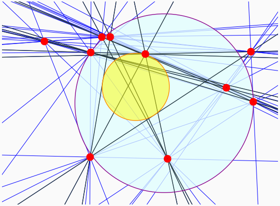
Fast Algorithms for Geometric Consensuses

Sariel Har-Peled[1] Mitchell Jones[1]

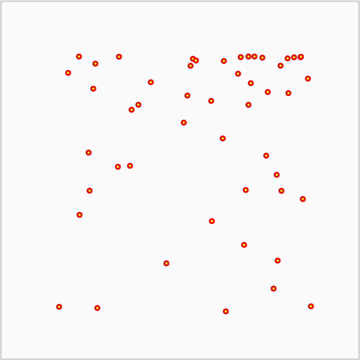
SoCG 2020, June 23–26

[1]University of Illinois at Urbana-Champaign

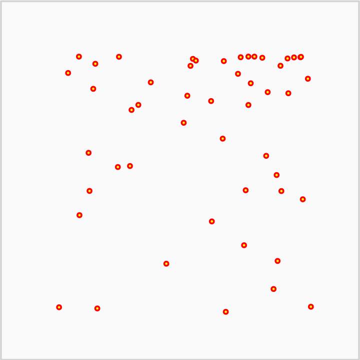
Student Presentation



The yolk: problem setup

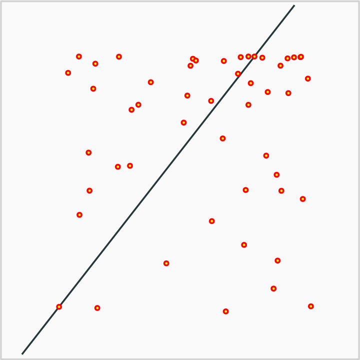


The yolk: problem setup



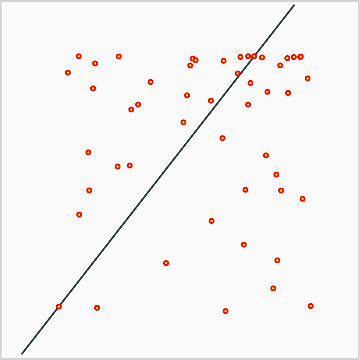
- ℓ is a median line if $\geq \lceil n/2 \rceil$ points of P lie on either side

The yolk: problem setup



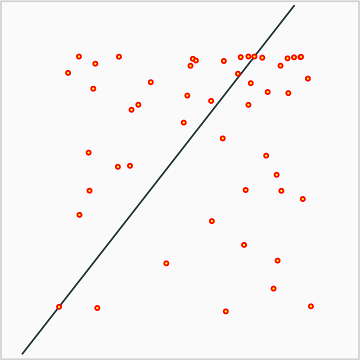
- ℓ is a median line if $\geq \lceil n/2 \rceil$ points of P lie on either side

The yolk: problem setup



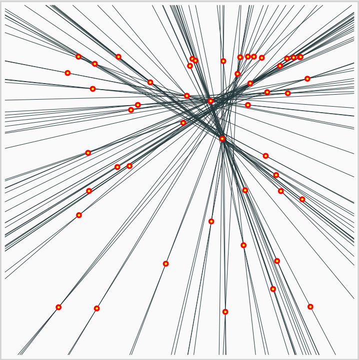
- ℓ is a median line if $\geq \lceil n/2 \rceil$ points of P lie on either side
- ℓ is extremal if it passes through 2 points of P

The yolk: problem setup



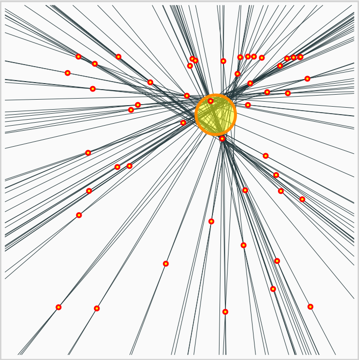
- ℓ is a median line if $\geq \lceil n/2 \rceil$ points of P lie on either side
- ℓ is extremal if it passes through 2 points of P
- Extremal yolk is smallest radius ball intersecting all extremal median lines

The yolk: problem setup



- ℓ is a median line if $\geq \lceil n/2 \rceil$ points of P lie on either side
- ℓ is extremal if it passes through 2 points of P
- Extremal yolk is smallest radius ball intersecting all extremal median lines

