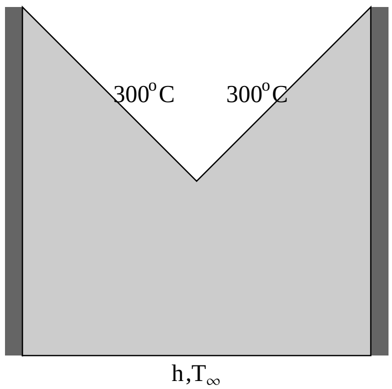
CHAPTER EIGHT

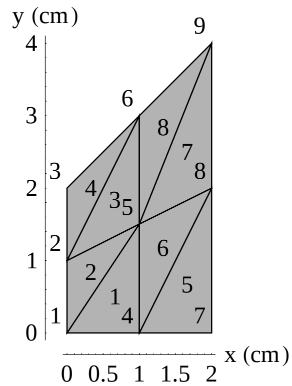
Transient Problems

Example 8.1: Transient heat flow (p. 552)

The cross-section of a long V-grooved ceramic strip is shown in Figure. The sides are insulated while convection heat loss takes place at the bottom with $h = 200\ W/m^2 \cdot °C$ and $T_\infty = 50\ °C$. The grooved surface is maintained at a constant temperature of 300°C. For ceramic the thermal conductivity is $k = 3\ W/m \cdot °C$, density $\rho = 1600\ kg/m^3$, and specific heat $c_p = 0.8\ kJ/kg \cdot °C$. Assuming the entire strip to be at 50°C initially, determine the time history of temperature distribution. Estimate the time it will take to reach steady state conditions.

Taking advantage of symmetry, we need to model only half of the cross-section.





The complete calculations are as follows.

Global equations at the start of the assembly of element equations

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} \dot{T}_1\\ \dot{T}_2\\ \dot{T}_3\\ \dot{T}_4\\ \dot{T}_5\\ \dot{T}_6\\ \dot{T}_7\\ \dot{T}_8\\ \dot{T}_9 \end{pmatrix} + \begin{pmatrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} T_1\\ T_2\\ T_3\\ T_4\\ T_5\\ T_6\\ T_7\\ T_8\\ T_9 \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}$$

Equations for element 1

$k_x = 3;$ $k_y = 3;$ $q = 0;$ $m = \rho C_p = 1280000$

$$\boldsymbol{C} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	1	0	0
2	4	$\frac{1}{100}$	0
3	5	$\frac{1}{100}$	$\frac{3}{200}$

$x_1 = 0 \qquad x_2 = \frac{1}{100} \qquad x_3 = \frac{1}{100}$

$y_1 = 0 \qquad y_2 = 0 \qquad y_3 = \frac{3}{200}$

Using these values we get

$$b_1 = -\frac{3}{200} \qquad b_2 = \frac{3}{200} \qquad b_3 = 0$$

$$c_1 = 0 \qquad c_2 = -\frac{1}{100} \qquad c_3 = \frac{1}{100}$$

$$f_1 = \frac{3}{20000} \qquad f_2 = 0 \qquad f_3 = 0$$

Element area, $A = \dfrac{3}{40000}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -\frac{3}{200} & \frac{3}{200} & 0 \\ 0 & -\frac{1}{100} & \frac{1}{100} \end{pmatrix}$$

$$\boldsymbol{m} = \begin{pmatrix} 16 & 8 & 8 \\ 8 & 16 & 8 \\ 8 & 8 & 16 \end{pmatrix}; \qquad \boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} \frac{9}{4} & -\frac{9}{4} & 0 \\ -\frac{9}{4} & \frac{13}{4} & -1 \\ 0 & -1 & 1 \end{pmatrix}; \qquad \boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

NBC on side 1 (nodes {1, 4})

End nodal coordinates: $\left(\{0, 0\} \quad \left\{\frac{1}{100}, 0\right\} \right)$ giving side length, $L = \dfrac{1}{100}$

Specified parameter values: $\alpha = -200$; $\beta = 10000$

$$\boldsymbol{k}_{\alpha} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_{\beta} = \begin{pmatrix} 50 \\ 50 \\ 0 \end{pmatrix}$$

Complete equations for element 1

$$\begin{pmatrix} 16 & 8 & 8 \\ 8 & 16 & 8 \\ 8 & 8 & 16 \end{pmatrix}\begin{pmatrix} \dot{T}_1 \\ \dot{T}_2 \\ \dot{T}_3 \\ \dot{T}_4 \\ \dot{T}_5 \\ \dot{T}_6 \\ \dot{T}_7 \\ \dot{T}_8 \\ \dot{T}_9 \end{pmatrix} + \begin{pmatrix} \frac{35}{12} & -\frac{23}{12} & 0 \\ -\frac{23}{12} & \frac{47}{12} & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{pmatrix} = \begin{pmatrix} 50 \\ 50 \\ 0 \end{pmatrix}$$

The element contributes to the following degrees of freedom.

The element contributes to {1, 4, 5} global degrees of freedom.

Adding element equations into appropriate locations,
the global equations after assembly of this element are as follows.

$$\begin{pmatrix} 16 & 0 & 0 & 8 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 16 & 8 & 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 8 & 16 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \dot{T}_1 \\ \dot{T}_2 \\ \dot{T}_3 \\ \dot{T}_4 \\ \dot{T}_5 \\ \dot{T}_6 \\ \dot{T}_7 \\ \dot{T}_8 \\ \dot{T}_9 \end{pmatrix} + \begin{pmatrix} \frac{35}{12} & 0 & 0 & -\frac{23}{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{23}{12} & 0 & 0 & \frac{47}{12} & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{pmatrix} = \begin{pmatrix} 50 \\ 0 \\ 0 \\ 50 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 2

$k_x = 3;$ $k_y = 3;$ $q = 0;$ $m = \rho C_p = 1280000$

$$\boldsymbol{C} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	5	$\frac{1}{100}$	$\frac{3}{200}$
2	2	0	$\frac{1}{100}$
3	1	0	0

