Clearly, $x_{\text{ep}}[n]$ and $x_{\text{op}}[n]$ are not equivalent to $x_e[n]$ and $x_o[n]$ as defined by Eqs. (2.149a) and (2.149b). However, it can be shown (see Problem 8.59) that

$$x_{\text{ep}}[n] = \{x_e[n] + x_e[n-N]\}, \qquad 0 \le n \le N-1, \tag{8.103}$$

and

$$x_{\text{op}}[n] = \{x_o[n] + x_o[n-N]\}, \qquad 0 \le n \le N-1. \tag{8.104}$$

In other words, $x_{\text{ep}}[n]$ and $x_{\text{op}}[n]$ can be generated by time-aliasing $x_e[n]$ and $x_o[n]$ into the interval $0 \le n \le N-1$. The sequences $x_{\text{ep}}[n]$ and $x_{\text{op}}[n]$ will be referred to as *the periodic conjugate-symmetric* and *periodic conjugate-antisymmetric* components, respectively, of $x[n]$. When $x_{\text{ep}}[n]$ and $x_{\text{op}}[n]$ are real, they will be referred to as the *periodic even* and *periodic odd* components, respectively. Note that the sequences $x_{\text{ep}}[n]$ and $x_{\text{op}}[n]$ are not periodic sequences; they are, however, finite-length sequences that are equal to one period of the periodic sequences $\tilde{x}_e[n]$ and $\tilde{x}_o[n]$, respectively.

Equations (8.101) and (8.102) define $x_{\text{ep}}[n]$ and $x_{\text{op}}[n]$ in terms of $x[n]$. The inverse relation, expressing $x[n]$ in terms of $x_{\text{ep}}[n]$ and $x_{\text{op}}[n]$, can be obtained by using Eqs. (8.97) and (8.98) to express $\tilde{x}[n]$ as

$$\tilde{x}[n] = \tilde{x}_e[n] + \tilde{x}_o[n]. \tag{8.105}$$

Thus,

$$x[n] = \tilde{x}[n] = \tilde{x}_e[n] + \tilde{x}_o[n], \qquad 0 \le n \le N-1. \tag{8.106}$$

Combining Eqs. (8.106) with Eqs. (8.99) and (8.100), we obtain

$$x[n] = x_{\text{ep}}[n] + x_{\text{op}}[n]. \tag{8.107}$$

Alternatively, Eqs. (8.102), when added, also lead to Eq. (8.107). The symmetry properties of the DFT associated with properties 11–14 in Table 8.1 now follow in a straightforward way:

$$\mathcal{R}e\{x[n]\} \overset{\mathcal{DFT}}{\longleftrightarrow} X_{\text{ep}}[k], \tag{8.108}$$

$$j\mathcal{I}m\{x[n]\} \overset{\mathcal{DFT}}{\longleftrightarrow} X_{\text{op}}[k], \tag{8.109}$$

$$x_{\text{ep}}[n] \overset{\mathcal{DFT}}{\longleftrightarrow} \mathcal{R}e\{X[k]\}, \tag{8.110}$$

$$x_{\text{op}}[n] \overset{\mathcal{DFT}}{\longleftrightarrow} j\mathcal{I}m\{X[k]\}. \tag{8.111}$$

8.6.5 Circular Convolution

In Section 8.2.5, we showed that multiplication of the DFS coefficients of two periodic sequences corresponds to a periodic convolution of the sequences. Here, we consider two *finite-duration* sequences $x_1[n]$ and $x_2[n]$, both of length N, with DFTs $X_1[k]$ and $X_2[k]$, respectively, and we wish to determine the sequence $x_3[n]$, for which the DFT is $X_3[k] = X_1[k]X_2[k]$. To determine $x_3[n]$, we can apply the results of Section 8.2.5. Specifically, $x_3[n]$ corresponds to one period of $\tilde{x}_3[n]$, which is given by Eq. (8.27). Thus,

$$x_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m], \qquad 0 \le n \le N-1, \tag{8.112}$$

or, equivalently,

$$x_3[n] = \sum_{m=0}^{N-1} x_1[((m))_N]x_2[((n-m))_N], \qquad 0 \le n \le N-1. \tag{8.113}$$

Since $((m))_N = m$ for $0 \le m \le N-1$, Eq. (8.113) can be written

$$x_3[n] = \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N], \qquad 0 \le n \le N-1. \tag{8.114}$$

Equation (8.114) differs from a linear convolution of $x_1[n]$ and $x_2[n]$ as defined by Eq. (2.49) in some important respects. In linear convolution, the computation of the sequence value $x_3[n]$ involves multiplying one sequence by a time-reversed and linearly shifted version of the other and then summing the values of the product $x_1[m]x_2[n-m]$ over all m. To obtain successive values of the sequence formed by the convolution operation, the two sequences are successively shifted relative to each other along a linear axis. In contrast, for the convolution defined by Eq. (8.114), the second sequence is circularly time reversed and circularly shifted with respect to the first. For this reason, the operation of combining two finite-length sequences according to Eq. (8.114) is called *circular convolution.* More specifically, we refer to Eq. (8.114) as an N-point circular convolution, explicitly identifying the fact that both sequences have length N (or less) and that the sequences are shifted modulo N. Sometimes, the operation of forming a sequence $x_3[n]$ for $0 \le n \le N-1$ using Eq. (8.114) will be denoted

$$x_3[n] = x_1[n] \; Ⓝ \; x_2[n], \tag{8.115}$$

i.e., the symbol Ⓝ denotes N-point circular convolution.

