CHAPTER FIVE

Two Dimensional Elements

Example 5.2: Laplace equation over a square domain (p. 334)

Consider solution of the Laplace equation over a square domain.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0; \quad 0 < x < 1; \quad 0 < y < 1$$

with the following boundary conditions

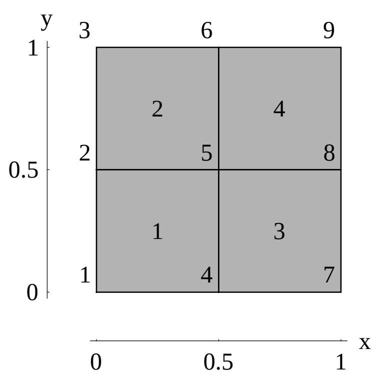
$$u(0, y) = 0; \quad u(1, y) = 0$$

$$u(x, 0) = x(1 - x); \quad u(x, 1) = 0$$

Comparing this problem to the general form of the 2D BVP we can see that here we have $k_x = k_y = 1$ and $p = q = 0$. The boundary conditions on all four sides are of the essential type.

An exact solution of the problem is known and is given by the sum of the following infinite series.

$$\text{Exact } u(x,\, y) = \sum_{n=1}^{\infty} -\frac{4\sin(n\,\pi\,x)\,((-1)^n - 1)\sinh(n\,\pi\,(1-y))}{\sinh(n\,\pi)\,n^3\,\pi^3}$$



From the given essential boundary conditions the following nodal values are known.

$$\text{Essential boundary conditions \{node, value\}: } \left(\{1,0\} \quad \{2,0\} \quad \{3,0\} \quad \left\{4, \tfrac{1}{4}\right\} \quad \{6,0\} \quad \{7,0\} \quad \{8,0\} \quad \{9,0\} \right)$$

The complete finite element solution is as follows.

Global equations at start of the element assembly process

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} u_1\\ u_2\\ u_3\\ u_4\\ u_5\\ u_6\\ u_7\\ u_8\\ u_9 \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}$$

Equations for element 1

Element dimensions: $a = \frac{1}{4}$; $b = \frac{1}{4}$

$k_x = 1$; $k_y = 1$; $p = 0$; $q = 0$

$$\boldsymbol{k}_k = \begin{pmatrix} \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{pmatrix}; \qquad \boldsymbol{k}_p = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_q = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} u_1 \\ u_4 \\ u_5 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to {1, 4, 5, 2} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} \frac{2}{3} & -\frac{1}{6} & 0 & -\frac{1}{6} & -\frac{1}{3} & 0 & 0 & 0 & 0 \\ -\frac{1}{6} & \frac{2}{3} & 0 & -\frac{1}{3} & -\frac{1}{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{6} & -\frac{1}{3} & 0 & \frac{2}{3} & -\frac{1}{6} & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{6} & 0 & -\frac{1}{6} & \frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 2

Element dimensions: $a = \frac{1}{4}$; $b = \frac{1}{4}$

$k_x = 1$; $k_y = 1$; $p = 0$; $q = 0$

$$\boldsymbol{k}_\mathrm{k} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{pmatrix}; \qquad \boldsymbol{k}_\mathrm{p} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_\mathrm{q} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} u_2 \\ u_5 \\ u_6 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to {2, 5, 6, 3} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} \frac{2}{3} & -\frac{1}{6} & 0 & -\frac{1}{6} & -\frac{1}{3} & 0 & 0 & 0 & 0 \\ -\frac{1}{6} & \frac{4}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & -\frac{1}{6} & \frac{2}{3} & 0 & -\frac{1}{3} & -\frac{1}{6} & 0 & 0 & 0 \\ -\frac{1}{6} & -\frac{1}{3} & 0 & \frac{2}{3} & -\frac{1}{6} & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{6} & \frac{4}{3} & -\frac{1}{6} & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & -\frac{1}{6} & 0 & -\frac{1}{6} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 3

Element dimensions: $a = \frac{1}{4}$; $\qquad b = \frac{1}{4}$

$k_x = 1$; $\qquad k_y = 1$; $\qquad p = 0$; $\qquad q = 0$

$$\boldsymbol{k}_\mathrm{k} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{pmatrix}; \qquad \boldsymbol{k}_\mathrm{p} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_\mathrm{q} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} u_4 \\ u_7 \\ u_8 \\ u_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to {4, 7, 8, 5} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} \frac{2}{3} & -\frac{1}{6} & 0 & -\frac{1}{6} & -\frac{1}{3} & 0 & 0 & 0 & 0 \\ -\frac{1}{6} & \frac{4}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & -\frac{1}{6} & \frac{2}{3} & 0 & -\frac{1}{3} & -\frac{1}{6} & 0 & 0 & 0 \\ -\frac{1}{6} & -\frac{1}{3} & 0 & \frac{4}{3} & -\frac{1}{3} & 0 & -\frac{1}{6} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 2 & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & -\frac{1}{3} & -\frac{1}{6} & 0 & -\frac{1}{6} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{6} & -\frac{1}{3} & 0 & \frac{2}{3} & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{6} & 0 & -\frac{1}{6} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 4