$x_1 = \frac{1}{100} \qquad x_2 = 0 \qquad x_3 = 0$

$y_1 = \frac{3}{200} \qquad y_2 = \frac{1}{100} \qquad y_3 = 0$

Using these values we get

$$b_1 = \frac{1}{100} \qquad b_2 = -\frac{3}{200} \qquad b_3 = \frac{1}{200}$$

$$c_1 = 0 \qquad c_2 = \frac{1}{100} \qquad c_3 = -\frac{1}{100}$$

$$f_1 = 0 \qquad f_2 = 0 \qquad f_3 = \frac{1}{10000}$$

Element area, $A = \dfrac{1}{20000}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} \frac{1}{100} & -\frac{3}{200} & \frac{1}{200} \\ 0 & \frac{1}{100} & -\frac{1}{100} \end{pmatrix}$$

$$\boldsymbol{m} = \begin{pmatrix} \frac{32}{3} & \frac{16}{3} & \frac{16}{3} \\ \frac{16}{3} & \frac{32}{3} & \frac{16}{3} \\ \frac{16}{3} & \frac{16}{3} & \frac{32}{3} \end{pmatrix}; \qquad \boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} \frac{3}{2} & -\frac{9}{4} & \frac{3}{4} \\ -\frac{9}{4} & \frac{39}{8} & -\frac{21}{8} \\ \frac{3}{4} & -\frac{21}{8} & \frac{15}{8} \end{pmatrix}; \qquad \boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete equations for element 2

$$\begin{pmatrix} \frac{32}{3} & \frac{16}{3} & \frac{16}{3} \\ \frac{16}{3} & \frac{32}{3} & \frac{16}{3} \\ \frac{16}{3} & \frac{16}{3} & \frac{32}{3} \end{pmatrix} \begin{pmatrix} \dot{T}_1 \\ \dot{T}_2 \\ \dot{T}_3 \\ \dot{T}_4 \\ \dot{T}_5 \\ \dot{T}_6 \\ \dot{T}_7 \\ \dot{T}_8 \\ \dot{T}_9 \end{pmatrix} + \begin{pmatrix} \frac{3}{2} & -\frac{9}{4} & \frac{3}{4} \\ -\frac{9}{4} & \frac{39}{8} & -\frac{21}{8} \\ \frac{3}{4} & -\frac{21}{8} & \frac{15}{8} \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to the following degrees of freedom.

The element contributes to {5, 2, 1} global degrees of freedom.

Adding element equations into appropriate locations,
the global equations after assembly of this element are as follows.

$$\begin{pmatrix} \frac{80}{3} & \frac{16}{3} & 0 & 8 & \frac{40}{3} & 0 & 0 & 0 & 0 \\ \frac{16}{3} & \frac{32}{3} & 0 & 0 & \frac{16}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 16 & 8 & 0 & 0 & 0 & 0 \\ \frac{40}{3} & \frac{16}{3} & 0 & 8 & \frac{80}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{T}_1 \\ \dot{T}_2 \\ \dot{T}_3 \\ \dot{T}_4 \\ \dot{T}_5 \\ \dot{T}_6 \\ \dot{T}_7 \\ \dot{T}_8 \\ \dot{T}_9 \end{pmatrix} + \begin{pmatrix} \frac{115}{24} & -\frac{21}{8} & 0 & -\frac{23}{12} & \frac{3}{4} & 0 & 0 & 0 & 0 \\ -\frac{21}{8} & \frac{39}{8} & 0 & 0 & -\frac{9}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{23}{12} & 0 & 0 & \frac{47}{12} & -1 & 0 & 0 & 0 & 0 \\ \frac{3}{4} & -\frac{9}{4} & 0 & -1 & \frac{5}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{pmatrix} = \begin{pmatrix} 50 \\ 0 \\ 0 \\ 50 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 3

$k_x = 3$; $k_y = 3$; $q = 0$; $m = \rho C_p = 1280000$

$$\boldsymbol{C} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	2	0	$\frac{1}{100}$
2	5	$\frac{1}{100}$	$\frac{3}{200}$
3	6	$\frac{1}{100}$	$\frac{3}{100}$

$x_1 = 0$ $x_2 = \frac{1}{100}$ $x_3 = \frac{1}{100}$

$y_1 = \frac{1}{100}$ $y_2 = \frac{3}{200}$ $y_3 = \frac{3}{100}$

Using these values we get

$$b_1 = -\frac{3}{200} \qquad b_2 = \frac{1}{50} \qquad b_3 = -\frac{1}{200}$$

$$c_1 = 0 \qquad c_2 = -\frac{1}{100} \qquad c_3 = \frac{1}{100}$$

$f_1 = \frac{3}{20000}$ $\quad f_2 = \frac{1}{10000}$ $\quad f_3 = -\frac{1}{10000}$

Element area, $A = \frac{3}{40000}$

$$\boldsymbol{B}^T = \begin{pmatrix} -\frac{3}{200} & \frac{1}{50} & -\frac{1}{200} \\ 0 & -\frac{1}{100} & \frac{1}{100} \end{pmatrix}$$

$$\boldsymbol{m} = \begin{pmatrix} 16 & 8 & 8 \\ 8 & 16 & 8 \\ 8 & 8 & 16 \end{pmatrix}; \qquad \boldsymbol{k}_k = \begin{pmatrix} \frac{9}{4} & -3 & \frac{3}{4} \\ -3 & 5 & -2 \\ \frac{3}{4} & -2 & \frac{5}{4} \end{pmatrix}; \qquad \boldsymbol{r}_q = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete equations for element 3

$$\begin{pmatrix} 16 & 8 & 8 \\ 8 & 16 & 8 \\ 8 & 8 & 16 \end{pmatrix} \begin{pmatrix} \dot{T}_1 \\ \dot{T}_2 \\ \dot{T}_3 \\ \dot{T}_4 \\ \dot{T}_5 \\ \dot{T}_6 \\ \dot{T}_7 \\ \dot{T}_8 \\ \dot{T}_9 \end{pmatrix} + \begin{pmatrix} \frac{9}{4} & -3 & \frac{3}{4} \\ -3 & 5 & -2 \\ \frac{3}{4} & -2 & \frac{5}{4} \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to the following degrees of freedom.

