**Example 5.2: Laplace equation over a square domain (p. 334)**

Consider solution of the Laplace equation over a square domain.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0; \quad 0 < x < 1; \quad 0 < y < 1$$

with the following boundary conditions

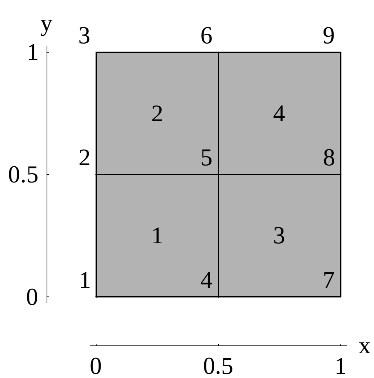
$$u(0, y) = 0; \quad u(1, y) = 0$$

$$u(x, 0) = x(1 - x); \quad u(x, 1) = 0$$

Comparing this problem to the general form of the 2D BVP we can see that here we have $k_x = k_y = 1$ and $p = q = 0$. The boundary conditions on all four sides are of the essential type.

An exact solution of the problem is known and is given by the sum of the following infinite series.

$$\text{Exact } u(x, y) = \sum_{n=1}^{\infty} -\frac{4 \sin(n \pi x) ((-1)^n - 1) \sinh(n \pi (1 - y))}{\sinh(n \pi) n^3 \pi^3}$$




From the given essential boundary conditions the following nodal values are known.

Essential boundary conditions {node, value}: $\left( \{1, 0\} \quad \{2, 0\} \quad \{3, 0\} \quad \left\{4, \frac{1}{4}\right\} \quad \{6, 0\} \quad \{7, 0\} \quad \{8, 0\} \quad \{9, 0\} \right)$

The complete finite element solution is as follows.

Global equations at start of the element assembly process

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 1

Element dimensions: $a = \frac{1}{4}$; $\quad b = \frac{1}{4}$

$k_x = 1$; $\quad k_y = 1$; $\quad p = 0$; $\quad q = 0$

$$\boldsymbol{k}_k = \begin{pmatrix} \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{pmatrix}; \qquad \boldsymbol{k}_p = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_q = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} u_1 \\ u_4 \\ u_5 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to {1, 4, 5, 2} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} \frac{2}{3} & -\frac{1}{6} & 0 & -\frac{1}{6} & -\frac{1}{3} & 0 & 0 & 0 & 0 \\ -\frac{1}{6} & \frac{2}{3} & 0 & -\frac{1}{3} & -\frac{1}{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{6} & -\frac{1}{3} & 0 & \frac{2}{3} & -\frac{1}{6} & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{6} & 0 & -\frac{1}{6} & \frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

## Equations for element 2

Element dimensions: $a = \frac{1}{4}$; $b = \frac{1}{4}$

$k_x = 1$; $k_y = 1$; $p = 0$; $q = 0$

$$\boldsymbol{k}_k = \begin{pmatrix} \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{pmatrix}; \qquad \boldsymbol{k}_p = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_q = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} u_2 \\ u_5 \\ u_6 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to {2, 5, 6, 3} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} \frac{2}{3} & -\frac{1}{6} & 0 & -\frac{1}{6} & -\frac{1}{3} & 0 & 0 & 0 & 0 \\ -\frac{1}{6} & \frac{4}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & -\frac{1}{6} & \frac{2}{3} & 0 & -\frac{1}{3} & -\frac{1}{6} & 0 & 0 & 0 \\ -\frac{1}{6} & -\frac{1}{3} & 0 & \frac{2}{3} & -\frac{1}{6} & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{6} & \frac{4}{3} & -\frac{1}{6} & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & -\frac{1}{6} & 0 & -\frac{1}{6} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 3

Element dimensions: $a = \frac{1}{4}$; $b = \frac{1}{4}$

$k_x = 1$; $k_y = 1$; $p = 0$; $q = 0$

$$\boldsymbol{k}_k = \begin{pmatrix} \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{pmatrix}; \qquad \boldsymbol{k}_p = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_q = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} u_4 \\ u_7 \\ u_8 \\ u_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to {4, 7, 8, 5} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} \frac{2}{3} & -\frac{1}{6} & 0 & -\frac{1}{6} & -\frac{1}{3} & 0 & 0 & 0 & 0 \\ -\frac{1}{6} & \frac{4}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & -\frac{1}{6} & \frac{2}{3} & 0 & -\frac{1}{3} & -\frac{1}{6} & 0 & 0 & 0 \\ -\frac{1}{6} & -\frac{1}{3} & 0 & \frac{4}{3} & -\frac{1}{3} & 0 & -\frac{1}{6} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 2 & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & -\frac{1}{3} & -\frac{1}{6} & 0 & -\frac{1}{6} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{6} & -\frac{1}{3} & 0 & \frac{2}{3} & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{6} & 0 & -\frac{1}{6} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 4