Element dimensions: $a = \frac{1}{4}$; $b = \frac{1}{4}$

$k_x = 1$; $k_y = 1$; $p = 0$; $q = 0$

$$\boldsymbol{k}_k = \begin{pmatrix} \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{pmatrix}; \qquad \boldsymbol{k}_p = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_q = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} u_5 \\ u_8 \\ u_9 \\ u_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to {5, 8, 9, 6} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} \frac{2}{3} & -\frac{1}{6} & 0 & -\frac{1}{6} & -\frac{1}{3} & 0 & 0 & 0 & 0 \\ -\frac{1}{6} & \frac{4}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & -\frac{1}{6} & \frac{2}{3} & 0 & -\frac{1}{3} & -\frac{1}{6} & 0 & 0 & 0 \\ -\frac{1}{6} & -\frac{1}{3} & 0 & \frac{4}{3} & -\frac{1}{3} & 0 & -\frac{1}{6} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & -\frac{1}{6} & 0 & -\frac{1}{3} & \frac{4}{3} & 0 & -\frac{1}{3} & -\frac{1}{6} \\ 0 & 0 & 0 & -\frac{1}{6} & -\frac{1}{3} & 0 & \frac{2}{3} & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{6} & \frac{4}{3} & -\frac{1}{6} \\ 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{6} & 0 & -\frac{1}{6} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Essential boundary conditions

Node	dof	Value
1	u_1	0
2	u_2	0
3	u_3	0
4	u_4	$\frac{1}{4}$
6	u_6	0
7	u_7	0
8	u_8	0
9	u_9	0

Delete equations {1, 2, 3, 4, 6, 7, 8, 9}.

$$\begin{pmatrix} -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{4} \\ u_5 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = (0)$$

Extract columns {1, 2, 3, 4, 6, 7, 8, 9}.

Multiply each column by its respective known value $\left\{0, 0, 0, \frac{1}{4}, 0, 0, 0, 0\right\}$.

Move all resulting vectors to the rhs.

After adjusting for essential boundary conditions we have

$\left(\frac{8}{3}\right)\left(u_5\right) = \left(\frac{1}{12}\right)$

Solving the final system of global equations we get

$$\left\{u_5 = \frac{1}{32}\right\}$$

Complete table of nodal values

	u
1	0
2	0
3	0
4	$\frac{1}{4}$
5	$\frac{1}{32}$
6	0
7	0
8	0
9	0

Solution for element 1

Coordinates of element center

$$x_c = \frac{1}{4}; \qquad y_c = \frac{1}{4}$$

Element dimensions: $a = \frac{1}{4}$; $\qquad b = \frac{1}{4}$

Interpolation functions in local element coordinates

$$\boldsymbol{N}^T = \left\{4\,t\,s - s - t + \frac{1}{4},\ -4\,t\,s + s - t + \frac{1}{4},\ 4\,t\,s + s + t + \frac{1}{4},\ -4\,t\,s - s + t + \frac{1}{4}\right\}$$

Shift for global coordinates: $s = x - \frac{1}{4}$; $\qquad t = y - \frac{1}{4}$

Interpolation functions in global coordinates

$$\boldsymbol{N}^T = \{4\,y\,x - 2\,x - 2\,y + 1,\ 2\,x - 4\,x\,y,\ 4\,x\,y,\ 2\,y - 4\,x\,y\}$$

Nodal values, $\boldsymbol{d}^T = \left\{0,\ \frac{1}{4},\ \frac{1}{32},\ 0\right\}$

$$u(x, y) = \boldsymbol{N}^{T}\boldsymbol{d} = \frac{x}{2} - \frac{7\,x\,y}{8}$$

$$\partial u/\partial x = \frac{1}{2} - \frac{7\,y}{8}; \qquad \partial u/\partial y = -\frac{7\,x}{8}$$

