The first order wave system

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August 18, 2018

Abstract

We study the first order wave equation in order to analyse and improve the behaviour of finite volume methods applied to the Euler equations in low Mach number regimes.

1 Introduction

The term *wave equation* usually refers to the second order PDE $\partial_{tt}u+c^2\triangle u=0$, where u is the scalar variable propagating at velocity c. It is however always possible to rewrite the wave equation as a first order system by introducing the new variable $U=(v,\vec{w})=(\partial_t u,\vec{\nabla}u)$:

$$\partial_t\left(\begin{array}{c} v\\ \vec{w}\end{array}\right)+\left(\begin{array}{cc} 0 & c^2\nabla\cdot\\ \vec{\nabla} & 0\end{array}\right)\left(\begin{array}{c} v\\ \vec{w}\end{array}\right)=\left(\begin{array}{c} 0\\ \vec{0}\end{array}\right) \tag{1}$$

We are interested in the study of waves propagating in fluid media. For this reason we consider the following Euler equations :

$$\begin{array}{rcl}
\partial_t\rho & + & \nabla\cdot(\rho\vec{u})=0\\
\partial_t(\rho\vec{u}) & + & \nabla\cdot(\rho\vec{u}\otimes\vec{u})+\vec{\nabla}p=0\\
\partial_t(\rho(e+\dfrac{1}{2}||\vec{u}||^2)) & + & \nabla\cdot((\rho e+\dfrac{1}{2}\rho||\vec{u}||^2+p)\vec{u})=0
\end{array}$$

that should be complemented by a state law $p=f(\rho,e)$ synthetising the thermodynamic properties of the medium.

In order to identify the wave operator contained in the Euler equations, we rewrite the Euler system using the variables $\rho,\vec{u},e$ instead of the conservative variables $\rho,\rho\vec{u},\rho E$:

$$\begin{array}{ccccc}
\partial_t\rho & + & \vec{u}\cdot\vec{\nabla}\rho & + & \rho\nabla\cdot\vec{u}=0\\
\partial_t\vec{u} & + & (\vec{u}\cdot\nabla)\vec{u} & + & \frac{1}{\rho}\vec{\nabla}p=0\\
\partial_t e & + & \vec{u}\cdot\vec{\nabla}e & + & p\nabla\cdot\vec{u}=0.
\end{array}$$

Each equation of the new system is the sum of an advection term (with velocity $\vec{u}$) and an acoustic term.

In the case of an isentropic flow, we can neglect the last equation of the Euler system and focus on the coupling between velocity and pressure. In this case we use the simple equation of state $p = \rho c^2$ where c is the sound speed assumed to be constant.

$$\begin{array}{ccccc}\partial_t p & + & \vec{u}\cdot\vec{\nabla} p & + & p\nabla\cdot\vec{u} = 0\\ \partial_t \vec{u} & + & (\vec{u}\cdot\nabla)\vec{u} & + & \frac{1}{\rho}\vec{\nabla} p = 0.\end{array}$$

We now focus on the acoustic part of the system and neglect the advection terms :

$$\begin{array}{ccc}\partial_t p & + & p\nabla\cdot\vec{u} = 0\\ \partial_t \vec{u} & + & \frac{1}{\rho}\vec{\nabla} p = 0.\end{array}$$

If we linearise the above system around a state $p_0 = \rho_0 c^2$, we recover the first order wave system (1) with the variables $v = p$ and $\vec{w} = \rho_0\vec{u}$.

$$\begin{array}{ccc}\partial_t p & + & c^2\nabla\cdot(\rho_0\vec{u}) = 0\\ \partial_t(\rho_0\vec{u}) & + & \vec{\nabla} p = 0.\end{array} \tag{2}$$

In the stationary regime, we expect find solutions of the following equations

$$\begin{array}{c}c^2\nabla\cdot(\rho_0\vec{u}) = 0\\ \vec{\nabla} p = 0,\end{array} \tag{3}$$