Since the DFT of $x_3[n]$ is $X_3[k] = X_1[k]X_2[k]$ and since $X_1[k]X_2[k] = X_2[k]X_1[k]$, it follows with no further analysis that

$$x_3[n] = x_2[n] \; Ⓝ \; x_1[n], \tag{8.116}$$

or, more specifically,

$$x_3[n] = \sum_{m=0}^{N-1} x_2[m]x_1[((n-m))_N]. \tag{8.117}$$

That is, circular convolution, like linear convolution, is a commutative operation.

Since circular convolution is really just periodic convolution, Example 8.4 and Figure 8.3 are also illustrative of circular convolution. However, if we use the notion of circular shifting, it is not necessary to construct the underlying periodic sequences as in Figure 8.3. This is illustrated in the following examples.

Example 8.10 Circular Convolution with a Delayed Impulse Sequence

An example of circular convolution is provided by the result of Section 8.6.2. Let $x_2[n]$ be a finite-duration sequence of length N and

$$x_1[n] = \delta[n-n_0], \tag{8.118}$$

where $0 < n_0 < N$. Clearly, $x_1[n]$ can be considered as the finite-duration sequence

$$x_1[n] = \begin{cases} 0, & 0 \le n < n_0, \\ 1, & n = n_0, \\ 0, & n_0 < n \le N-1. \end{cases} \tag{8.119}$$

as depicted in Figure 8.14 for $n_0 = 1$.

The DFT of $x_1[n]$ is

$$X_1[k] = W_N^{kn_0}. \tag{8.120}$$

If we form the product

$$X_3[k] = W_N^{kn_0} X_2[k], \tag{8.121}$$

we see from Section 8.6.2 that the finite-duration sequence corresponding to $X_3[k]$ is the sequence $x_2[n]$ rotated to the right by n_0 samples in the interval $0 \le n \le N-1$. That is, the circular convolution of a sequence $x_2[n]$ with a single delayed unit impulse results in a rotation of $x_2[n]$ in the interval $0 \le n \le N-1$. This example is illustrated in Figure 8.14 for $N = 5$ and $n_0 = 1$. Here, we show the sequences $x_2[m]$

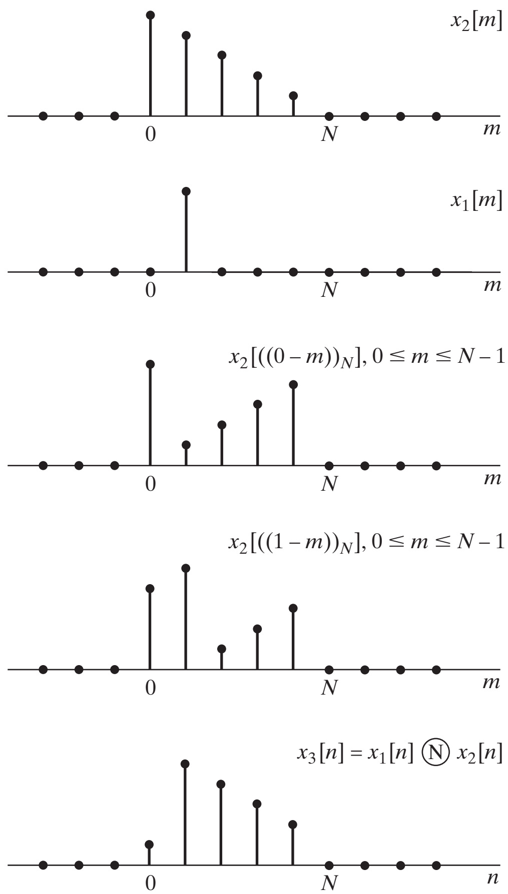

Figure 8.14 Circular convolution of a finite-length sequence $x_2[n]$ with a single delayed impulse, $x_1[n] = \delta[n-1]$.

and $x_1[m]$ and then $x_2[((0-m))_N]$ and $x_2[((1-m))_N]$. It is clear from these two cases that the result of circular convolution of $x_2[n]$ with a single shifted unit impulse will be to circularly shift $x_2[n]$. The last sequence shown is $x_3[n]$, the result of the circular convolution of $x_1[n]$ and $x_2[n]$.

Example 8.11 Circular Convolution of Two Rectangular Pulses

As another example of circular convolution, let

$$x_1[n] = x_2[n] = \begin{cases} 1, & 0 \le n \le L-1, \\ 0, & \text{otherwise}, \end{cases} \tag{8.122}$$

where, in Figure 8.15, $L = 6$. If we let N denote the DFT length, then, for $N = L$, the N-point DFTs are

$$X_1[k] = X_2[k] = \sum_{n=0}^{N-1} W_N^{kn} = \begin{cases} N, & k = 0, \\ 0, & \text{otherwise}. \end{cases} \tag{8.123}$$

If we explicitly multiply $X_1[k]$ and $X_2[k]$, we obtain

$$X_3[k] = X_1[k]X_2[k] = \begin{cases} N^2, & k = 0, \\ 0, & \text{otherwise}, \end{cases} \tag{8.124}$$

from which it follows that

$$x_3[n] = N, \qquad 0 \le n \le N-1. \tag{8.125}$$

This result is depicted in Figure 8.15. Clearly, as the sequence $x_2[((n-m))_N]$ is rotated with respect to $x_1[m]$, the sum of products $x_1[m]x_2[((n-m))_N]$ will always be equal to N.