Solution for element 2

Coordinates of element center

$$x_c = \frac{1}{4}; \qquad y_c = \frac{3}{4}$$

Element dimensions: $a = \frac{1}{4}$; $\qquad b = \frac{1}{4}$

Interpolation functions in local element coordinates

$$\boldsymbol{N}^{T} = \left\{4\,t\,s - s - t + \frac{1}{4},\, -4\,t\,s + s - t + \frac{1}{4},\, 4\,t\,s + s + t + \frac{1}{4},\, -4\,t\,s - s + t + \frac{1}{4}\right\}$$

Shift for global coordinates: $s = x - \frac{1}{4}$; $\qquad t = y - \frac{3}{4}$

Interpolation functions in global coordinates

$$\boldsymbol{N}^{T} = \{4\,y\,x - 4\,x - 2\,y + 2,\, 4\,x - 4\,x\,y,\, 4\,x\,y - 2\,x,\, -4\,y\,x + 2\,x + 2\,y - 1\}$$

Nodal values, $\boldsymbol{d}^{T} = \left\{0,\, \frac{1}{32},\, 0,\, 0\right\}$

$$u(x, y) = \boldsymbol{N}^{T}\boldsymbol{d} = \frac{x}{8} - \frac{x\,y}{8}$$

$$\partial u/\partial x = \frac{1}{8} - \frac{y}{8}; \qquad \partial u/\partial y = -\frac{x}{8}$$

Solution for element 3

Coordinates of element center

$$x_c = \frac{3}{4}; \qquad y_c = \frac{1}{4}$$

Element dimensions: $a = \frac{1}{4}$; $\qquad b = \frac{1}{4}$

Interpolation functions in local element coordinates

$$\boldsymbol{N}^{T} = \left\{4\,t\,s - s - t + \frac{1}{4},\, -4\,t\,s + s - t + \frac{1}{4},\, 4\,t\,s + s + t + \frac{1}{4},\, -4\,t\,s - s + t + \frac{1}{4}\right\}$$

Shift for global coordinates: $s = x - \frac{3}{4}$; $\qquad t = y - \frac{1}{4}$

Interpolation functions in global coordinates

$$\boldsymbol{N}^{T} = \{4\,y\,x - 2\,x - 4\,y + 2,\ -4\,y\,x + 2\,x + 2\,y - 1,\ 4\,x\,y - 2\,y,\ 4\,y - 4\,x\,y\}$$

Nodal values, $\boldsymbol{d}^{T} = \left\{\frac{1}{4}, 0, 0, \frac{1}{32}\right\}$

$u(x, y) = \boldsymbol{N}^{T}\boldsymbol{d} = \frac{7\,y\,x}{8} - \frac{x}{2} - \frac{7\,y}{8} + \frac{1}{2}$

$\partial u/\partial x = \frac{7\,y}{8} - \frac{1}{2}$; $\partial u/\partial y = \frac{7\,x}{8} - \frac{7}{8}$

Solution for element 4

Coordinates of element center

$$x_c = \frac{3}{4}; \qquad y_c = \frac{3}{4}$$

Element dimensions: $a = \frac{1}{4}$; $b = \frac{1}{4}$

Interpolation functions in local element coordinates

$$\boldsymbol{N}^{T} = \left\{4\,t\,s - s - t + \frac{1}{4},\ -4\,t\,s + s - t + \frac{1}{4},\ 4\,t\,s + s + t + \frac{1}{4},\ -4\,t\,s - s + t + \frac{1}{4}\right\}$$

Shift for global coordinates: $s = x - \frac{3}{4}$; $t = y - \frac{3}{4}$

Interpolation functions in global coordinates

$$\boldsymbol{N}^{T} = \{4\,y\,x - 4\,x - 4\,y + 4,\ -4\,y\,x + 4\,x + 2\,y - 2,\ 4\,y\,x - 2\,x - 2\,y + 1,\ -4\,y\,x + 2\,x + 4\,y - 2\}$$

Nodal values, $\boldsymbol{d}^{T} = \left\{\frac{1}{32}, 0, 0, 0\right\}$

$u(x, y) = \boldsymbol{N}^{T}\boldsymbol{d} = \frac{y\,x}{8} - \frac{x}{8} - \frac{y}{8} + \frac{1}{8}$

$\partial u/\partial x = \frac{y}{8} - \frac{1}{8}$; $\partial u/\partial y = \frac{x}{8} - \frac{1}{8}$

Solution summary

Nodal solution

	x−coord	y−coord	u
1	0	0	0
2	0	$\frac{1}{2}$	0
3	0	1	0
4	$\frac{1}{2}$	0	$\frac{1}{4}$
5	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{32}$
6	$\frac{1}{2}$	1	0
7	1	0	0
8	1	$\frac{1}{2}$	0
9	1	1	0

