Example 9.9: Groundwater flow — Free surface problem (p. 633)

The flow of water through porous media gives rise to the so-called free surface problem. A typical situation, illustrated in Figure, considers flow of groundwater towards a well. At a sufficient distance away from the well the groundwater level is equal to the established water table in the area and is unaffected by the well. Closer to the well the groundwater level is lower because of the water flowing into the well. The exact shape and depth below ground of the top of the groundwater surface depends on the coefficient of permeability of the soil and is not known a priori. An iterative procedure is used in the analysis to establish this free surface.

The governing differential equation for the problem is as follows.

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial \phi}{\partial y}\right) = 0$$

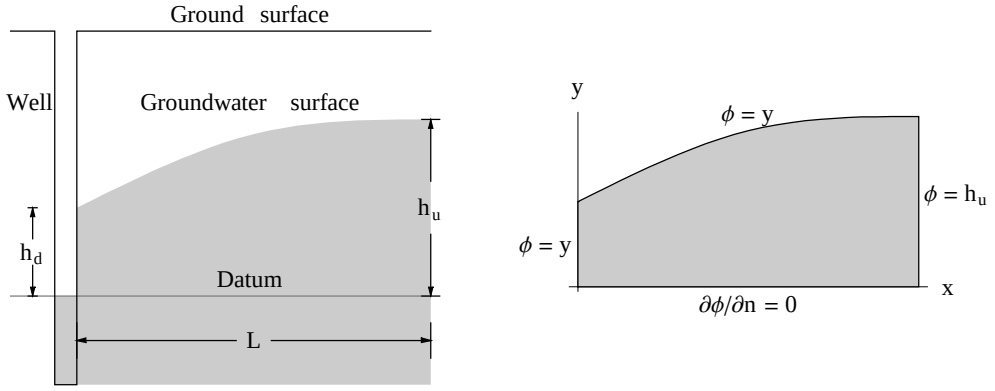
where $\phi(x, y)$ is the hydraulic head (or hydraulic potential) and k_x and k_y are coefficients of permeability in the x and y directions. Typical units for ϕ are meters and those for k_x and k_y are m / day. The fluid velocity components in the x and y directions are related to the hydraulic head as follows.

$$v_x = -k_x \frac{\partial \phi}{\partial x} \text{ and } v_y = -k_y \frac{\partial \phi}{\partial y}$$

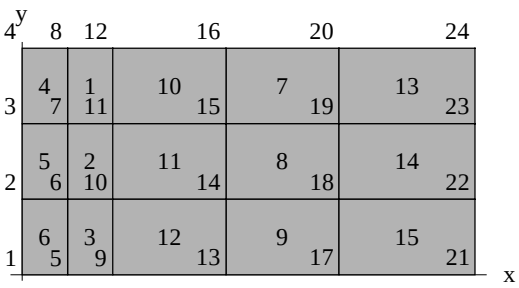
The computational domain consists of the region bounded by the *unknown* top of the groundwater suface, side of the well, line extending from the water line in the well and a vertical line at a sufficient distance away from the well. The boundary conditions on this computational doamin are as illustrated in Figure. Considering the water line in the well as datum for the hydraulic head, ϕ on the side away from the well is equal to the water depth h_u and that on the well side is equal to the elevation of the groundwater surface. On the bottom there is no flow and therefore the boundary condition is $\partial\phi/\partial n = 0$. On the top surface $\phi = y$ the height of the surface above the datum and also $\partial\phi/\partial n = 0$. It is unusual to have two specified boundary conditions for one boundary. However since the boundary itself is not known, there is no inconsistency. One of the conditions is used to establish the boundary while the other is used as a usual boundary condition. In finite elements a zero natural boundary condition does not require any work. Thus computationally we simply have to establish the free surface.

In an iterative procdure we arbitrarily choose the height of the free surface, say equal to the water level h_u. The finite model is now established with zero natural boundary conditions at the top and bottom, essential boundary condition on the far side and essential and natural boundary conditions on the well side. The problem is solved in the usual manner. In general the computed ϕ values on the top suface will not be equal to the assumed height. The top surface is now re-established by setting y equal to the computed ϕ values and the process is repeated until the difference between the computed ϕ values for the nodes on the top surface are equal to the y coordinate for these nodes. The procedure is illustrated using the following numerical data.

$$k_x = k_y = 1\,m/s; \quad h_u = 1\,m; \quad L = 2\,m$$



Initially the groundwater surface is assumed to be horizontal. Thus the computational domain is an $L \times h_u$ rectangle as shown in Figure. It is discretized into 15 elements. Essential boundary condition of $\phi = h_u$ is specified for element 13 side 23-24, element 14 side 22-23 and element 15 side 21-22. For element 5 side 2-3 and element 6 side 1-2 the specified boundary condition is $\phi = y$. It is important not to specify ϕ value at node 4 otherwise its elevation will not change and we will not be able to adjust the groundwater surface.



A linear ($n = 1$) element is used to solve the problem. The computed nodal ϕ values are as follows.

Equations for element 1

$$\begin{pmatrix} 0.755556 & -0.455556 & -0.377778 & 0.0777778 \\ -0.455556 & 0.755556 & 0.0777778 & -0.377778 \\ -0.377778 & 0.0777778 & 0.755556 & -0.455556 \\ 0.0777778 & -0.377778 & -0.455556 & 0.755556 \end{pmatrix} \begin{pmatrix} \phi_{12} \\ \phi_8 \\ \phi_7 \\ \phi_{11} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 2

$$\begin{pmatrix} 0.755556 & -0.455556 & -0.377778 & 0.0777778 \\ -0.455556 & 0.755556 & 0.0777778 & -0.377778 \\ -0.377778 & 0.0777778 & 0.755556 & -0.455556 \\ 0.0777778 & -0.377778 & -0.455556 & 0.755556 \end{pmatrix} \begin{pmatrix} \phi_{11} \\ \phi_7 \\ \phi_6 \\ \phi_{10} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 3

$$\begin{pmatrix} 0.755556 & -0.455556 & -0.377778 & 0.0777778 \\ -0.455556 & 0.755556 & 0.0777778 & -0.377778 \\ -0.377778 & 0.0777778 & 0.755556 & -0.455556 \\ 0.0777778 & -0.377778 & -0.455556 & 0.755556 \end{pmatrix} \begin{pmatrix} \phi_{10} \\ \phi_6 \\ \phi_5 \\ \phi_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 4

$$\begin{pmatrix} 0.755556 & -0.455556 & -0.377778 & 0.0777778 \\ -0.455556 & 0.755556 & 0.0777778 & -0.377778 \\ -0.377778 & 0.0777778 & 0.755556 & -0.455556 \\ 0.0777778 & -0.377778 & -0.455556 & 0.755556 \end{pmatrix} \begin{pmatrix} \phi_8 \\ \phi_4 \\ \phi_3 \\ \phi_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 5

$$\begin{pmatrix} 0.755556 & -0.455556 & -0.377778 & 0.0777778 \\ -0.455556 & 0.755556 & 0.0777778 & -0.377778 \\ -0.377778 & 0.0777778 & 0.755556 & -0.455556 \\ 0.0777778 & -0.377778 & -0.455556 & 0.755556 \end{pmatrix} \begin{pmatrix} \phi_7 \\ \phi_3 \\ \phi_2 \\ \phi_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 6

$$\begin{pmatrix} 0.755556 & -0.455556 & -0.377778 & 0.0777778 \\ -0.455556 & 0.755556 & 0.0777778 & -0.377778 \\ -0.377778 & 0.0777778 & 0.755556 & -0.455556 \\ 0.0777778 & -0.377778 & -0.455556 & 0.755556 \end{pmatrix} \begin{pmatrix} \phi_6 \\ \phi_2 \\ \phi_1 \\ \phi_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 7

$$\begin{pmatrix} 0.722222 & 0.0277778 & -0.361111 & -0.388889 \\ 0.0277778 & 0.722222 & -0.388889 & -0.361111 \\ -0.361111 & -0.388889 & 0.722222 & 0.0277778 \\ -0.388889 & -0.361111 & 0.0277778 & 0.722222 \end{pmatrix} \begin{pmatrix} \phi_{20} \\ \phi_{16} \\ \phi_{15} \\ \phi_{19} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 8

$$\begin{pmatrix} 0.722222 & 0.0277778 & -0.361111 & -0.388889 \\ 0.0277778 & 0.722222 & -0.388889 & -0.361111 \\ -0.361111 & -0.388889 & 0.722222 & 0.0277778 \\ -0.388889 & -0.361111 & 0.0277778 & 0.722222 \end{pmatrix} \begin{pmatrix} \phi_{19} \\ \phi_{15} \\ \phi_{14} \\ \phi_{18} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 9

$$\begin{pmatrix} 0.722222 & 0.0277778 & -0.361111 & -0.388889 \\ 0.0277778 & 0.722222 & -0.388889 & -0.361111 \\ -0.361111 & -0.388889 & 0.722222 & 0.0277778 \\ -0.388889 & -0.361111 & 0.0277778 & 0.722222 \end{pmatrix} \begin{pmatrix} \phi_{18} \\ \phi_{14} \\ \phi_{13} \\ \phi_{17} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 10

