Journey to the Center of the Point Set

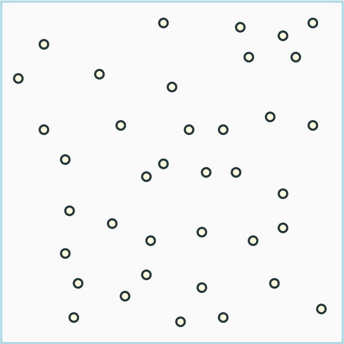
Sariel Har-Peled and Mitchell Jones (UIUC)

70th Midwest Theory Day, November 23rd, 2019

Definition: Centerpoints

$P \subset \mathbb{R}^d$: set of n points.

$c \in \mathbb{R}^d$ centerpoint for P if for every closed halfspace h^+:
$c \in h^+ \implies |P \cap h^+| \geqslant n/(d+1)$.



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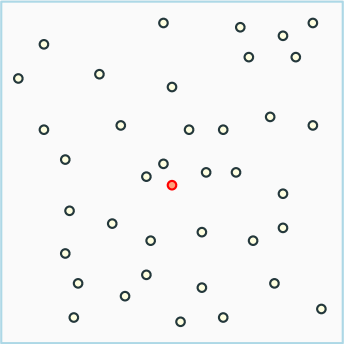

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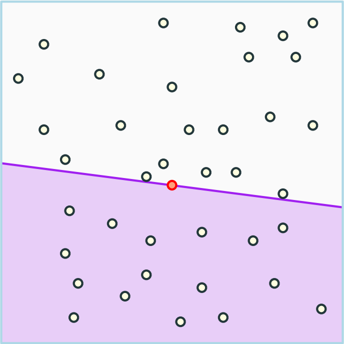

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- α-centerpoints for $\alpha \in (0, 1/(d+1)]$

Previous work: computing centerpoints

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- **Open:** $\approx 1/(d+1)$-centerpoint in $O(\mathrm{poly}(d))$ time?

A polynomial algorithm

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With random sampling, $\approx 1/(d+2)^2$-centerpoint in time $O(d^7 \log^3 d)$.

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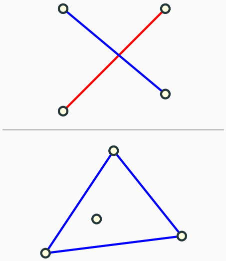
- Approximate centerpoint for $d + O(1)$ points in $\mathbb{R}^d$?
- Yes! Radon's Theorem.

A detour: Radon's Theorem

Radon's Theorem

$P \subset \mathbb{R}^d$: set of $d+2$ points.

$\exists$ partition $P = Q \sqcup R$ s.t. $\mathrm{conv}(Q) \cap \mathrm{conv}(R) \neq \varnothing$.



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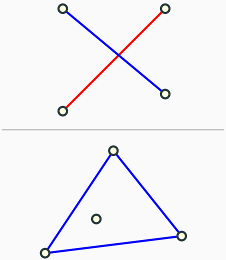
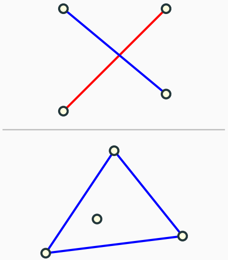


- Radon point: compute in $O(d^3)$ time.



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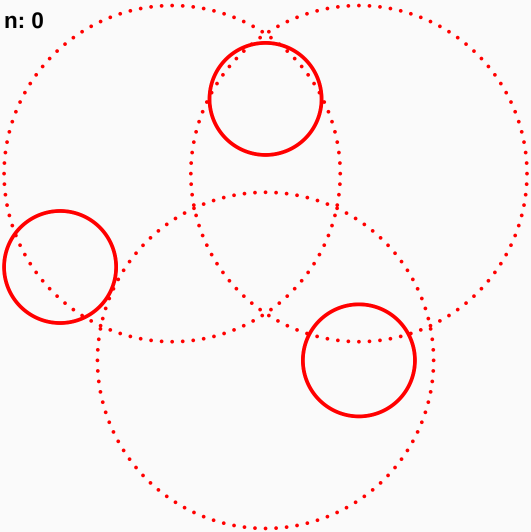
- Radon point: compute in $O(d^3)$ time.
- Radon point: $2/(d+2)$-centerpoint for P.

Our contribution

A simplified variant of [Clarkson, Eppstein, et al., 1996].