Of course, it is possible to consider $x_1[n]$ and $x_2[n]$ as $2L$-point sequences by augmenting them with L zeros. If we then perform a $2L$-point circular convolution of the augmented sequences, we obtain the sequence in Figure 8.16, which can be seen to be identical to the linear convolution of the finite-duration sequences $x_1[n]$ and $x_2[n]$. This important observation will be discussed in much more detail in Section 8.7.

Note that for $N = 2L$, as in Figure 8.16,

$$X_1[k] = X_2[k] = \frac{1 - W_N^{Lk}}{1 - W_N^k},$$

so the DFT of the triangular-shaped sequence $x_3[n]$ in Figure 8.16(e) is

$$X_3[k] = \left(\frac{1 - W_N^{Lk}}{1 - W_N^k}\right)^2,$$

with $N = 2L$.

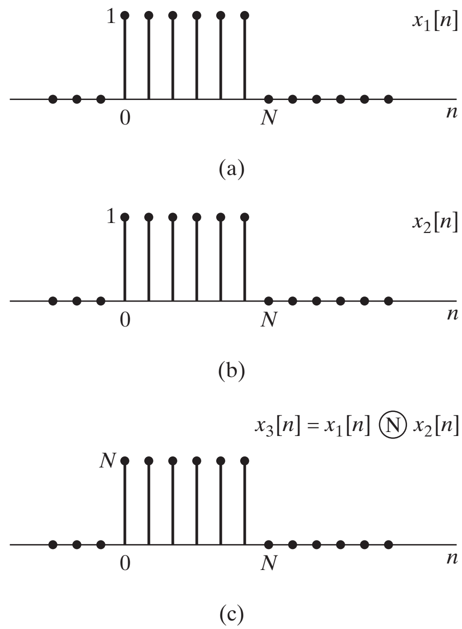

Figure 8.15 N-point circular convolution of two constant sequences of length N.

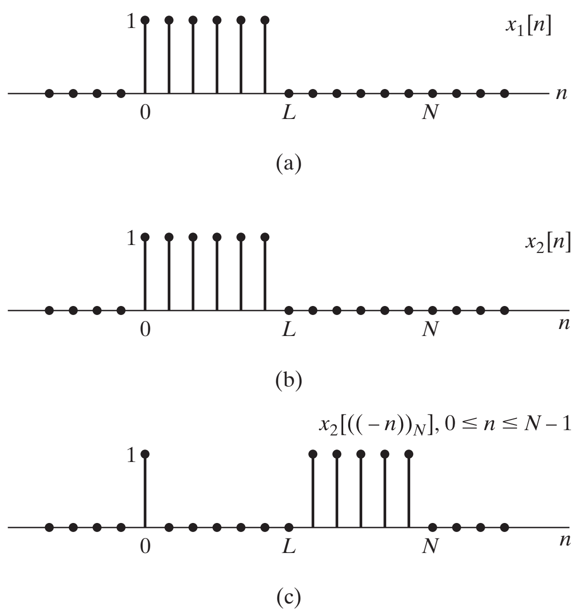

Figure 8.16 $2L$-point circular convolution of two constant sequences of length L.

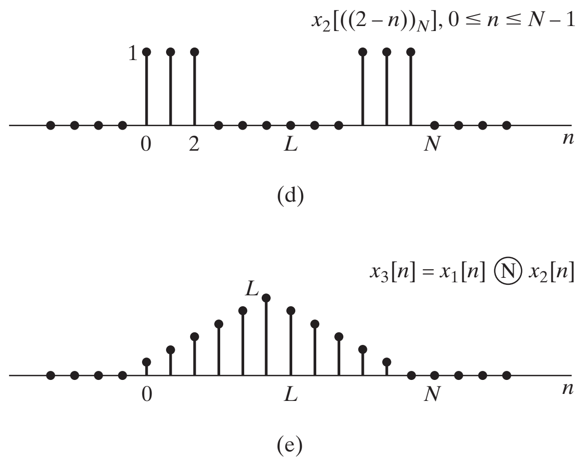

Figure 8.16 (*continued*)

The circular convolution property is represented as

$$x_1[n] \circledN x_2[n] \stackrel{\mathcal{DFT}}{\longleftrightarrow} X_1[k]X_2[k]. \tag{8.126}$$

In view of the duality of the DFT relations, it is not surprising that the DFT of a product of two N-point sequences is the circular convolution of their respective DFTs. Specifically, if $x_3[n] = x_1[n]x_2[n]$, then

$$X_3[k] = \frac{1}{N}\sum_{\ell=0}^{N-1} X_1[\ell]X_2[((k-\ell))_N] \tag{8.127}$$

or

$$x_1[n]x_2[n] \stackrel{\mathcal{DFT}}{\longleftrightarrow} \frac{1}{N} X_1[k] \circledN X_2[k]. \tag{8.128}$$

8.6.6 Summary of Properties of the DFT

The properties of the DFT that we discussed in Section 8.6 are summarized in Table 8.2. Note that for all of the properties, the expressions given specify $x[n]$ for $0 \le n \le N-1$ and $X[k]$ for $0 \le k \le N-1$. Both $x[n]$ and $X[k]$ are equal to zero outside those ranges.

