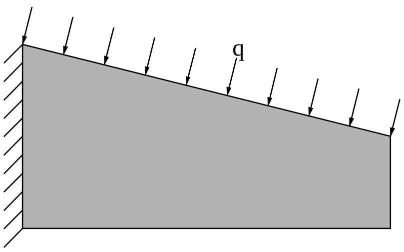
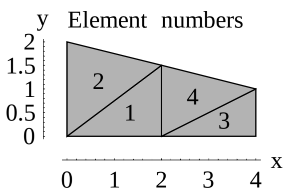
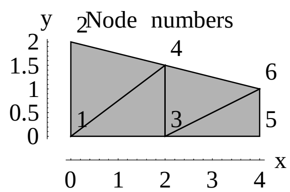
Stress analysis of a bracket: Examples 1.6 p. 32, 1.9 p. 46, and 1.12 p. 55

Top surface of a thin cantilever bracket is subjected to normal pressure $q = 20\,\text{lb}/\text{in}^2$ as shown in Figure. The bracket is 4 in long and is 2 in wide at the base and 1 in wide at the free end. The thickness of the bracket perpendicular to the plane of paper is $1/4$ in. The material properties are $E = 10^4\,\text{lb}/\text{in}^2$ and $\nu = 0.2$.







Global equations at start of the element assembly process

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} u_1\\ v_1\\ u_2\\ v_2\\ u_3\\ v_3\\ u_4\\ v_4\\ u_5\\ v_5\\ u_6\\ v_6 \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}$$

Equations for element 1

h = 0.25;   E = 10000; $\nu = 0.2$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 10416.7 & 2083.33 & 0 \\ 2083.33 & 10416.7 & 0 \\ 0 & 0 & 4166.67 \end{pmatrix}$

Nodal coordinates

Element node	Global node number	x	y
1	1	0.	0.
2	3	2.	0.
3	4	2.	1.5

$x_1 = 0. \quad x_2 = 2. \quad x_3 = 2.$
$y_1 = 0. \quad y_2 = 0. \quad y_3 = 1.5$

Using these values we get

$b_1 = -1.5 \quad b_2 = 1.5 \quad b_3 = 0.$

$c_1 = 0. \quad c_2 = -2. \quad c_3 = 2.$

$f_1 = 3. \quad f_2 = 0. \quad f_3 = 0.$

Element area, A = 1.5

$$\boldsymbol{B}^T = \begin{pmatrix} -0.5 & 0 & 0.5 & 0 & 0. & 0 \\ 0 & 0. & 0 & -0.666667 & 0 & 0.666667 \\ 0. & -0.5 & -0.666667 & 0.5 & 0.666667 & 0. \end{pmatrix}$$

Thus the element stiffness matrix is

$$\boldsymbol{k} = hA\boldsymbol{BCB}^T = \begin{pmatrix} 976.563 & 0 & -976.563 & 260.417 & 0 & -260.417 \\ 0 & 390.625 & 520.833 & -390.625 & -520.833 & 0 \\ -976.563 & 520.833 & 1671.01 & -781.25 & -694.444 & 260.417 \\ 260.417 & -390.625 & -781.25 & 2126.74 & 520.833 & -1736.11 \\ 0 & -520.833 & -694.444 & 520.833 & 694.444 & 0 \\ -260.417 & 0 & 260.417 & -1736.11 & 0 & 1736.11 \end{pmatrix}$$

Complete equations for element 1

$$\begin{pmatrix} 976.563 & 0 & -976.563 & 260.417 & 0 & -260.417 \\ 0 & 390.625 & 520.833 & -390.625 & -520.833 & 0 \\ -976.563 & 520.833 & 1671.01 & -781.25 & -694.444 & 260.417 \\ 260.417 & -390.625 & -781.25 & 2126.74 & 520.833 & -1736.11 \\ 0 & -520.833 & -694.444 & 520.833 & 694.444 & 0 \\ -260.417 & 0 & 260.417 & -1736.11 & 0 & 1736.11 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {1, 2, 5, 6, 7, 8} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 1 \\ 2 \\ 5 \\ 6 \\ 7 \\ 8 \end{pmatrix}$

and to a global matrix: $\begin{pmatrix} [1,1] & [1,2] & [1,5] & [1,6] & [1,7] & [1,8] \\ [2,1] & [2,2] & [2,5] & [2,6] & [2,7] & [2,8] \\ [5,1] & [5,2] & [5,5] & [5,6] & [5,7] & [5,8] \\ [6,1] & [6,2] & [6,5] & [6,6] & [6,7] & [6,8] \\ [7,1] & [7,2] & [7,5] & [7,6] & [7,7] & [7,8] \\ [8,1] & [8,2] & [8,5] & [8,6] & [8,7] & [8,8] \end{pmatrix}$