1. $Q \subseteq P$ sample of size $\approx O(d^3 \log d)$ [Li, Long, et al., 2001]
2. For $i = 1, \ldots, O(d|Q|)$:
   2.1 Sample $d + 2$ points of Q
   2.2 Compute their radon point r
   2.3 Add r to Q
   2.4 Delete a random point from Q (which isn't r)

Visualization



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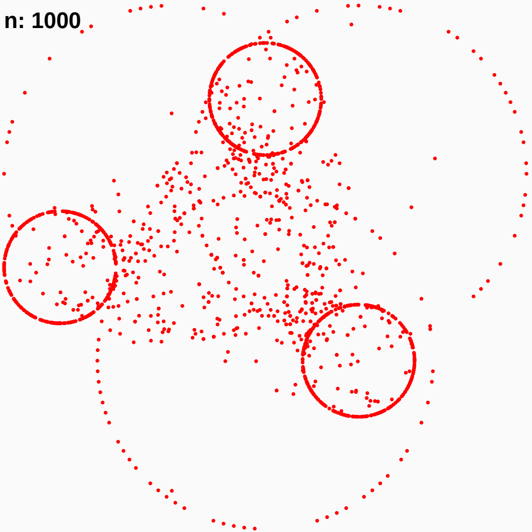

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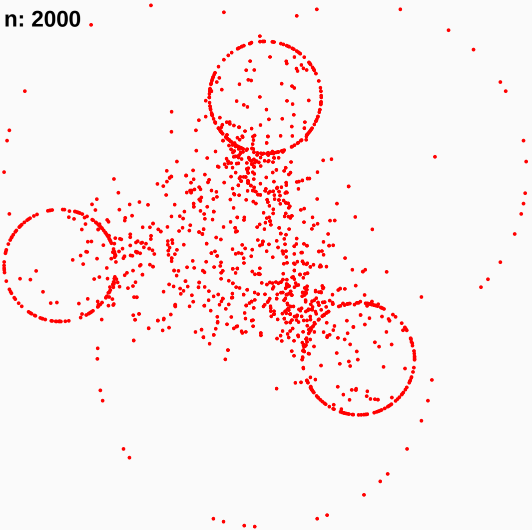

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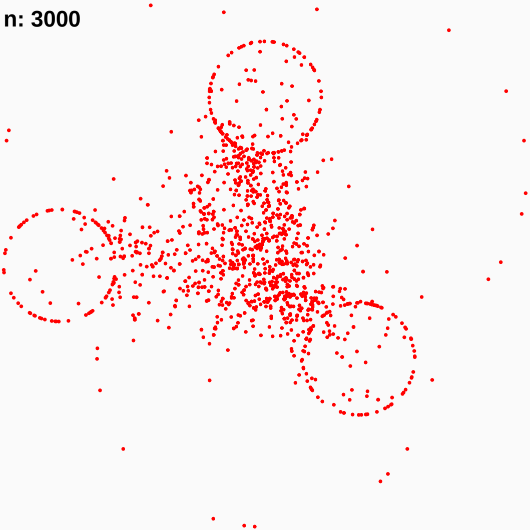

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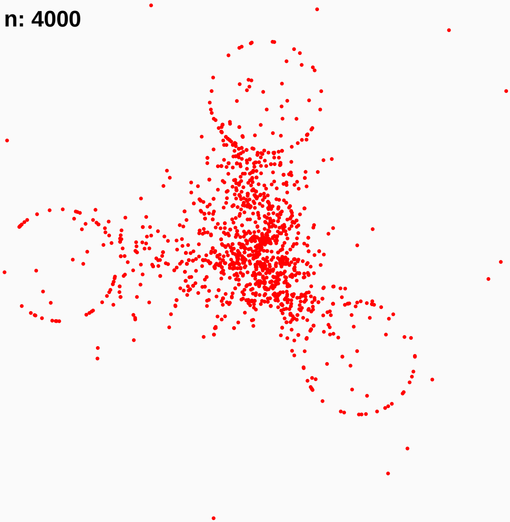

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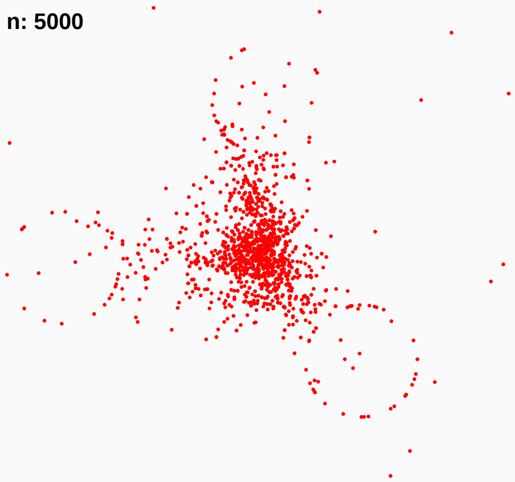

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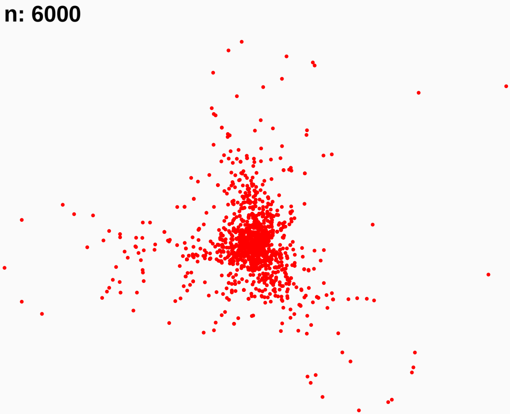