Element dimensions: $a = \frac{1}{4}$; $b = \frac{1}{4}$

$k_x = 1$; $k_y = 1$; $p = 0$; $q = 0$

$$\boldsymbol{k}_k = \begin{pmatrix} \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{pmatrix}; \qquad \boldsymbol{k}_p = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_q = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} u_5 \\ u_8 \\ u_9 \\ u_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to {5, 8, 9, 6} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} \frac{2}{3} & -\frac{1}{6} & 0 & -\frac{1}{6} & -\frac{1}{3} & 0 & 0 & 0 & 0 \\ -\frac{1}{6} & \frac{4}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & -\frac{1}{6} & \frac{2}{3} & 0 & -\frac{1}{3} & -\frac{1}{6} & 0 & 0 & 0 \\ -\frac{1}{6} & -\frac{1}{3} & 0 & \frac{4}{3} & -\frac{1}{3} & 0 & -\frac{1}{6} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & -\frac{1}{6} & 0 & -\frac{1}{3} & \frac{4}{3} & 0 & -\frac{1}{3} & -\frac{1}{6} \\ 0 & 0 & 0 & -\frac{1}{6} & -\frac{1}{3} & 0 & \frac{2}{3} & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{6} & \frac{4}{3} & -\frac{1}{6} \\ 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{6} & 0 & -\frac{1}{6} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Essential boundary conditions

| Node | dof | Value |
|---|---|---|
| 1 | $u_1$ | 0 |
| 2 | $u_2$ | 0 |
| 3 | $u_3$ | 0 |
| 4 | $u_4$ | $\frac{1}{4}$ |
| 6 | $u_6$ | 0 |
| 7 | $u_7$ | 0 |
| 8 | $u_8$ | 0 |
| 9 | $u_9$ | 0 |

Delete equations {1, 2, 3, 4, 6, 7, 8, 9}.

$$\begin{pmatrix} -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{4} \\ u_5 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = (0)$$

Extract columns {1, 2, 3, 4, 6, 7, 8, 9}.

Multiply each column by its respective known value $\left\{0, 0, 0, \frac{1}{4}, 0, 0, 0, 0\right\}$.

Move all resulting vectors to the rhs.

After adjusting for essential boundary conditions we have

$$\left(\tfrac{8}{3}\right)\left(u_5\right) = \left(\tfrac{1}{12}\right)$$

Solving the final system of global equations we get

$$\left\{u_5 = \frac{1}{32}\right\}$$

Complete table of nodal values

| | u |
|---|---|
| 1 | 0 |
| 2 | 0 |
| 3 | 0 |
| 4 | $\frac{1}{4}$ |
| 5 | $\frac{1}{32}$ |
| 6 | 0 |
| 7 | 0 |
| 8 | 0 |
| 9 | 0 |

Solution for element 1

Coordinates of element center

$$x_c = \frac{1}{4}; \qquad y_c = \frac{1}{4}$$

Element dimensions: $a = \frac{1}{4}$; $\qquad b = \frac{1}{4}$

Interpolation functions in local element coordinates

$$\boldsymbol{N}^T = \left\{4\,t\,s - s - t + \frac{1}{4},\ -4\,t\,s + s - t + \frac{1}{4},\ 4\,t\,s + s + t + \frac{1}{4},\ -4\,t\,s - s + t + \frac{1}{4}\right\}$$

Shift for global coordinates: $s = x - \frac{1}{4}$; $\qquad t = y - \frac{1}{4}$

Interpolation functions in global coordinates

$$\boldsymbol{N}^T = \{4\,y\,x - 2\,x - 2\,y + 1,\ 2\,x - 4\,x\,y,\ 4\,x\,y,\ 2\,y - 4\,x\,y\}$$

Nodal values, $\boldsymbol{d}^T = \left\{0,\ \frac{1}{4},\ \frac{1}{32},\ 0\right\}$

$$u(x, y) = \boldsymbol{N}^T\boldsymbol{d} = \frac{x}{2} - \frac{7\,x\,y}{8}$$

$\partial u/\partial x = \frac{1}{2} - \frac{7\,y}{8}$; $\quad\partial u/\partial y = -\frac{7\,x}{8}$