Solution at element centroids

	x−coord	y−coord	u	$\partial u/\partial x$	$\partial u/\partial y$
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{9}{128}$	$\frac{9}{32}$	$-\frac{7}{32}$
2	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{128}$	$\frac{1}{32}$	$-\frac{1}{32}$
3	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{9}{128}$	$-\frac{9}{32}$	$-\frac{7}{32}$
4	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{128}$	$-\frac{1}{32}$	$-\frac{1}{32}$

4 Element Solution $\times 10^{-3}$

7.8125	7.8125
70.3125	70.3125

Exact Solution $\times 10^{-3}$ (5 terms)

13.7286	13.7286
83.201	83.201

16 Element solution

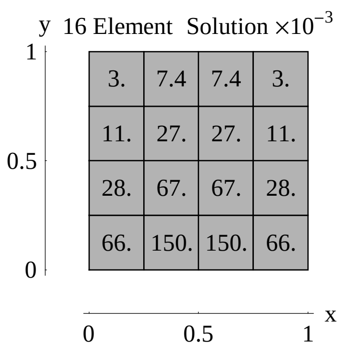
Solution summary

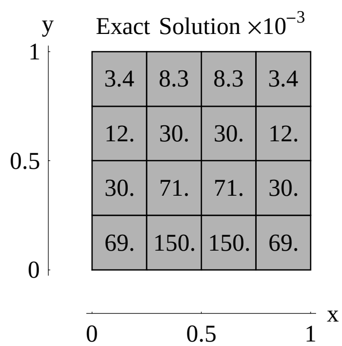
Nodal solution

	x−coord	y−coord	u
1	0.	0.	0
2	0.	0.25	0
3	0.	0.5	0
4	0.	0.75	0
5	0.	1.	0
6	0.25	0.	$\frac{3}{16}$
7	0.25	0.25	0.0788018
8	0.25	0.5	0.0334821
9	0.25	0.75	0.0122696
10	0.25	1.	0
11	0.5	0.	$\frac{1}{4}$
12	0.5	0.25	0.112111
13	0.5	0.5	0.0473214
14	0.5	0.75	0.0173531
15	0.5	1.	0
16	0.75	0.	$\frac{3}{16}$
17	0.75	0.25	0.0788018
18	0.75	0.5	0.0334821
19	0.75	0.75	0.0122696
20	0.75	1.	0
21	1.	0.	0
22	1.	0.25	0
23	1.	0.5	0
24	1.	0.75	0
25	1.	1.	0

Solution at element centroids

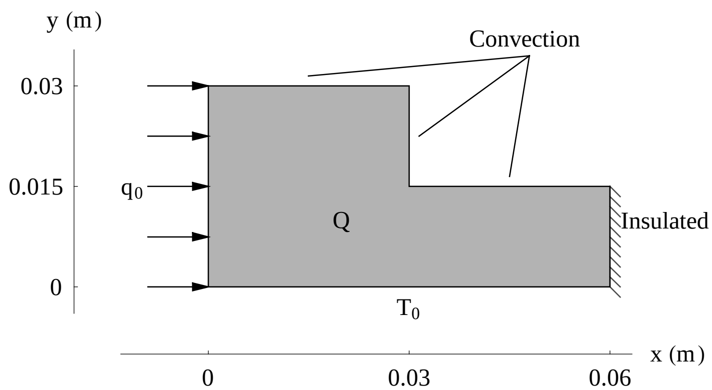
	x−coord	y−coord	u	∂u/∂x	∂u/∂y
1	0.125	0.125	0.0665755	0.532604	−0.217396
2	0.125	0.375	0.028071	0.224568	−0.0906394
3	0.125	0.625	0.0114379	0.0915035	−0.0424251
4	0.125	0.875	0.0030674	0.0245392	−0.0245392
5	0.375	0.125	0.157103	0.191619	−0.493174
6	0.375	0.375	0.0679291	0.0942972	−0.220219
7	0.375	0.625	0.0276066	0.0378456	−0.102362
8	0.375	0.875	0.00740567	0.0101671	−0.0592454
9	0.625	0.125	0.157103	−0.191619	−0.493174
10	0.625	0.375	0.0679291	−0.0942972	−0.220219
11	0.625	0.625	0.0276066	−0.0378456	−0.102362
12	0.625	0.875	0.00740567	−0.0101671	−0.0592454
13	0.875	0.125	0.0665755	−0.532604	−0.217396
14	0.875	0.375	0.028071	−0.224568	−0.0906394
15	0.875	0.625	0.0114379	−0.0915035	−0.0424251
16	0.875	0.875	0.0030674	−0.0245392	−0.0245392





Example 5.3: Heat flow in an L-shaped body (p. 337)

Consider two dimensional heat flow over an L-shaped body with thermal conductivity $k = 45\ W/m.°C$ shown in Figure. The bottom is maintained at $T_0 = 110\,°C$. Convection heat loss takes place on the top where the ambient air temperature is 20 °C and the convection heat transfer coefficient is $h = 55\ W/m^2.°C$. The right side is insulated. The left side is subjected to heat flux at a uniform rate of $q_L = 8000\ W/m^2$. Heat is generated in the body at a rate of $Q = 5 \times 10^6\ W/m^3$. Determine temperature distribution in the body.