The yolk: problem setup



- ℓ is a median line if $\geq \lceil n/2 \rceil$ points of P lie on either side
- ℓ is extremal if it passes through 2 points of P
- Extremal yolk is smallest radius ball intersecting all extremal median lines

The yolk

- Yolk is smallest radius ball intersecting all median lines

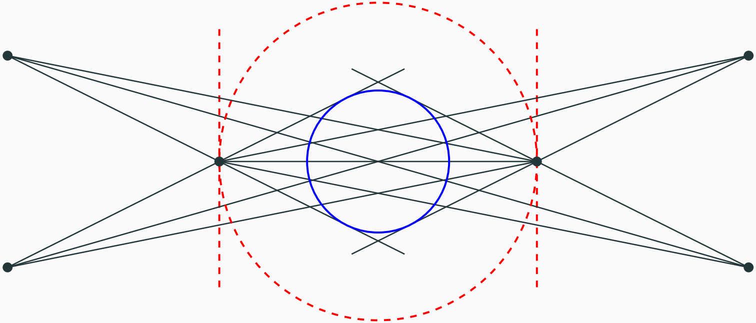
The yolk

- Yolk is smallest radius ball intersecting all median lines
- Motivation: voting theory (generalizes plurality point)

The yolk

- Yolk is smallest radius ball intersecting all median lines
- Motivation: voting theory (generalizes plurality point)

yolk $\neq$ extremal yolk [Stone and Tovey, 1992]



Previous work & our results

Exact:

- $O_d(n^{(d+1)^2})$ for yolk (improved to $O(n^4)$ in $\mathbb{R}^2$) [Tovey, 1992]

Previous work & our results

Exact:

- $O_d(n^{(d+1)^2})$ for yolk (improved to $O(n^4)$ in $\mathbb{R}^2$) [Tovey, 1992]
- $O(n^{4/3} \log^{O(1)} n)$ in $\mathbb{R}^2$ for yolk and extremal yolk [Berg, Gudmundsson, et al., 2018]

Previous work & our results

Exact:

- $O_d(n^{(d+1)^2})$ for yolk (improved to $O(n^4)$ in $\mathbb{R}^2$) [Tovey, 1992]
- $O(n^{4/3} \log^{O(1)} n)$ in $\mathbb{R}^2$ for yolk and extremal yolk [Berg, Gudmundsson, et al., 2018]

$(1+\varepsilon)$-apx in $\mathbb{R}^2$:

Previous work & our results

Exact:

- $O_d(n^{(d+1)^2})$ for yolk (improved to $O(n^4)$ in $\mathbb{R}^2$) [Tovey, 1992]
- $O(n^{4/3} \log^{O(1)} n)$ in $\mathbb{R}^2$ for yolk and extremal yolk [Berg, Gudmundsson, et al., 2018]

$(1+\varepsilon)$-apx in $\mathbb{R}^2$:

- $O(n \log^7 n \log^4(1/\varepsilon))$ for yolk [Gudmundsson and Wong, 2019]

Previous work & our results

Exact:

- $O_d(n^{(d+1)^2})$ for yolk (improved to $O(n^4)$ in $\mathbb{R}^2$) [Tovey, 1992]
- $O(n^{4/3}\log^{O(1)} n)$ in $\mathbb{R}^2$ for yolk and extremal yolk [Berg, Gudmundsson, et al., 2018]

$(1+\varepsilon)$-apx in $\mathbb{R}^2$:

- $O(n\log^7 n\log^4(1/\varepsilon))$ for yolk [Gudmundsson and Wong, 2019]
- $O((n/\varepsilon^3)\log^3 n)$ for extremal yolk [Berg, Chung, et al., 2019]

Previous work & our results

Exact:

- $O_d(n^{(d+1)^2})$ for yolk (improved to $O(n^4)$ in $\mathbb{R}^2$) [Tovey, 1992]
- $O(n^{4/3} \log^{O(1)} n)$ in $\mathbb{R}^2$ for yolk and extremal yolk [Berg, Gudmundsson, et al., 2018]

$(1+\varepsilon)$-apx in $\mathbb{R}^2$:

- $O(n \log^7 n \log^4(1/\varepsilon))$ for yolk [Gudmundsson and Wong, 2019]
- $O((n/\varepsilon^3) \log^3 n)$ for extremal yolk [Berg, Chung, et al., 2019]

Our result

In $\mathbb{R}^2$: $O(n \log n)$ expected time for yolk/extremal yolk

Previous work & our results

Exact:

- $O_d(n^{(d+1)^2})$ for yolk (improved to $O(n^4)$ in $\mathbb{R}^2$) [Tovey, 1992]
- $O(n^{4/3} \log^{O(1)} n)$ in $\mathbb{R}^2$ for yolk and extremal yolk [Berg, Gudmundsson, et al., 2018]

$(1+\varepsilon)$-apx in $\mathbb{R}^2$:

- $O(n \log^7 n \log^4(1/\varepsilon))$ for yolk [Gudmundsson and Wong, 2019]
- $O((n/\varepsilon^3) \log^3 n)$ for extremal yolk [Berg, Chung, et al., 2019]

Our result

In $\mathbb{R}^2$: $O(n \log n)$ expected time for yolk/extremal yolk, and $O_d(n^{d-1} \log n)$ for $\mathbb{R}^d$

Implicit LPs

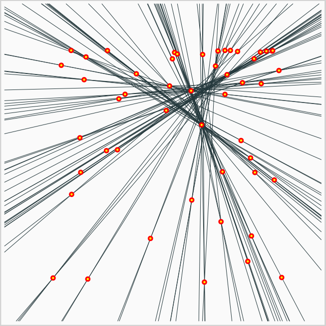
- Extremal median lines induced by 2 points of P

  $\implies \leq \binom{n}{2}$ implicit constraints

Implicit LPs

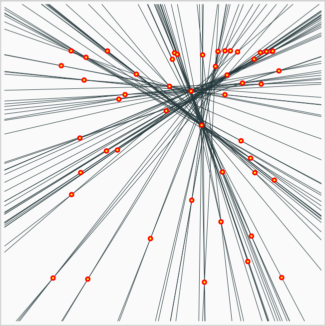
- Extremal median lines induced by 2 points of P

  $\implies \leq \binom{n}{2}$ implicit constraints



Implicit LPs

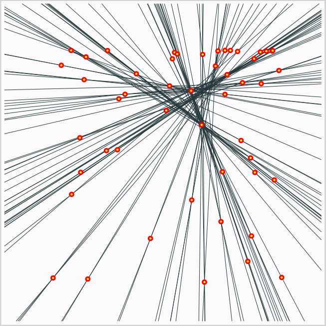
- Extremal median lines induced by 2 points of P
  $\implies \leq \binom{n}{2}$ implicit constraints



- Solve LP in $O(n^2)$ time

Implicit LPs

- Extremal median lines induced by 2 points of P
  $\implies \leq \binom{n}{2}$ implicit constraints



- Solve LP in $O(n^2)$ time
- Faster?

Solving the implicit LP

- Uses Chan's framework for implicit LPs [Chan, 2004]

Solving the implicit LP

- Uses Chan's framework for implicit LPs [Chan, 2004]
- P: a set of points; $n = |P|$

Solving the implicit LP

- Uses Chan's framework for implicit LPs [Chan, 2004]
- P: a set of points; $n = |P|$
- $L(P)$: extremal median lines induced by P

Solving the implicit LP

- Uses Chan's framework for implicit LPs [Chan, 2004]
- P: a set of points; $n = |P|$
- $L(P)$: extremal median lines induced by P
- **Goal:** compute a solution for $L(P)$

Solving the implicit LP

- Uses Chan's framework for implicit LPs [Chan, 2004]
- P: a set of points; $n = |P|$
- $L(P)$: extremal median lines induced by P
- **Goal:** compute a solution for $L(P)$

For any $Q \subseteq P$, $m = |Q|$, assume in $D(m)$ time,

Solving the implicit LP

- Uses Chan's framework for implicit LPs [Chan, 2004]
- P: a set of points; $n = |P|$
- $L(P)$: extremal median lines induced by P
- **Goal:** compute a solution for $L(P)$

For any $Q \subseteq P$, $m = |Q|$, assume in $D(m)$ time,

(a) Given Q, candidate ball b: does b violate $L(Q)$?