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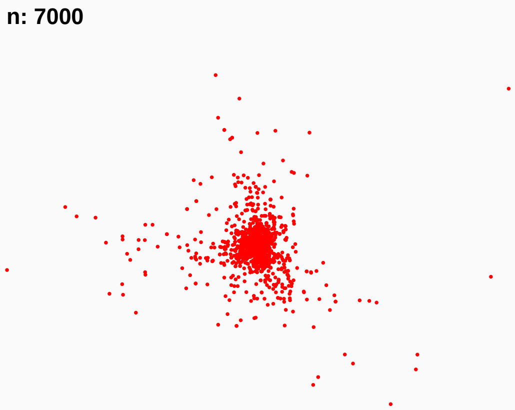

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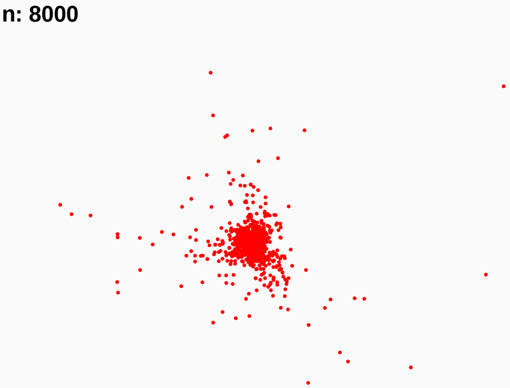

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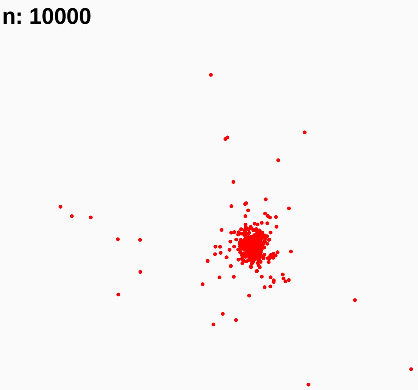

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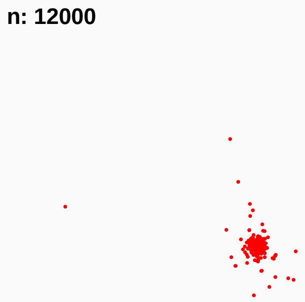

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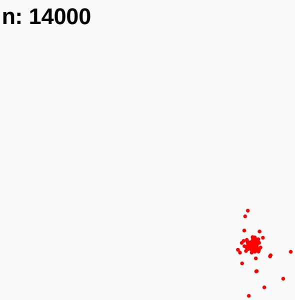

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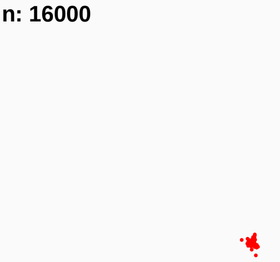


Visualization

n: 18000

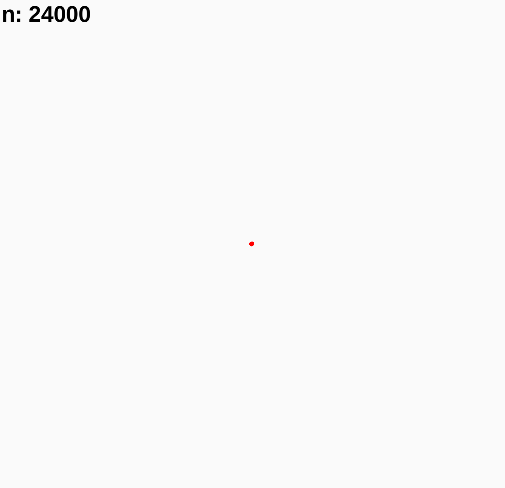
Visualization

n: 20000

Visualization

n: 22000

Visualization



Visualization

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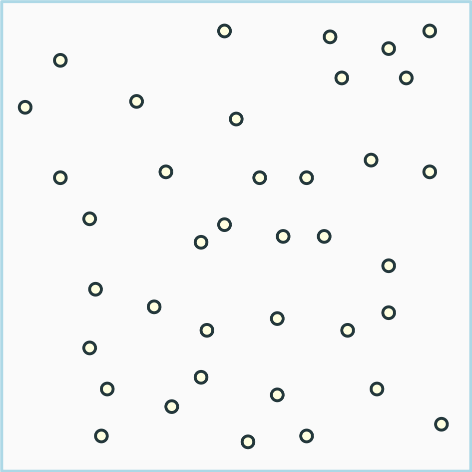
Why does this work?

q is an α-centerpoint for P $\iff$ all halfspaces h^+ with $|P \cap h^+| > (1-\alpha)|P|$ contain q



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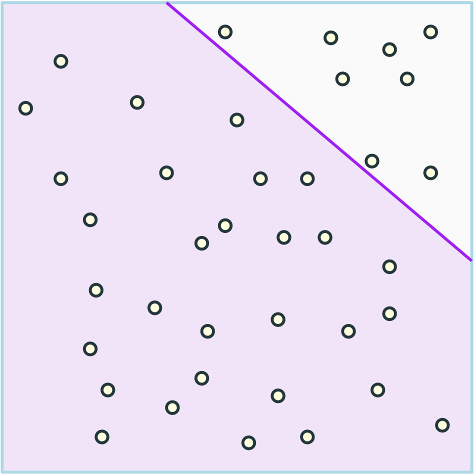