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 976.563 & 0 & 0 & 0 & -976.563 & 260.417 & 0 & -260.417 & 0 & 0 & 0 & 0 \\ 0 & 390.625 & 0 & 0 & 520.833 & -390.625 & -520.833 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -976.563 & 520.833 & 0 & 0 & 1671.01 & -781.25 & -694.444 & 260.417 & 0 & 0 & 0 & 0 \\ 260.417 & -390.625 & 0 & 0 & -781.25 & 2126.74 & 520.833 & -1736.11 & 0 & 0 & 0 & 0 \\ 0 & -520.833 & 0 & 0 & -694.444 & 520.833 & 694.444 & 0 & 0 & 0 & 0 & 0 \\ -260.417 & 0 & 0 & 0 & 260.417 & -1736.11 & 0 & 1736.11 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 2

$h = 0.25;$ $E = 10000;$ $\nu = 0.2$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 10416.7 & 2083.33 & 0 \\ 2083.33 & 10416.7 & 0 \\ 0 & 0 & 4166.67 \end{pmatrix}$

Nodal coordinates

Element node	Global node number	x	y
1	4	2.	1.5
2	2	0.	2.
3	1	0.	0.

$x_1 = 2.$ $x_2 = 0.$ $x_3 = 0.$
$y_1 = 1.5$ $y_2 = 2.$ $y_3 = 0.$

Using these values we get

$b_1 = 2.$ $b_2 = -1.5$ $b_3 = -0.5$

$c_1 = 0.$ $c_2 = 2.$ $c_3 = -2.$

$f_1 = 0.$ $f_2 = 0.$ $f_3 = 4.$

Element area, $A = 2.$

$$\boldsymbol{B}^T = \begin{pmatrix} 0.5 & 0 & -0.375 & 0 & -0.125 & 0 \\ 0 & 0. & 0 & 0.5 & 0 & -0.5 \\ 0. & 0.5 & 0.5 & -0.375 & -0.5 & -0.125 \end{pmatrix}$$

Thus the element stiffness matrix is

$$\boldsymbol{k} = hA\boldsymbol{BCB}^T = \begin{pmatrix} 1302.08 & 0 & -976.563 & 260.417 & -325.521 & -260.417 \\ 0 & 520.833 & 520.833 & -390.625 & -520.833 & -130.208 \\ -976.563 & 520.833 & 1253.26 & -585.938 & -276.693 & 65.1042 \\ 260.417 & -390.625 & -585.938 & 1595.05 & 325.521 & -1204.43 \\ -325.521 & -520.833 & -276.693 & 325.521 & 602.214 & 195.313 \\ -260.417 & -130.208 & 65.1042 & -1204.43 & 195.313 & 1334.64 \end{pmatrix}$$

Load vector due to distributed load on side 1 (nodes {4, 2})

Specified load components: $q_n = -20;$ $q_t = 0$

End nodal coordinates: ({2., 1.5} {0., 2.}) giving side length, $L = 2.06155$

Components of unit normal to the side: $n_x = 0.242536$; $n_y = 0.970143$

Using these values we get
$\boldsymbol{r}_q^T = (\,-1.25 \quad -5. \quad -1.25 \quad -5. \quad 0 \quad 0\,)$

Complete equations for element 2

$$\begin{pmatrix} 1302.08 & 0 & -976.563 & 260.417 & -325.521 & -260.417 \\ 0 & 520.833 & 520.833 & -390.625 & -520.833 & -130.208 \\ -976.563 & 520.833 & 1253.26 & -585.938 & -276.693 & 65.1042 \\ 260.417 & -390.625 & -585.938 & 1595.05 & 325.521 & -1204.43 \\ -325.521 & -520.833 & -276.693 & 325.521 & 602.214 & 195.313 \\ -260.417 & -130.208 & 65.1042 & -1204.43 & 195.313 & 1334.64 \end{pmatrix} \begin{pmatrix} u_4 \\ v_4 \\ u_2 \\ v_2 \\ u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} -1.25 \\ -5. \\ -1.25 \\ -5. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {7, 8, 3, 4, 1, 2} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 7 \\ 8 \\ 3 \\ 4 \\ 1 \\ 2 \end{pmatrix}$

and to a global matrix: $\begin{pmatrix} [7,7] & [7,8] & [7,3] & [7,4] & [7,1] & [7,2] \\ [8,7] & [8,8] & [8,3] & [8,4] & [8,1] & [8,2] \\ [3,7] & [3,8] & [3,3] & [3,4] & [3,1] & [3,2] \\ [4,7] & [4,8] & [4,3] & [4,4] & [4,1] & [4,2] \\ [1,7] & [1,8] & [1,3] & [1,4] & [1,1] & [1,2] \\ [2,7] & [2,8] & [2,3] & [2,4] & [2,1] & [2,2] \end{pmatrix}$