The element contributes to {2, 5, 6} global degrees of freedom.

Adding element equations into appropriate locations,
the global equations after assembly of this element are as follows.

$$\begin{pmatrix} \frac{80}{3} & \frac{16}{3} & 0 & 8 & \frac{40}{3} & 0 & 0 & 0 & 0 \\ \frac{16}{3} & \frac{80}{3} & 0 & 0 & \frac{40}{3} & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 16 & 8 & 0 & 0 & 0 & 0 \\ \frac{40}{3} & \frac{40}{3} & 0 & 8 & \frac{128}{3} & 8 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 8 & 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{T}_1 \\ \dot{T}_2 \\ \dot{T}_3 \\ \dot{T}_4 \\ \dot{T}_5 \\ \dot{T}_6 \\ \dot{T}_7 \\ \dot{T}_8 \\ \dot{T}_9 \end{pmatrix} + \begin{pmatrix} \frac{115}{24} & -\frac{21}{8} & 0 & -\frac{23}{12} & \frac{3}{4} & 0 & 0 & 0 & 0 \\ -\frac{21}{8} & \frac{57}{8} & 0 & 0 & -\frac{21}{4} & \frac{3}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{23}{12} & 0 & 0 & \frac{47}{12} & -1 & 0 & 0 & 0 & 0 \\ \frac{3}{4} & -\frac{21}{4} & 0 & -1 & \frac{15}{2} & -2 & 0 & 0 & 0 \\ 0 & \frac{3}{4} & 0 & 0 & -2 & \frac{5}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{pmatrix} = \begin{pmatrix} 50 \\ 0 \\ 0 \\ 50 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 4

$k_x = 3;$ $\quad k_y = 3;$ $\quad q = 0;$ $\quad m = \rho C_p = 1280000$

$$\boldsymbol{C} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	6	$\frac{1}{100}$	$\frac{3}{100}$
2	3	0	$\frac{1}{50}$
3	2	0	$\frac{1}{100}$

$x_1 = \frac{1}{100}$ $\quad x_2 = 0$ $\quad x_3 = 0$

$y_1 = \frac{3}{100}$ $\quad y_2 = \frac{1}{50}$ $\quad y_3 = \frac{1}{100}$

Using these values we get

$$b_1 = \frac{1}{100} \qquad b_2 = -\frac{1}{50} \qquad b_3 = \frac{1}{100}$$

$$c_1 = 0 \qquad c_2 = \frac{1}{100} \qquad c_3 = -\frac{1}{100}$$

$$f_1 = 0 \qquad f_2 = -\frac{1}{10000} \qquad f_3 = \frac{1}{5000}$$

Element area, $A = \frac{1}{20000}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} \frac{1}{100} & -\frac{1}{50} & \frac{1}{100} \\ 0 & \frac{1}{100} & -\frac{1}{100} \end{pmatrix}$$

$$\boldsymbol{m} = \begin{pmatrix} \frac{32}{3} & \frac{16}{3} & \frac{16}{3} \\ \frac{16}{3} & \frac{32}{3} & \frac{16}{3} \\ \frac{16}{3} & \frac{16}{3} & \frac{32}{3} \end{pmatrix}; \qquad \boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} \frac{3}{2} & -3 & \frac{3}{2} \\ -3 & \frac{15}{2} & -\frac{9}{2} \\ \frac{3}{2} & -\frac{9}{2} & 3 \end{pmatrix}; \qquad \boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete equations for element 4

$$\begin{pmatrix} \frac{32}{3} & \frac{16}{3} & \frac{16}{3} \\ \frac{16}{3} & \frac{32}{3} & \frac{16}{3} \\ \frac{16}{3} & \frac{16}{3} & \frac{32}{3} \end{pmatrix} \begin{pmatrix} \dot{\mathrm{T}}_1 \\ \dot{\mathrm{T}}_2 \\ \dot{\mathrm{T}}_3 \\ \dot{\mathrm{T}}_4 \\ \dot{\mathrm{T}}_5 \\ \dot{\mathrm{T}}_6 \\ \dot{\mathrm{T}}_7 \\ \dot{\mathrm{T}}_8 \\ \dot{\mathrm{T}}_9 \end{pmatrix} + \begin{pmatrix} \frac{3}{2} & -3 & \frac{3}{2} \\ -3 & \frac{15}{2} & -\frac{9}{2} \\ \frac{3}{2} & -\frac{9}{2} & 3 \end{pmatrix} \begin{pmatrix} \mathrm{T}_1 \\ \mathrm{T}_2 \\ \mathrm{T}_3 \\ \mathrm{T}_4 \\ \mathrm{T}_5 \\ \mathrm{T}_6 \\ \mathrm{T}_7 \\ \mathrm{T}_8 \\ \mathrm{T}_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to the following degrees of freedom.

The element contributes to {6, 3, 2} global degrees of freedom.

Adding element equations into appropriate locations,
the global equations after assembly of this element are as follows.

$$\begin{pmatrix} \frac{80}{3} & \frac{16}{3} & 0 & 8 & \frac{40}{3} & 0 & 0 & 0 & 0 \\ \frac{16}{3} & \frac{112}{3} & \frac{16}{3} & 0 & \frac{40}{3} & \frac{40}{3} & 0 & 0 & 0 \\ 0 & \frac{16}{3} & \frac{32}{3} & 0 & 0 & \frac{16}{3} & 0 & 0 & 0 \\ 8 & 0 & 0 & 16 & 8 & 0 & 0 & 0 & 0 \\ \frac{40}{3} & \frac{40}{3} & 0 & 8 & \frac{128}{3} & 8 & 0 & 0 & 0 \\ 0 & \frac{40}{3} & \frac{16}{3} & 0 & 8 & \frac{80}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{T}_1 \\ \dot{T}_2 \\ \dot{T}_3 \\ \dot{T}_4 \\ \dot{T}_5 \\ \dot{T}_6 \\ \dot{T}_7 \\ \dot{T}_8 \\ \dot{T}_9 \end{pmatrix} +$$