$$\begin{pmatrix} 0.722222 & 0.0277778 & -0.361111 & -0.388889 \\ 0.0277778 & 0.722222 & -0.388889 & -0.361111 \\ -0.361111 & -0.388889 & 0.722222 & 0.0277778 \\ -0.388889 & -0.361111 & 0.0277778 & 0.722222 \end{pmatrix} \begin{pmatrix} \phi_{16} \\ \phi_{12} \\ \phi_{11} \\ \phi_{15} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 11

$$\begin{pmatrix} 0.722222 & 0.0277778 & -0.361111 & -0.388889 \\ 0.0277778 & 0.722222 & -0.388889 & -0.361111 \\ -0.361111 & -0.388889 & 0.722222 & 0.0277778 \\ -0.388889 & -0.361111 & 0.0277778 & 0.722222 \end{pmatrix} \begin{pmatrix} \phi_{15} \\ \phi_{11} \\ \phi_{10} \\ \phi_{14} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 12

$$\begin{pmatrix} 0.722222 & 0.0277778 & -0.361111 & -0.388889 \\ 0.0277778 & 0.722222 & -0.388889 & -0.361111 \\ -0.361111 & -0.388889 & 0.722222 & 0.0277778 \\ -0.388889 & -0.361111 & 0.0277778 & 0.722222 \end{pmatrix} \begin{pmatrix} \phi_{14} \\ \phi_{10} \\ \phi_{9} \\ \phi_{13} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 13

$$\begin{pmatrix} 0.785185 & 0.114815 & -0.392593 & -0.507407 \\ 0.114815 & 0.785185 & -0.507407 & -0.392593 \\ -0.392593 & -0.507407 & 0.785185 & 0.114815 \\ -0.507407 & -0.392593 & 0.114815 & 0.785185 \end{pmatrix} \begin{pmatrix} \phi_{24} \\ \phi_{20} \\ \phi_{19} \\ \phi_{23} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 14

$$\begin{pmatrix} 0.785185 & 0.114815 & -0.392593 & -0.507407 \\ 0.114815 & 0.785185 & -0.507407 & -0.392593 \\ -0.392593 & -0.507407 & 0.785185 & 0.114815 \\ -0.507407 & -0.392593 & 0.114815 & 0.785185 \end{pmatrix} \begin{pmatrix} \phi_{23} \\ \phi_{19} \\ \phi_{18} \\ \phi_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 15

$$\begin{pmatrix} 0.785185 & 0.114815 & -0.392593 & -0.507407 \\ 0.114815 & 0.785185 & -0.507407 & -0.392593 \\ -0.392593 & -0.507407 & 0.785185 & 0.114815 \\ -0.507407 & -0.392593 & 0.114815 & 0.785185 \end{pmatrix} \begin{pmatrix} \phi_{22} \\ \phi_{18} \\ \phi_{17} \\ \phi_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Global equations after assembling all elements

$$\begin{pmatrix}
0.755556 & 0.0777778 & 0 & 0 & -0.455556 & -0.377778 & 0 & 0 & 0 \\
0.0777778 & 1.51111 & 0.0777778 & 0 & -0.377778 & -0.911111 & -0.377778 & 0 & 0 \\
0 & 0.0777778 & 1.51111 & 0.0777778 & 0 & -0.377778 & -0.911111 & -0.377778 & 0 \\
0 & 0 & 0.0777778 & 0.755556 & 0 & 0 & -0.377778 & -0.455556 & 0 \\
-0.455556 & -0.377778 & 0 & 0 & 1.51111 & 0.155556 & 0 & 0 & -0.4 \\
-0.377778 & -0.911111 & -0.377778 & 0 & 0.155556 & 3.02222 & 0.155556 & 0 & -0.3 \\
0 & -0.377778 & -0.911111 & -0.377778 & 0 & 0.155556 & 3.02222 & 0.155556 & 0 \\
0 & 0 & -0.377778 & -0.455556 & 0 & 0 & 0.155556 & 1.51111 & 0 \\
0 & 0 & 0 & 0 & -0.455556 & -0.377778 & 0 & 0 & 1.4 \\
0 & 0 & 0 & 0 & -0.377778 & -0.911111 & -0.377778 & 0 & -0.3 \\
0 & 0 & 0 & 0 & 0 & -0.377778 & -0.911111 & -0.377778 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.377778 & -0.455556 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Essential boundary conditions

On element 13, side 4, specified value = 1

$$\{\phi_{23}, \phi_{24}\} = \{1, 1\}$$

On element 14, side 4, specified value = 1

$$\{\phi_{22}, \phi_{23}\} = \{1, 1\}$$

On element 15, side 4, specified value = 1

$$\{\phi_{21}, \phi_{22}\} = \{1, 1\}$$

On element 5, side 2, specified value = y

$$\{\phi_3, \phi_2\} = \left\{\frac{2}{3}, \frac{1}{3}\right\}$$

On element 6, side 2, specified value = y

$$\{\phi_2, \phi_1\} = \left\{\frac{1}{3}, 0\right\}$$

Known values from EBC

$$\left\{\phi_1 = 0, \phi_2 = \frac{1}{3}, \phi_3 = \frac{2}{3}, \phi_{21} = 1, \phi_{22} = 1, \phi_{23} = 1, \phi_{24} = 1\right\}$$

Global equations after EBC

$$\begin{pmatrix}
0.755556 & 0 & 0 & -0.377778 & -0.455556 & 0 & 0 & 0 & 0 \\
0 & 1.51111 & 0.155556 & 0 & 0 & -0.455556 & -0.377778 & 0 & 0 \\
0 & 0.155556 & 3.02222 & 0.155556 & 0 & -0.377778 & -0.911111 & -0.377778 & 0 \\
-0.377778 & 0 & 0.155556 & 3.02222 & 0.155556 & 0 & -0.377778 & -0.911111 & -0.377 \\
-0.455556 & 0 & 0 & 0.155556 & 1.51111 & 0 & 0 & -0.377778 & -0.45 \\
0 & -0.455556 & -0.377778 & 0 & 0 & 1.47778 & -0.311111 & 0 & 0 \\
0 & -0.377778 & -0.911111 & -0.377778 & 0 & -0.311111 & 2.95556 & -0.311111 & 0 \\
0 & 0 & -0.377778 & -0.911111 & -0.377778 & 0 & -0.311111 & 2.95556 & -0.31 \\
0 & 0 & 0 & -0.377778 & -0.455556 & 0 & 0 & -0.311111 & 1.47 \\
0 & 0 & 0 & 0 & 0 & 0.0277778 & -0.361111 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -0.361111 & 0.0555556 & -0.361111 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.361111 & 0.0555556 & -0.36 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.361111 & 0.02 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Solving the final system of global equations we get

$$\{\phi_4 = 0.596157, \phi_5 = 0.306957, \phi_6 = 0.423017, \phi_7 = 0.581671, \phi_8 = 0.620208, \phi_9 = 0.463903,$$
$$\phi_{10} = 0.509264, \phi_{11} = 0.587326, \phi_{12} = 0.619842, \phi_{13} = 0.676938, \phi_{14} = 0.681975, \phi_{15} = 0.693494,$$
$$\phi_{16} = 0.700009, \phi_{17} = 0.828093, \phi_{18} = 0.829002, \phi_{19} = 0.830666, \phi_{20} = 0.831418\}$$

Solution for element 1

DOF values for the element

$$\{\phi_{12} = 0.619842, \phi_8 = 0.620208, \phi_7 = 0.581671, \phi_{11} = 0.587326\}$$

$\boldsymbol{d}^{\mathrm{T}} = (\ 0.619842 \quad 0.620208 \quad 0.581671 \quad 0.587326\ )$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	0.3	0.833333	0.602262	0.0132221	0.10658

Solution for element 2

DOF values for the element

$\{\phi_{11} = 0.587326,\ \phi_7 = 0.581671,\ \phi_6 = 0.423017,\ \phi_{10} = 0.509264\}$

$\boldsymbol{d}^{\mathrm{T}} = (\ 0.587326 \quad 0.581671 \quad 0.423017 \quad 0.509264\ )$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	0.3	0.5	0.52532	0.229755	0.355073

Solution for element 3

DOF values for the element

$\{\phi_{10} = 0.509264,\ \phi_6 = 0.423017,\ \phi_5 = 0.306957,\ \phi_9 = 0.463903\}$

$\boldsymbol{d}^{\mathrm{T}} = (\ 0.509264 \quad 0.423017 \quad 0.306957 \quad 0.463903\ )$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	0.3	0.166667	0.425785	0.607983	0.242133

Solution for element 4

DOF values for the element

$\left\{\phi_8 = 0.620208,\ \phi_4 = 0.596157,\ \phi_3 = \frac{2}{3},\ \phi_7 = 0.581671\right\}$