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 1578.78 & 195.313 & -276.693 & 325.521 & -976.563 & 260.417 & -325.521 & -781.25 & 0 & 0 & 0 & 0 \\ 195.313 & 1725.26 & 65.1042 & -1204.43 & 520.833 & -390.625 & -781.25 & -130.208 & 0 & 0 & 0 & 0 \\ -276.693 & 65.1042 & 1253.26 & -585.938 & 0 & 0 & -976.563 & 520.833 & 0 & 0 & 0 & 0 \\ 325.521 & -1204.43 & -585.938 & 1595.05 & 0 & 0 & 260.417 & -390.625 & 0 & 0 & 0 & 0 \\ -976.563 & 520.833 & 0 & 0 & 1671.01 & -781.25 & -694.444 & 260.417 & 0 & 0 & 0 & 0 \\ 260.417 & -390.625 & 0 & 0 & -781.25 & 2126.74 & 520.833 & -1736.11 & 0 & 0 & 0 & 0 \\ -325.521 & -781.25 & -976.563 & 260.417 & -694.444 & 520.833 & 1996.53 & 0 & 0 & 0 & 0 & 0 \\ -781.25 & -130.208 & 520.833 & -390.625 & 260.417 & -1736.11 & 0 & 2256.94 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1.25 \\ -5. \\ 0 \\ 0 \\ -1.25 \\ -5. \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 3

$h = 0.25;$ $E = 10000;$ $\nu = 0.2$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 10416.7 & 2083.33 & 0 \\ 2083.33 & 10416.7 & 0 \\ 0 & 0 & 4166.67 \end{pmatrix}$

Nodal coordinates

Element node	Global node number	x	y
1	3	2.	0.
2	5	4.	0.
3	6	4.	1.

$x_1 = 2.$ $x_2 = 4.$ $x_3 = 4.$
$y_1 = 0.$ $y_2 = 0.$ $y_3 = 1.$

Using these values we get

$b_1 = -1.$ $b_2 = 1.$ $b_3 = 0.$

$c_1 = 0.$ $c_2 = -2.$ $c_3 = 2.$

$f_1 = 4.$ $f_2 = -2.$ $f_3 = 0.$

Element area, $A = 1.$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -0.5 & 0 & 0.5 & 0 & 0. & 0 \\ 0 & 0. & 0 & -1. & 0 & 1. \\ 0. & -0.5 & -1. & 0.5 & 1. & 0. \end{pmatrix}$$

Thus the element stiffness matrix is

$$\boldsymbol{k} = \mathrm{hA}\boldsymbol{BCB}^{\mathrm{T}} = \begin{pmatrix} 651.042 & 0 & -651.042 & 260.417 & 0 & -260.417 \\ 0 & 260.417 & 520.833 & -260.417 & -520.833 & 0 \\ -651.042 & 520.833 & 1692.71 & -781.25 & -1041.67 & 260.417 \\ 260.417 & -260.417 & -781.25 & 2864.58 & 520.833 & -2604.17 \\ 0 & -520.833 & -1041.67 & 520.833 & 1041.67 & 0 \\ -260.417 & 0 & 260.417 & -2604.17 & 0 & 2604.17 \end{pmatrix}$$

Complete equations for element 3

$$\begin{pmatrix} 651.042 & 0 & -651.042 & 260.417 & 0 & -260.417 \\ 0 & 260.417 & 520.833 & -260.417 & -520.833 & 0 \\ -651.042 & 520.833 & 1692.71 & -781.25 & -1041.67 & 260.417 \\ 260.417 & -260.417 & -781.25 & 2864.58 & 520.833 & -2604.17 \\ 0 & -520.833 & -1041.67 & 520.833 & 1041.67 & 0 \\ -260.417 & 0 & 260.417 & -2604.17 & 0 & 2604.17 \end{pmatrix} \begin{pmatrix} u_3 \\ v_3 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {5, 6, 9, 10, 11, 12} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 5 \\ 6 \\ 9 \\ 10 \\ 11 \\ 12 \end{pmatrix}$

and to a global matrix: $\begin{pmatrix} [5,5] & [5,6] & [5,9] & [5,10] & [5,11] & [5,12] \\ [6,5] & [6,6] & [6,9] & [6,10] & [6,11] & [6,12] \\ [9,5] & [9,6] & [9,9] & [9,10] & [9,11] & [9,12] \\ [10,5] & [10,6] & [10,9] & [10,10] & [10,11] & [10,12] \\ [11,5] & [11,6] & [11,9] & [11,10] & [11,11] & [11,12] \\ [12,5] & [12,6] & [12,9] & [12,10] & [12,11] & [12,12] \end{pmatrix}$