Solving the implicit LP

- Uses Chan's framework for implicit LPs [Chan, 2004]
- P: a set of points; $n = |P|$
- $L(P)$: extremal median lines induced by P
- **Goal:** compute a solution for $L(P)$

For any $Q \subseteq P$, $m = |Q|$, assume in $D(m)$ time,

(a) Given Q, candidate ball b: does b violate $L(Q)$?

(b) Given Q: construct $O(1)$ sets $Q_1, \ldots, Q_r$, $|Q_i| \leq \alpha m$, $L(Q) = \cup_{i=1}^{r} L(Q_i)$

Solving the implicit LP

- Uses Chan's framework for implicit LPs [Chan, 2004]
- P: a set of points; $n = |P|$
- $L(P)$: extremal median lines induced by P
- **Goal:** compute a solution for $L(P)$

For any $Q \subseteq P$, $m = |Q|$, assume in $D(m)$ time,

(a) Given Q, candidate ball b: does b violate $L(Q)$?

(b) Given Q: construct $O(1)$ sets $Q_1, \ldots, Q_r$, $|Q_i| \leq \alpha m$, $L(Q) = \cup_{i=1}^{r} L(Q_i)$

Result (using [Chan, 2004])

Given P, can compute extremal yolk in $O(D(n))$ expected time.

Solving the implicit LP

- Uses Chan's framework for implicit LPs [Chan, 2004]
- P: a set of points; $n = |P|$
- $L(P)$: extremal median lines induced by P
- **Goal:** compute a solution for $L(P)$

For any $Q \subseteq P$, $m = |Q|$, assume in $D(m)$ time,

(a) Given Q, candidate ball b: does b violate $L(Q)$?

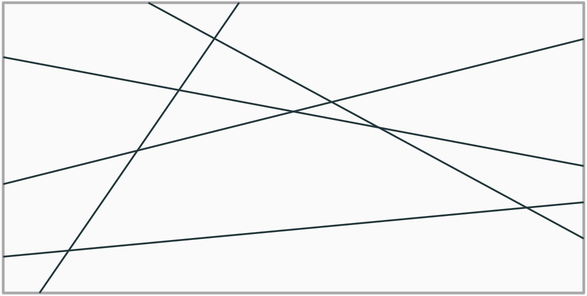
(b) Given Q: construct $O(1)$ sets $Q_1, \ldots, Q_r$, $|Q_i| \leq \alpha m$, $L(Q) = \cup_{i=1}^{r} L(Q_i)$

Result (using [Chan, 2004])

Given P, can compute extremal yolk in $O(D(n))$ expected time.

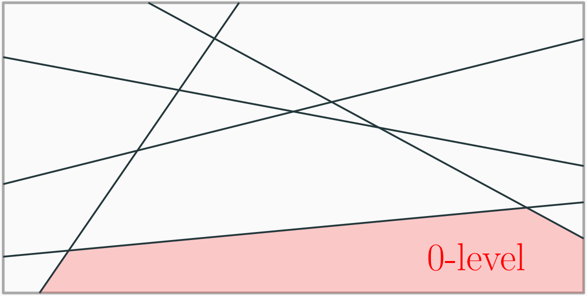
Definition: k-levels

Given lines L, k-level $=$ {points lying above or on k lines of L}.



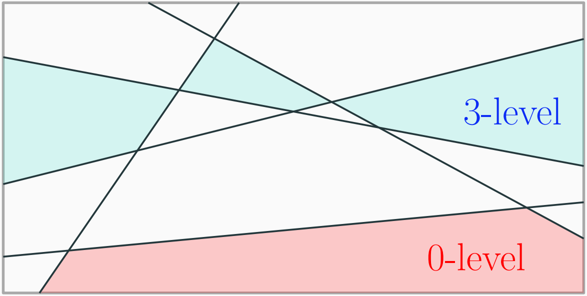
Definition: k-levels

Given lines L, k-level $=$ {points lying above or on k lines of L}.



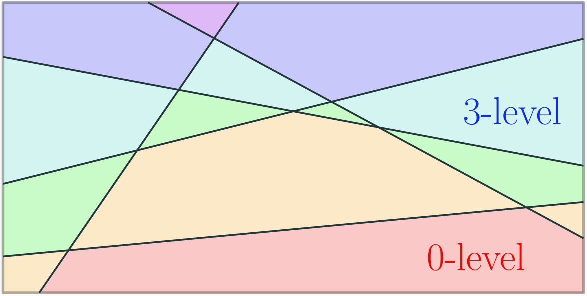
Definition: k-levels

Given lines L, k-level $= \{$points lying above or on k lines of $L\}$.



