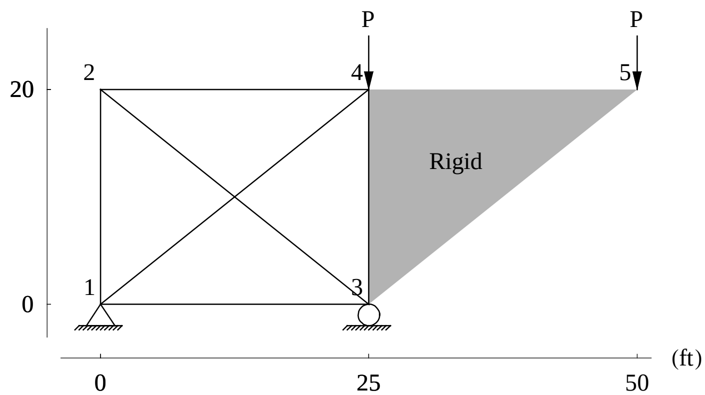
Example 1.18: Truss supporting a rigid plate (p. 80)

A plane truss is designed to support a rigid triangular plate as shown in Figure. All members have the same cross-sectional area $A = 1\,\mathrm{in}^2$ and are of the same material, $E = 29,\,000\,\mathrm{ksi}$. The load $P = 20\,\mathrm{kip}$. The dimensions in ft are shown in the figure. Note there is no connection between the diagonal members where they cross each other.



The model consists of 5 nodes and thus the global system of equations before boundary conditions will be 10×10. The equations for the six truss elements are written as in the previous examples and assembled in the usual manner to give the following system of equations.

$$\text{Element 1, k} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 120.833 & 0 & -120.833 \\ 0 & 0 & 0 & 0 \\ 0 & -120.833 & 0 & 120.833 \end{pmatrix}$$

$$\text{Global } \mathrm{K} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 120.833 & 0 & -120.833 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -120.833 & 0 & 120.833 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Element 2, k} = \begin{pmatrix} 96.6667 & 0 & -96.6667 & 0 \\ 0 & 0 & 0 & 0 \\ -96.6667 & 0 & 96.6667 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Global } K = \begin{pmatrix} 96.6667 & 0 & 0 & 0 & -96.6667 & 0 & 0 & 0 & 0 & 0 \\ 0 & 120.833 & 0 & -120.833 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -120.833 & 0 & 120.833 & 0 & 0 & 0 & 0 & 0 & 0 \\ -96.6667 & 0 & 0 & 0 & 96.6667 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Element 3, } k = \begin{pmatrix} 46.0268 & 36.8215 & -46.0268 & -36.8215 \\ 36.8215 & 29.4572 & -36.8215 & -29.4572 \\ -46.0268 & -36.8215 & 46.0268 & 36.8215 \\ -36.8215 & -29.4572 & 36.8215 & 29.4572 \end{pmatrix}$$

$$\text{Global } K = \begin{pmatrix} 142.693 & 36.8215 & 0 & 0 & -96.6667 & 0 & -46.0268 & -36.8215 & 0 & 0 \\ 36.8215 & 150.29 & 0 & -120.833 & 0 & 0 & -36.8215 & -29.4572 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -120.833 & 0 & 120.833 & 0 & 0 & 0 & 0 & 0 & 0 \\ -96.6667 & 0 & 0 & 0 & 96.6667 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -46.0268 & -36.8215 & 0 & 0 & 0 & 0 & 46.0268 & 36.8215 & 0 & 0 \\ -36.8215 & -29.4572 & 0 & 0 & 0 & 0 & 36.8215 & 29.4572 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Element 4, } k = \begin{pmatrix} 96.6667 & 0. & -96.6667 & 0. \\ 0. & 0. & 0. & 0. \\ -96.6667 & 0. & 96.6667 & 0. \\ 0. & 0. & 0. & 0. \end{pmatrix}$$