Adding element equations into appropriate locations we have

$$\left(\begin{array}{cccccccccc} 1578.78 & 195.313 & -276.693 & 325.521 & -976.563 & 260.417 & -325.521 & -781.25 & 0 & 0 \\ 195.313 & 1725.26 & 65.1042 & -1204.43 & 520.833 & -390.625 & -781.25 & -130.208 & 0 & 0 \\ -276.693 & 65.1042 & 1253.26 & -585.938 & 0 & 0 & -976.563 & 520.833 & 0 & 0 \\ 325.521 & -1204.43 & -585.938 & 1595.05 & 0 & 0 & 260.417 & -390.625 & 0 & 0 \\ -976.563 & 520.833 & 0 & 0 & 2322.05 & -781.25 & -694.444 & 260.417 & -651.042 & 260. \\ 260.417 & -390.625 & 0 & 0 & -781.25 & 2387.15 & 520.833 & -1736.11 & 520.833 & -260. \\ -325.521 & -781.25 & -976.563 & 260.417 & -694.444 & 520.833 & 1996.53 & 0 & 0 & 0 \\ -781.25 & -130.208 & 520.833 & -390.625 & 260.417 & -1736.11 & 0 & 2256.94 & 0 & 0 \\ 0 & 0 & 0 & 0 & -651.042 & 520.833 & 0 & 0 & 1692.71 & -781. \\ 0 & 0 & 0 & 0 & 260.417 & -260.417 & 0 & 0 & -781.25 & 2864. \\ 0 & 0 & 0 & 0 & 0 & -520.833 & 0 & 0 & -1041.67 & 520. \\ 0 & 0 & 0 & 0 & -260.417 & 0 & 0 & 0 & 260.417 & -2604. \end{array} \right.$$

Equations for element 4

$h = 0.25;$ $E = 10000;$ $\nu = 0.2$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 10416.7 & 2083.33 & 0 \\ 2083.33 & 10416.7 & 0 \\ 0 & 0 & 4166.67 \end{pmatrix}$

Nodal coordinates

Element node	Global node number	x	y
1	6	4.	1.
2	4	2.	1.5
3	3	2.	0.

$x_1 = 4.$ $x_2 = 2.$ $x_3 = 2.$
$y_1 = 1.$ $y_2 = 1.5$ $y_3 = 0.$

Using these values we get

$b_1 = 1.5$ $b_2 = -1.$ $b_3 = -0.5$

$c_1 = 0.$ $c_2 = 2.$ $c_3 = -2.$

$f_1 = -3.$ $f_2 = 2.$ $f_3 = 4.$

Element area, A = 1.5

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 0.5 & 0 & -0.333333 & 0 & -0.166667 & 0 \\ 0 & 0. & 0 & 0.666667 & 0 & -0.666667 \\ 0. & 0.5 & 0.666667 & -0.333333 & -0.666667 & -0.166667 \end{pmatrix}$$

Thus the element stiffness matrix is

$$\boldsymbol{k} = \mathrm{hA}\boldsymbol{BCB}^{\mathrm{T}} = \begin{pmatrix} 976.563 & 0 & -651.042 & 260.417 & -325.521 & -260.417 \\ 0 & 390.625 & 520.833 & -260.417 & -520.833 & -130.208 \\ -651.042 & 520.833 & 1128.47 & -520.833 & -477.431 & 0 \\ 260.417 & -260.417 & -520.833 & 1909.72 & 260.417 & -1649.31 \\ -325.521 & -520.833 & -477.431 & 260.417 & 802.951 & 260.417 \\ -260.417 & -130.208 & 0 & -1649.31 & 260.417 & 1779.51 \end{pmatrix}$$

Load vector due to distributed load on side 1 (nodes {6, 4})

Specified load components: $q_n = -20;$ $q_t = 0$

End nodal coordinates: ({4., 1.} {2., 1.5}) giving side length, L = 2.06155

Components of unit normal to the side: $n_x = 0.242536;$ $n_y = 0.970143$

Using these values we get
$\boldsymbol{r}_q^{\mathrm{T}} = (\,-1.25 \quad -5. \quad -1.25 \quad -5. \quad 0 \quad 0\,)$

Complete equations for element 4

$$\begin{pmatrix} 976.563 & 0 & -651.042 & 260.417 & -325.521 & -260.417 \\ 0 & 390.625 & 520.833 & -260.417 & -520.833 & -130.208 \\ -651.042 & 520.833 & 1128.47 & -520.833 & -477.431 & 0 \\ 260.417 & -260.417 & -520.833 & 1909.72 & 260.417 & -1649.31 \\ -325.521 & -520.833 & -477.431 & 260.417 & 802.951 & 260.417 \\ -260.417 & -130.208 & 0 & -1649.31 & 260.417 & 1779.51 \end{pmatrix} \begin{pmatrix} u_6 \\ v_6 \\ u_4 \\ v_4 \\ u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} -1.25 \\ -5. \\ -1.25 \\ -5. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {11, 12, 7, 8, 5, 6} global degrees of freedom.