$\boldsymbol{d}^{\mathrm{T}} = \left(\ 0.620208 \quad 0.596157 \quad \frac{2}{3} \quad 0.581671\ \right)$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	0.1	0.833333	0.616176	−0.152361	−0.0479582

Solution for element 5

DOF values for the element

$\left\{\phi_7 = 0.581671,\ \phi_3 = \frac{2}{3},\ \phi_2 = \frac{1}{3},\ \phi_6 = 0.423017\right\}$

$\boldsymbol{d}^{\mathrm{T}} = \left(\ 0.581671 \quad \frac{2}{3} \quad \frac{1}{3} \quad 0.423017\ \right)$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	0.1	0.5	0.501172	0.0117208	0.737981

Solution for element 6

DOF values for the element

$\left\{\phi_6 = 0.423017,\ \phi_2 = \frac{1}{3},\ \phi_1 = 0,\ \phi_5 = 0.306957\right\}$

$\boldsymbol{d}^{\mathrm{T}} = \left(\begin{array}{cccc} 0.423017 & \frac{1}{3} & 0 & 0.306957 \end{array}\right)$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	0.1	0.166667	0.265827	0.991601	0.674091

Solution for element 7

DOF values for the element

$\{\phi_{20} = 0.831418,\ \phi_{16} = 0.700009,\ \phi_{15} = 0.693494,\ \phi_{19} = 0.830666\}$

$\boldsymbol{d}^{\mathrm{T}} = (\ 0.831418 \quad 0.700009 \quad 0.693494 \quad 0.830666\ )$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	1.15	0.833333	0.763897	0.26858	0.0108989

Solution for element 8

DOF values for the element

$\{\phi_{19} = 0.830666,\ \phi_{15} = 0.693494,\ \phi_{14} = 0.681975,\ \phi_{18} = 0.829002\}$

$\boldsymbol{d}^{\mathrm{T}} = (\ 0.830666 \quad 0.693494 \quad 0.681975 \quad 0.829002\ )$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	1.15	0.5	0.758784	0.284199	0.019776

Solution for element 9

DOF values for the element

$\{\phi_{18} = 0.829002,\ \phi_{14} = 0.681975,\ \phi_{13} = 0.676938,\ \phi_{17} = 0.828093\}$

$\boldsymbol{d}^{\mathrm{T}} = (\ 0.829002 \quad 0.681975 \quad 0.676938 \quad 0.828093\ )$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	1.15	0.166667	0.754002	0.298182	0.00891866

Solution for element 10

DOF values for the element

$\{\phi_{16} = 0.700009,\ \phi_{12} = 0.619842,\ \phi_{11} = 0.587326,\ \phi_{15} = 0.693494\}$

$\boldsymbol{d}^{\mathrm{T}}$ = (0.700009 0.619842 0.587326 0.693494)

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	0.65	0.833333	0.650168	0.186335	0.058547

Solution for element 11

DOF values for the element

$\{\phi_{15} = 0.693494, \phi_{11} = 0.587326, \phi_{10} = 0.509264, \phi_{14} = 0.681975\}$

$\boldsymbol{d}^{\mathrm{T}}$ = (0.693494 0.587326 0.509264 0.681975)

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	0.65	0.5	0.618015	0.278879	0.134371

Solution for element 12

DOF values for the element

$\{\phi_{14} = 0.681975, \phi_{10} = 0.509264, \phi_{9} = 0.463903, \phi_{13} = 0.676938\}$

$\boldsymbol{d}^{\mathrm{T}}$ = (0.681975 0.509264 0.463903 0.676938)

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	0.65	0.166667	0.58302	0.385746	0.0755986

Solution for element 13

DOF values for the element

$\{\phi_{24} = 1, \phi_{20} = 0.831418, \phi_{19} = 0.830666, \phi_{23} = 1\}$

$\boldsymbol{d}^{\mathrm{T}}$ = (1 0.831418 0.830666 1)

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	1.7	0.833333	0.915521	0.281597	0.00112653

Solution for element 14

DOF values for the element

$\{\phi_{23} = 1, \phi_{19} = 0.830666, \phi_{18} = 0.829002, \phi_{22} = 1\}$

$\boldsymbol{d}^{\mathrm{T}}$ = (1 0.830666 0.829002 1)

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	1.7	0.5	0.914917	0.28361	0.0024972

Solution for element 15

DOF values for the element

$\{\phi_{22} = 1, \phi_{18} = 0.829002, \phi_{17} = 0.828093, \phi_{21} = 1\}$

$\boldsymbol{d}^{\mathrm{T}} = (\ 1 \quad 0.829002 \quad 0.828093 \quad 1\ )$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	1.7	0.166667	0.914274	0.285754	0.00136265

Nodal solution summary

dof	x	y	Value
ϕ_1	0	0	0
ϕ_2	0	$\frac{1}{3}$	$\frac{1}{3}$
ϕ_3	0	$\frac{2}{3}$	$\frac{2}{3}$
ϕ_4	0	1	0.596157
ϕ_5	0.2	0	0.306957
ϕ_6	0.2	$\frac{1}{3}$	0.423017
ϕ_7	0.2	$\frac{2}{3}$	0.581671
ϕ_8	0.2	1	0.620208
ϕ_9	0.4	0	0.463903
ϕ_{10}	0.4	$\frac{1}{3}$	0.509264
ϕ_{11}	0.4	$\frac{2}{3}$	0.587326
ϕ_{12}	0.4	1	0.619842
ϕ_{13}	0.9	0	0.676938
ϕ_{14}	0.9	$\frac{1}{3}$	0.681975
ϕ_{15}	0.9	$\frac{2}{3}$	0.693494
ϕ_{16}	0.9	1	0.700009
ϕ_{17}	1.4	0	0.828093
ϕ_{18}	1.4	$\frac{1}{3}$	0.829002
ϕ_{19}	1.4	$\frac{2}{3}$	0.830666
ϕ_{20}	1.4	1	0.831418
ϕ_{21}	2	0	1
ϕ_{22}	2	$\frac{1}{3}$	1
ϕ_{23}	2	$\frac{2}{3}$	1
ϕ_{24}	2	1	1

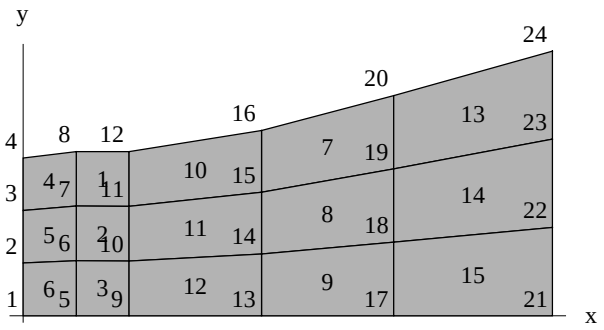
Element solution summary

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	0.3	0.833333	0.602262	0.0132221	0.10658
2	0.3	0.5	0.52532	0.229755	0.355073
3	0.3	0.166667	0.425785	0.607983	0.242133
4	0.1	0.833333	0.616176	−0.152361	−0.0479582
5	0.1	0.5	0.501172	0.0117208	0.737981
6	0.1	0.166667	0.265827	0.991601	0.674091
7	1.15	0.833333	0.763897	0.26858	0.0108989
8	1.15	0.5	0.758784	0.284199	0.019776
9	1.15	0.166667	0.754002	0.298182	0.00891866
10	0.65	0.833333	0.650168	0.186335	0.058547
11	0.65	0.5	0.618015	0.278879	0.134371
12	0.65	0.166667	0.58302	0.385746	0.0755986
13	1.7	0.833333	0.915521	0.281597	0.00112653
14	1.7	0.5	0.914917	0.28361	0.0024972
15	1.7	0.166667	0.914274	0.285754	0.00136265

For the next iteration, the y coordinates of the nodes on the top surface are set equal to the computed ϕ values at nodes 4, 8, 12, 16, 20, and 24.

New y coordinates of top surface = {0.596157, 0.620208, 0.619842, 0.700009, 0.831418, 1}

To avoid badly shaped elements a completely new finite element mesh, shown in Figure, is created.



Using this new finite element model we get the following nodal solution.