$$\begin{pmatrix} \frac{115}{24} & -\frac{21}{8} & 0 & -\frac{23}{12} & \frac{3}{4} & 0 & 0 & 0 & 0 \\ -\frac{21}{8} & \frac{81}{8} & -\frac{9}{2} & 0 & -\frac{21}{4} & \frac{9}{4} & 0 & 0 & 0 \\ 0 & -\frac{9}{2} & \frac{15}{2} & 0 & 0 & -3 & 0 & 0 & 0 \\ -\frac{23}{12} & 0 & 0 & \frac{47}{12} & -1 & 0 & 0 & 0 & 0 \\ \frac{3}{4} & -\frac{21}{4} & 0 & -1 & \frac{15}{2} & -2 & 0 & 0 & 0 \\ 0 & \frac{9}{4} & -3 & 0 & -2 & \frac{11}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{pmatrix} = \begin{pmatrix} 50 \\ 0 \\ 0 \\ 50 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 5

$k_x = 3;$ $k_y = 3;$ $q = 0;$ $m = \rho C_p = 1280000$

$$\boldsymbol{C} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	4	$\frac{1}{100}$	0
2	7	$\frac{1}{50}$	0
3	8	$\frac{1}{50}$	$\frac{1}{50}$

$x_1 = \frac{1}{100}$ $\quad x_2 = \frac{1}{50}$ $\quad x_3 = \frac{1}{50}$

$y_1 = 0$ $\quad y_2 = 0$ $\quad y_3 = \frac{1}{50}$

Using these values we get

$b_1 = -\frac{1}{50}$ $\quad b_2 = \frac{1}{50}$ $\quad b_3 = 0$

$c_1 = 0$ $\quad c_2 = -\frac{1}{100}$ $\quad c_3 = \frac{1}{100}$

$f_1 = \frac{1}{2500}$ $\quad f_2 = -\frac{1}{5000}$ $\quad f_3 = 0$

Element area, $A = \frac{1}{10000}$

$$\boldsymbol{B}^{T} = \begin{pmatrix} -\frac{1}{50} & \frac{1}{50} & 0 \\ 0 & -\frac{1}{100} & \frac{1}{100} \end{pmatrix}$$

$$\boldsymbol{m} = \begin{pmatrix} \frac{64}{3} & \frac{32}{3} & \frac{32}{3} \\ \frac{32}{3} & \frac{64}{3} & \frac{32}{3} \\ \frac{32}{3} & \frac{32}{3} & \frac{64}{3} \end{pmatrix}; \qquad \boldsymbol{k}_{k} = \begin{pmatrix} 3 & -3 & 0 \\ -3 & \frac{15}{4} & -\frac{3}{4} \\ 0 & -\frac{3}{4} & \frac{3}{4} \end{pmatrix}; \qquad \boldsymbol{r}_{q} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

NBC on side 1 (nodes {4, 7})

End nodal coordinates: $\left(\left\{ \frac{1}{100}, 0 \right\} \quad \left\{ \frac{1}{50}, 0 \right\} \right)$ giving side length, $L = \frac{1}{100}$

Specified parameter values: $\quad \alpha = -200; \quad \beta = 10000$

$$\boldsymbol{k}_{\alpha} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_{\beta} = \begin{pmatrix} 50 \\ 50 \\ 0 \end{pmatrix}$$

Complete equations for element 5

$$\begin{pmatrix} \frac{64}{3} & \frac{32}{3} & \frac{32}{3} \\ \frac{32}{3} & \frac{64}{3} & \frac{32}{3} \\ \frac{32}{3} & \frac{32}{3} & \frac{64}{3} \end{pmatrix} \begin{pmatrix} \dot{T}_1 \\ \dot{T}_2 \\ \dot{T}_3 \\ \dot{T}_4 \\ \dot{T}_5 \\ \dot{T}_6 \\ \dot{T}_7 \\ \dot{T}_8 \\ \dot{T}_9 \end{pmatrix} + \begin{pmatrix} \frac{11}{3} & -\frac{8}{3} & 0 \\ -\frac{8}{3} & \frac{53}{12} & -\frac{3}{4} \\ 0 & -\frac{3}{4} & \frac{3}{4} \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{pmatrix} = \begin{pmatrix} 50 \\ 50 \\ 0 \end{pmatrix}$$

The element contributes to the following degrees of freedom.

The element contributes to {4, 7, 8} global degrees of freedom.

Adding element equations into appropriate locations,
the global equations after assembly of this element are as follows.

$$\begin{pmatrix} \frac{80}{3} & \frac{16}{3} & 0 & 8 & \frac{40}{3} & 0 & 0 & 0 & 0 \\ \frac{16}{3} & \frac{112}{3} & \frac{16}{3} & 0 & \frac{40}{3} & \frac{40}{3} & 0 & 0 & 0 \\ 0 & \frac{16}{3} & \frac{32}{3} & 0 & 0 & \frac{16}{3} & 0 & 0 & 0 \\ 8 & 0 & 0 & \frac{112}{3} & 8 & 0 & \frac{32}{3} & \frac{32}{3} & 0 \\ \frac{40}{3} & \frac{40}{3} & 0 & 8 & \frac{128}{3} & 8 & 0 & 0 & 0 \\ 0 & \frac{40}{3} & \frac{16}{3} & 0 & 8 & \frac{80}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{32}{3} & 0 & 0 & \frac{64}{3} & \frac{32}{3} & 0 \\ 0 & 0 & 0 & \frac{32}{3} & 0 & 0 & \frac{32}{3} & \frac{64}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{T}_1 \\ \dot{T}_2 \\ \dot{T}_3 \\ \dot{T}_4 \\ \dot{T}_5 \\ \dot{T}_6 \\ \dot{T}_7 \\ \dot{T}_8 \\ \dot{T}_9 \end{pmatrix} +$$