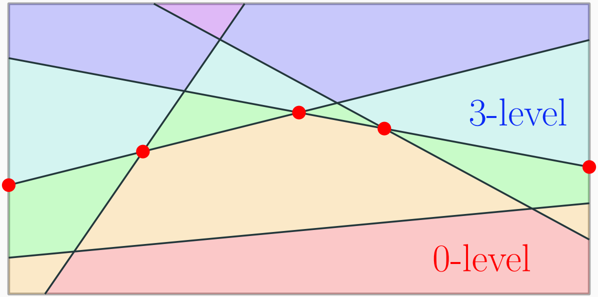
Definition: k-levels

Given lines L, k-level $=$ {points lying above or on k lines of L}.



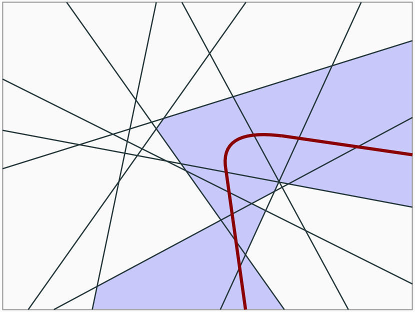
Definition: k-levels

Given lines L, k-level $=$ {points lying above or on k lines of L}.



Definition: Zone of a surface

Given lines L, curve γ, the zone $\mathcal{Z}(\gamma, L)$ are cells of $\mathcal{A}(L)$ intersecting γ.



Lemma [Aronov et al., 1993, Berg, Dobrindt, et al., 1995]

$\mathcal{Z}(\gamma, L)$ can be computed in $O(n \log n)$ expected time.

Computing the extremal yolk

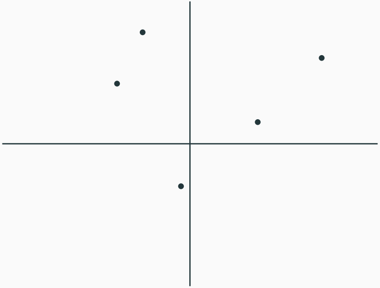
- P points, $L(P)$ extremal median lines

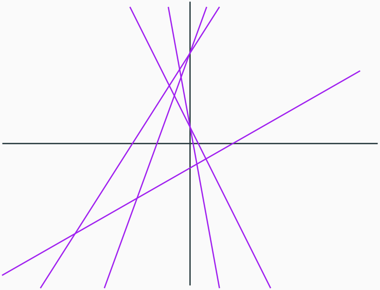
Computing the extremal yolk

- P points, $L(P)$ extremal median lines
- In the dual:

Computing the extremal yolk

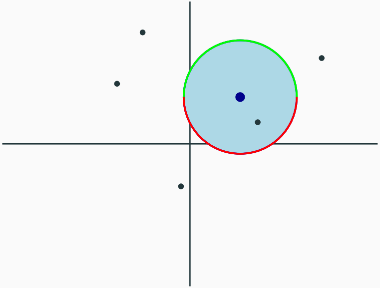
- P points, $L(P)$ extremal median lines
- In the dual:
  - Points $\iff$ lines

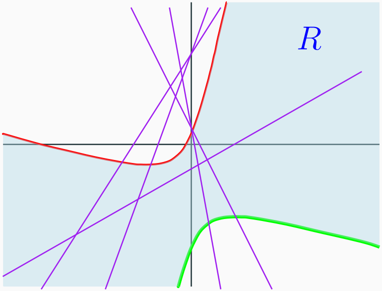




Computing the extremal yolk

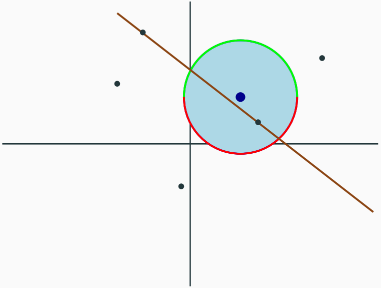
- P points, $L(P)$ extremal median lines
- In the dual:
  - Points $\iff$ lines
  - Ball $\iff$ region R between hyperbola