Solution for element 2

Coordinates of element center

$$x_c = \frac{1}{4}; \qquad y_c = \frac{3}{4}$$

Element dimensions: $a = \frac{1}{4}$; $\quad b = \frac{1}{4}$

Interpolation functions in local element coordinates

$$\boldsymbol{N}^T = \left\{4\,t\,s - s - t + \frac{1}{4},\ -4\,t\,s + s - t + \frac{1}{4},\ 4\,t\,s + s + t + \frac{1}{4},\ -4\,t\,s - s + t + \frac{1}{4}\right\}$$

Shift for global coordinates: $s = x - \frac{1}{4}$; $\quad t = y - \frac{3}{4}$

Interpolation functions in global coordinates

$$\boldsymbol{N}^T = \{4\,y\,x - 4\,x - 2\,y + 2,\ 4\,x - 4\,x\,y,\ 4\,x\,y - 2\,x,\ -4\,y\,x + 2\,x + 2\,y - 1\}$$

Nodal values, $\boldsymbol{d}^T = \left\{0,\ \frac{1}{32},\ 0,\ 0\right\}$

$u(x, y) = \boldsymbol{N}^T\boldsymbol{d} = \frac{x}{8} - \frac{x\,y}{8}$

$\partial u/\partial x = \frac{1}{8} - \frac{y}{8}$; $\quad\partial u/\partial y = -\frac{x}{8}$

Solution for element 3

Coordinates of element center

$$x_c = \frac{3}{4}; \qquad y_c = \frac{1}{4}$$

Element dimensions: $a = \frac{1}{4}$; $\quad b = \frac{1}{4}$

Interpolation functions in local element coordinates

$$\boldsymbol{N}^T = \left\{4\,t\,s - s - t + \frac{1}{4},\ -4\,t\,s + s - t + \frac{1}{4},\ 4\,t\,s + s + t + \frac{1}{4},\ -4\,t\,s - s + t + \frac{1}{4}\right\}$$

Shift for global coordinates: $s = x - \frac{3}{4}$; $\quad t = y - \frac{1}{4}$

Interpolation functions in global coordinates

$$\boldsymbol{N}^T = \{4\,y\,x - 2\,x - 4\,y + 2,\ -4\,y\,x + 2\,x + 2\,y - 1,\ 4\,x\,y - 2\,y,\ 4\,y - 4\,x\,y\}$$

Nodal values, $\boldsymbol{d}^{\mathrm{T}} = \left\{\frac{1}{4}, 0, 0, \frac{1}{32}\right\}$

$u(x, y) = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{d} = \frac{7\,y\,x}{8} - \frac{x}{2} - \frac{7\,y}{8} + \frac{1}{2}$

$\partial u/\partial x = \frac{7\,y}{8} - \frac{1}{2};\qquad \partial u/\partial y = \frac{7\,x}{8} - \frac{7}{8}$

## Solution for element 4

Coordinates of element center

$x_c = \frac{3}{4};\qquad y_c = \frac{3}{4}$

Element dimensions: $a = \frac{1}{4};\qquad b = \frac{1}{4}$

Interpolation functions in local element coordinates

$$\boldsymbol{N}^{\mathrm{T}} = \left\{4\,t\,s - s - t + \frac{1}{4}, -4\,t\,s + s - t + \frac{1}{4}, 4\,t\,s + s + t + \frac{1}{4}, -4\,t\,s - s + t + \frac{1}{4}\right\}$$

Shift for global coordinates: $s = x - \frac{3}{4};\qquad t = y - \frac{3}{4}$

Interpolation functions in global coordinates

$$\boldsymbol{N}^{\mathrm{T}} = \{4\,y\,x - 4\,x - 4\,y + 4, -4\,y\,x + 4\,x + 2\,y - 2, 4\,y\,x - 2\,x - 2\,y + 1, -4\,y\,x + 2\,x + 4\,y - 2\}$$

Nodal values, $\boldsymbol{d}^{\mathrm{T}} = \left\{\frac{1}{32}, 0, 0, 0\right\}$

$u(x, y) = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{d} = \frac{y\,x}{8} - \frac{x}{8} - \frac{y}{8} + \frac{1}{8}$

$\partial u/\partial x = \frac{y}{8} - \frac{1}{8};\qquad \partial u/\partial y = \frac{x}{8} - \frac{1}{8}$

## Solution summary

Nodal solution

| | x−coord | y−coord | u |
|---|---|---|---|
| 1 | 0 | 0 | 0 |
| 2 | 0 | $\frac{1}{2}$ | 0 |
| 3 | 0 | 1 | 0 |
| 4 | $\frac{1}{2}$ | 0 | $\frac{1}{4}$ |
| 5 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{32}$ |
| 6 | $\frac{1}{2}$ | 1 | 0 |
| 7 | 1 | 0 | 0 |
| 8 | 1 | $\frac{1}{2}$ | 0 |
| 9 | 1 | 1 | 0 |