$$\text{Global } K = \begin{pmatrix} 142.693 & 36.8215 & 0 & 0 & -96.6667 & 0 & -46.0268 & -36.8215 & 0 & 0 \\ 36.8215 & 150.29 & 0 & -120.833 & 0 & 0 & -36.8215 & -29.4572 & 0 & 0 \\ 0 & 0 & 96.6667 & 0. & 0 & 0 & -96.6667 & 0. & 0 & 0 \\ 0 & -120.833 & 0. & 120.833 & 0 & 0 & 0. & 0. & 0 & 0 \\ -96.6667 & 0 & 0 & 0 & 96.6667 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -46.0268 & -36.8215 & -96.6667 & 0. & 0 & 0 & 142.693 & 36.8215 & 0 & 0 \\ -36.8215 & -29.4572 & 0. & 0. & 0 & 0 & 36.8215 & 29.4572 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Element 5, } k = \begin{pmatrix} 46.0268 & -36.8215 & -46.0268 & 36.8215 \\ -36.8215 & 29.4572 & 36.8215 & -29.4572 \\ -46.0268 & 36.8215 & 46.0268 & -36.8215 \\ 36.8215 & -29.4572 & -36.8215 & 29.4572 \end{pmatrix}$$

$$\text{Global K} = \begin{pmatrix} 142.693 & 36.8215 & 0 & 0 & -96.6667 & 0 & -46.0268 & -36.8215 & 0 & 0 \\ 36.8215 & 150.29 & 0 & -120.833 & 0 & 0 & -36.8215 & -29.4572 & 0 & 0 \\ 0 & 0 & 142.693 & -36.8215 & -46.0268 & 36.8215 & -96.6667 & 0. & 0 & 0 \\ 0 & -120.833 & -36.8215 & 150.29 & 36.8215 & -29.4572 & 0. & 0. & 0 & 0 \\ -96.6667 & 0 & -46.0268 & 36.8215 & 142.693 & -36.8215 & 0 & 0 & 0 & 0 \\ 0 & 0 & 36.8215 & -29.4572 & -36.8215 & 29.4572 & 0 & 0 & 0 & 0 \\ -46.0268 & -36.8215 & -96.6667 & 0. & 0 & 0 & 142.693 & 36.8215 & 0 & 0 \\ -36.8215 & -29.4572 & 0. & 0. & 0 & 0 & 36.8215 & 29.4572 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Element 6, k} = \begin{pmatrix} 0. & 0. & 0. & 0. \\ 0. & 120.833 & 0. & -120.833 \\ 0. & 0. & 0. & 0. \\ 0. & -120.833 & 0. & 120.833 \end{pmatrix}$$

$$\text{Global K} = \begin{pmatrix} 142.693 & 36.8215 & 0 & 0 & -96.6667 & 0 & -46.0268 & -36.8215 & 0 & 0 \\ 36.8215 & 150.29 & 0 & -120.833 & 0 & 0 & -36.8215 & -29.4572 & 0 & 0 \\ 0 & 0 & 142.693 & -36.8215 & -46.0268 & 36.8215 & -96.6667 & 0. & 0 & 0 \\ 0 & -120.833 & -36.8215 & 150.29 & 36.8215 & -29.4572 & 0. & 0. & 0 & 0 \\ -96.6667 & 0 & -46.0268 & 36.8215 & 142.693 & -36.8215 & 0. & 0. & 0 & 0 \\ 0 & 0 & 36.8215 & -29.4572 & -36.8215 & 150.29 & 0. & -120.833 & 0 & 0 \\ -46.0268 & -36.8215 & -96.6667 & 0. & 0. & 0. & 142.693 & 36.8215 & 0 & 0 \\ -36.8215 & -29.4572 & 0. & 0. & 0. & -120.833 & 36.8215 & 150.29 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 142.693 & 36.8215 & 0 & 0 & -96.6667 & 0 & -46.0268 & -36.8215 & 0 & 0 \\ 36.8215 & 150.29 & 0 & -120.833 & 0 & 0 & -36.8215 & -29.4572 & 0 & 0 \\ 0 & 0 & 142.693 & -36.8215 & -46.0268 & 36.8215 & -96.6667 & 0. & 0 & 0 \\ 0 & -120.833 & -36.8215 & 150.29 & 36.8215 & -29.4572 & 0. & 0. & 0 & 0 \\ -96.6667 & 0 & -46.0268 & 36.8215 & 142.693 & -36.8215 & 0. & 0. & 0 & 0 \\ 0 & 0 & 36.8215 & -29.4572 & -36.8215 & 150.29 & 0. & -120.833 & 0 & 0 \\ -46.0268 & -36.8215 & -96.6667 & 0. & 0. & 0. & 142.693 & 36.8215 & 0 & 0 \\ -36.8215 & -29.4572 & 0. & 0. & 0. & -120.833 & 36.8215 & 150.29 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -20 \\ 0 \\ -20 \end{pmatrix}$$

