Example 2.9: Tapered bar (p. 158)

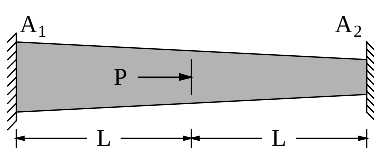
Consider solution of a tapered axially loaded bar shown in Figure. Use two node uniform axial deformation element to model the bar and determine the axial stress and force distribution in the bar. Compute the support reactions from the axial force and see whether the overall equilibrium is satisfied. Comment on the quality of the finite element solution. Use the following numerical data.

$$E = 70\,\text{GPa}; \quad F = 20\,\text{kN}; \quad L = 300\,\text{mm}; \quad A_1 = 2400\,\text{mm}^2; \quad A_2 = 600\,\text{mm}^2$$

where A_1 and A_2 are the areas of cross-section at the two ends of the bar. The area of cross-section can be expressed as a linear function of x using the Lagrange interpolation formula as follows.

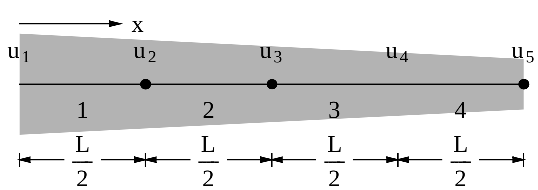
$$A(x) = \frac{x-2L}{-2L} A_1 + \frac{x}{2L} A_2$$

Since the displacements are generally small, numerically it is convenient to use $N-\text{mm}$ units. Then the computed nodal displacements are in mm and the stresses in MPa.



Denoting area of cross-section at the left end of an element by A_l and that at the right end by A_r, the average area for each element is $(A_l + A_r)/2$. The concentrated nodal loads will be added directly to the global equations after assembly. Thus the element equations are as follows.

$$\frac{(A_l+A_r)E}{2\ell} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u_l \\ u_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



Nodal locations: {0, 150., 300., 450., 600.}

Areas at the nodes: {2400., 1950., 1500., 1050., 600.}

Average area for each element: {2175., 1725., 1275., 825.}

4 element solution

Nodal locations: {0, 150., 300., 450., 600.}

Element 1

Element nodes: $\{x_1 \to 0,\ x_2 \to 150.\}$

$$\begin{pmatrix} 1.015\times10^6 & -1.015\times10^6 \\ -1.015\times10^6 & 1.015\times10^6 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Global equations after assembly of this element

$$\begin{pmatrix} 1.015\times10^6 & -1.015\times10^6 & 0 & 0 & 0 \\ -1.015\times10^6 & 1.015\times10^6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Element 2

Element nodes: $\{x_2 \to 150.,\ x_3 \to 300.\}$

$$\begin{pmatrix} 805000. & -805000. \\ -805000. & 805000. \end{pmatrix}\begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Global equations after assembly of this element

$$\begin{pmatrix} 1.015\times10^6 & -1.015\times10^6 & 0 & 0 & 0 \\ -1.015\times10^6 & 1.82\times10^6 & -805000. & 0 & 0 \\ 0 & -805000. & 805000. & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Element 3

Element nodes: $\{x_3 \to 300., x_4 \to 450.\}$

$$\begin{pmatrix} 595000. & -595000. \\ -595000. & 595000. \end{pmatrix} \begin{pmatrix} u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Global equations after assembly of this element

$$\begin{pmatrix} 1.015\times10^6 & -1.015\times10^6 & 0 & 0 & 0 \\ -1.015\times10^6 & 1.82\times10^6 & -805000. & 0 & 0 \\ 0 & -805000. & 1.4\times10^6 & -595000. & 0 \\ 0 & 0 & -595000. & 595000. & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Element 4

Element nodes: $\{x_4 \to 450., x_5 \to 600.\}$

$$\begin{pmatrix} 385000. & -385000. \\ -385000. & 385000. \end{pmatrix} \begin{pmatrix} u_4 \\ u_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Global equations after assembly of this element

$$\begin{pmatrix} 1.015\times10^6 & -1.015\times10^6 & 0 & 0 & 0 \\ -1.015\times10^6 & 1.82\times10^6 & -805000. & 0 & 0 \\ 0 & -805000. & 1.4\times10^6 & -595000. & 0 \\ 0 & 0 & -595000. & 980000. & -385000. \\ 0 & 0 & 0 & -385000. & 385000. \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Global equations before boundary conditions

$$\begin{pmatrix} 1.015\times10^6 & -1.015\times10^6 & 0 & 0 & 0 \\ -1.015\times10^6 & 1.82\times10^6 & -805000. & 0 & 0 \\ 0 & -805000. & 1.4\times10^6 & -595000. & 0 \\ 0 & 0 & -595000. & 980000. & -385000. \\ 0 & 0 & 0 & -385000. & 385000. \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Natural boundary conditions