As shown earlier the governing differential equation for a heat flow problem is a special case of the general form. With the numerical values given for this example

$$k_x = k_y = 45; \quad p = 0; \quad q = 5 \times 10^6$$

The boundary conditions are as follows.

For all nodes on the bottom side, $T = 110$

On the left side ($n_x = -1$, $n_y = 0$):

$$-k\ \frac{\partial T}{\partial n} = k\ \frac{\partial T}{\partial x} = q_L \quad \Longrightarrow \quad \alpha = 0; \beta = 8000$$

On the right side, $\alpha = 0; \beta = 0$

For convection on horizontal portions of the top side ($n_x = 0$, $n_y = 1$):

$$-k\ \frac{\partial T}{\partial n} = -k\ \frac{\partial T}{\partial y} = h(T - T_\infty)$$

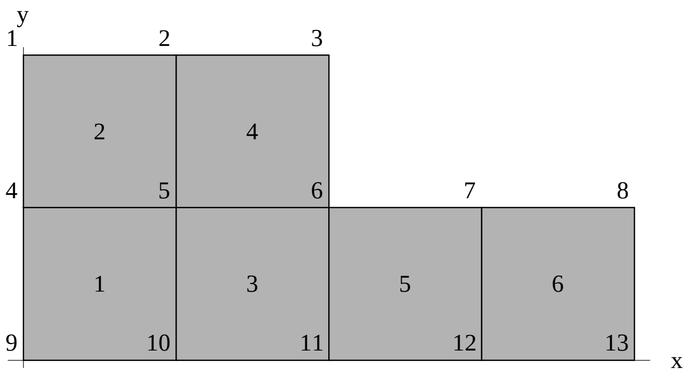
$$\Longrightarrow \quad \alpha = -h = -55; \beta = h\, T_\infty = 55 \times 20 = 1100$$

For convection on vertical portion of the top side ($n_x = 1$, $n_y = 0$):

$$-k\ \frac{\partial T}{\partial n} = -k\ \frac{\partial T}{\partial x} = h(T - T_\infty)$$

$$\Longrightarrow \quad \alpha = -h = -55; \beta = h\, T_\infty = 55 \times 20 = 1100$$

The complete finite element solution is as follows.



Global equations at start of the element assembly process

$$\left(\begin{array}{ccccccccccccc} 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0 \end{array}\right)\left(\begin{array}{c} T_1\\ T_2\\ T_3\\ T_4\\ T_5\\ T_6\\ T_7\\ T_8\\ T_9\\ T_{10}\\ T_{11}\\ T_{12}\\ T_{13} \end{array}\right)=\left(\begin{array}{c} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{array}\right)$$

Equations for element 1

Element dimensions: $a = 0.0075$; $b = 0.0075$

$k_x = 45$; $k_y = 45$; $p = 0$; $q = 5000000$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 30. & -7.5 & -15. & -7.5 \\ -7.5 & 30. & -7.5 & -15. \\ -15. & -7.5 & 30. & -7.5 \\ -7.5 & -15. & -7.5 & 30. \end{pmatrix}; \qquad \boldsymbol{k}_{\mathrm{p}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 281.25 \\ 281.25 \\ 281.25 \\ 281.25 \end{pmatrix}$$

NBC on side 4

L = 0.015; $\alpha = 0$; $\beta = 8000$

$$\boldsymbol{k}_{\alpha} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_{\beta} = \begin{pmatrix} 60. \\ 0 \\ 0 \\ 60. \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 30. & -7.5 & -15. & -7.5 \\ -7.5 & 30. & -7.5 & -15. \\ -15. & -7.5 & 30. & -7.5 \\ -7.5 & -15. & -7.5 & 30. \end{pmatrix} \begin{pmatrix} T_9 \\ T_{10} \\ T_5 \\ T_4 \end{pmatrix} = \begin{pmatrix} 341.25 \\ 281.25 \\ 281.25 \\ 341.25 \end{pmatrix}$$