TABLE 8.2 SUMMARY OF PROPERTIES OF THE DFT

Finite-Length Sequence (Length N)	N-point DFT (Length N)
1. $x[n]$	$X[k]$
2. $x_1[n], x_2[n]$	$X_1[k], X_2[k]$
3. $ax_1[n] + bx_2[n]$	$aX_1[k] + bX_2[k]$
4. $X[n]$	$Nx[((-k))_N]$
5. $x[((n-m))_N]$	$W_N^{km} X[k]$
6. $W_N^{-\ell n} x[n]$	$X[((k-\ell))_N]$
7. $\sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N]$	$X_1[k]X_2[k]$
8. $x_1[n]x_2[n]$	$\frac{1}{N}\sum_{\ell=0}^{N-1} X_1[\ell]X_2[((k-\ell))_N]$
9. $x^*[n]$	$X^*[((-k))_N]$
10. $x^*[((-n))_N]$	$X^*[k]$
11. $\mathcal{R}e\{x[n]\}$	$X_{\text{ep}}[k] = \frac{1}{2}\{X[((k))_N] + X^*[((-k))_N]\}$
12. $j\mathcal{I}m\{x[n]\}$	$X_{\text{op}}[k] = \frac{1}{2}\{X[((k))_N] - X^*[((-k))_N]\}$
13. $x_{\text{ep}}[n] = \frac{1}{2}\{x[n] + x^*[((-n))_N]\}$	$\mathcal{R}e\{X[k]\}$
14. $x_{\text{op}}[n] = \frac{1}{2}\{x[n] - x^*[((-n))_N]\}$	$j\mathcal{I}m\{X[k]\}$
Properties 15–17 apply only when $x[n]$ is real.	
15. Symmetry properties	$\begin{cases} X[k] = X^*[((-k))_N] \\ \mathcal{R}e\{X[k]\} = \mathcal{R}e\{X[((-k))_N]\} \\ \mathcal{I}m\{X[k]\} = -\mathcal{I}m\{X[((-k))_N]\} \\ \lvert X[k]\rvert = \lvert X[((-k))_N]\rvert \\ \angle\{X[k]\} = -\angle\{X[((-k))_N]\} \end{cases}$
16. $x_{\text{ep}}[n] = \frac{1}{2}\{x[n] + x[((-n))_N]\}$	$\mathcal{R}e\{X[k]\}$
17. $x_{\text{op}}[n] = \frac{1}{2}\{x[n] - x[((-n))_N]\}$	$j\mathcal{I}m\{X[k]\}$

8.7 COMPUTING LINEAR CONVOLUTION USING THE DFT

We will show in Chapter 9 that efficient algorithms are available for computing the DFT of a finite-duration sequence. These are known collectively as FFT algorithms. Because these algorithms are available, it is computationally efficient to implement a convolution of two sequences by the following procedure:

(a) Compute the N-point DFTs $X_1[k]$ and $X_2[k]$ of the two sequences $x_1[n]$ and $x_2[n]$, respectively.

(b) Compute the product $X_3[k] = X_1[k]X_2[k]$ for $0 \le k \le N-1$.

(c) Compute the sequence $x_3[n] = x_1[n] \; Ⓝ \; x_2[n]$ as the inverse DFT of $X_3[k]$.

In many DSP applications, we are interested in implementing a linear convolution of two sequences; i.e., we wish to implement an LTI system. This is certainly true, for example, in filtering a sequence such as a speech waveform or a radar signal or in computing the autocorrelation function of such signals. As we saw in Section 8.6.5, the multiplication of DFTs corresponds to a circular convolution of the sequences. To obtain a linear convolution, we must ensure that circular convolution has the effect of linear convolution. The discussion at the end of Example 8.11 hints at how this might be done. We now present a more detailed analysis.

8.7.1 Linear Convolution of Two Finite-Length Sequences

Consider a sequence $x_1[n]$ whose length is L points and a sequence $x_2[n]$ whose length is P points, and suppose that we wish to combine these two sequences by linear convolution to obtain a third sequence

$$x_3[n] = \sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m]. \tag{8.129}$$

Figure 8.17(a) shows a typical sequence $x_1[m]$ and Figure 8.17(b) shows a typical sequence $x_2[n-m]$ for the three cases $n = -1$, for $0 \le n \le L-1$, and $n = L+P-1$. Clearly, the product $x_1[m]x_2[n-m]$ is zero for all m whenever $n < 0$ and $n > L+P-2$; i.e., $x_3[n] \neq 0$ for $0 \le n \le L+P-2$. Therefore, $(L+P-1)$ is the maximum length of the sequence $x_3[n]$ resulting from the linear convolution of a sequence of length L with a sequence of length P.

8.7.2 Circular Convolution as Linear Convolution with Aliasing

As Examples 8.10 and 8.11 show, whether a circular convolution corresponding to the product of two N-point DFTs is the same as the linear convolution of the corresponding finite-length sequences depends on the length of the DFT in relation to the length of the finite-length sequences. An extremely useful interpretation of the relationship between circular convolution and linear convolution is in terms of time aliasing. Since this interpretation is so important and useful in understanding circular convolution, we will develop it in several ways.