DOF	α	β
u_3	0	20000

DOF	k(x)	$-$k(x) α	k(x) β
u_3	1	0	20000

Global equations after incorporating NBC

$$\begin{pmatrix} 1.015\times 10^6 & -1.015\times 10^6 & 0 & 0 & 0 \\ -1.015\times 10^6 & 1.82\times 10^6 & -805000. & 0 & 0 \\ 0 & -805000. & 1.4\times 10^6 & -595000. & 0 \\ 0 & 0 & -595000. & 980000. & -385000. \\ 0 & 0 & 0 & -385000. & 385000. \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 20000 \\ 0 \\ 0 \end{pmatrix}$$

Essential boundary conditions

DOF	Value
u_1	0
u_5	0

Incorporating EBC the final system of equations is

$$\begin{pmatrix} 1.82\times 10^6 & -805000. & 0 \\ -805000. & 1.4\times 10^6 & -595000. \\ 0 & -595000. & 980000. \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0. \\ 20000. \\ 0. \end{pmatrix}$$

Solution for nodal unknowns

DOF	x	Solution
u_1	0	0
u_2	150.	0.0129577
u_3	300.	0.0292958
u_4	450.	0.0177867
u_5	600.	0

Solution over elements

Element 1

Nodes: $\{x_1 \to 0,\ x_2 \to 150.\}$

Interpolation functions: $\boldsymbol{N}^{\mathrm{T}} = \{1. - 0.00666667\,x,\ 0.00666667\,x\}$

Nodal values: $\boldsymbol{d}^{\mathrm{T}} = \{0,\ 0.0129577\}$

Solution: $u(x) = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{d} = 0.000086385\,x$

Element 2

Nodes: $\{x_1 \to 150.,\ x_2 \to 300.\}$

Interpolation functions: $\boldsymbol{N}^{\mathrm{T}} = \{2. - 0.00666667\,x,\ 0.00666667\,x - 1.\}$

Nodal values: $\boldsymbol{d}^{\mathrm{T}} = \{0.0129577,\ 0.0292958\}$

Solution: $u(x) = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{d} = 0.00010892\,x - 0.00338028$

Element 3

Nodes: $\{x_1 \to 300.,\ x_2 \to 450.\}$

Interpolation functions: $\boldsymbol{N}^{\mathrm{T}} = \{3. - 0.00666667\,x,\ 0.00666667\,x - 2.\}$

Nodal values: $\boldsymbol{d}^{\mathrm{T}} = \{0.0292958,\ 0.0177867\}$

Solution: $u(x) = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{d} = 0.0523139 - 0.000076727\,x$

Element 4

Nodes: $\{x_1 \to 450.,\ x_2 \to 600.\}$

Interpolation functions: $\boldsymbol{N}^{\mathrm{T}} = \{4. - 0.00666667\,x,\ 0.00666667\,x - 3.\}$

Nodal values: $\boldsymbol{d}^{\mathrm{T}} = \{0.0177867,\ 0\}$

Solution: $u(x) = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{d} = 0.0711469 - 0.000118578\,x$

Solution summary

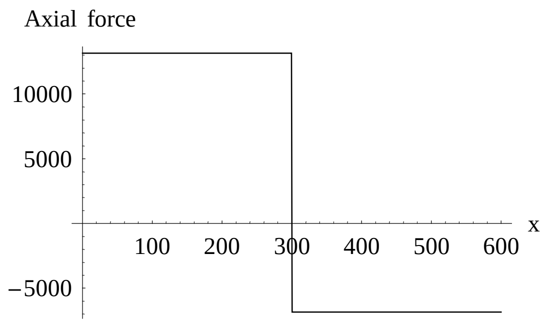
	Range	Solution
1	$0 \le x \le 150.$	$0.000086385\,x$
2	$150. \le x \le 300.$	$0.00010892\,x - 0.00338028$
3	$300. \le x \le 450.$	$0.0523139 - 0.000076727\,x$
4	$450. \le x \le 600.$	$0.0711469 - 0.000118578\,x$