Equations for element 1

$$\begin{pmatrix} 0.667888 & -0.183174 & -0.334276 & -0.150437 \\ -0.183174 & 0.666166 & -0.15024 & -0.332751 \\ -0.334276 & -0.15024 & 0.667691 & -0.183174 \\ -0.150437 & -0.332751 & -0.183174 & 0.666363 \end{pmatrix} \begin{pmatrix} \phi_{12} \\ \phi_8 \\ \phi_7 \\ \phi_{11} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 2

$$\begin{pmatrix} 0.667582 & -0.183175 & -0.333971 & -0.150437 \\ -0.183175 & 0.66647 & -0.15024 & -0.333056 \\ -0.333971 & -0.15024 & 0.667385 & -0.183175 \\ -0.150437 & -0.333056 & -0.183175 & 0.666667 \end{pmatrix} \begin{pmatrix} \phi_{11} \\ \phi_7 \\ \phi_6 \\ \phi_{10} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 3

$$\begin{pmatrix} 0.667277 & -0.183175 & -0.333666 & -0.150437 \\ -0.183175 & 0.666775 & -0.15024 & -0.333361 \\ -0.333666 & -0.15024 & 0.66708 & -0.183175 \\ -0.150437 & -0.333361 & -0.183175 & 0.666972 \end{pmatrix} \begin{pmatrix} \phi_{10} \\ \phi_6 \\ \phi_5 \\ \phi_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 4

$$\begin{pmatrix} 0.613386 & -0.17176 & -0.284952 & -0.156674 \\ -0.17176 & 0.726849 & -0.169924 & -0.385165 \\ -0.284952 & -0.169924 & 0.626636 & -0.17176 \\ -0.156674 & -0.385165 & -0.17176 & 0.713599 \end{pmatrix} \begin{pmatrix} \phi_8 \\ \phi_4 \\ \phi_3 \\ \phi_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 5

$$\begin{pmatrix} 0.631336 & -0.172817 & -0.303938 & -0.154581 \\ -0.172817 & 0.704672 & -0.167789 & -0.364066 \\ -0.303938 & -0.167789 & 0.644544 & -0.172817 \\ -0.154581 & -0.364066 & -0.172817 & 0.691463 \end{pmatrix} \begin{pmatrix} \phi_7 \\ \phi_3 \\ \phi_2 \\ \phi_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 6

$$\begin{pmatrix} 0.650332 & -0.173346 & -0.323452 & -0.153534 \\ -0.173346 & 0.683562 & -0.166722 & -0.343495 \\ -0.323452 & -0.166722 & 0.663519 & -0.173346 \\ -0.153534 & -0.343495 & -0.173346 & 0.670374 \end{pmatrix} \begin{pmatrix} \phi_6 \\ \phi_2 \\ \phi_1 \\ \phi_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 7

$$\begin{pmatrix} 0.710312 & 0.173046 & -0.318777 & -0.56458 \\ 0.173046 & 1.00283 & -0.638081 & -0.537791 \\ -0.318777 & -0.638081 & 0.783812 & 0.173046 \\ -0.56458 & -0.537791 & 0.173046 & 0.929325 \end{pmatrix} \begin{pmatrix} \phi_{20} \\ \phi_{16} \\ \phi_{15} \\ \phi_{19} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 8

$$\begin{pmatrix} 0.734881 & 0.162999 & -0.352532 & -0.545347 \\ 0.162999 & 0.938065 & -0.617123 & -0.483941 \\ -0.352532 & -0.617123 & 0.806657 & 0.162999 \\ -0.545347 & -0.483941 & 0.162999 & 0.866289 \end{pmatrix} \begin{pmatrix} \phi_{19} \\ \phi_{15} \\ \phi_{14} \\ \phi_{18} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 9

$$\begin{pmatrix} 0.769067 & 0.157975 & -0.391311 & -0.535731 \\ 0.157975 & 0.883784 & -0.606645 & -0.435114 \\ -0.391311 & -0.606645 & 0.839981 & 0.157975 \\ -0.535731 & -0.435114 & 0.157975 & 0.81287 \end{pmatrix} \begin{pmatrix} \phi_{18} \\ \phi_{14} \\ \phi_{13} \\ \phi_{17} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 10

$$\begin{pmatrix} 0.824252 & 0.239508 & -0.392678 & -0.671083 \\ 0.239508 & 1.01368 & -0.7269 & -0.526289 \\ -0.392678 & -0.7269 & 0.88007 & 0.239508 \\ -0.671083 & -0.526289 & 0.239508 & 0.957864 \end{pmatrix} \begin{pmatrix} \phi_{16} \\ \phi_{12} \\ \phi_{11} \\ \phi_{15} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 11

$$\begin{pmatrix} 0.842571 & 0.235175 & -0.415066 & -0.662679 \\ 0.235175 & 0.978029 & -0.71797 & -0.495233 \\ -0.415066 & -0.71797 & 0.897862 & 0.235175 \\ -0.662679 & -0.495233 & 0.235175 & 0.922738 \end{pmatrix} \begin{pmatrix} \phi_{15} \\ \phi_{11} \\ \phi_{10} \\ \phi_{14} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 12

$$\begin{pmatrix} 0.865091 & 0.233008 & -0.439622 & -0.658477 \\ 0.233008 & 0.946841 & -0.713505 & -0.466344 \\ -0.439622 & -0.713505 & 0.920119 & 0.233008 \\ -0.658477 & -0.466344 & 0.233008 & 0.891813 \end{pmatrix} \begin{pmatrix} \phi_{14} \\ \phi_{10} \\ \phi_{9} \\ \phi_{13} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 13

$$\begin{pmatrix} 0.706367 & 0.177219 & -0.314511 & -0.569074 \\ 0.177219 & 1.01996 & -0.648529 & -0.548654 \\ -0.314511 & -0.648529 & 0.785821 & 0.177219 \\ -0.569074 & -0.548654 & 0.177219 & 0.940509 \end{pmatrix} \begin{pmatrix} \phi_{24} \\ \phi_{20} \\ \phi_{19} \\ \phi_{23} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 14

$$\begin{pmatrix} 0.731202 & 0.165692 & -0.349813 & -0.547081 \\ 0.165692 & 0.949019 & -0.624413 & -0.490298 \\ -0.349813 & -0.624413 & 0.808534 & 0.165692 \\ -0.547081 & -0.490298 & 0.165692 & 0.871687 \end{pmatrix} \begin{pmatrix} \phi_{23} \\ \phi_{19} \\ \phi_{18} \\ \phi_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 15

$$\begin{pmatrix} 0.767033 & 0.159928 & -0.390877 & -0.536084 \\ 0.159928 & 0.890133 & -0.612355 & -0.437706 \\ -0.390877 & -0.612355 & 0.843304 & 0.159928 \\ -0.536084 & -0.437706 & 0.159928 & 0.813862 \end{pmatrix} \begin{pmatrix} \phi_{22} \\ \phi_{18} \\ \phi_{17} \\ \phi_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Global equations after assembling all elements

$$
\left(\begin{array}{ccccccccc}
0.663519 & -0.166722 & 0 & 0 & -0.173346 & -0.323452 & 0 & 0 & 0 \\
-0.166722 & 1.32811 & -0.167789 & 0 & -0.343495 & -0.346163 & -0.303938 & 0 & 0 \\
0 & -0.167789 & 1.33131 & -0.169924 & 0 & -0.364066 & -0.344577 & -0.284952 & 0 \\
0 & 0 & -0.169924 & 0.726849 & 0 & 0 & -0.385165 & -0.17176 & 0 \\
-0.173346 & -0.343495 & 0 & 0 & 1.33745 & -0.303774 & 0 & 0 & -0.183175 \\
-0.323452 & -0.346163 & -0.364066 & 0 & -0.303774 & 2.67596 & -0.304821 & 0 & -0.333361 \\
0 & -0.303938 & -0.344577 & -0.385165 & 0 & -0.304821 & 2.6791 & -0.306914 & 0 \\
0 & 0 & -0.284952 & -0.17176 & 0 & 0 & -0.306914 & 1.27955 & 0 \\
0 & 0 & 0 & 0 & -0.183175 & -0.333361 & 0 & 0 & 1.58709 \\
0 & 0 & 0 & 0 & -0.333666 & -0.36635 & -0.333056 & 0 & -0.863941 \\
0 & 0 & 0 & 0 & 0 & -0.333971 & -0.366349 & -0.332751 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.334276 & -0.183174 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.233008 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.439622 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right.
$$

Essential boundary conditions

On element 13, side 4, specified value = 1

$\{\phi_{23}, \phi_{24}\} = \{1, 1\}$

On element 14, side 4, specified value = 1

$\{\phi_{22}, \phi_{23}\} = \{1, 1\}$

On element 15, side 4, specified value = 1

$\{\phi_{21}, \phi_{22}\} = \{1, 1\}$

On element 5, side 2, specified value = y

$\{\phi_3, \phi_2\} = \{0.397438, 0.198719\}$

On element 6, side 2, specified value = y

$\{\phi_2, \phi_1\} = \{0.198719, 0.\}$

Known values from EBC

$\{\phi_1 = 0., \phi_2 = 0.198719, \phi_3 = 0.397438, \phi_{21} = 1, \phi_{22} = 1, \phi_{23} = 1, \phi_{24} = 1\}$