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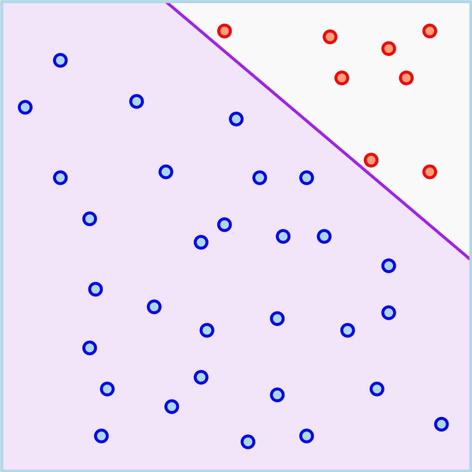

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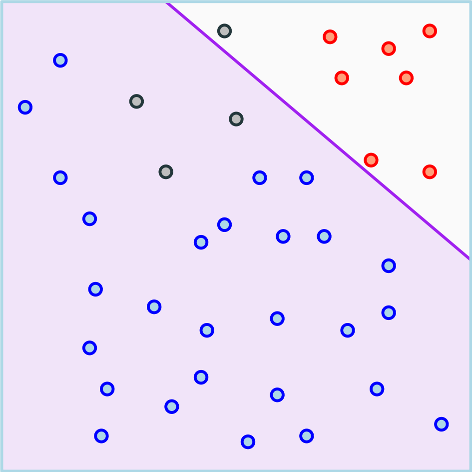

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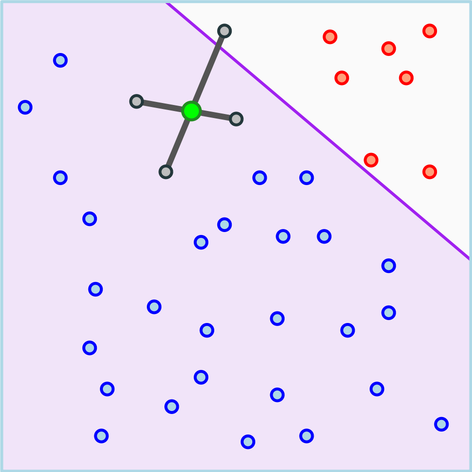

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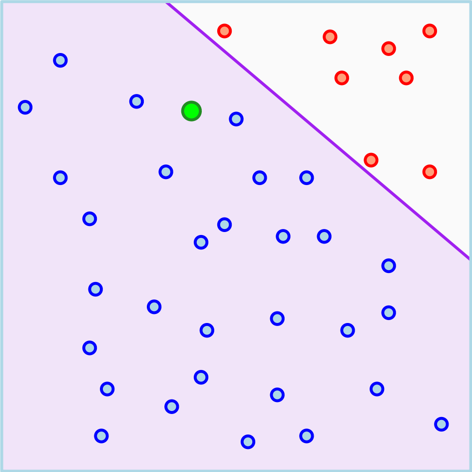

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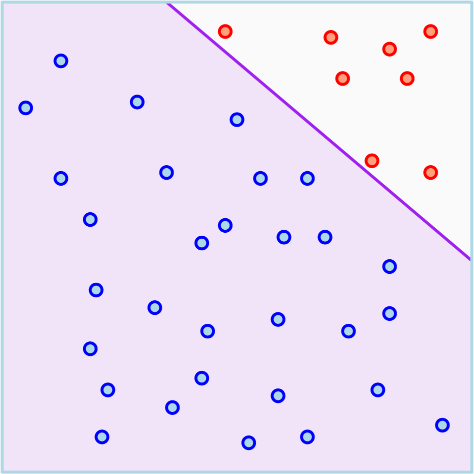

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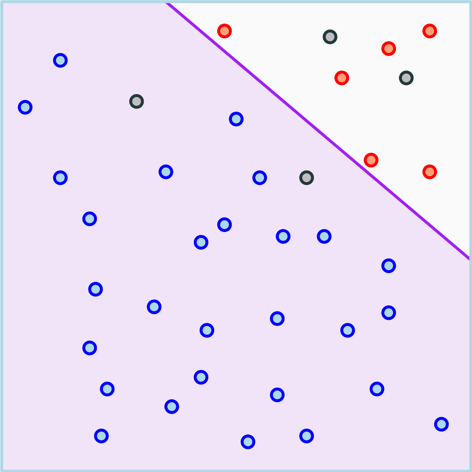