Solution at element centroids

| | x−coord | y−coord | u | ∂u/∂x | ∂u/∂y |
|---|---|---|---|---|---|
| 1 | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{9}{128}$ | $\frac{9}{32}$ | $-\frac{7}{32}$ |
| 2 | $\frac{1}{4}$ | $\frac{3}{4}$ | $\frac{1}{128}$ | $\frac{1}{32}$ | $-\frac{1}{32}$ |
| 3 | $\frac{3}{4}$ | $\frac{1}{4}$ | $\frac{9}{128}$ | $-\frac{9}{32}$ | $-\frac{7}{32}$ |
| 4 | $\frac{3}{4}$ | $\frac{3}{4}$ | $\frac{1}{128}$ | $-\frac{1}{32}$ | $-\frac{1}{32}$ |

4 Element Solution $\times 10^{-3}$

| | |
|---|---|
| 7.8125 | 7.8125 |
| 70.3125 | 70.3125 |

Exact Solution $\times 10^{-3}$ (5 terms )

| | |
|---|---|
| 13.7286 | 13.7286 |
| 83.201 | 83.201 |

16 Element solution

Solution summary

Nodal solution

| | x−coord | y−coord | u |
|---|---|---|---|
| 1 | 0. | 0. | 0 |
| 2 | 0. | 0.25 | 0 |
| 3 | 0. | 0.5 | 0 |
| 4 | 0. | 0.75 | 0 |
| 5 | 0. | 1. | 0 |
| 6 | 0.25 | 0. | $\frac{3}{16}$ |
| 7 | 0.25 | 0.25 | 0.0788018 |
| 8 | 0.25 | 0.5 | 0.0334821 |
| 9 | 0.25 | 0.75 | 0.0122696 |
| 10 | 0.25 | 1. | 0 |
| 11 | 0.5 | 0. | $\frac{1}{4}$ |
| 12 | 0.5 | 0.25 | 0.112111 |
| 13 | 0.5 | 0.5 | 0.0473214 |
| 14 | 0.5 | 0.75 | 0.0173531 |
| 15 | 0.5 | 1. | 0 |
| 16 | 0.75 | 0. | $\frac{3}{16}$ |
| 17 | 0.75 | 0.25 | 0.0788018 |
| 18 | 0.75 | 0.5 | 0.0334821 |
| 19 | 0.75 | 0.75 | 0.0122696 |
| 20 | 0.75 | 1. | 0 |
| 21 | 1. | 0. | 0 |
| 22 | 1. | 0.25 | 0 |
| 23 | 1. | 0.5 | 0 |
| 24 | 1. | 0.75 | 0 |
| 25 | 1. | 1. | 0 |

Solution at element centroids

| | x−coord | y−coord | u | ∂u/∂x | ∂u/∂y |
|---|---|---|---|---|---|
| 1 | 0.125 | 0.125 | 0.0665755 | 0.532604 | −0.217396 |
| 2 | 0.125 | 0.375 | 0.028071 | 0.224568 | −0.0906394 |
| 3 | 0.125 | 0.625 | 0.0114379 | 0.0915035 | −0.0424251 |
| 4 | 0.125 | 0.875 | 0.0030674 | 0.0245392 | −0.0245392 |
| 5 | 0.375 | 0.125 | 0.157103 | 0.191619 | −0.493174 |
| 6 | 0.375 | 0.375 | 0.0679291 | 0.0942972 | −0.220219 |
| 7 | 0.375 | 0.625 | 0.0276066 | 0.0378456 | −0.102362 |
| 8 | 0.375 | 0.875 | 0.00740567 | 0.0101671 | −0.0592454 |
| 9 | 0.625 | 0.125 | 0.157103 | −0.191619 | −0.493174 |
| 10 | 0.625 | 0.375 | 0.0679291 | −0.0942972 | −0.220219 |
| 11 | 0.625 | 0.625 | 0.0276066 | −0.0378456 | −0.102362 |
| 12 | 0.625 | 0.875 | 0.00740567 | −0.0101671 | −0.0592454 |
| 13 | 0.875 | 0.125 | 0.0665755 | −0.532604 | −0.217396 |
| 14 | 0.875 | 0.375 | 0.028071 | −0.224568 | −0.0906394 |
| 15 | 0.875 | 0.625 | 0.0114379 | −0.0915035 | −0.0424251 |
| 16 | 0.875 | 0.875 | 0.0030674 | −0.0245392 | −0.0245392 |