The element contributes to {9, 10, 5, 4} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 30. & -7.5 & 0 & 0 & 0 & -7.5 & -15. & 0 & 0 & 0 \\ 0 & 0 & 0 & -7.5 & 30. & 0 & 0 & 0 & -15. & -7.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7.5 & -15. & 0 & 0 & 0 & 30. & -7.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -15. & -7.5 & 0 & 0 & 0 & -7.5 & 30. & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \\ T_{10} \\ T_{11} \\ T_{12} \\ T_{13} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 341.25 \\ 281.25 \\ 0 \\ 0 \\ 0 \\ 341.25 \\ 281.25 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 2

Element dimensions: a = 0.0075; b = 0.0075

$k_x = 45$; $k_y = 45$; p = 0; q = 5000000

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 30. & -7.5 & -15. & -7.5 \\ -7.5 & 30. & -7.5 & -15. \\ -15. & -7.5 & 30. & -7.5 \\ -7.5 & -15. & -7.5 & 30. \end{pmatrix}; \qquad \boldsymbol{k}_{\mathrm{p}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 281.25 \\ 281.25 \\ 281.25 \\ 281.25 \end{pmatrix}$$

NBC on side 3

$$\mathrm{L} = 0.015; \qquad \alpha = -55; \qquad \beta = 1100$$

$$\boldsymbol{k}_{\alpha} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.275 & 0.1375 \\ 0 & 0 & 0.1375 & 0.275 \end{pmatrix}; \qquad \boldsymbol{r}_{\beta} = \begin{pmatrix} 0 \\ 0 \\ 8.25 \\ 8.25 \end{pmatrix}$$

NBC on side 4

$$\mathrm{L} = 0.015; \qquad \alpha = 0; \qquad \beta = 8000$$

$$\boldsymbol{k}_{\alpha} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_{\beta} = \begin{pmatrix} 60. \\ 0 \\ 0 \\ 60. \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 30. & -7.5 & -15. & -7.5 \\ -7.5 & 30. & -7.5 & -15. \\ -15. & -7.5 & 30.275 & -7.3625 \\ -7.5 & -15. & -7.3625 & 30.275 \end{pmatrix} \begin{pmatrix} T_4 \\ T_5 \\ T_2 \\ T_1 \end{pmatrix} = \begin{pmatrix} 341.25 \\ 281.25 \\ 289.5 \\ 349.5 \end{pmatrix}$$

The element contributes to {4, 5, 2, 1} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 30.275 & -7.3625 & 0 & -7.5 & -15. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -7.3625 & 30.275 & 0 & -15. & -7.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -7.5 & -15. & 0 & 60. & -15. & 0 & 0 & 0 & -7.5 & -15. & 0 & 0 & 0 \\ -15. & -7.5 & 0 & -15. & 60. & 0 & 0 & 0 & -15. & -7.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7.5 & -15. & 0 & 0 & 0 & 30. & -7.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -15. & -7.5 & 0 & 0 & 0 & -7.5 & 30. & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \\ T_{10} \\ T_{11} \\ T_{12} \\ T_{13} \end{pmatrix} = \begin{pmatrix} 349.5 \\ 289.5 \\ 0 \\ 682.5 \\ 562.5 \\ 0 \\ 0 \\ 0 \\ 341.25 \\ 281.25 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 3

Element dimensions: $a = 0.0075$; $b = 0.0075$

$k_x = 45$; $k_y = 45$; $p = 0$; $q = 5000000$

$$\boldsymbol{k}_{\text{k}} = \begin{pmatrix} 30. & -7.5 & -15. & -7.5 \\ -7.5 & 30. & -7.5 & -15. \\ -15. & -7.5 & 30. & -7.5 \\ -7.5 & -15. & -7.5 & 30. \end{pmatrix}; \quad \boldsymbol{k}_{\text{p}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \boldsymbol{r}_{\text{q}} = \begin{pmatrix} 281.25 \\ 281.25 \\ 281.25 \\ 281.25 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 30. & -7.5 & -15. & -7.5 \\ -7.5 & 30. & -7.5 & -15. \\ -15. & -7.5 & 30. & -7.5 \\ -7.5 & -15. & -7.5 & 30. \end{pmatrix} \begin{pmatrix} T_{10} \\ T_{11} \\ T_6 \\ T_5 \end{pmatrix} = \begin{pmatrix} 281.25 \\ 281.25 \\ 281.25 \\ 281.25 \end{pmatrix}$$