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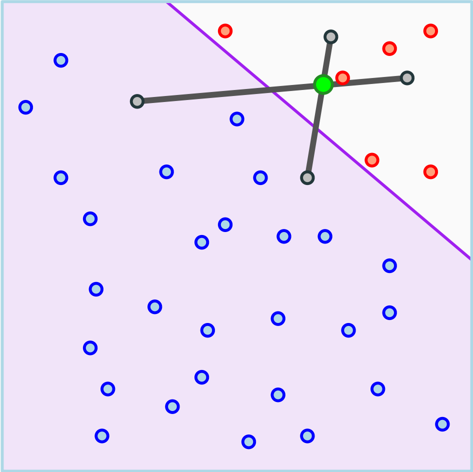
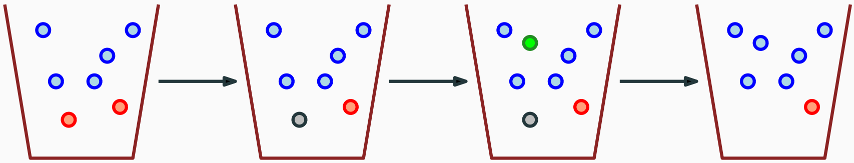


Radon's urn

- Urn with b blue balls, $r = m - b$ red balls.



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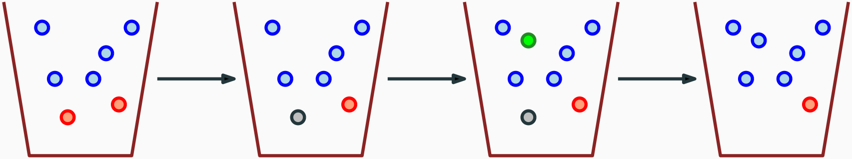

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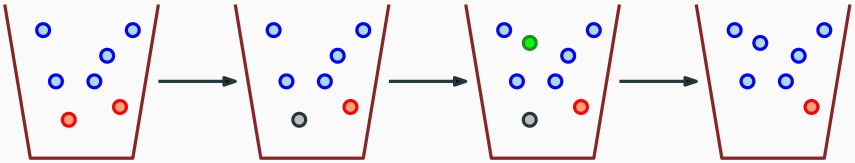

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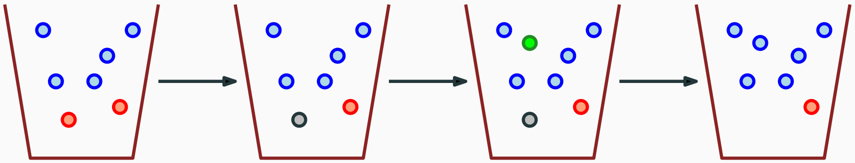

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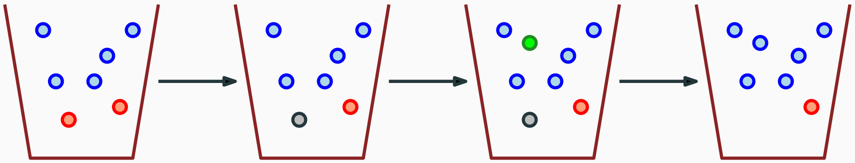

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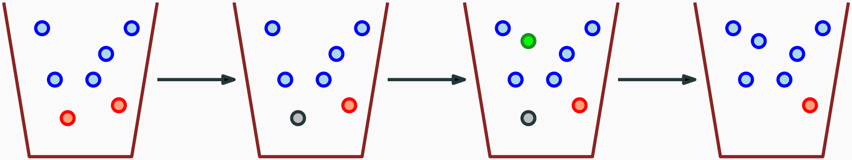
  4. Remove marked ball from urn.



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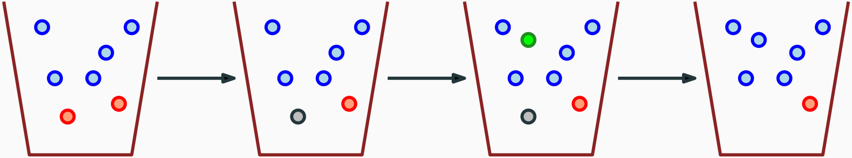
Random walk process



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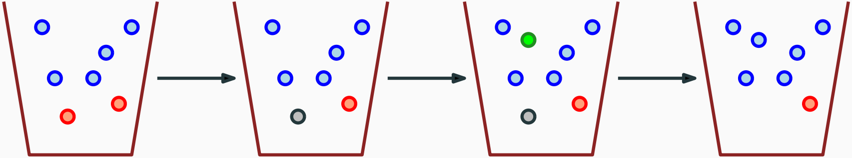

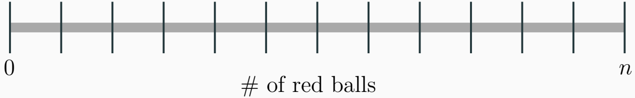

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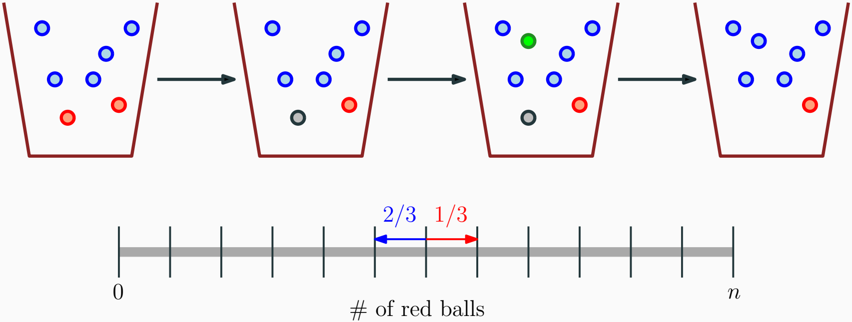