$$\text{Locations for element contributions to a global vector: } \begin{pmatrix} 11 \\ 12 \\ 7 \\ 8 \\ 5 \\ 6 \end{pmatrix}$$

$$\text{and to a global matrix: } \begin{pmatrix} [11, 11] & [11, 12] & [11, 7] & [11, 8] & [11, 5] & [11, 6] \\ [12, 11] & [12, 12] & [12, 7] & [12, 8] & [12, 5] & [12, 6] \\ [7, 11] & [7, 12] & [7, 7] & [7, 8] & [7, 5] & [7, 6] \\ [8, 11] & [8, 12] & [8, 7] & [8, 8] & [8, 5] & [8, 6] \\ [5, 11] & [5, 12] & [5, 7] & [5, 8] & [5, 5] & [5, 6] \\ [6, 11] & [6, 12] & [6, 7] & [6, 8] & [6, 5] & [6, 6] \end{pmatrix}$$

Adding element equations into appropriate locations we have

$$\left(\begin{array}{cccccccccc} 1578.78 & 195.313 & -276.693 & 325.521 & -976.563 & 260.417 & -325.521 & -781.25 & 0 & \\ 195.313 & 1725.26 & 65.1042 & -1204.43 & 520.833 & -390.625 & -781.25 & -130.208 & 0 & \\ -276.693 & 65.1042 & 1253.26 & -585.938 & 0 & 0 & -976.563 & 520.833 & 0 & \\ 325.521 & -1204.43 & -585.938 & 1595.05 & 0 & 0 & 260.417 & -390.625 & 0 & \\ -976.563 & 520.833 & 0 & 0 & 3125. & -520.833 & -1171.88 & 520.833 & -651.042 & 2 \\ 260.417 & -390.625 & 0 & 0 & -520.833 & 4166.67 & 520.833 & -3385.42 & 520.833 & -2 \\ -325.521 & -781.25 & -976.563 & 260.417 & -1171.88 & 520.833 & 3125. & -520.833 & 0 & \\ -781.25 & -130.208 & 520.833 & -390.625 & 520.833 & -3385.42 & -520.833 & 4166.67 & 0 & \\ 0 & 0 & 0 & 0 & -651.042 & 520.833 & 0 & 0 & 1692.71 & -7 \\ 0 & 0 & 0 & 0 & 260.417 & -260.417 & 0 & 0 & -781.25 & 28 \\ 0 & 0 & 0 & 0 & -325.521 & -781.25 & -651.042 & 260.417 & -1041.67 & 5 \\ 0 & 0 & 0 & 0 & -781.25 & -130.208 & 520.833 & -260.417 & 260.417 & -26 \end{array} \right.$$

Essential boundary conditions

Node	dof	Value
1	u_1	0
	v_1	0
2	u_2	0
	v_2	0

Remove {1, 2, 3, 4} rows and columns.

After adjusting for essential boundary conditions we have

$$\begin{pmatrix} 3125. & -520.833 & -1171.88 & 520.833 & -651.042 & 260.417 & -325.521 & -781.25 \\ -520.833 & 4166.67 & 520.833 & -3385.42 & 520.833 & -260.417 & -781.25 & -130.208 \\ -1171.88 & 520.833 & 3125. & -520.833 & 0 & 0 & -651.042 & 520.833 \\ 520.833 & -3385.42 & -520.833 & 4166.67 & 0 & 0 & 260.417 & -260.417 \\ -651.042 & 520.833 & 0 & 0 & 1692.71 & -781.25 & -1041.67 & 260.417 \\ 260.417 & -260.417 & 0 & 0 & -781.25 & 2864.58 & 520.833 & -2604.17 \\ -325.521 & -781.25 & -651.042 & 260.417 & -1041.67 & 520.833 & 2018.23 & 0 \\ -781.25 & -130.208 & 520.833 & -260.417 & 260.417 & -2604.17 & 0 & 2994.79 \end{pmatrix}$$

$$\begin{pmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2.5 \\ -10. \\ 0 \\ 0 \\ -1.25 \\ -5. \end{pmatrix}$$

Solving the final system of global equations we get

$$\{u_3 = -0.0103553,\ v_3 = -0.0255297,\ u_4 = 0.00472765,\ v_4 = -0.0247357,$$
$$u_5 = -0.0131394,\ v_5 = -0.0554931,\ u_6 = 0.0000838902,\ v_6 = -0.0555664\}$$

Complete table of nodal values

	u	v
1	0	0
2	0	0
3	−0.0103553	−0.0255297
4	0.00472765	−0.0247357
5	−0.0131394	−0.0554931
6	0.0000838902	−0.0555664