The element contributes to {10, 11, 6, 5} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 30.275 & -7.3625 & 0 & -7.5 & -15. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -7.3625 & 30.275 & 0 & -15. & -7.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -7.5 & -15. & 0 & 60. & -15. & 0 & 0 & 0 & -7.5 & -15. & 0 & 0 & 0 \\ -15. & -7.5 & 0 & -15. & 90. & -7.5 & 0 & 0 & -15. & -15. & -15. & 0 & 0 \\ 0 & 0 & 0 & 0 & -7.5 & 30. & 0 & 0 & 0 & -15. & -7.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7.5 & -15. & 0 & 0 & 0 & 30. & -7.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -15. & -15. & -15. & 0 & 0 & -7.5 & 60. & -7.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & -15. & -7.5 & 0 & 0 & 0 & -7.5 & 30. & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \\ T_{10} \\ T_{11} \\ T_{12} \\ T_{13} \end{pmatrix} = \begin{pmatrix} 349.5 \\ 289.5 \\ 0 \\ 682.5 \\ 843.75 \\ 281.25 \\ 0 \\ 0 \\ 341.25 \\ 562.5 \\ 281.25 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 4

Element dimensions: a = 0.0075; b = 0.0075

$k_x = 45$; $k_y = 45$; p = 0; q = 5000000

$$\boldsymbol{k}_\text{k} = \begin{pmatrix} 30. & -7.5 & -15. & -7.5 \\ -7.5 & 30. & -7.5 & -15. \\ -15. & -7.5 & 30. & -7.5 \\ -7.5 & -15. & -7.5 & 30. \end{pmatrix}; \quad \boldsymbol{k}_\text{p} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \boldsymbol{r}_\text{q} = \begin{pmatrix} 281.25 \\ 281.25 \\ 281.25 \\ 281.25 \end{pmatrix}$$

NBC on side 2

L = 0.015; $\alpha = -55$; $\beta = 1100$

$$\boldsymbol{k}_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.275 & 0.1375 & 0 \\ 0 & 0.1375 & 0.275 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \boldsymbol{r}_\beta = \begin{pmatrix} 0 \\ 8.25 \\ 8.25 \\ 0 \end{pmatrix}$$

NBC on side 3

L = 0.015; $\alpha = -55$; $\beta = 1100$

$$\boldsymbol{k}_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.275 & 0.1375 \\ 0 & 0 & 0.1375 & 0.275 \end{pmatrix}; \quad \boldsymbol{r}_\beta = \begin{pmatrix} 0 \\ 0 \\ 8.25 \\ 8.25 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 30. & -7.5 & -15. & -7.5 \\ -7.5 & 30.275 & -7.3625 & -15. \\ -15. & -7.3625 & 30.55 & -7.3625 \\ -7.5 & -15. & -7.3625 & 30.275 \end{pmatrix} \begin{pmatrix} T_5 \\ T_6 \\ T_3 \\ T_2 \end{pmatrix} = \begin{pmatrix} 281.25 \\ 289.5 \\ 297.75 \\ 289.5 \end{pmatrix}$$

The element contributes to {5, 6, 3, 2} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 30.275 & -7.3625 & 0 & -7.5 & -15. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -7.3625 & 60.55 & -7.3625 & -15. & -15. & -15. & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -7.3625 & 30.55 & 0 & -15. & -7.3625 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -7.5 & -15. & 0 & 60. & -15. & 0 & 0 & 0 & -7.5 & -15. & 0 & 0 & 0 \\ -15. & -15. & -15. & -15. & 120. & -15. & 0 & 0 & -15. & -15. & -15. & 0 & 0 \\ 0 & -15. & -7.3625 & 0 & -15. & 60.275 & 0 & 0 & 0 & -15. & -7.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7.5 & -15. & 0 & 0 & 0 & 30. & -7.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -15. & -15. & -15. & 0 & 0 & -7.5 & 60. & -7.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & -15. & -7.5 & 0 & 0 & 0 & -7.5 & 30. & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \\ T_{10} \\ T_{11} \\ T_{12} \\ T_{13} \end{pmatrix} = \begin{pmatrix} 349.5 \\ 579. \\ 297.75 \\ 682.5 \\ 1125. \\ 570.75 \\ 0 \\ 0 \\ 341.25 \\ 562.5 \\ 281.25 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 5

Element dimensions: a = 0.0075; b = 0.0075

$k_x = 45$; $k_y = 45$; $p = 0$; $q = 5000000$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 30. & -7.5 & -15. & -7.5 \\ -7.5 & 30. & -7.5 & -15. \\ -15. & -7.5 & 30. & -7.5 \\ -7.5 & -15. & -7.5 & 30. \end{pmatrix}; \quad \boldsymbol{k}_{\mathrm{p}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 281.25 \\ 281.25 \\ 281.25 \\ 281.25 \end{pmatrix}$$