$$\begin{pmatrix} \frac{115}{24} & -\frac{21}{8} & 0 & -\frac{23}{12} & \frac{3}{4} & 0 & 0 & 0 & 0 \\ -\frac{21}{8} & \frac{81}{8} & -\frac{9}{2} & 0 & -\frac{21}{4} & \frac{9}{4} & 0 & 0 & 0 \\ 0 & -\frac{9}{2} & \frac{15}{2} & 0 & 0 & -3 & 0 & 0 & 0 \\ -\frac{23}{12} & 0 & 0 & \frac{91}{12} & -1 & 0 & -\frac{8}{3} & 0 & 0 \\ \frac{3}{4} & -\frac{21}{4} & 0 & -1 & \frac{15}{2} & -2 & 0 & 0 & 0 \\ 0 & \frac{9}{4} & -3 & 0 & -2 & \frac{11}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{8}{3} & 0 & 0 & \frac{53}{12} & -\frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{pmatrix} = \begin{pmatrix} 50 \\ 0 \\ 0 \\ 100 \\ 0 \\ 0 \\ 50 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 6

$k_x = 3;$ $k_y = 3;$ $q = 0;$ $m = \rho C_p = 1280000$

$$\boldsymbol{C} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	8	$\frac{1}{50}$	$\frac{1}{50}$
2	5	$\frac{1}{100}$	$\frac{3}{200}$
3	4	$\frac{1}{100}$	0

$x_1 = \frac{1}{50}$ $\quad x_2 = \frac{1}{100}$ $\quad x_3 = \frac{1}{100}$

$y_1 = \frac{1}{50}$ $\quad y_2 = \frac{3}{200}$ $\quad y_3 = 0$

Using these values we get

$b_1 = \frac{3}{200}$ $\quad b_2 = -\frac{1}{50}$ $\quad b_3 = \frac{1}{200}$

$c_1 = 0$ $\quad c_2 = \frac{1}{100}$ $\quad c_3 = -\frac{1}{100}$

$f_1 = -\frac{3}{20000}$ $\quad f_2 = \frac{1}{5000}$ $\quad f_3 = \frac{1}{10000}$

Element area, $A = \frac{3}{40000}$

$$\boldsymbol{B}^T = \begin{pmatrix} \frac{3}{200} & -\frac{1}{50} & \frac{1}{200} \\ 0 & \frac{1}{100} & -\frac{1}{100} \end{pmatrix}$$

$$\boldsymbol{m} = \begin{pmatrix} 16 & 8 & 8 \\ 8 & 16 & 8 \\ 8 & 8 & 16 \end{pmatrix}; \qquad \boldsymbol{k}_k = \begin{pmatrix} \frac{9}{4} & -3 & \frac{3}{4} \\ -3 & 5 & -2 \\ \frac{3}{4} & -2 & \frac{5}{4} \end{pmatrix}; \qquad \boldsymbol{r}_q = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete equations for element 6

$$\begin{pmatrix} 16 & 8 & 8 \\ 8 & 16 & 8 \\ 8 & 8 & 16 \end{pmatrix} \begin{pmatrix} \dot{T}_1 \\ \dot{T}_2 \\ \dot{T}_3 \\ \dot{T}_4 \\ \dot{T}_5 \\ \dot{T}_6 \\ \dot{T}_7 \\ \dot{T}_8 \\ \dot{T}_9 \end{pmatrix} + \begin{pmatrix} \frac{9}{4} & -3 & \frac{3}{4} \\ -3 & 5 & -2 \\ \frac{3}{4} & -2 & \frac{5}{4} \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to the following degrees of freedom.

The element contributes to {8, 5, 4} global degrees of freedom.

Adding element equations into appropriate locations,
the global equations after assembly of this element are as follows.

$$\begin{pmatrix} \frac{80}{3} & \frac{16}{3} & 0 & 8 & \frac{40}{3} & 0 & 0 & 0 & 0 \\ \frac{16}{3} & \frac{112}{3} & \frac{16}{3} & 0 & \frac{40}{3} & \frac{40}{3} & 0 & 0 & 0 \\ 0 & \frac{16}{3} & \frac{32}{3} & 0 & 0 & \frac{16}{3} & 0 & 0 & 0 \\ 8 & 0 & 0 & \frac{160}{3} & 16 & 0 & \frac{32}{3} & \frac{56}{3} & 0 \\ \frac{40}{3} & \frac{40}{3} & 0 & 16 & \frac{176}{3} & 8 & 0 & 8 & 0 \\ 0 & \frac{40}{3} & \frac{16}{3} & 0 & 8 & \frac{80}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{32}{3} & 0 & 0 & \frac{64}{3} & \frac{32}{3} & 0 \\ 0 & 0 & 0 & \frac{56}{3} & 8 & 0 & \frac{32}{3} & \frac{112}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{T}_1 \\ \dot{T}_2 \\ \dot{T}_3 \\ \dot{T}_4 \\ \dot{T}_5 \\ \dot{T}_6 \\ \dot{T}_7 \\ \dot{T}_8 \\ \dot{T}_9 \end{pmatrix} +$$

$$\begin{pmatrix} \frac{115}{24} & -\frac{21}{8} & 0 & -\frac{23}{12} & \frac{3}{4} & 0 & 0 & 0 & 0 \\ -\frac{21}{8} & \frac{81}{8} & -\frac{9}{2} & 0 & -\frac{21}{4} & \frac{9}{4} & 0 & 0 & 0 \\ 0 & -\frac{9}{2} & \frac{15}{2} & 0 & 0 & -3 & 0 & 0 & 0 \\ -\frac{23}{12} & 0 & 0 & \frac{53}{6} & -3 & 0 & -\frac{8}{3} & \frac{3}{4} & 0 \\ \frac{3}{4} & -\frac{21}{4} & 0 & -3 & \frac{25}{2} & -2 & 0 & -3 & 0 \\ 0 & \frac{9}{4} & -3 & 0 & -2 & \frac{11}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{8}{3} & 0 & 0 & \frac{53}{12} & -\frac{3}{4} & 0 \\ 0 & 0 & 0 & \frac{3}{4} & -3 & 0 & -\frac{3}{4} & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{pmatrix} = \begin{pmatrix} 50 \\ 0 \\ 0 \\ 100 \\ 0 \\ 0 \\ 50 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 7