In Section 8.4, we observed that if the Fourier transform $X(e^{j\omega})$ of a sequence $x[n]$ is sampled at frequencies $\omega_k = 2\pi k/N$, then the resulting sequence corresponds to the DFS coefficients of the periodic sequence

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n-rN]. \tag{8.130}$$

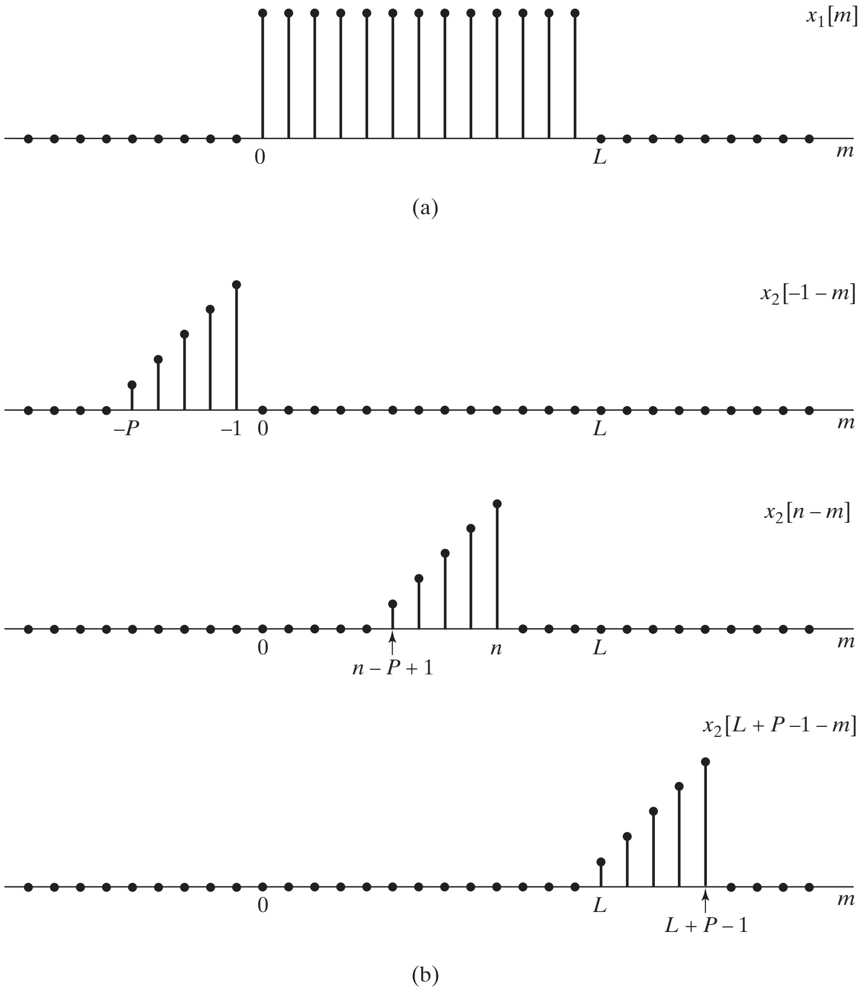

Figure 8.17 Example of linear convolution of two finite-length sequences showing that the result is such that $x_3[n] = 0$ for $n \leq -1$ and for $n \geq L+P-1$. (a) Finite-length sequence $x_1[m]$. (b) $x_2[n-m]$ for several values of n.

From our discussion of the DFT, it follows that the finite-length sequence

$$X[k] = \begin{cases} X(e^{j(2\pi k/N)}), & 0 \leq k \leq N-1, \\ 0, & \text{otherwise}, \end{cases} \tag{8.131}$$

is the DFT of one period of $\tilde{x}[n]$, as given by Eq. (8.130); i.e.,

$$x_p[n] = \begin{cases} \tilde{x}[n], & 0 \leq n \leq N-1, \\ 0, & \text{otherwise}. \end{cases} \tag{8.132}$$

Obviously, if $x[n]$ has length less than or equal to N, no time aliasing occurs and $x_p[n] = x[n]$. However, if the length of $x[n]$ is greater than N, $x_p[n]$ may not be equal to $x[n]$ for some or all values of n. We will henceforth use the subscript p to denote that a sequence is one period of a periodic sequence resulting from an inverse DFT of a sampled Fourier transform. The subscript can be dropped if it is clear that time aliasing is avoided.

The sequence $x_3[n]$ in Eq. (8.129) has Fourier transform

$$X_3(e^{j\omega}) = X_1(e^{j\omega})X_2(e^{j\omega}). \tag{8.133}$$

If we define a DFT

$$X_3[k] = X_3(e^{j(2\pi k/N)}), \qquad 0 \le k \le N-1, \tag{8.134}$$

then it is clear from Eqs. (8.133) and (8.134) that, also

$$X_3[k] = X_1(e^{j(2\pi k/N)})X_2(e^{j(2\pi k/N)}), \qquad 0 \le k \le N-1, \tag{8.135}$$

and therefore,

$$X_3[k] = X_1[k]X_2[k]. \tag{8.136}$$

That is, the sequence resulting as the inverse DFT of $X_3[k]$ is

$$x_{3p}[n] = \begin{cases} \displaystyle\sum_{r=-\infty}^{\infty} x_3[n-rN], & 0 \le n \le N-1, \\ 0, & \text{otherwise,} \end{cases} \tag{8.137}$$

and from Eq. (8.136), it follows that

$$x_{3p}[n] = x_1[n] \; Ⓝ \; x_2[n]. \tag{8.138}$$

Thus, the circular convolution of two finite-length sequences is equivalent to linear convolution of the two sequences, followed by time aliasing according to Eq. (8.137).

Note that if N is greater than or equal to either L or P, $X_1[k]$ and $X_2[k]$ represent $x_1[n]$ and $x_2[n]$ exactly, but $x_{3p}[n] = x_3[n]$ for all n only if N is greater than or equal to the length of the sequence $x_3[n]$. As we showed in Section 8.7.1, if $x_1[n]$ has length L and $x_2[n]$ has length P, then $x_3[n]$ has maximum length $(L + P - 1)$. Therefore, the circular convolution corresponding to $X_1[k]X_2[k]$ is identical to the linear convolution corresponding to $X_1(e^{j\omega})X_2(e^{j\omega})$ if N, the length of the DFTs, satisfies $N \ge L + P - 1$.