That are constant pressure $p = \textit{constant}$ and divergence free velocity fields $\nabla\cdot\vec{u} = 0$.
The space $H(div_0)(\mathbb{R}^d)$ of weak divergence free vectors is the subspace of $(H^1(\mathbb{R}^d))^d$ containing vectors $\vec{u} = (u_1, \dots, u_d)$ having their components related by the single constraint

$$\sum_{k=1}^{d} \partial_k u_k = 0.$$

$H(div_0)(\mathbb{R}^d)$ is therefore a very large subspace of $(H^1(\mathbb{R}^d))^d$. However as we will see later, at the discrete level, numerical methods often capture stationary regimes that are a tiny subspace of $(H^1(\mathbb{R}^d))^d$. For instance, the upwind scheme can only capture stationary velocity vectors satisfying the d constraints

$$\partial_k u_k = 0, \qquad k = 1, \dots, d.$$

We can identify the vectors of $(H^1(\mathbb{R}^d))^d$ with their $d\times d$ jacobian matrix. The divergence free vector correspond to those jacobian matrices that have null trace. The discrete stationary state of the upwind scheme correspond to the jacobian matrices with null diagonal (d diagonal coefficients equal zero).

2 Spectrum of the wave system

A linear system of partial differential equations taking the form

$$\partial_t U + \sum_{i=1}^{d} A_i \partial_{x_i} U = 0,$$

is said to be hyperbolic if the jacobian matrix

$$A(\vec{n}) = \sum_{i=1}^{d} n_i A_i,$$

is diagonalisable with real eigenvalues for any vector $\vec{n} \in \mathbb{R}^d$.

In the case of the first order wave system, the jacobian matrix is the following $(d+1) \times (d+1)$ matrix

$$A_{wav}(\vec{n}) = \begin{pmatrix} 0 & c^2\,{}^t\vec{n} \\ \vec{n} & 0 \end{pmatrix}.$$

Whatever the value of $\vec{n} \neq 0$, A_{wav} has rank two and $d-1$ vectors in its kernel.

2.1 Eigenvalue 0

Let $\vec{\omega}_1, \dots, \vec{\omega}_{d-1}$ be an orthogonal completion of $\vec{n}$: $\vec{n}, \vec{\omega}_1, \dots, \vec{\omega}_{d-1}$ is an orthogonal basis of $\mathbb{R}^d$.
Let $V_1^0, \dots, V_{d-1}^0 \in \mathbb{R}^{d+1}$ be defined by

$$V_k^0 = \begin{pmatrix} 0 \\ \vec{\omega}_k \end{pmatrix}, \quad k = 1, \dots, d-1.$$

The vectors V_k^0 are left and right eigenvectors of A_{wav} :

$$A_{wav} V_k^0 = 0, \quad {}^t A_{wav} V_k^0 = 0, \quad k = 1, \dots, d-1.$$

We remark also that we have the following useful characterisation of the kernel of the wave system :

$$A_{wav} X = 0 \Leftrightarrow \begin{cases} X \cdot (0, \vec{n}) = 0 \\ X \cdot (1, \vec{0}) = 0 \end{cases}. \tag{4}$$

2.2 Eigenvalues $\pm c$

Let the vectors $\vec{r}_+, \vec{r}_-, \vec{l}_+, \vec{l}_- \in \mathbb{R}^{d+1}$ be defined by

$$\vec{r}_\pm = \begin{pmatrix} c||\vec{n}|| \\ \pm\vec{n} \end{pmatrix}, \quad \vec{l}_\pm = \begin{pmatrix} ||\vec{n}|| \\ \pm c\vec{n} \end{pmatrix}.$$

the two vectors $\vec{r}_\pm$ (resp. $\vec{l}_\pm$) are right (resp. left) eigenvectors of A_{wav} :

$$A_{wav}\vec{r}_\pm = \begin{pmatrix} \pm c^2||\vec{n}||^2 \\ c||\vec{n}||\vec{n} \end{pmatrix} = \pm c||\vec{n}||\vec{r}_\pm, \quad {}^t A_{wav}\vec{l}_\pm = \begin{pmatrix} \pm c||\vec{n}||^2 \\ c^2||\vec{n}||\vec{n} \end{pmatrix} = \pm c||\vec{n}||\vec{l}_\pm.$$

2.3 Condition number of $\epsilon I_d + A_{wav}$

We consider the matrix

$$B(\epsilon) = \epsilon I_d + A_{wav}$$

where ϵ is a small parameter. This type of matrices appear after discretisation of the wave system with $\epsilon = \frac{\Delta x}{\Delta t}$ the ratio of space and time steps. On a given spatial mesh, in order to save simulation time, one is usually interested in large time step making ϵ as small as possible. This is usually done with implicit numerical schemes where one needs to solve a linear system with a system matrix sharing similarities with B. However since A_{wav} is not invertible we know that

$$\lim_{\epsilon \to 0} \kappa(B(\epsilon)) = \infty.$$

Hence linear systems will be more and more difficult to solve as ϵ get smaller and smaller.