Computation of reactions

Equation numbers of dof with specified values: {1, 2, 3, 4}

Extracting equations {1, 2, 3, 4} from the global system we have

$$\begin{pmatrix} 1578.78 & 195.313 & -276.693 & 325.521 & -976.563 & 260.417 & -325.521 & -781.25 & 0 & 0 & 0 & 0 \\ 195.313 & 1725.26 & 65.1042 & -1204.43 & 520.833 & -390.625 & -781.25 & -130.208 & 0 & 0 & 0 & 0 \\ -276.693 & 65.1042 & 1253.26 & -585.938 & 0 & 0 & -976.563 & 520.833 & 0 & 0 & 0 & 0 \\ 325.521 & -1204.43 & -585.938 & 1595.05 & 0 & 0 & 260.417 & -390.625 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \end{pmatrix} =$$

$$\begin{pmatrix} R_1 + 0. \\ R_2 + 0. \\ R_3 - 1.25 \\ R_4 - 5. \end{pmatrix}$$

Substituting the nodal values and re−arranging

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{pmatrix} = \begin{pmatrix} 1578.78 & 195.313 & -276.693 & 325.521 & -976.563 & 260.417 & -325.521 & -781.25 & 0 & 0 & 0 & 0 \\ 195.313 & 1725.26 & 65.1042 & -1204.43 & 520.833 & -390.625 & -781.25 & -130.208 & 0 & 0 & 0 & 0 \\ -276.693 & 65.1042 & 1253.26 & -585.938 & 0 & 0 & -976.563 & 520.833 & 0 & 0 & 0 & 0 \\ 325.521 & -1204.43 & -585.938 & 1595.05 & 0 & 0 & 260.417 & -390.625 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -0.0103553 \\ -0.0255297 \\ 0.00472765 \\ -0.0247357 \\ -0.0131394 \\ -0.0554931 \\ 0.0000838902 \\ -0.0555664 \end{pmatrix} - \begin{pmatrix} 0. \\ 0. \\ -1.25 \\ -5. \end{pmatrix}$$

Carrying out computations, the reactions are as follows.

Label	dof	Reaction
R_1	u_1	21.25
R_2	v_1	4.10648
R_3	u_2	−16.25
R_4	v_2	15.8935

Sum of Reactions

dof: u	5.
dof: v	20.

Solution for element 1

h = 0.25; E = 10000; $\nu = 0.2$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 10416.7 & 2083.33 & 0 \\ 2083.33 & 10416.7 & 0 \\ 0 & 0 & 4166.67 \end{pmatrix}$

Element nodes: First node (node # 1): {0., 0.}
Second node (node # 3): {2., 0.} Third node (node # 4): {2., 1.5}

$x_1 = 0.$ $x_2 = 2.$ $x_3 = 2.$
$y_1 = 0.$ $y_2 = 0.$ $y_3 = 1.5$

Using these values we get

$b_1 = -1.5$ $b_2 = 1.5$ $b_3 = 0.$

$c_1 = 0.$ $c_2 = -2.$ $c_3 = 2.$

$f_1 = 3.$ $f_2 = 0.$ $f_3 = 0.$

Element area, A = 1.5

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -0.5 & 0 & 0.5 & 0 & 0. & 0 \\ 0 & 0. & 0 & -0.666667 & 0 & 0.666667 \\ 0. & -0.5 & -0.666667 & 0.5 & 0.666667 & 0. \end{pmatrix}$$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\{1. - 0.5\,x,\ 0.5\,x - 0.666667\,y,\ 0.666667\,y\}$

$$\boldsymbol{N}^{\mathrm{T}} = \begin{pmatrix} 1. - 0.5\,x & 0 & 0.5\,x - 0.666667\,y & 0 & 0.666667\,y & 0 \\ 0 & 1. - 0.5\,x & 0 & 0.5\,x - 0.666667\,y & 0 & 0.666667\,y \end{pmatrix}$$

From global solution the displacements at the element nodes are
(displacements at nodes {1, 3, 4}):
$\boldsymbol{d}^{\mathrm{T}} = \{0, 0, -0.0103553, -0.0255297, 0.00472765, -0.0247357\}$

The displacement distribution over the element is

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{d} = \begin{pmatrix} 0.0100553\,y - 0.00517764\,x \\ 0.000529362\,y - 0.0127648\,x \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{d} = (\,-0.00517764 \quad 0.000529362 \quad -0.00270956\,)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\,-52.8309 \quad -5.27256 \quad -11.2898\,)$

Computing out−of−plane strain and stress components
using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{\mathrm{T}} = (\,-0.00517764 \quad 0.000529362 \quad 0.00116207 \quad -0.00270956 \quad 0 \quad 0\,)$

$\boldsymbol{\sigma}^{\mathrm{T}} = (\,-52.8309 \quad -5.27256 \quad 0 \quad -11.2898 \quad 0 \quad 0\,)$

Substituting these stress components into appropriate formulas

Principal stresses = $(\,0 \quad -2.72856 \quad -55.3749\,)$

Effective stress (von Mises) = 54.0623

Solution for element 2

h = 0.25; E = 10000; $\nu = 0.2$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 10416.7 & 2083.33 & 0 \\ 2083.33 & 10416.7 & 0 \\ 0 & 0 & 4166.67 \end{pmatrix}$

Element nodes: First node (node # 4): {2., 1.5}
Second node (node # 2): {0., 2.} Third node (node # 1): {0., 0.}

$x_1 = 2.$ $x_2 = 0.$ $x_3 = 0.$
$y_1 = 1.5$ $y_2 = 2.$ $y_3 = 0.$

Using these values we get

$b_1 = 2.$ $b_2 = -1.5$ $b_3 = -0.5$

$c_1 = 0.$ $c_2 = 2.$ $c_3 = -2.$

$f_1 = 0.$ $f_2 = 0.$ $f_3 = 4.$

Element area, A = 2.