Example 8.12 Circular Convolution as Linear Convolution with Aliasing

The results of Example 8.11 are easily understood in light of the interpretation just discussed. Note that $x_1[n]$ and $x_2[n]$ are identical constant sequences of length $L = P = 6$, as shown in Figure 8.18(a). The linear convolution of $x_1[n]$ and $x_2[n]$ is of length $L+P-1 = 11$ and has the triangular shape shown in Figure 8.18(b). In Figures 8.18(c) and (d) are shown two of the shifted versions $x_3[n - rN]$ in Eq. (8.137), $x_3[n - N]$ and $x_3[n + N]$ for $N = 6$. The N-point circular convolution of $x_1[n]$ and $x_2[n]$ can be formed by using Eq. (8.137). This is shown in Figure 8.18(e) for $N = L = 6$ and in Figure 8.18(f) for $N = 2L = 12$. Note that for $N = L = 6$, only $x_3[n]$ and $x_3[n + N]$ contribute to the result. For $N = 2L = 12$, only $x_3[n]$ contributes to the result. Since the length of the linear convolution is $(2L - 1)$, the result of the circular convolution for $N = 2L$ is identical to the result of linear convolution for all $0 \le n \le N - 1$. In fact, this would be true for $N = 2L - 1 = 11$ as well.

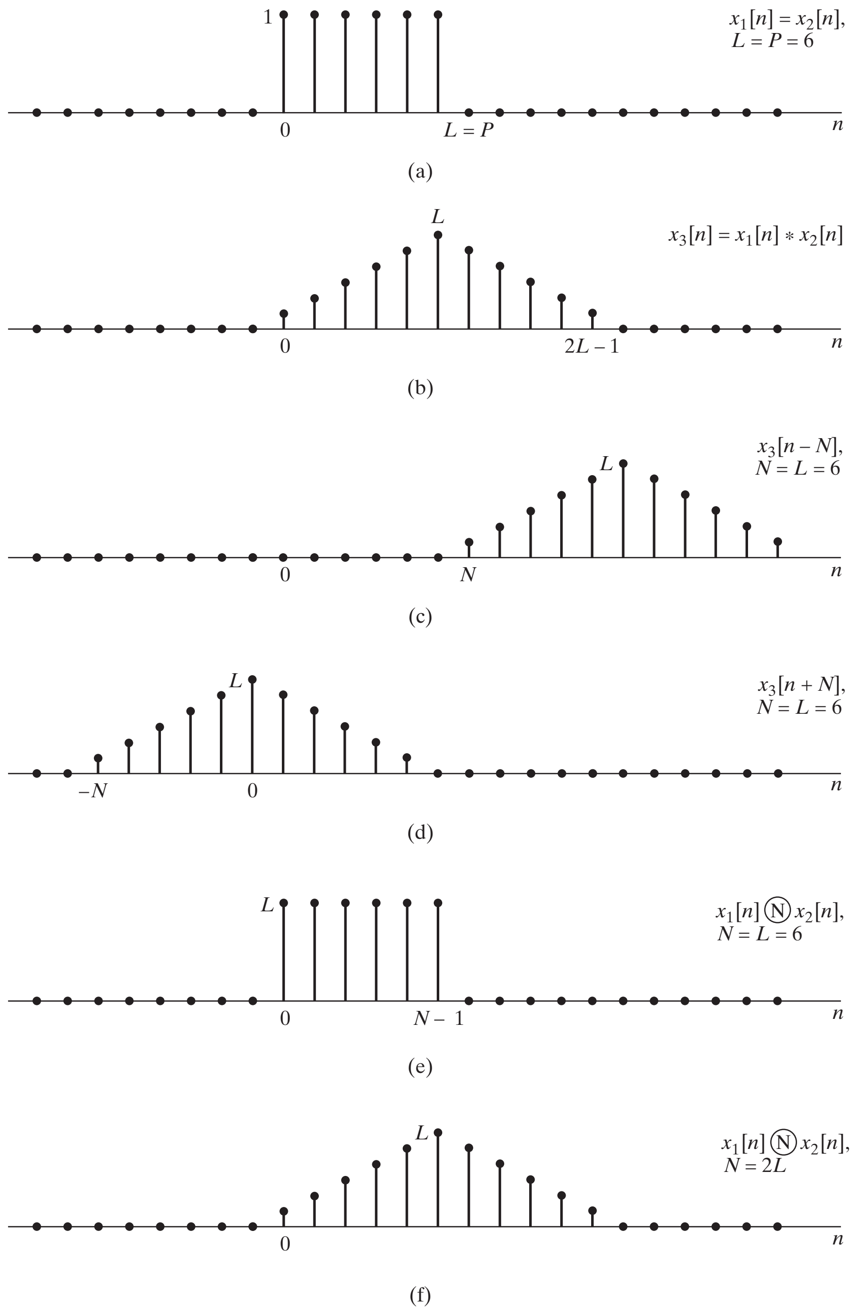

Figure 8.18 Illustration that circular convolution is equivalent to linear convolution followed by aliasing. (a) The sequences $x_1[n]$ and $x_2[n]$ to be convolved. (b) The linear convolution of $x_1[n]$ and $x_2[n]$. (c) $x_3[n - N]$ for $N = 6$. (d) $x_3[n + N]$ for $N = 6$. (e) $x_1[n]$ ⑥ $x_2[n]$, which is equal to the sum of (b), (c), and (d) in the interval $0 \leq n \leq 5$. (f) $x_1[n]$ ⑫ $x_2[n]$.