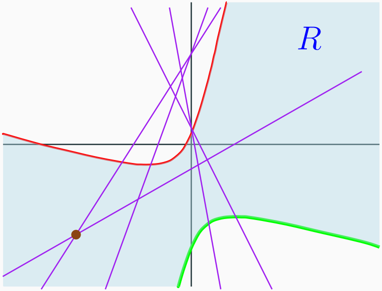




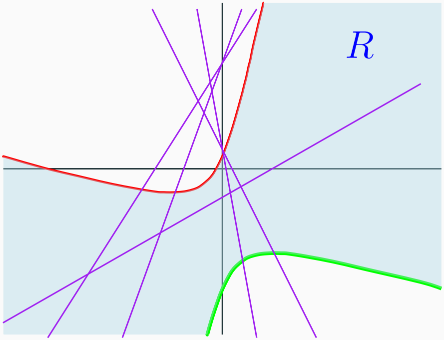
Computing the extremal yolk

- P points, $L(P)$ extremal median lines
- In the dual:
  - Points $\iff$ lines
  - Ball $\iff$ region R between hyperbola
  - Extremal median line $\iff$ vertex of the $n/2$-level

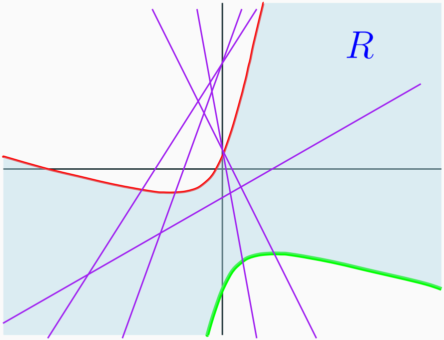




The key subproblem



Is there a vertex of the $n/2$-level outside R?

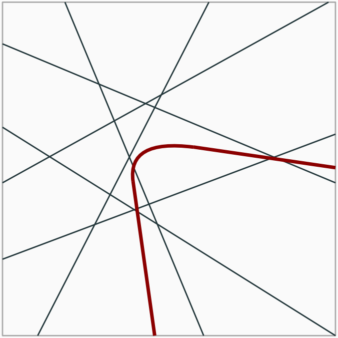


Is there a vertex of the $n/2$-level outside R?

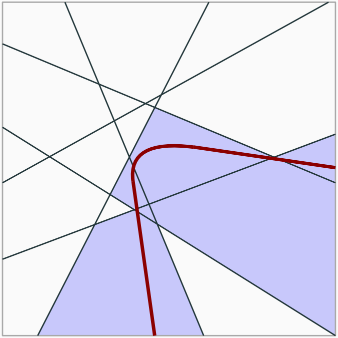
Check vertices of $\mathcal{A}(L(P))$ near boundary of R!

The key subproblem

Task: Is there a vertex of $n/2$-level in $\mathbb{R}^2 \setminus R$?



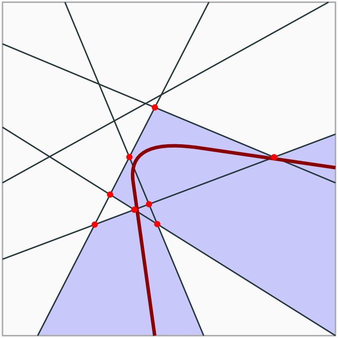
Task: Is there a vertex of $n/2$-level in $\mathbb{R}^2 \setminus R$?



Idea: Compute zone of ∂R

The key subproblem

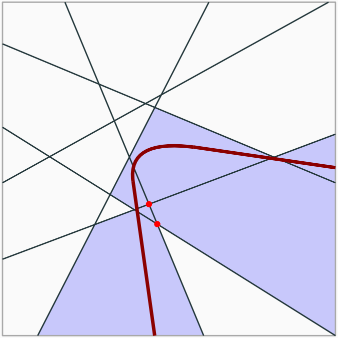
Task: Is there a vertex of $n/2$-level in $\mathbb{R}^2 \setminus R$?