The essential boundary conditions at nodes 1 ($u_1 = v_1 = 0$) are incorporated by removing the corresponding rows and columns in the usual way. Node 3 also has zero vertical displacement. However since this node is connected to the rigid plate as well, the boundary condition $v_3 = 0$, will be imposed later as part of the multi-point constraints. Removing the first two rows and columns, the global system of equations is as follows.

$\boldsymbol{K}\,\boldsymbol{d} = \boldsymbol{R} \Longrightarrow$

$$\begin{pmatrix} 142.693 & -36.8215 & -46.0268 & 36.8215 & -96.6667 & 0. & 0 & 0 \\ -36.8215 & 150.29 & 36.8215 & -29.4572 & 0. & 0. & 0 & 0 \\ -46.0268 & 36.8215 & 142.693 & -36.8215 & 0. & 0. & 0 & 0 \\ 36.8215 & -29.4572 & -36.8215 & 150.29 & 0. & -120.833 & 0 & 0 \\ -96.6667 & 0. & 0. & 0. & 142.693 & 36.8215 & 0 & 0 \\ 0. & 0. & 0. & -120.833 & 36.8215 & 150.29 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ -20. \\ 0. \\ -20. \end{pmatrix}$$

The rigid plate is connected between nodes 3, 5 and 4. Treating u_3, v_3, and u_5 as independent degrees of freedom, the multi-point constraints are as follows.

$$\begin{pmatrix} v_5 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} \frac{x_3-x_5}{y_3-y_5} & 1 & \frac{x_5-x_3}{y_3-y_5} \\ \frac{y_4-y_5}{y_3-y_5} & 0 & \frac{y_3-y_4}{y_3-y_5} \\ \frac{x_3-x_4}{y_3-y_5} & 1 & \frac{x_4-x_3}{y_3-y_5} \end{pmatrix} \begin{pmatrix} u_3 \\ v_3 \\ u_5 \end{pmatrix} = \begin{pmatrix} \frac{5}{4} & 1 & -\frac{5}{4} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} u_3 \\ v_3 \\ u_5 \end{pmatrix}$$

Expanding and re-arranging we have

$$-\frac{5\,u_3}{4} + \frac{5\,u_5}{4} - v_3 + v_5 = 0$$
$$u_4 - u_5 = 0$$
$$v_4 - v_3 = 0$$

To this list we must also add the roller support constraint that $v_3 = 0$. Thus the complete set of constraint equations, expanded to include all degrees of freedom present in the global equations, we have

$$\boldsymbol{C}\,\boldsymbol{d} = \boldsymbol{q} \Longrightarrow \begin{pmatrix} 0 & 0 & -\frac{5}{4} & -1 & 0 & 0 & \frac{5}{4} & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

For using the penalty function approach we choose the penalty parameter μ equal to 10^5 times the largest number in the global $\boldsymbol{K}$ matrix.

$$\mu = 150.29 \times 10^5 = 1.5029 \times 10^7$$

Incorporating the constraints into the global equations with this value of μ, the final system of equations is as follows.