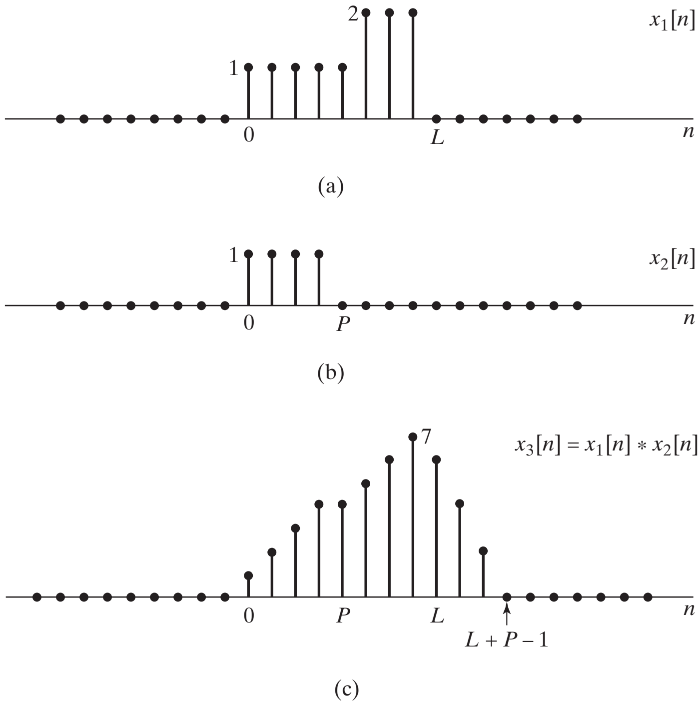

Figure 8.19 An example of linear convolution of two finite-length sequences.

As Example 8.12 points out, time aliasing in the circular convolution of two finite-length sequences can be avoided if $N \geq L + P - 1$. Also, it is clear that if $N = L = P$, all of the sequence values of the circular convolution may be different from those of the linear convolution. However, if $P < L$, some of the sequence values in an L-point circular convolution will be equal to the corresponding sequence values of the linear convolution. The time-aliasing interpretation is useful for showing this.

Consider two finite-duration sequences $x_1[n]$ and $x_2[n]$, with $x_1[n]$ of length L and $x_2[n]$ of length P, where $P < L$, as indicated in Figures 8.19(a) and 8.19(b), respectively. Let us first consider the L-point circular convolution of $x_1[n]$ and $x_2[n]$ and inquire as to which sequence values in the circular convolution are identical to values that would be obtained from a linear convolution and which are not. The linear convolution of $x_1[n]$ with $x_2[n]$ will be a finite-length sequence of length $(L + P - 1)$, as indicated in Figure 8.19(c). To determine the L-point circular convolution, we use Eqs. (8.137) and (8.138) so that

$$x_{3p}[n] = \begin{cases} x_1[n] \text{ Ⓛ } x_2[n] = \displaystyle\sum_{r=-\infty}^{\infty} x_3[n - rL], & 0 \leq n \leq L - 1, \\ 0, & \text{otherwise.} \end{cases} \tag{8.139}$$

Figure 8.20(a) shows the term in Eq. (8.139) for $r = 0$, and Figures 8.20(b) and 8.20(c) show the terms for $r = -1$ and $r = +1$, respectively. From Figure 8.20, it should be clear that in the interval $0 \leq n \leq L - 1$, $x_{3p}[n]$ is influenced only by $x_3[n]$ and $x_3[n + L]$.