$k_x = 3;$ $\quad k_y = 3;$ $\quad q = 0;$ $\quad m = \rho C_p = 1280000$

$$\boldsymbol{C} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	5	$\frac{1}{100}$	$\frac{3}{200}$
2	8	$\frac{1}{50}$	$\frac{1}{50}$
3	9	$\frac{1}{50}$	$\frac{1}{25}$

$x_1 = \frac{1}{100}$ $x_2 = \frac{1}{50}$ $x_3 = \frac{1}{50}$

$y_1 = \frac{3}{200}$ $y_2 = \frac{1}{50}$ $y_3 = \frac{1}{25}$

Using these values we get

$b_1 = -\frac{1}{50}$ $b_2 = \frac{1}{40}$ $b_3 = -\frac{1}{200}$

$c_1 = 0$ $c_2 = -\frac{1}{100}$ $c_3 = \frac{1}{100}$

$f_1 = \frac{1}{2500}$ $f_2 = -\frac{1}{10000}$ $f_3 = -\frac{1}{10000}$

Element area, $A = \frac{1}{10000}$

$$\boldsymbol{B}^T = \begin{pmatrix} -\frac{1}{50} & \frac{1}{40} & -\frac{1}{200} \\ 0 & -\frac{1}{100} & \frac{1}{100} \end{pmatrix}$$

$$\boldsymbol{m} = \begin{pmatrix} \frac{64}{3} & \frac{32}{3} & \frac{32}{3} \\ \frac{32}{3} & \frac{64}{3} & \frac{32}{3} \\ \frac{32}{3} & \frac{32}{3} & \frac{64}{3} \end{pmatrix}; \qquad \boldsymbol{k}_k = \begin{pmatrix} 3 & -\frac{15}{4} & \frac{3}{4} \\ -\frac{15}{4} & \frac{87}{16} & -\frac{27}{16} \\ \frac{3}{4} & -\frac{27}{16} & \frac{15}{16} \end{pmatrix}; \qquad \boldsymbol{r}_q = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete equations for element 7

$$\begin{pmatrix} \frac{64}{3} & \frac{32}{3} & \frac{32}{3} \\ \frac{32}{3} & \frac{64}{3} & \frac{32}{3} \\ \frac{32}{3} & \frac{32}{3} & \frac{64}{3} \end{pmatrix} \begin{pmatrix} \dot{T}_1 \\ \dot{T}_2 \\ \dot{T}_3 \\ \dot{T}_4 \\ \dot{T}_5 \\ \dot{T}_6 \\ \dot{T}_7 \\ \dot{T}_8 \\ \dot{T}_9 \end{pmatrix} + \begin{pmatrix} 3 & -\frac{15}{4} & \frac{3}{4} \\ -\frac{15}{4} & \frac{87}{16} & -\frac{27}{16} \\ \frac{3}{4} & -\frac{27}{16} & \frac{15}{16} \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to the following degrees of freedom.

The element contributes to {5, 8, 9} global degrees of freedom.

Adding element equations into appropriate locations,
the global equations after assembly of this element are as follows.

$$
\begin{pmatrix}
\frac{80}{3} & \frac{16}{3} & 0 & 8 & \frac{40}{3} & 0 & 0 & 0 & 0 \\
\frac{16}{3} & \frac{112}{3} & \frac{16}{3} & 0 & \frac{40}{3} & \frac{40}{3} & 0 & 0 & 0 \\
0 & \frac{16}{3} & \frac{32}{3} & 0 & 0 & \frac{16}{3} & 0 & 0 & 0 \\
8 & 0 & 0 & \frac{160}{3} & 16 & 0 & \frac{32}{3} & \frac{56}{3} & 0 \\
\frac{40}{3} & \frac{40}{3} & 0 & 16 & 80 & 8 & 0 & \frac{56}{3} & \frac{32}{3} \\
0 & \frac{40}{3} & \frac{16}{3} & 0 & 8 & \frac{80}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{32}{3} & 0 & 0 & \frac{64}{3} & \frac{32}{3} & 0 \\
0 & 0 & 0 & \frac{56}{3} & \frac{56}{3} & 0 & \frac{32}{3} & \frac{176}{3} & \frac{32}{3} \\
0 & 0 & 0 & 0 & \frac{32}{3} & 0 & 0 & \frac{32}{3} & \frac{64}{3}
\end{pmatrix}
\begin{pmatrix} \dot{T}_1 \\ \dot{T}_2 \\ \dot{T}_3 \\ \dot{T}_4 \\ \dot{T}_5 \\ \dot{T}_6 \\ \dot{T}_7 \\ \dot{T}_8 \\ \dot{T}_9 \end{pmatrix} +
$$

$$
\begin{pmatrix}
\frac{115}{24} & -\frac{21}{8} & 0 & -\frac{23}{12} & \frac{3}{4} & 0 & 0 & 0 & 0 \\
-\frac{21}{8} & \frac{81}{8} & -\frac{9}{2} & 0 & -\frac{21}{4} & \frac{9}{4} & 0 & 0 & 0 \\
0 & -\frac{9}{2} & \frac{15}{2} & 0 & 0 & -3 & 0 & 0 & 0 \\
-\frac{23}{12} & 0 & 0 & \frac{53}{6} & -3 & 0 & -\frac{8}{3} & \frac{3}{4} & 0 \\
\frac{3}{4} & -\frac{21}{4} & 0 & -3 & \frac{31}{2} & -2 & 0 & -\frac{27}{4} & \frac{3}{4} \\
0 & \frac{9}{4} & -3 & 0 & -2 & \frac{11}{4} & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{8}{3} & 0 & 0 & \frac{53}{12} & -\frac{3}{4} & 0 \\
0 & 0 & 0 & \frac{3}{4} & -\frac{27}{4} & 0 & -\frac{3}{4} & \frac{135}{16} & -\frac{27}{16} \\
0 & 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & -\frac{27}{16} & \frac{15}{16}
\end{pmatrix}
\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{pmatrix} =
\begin{pmatrix} 50 \\ 0 \\ 0 \\ 100 \\ 0 \\ 0 \\ 50 \\ 0 \\ 0 \end{pmatrix}
$$