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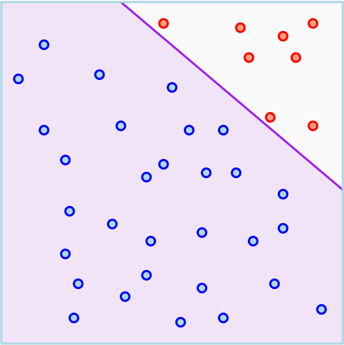

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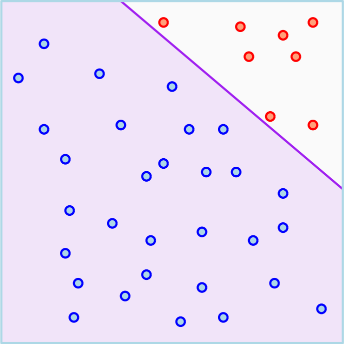

Our result

When # of balls m is sufficiently large: $O(m \log^2 m)$ rounds.

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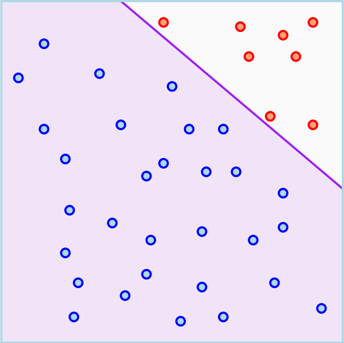
- Simulate random walk process in parallel for all $O(n^d)$ halfspaces.



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Problem

$\approx 1/d$-centerpoints in $O(\mathrm{poly}(d))$ time?

Application: Center nets

Definition: center nets

$P \subset \mathbb{R}^d$: set of n points.

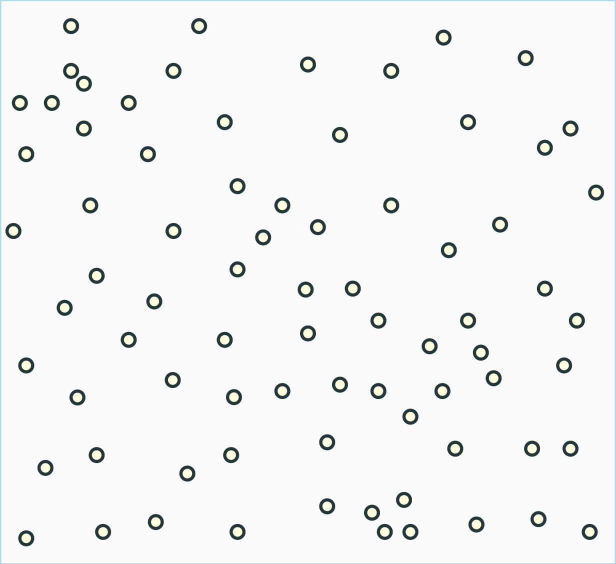
$Q \subset \mathbb{R}^d$, (ε, α)-center net if $\forall$ convex bodies $C \subseteq \mathbb{R}^d$:

$|P \cap C| \geqslant \varepsilon n \implies \exists q \in Q \cap C$, q an α-centerpoint of $P \cap C$.

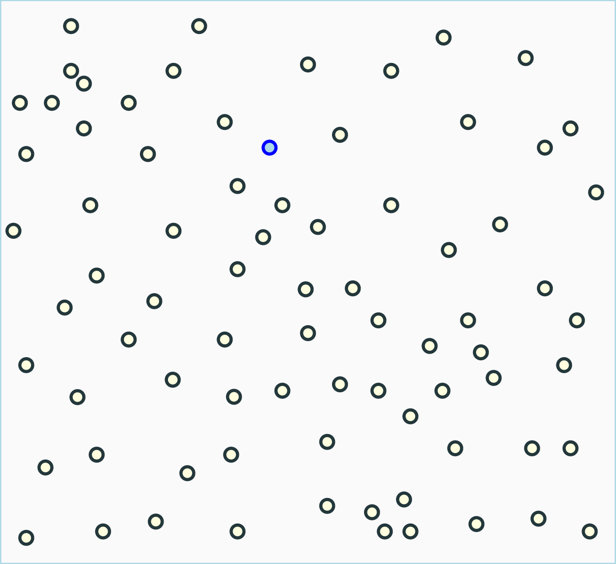
Our result

There exists an $\left(\varepsilon, \Omega\left(\frac{1}{d \log \varepsilon^{-1}}\right)\right)$-center net for P of size $\widetilde{O}\left((d^2/\varepsilon)^{d^2}\right)$.