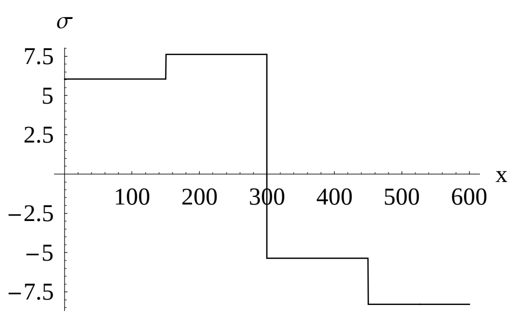
The axial forces at the ends must balance the support reactions. With the sign convention for axial forces discussed earlier, the reaction at the left support is the negative of the axial force at this point and that at the right support is equal to the axial force. Thus from the axial forces we get the support reactions as

Reactions = $\{-13152.1,\ -6847.89\}$

Sum of reactions = $-20000.$

The sum of reactions is equal and opposite to the applied load and therefore the overall equilibrium is satisfied. Plots of the axial stress and axial force are as follows. The axial force plot looks reasonable. In the stress plot we expect a discontinuity at the middle because of the concentrated applied force. However the stress at nodes 2 and 4 should be continuous. A large discontinuity in the stress at these locations indicates that the solution is not very accurate.





The following solution is obtained using 8 equal length elements.

8 element solution

Nodal locations: {0, 75., 150., 225., 300., 375., 450., 525., 600.}

Element 1

Element nodes: $\{x_1 \to 0, x_2 \to 75.\}$

$$\begin{pmatrix} 2.135\times10^6 & -2.135\times10^6 \\ -2.135\times10^6 & 2.135\times10^6 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Global equations after assembly of this element

$$\begin{pmatrix} 2.135\times10^6 & -2.135\times10^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2.135\times10^6 & 2.135\times10^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Element 2

Element nodes: $\{x_2 \to 75., x_3 \to 150.\}$

$$\begin{pmatrix} 1.925\times10^6 & -1.925\times10^6 \\ -1.925\times10^6 & 1.925\times10^6 \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Global equations after assembly of this element

$$\begin{pmatrix} 2.135\times10^6 & -2.135\times10^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2.135\times10^6 & 4.06\times10^6 & -1.925\times10^6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.925\times10^6 & 1.925\times10^6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Element 3

Element nodes: $\{x_3 \to 150., x_4 \to 225.\}$

$$\begin{pmatrix} 1.715\times10^6 & -1.715\times10^6 \\ -1.715\times10^6 & 1.715\times10^6 \end{pmatrix} \begin{pmatrix} u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Global equations after assembly of this element

$$\begin{pmatrix} 2.135\times10^6 & -2.135\times10^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2.135\times10^6 & 4.06\times10^6 & -1.925\times10^6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.925\times10^6 & 3.64\times10^6 & -1.715\times10^6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1.715\times10^6 & 1.715\times10^6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Element 4

Element nodes: $\{x_4 \to 225., x_5 \to 300.\}$

$$\begin{pmatrix} 1.505\times10^6 & -1.505\times10^6 \\ -1.505\times10^6 & 1.505\times10^6 \end{pmatrix} \begin{pmatrix} u_4 \\ u_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Global equations after assembly of this element

$$\begin{pmatrix} 2.135\times10^6 & -2.135\times10^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2.135\times10^6 & 4.06\times10^6 & -1.925\times10^6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.925\times10^6 & 3.64\times10^6 & -1.715\times10^6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1.715\times10^6 & 3.22\times10^6 & -1.505\times10^6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.505\times10^6 & 1.505\times10^6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Element 5

Element nodes: $\{x_5 \to 300., x_6 \to 375.\}$

$$\begin{pmatrix} 1.295\times10^6 & -1.295\times10^6 \\ -1.295\times10^6 & 1.295\times10^6 \end{pmatrix} \begin{pmatrix} u_5 \\ u_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Global equations after assembly of this element

$$\begin{pmatrix} 2.135\times10^6 & -2.135\times10^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2.135\times10^6 & 4.06\times10^6 & -1.925\times10^6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.925\times10^6 & 3.64\times10^6 & -1.715\times10^6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1.715\times10^6 & 3.22\times10^6 & -1.505\times10^6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.505\times10^6 & 2.8\times10^6 & -1.295\times10^6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.295\times10^6 & 1.295\times10^6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Element 6

