A Proof of Dudley's Convex Approximation

Sariel Har-Peled* Mitchell Jones†

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Abstract

We provide a self contained proof of a result of Dudley [Dud74], which shows that a bounded convex-body in $\mathbb{R}^d$ can be ε-approximated, by the intersection of $O_d\bigl(\varepsilon^{-(d-1)/2}\bigr)$ halfspaces, where O_d hides constants that depends on d.

1. Statement and proof

For a convex body $\mathsf{C} \subseteq \mathbb{R}^d$, let $\mathsf{C}_{\oplus\varepsilon}$ denote the set of all points in $\mathbb{R}^d$ in distance at most $\leq \varepsilon$ from C. In particular, $\mathsf{C} \subseteq \mathsf{C}_{\oplus\varepsilon}$, and the Hausdorff distance between C and $\mathsf{C}_{\oplus\varepsilon}$ is ε.

Theorem 1.1 ([Dud74]). *Let C be a (closed) convex body in $\mathbb{R}^d$, containing the unit ball of radius one centered at the origin, such that C is contained in a ball of radius d centered at the origin. For a parameter $\varepsilon > 0$, one can compute a convex body D, which is the intersection of $O_d(1/\varepsilon^{(d-1)/2})$ halfspaces, such that $\mathsf{C} \subseteq D \subseteq \mathsf{C}_{\oplus\varepsilon}$.*

Proof: Let S be the sphere of radius $2d$ centered at the origin, and let Q be a maximal δ-packing of S, where $\delta = \sqrt{d\varepsilon/8}$. We remind the reader that a set $Q \subseteq S$ is a ***δ-packing***, if

(i) for any point $p \in S$, there is a point $q \in Q$, such that $\|p - q\| \leq \delta$, and
(ii) for any two points $q, q' \in Q$, we have that $\|q - q'\| \leq \delta$.

In particular, it is easy to verify that $|Q| = O_d((d/\delta)^{d-1}) = O_d(\varepsilon^{-(d-1)/2})$.

Next, for every point $q \in Q$, let $n(q)$ be its nearest neighbor in C (which naturally lies on $\partial\mathsf{C}$), and consider the halfspace that passes through $n(q)$, contains C, and is orthogonal to the vector $q - n(q)$. Let $h_{\mathsf{C}}(q)$ denote this halfspace. Let $D = \bigcap_{q \in Q} h_{\mathsf{C}}(q)$. We claim that D is the desired approximation.

First, it is clear that $\mathsf{C} \subseteq D$. As for the other direction, consider any point $p \in \partial\mathsf{C}$, and consider a normal to C at p, denoted by v. Consider the ray emanating from p in the direction of v. It hits S at a point p', and let $q \in Q$, be the nearest point in the packing Q to it. Next, consider $n(q)$. It is easy to verify that $\|p - n(q)\| \leq \|p' - q\| \leq \delta$ (because projecting to nearest-neighbor is a contraction).

*Department of Computer Science; University of Illinois; 201 N. Goodwin Avenue; Urbana, IL, 61801, USA; sariel@illinois.edu; http://sarielhp.org/. Work on this paper was partially supported by a NSF AF awards CCF-1421231, and CCF-1217462.

†Department of Computer Science; University of Illinois; 201 N. Goodwin Avenue; Urbana, IL, 61801, USA; mfjones2@illinois.edu; http://mfjones2.web.engr.illinois.edu/.

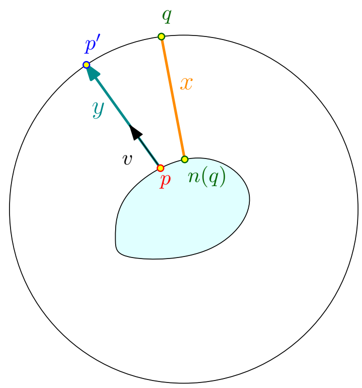
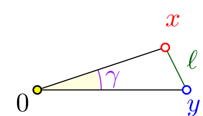


Figure 1.1

We are interested in angle between v and $n(q)-q$. To this end, observe that $\|p'-q\| \leq \delta$, $\|p-n(q)\| \leq \delta$, $\|p-p'\| \geq d$, and $\|q-n(q)\| \geq d$. Let $x = q-n(q)$ and $y = p'-p$. Notice that

$$\begin{aligned}\ell = \|x-y\| &= \Big\|q-n(q)-(p'-p)\Big\| = \Big\|(p-n(q))+(q-p')\Big\| \\ &\leq \|p-n(q)\| + \|q-p'\| \leq 2\delta.\end{aligned}$$

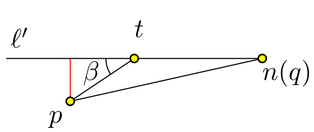


Observe that $\|x\| \leq \mathrm{radius}(S) = 2d$ and $\|y\| \leq 2d$. Let $\triangle$ be the triangle formed by the origin, x and y. The height of $\triangle$ at x is bounded by ℓ. As such, $\mathrm{area}(\triangle) \leq \frac{1}{2}\max(\|x\|, \|y\|)\mathrm{height}(\triangle) \leq d\mathrm{height}(\triangle) \leq d\ell \leq 2d\delta$.

Let γ denote the angle between x and y. We have that

$$\mathrm{area}(\triangle) = \frac{1}{2}\|x\|\,\|y\|\sin\gamma \leq 2d\delta \implies \sin\gamma \leq \frac{4\delta}{d}.$$

Let h be the three dimensional affine subspace that is spanned by the vectors $x, y, p-n(q)$, and passes through $n(q)$. Clearly, $p \in h$. Now, $H_q = h \cap h_{\mathsf{C}}(q)$ and $H_{p'} = h \cap h_{\mathsf{C}}(p')$ are two halfspaces contained in h. The angle between their bounding planes is exactly γ (as their normals are x and y). In particular, let $f \subseteq h$ be the two dimensional plane that contains the points $n(q), q, p$. Let ℓ be the line $\partial H_q \cap \partial H_{p'}$, and let t be the intersection of f with ℓ.



The distance of p from $\partial h_{\mathsf{C}}(q)$ bounds the distance of p from the boundary of D. This distance in turn is bounded by the distance from p to the line ℓ' spanned by $n(q)$ and t. Let β be the angle between ℓ' and pt (see figure). It is easy to verify that as f contains the vector x, this implies that $\beta \leq \gamma$. This in turn implies that $\angle tn(q)p \leq \beta \leq \gamma$. Using the packing property that $\delta \leq \sqrt{d\varepsilon/8}$, we have

$$\mathrm{dist}(p, \partial D) \leq \mathrm{dist}(p, \ell') \leq \|p-n(q)\|\sin\beta \leq \delta\sin\gamma \leq \delta\frac{4\delta}{d} \leq \frac{\varepsilon}{2}.$$

The distance of any point of $\partial\mathsf{C}_{\oplus\varepsilon}$ from C is at least $\varepsilon/2$. It follows that $D \subseteq \mathsf{C}_{\oplus\varepsilon}$. ■

References

[Dud74] R. M. Dudley. Metric entropy of some classes of sets with differentiable boundaries. *J. Approx. Theory*, 10(3):227–236, 1974.