Global equations after EBC

$$\begin{pmatrix}
0.726849 & 0 & 0 & -0.385165 & -0.17176 & 0 & 0 & 0 & 0 \\
0 & 1.33745 & -0.303774 & 0 & 0 & -0.183175 & -0.333666 & 0 & 0 \\
0 & -0.303774 & 2.67596 & -0.304821 & 0 & -0.333361 & -0.36635 & -0.333971 & 0 \\
-0.385165 & 0 & -0.304821 & 2.6791 & -0.306914 & 0 & -0.333056 & -0.366349 & -0.334276 \\
-0.17176 & 0 & 0 & -0.306914 & 1.27955 & 0 & 0 & -0.332751 & -0.183174 \\
0 & -0.183175 & -0.333361 & 0 & 0 & 1.58709 & -0.863941 & 0 & 0 \\
0 & -0.333666 & -0.36635 & -0.333056 & 0 & -0.863941 & 3.17865 & -0.868407 & 0 \\
0 & 0 & -0.333971 & -0.366349 & -0.332751 & 0 & -0.868407 & 3.19204 & -0.877338 \\
0 & 0 & 0 & -0.334276 & -0.183174 & 0 & 0 & -0.877338 & 1.68157 \\
0 & 0 & 0 & 0 & 0 & 0.233008 & -0.466344 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -0.439622 & 0.468182 & -0.495233 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.415066 & 0.474683 & -0.526289 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.392678 & 0.239508 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Solving the final system of global equations we get

$\{\phi_4 = 0.398856, \phi_5 = 0.289648, \phi_6 = 0.32478, \phi_7 = 0.389915, \phi_8 = 0.420311, \phi_9 = 0.419679,$
$\phi_{10} = 0.430365, \phi_{11} = 0.452748, \phi_{12} = 0.46802, \phi_{13} = 0.637184, \phi_{14} = 0.640101, \phi_{15} = 0.648756,$
$\phi_{16} = 0.663723, \phi_{17} = 0.812016, \phi_{18} = 0.815298, \phi_{19} = 0.825998, \phi_{20} = 0.846926\}$

Solution for element 1

DOF values for the element

$\{\phi_{12} = 0.46802, \phi_8 = 0.420311, \phi_7 = 0.389915, \phi_{11} = 0.452748\}$

$\boldsymbol{d}^{\mathrm{T}} = (\ 0.46802 \quad 0.420311 \quad 0.389915 \quad 0.452748\ )$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	0.3	0.516688	0.432748	0.276523	0.110486

Solution for element 2

DOF values for the element

$\{\phi_{11} = 0.452748,\ \phi_7 = 0.389915,\ \phi_6 = 0.32478,\ \phi_{10} = 0.430365\}$

$\boldsymbol{d}^{\mathrm{T}} = (\ 0.452748 \quad 0.389915 \quad 0.32478 \quad 0.430365\ )$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	0.3	0.310013	0.399452	0.421239	0.211725

Solution for element 3

DOF values for the element

$\{\phi_{10} = 0.430365,\ \phi_6 = 0.32478,\ \phi_5 = 0.289648,\ \phi_9 = 0.419679\}$

$\boldsymbol{d}^{\mathrm{T}} = (\ 0.430365 \quad 0.32478 \quad 0.289648 \quad 0.419679\ )$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	0.3	0.103338	0.366118	0.589074	0.110846

Solution for element 4

DOF values for the element

$\{\phi_8 = 0.420311,\ \phi_4 = 0.398856,\ \phi_3 = 0.397438,\ \phi_7 = 0.389915\}$

$\boldsymbol{d}^{\mathrm{T}} = (\ 0.420311 \quad 0.398856 \quad 0.397438 \quad 0.389915\ )$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	0.1	0.506819	0.40163	0.0269648	0.0784678

Solution for element 5

DOF values for the element

$\{\phi_7 = 0.389915,\ \phi_3 = 0.397438,\ \phi_2 = 0.198719,\ \phi_6 = 0.32478\}$

$\boldsymbol{d}^{\mathrm{T}} = (\ 0.389915 \quad 0.397438 \quad 0.198719 \quad 0.32478\ )$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	0.1	0.304091	0.327713	0.257215	0.650759

Solution for element 6

DOF values for the element

$\{\phi_6 = 0.32478,\ \phi_2 = 0.198719,\ \phi_1 = 0.,\ \phi_5 = 0.289648\}$

$\boldsymbol{d}^{\mathrm{T}} = (\ 0.32478 \quad 0.198719 \quad 0. \quad 0.289648\ )$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	0.1	0.101364	0.203287	1.02771	0.576762

Solution for element 7

DOF values for the element

$\{\phi_{20} = 0.846926, \phi_{16} = 0.663723, \phi_{15} = 0.648756, \phi_{19} = 0.825998\}$

$\boldsymbol{d}^{\mathrm{T}} = (\ 0.846926 \quad 0.663723 \quad 0.648756 \quad 0.825998\ )$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	1.15	0.638095	0.746351	0.345044	0.0703166

Solution for element 8

DOF values for the element

$\{\phi_{19} = 0.825998, \phi_{15} = 0.648756, \phi_{14} = 0.640101, \phi_{18} = 0.815298\}$

$\boldsymbol{d}^{\mathrm{T}} = (\ 0.825998 \quad 0.648756 \quad 0.640101 \quad 0.815298\ )$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	1.15	0.382857	0.732538	0.347457	0.0379155

Solution for element 9

DOF values for the element

$\{\phi_{18} = 0.815298, \phi_{14} = 0.640101, \phi_{13} = 0.637184, \phi_{17} = 0.812016\}$

$\boldsymbol{d}^{\mathrm{T}} = (\ 0.815298 \quad 0.640101 \quad 0.637184 \quad 0.812016\ )$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	1.15	0.127619	0.72615	0.349497	0.0121435

Solution for element 10

DOF values for the element

$\{\phi_{16} = 0.663723, \phi_{12} = 0.46802, \phi_{11} = 0.452748, \phi_{15} = 0.648756\}$

$\boldsymbol{d}^{\mathrm{T}} = (\ 0.663723 \quad 0.46802 \quad 0.452748 \quad 0.648756\ )$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	0.65	0.549938	0.558312	0.382528	0.068734

Solution for element 11

DOF values for the element

$\{\phi_{15} = 0.648756,\ \phi_{11} = 0.452748,\ \phi_{10} = 0.430365,\ \phi_{14} = 0.640101\}$

$\boldsymbol{d}^{\mathrm{T}} = (\ 0.648756 \quad 0.452748 \quad 0.430365 \quad 0.640101\ )$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	0.65	0.329963	0.542992	0.400088	0.0705481

Solution for element 12

DOF values for the element

$\{\phi_{14} = 0.640101,\ \phi_{10} = 0.430365,\ \phi_{9} = 0.419679,\ \phi_{13} = 0.637184\}$

$\boldsymbol{d}^{\mathrm{T}} = (\ 0.640101 \quad 0.430365 \quad 0.419679 \quad 0.637184\ )$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	0.65	0.109988	0.531832	0.426415	0.0309182

Solution for element 13

DOF values for the element

$\{\phi_{24} = 1,\ \phi_{20} = 0.846926,\ \phi_{19} = 0.825998,\ \phi_{23} = 1\}$

$\boldsymbol{d}^{\mathrm{T}} = (\ 1 \quad 0.846926 \quad 0.825998 \quad 1\ )$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	1.7	0.763091	0.918231	0.264537	0.0342814

Solution for element 14

DOF values for the element

$\{\phi_{23} = 1,\ \phi_{19} = 0.825998,\ \phi_{18} = 0.815298,\ \phi_{22} = 1\}$

$\boldsymbol{d}^{\mathrm{T}} = (\ 1 \quad 0.825998 \quad 0.815298 \quad 1\ )$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	1.7	0.457854	0.910324	0.296458	0.0175266

Solution for element 15

DOF values for the element

$\{\phi_{22} = 1,\ \phi_{18} = 0.815298,\ \phi_{17} = 0.812016,\ \phi_{21} = 1\}$

$\boldsymbol{d}^{\mathrm{T}} = (\ 1 \quad 0.815298 \quad 0.812016 \quad 1\ )$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	1.7	0.152618	0.906828	0.31032	0.00537721

Nodal solution summary

dof	x	y	Value
ϕ_1	0	0	0.
ϕ_2	0	0.198719	0.198719
ϕ_3	0	0.397438	0.397438
ϕ_4	0	0.596157	0.398856
ϕ_5	0.2	0	0.289648
ϕ_6	0.2	0.206736	0.32478
ϕ_7	0.2	0.413472	0.389915
ϕ_8	0.2	0.620208	0.420311
ϕ_9	0.4	0	0.419679
ϕ_{10}	0.4	0.206614	0.430365
ϕ_{11}	0.4	0.413228	0.452748
ϕ_{12}	0.4	0.619842	0.46802
ϕ_{13}	0.9	0	0.637184
ϕ_{14}	0.9	0.233336	0.640101
ϕ_{15}	0.9	0.466673	0.648756
ϕ_{16}	0.9	0.700009	0.663723
ϕ_{17}	1.4	0	0.812016
ϕ_{18}	1.4	0.277139	0.815298
ϕ_{19}	1.4	0.554278	0.825998
ϕ_{20}	1.4	0.831418	0.846926
ϕ_{21}	2	0	1
ϕ_{22}	2	$\frac{1}{3}$	1
ϕ_{23}	2	$\frac{2}{3}$	1
ϕ_{24}	2	1	1