Element nodes: $\{x_6 \to 375., x_7 \to 450.\}$

$$\begin{pmatrix} 1.085\times10^6 & -1.085\times10^6 \\ -1.085\times10^6 & 1.085\times10^6 \end{pmatrix} \begin{pmatrix} u_6 \\ u_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Global equations after assembly of this element

$$\begin{pmatrix} 2.135\times10^6 & -2.135\times10^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2.135\times10^6 & 4.06\times10^6 & -1.925\times10^6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.925\times10^6 & 3.64\times10^6 & -1.715\times10^6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1.715\times10^6 & 3.22\times10^6 & -1.505\times10^6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.505\times10^6 & 2.8\times10^6 & -1.295\times10^6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.295\times10^6 & 2.38\times10^6 & -1.085\times10^6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1.085\times10^6 & 1.085\times10^6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Element 7

Element nodes: $\{x_7 \to 450., x_8 \to 525.\}$

$$\begin{pmatrix} 875000. & -875000. \\ -875000. & 875000. \end{pmatrix} \begin{pmatrix} u_7 \\ u_8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Global equations after assembly of this element

$$\begin{pmatrix}
2.135\times10^6 & -2.135\times10^6 & 0 & 0 & 0 & 0 & 0 & \\
-2.135\times10^6 & 4.06\times10^6 & -1.925\times10^6 & 0 & 0 & 0 & 0 & \\
0 & -1.925\times10^6 & 3.64\times10^6 & -1.715\times10^6 & 0 & 0 & 0 & \\
0 & 0 & -1.715\times10^6 & 3.22\times10^6 & -1.505\times10^6 & 0 & 0 & \\
0 & 0 & 0 & -1.505\times10^6 & 2.8\times10^6 & -1.295\times10^6 & 0 & \\
0 & 0 & 0 & 0 & -1.295\times10^6 & 2.38\times10^6 & -1.085\times10^6 & \\
0 & 0 & 0 & 0 & 0 & -1.085\times10^6 & 1.96\times10^6 & - \\
0 & 0 & 0 & 0 & 0 & 0 & -875000. & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 &
\end{pmatrix}$$

Element 8

Element nodes: $\{x_8 \to 525., x_9 \to 600.\}$

$$\begin{pmatrix} 665000. & -665000. \\ -665000. & 665000. \end{pmatrix} \begin{pmatrix} u_8 \\ u_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Global equations after assembly of this element

$$\begin{pmatrix}
2.135\times10^6 & -2.135\times10^6 & 0 & 0 & 0 & 0 & 0 & \\
-2.135\times10^6 & 4.06\times10^6 & -1.925\times10^6 & 0 & 0 & 0 & 0 & \\
0 & -1.925\times10^6 & 3.64\times10^6 & -1.715\times10^6 & 0 & 0 & 0 & \\
0 & 0 & -1.715\times10^6 & 3.22\times10^6 & -1.505\times10^6 & 0 & 0 & \\
0 & 0 & 0 & -1.505\times10^6 & 2.8\times10^6 & -1.295\times10^6 & 0 & \\
0 & 0 & 0 & 0 & -1.295\times10^6 & 2.38\times10^6 & -1.085\times10^6 & \\
0 & 0 & 0 & 0 & 0 & -1.085\times10^6 & 1.96\times10^6 & - \\
0 & 0 & 0 & 0 & 0 & 0 & -875000. & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -
\end{pmatrix}$$

Global equations before boundary conditions

$$\left(\begin{array}{llllllll}
2.135\times10^{6} & -2.135\times10^{6} & 0 & 0 & 0 & 0 & 0 & \\
-2.135\times10^{6} & 4.06\times10^{6} & -1.925\times10^{6} & 0 & 0 & 0 & 0 & \\
0 & -1.925\times10^{6} & 3.64\times10^{6} & -1.715\times10^{6} & 0 & 0 & 0 & \\
0 & 0 & -1.715\times10^{6} & 3.22\times10^{6} & -1.505\times10^{6} & 0 & 0 & \\
0 & 0 & 0 & -1.505\times10^{6} & 2.8\times10^{6} & -1.295\times10^{6} & 0 & \\
0 & 0 & 0 & 0 & -1.295\times10^{6} & 2.38\times10^{6} & -1.085\times10^{6} & \\
0 & 0 & 0 & 0 & 0 & -1.085\times10^{6} & 1.96\times10^{6} & - \\
0 & 0 & 0 & 0 & 0 & 0 & -875000. & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -
\end{array}\right.$$