In order to evaluate the condition number of B, let's evaluate

$$\begin{array}{rcl}
B^t B & = & (\epsilon I_d + A_{wav})^t(\epsilon I_d + A_{wav}) \\
& = & \epsilon^2 I_d + \epsilon(A_{wav} + {}^tA_{wav}) + A_{wav}\,{}^tA_{wav} \\
& = & \epsilon^2 I_d + \epsilon \left[\begin{pmatrix} 0 & c^2\,{}^t\vec{n} \\ \vec{n} & 0 \end{pmatrix} + \begin{pmatrix} 0 & {}^t\vec{n} \\ c^2\vec{n} & 0 \end{pmatrix} \right] + \begin{pmatrix} 0 & c^2\,{}^t\vec{n} \\ \vec{n} & 0 \end{pmatrix} \begin{pmatrix} 0 & {}^t\vec{n} \\ c^2\vec{n} & 0 \end{pmatrix} \\
& = & \epsilon^2 I_d + \epsilon \begin{pmatrix} 0 & (1+c^2)\,{}^t\vec{n} \\ (1+c^2)\vec{n} & 0 \end{pmatrix} + \begin{pmatrix} c^4||\vec{n}||^2 & 0 \\ 0 & \vec{n}\otimes\vec{n} \end{pmatrix} \\
& = & \begin{pmatrix} \epsilon^2 + c^4||\vec{n}||^2 & \epsilon(1+c^2)\,{}^t\vec{n} \\ \epsilon(1+c^2)\vec{n} & \epsilon^2 I_{d-1} + \vec{n}\otimes\vec{n} \end{pmatrix}.
\end{array}$$

Here again, the family

$$V_k^0, \quad k = 1, \dots, d-1$$

are left and right eigenvectors of $B^t B$ associated to the eigenvalue ϵ^2.

2.4 Computation of $(\epsilon I_d + A_{wav})^{-1}$

In a first step to design preconditioners for the discrete wave system, we compute B^{-1}. From the spectral decomposition of a $N \times N$ matrix A having eigenvalues $\lambda_1, \dots, \lambda_N$ and left and right eigenvectors $\vec{r}_1, \dots, \vec{r}_N$ and $\vec{l}_1, \dots, \vec{l}_N$.
we have

$$\begin{array}{rcl}
I_d & = & \displaystyle\sum_{k=1}^{N} \frac{1}{\vec{r}_k \cdot \vec{l}_k} \vec{r}_k \otimes \vec{l}_k \\
A & = & \displaystyle\sum_{k=1}^{N} \frac{\lambda_k}{\vec{r}_k \cdot \vec{l}_k} \vec{r}_k \otimes \vec{l}_k,
\end{array}$$

from which we deduce

$$\begin{aligned}
\epsilon I_d + A &= \sum_{k=1}^{N} (\epsilon + \lambda_k) \frac{1}{\vec{r}_k \cdot \vec{l}_k} \vec{r}_k \otimes \vec{l}_k \\
(\epsilon I_d + A)^{-1} &= \sum_{k=1}^{N} \frac{1}{\epsilon + \lambda_k} \frac{1}{\vec{r}_k \cdot \vec{l}_k} \vec{r}_k \otimes \vec{l}_k
\end{aligned}$$

Taking $A = A_{wav}$ we deduce

$$(\epsilon I_d + A_{wav})^{-1} = \frac{1}{\epsilon} \sum_{k=1}^{d-1} \frac{1}{||\vec{\omega}_k||^2} \vec{V}_k^0 \otimes \vec{V}_k^0 + \frac{1}{2c||\vec{n}||^2} \left(\frac{1}{\epsilon + c} \vec{r}_+ \otimes \vec{l}_+ + \frac{1}{\epsilon - c} \vec{r}_- \otimes \vec{l}_- \right),$$

where we used $\vec{r}_\pm \cdot \vec{l}_\pm = 2c||\vec{n}||^2$.
Direct computation gives

$$\begin{aligned}
V_k^0 \otimes \vec{V}_k^0 &= \begin{pmatrix} 0 & 0 \\ 0 & \vec{\omega}_k \otimes \vec{\omega}_k \end{pmatrix} \\
\vec{r}_\pm \otimes \vec{l}_\pm &= \begin{pmatrix} c||\vec{n}|| \\ \pm \vec{n} \end{pmatrix} \otimes \begin{pmatrix} ||\vec{n}|| \\ \pm c\vec{n} \end{pmatrix} . = \begin{pmatrix} c||\vec{n}||^2 & \pm c^2 ||\vec{n}||^t \vec{n} \\ \pm ||\vec{n}|| \vec{n} & c\vec{n} \otimes \vec{n} \end{pmatrix}.
\end{aligned}$$