Idea: Compute zone of ∂R, walk around vertices in zone

The key subproblem

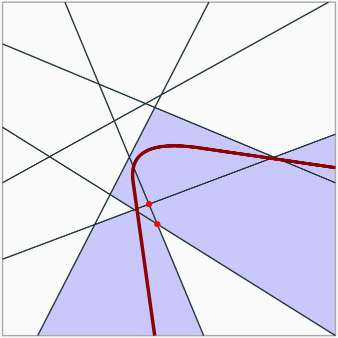
Task: Is there a vertex of $n/2$-level in $\mathbb{R}^2 \setminus R$?



Idea: Compute zone of ∂R, walk around vertices in zone, check if any vertex in $\mathbb{R}^2 \setminus R$ has level $n/2$.

The key subproblem

Task: Is there a vertex of $n/2$-level in $\mathbb{R}^2 \setminus R$?



Idea: Compute zone of ∂R, walk around vertices in zone, check if any vertex in $\mathbb{R}^2 \setminus R$ has level $n/2$.

Result: Decider takes $O(n \log n)$ time.

Solving the implicit LP

- P: a set of points; $n = |P|$
- $L(P)$: extremal median lines induced by P
- Goal: compute a solution for $L(P)$

For any $Q \subseteq P$, $m = |Q|$, assume in $D(m)$ time,

(a) Given Q, candidate ball b: does b violate $L(Q)$?

(b) Given Q: construct $O(1)$ sets $Q_1, \ldots, Q_r$, $|Q_i| \leq \alpha m$, $L(Q) = \cup_{i=1}^{r} L(Q_i)$

Solving the implicit LP

- P: a set of points; $n = |P|$
- $L(P)$: extremal median lines induced by P
- Goal: compute a solution for $L(P)$

For any $Q \subseteq P$, $m = |Q|$, assume in $D(m)$ time,

(a) Given Q, candidate ball b: does b violate $L(Q)$? ✓

(b) Given Q: construct $O(1)$ sets $Q_1, \ldots, Q_r$, $|Q_i| \leq \alpha m$, $L(Q) = \cup_{i=1}^{r} L(Q_i)$

Solving the implicit LP

- P: a set of points; $n = |P|$
- $L(P)$: extremal median lines induced by P
- Goal: compute a solution for $L(P)$

For any $Q \subseteq P$, $m = |Q|$, assume in $D(m)$ time,

(a) Given Q, candidate ball b: does b violate $L(Q)$? ✓

(b) Given Q: construct $O(1)$ sets $Q_1, \ldots, Q_r$, $|Q_i| \leq \alpha m$, $H(Q) = \cup_{i=1}^{r} H(Q_i)$ ✓

Solving the implicit LP

- P: a set of points; $n = |P|$
- $L(P)$: extremal median lines induced by P
- Goal: compute a solution for $L(P)$

For any $Q \subseteq P$, $m = |Q|$, assume in $D(m)$ time,

(a) Given Q, candidate ball b: does b violate $L(Q)$? ✓

(b) Given Q: construct $O(1)$ sets $Q_1, \ldots, Q_r$, $|Q_i| \leq \alpha m$, $H(Q) = \cup_{i=1}^{r} H(Q_i)$ ✓

Our result

$D(m) = O_d(m^{d-1} \log m) \implies$

extremal yolk in $O_d(n^{d-1} \log n)$ expected time.

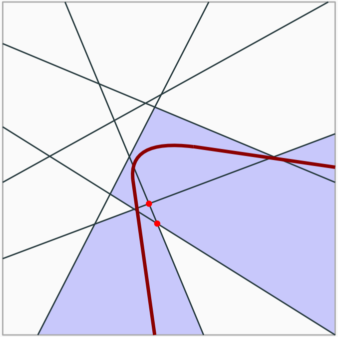
Other applications I

- Yolk: Smallest ball intersecting all median hyperplanes

Other applications I

- Yolk: Smallest ball intersecting all median hyperplanes
- Egg: Smallest ball intersecting all extremal hyperplanes

- Yolk: Smallest ball intersecting all median hyperplanes
- Egg: Smallest ball intersecting all extremal hyperplanes
  - Easy modification! Check if any vertex of $\mathcal{A}(L(P))$ lies outside R



$\implies$ $O_d(n^{d-1} \log n)$ expected time

Other applications II

- $H_k(P) = \{\text{open halfspaces containing } > n-k \text{ points of } P\}$
- $\mathcal{T}_k = \bigcap\{h \mid h \in H_k(P)\}$

