L(f,P) = 0$$

Upper sum:

$$U(f,P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \leq 1(x_i - x_{i-1})$$

We need to show that the upper sum becomes 0 as the subinterval

We need to show that the upper sum becomes 0 as the subinterval can be made arbitrarily small by choosing a finer partition.

Then for some partition P_ε where $U(f, P_\varepsilon) < \varepsilon$ for any $\varepsilon > 0$,

$$U(f) = \inf_P U(f, P) = 0$$

Thus $U(f) = L(f)$, $f(x)$ is integrable, and $\int_{-1}^{1} f(x)\,dx = L(f) = U(f) = 0$

Problem 8: Suppose that (f_n) is a sequence of functions converging uniformly to a function f as $n \to \infty$.

Let $g_n(x) = f_n(x) - f_{n-1}(x)$ be another sequence of functions. Prove that $g_n \to 0$ uniformly on $\mathbb{R}$ as $n \to \infty$.

We are told that (f_n) converges uniformly to f as $n \to \infty$. Thus for any $\varepsilon > 0$, there exists some N_1 s.t. for all $n \geq N_1$, $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$.

Then there is also some N_2 where when $n \geq N_2$, $|f_{n-1}(x) - f(x)| < \varepsilon/2$

Choose $N = \max\{N_1, N_2\}$

Let $n \geq N$

Thus

Let $n \geq N$

Thus

$$|g_n(x)| = |f_n(x) - f_{n-1}(x)|$$

$$= |f_n(x) - f(x) + f(x) - f_{n-1}(x)| \leq |f_n(x) - f(x)| + |f(x) - f_{n-1}(x)|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$