Hence

$$\begin{aligned}
(\epsilon I_d + A_{wav})^{-1} &= \frac{1}{\epsilon} \sum_{k=1}^{d-1} \frac{1}{||\vec{\omega}_k||^2} \begin{pmatrix} 0 & 0 \\ 0 & \vec{\omega}_k \otimes \vec{\omega}_k \end{pmatrix} + \frac{1}{2c||\vec{n}||^2} \left(\frac{1}{\epsilon + c} \begin{pmatrix} c||\vec{n}||^2 & c^2 ||\vec{n}||^t \vec{n} \\ ||\vec{n}|| \vec{n} & c\vec{n} \otimes \vec{n} \end{pmatrix} + \frac{1}{\epsilon - c} \left(\begin{matrix} c||\vec{n}|| \\ -||\vec{n}| \end{matrix} \right. \right. \\
&= \frac{1}{\epsilon} \sum_{k=1}^{d-1} \frac{1}{||\vec{\omega}_k||^2} \begin{pmatrix} 0 & 0 \\ 0 & \vec{\omega}_k \otimes \vec{\omega}_k \end{pmatrix} + \frac{1}{||\vec{n}||^2 (\epsilon^2 - c^2)} \begin{pmatrix} \epsilon c ||\vec{n}||^2 & -c^2 ||\vec{n}||^t \vec{n} \\ -||\vec{n}|| \vec{n} & \epsilon \vec{n} \otimes \vec{n} \end{pmatrix}.
\end{aligned}$$

We deduce that as ϵ goes to zero, $(\epsilon I_d + A_{wav})^{-1}$ behaves like $\frac{1}{\epsilon}$:

$$B(\epsilon) = \frac{1}{\epsilon} \begin{pmatrix} 0 & 0 \\ 0 & \sum_{k=1}^{d-1} \frac{1}{||\vec{\omega}_k||^2} \vec{\omega}_k \otimes \vec{\omega}_k \end{pmatrix} + \frac{1}{||\vec{n}||} \begin{pmatrix} 0 & {}^t\vec{n} \\ \frac{1}{c^2} \vec{n} & 0 \end{pmatrix} + O(1).$$

2.5 Computation of $|A_{wav}|$

We recall that we have

$$A_{wav} = \sum_{i=1}^{N} \frac{\lambda_i}{\vec{r}_i \cdot \vec{l}_i} \vec{r}_i \otimes \vec{l}_i$$

$$|A_{wav}| = \sum_{i=1}^{N} \frac{|\lambda_i|}{\vec{r}_i \cdot \vec{l}_i} \vec{r}_i \otimes \vec{l}_i$$

for $N \times N$ matrix A having eigenvalues $\lambda_1, \ldots, \lambda_N$ and left and right eigenvectors $\vec{r}_1, \ldots, \vec{r}_N$ and $\vec{l}_1, \ldots, \vec{l}_N$.
Direct computation gives

$$\begin{aligned}
\vec{r}_\pm \cdot \vec{l}_\pm &= 2c||\vec{n}||^2, \\
\vec{r}_\pm \otimes \vec{l}_\pm &= \begin{pmatrix} c||\vec{n}|| \\ \pm \vec{n} \end{pmatrix} \otimes \begin{pmatrix} ||\vec{n}|| \\ \pm c\vec{n} \end{pmatrix} . = \begin{pmatrix} c||\vec{n}||^2 & \pm c^2 ||\vec{n}||^t \vec{n} \\ \pm ||\vec{n}|| \vec{n} & c\vec{n} \otimes \vec{n} \end{pmatrix}.
\end{aligned}$$

We can check that our computations are correct with

$$c\frac{\vec{r}_+ \otimes \vec{l}_+}{2c||\vec{n}||^2} - c\frac{\vec{r}_- \otimes \vec{l}_-}{2c||\vec{n}||^2} = \frac{1}{2||\vec{n}||^2}\begin{pmatrix} 0 & 2c^2||\vec{n}||^t\vec{n} \\ 2||\vec{n}||\vec{n} & 0 \end{pmatrix} = A_{wav}.$$