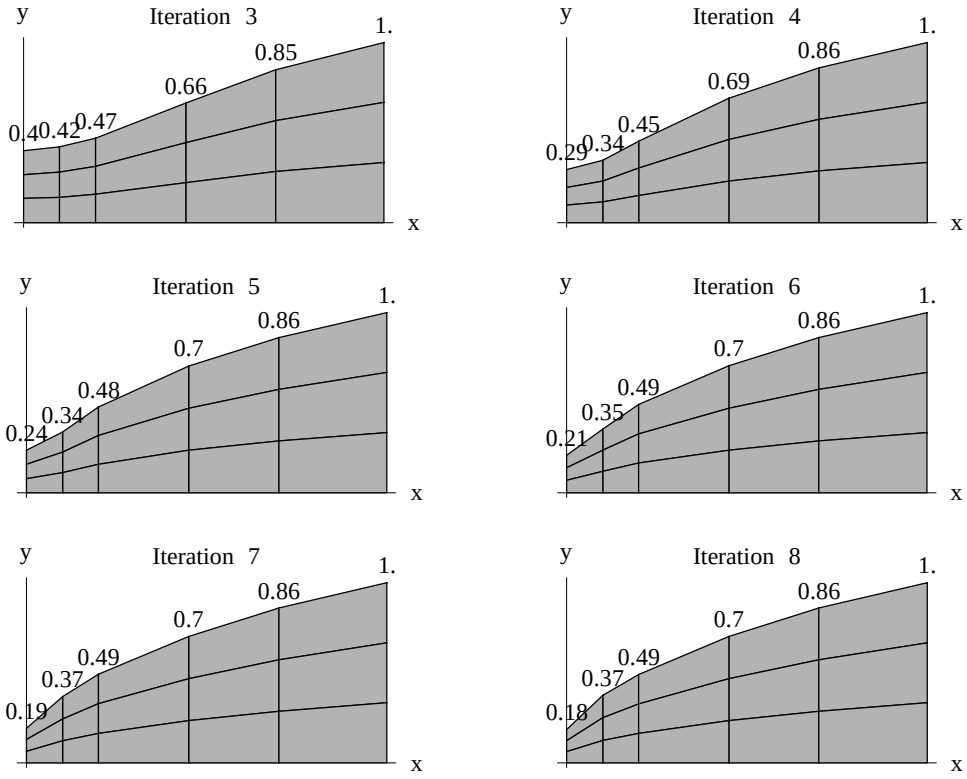
Element solution summary

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	0.3	0.516688	0.432748	0.276523	0.110486
2	0.3	0.310013	0.399452	0.421239	0.211725
3	0.3	0.103338	0.366118	0.589074	0.110846
4	0.1	0.506819	0.40163	0.0269648	0.0784678
5	0.1	0.304091	0.327713	0.257215	0.650759
6	0.1	0.101364	0.203287	1.02771	0.576762
7	1.15	0.638095	0.746351	0.345044	0.0703166
8	1.15	0.382857	0.732538	0.347457	0.0379155
9	1.15	0.127619	0.72615	0.349497	0.0121435
10	0.65	0.549938	0.558312	0.382528	0.068734
11	0.65	0.329963	0.542992	0.400088	0.0705481
12	0.65	0.109988	0.531832	0.426415	0.0309182
13	1.7	0.763091	0.918231	0.264537	0.0342814
14	1.7	0.457854	0.910324	0.296458	0.0175266
15	1.7	0.152618	0.906828	0.31032	0.00537721

A new finite element mesh is created using the new coordinate values from this solution.

New y coordinates of top surface = {0.398856, 0.420311, 0.46802, 0.663723, 0.846926, 1}

The process is repeated several more times. The finite element meshes and the coordinates of the groundwater surface are shown in the following figures.



The coordinates of the top surface have converged now. We can use the solution from this mesh to compute the flow rate and fluid velocity.

Equations for element 1

$$\begin{pmatrix} 0.520071 & 0.0476235 & -0.167023 & -0.400671 \\ 0.0476235 & 1.10997 & -0.509595 & -0.647996 \\ -0.167023 & -0.509595 & 0.628995 & 0.0476235 \\ -0.400671 & -0.647996 & 0.0476235 & 1.00104 \end{pmatrix} \begin{pmatrix} \phi_{12} \\ \phi_8 \\ \phi_7 \\ \phi_{11} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 2

$$\begin{pmatrix} 0.551899 & 0.0131356 & -0.22873 & -0.336304 \\ 0.0131356 & 0.940188 & -0.43601 & -0.517313 \\ -0.22873 & -0.43601 & 0.651604 & 0.0131356 \\ -0.336304 & -0.517313 & 0.0131356 & 0.840482 \end{pmatrix} \begin{pmatrix} \phi_{11} \\ \phi_7 \\ \phi_6 \\ \phi_{10} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 3

$$\begin{pmatrix} 0.61591 & -0.00410842 & -0.30768 & -0.304121 \\ -0.00410842 & 0.807201 & -0.399217 & -0.403875 \\ -0.30768 & -0.399217 & 0.711006 & -0.00410842 \\ -0.304121 & -0.403875 & -0.00410842 & 0.712104 \end{pmatrix} \begin{pmatrix} \phi_{10} \\ \phi_6 \\ \phi_5 \\ \phi_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 4

$$\begin{pmatrix} 0.743189 & 0.452772 & -0.290403 & -0.905557 \\ 0.452772 & 1.99895 & -1.37105 & -1.08067 \\ -0.290403 & -1.37105 & 1.20869 & 0.452772 \\ -0.905557 & -1.08067 & 0.452772 & 1.53345 \end{pmatrix} \begin{pmatrix} \phi_8 \\ \phi_4 \\ \phi_3 \\ \phi_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 5

$$\begin{pmatrix} 0.654138 & 0.303958 & -0.299642 & -0.658454 \\ 0.303958 & 1.49274 & -1.0229 & -0.773799 \\ -0.299642 & -1.0229 & 1.01859 & 0.303958 \\ -0.658454 & -0.773799 & 0.303958 & 1.12829 \end{pmatrix} \begin{pmatrix} \phi_7 \\ \phi_3 \\ \phi_2 \\ \phi_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 6

$$\begin{pmatrix} 0.688638 & 0.229551 & -0.383287 & -0.534902 \\ 0.229551 & 1.16061 & -0.848826 & -0.541339 \\ -0.383287 & -0.848826 & 1.00256 & 0.229551 \\ -0.534902 & -0.541339 & 0.229551 & 0.846691 \end{pmatrix} \begin{pmatrix} \phi_6 \\ \phi_2 \\ \phi_1 \\ \phi_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 7

$$\begin{pmatrix} 0.686181 & 0.171168 & -0.298912 & -0.558437 \\ 0.171168 & 1.03831 & -0.646143 & -0.563331 \\ -0.298912 & -0.646143 & 0.773887 & 0.171168 \\ -0.558437 & -0.563331 & 0.171168 & 0.9506 \end{pmatrix} \begin{pmatrix} \phi_{20} \\ \phi_{16} \\ \phi_{15} \\ \phi_{19} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 8

$$\begin{pmatrix} 0.711741 & 0.156774 & -0.337402 & -0.531113 \\ 0.156774 & 0.955169 & -0.61589 & -0.496053 \\ -0.337402 & -0.61589 & 0.796518 & 0.156774 \\ -0.531113 & -0.496053 & 0.156774 & 0.870392 \end{pmatrix} \begin{pmatrix} \phi_{19} \\ \phi_{15} \\ \phi_{14} \\ \phi_{18} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 9

$$\begin{pmatrix} 0.750962 & 0.149577 & -0.383089 & -0.517451 \\ 0.149577 & 0.887159 & -0.600764 & -0.435972 \\ -0.383089 & -0.600764 & 0.834275 & 0.149577 \\ -0.517451 & -0.435972 & 0.149577 & 0.803846 \end{pmatrix} \begin{pmatrix} \phi_{18} \\ \phi_{14} \\ \phi_{13} \\ \phi_{17} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 10

$$\begin{pmatrix} 0.815551 & 0.345523 & -0.36808 & -0.792994 \\ 0.345523 & 1.35995 & -0.985804 & -0.719671 \\ -0.36808 & -0.985804 & 1.00836 & 0.345523 \\ -0.792994 & -0.719671 & 0.345523 & 1.16714 \end{pmatrix} \begin{pmatrix} \phi_{16} \\ \phi_{12} \\ \phi_{11} \\ \phi_{15} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 11

$$\begin{pmatrix} 0.824643 & 0.311933 & -0.404809 & -0.731768 \\ 0.311933 & 1.2165 & -0.91267 & -0.615763 \\ -0.404809 & -0.91267 & 1.00555 & 0.311933 \\ -0.731768 & -0.615763 & 0.311933 & 1.0356 \end{pmatrix} \begin{pmatrix} \phi_{15} \\ \phi_{11} \\ \phi_{10} \\ \phi_{14} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 12