Natural boundary conditions

DOF	α	β
u_5	0	20000

DOF	k(x)	$-$k(x) α	k(x) β
u_5	1	0	20000

Global equations after incorporating NBC

$$\left(\begin{array}{llllllll}
2.135\times10^{6} & -2.135\times10^{6} & 0 & 0 & 0 & 0 & 0 & \\
-2.135\times10^{6} & 4.06\times10^{6} & -1.925\times10^{6} & 0 & 0 & 0 & 0 & \\
0 & -1.925\times10^{6} & 3.64\times10^{6} & -1.715\times10^{6} & 0 & 0 & 0 & \\
0 & 0 & -1.715\times10^{6} & 3.22\times10^{6} & -1.505\times10^{6} & 0 & 0 & \\
0 & 0 & 0 & -1.505\times10^{6} & 2.8\times10^{6} & -1.295\times10^{6} & 0 & \\
0 & 0 & 0 & 0 & -1.295\times10^{6} & 2.38\times10^{6} & -1.085\times10^{6} & \\
0 & 0 & 0 & 0 & 0 & -1.085\times10^{6} & 1.96\times10^{6} & - \\
0 & 0 & 0 & 0 & 0 & 0 & -875000. & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -
\end{array}\right.$$

Essential boundary conditions

DOF	Value
u_1	0
u_9	0

Incorporating EBC the final system of equations is

$$\begin{pmatrix} 4.06\times10^6 & -1.925\times10^6 & 0 & 0 & 0 & 0 & 0 \\ -1.925\times10^6 & 3.64\times10^6 & -1.715\times10^6 & 0 & 0 & 0 & 0 \\ 0 & -1.715\times10^6 & 3.22\times10^6 & -1.505\times10^6 & 0 & 0 & 0 \\ 0 & 0 & -1.505\times10^6 & 2.8\times10^6 & -1.295\times10^6 & 0 & 0 \\ 0 & 0 & 0 & -1.295\times10^6 & 2.38\times10^6 & -1.085\times10^6 & 0 \\ 0 & 0 & 0 & 0 & -1.085\times10^6 & 1.96\times10^6 & -875000. \\ 0 & 0 & 0 & 0 & 0 & -875000. & 1.54\times10 \end{pmatrix}$$

Solution for nodal unknowns

DOF	x	Solution
u_1	0	0
u_2	75.	0.00618323
u_3	150.	0.013041
u_4	225.	0.0207385
u_5	300.	0.02951
u_6	375.	0.02426
u_7	450.	0.0179938
u_8	525.	0.0102238
u_9	600.	0

Solution over elements

Element 1

Nodes: $\{x_1 \to 0,\ x_2 \to 75.\}$

Interpolation functions: $\boldsymbol{N}^{\mathrm{T}} = \{1. - 0.0133333\,x,\ 0.0133333\,x\}$

Nodal values: $\boldsymbol{d}^{\mathrm{T}} = \{0,\ 0.00618323\}$

Solution: $u(x) = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{d} = 0.0000824431\,x$

Element 2

Nodes: $\{x_1 \to 75.,\ x_2 \to 150.\}$

Interpolation functions: $\boldsymbol{N}^{\mathrm{T}} = \{2. - 0.0133333\,x,\ 0.0133333\,x - 1.\}$

Nodal values: $\boldsymbol{d}^{\mathrm{T}} = \{0.00618323,\ 0.013041\}$

Solution: $u(x) = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{d} = 0.0000914369\,x - 0.000674534$

Element 3

Nodes: $\{x_1 \to 150.,\ x_2 \to 225.\}$

Interpolation functions: $\boldsymbol{N}^{\mathrm{T}} = \{3. - 0.0133333\,x,\ 0.0133333\,x - 2.\}$

Nodal values: $\boldsymbol{d}^{\mathrm{T}} = \{0.013041,\ 0.0207385\}$

Solution: $u(x) = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{d} = 0.000102633\,x - 0.00235399$

Element 4

Nodes: $\{x_1 \to 225.,\ x_2 \to 300.\}$

Interpolation functions: $\boldsymbol{N}^{\mathrm{T}} = \{4. - 0.0133333\,x,\ 0.0133333\,x - 3.\}$

Nodal values: $\boldsymbol{d}^{\mathrm{T}} = \{0.0207385,\ 0.02951\}$

Solution: $u(x) = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{d} = 0.000116954\,x - 0.00557619$