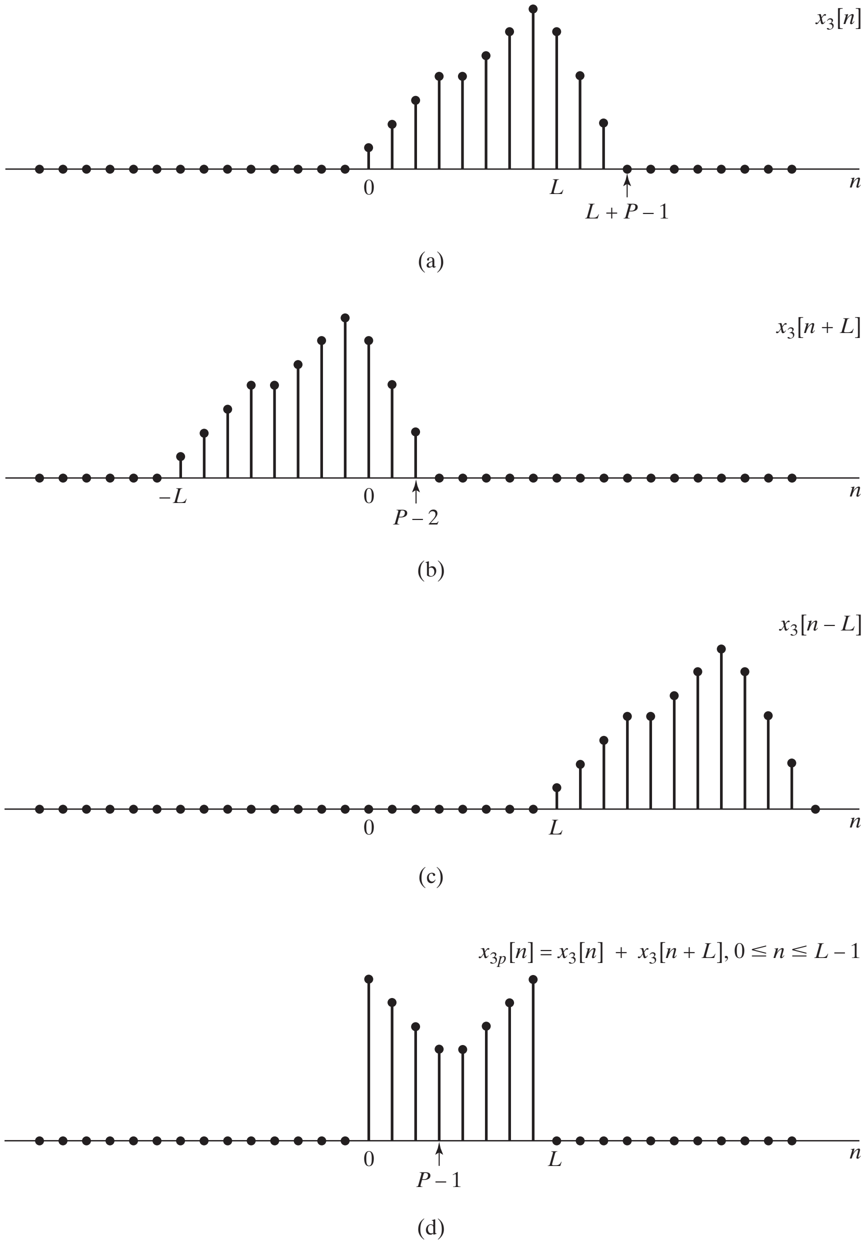

Figure 8.20 Interpretation of circular convolution as linear convolution followed by aliasing for the circular convolution of the two sequences $x_1[n]$ and $x_2[n]$ in Figure 8.19.

In general, whenever $P < L$, only the term $x_3[n + L]$ will alias into the interval $0 \leq n \leq L-1$. More specifically, when these terms are summed, the last $(P-1)$ points of $x_3[n+L]$, which extend from $n = 0$ to $n = P-2$, will be added to the first $(P-1)$ points of $x_3[n]$, and the last $(P-1)$ points of $x_3[n]$, extending from $n = L$ to $n = L+P-2$, will contribute only to the next period of the underlying periodic result $\tilde{x}_3[n]$. Then, $x_{3p}[n]$ is formed by extracting the portion for $0 \leq n \leq L-1$. Since the last $(P-1)$ points of $x_3[n+L]$ and the last $(P-1)$ points of $x_3[n]$ are identical, we can alternatively view the process of forming the circular convolution $x_{3p}[n]$ through linear convolution plus aliasing, as taking the $(P-1)$ values of $x_3[n]$ from $n = L$ to $n = L+P-2$ and adding them to the first $(P-1)$ values of $x_3[n]$. This process is illustrated in Figure 8.21 for the case $P = 4$ and $L = 8$. Figure 8.21(a) shows the linear convolution $x_3[n]$, with the points for $n \geq L$ denoted by open symbols. Note that only $(P-1)$ points for $n \geq L$ are nonzero. Figure 8.21(b) shows the formation of $x_{3p}[n]$ by "wrapping $x_3[n]$ around on itself." The first $(P-1)$ points are corrupted by the time aliasing, and the remaining points from $n = P-1$ to $n = L-1$ (i.e., the last $L-P+1$ points) are not corrupted; that is, they are identical to what would be obtained with a linear convolution.

From this discussion, it should be clear that if the circular convolution is of sufficient length relative to the lengths of the sequences $x_1[n]$ and $x_2[n]$, then aliasing with nonzero values can be avoided, in which case the circular convolution and linear convolution will be identical. Specifically, if, for the case just considered, $x_3[n]$ is replicated with period $N \geq L+P-1$, then no nonzero overlap will occur. Figures 8.21(c) and 8.21(d) illustrate this case, again for $P = 4$ and $L = 8$, with $N = 11$.

8.7.3 Implementing Linear Time-Invariant Systems Using the DFT

The previous discussion focused on ways of obtaining a linear convolution from a circular convolution. Since LTI systems can be implemented by convolution, this implies that circular convolution (implemented by the procedure suggested at the beginning of Section 8.7) can be used to implement these systems. To see how this can be done, let us first consider an L-point input sequence $x[n]$ and a P-point impulse response $h[n]$. The linear convolution of these two sequences, which will be denoted by $y[n]$, has finite duration with length $(L+P-1)$. Consequently, as discussed in Section 8.7.2, for the circular convolution and linear convolution to be identical, the circular convolution must have a length of at least $(L+P-1)$ points. The circular convolution can be achieved by multiplying the DFTs of $x[n]$ and $h[n]$. Since we want the product to represent the DFT of the linear convolution of $x[n]$ and $h[n]$, which has length $(L+P-1)$, the DFTs that we compute must also be of at least that length, i.e., both $x[n]$ and $h[n]$ must be augmented with sequence values of zero amplitude. This process is often referred to as *zero-padding*.

This procedure permits the computation of the linear convolution of two finite-length sequences using the DFT; i.e., the output of an FIR system whose input also has finite length can be computed with the DFT. In many applications, such as filtering a speech waveform, the input signal is of indefinite duration. Theoretically, while we might be able to store the entire waveform and then implement the procedure just discussed using a DFT for a large number of points, such a DFT might be impractical to compute.