$$\begin{pmatrix} 0.864349 & 0.295139 & -0.458332 & -0.701155 \\ 0.295139 & 1.10961 & -0.876103 & -0.52865 \\ -0.458332 & -0.876103 & 1.0393 & 0.295139 \\ -0.701155 & -0.52865 & 0.295139 & 0.934667 \end{pmatrix} \begin{pmatrix} \phi_{14} \\ \phi_{10} \\ \phi_{9} \\ \phi_{13} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 13

$$\begin{pmatrix} 0.714147 & 0.163726 & -0.324158 & -0.553714 \\ 0.163726 & 0.973638 & -0.61763 & -0.519734 \\ -0.324158 & -0.61763 & 0.778063 & 0.163726 \\ -0.553714 & -0.519734 & 0.163726 & 0.909722 \end{pmatrix} \begin{pmatrix} \phi_{24} \\ \phi_{20} \\ \phi_{19} \\ \phi_{23} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 14

$$\begin{pmatrix} 0.738031 & 0.155811 & -0.355358 & -0.538483 \\ 0.155811 & 0.918093 & -0.6012 & -0.472704 \\ -0.355358 & -0.6012 & 0.800748 & 0.155811 \\ -0.538483 & -0.472704 & 0.155811 & 0.855376 \end{pmatrix} \begin{pmatrix} \phi_{23} \\ \phi_{19} \\ \phi_{18} \\ \phi_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 15

$$\begin{pmatrix} 0.76953 & 0.151853 & -0.390516 & -0.530867 \\ 0.151853 & 0.870763 & -0.592985 & -0.429631 \\ -0.390516 & -0.592985 & 0.831648 & 0.151853 \\ -0.530867 & -0.429631 & 0.151853 & 0.808645 \end{pmatrix} \begin{pmatrix} \phi_{22} \\ \phi_{18} \\ \phi_{17} \\ \phi_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Global equations after assembling all elements

$$\left(\begin{array}{ccccccccc} 1.00256 & -0.848826 & 0 & 0 & 0.229551 & -0.383287 & 0 & 0 & 0 \\ -0.848826 & 2.1792 & -1.0229 & 0 & -0.541339 & 0.533508 & -0.299642 & 0 & 0 \\ 0 & -1.0229 & 2.70143 & -1.37105 & 0 & -0.773799 & 0.756729 & -0.290403 & 0 \\ 0 & 0 & -1.37105 & 1.99895 & 0 & 0 & -1.08067 & 0.452772 & 0 \\ 0.229551 & -0.541339 & 0 & 0 & 1.5577 & -0.934119 & 0 & 0 & -0. \\ -0.383287 & 0.533508 & -0.773799 & 0 & -0.934119 & 3.27574 & -1.09446 & 0 & -0. \\ 0 & -0.299642 & 0.756729 & -1.08067 & 0 & -1.09446 & 3.75677 & -1.41515 & 0 \\ 0 & 0 & -0.290403 & 0.452772 & 0 & 0 & -1.41515 & 1.85316 & 0 \\ 0 & 0 & 0 & 0 & -0.00410842 & -0.403875 & 0 & 0 & 1. \\ 0 & 0 & 0 & 0 & -0.30768 & 0.00902714 & -0.517313 & 0 & -1. \\ 0 & 0 & 0 & 0 & 0 & -0.22873 & 0.0607591 & -0.647996 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.167023 & 0.0476235 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0. \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0. \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right.$$

Essential boundary conditions

On element 13, side 4, specified value = 1

$$\{\phi_{23}, \phi_{24}\} = \{1, 1\}$$

On element 14, side 4, specified value = 1

$$\{\phi_{22}, \phi_{23}\} = \{1, 1\}$$

On element 15, side 4, specified value = 1

$$\{\phi_{21}, \phi_{22}\} = \{1, 1\}$$

On element 5, side 2, specified value = y

$$\{\phi_3, \phi_2\} = \{0.122989, 0.0614946\}$$

On element 6, side 2, specified value = y

$$\{\phi_2, \phi_1\} = \{0.0614946, 0.\}$$

Known values from EBC

$$\{\phi_1 = 0., \phi_2 = 0.0614946, \phi_3 = 0.122989, \phi_{21} = 1, \phi_{22} = 1, \phi_{23} = 1, \phi_{24} = 1\}$$

Global equations after EBC

$$\begin{pmatrix}
1.99895 & 0 & 0 & -1.08067 & 0.452772 & 0 & 0 & 0 \\
0 & 1.5577 & -0.934119 & 0 & 0 & -0.00410842 & -0.30768 & 0 \\
0 & -0.934119 & 3.27574 & -1.09446 & 0 & -0.403875 & 0.00902714 & -0.22873 \\
-1.08067 & 0 & -1.09446 & 3.75677 & -1.41515 & 0 & -0.517313 & 0.060759 \\
0.452772 & 0 & 0 & -1.41515 & 1.85316 & 0 & 0 & -0.647996 \\
0 & -0.00410842 & -0.403875 & 0 & 0 & 1.7514 & -1.18022 & 0 \\
0 & -0.30768 & 0.00902714 & -0.517313 & 0 & -1.18022 & 3.57155 & -1.24897 \\
0 & 0 & -0.22873 & 0.0607591 & -0.647996 & 0 & -1.24897 & 3.7778 \\
0 & 0 & 0 & -0.167023 & 0.0476235 & 0 & 0 & -1.38648 \\
0 & 0 & 0 & 0 & 0 & 0.295139 & -0.52865 & 0 \\
0 & 0 & 0 & 0 & 0 & -0.458332 & 0.607072 & -0.615763 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.404809 & 0.657457 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.36808 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Solving the final system of global equations we get

$\{\phi_4 = 0.179251,\ \phi_5 = 0.284834,\ \phi_6 = 0.294747,\ \phi_7 = 0.333891,\ \phi_8 = 0.377971,\ \phi_9 = 0.423078,$
$\phi_{10} = 0.433334,\ \phi_{11} = 0.457718,\ \phi_{12} = 0.487647,\ \phi_{13} = 0.654265,\ \phi_{14} = 0.659864,\ \phi_{15} = 0.676055,$
$\phi_{16} = 0.700769,\ \phi_{17} = 0.824487,\ \phi_{18} = 0.828468,\ \phi_{19} = 0.840155,\ \phi_{20} = 0.859261\}$

Solution for element 1

DOF values for the element

$\{\phi_{12} = 0.487647,\ \phi_8 = 0.377971,\ \phi_7 = 0.333891,\ \phi_{11} = 0.457718\}$

$\boldsymbol{d}^{\mathrm{T}} = (\ 0.487647 \quad 0.377971 \quad 0.333891 \quad 0.457718\ )$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	0.357735	0.432418	0.457513	0.449434	0.212296
2	0.357735	0.342892	0.438507	0.513867	0.212296
3	0.242265	0.370467	0.392465	0.397391	0.309299
4	0.242265	0.293768	0.368742	0.472599	0.309299

Solution for element 2

DOF values for the element

$\{\phi_{11} = 0.457718,\ \phi_7 = 0.333891,\ \phi_6 = 0.294747,\ \phi_{10} = 0.433334\}$

$\boldsymbol{d}^{\mathrm{T}} = (\ 0.457718 \quad 0.333891 \quad 0.294747 \quad 0.433334\ )$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	0.357735	0.277356	0.425738	0.573696	0.177371
2	0.357735	0.18783	0.409859	0.636004	0.177371
3	0.242265	0.23762	0.352446	0.541416	0.271175
4	0.242265	0.160921	0.331647	0.614143	0.271175

Solution for element 3

DOF values for the element

$\{\phi_{10} = 0.433334,\ \phi_6 = 0.294747,\ \phi_5 = 0.284834,\ \phi_9 = 0.423078\}$

$\boldsymbol{d}^{\mathrm{T}} = (\ 0.433334 \quad 0.294747 \quad 0.284834 \quad 0.423078\ )$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	0.357735	0.122294	0.401895	0.682608	0.0656707
2	0.357735	0.0327685	0.396016	0.688915	0.0656707
3	0.242265	0.104773	0.321924	0.681168	0.0751652
4	0.242265	0.0280738	0.316158	0.688529	0.0751652

Solution for element 4

DOF values for the element

$\{\phi_8 = 0.377971,\ \phi_4 = 0.179251,\ \phi_3 = 0.122989,\ \phi_7 = 0.333891\}$

$\boldsymbol{d}^{\mathrm{T}} = (\ 0.377971 \quad 0.179251 \quad 0.122989 \quad 0.333891\ )$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	0.157735	0.310534	0.326117	0.637141	0.418969
2	0.157735	0.246243	0.299181	0.748772	0.418969
3	0.042265	0.208745	0.2099	0.374221	0.717228
4	0.042265	0.165528	0.178903	0.540285	0.717228