Equations for element 8

$k_x = 3;$ $k_y = 3;$ $q = 0;$ $m = \rho C_p = 1280000$

$$\boldsymbol{C} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	9	$\frac{1}{50}$	$\frac{1}{25}$
2	6	$\frac{1}{100}$	$\frac{3}{100}$
3	5	$\frac{1}{100}$	$\frac{3}{200}$

$x_1 = \frac{1}{50}$ $\quad x_2 = \frac{1}{100}$ $\quad x_3 = \frac{1}{100}$

$y_1 = \frac{1}{25}$ $\quad y_2 = \frac{3}{100}$ $\quad y_3 = \frac{3}{200}$

Using these values we get

$b_1 = \frac{3}{200}$ $\quad b_2 = -\frac{1}{40}$ $\quad b_3 = \frac{1}{100}$

$c_1 = 0$ $\quad c_2 = \frac{1}{100}$ $\quad c_3 = -\frac{1}{100}$

$f_1 = -\frac{3}{20000}$ $\quad f_2 = \frac{1}{10000}$ $\quad f_3 = \frac{1}{5000}$

Element area, $A = \frac{3}{40000}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} \frac{3}{200} & -\frac{1}{40} & \frac{1}{100} \\ 0 & \frac{1}{100} & -\frac{1}{100} \end{pmatrix}$$

$$\boldsymbol{m} = \begin{pmatrix} 16 & 8 & 8 \\ 8 & 16 & 8 \\ 8 & 8 & 16 \end{pmatrix}; \qquad \boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} \frac{9}{4} & -\frac{15}{4} & \frac{3}{2} \\ -\frac{15}{4} & \frac{29}{4} & -\frac{7}{2} \\ \frac{3}{2} & -\frac{7}{2} & 2 \end{pmatrix}; \qquad \boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete equations for element 8

$$\begin{pmatrix} 16 & 8 & 8 \\ 8 & 16 & 8 \\ 8 & 8 & 16 \end{pmatrix} \begin{pmatrix} \dot{T}_1 \\ \dot{T}_2 \\ \dot{T}_3 \\ \dot{T}_4 \\ \dot{T}_5 \\ \dot{T}_6 \\ \dot{T}_7 \\ \dot{T}_8 \\ \dot{T}_9 \end{pmatrix} + \begin{pmatrix} \frac{9}{4} & -\frac{15}{4} & \frac{3}{2} \\ -\frac{15}{4} & \frac{29}{4} & -\frac{7}{2} \\ \frac{3}{2} & -\frac{7}{2} & 2 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to the following degrees of freedom.

The element contributes to {9, 6, 5} global degrees of freedom.

Adding element equations into appropriate locations,
the global equations after assembly of this element are as follows.

$$\begin{pmatrix} \frac{80}{3} & \frac{16}{3} & 0 & 8 & \frac{40}{3} & 0 & 0 & 0 & 0 \\ \frac{16}{3} & \frac{112}{3} & \frac{16}{3} & 0 & \frac{40}{3} & \frac{40}{3} & 0 & 0 & 0 \\ 0 & \frac{16}{3} & \frac{32}{3} & 0 & 0 & \frac{16}{3} & 0 & 0 & 0 \\ 8 & 0 & 0 & \frac{160}{3} & 16 & 0 & \frac{32}{3} & \frac{56}{3} & 0 \\ \frac{40}{3} & \frac{40}{3} & 0 & 16 & 96 & 16 & 0 & \frac{56}{3} & \frac{56}{3} \\ 0 & \frac{40}{3} & \frac{16}{3} & 0 & 16 & \frac{128}{3} & 0 & 0 & 8 \\ 0 & 0 & 0 & \frac{32}{3} & 0 & 0 & \frac{64}{3} & \frac{32}{3} & 0 \\ 0 & 0 & 0 & \frac{56}{3} & \frac{56}{3} & 0 & \frac{32}{3} & \frac{176}{3} & \frac{32}{3} \\ 0 & 0 & 0 & 0 & \frac{56}{3} & 8 & 0 & \frac{32}{3} & \frac{112}{3} \end{pmatrix} \begin{pmatrix} \dot{T}_1 \\ \dot{T}_2 \\ \dot{T}_3 \\ \dot{T}_4 \\ \dot{T}_5 \\ \dot{T}_6 \\ \dot{T}_7 \\ \dot{T}_8 \\ \dot{T}_9 \end{pmatrix} +$$

$$\begin{pmatrix} \frac{115}{24} & -\frac{21}{8} & 0 & -\frac{23}{12} & \frac{3}{4} & 0 & 0 & 0 & 0 \\ -\frac{21}{8} & \frac{81}{8} & -\frac{9}{2} & 0 & -\frac{21}{4} & \frac{9}{4} & 0 & 0 & 0 \\ 0 & -\frac{9}{2} & \frac{15}{2} & 0 & 0 & -3 & 0 & 0 & 0 \\ -\frac{23}{12} & 0 & 0 & \frac{53}{6} & -3 & 0 & -\frac{8}{3} & \frac{3}{4} & 0 \\ \frac{3}{4} & -\frac{21}{4} & 0 & -3 & \frac{35}{2} & -\frac{11}{2} & 0 & -\frac{27}{4} & \frac{9}{4} \\ 0 & \frac{9}{4} & -3 & 0 & -\frac{11}{2} & 10 & 0 & 0 & -\frac{15}{4} \\ 0 & 0 & 0 & -\frac{8}{3} & 0 & 0 & \frac{53}{12} & -\frac{3}{4} & 0 \\ 0 & 0 & 0 & \frac{3}{4} & -\frac{27}{4} & 0 & -\frac{3}{4} & \frac{135}{16} & -\frac{27}{16} \\ 0 & 0 & 0 & 0 & \frac{9}{4} & -\frac{15}{4} & 0 & -\frac{27}{16} & \frac{51}{16} \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{pmatrix} = \begin{pmatrix} 50 \\ 0 \\ 0 \\ 100 \\ 0 \\ 0 \\ 50 \\ 0 \\ 0 \end{pmatrix}$$