We compute

$$\begin{aligned} |A_{wav}| &= c\frac{\vec{r}_+ \otimes \vec{l}_+}{2c||\vec{n}||^2} + c\frac{\vec{r}_- \otimes \vec{l}_-}{2c||\vec{n}||^2} = \frac{1}{2||\vec{n}||^2}\begin{pmatrix} 2c||\vec{n}||^2 & 0 \\ 0 & 2c\vec{n}\otimes\vec{n} \end{pmatrix} \\ &= \frac{c}{||\vec{n}||^2}\begin{pmatrix} ||\vec{n}||^2 & 0 \\ 0 & \vec{n}\otimes\vec{n} \end{pmatrix}. \end{aligned}$$

3 Finite volume discretisation

We consider finite volume semi-discretisation of the first order wave system on unstructured meshes using flux schemes of the form :

$$\partial_t U_i + \frac{1}{v_i}\sum_{j\in\nu(i)} s_{ij}F_{ij}$$

where the numerical flux takes the form

$$F_{ij} = \frac{F(U_i)+F(U_j)}{2} + D_{ij}\frac{U_i - U_j}{2}.$$

From the identity

$$\partial_t\left(\sum_i v_i||U_i||^2\right) = -\sum_{f_{ij}} {}^t s_{ij}(U_i - U_j)D_{ij}(U_i - U_j) \tag{5}$$

we can deduce that the scheme is stable if and only if all interfacial matrices are positive matrices.
A classical example of finite volume scheme is the upwind scheme, where $D_{ij} = |A(\vec{n}_{ij})|$ which is a positive matrix.

Again from (5), since $D_{ij} \geq 0$, we can conclude that the discrete stationary solutions are given by the condition

$$D_{ij}(U_i - U_j) = 0,$$

which means that the coss-interface variation must belong to the kernel of D_{ij}.

3.1 Stationary states of the upwind scheme

For the upwind scheme, $D_{ij} = |A(\vec{n}_{ij})|$ therefore the kernel of D_{ij} is identical to the kernel of $|A(\vec{n}_{ij})|$. The kernel of $|A(\vec{n}_{ij})|$ was computed in section 2.1. The characterisation 4 becomes here:

$$\begin{aligned} D_{ij}(U_i - U_j) = 0 \quad &\Leftrightarrow \quad A(\vec{n}_{ij})(U_i - U_j) = 0 \\ &\Leftrightarrow \quad \begin{cases} (U_i - U_j)\cdot(0,\vec{n}_{ij}) = 0 \\ (U_i - U_j)\cdot(1,\vec{0}) = 0 \end{cases} \\ &\Leftrightarrow \quad \begin{cases} (\vec{u}_i - \vec{u}_j)\cdot\vec{n}_{ij} = 0 \\ p_i = p_j \end{cases} \end{aligned}$$

The second condition $p_i = p_j$ tells us that the pressure is constant in a stationary state, which is expected since we are discretising $\vec{\nabla} p = 0$. However we need to check whether the first condition $(\vec{u}_i - \vec{u}_j) \cdot \vec{n}_{ij} = 0$ yields a precise discretisation of $\nabla \cdot \vec{u} = 0$.

3.1.1 Structured meshes

In the case of a structured grid, the second condition tells us that the velocity in the direction x is constant on all horizontal lines $x = constant$. More generally, we capture stationary velocity of the form

$$\vec{u} = (u_1(y,z), u_2(x,z), u_3(x,y)). \tag{6}$$

So if $u_1 = 0$ at some point (for example a boundary condition, then it is 0 on the entire horizontal line.
The upwind scheme on a structured mesh can only capture the projection $P(\vec{u})$ of a divergence free field $\vec{u}$ on the space of function of the form (6). P has a large kernel : there are divergence free velocity fields that projects to zero.
Below we give an example of vortex function in the kernel of P that can not be captured by the upwind scheme on a structured grid.

Example We consider the 2D divergence free vector field on the square $[0,1] \times [0,1]$

$$\vec{u} = (sin(\pi x)cos(\pi y), -sin(\pi y)cos(\pi x)).$$

The projection $P(\vec{u})$ of $\vec{u}$ on the space (6) is

$$P(\vec{u}) = \left(\int_0^1 sin(\pi x)cos(\pi y)\mathrm{d}x, \int_0^1 -sin(\pi y)cos(\pi x)\mathrm{d}y \right) = (0,0).$$

Therefore the upwind scheme will capture a constant null velocity field instead of a vortex on a structured grid.