Application: Functional nets



Application: Functional nets



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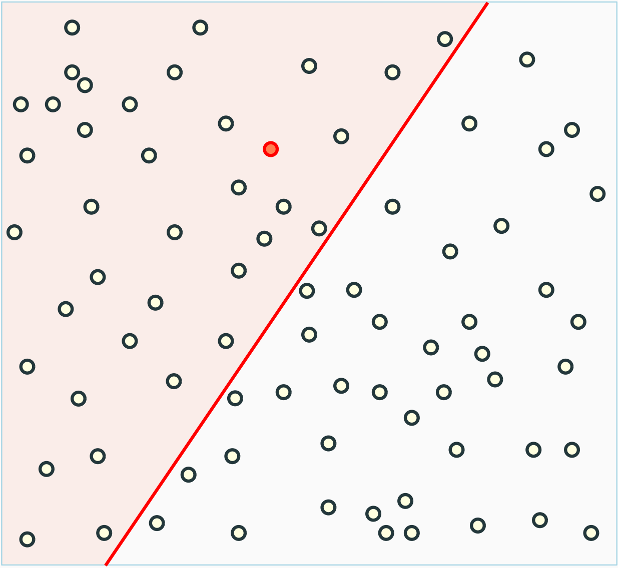

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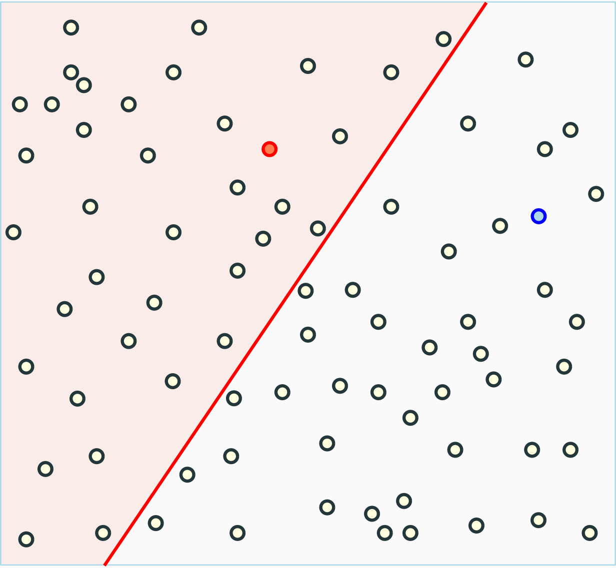

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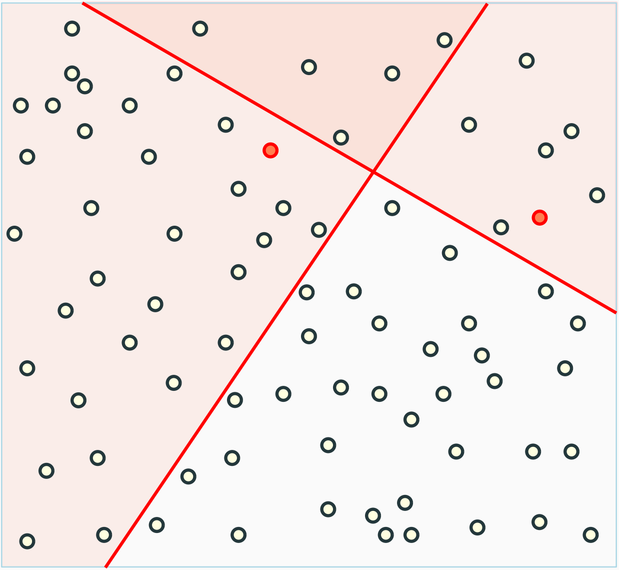

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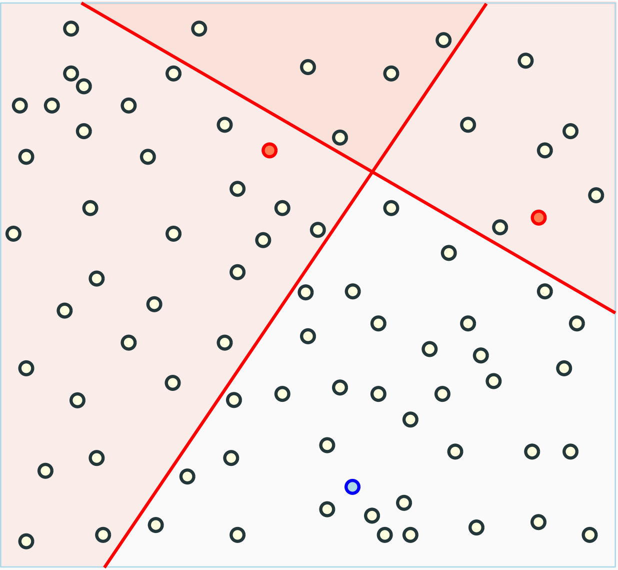