Node	dof	Value
3	T_3	300
6	T_6	300
9	T_9	300

Essential boundary conditions

Delete equations {3, 6, 9}.

$$\begin{pmatrix} \frac{80}{3} & \frac{16}{3} & 0 & 8 & \frac{40}{3} & 0 & 0 & 0 & 0 \\ \frac{16}{3} & \frac{112}{3} & \frac{16}{3} & 0 & \frac{40}{3} & \frac{40}{3} & 0 & 0 & 0 \\ 8 & 0 & 0 & \frac{160}{3} & 16 & 0 & \frac{32}{3} & \frac{56}{3} & 0 \\ \frac{40}{3} & \frac{40}{3} & 0 & 16 & 96 & 16 & 0 & \frac{56}{3} & \frac{56}{3} \\ 0 & 0 & 0 & \frac{32}{3} & 0 & 0 & \frac{64}{3} & \frac{32}{3} & 0 \\ 0 & 0 & 0 & \frac{56}{3} & \frac{56}{3} & 0 & \frac{32}{3} & \frac{176}{3} & \frac{32}{3} \end{pmatrix} \begin{pmatrix} \dot{T}_1 \\ \dot{T}_2 \\ 0 \\ \dot{T}_4 \\ \dot{T}_5 \\ 0 \\ \dot{T}_7 \\ \dot{T}_8 \\ 0 \end{pmatrix} +$$

$$\begin{pmatrix} \frac{115}{24} & -\frac{21}{8} & 0 & -\frac{23}{12} & \frac{3}{4} & 0 & 0 & 0 & 0 \\ -\frac{21}{8} & \frac{81}{8} & -\frac{9}{2} & 0 & -\frac{21}{4} & \frac{9}{4} & 0 & 0 & 0 \\ -\frac{23}{12} & 0 & 0 & \frac{53}{6} & -3 & 0 & -\frac{8}{3} & \frac{3}{4} & 0 \\ \frac{3}{4} & -\frac{21}{4} & 0 & -3 & \frac{35}{2} & -\frac{11}{2} & 0 & -\frac{27}{4} & \frac{9}{4} \\ 0 & 0 & 0 & -\frac{8}{3} & 0 & 0 & \frac{53}{12} & -\frac{3}{4} & 0 \\ 0 & 0 & 0 & \frac{3}{4} & -\frac{27}{4} & 0 & -\frac{3}{4} & \frac{135}{16} & -\frac{27}{16} \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ 300 \\ T_4 \\ T_5 \\ 300 \\ T_7 \\ T_8 \\ 300 \end{pmatrix} = \begin{pmatrix} 50 \\ 0 \\ 100 \\ 0 \\ 50 \\ 0 \end{pmatrix}$$

Extract columns {3, 6, 9}, multiply by the corresponding known values, and move to the right hand side.

The final global system of equations after adjusting for essential boundary conditions is as follows

$$\begin{pmatrix} \frac{80}{3} & \frac{16}{3} & 8 & \frac{40}{3} & 0 & 0 \\ \frac{16}{3} & \frac{112}{3} & 0 & \frac{40}{3} & 0 & 0 \\ 8 & 0 & \frac{160}{3} & 16 & \frac{32}{3} & \frac{56}{3} \\ \frac{40}{3} & \frac{40}{3} & 16 & 96 & 0 & \frac{56}{3} \\ 0 & 0 & \frac{32}{3} & 0 & \frac{64}{3} & \frac{32}{3} \\ 0 & 0 & \frac{56}{3} & \frac{56}{3} & \frac{32}{3} & \frac{176}{3} \end{pmatrix} \begin{pmatrix} \dot{T}_1 \\ \dot{T}_2 \\ \dot{T}_4 \\ \dot{T}_5 \\ \dot{T}_7 \\ \dot{T}_8 \end{pmatrix} + \begin{pmatrix} \frac{115}{24} & -\frac{21}{8} & -\frac{23}{12} & \frac{3}{4} & 0 & 0 \\ -\frac{21}{8} & \frac{81}{8} & 0 & -\frac{21}{4} & 0 & 0 \\ -\frac{23}{12} & 0 & \frac{53}{6} & -3 & -\frac{8}{3} & \frac{3}{4} \\ \frac{3}{4} & -\frac{21}{4} & -3 & \frac{35}{2} & 0 & -\frac{27}{4} \\ 0 & 0 & -\frac{8}{3} & 0 & \frac{53}{12} & -\frac{3}{4} \\ 0 & 0 & \frac{3}{4} & -\frac{27}{4} & -\frac{3}{4} & \frac{135}{16} \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_4 \\ T_5 \\ T_7 \\ T_8 \end{pmatrix} = \begin{pmatrix} 50 \\ 675 \\ 100 \\ 975 \\ 50 \\ \frac{2025}{4} \end{pmatrix}$$

Transient response analysis

Initial values: T ⟹ for all nodes = 50

Time history of nodal solution

Time	T_1	T_2	T_3	T_4	T_5	T_6	T_7
0	50.	50.	300	50.	50.	300	50.
25	73.6209	151.427	300	59.2213	147.405	300	55.7964
50	108.643	186.212	300	95.1527	187.632	300	90.3225
75	130.19	206.039	300	118.061	210.386	300	112.919
100	142.974	217.714	300	131.687	223.768	300	126.387
125	150.528	224.607	300	139.741	231.668	300	134.348
150	154.99	228.679	300	144.498	236.334	300	139.05
175	157.626	231.083	300	147.307	239.09	300	141.827
200	159.182	232.503	300	148.967	240.717	300	143.467
225	160.102	233.342	300	149.947	241.679	300	144.436
250	160.645	233.838	300	150.526	242.247	300	145.008
275	160.965	234.13	300	150.868	242.582	300	145.346
300	161.155	234.303	300	151.069	242.78	300	145.546