Element 5

Nodes: $\{x_1 \to 300.,\ x_2 \to 375.\}$

Interpolation functions: $\boldsymbol{N}^{\mathrm{T}} = \{5. - 0.0133333\,x,\ 0.0133333\,x - 4.\}$

Nodal values: $\boldsymbol{d}^{\mathrm{T}} = \{0.02951,\ 0.02426\}$

Solution: $u(x) = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{d} = 0.0505102 - 0.0000700005\,x$

Element 6

Nodes: $\{x_1 \to 375.,\ x_2 \to 450.\}$

Interpolation functions: $\boldsymbol{N}^{\mathrm{T}} = \{6. - 0.0133333\,x,\ 0.0133333\,x - 5.\}$

Nodal values: $\boldsymbol{d}^{\mathrm{T}} = \{0.02426,\ 0.0179938\}$

Solution: $u(x) = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{d} = 0.0555909 - 0.000083549\,x$

Element 7

Nodes: $\{x_1 \to 450.,\ x_2 \to 525.\}$

Interpolation functions: $\boldsymbol{N}^{\mathrm{T}} = \{7. - 0.0133333\,x,\ 0.0133333\,x - 6.\}$

Nodal values: $\boldsymbol{d}^{\mathrm{T}} = \{0.0179938,\ 0.0102238\}$

Solution: $u(x) = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{d} = 0.0646142 - 0.000103601\,x$

Element 8

Nodes: $\{x_1 \to 525.,\ x_2 \to 600.\}$

Interpolation functions: $\boldsymbol{N}^{\mathrm{T}} = \{8. - 0.0133333\,x,\ 0.0133333\,x - 7.\}$

Nodal values: $\boldsymbol{d}^{\mathrm{T}} = \{0.0102238,\ 0\}$

Solution: $u(x) = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{d} = 0.0817901 - 0.000136317\,x$

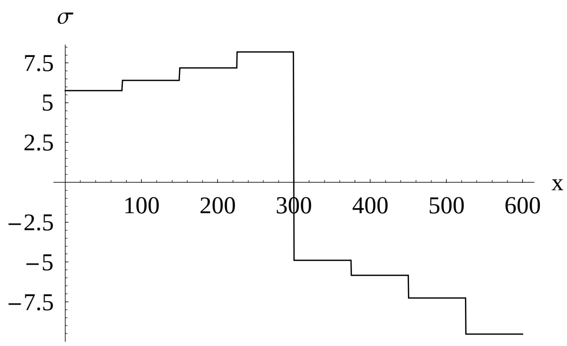
Solution summary

	Range	Solution
1	$0 \le x \le 75.$	$0.0000824431\,x$
2	$75. \le x \le 150.$	$0.0000914369\,x - 0.000674534$
3	$150. \le x \le 225.$	$0.000102633\,x - 0.00235399$
4	$225. \le x \le 300.$	$0.000116954\,x - 0.00557619$
5	$300. \le x \le 375.$	$0.0505102 - 0.0000700005\,x$
6	$375. \le x \le 450.$	$0.0555909 - 0.000083549\,x$
7	$450. \le x \le 525.$	$0.0646142 - 0.000103601\,x$
8	$525. \le x \le 600.$	$0.0817901 - 0.000136317\,x$

	Range	ϵ	σ	F
1	$0 \le x \le 75.$	0.0000824431	5.77101	13201.2
2	$75. \le x \le 150.$	0.0000914369	6.40058	13201.2
3	$150. \le x \le 225.$	0.000102633	7.18432	13201.2
4	$225. \le x \le 300.$	0.000116954	8.18679	13201.2
5	$300. \le x \le 375.$	−0.0000700005	−4.90004	−6798.8
6	$375. \le x \le 450.$	−0.000083549	−5.84843	−6798.8
7	$450. \le x \le 525.$	−0.000103601	−7.25206	−6798.8
8	$525. \le x \le 600.$	−0.000136317	−9.54218	−6798.8

The stresses are plotted in the following figure. Because of the constant area assumption over each element, we still have discontinuities in the stress. However if we take the average of the stresses from the two elements at common nodes the solution is very close to the exact solution.

{6.0858, 6.79245, 7.68556, −5.37424, −6.55024, −8.39712}


σ
7.5
5
2.5
−2.5
−5
−7.5
100
200
300
400
500
600
x