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 0.5 & 0 & -0.375 & 0 & -0.125 & 0 \\ 0 & 0. & 0 & 0.5 & 0 & -0.5 \\ 0. & 0.5 & 0.5 & -0.375 & -0.5 & -0.125 \end{pmatrix}$$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\{0.5\,x,\ 0.5\,y - 0.375\,x,\ -0.125\,x - 0.5\,y + 1.\}$

$$\boldsymbol{N}^{\mathrm{T}} = \begin{pmatrix} 0.5\,x & 0 & 0.5\,y - 0.375\,x & 0 & -0.125\,x - 0.5\,y + 1. & 0 \\ 0 & 0.5\,x & 0 & 0.5\,y - 0.375\,x & 0 & -0.125\,x - 0.5\,y + 1. \end{pmatrix}$$

From global solution the displacements at the element nodes are
(displacements at nodes {4, 2, 1}):
$\boldsymbol{d}^{\mathrm{T}} = \{0.00472765, -0.0247357, 0, 0, 0, 0\}$

The displacement distribution over the element is

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{d} = \begin{pmatrix} 0.00236383\,x \\ -0.0123678\,x \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{d} = (\ 0.00236383 \quad 0 \quad -0.0123678\ )$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\ 24.6232 \quad 4.92464 \quad -51.5326\ )$

Computing out−of−plane strain and stress components
using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{\mathrm{T}} = (\ 0.00236383 \quad 0 \quad -0.000590956 \quad -0.0123678 \quad 0 \quad 0\ )$

$\boldsymbol{\sigma}^{\mathrm{T}} = (\ 24.6232 \quad 4.92464 \quad 0 \quad -51.5326 \quad 0 \quad 0\ )$

Substituting these stress components into appropriate formulas

Principal stresses = (67.2393   0   −37.6915)

Effective stress (von Mises) = 92.0659

Solution for element 3

$h = 0.25;$ $E = 10000;$ $\nu = 0.2$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 10416.7 & 2083.33 & 0 \\ 2083.33 & 10416.7 & 0 \\ 0 & 0 & 4166.67 \end{pmatrix}$

Element nodes: First node (node # 3): {2., 0.}
Second node (node # 5): {4., 0.} Third node (node # 6): {4., 1.}

$x_1 = 2.$ $x_2 = 4.$ $x_3 = 4.$
$y_1 = 0.$ $y_2 = 0.$ $y_3 = 1.$

Using these values we get

$b_1 = -1.$ $b_2 = 1.$ $b_3 = 0.$

$c_1 = 0.$ $c_2 = -2.$ $c_3 = 2.$

$f_1 = 4.$ $f_2 = -2.$ $f_3 = 0.$

Element area, A = 1.

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -0.5 & 0 & 0.5 & 0 & 0. & 0 \\ 0 & 0. & 0 & -1. & 0 & 1. \\ 0. & -0.5 & -1. & 0.5 & 1. & 0. \end{pmatrix}$$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\{2. - 0.5\,x,\ 0.5\,x - 1.\,y - 1.,\ 1.\,y\}$

$$\boldsymbol{N}^{\mathrm{T}} = \begin{pmatrix} 2. - 0.5\,x & 0 & 0.5\,x - 1.\,y - 1. & 0 & 1.\,y & 0 \\ 0 & 2. - 0.5\,x & 0 & 0.5\,x - 1.\,y - 1. & 0 & 1.\,y \end{pmatrix}$$

From global solution the displacements at the element nodes are
(displacements at nodes {3, 5, 6}):
$\boldsymbol{d}^{\mathrm{T}} = \{-0.0103553, -0.0255297, -0.0131394, -0.0554931, 0.0000838902, -0.0555664\}$

The displacement distribution over the element is

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{d} = \begin{pmatrix} -0.00139207\,x + 0.0132233\,y - 0.00757114 \\ -0.0149817\,x - 0.0000732667\,y + 0.00443371 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{d} = (\,-0.00139207 \quad -0.0000732667 \quad -0.0017584\,)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\,-14.6533 \quad -3.66334 \quad -7.32667\,)$