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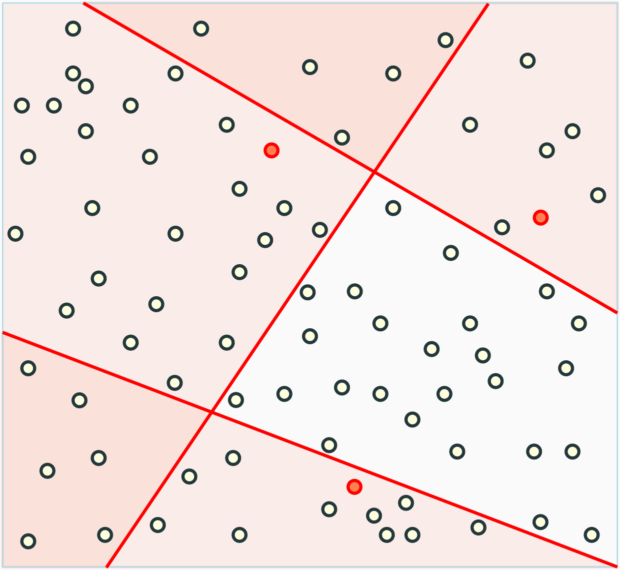

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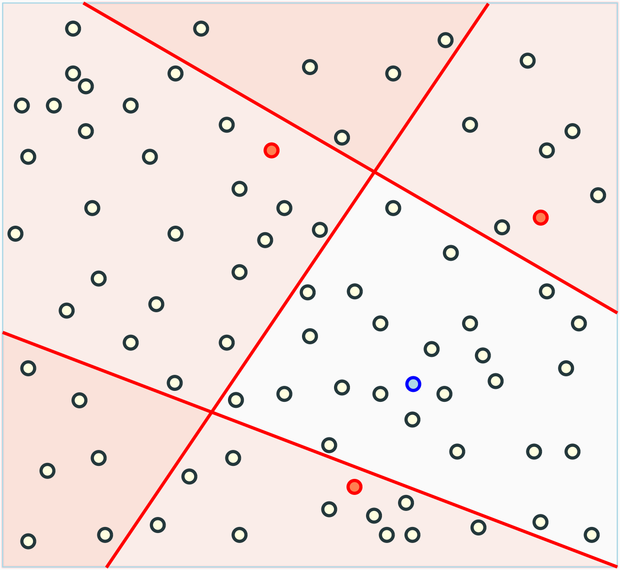

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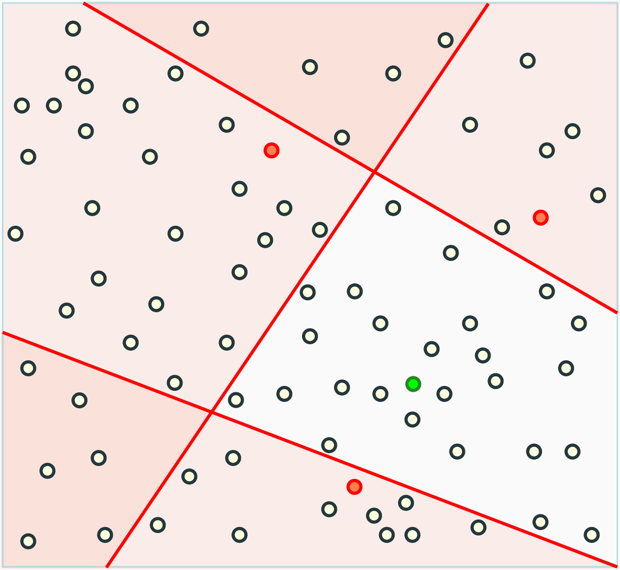

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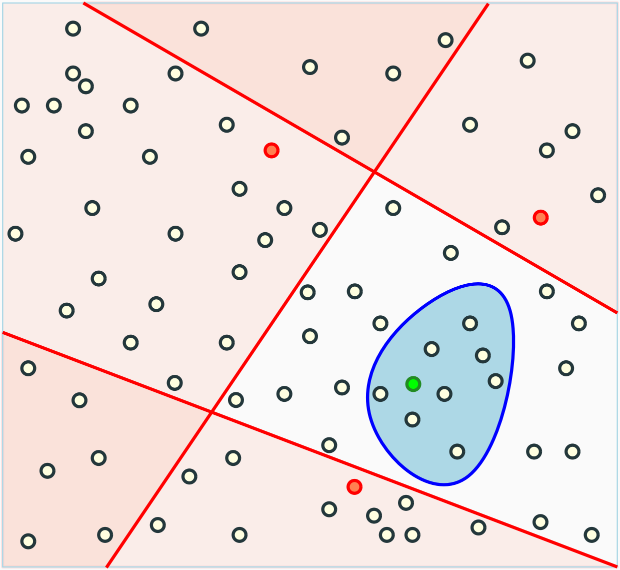

Application: Functional nets

Our result

Can verify if $|C \cap P| \leqslant \varepsilon n$ with $O(d^2 \log \varepsilon^{-1})$ oracle queries to C, in $\widetilde{O}(d^9/\varepsilon)$ randomized time.

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- Weak ε-nets in a different model
- Weak ε-nets have exponential dependency on d [Matoušek and Wagner, 2004] [Mustafa and Ray, 2008]
- What models can we obtain similar results with better dependency on d?

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