Solution for element 5

DOF values for the element

$\{\phi_7 = 0.333891,\ \phi_3 = 0.122989,\ \phi_2 = 0.0614946,\ \phi_6 = 0.294747\}$

$\boldsymbol{d}^{\mathrm{T}} = (\ 0.333891 \quad 0.122989 \quad 0.0614946 \quad 0.294747\ )$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	0.157735	0.199179	0.280052	0.855389	0.393937
2	0.157735	0.134888	0.254725	0.991805	0.393937
3	0.042265	0.133891	0.155561	0.649307	0.758419
4	0.042265	0.0906734	0.122784	0.852242	0.758419

Solution for element 6

DOF values for the element

$\{\phi_6 = 0.294747,\ \phi_2 = 0.0614946,\ \phi_1 = 0.,\ \phi_5 = 0.284834\}$

$\boldsymbol{d}^{\mathrm{T}} = (\ 0.294747 \quad 0.0614946 \quad 0. \quad 0.284834\ )$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	0.157735	0.0878232	0.241056	1.17417	0.186911
2	0.157735	0.0235322	0.22904	1.35718	0.186911
3	0.042265	0.0590361	0.100095	1.05226	0.675898
4	0.042265	0.0158187	0.0708843	1.32452	0.675898

Solution for element 7

DOF values for the element

$\{\phi_{20} = 0.859261,\ \phi_{16} = 0.700769,\ \phi_{15} = 0.676055,\ \phi_{19} = 0.840155\}$

$\boldsymbol{d}^{\mathrm{T}}$ = (0.859261 0.700769 0.676055 0.840155)

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	1.29434	0.767498	0.82148	0.297608	0.0737282
2	1.29434	0.6086	0.809765	0.308586	0.0737282
3	1.00566	0.682353	0.72929	0.290991	0.0961616
4	1.00566	0.541083	0.715706	0.303339	0.0961616

Solution for element 8

DOF values for the element

$\{\phi_{19} = 0.840155, \phi_{15} = 0.676055, \phi_{14} = 0.659864, \phi_{18} = 0.828468\}$

$\boldsymbol{d}^{\mathrm{T}}$ = (0.840155 0.676055 0.659864 0.828468)

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	1.29434	0.492278	0.802806	0.321416	0.0459218
2	1.29434	0.33338	0.795509	0.329422	0.0459218
3	1.00566	0.437666	0.707513	0.318322	0.0622798
4	1.00566	0.296396	0.698714	0.327326	0.0622798

Solution for element 9

DOF values for the element

$\{\phi_{18} = 0.828468, \phi_{14} = 0.659864, \phi_{13} = 0.654265, \phi_{17} = 0.824487\}$

$\boldsymbol{d}^{\mathrm{T}}$ = (0.828468 0.659864 0.654265 0.824487)

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	1.29434	0.217059	0.791924	0.336582	0.0157086
2	1.29434	0.0581607	0.789428	0.339409	0.0157086
3	1.00566	0.192979	0.694383	0.3361	0.0214848
4	1.00566	0.0517085	0.691348	0.33928	0.0214848

Solution for element 10

DOF values for the element

$\{\phi_{16} = 0.700769, \phi_{12} = 0.487647, \phi_{11} = 0.457718, \phi_{15} = 0.676055\}$

$\boldsymbol{d}^{\mathrm{T}}$ = (0.700769 0.487647 0.457718 0.676055)

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	0.794338	0.609748	0.650276	0.382142	0.118072
2	0.794338	0.48351	0.63537	0.39775	0.118072
3	0.505662	0.496533	0.526593	0.364954	0.161899
4	0.505662	0.393734	0.50995	0.38412	0.161899

Solution for element 11

DOF values for the element

$\{\phi_{15} = 0.676055, \phi_{11} = 0.457718, \phi_{10} = 0.433334, \phi_{14} = 0.659864\}$

$\boldsymbol{d}^{\mathrm{T}} = (\ 0.676055 \quad 0.457718 \quad 0.433334 \quad 0.659864\ )$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	0.794338	0.391097	0.626127	0.419517	0.081968
2	0.794338	0.264858	0.61578	0.435633	0.081968
3	0.505662	0.31848	0.499071	0.408132	0.127226
4	0.505662	0.21568	0.485992	0.427923	0.127226

Solution for element 12

DOF values for the element

$\{\phi_{14} = 0.659864, \phi_{10} = 0.433334, \phi_{9} = 0.423078, \phi_{13} = 0.654265\}$

$\boldsymbol{d}^{\mathrm{T}} = (\ 0.659864 \quad 0.433334 \quad 0.423078 \quad 0.654265\ )$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	0.794338	0.172445	0.610601	0.451689	0.0301067
2	0.794338	0.0462065	0.606801	0.45951	0.0301067
3	0.505662	0.140426	0.479246	0.449253	0.0520705
4	0.505662	0.0376271	0.473893	0.458857	0.0520705

Solution for element 13

DOF values for the element

$\{\phi_{24} = 1, \phi_{20} = 0.859261, \phi_{19} = 0.840155, \phi_{23} = 1\}$

$\boldsymbol{d}^{\mathrm{T}} = (\ 1 \quad 0.859261 \quad 0.840155 \quad 1\ )$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	1.87321	0.901897	0.969405	0.238571	0.0124844
2	1.87321	0.715174	0.967074	0.257519	0.0124844
3	1.52679	0.826325	0.885818	0.2302	0.0508535
4	1.52679	0.655247	0.877119	0.250882	0.0508535

Solution for element 14

DOF values for the element

$\{\phi_{23} = 1, \phi_{19} = 0.840155, \phi_{18} = 0.828468, \phi_{22} = 1\}$

$\boldsymbol{d}^{\mathrm{T}} = (\ 1 \quad 0.840155 \quad 0.828468 \quad 1\ )$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	1.87321	0.578483	0.965699	0.269456	0.00763633
2	1.87321	0.39176	0.964273	0.281047	0.00763633
3	1.52679	0.53001	0.871986	0.266172	0.0311056
4	1.52679	0.358933	0.866665	0.278823	0.0311056

Solution for element 15

DOF values for the element

$\{\phi_{22} = 1, \phi_{18} = 0.828468, \phi_{17} = 0.824487, \phi_{21} = 1\}$

$\boldsymbol{d}^{\mathrm{T}} = (\ 1 \quad 0.828468 \quad 0.824487 \quad 1\ )$

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	1.87321	0.255069	0.963573	0.287128	0.00260162
2	1.87321	0.0683454	0.963087	0.291077	0.00260162
3	1.52679	0.233696	0.864054	0.286635	0.0105973
4	1.52679	0.0626186	0.862241	0.290945	0.0105973

Nodal solution summary

dof	x	y	Value
ϕ_1	0	0	0.
ϕ_2	0	0.0614946	0.0614946
ϕ_3	0	0.122989	0.122989
ϕ_4	0	0.184484	0.179251
ϕ_5	0.2	0	0.284834
ϕ_6	0.2	0.124716	0.294747
ϕ_7	0.2	0.249431	0.333891
ϕ_8	0.2	0.374147	0.377971
ϕ_9	0.4	0	0.423078
ϕ_{10}	0.4	0.163193	0.433334
ϕ_{11}	0.4	0.326387	0.457718
ϕ_{12}	0.4	0.48958	0.487647
ϕ_{13}	0.9	0	0.654265
ϕ_{14}	0.9	0.233512	0.659864
ϕ_{15}	0.9	0.467023	0.676055
ϕ_{16}	0.9	0.700535	0.700769
ϕ_{17}	1.4	0	0.824487
ϕ_{18}	1.4	0.286395	0.828468
ϕ_{19}	1.4	0.57279	0.840155
ϕ_{20}	1.4	0.859186	0.859261
ϕ_{21}	2	0	1
ϕ_{22}	2	$\frac{1}{3}$	1
ϕ_{23}	2	$\frac{2}{3}$	1
ϕ_{24}	2	1	1

Element solution summary

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	0.357735	0.432418	0.457513	0.449434	0.212296
2	0.357735	0.277356	0.425738	0.573696	0.177371
3	0.357735	0.122294	0.401895	0.682608	0.0656707
4	0.157735	0.310534	0.326117	0.637141	0.418969
5	0.157735	0.199179	0.280052	0.855389	0.393937
6	0.157735	0.0878232	0.241056	1.17417	0.186911
7	1.29434	0.767498	0.82148	0.297608	0.0737282
8	1.29434	0.492278	0.802806	0.321416	0.0459218
9	1.29434	0.217059	0.791924	0.336582	0.0157086
10	0.794338	0.609748	0.650276	0.382142	0.118072
11	0.794338	0.391097	0.626127	0.419517	0.081968
12	0.794338	0.172445	0.610601	0.451689	0.0301067
13	1.87321	0.901897	0.969405	0.238571	0.0124844
14	1.87321	0.578483	0.965699	0.269456	0.00763633
15	1.87321	0.255069	0.963573	0.287128	0.00260162