- $H_k(P) = \{$open halfspaces containing $> n - k$ points of $P\}$
- $\mathcal{T}_k = \bigcap \{h \mid h \in H_k(P)\}$
- E.g., $\mathcal{T}_1 = \mathrm{conv}(P)$, $\mathcal{T}_{n/(d+1)}$ contains centerpoint of P

Other applications II

- $H_k(P) = \{$open halfspaces containing $> n - k$ points of $P\}$
- $\mathcal{T}_k = \bigcap\{h \mid h \in H_k(P)\}$
- E.g., $\mathcal{T}_1 = \text{conv}(P)$, $\mathcal{T}_{n/(d+1)}$ contains centerpoint of P
- If $\mathcal{T}_k \neq \varnothing$, center ball = largest ball inside $\mathcal{T}_k$

Other applications II

- $H_k(P) = \{$open halfspaces containing $> n - k$ points of $P\}$
- $\mathcal{T}_k = \bigcap\{h \mid h \in H_k(P)\}$
- E.g., $\mathcal{T}_1 = \mathrm{conv}(P)$, $\mathcal{T}_{n/(d+1)}$ contains centerpoint of P
- If $\mathcal{T}_k \neq \varnothing$, center ball = largest ball inside $\mathcal{T}_k$
- If $\mathcal{T}_k = \varnothing$, Tukey ball = smallest ball intersecting all halfspaces $H_k(P)$

Other applications II

- $H_k(P) = \{$open halfspaces containing $> n - k$ points of $P\}$
- $\mathcal{T}_k = \bigcap\{h \mid h \in H_k(P)\}$
- E.g., $\mathcal{T}_1 = \mathrm{conv}(P)$, $\mathcal{T}_{n/(d+1)}$ contains centerpoint of P
- If $\mathcal{T}_k \neq \varnothing$, center ball = largest ball inside $\mathcal{T}_k$
- If $\mathcal{T}_k = \varnothing$, Tukey ball = smallest ball intersecting all halfspaces $H_k(P)$

Our result

Center ball and Tukey ball can be computed in $O_d(n^{d-1} \log n)$ expected time

Other applications II

- $H_k(P) = \{$open halfspaces containing $> n - k$ points of $P\}$
- $\mathcal{T}_k = \bigcap\{h \mid h \in H_k(P)\}$
- E.g., $\mathcal{T}_1 = \text{conv}(P)$, $\mathcal{T}_{n/(d+1)}$ contains centerpoint of P
- If $\mathcal{T}_k \neq \varnothing$, center ball = largest ball inside $\mathcal{T}_k$
- If $\mathcal{T}_k = \varnothing$, Tukey ball = smallest ball intersecting all halfspaces $H_k(P)$

Our result

Center ball and Tukey ball can be computed in $O_d(n^{d-1} \log n)$ expected time

Thank you!

References i

Richard E. Stone and Craig A. Tovey. *Limiting median lines do not suffice to determine the yolk*. *Social Choice and Welfare*, 9(1): 33–35, 1992.

Craig A. Tovey. *A polynomial-time algorithm for computing the yolk in fixed dimension*. *Math. Program.*, 57: 259–277, 1992.

Mark de Berg, Joachim Gudmundsson, and Mehran Mehr. *Faster algorithms for computing plurality points*. *ACM Trans. Algorithms*, 14(3): 36:1–36:23, 2018.

Joachim Gudmundsson and Sampson Wong. *Computing the yolk in spatial voting games without computing median lines*. *33th Conf. Artificial Intell.* (AAAI),

References ii

Mark de Berg, Jonathan Chung, and Joachim Gudmundsson. *Computing the Yolk in Spatial Voting Games*. 2019.

Timothy M. Chan. *An optimal randomized algorithm for maximum Tukey depth*. *Proc. 15th ACM-SIAM Sympos. Discrete Algs.* (SODA), 430–436, 2004.

Boris Aronov, Marco Pellegrini, and Micha Sharir. *On the zone of a surface in a hyperplane arrangement*. *Discrete Comput. Geom.*, 9: 177–186, 1993.

Mark de Berg, Katrin Dobrindt, and Otfried Schwarzkopf. *On lazy randomized incremental construction*. *Discrete Comput. Geom.*, 14(3): 261–286, 1995.