Computing out−of−plane strain and stress components
using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{\mathrm{T}} = (\,-0.00139207 \quad -0.0000732667 \quad 0.000366334 \quad -0.0017584 \quad 0 \quad 0\,)$

$\boldsymbol{\sigma}^{\mathrm{T}} = (\,-14.6533 \quad -3.66334 \quad 0 \quad -7.32667 \quad 0 \quad 0\,)$

Substituting these stress components into appropriate formulas

Principal stresses = $(\,0 \quad 0 \quad -18.3167\,)$

Effective stress (von Mises) = 18.3167

Solution for element 4

$h = 0.25;$ $E = 10000;$ $\nu = 0.2$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 10416.7 & 2083.33 & 0 \\ 2083.33 & 10416.7 & 0 \\ 0 & 0 & 4166.67 \end{pmatrix}$

Element nodes: First node (node # 6): {4., 1.}
Second node (node # 4): {2., 1.5} Third node (node # 3): {2., 0.}

$x_1 = 4.$ $\quad x_2 = 2.$ $\quad x_3 = 2.$
$y_1 = 1.$ $\quad y_2 = 1.5$ $\quad y_3 = 0.$

Using these values we get

$b_1 = 1.5 \quad b_2 = -1. \quad b_3 = -0.5$

$c_1 = 0. \quad c_2 = 2. \quad c_3 = -2.$

$f_1 = -3. \quad f_2 = 2. \quad f_3 = 4.$

Element area, A = 1.5

$$\boldsymbol{B}^T = \begin{pmatrix} 0.5 & 0 & -0.333333 & 0 & -0.166667 & 0 \\ 0 & 0. & 0 & 0.666667 & 0 & -0.666667 \\ 0. & 0.5 & 0.666667 & -0.333333 & -0.666667 & -0.166667 \end{pmatrix}$$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\{0.5\,x - 1.,\ -0.333333\,x + 0.666667\,y + 0.666667,\ -0.166667\,x - 0.666667\,y + 1.33333\}$

$$\boldsymbol{N}^T = \begin{pmatrix} 0.5\,x - 1. & 0 & -0.333333\,x + 0.666667\,y + 0.666667 & 0 & -0.1 \\ 0 & 0.5\,x - 1. & 0 & -0.333333\,x + 0.666667\,y + 0.666667 \end{pmatrix}$$

From global solution the displacements at the element nodes are
(displacements at nodes {6, 4, 3}):
$\boldsymbol{d}^T = \{0.0000838902,\ -0.0555664,\ 0.00472765,\ -0.0247357,\ -0.0103553,\ -0.0255297\}$

The displacement distribution over the element is

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \boldsymbol{N}^T\boldsymbol{d} = \begin{pmatrix} 0.000191941\,x + 0.0100553\,y - 0.0107392 \\ -0.015283\,x + 0.000529362\,y + 0.00503634 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^T\boldsymbol{d} = (\ 0.000191941 \quad 0.000529362 \quad -0.00522773\ )$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\ 3.10223 \quad 5.91407 \quad -21.7822\ )$

Computing out−of−plane strain and stress components
using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^T = (\ 0.000191941 \quad 0.000529362 \quad -0.000180326 \quad -0.00522773 \quad 0 \quad 0\ )$

$\boldsymbol{\sigma}^T = (\ 3.10223 \quad 5.91407 \quad 0 \quad -21.7822 \quad 0 \quad 0\ )$

Substituting these stress components into appropriate formulas

Principal stresses = $(\ 26.3357 \quad 0 \quad -17.3194\ )$

Effective stress (von Mises) = 38.0742

Solution summary

Nodal solution

	x	y	u	v
1	0.	0.	0	0
2	0.	2.	0	0
3	2.	0.	−0.0103553	−0.0255297
4	2.	1.5	0.00472765	−0.0247357
5	4.	0.	−0.0131394	−0.0554931
6	4.	1.	0.0000838902	−0.0555664

Solution at element centers

	Coord	Disp	Stresses	Principal stresses	Effective Stress
1	1.33333 0.5	−0.00187588 −0.0167551	−52.8309 −5.27256 0 −11.2898 0 0	0 −2.72856 −55.3749	54.0623
2	0.666667 1.16667	0.00157588 −0.00824522	24.6232 4.92464 0 −51.5326 0 0	67.2393 0 −37.6915	92.0659
3	3.33333 0.333333	−0.0078036 −0.0455297	−14.6533 −3.66334 0 −7.32667 0 0	0 0 −18.3167	18.3167
4	2.66667 0.833333	−0.00184791 −0.0352772	3.10223 5.91407 0 −21.7822 0 0	26.3357 0 −17.3194	38.0742

Support reactions

Node	dof	Reaction
1	1	21.25
1	2	4.10648
2	1	−16.25
2	2	15.8935

Sum of applied loads → $(-5. \quad -20.)$

Sum of support reactions → $(5. \quad 20.)$