$$(\boldsymbol{K} + \mu\, \boldsymbol{C}^T\, \boldsymbol{C})\, \boldsymbol{d} = \boldsymbol{R} + \mu\, \boldsymbol{C}^T \boldsymbol{q} \Longrightarrow$$

$$10^5 \left(\begin{array}{lllllll}
0.00142693 & -0.000368215 & -0.000460268 & 0.000368215 & -0.000966667 & 0. & 0 \\
-0.000368215 & 0.0015029 & 0.000368215 & -0.000294572 & 0. & 0. & 0 \\
-0.000460268 & 0.000368215 & -234.827 & -187.863 & 0. & 0. & 234.829 \\
0.000368215 & -0.000294572 & -187.863 & -450.87 & 0. & 150.289 & 187.863 \\
-0.000966667 & 0. & 0. & 0. & -150.289 & 0.000368215 & 150.29 \\
0. & 0. & 0. & 150.289 & 0.000368215 & -150.289 & 0 \\
0 & 0 & 234.829 & 187.863 & 150.29 & 0 & -385.119 \\
0 & 0 & 187.863 & 150.29 & 0 & 0 & -187.863
\end{array} \right.$$

Solving this system of linear equations we get

$$\{u_2 = 0.172845,\ v_2 = 0.076446,\ u_3 = -0.139174,\ v_3 = 3.99227 \times 10^{-6},$$
$$u_4 = 0.292292,\ v_4 = 6.03917 \times 10^{-6},\ u_5 = 0.29229,\ v_5 = -0.539324\}$$

Substituting these values into the constraint equations we can see that the constraints are reasonably satisfied.

$$\boldsymbol{C}\, \boldsymbol{d} = \begin{pmatrix} 1.33076 \times 10^{-6} \\ 1.66345 \times 10^{-6} \\ 2.0469 \times 10^{-6} \\ 3.99227 \times 10^{-6} \end{pmatrix} \approx \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Knowing the nodal values, the element solutions can be computed in the usual manner.

	Strain	Stress	Axial force
1	0.000318525	9.23722	9.23722
2	−0.000463913	−13.4535	−13.4535
3	0.000594098	17.2289	17.2289
4	0.000398156	11.5465	11.5465
5	−0.000509888	−14.7868	−14.7868
6	8.52877×10^{-9}	0.000247334	0.000247334

Using the Lagrange multipliers method, the solution is obtained as follows.

Augmented system of equations

$$\begin{pmatrix} 142.693 & -36.8215 & -46.0268 & 36.8215 & -96.6667 & 0. & 0 & 0 & 0 & 0 & 0 & 0 \\ -36.8215 & 150.29 & 36.8215 & -29.4572 & 0. & 0. & 0 & 0 & 0 & 0 & 0 & 0 \\ -46.0268 & 36.8215 & 142.693 & -36.8215 & 0. & 0. & 0 & 0 & -\frac{5}{4} & 0 & 0 & 0 \\ 36.8215 & -29.4572 & -36.8215 & 150.29 & 0. & -120.833 & 0 & 0 & -1 & 0 & -1 & 1 \\ -96.6667 & 0. & 0. & 0. & 142.693 & 36.8215 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0. & 0. & 0. & -120.833 & 36.8215 & 150.29 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{4} & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{5}{4} & -1 & 0 & 0 & \frac{5}{4} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ -20. \\ 0. \\ -20. \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solution

$$\{d_1 = 0.172849,\ d_2 = 0.0764461,\ d_3 = -0.139174,\ d_4 = -4.68418\times 10^{-18},\ d_5 = 0.292296,$$
$$d_6 = -1.62088\times 10^{-17},\ d_7 = 0.292296,\ d_8 = -0.539337,\ \lambda_1 = -20.,\ \lambda_2 = -25.,\ \lambda_3 = -30.7628,\ \lambda_4 = -60.\}$$