NBC on side 3

$L = 0.015$; $\alpha = -55$; $\beta = 1100$

$$\boldsymbol{k}_{\alpha} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.275 & 0.1375 \\ 0 & 0 & 0.1375 & 0.275 \end{pmatrix}; \quad \boldsymbol{r}_{\beta} = \begin{pmatrix} 0 \\ 0 \\ 8.25 \\ 8.25 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 30. & -7.5 & -15. & -7.5 \\ -7.5 & 30. & -7.5 & -15. \\ -15. & -7.5 & 30.275 & -7.3625 \\ -7.5 & -15. & -7.3625 & 30.275 \end{pmatrix} \begin{pmatrix} T_{11} \\ T_{12} \\ T_7 \\ T_6 \end{pmatrix} = \begin{pmatrix} 281.25 \\ 281.25 \\ 289.5 \\ 289.5 \end{pmatrix}$$

The element contributes to {11, 12, 7, 6} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 30.275 & -7.3625 & 0 & -7.5 & -15. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -7.3625 & 60.55 & -7.3625 & -15. & -15. & -15. & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -7.3625 & 30.55 & 0 & -15. & -7.3625 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -7.5 & -15. & 0 & 60. & -15. & 0 & 0 & 0 & -7.5 & -15. & 0 & 0 & 0 \\ -15. & -15. & -15. & -15. & 120. & -15. & 0 & 0 & -15. & -15. & -15. & 0 & 0 \\ 0 & -15. & -7.3625 & 0 & -15. & 90.55 & -7.3625 & 0 & 0 & -15. & -15. & -15. & 0 \\ 0 & 0 & 0 & 0 & 0 & -7.3625 & 30.275 & 0 & 0 & 0 & -15. & -7.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7.5 & -15. & 0 & 0 & 0 & 30. & -7.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -15. & -15. & -15. & 0 & 0 & -7.5 & 60. & -7.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & -15. & -15. & -15. & 0 & 0 & -7.5 & 60. & -7.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & -15. & -7.5 & 0 & 0 & 0 & -7.5 & 30. & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Equations for element 6

Element dimensions: a = 0.0075; b = 0.0075

$k_x = 45;\qquad k_y = 45;\qquad p = 0;\qquad q = 5000000$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 30. & -7.5 & -15. & -7.5 \\ -7.5 & 30. & -7.5 & -15. \\ -15. & -7.5 & 30. & -7.5 \\ -7.5 & -15. & -7.5 & 30. \end{pmatrix};\qquad \boldsymbol{k}_{\mathrm{p}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix};\qquad \boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 281.25 \\ 281.25 \\ 281.25 \\ 281.25 \end{pmatrix}$$

NBC on side 3

$L = 0.015;\qquad \alpha = -55;\qquad \beta = 1100$

$$\boldsymbol{k}_{\alpha} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.275 & 0.1375 \\ 0 & 0 & 0.1375 & 0.275 \end{pmatrix};\qquad \boldsymbol{r}_{\beta} = \begin{pmatrix} 0 \\ 0 \\ 8.25 \\ 8.25 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 30. & -7.5 & -15. & -7.5 \\ -7.5 & 30. & -7.5 & -15. \\ -15. & -7.5 & 30.275 & -7.3625 \\ -7.5 & -15. & -7.3625 & 30.275 \end{pmatrix} \begin{pmatrix} T_{12} \\ T_{13} \\ T_{8} \\ T_{7} \end{pmatrix} = \begin{pmatrix} 281.25 \\ 281.25 \\ 289.5 \\ 289.5 \end{pmatrix}$$

The element contributes to {12, 13, 8, 7} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\left(\begin{matrix} 30.275 & -7.3625 & 0 & -7.5 & -15. & 0 & 0 & 0 & 0 & 0 & 0 \\ -7.3625 & 60.55 & -7.3625 & -15. & -15. & -15. & 0 & 0 & 0 & 0 & 0 \\ 0 & -7.3625 & 30.55 & 0 & -15. & -7.3625 & 0 & 0 & 0 & 0 & 0 \\ -7.5 & -15. & 0 & 60. & -15. & 0 & 0 & 0 & -7.5 & -15. & 0 \\ -15. & -15. & -15. & -15. & 120. & -15. & 0 & 0 & -15. & -15. & -15. \\ 0 & -15. & -7.3625 & 0 & -15. & 90.55 & -7.3625 & 0 & 0 & -15. & -15. \\ 0 & 0 & 0 & 0 & 0 & -7.3625 & 60.55 & -7.3625 & 0 & 0 & -15. \\ 0 & 0 & 0 & 0 & 0 & 0 & -7.3625 & 30.275 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7.5 & -15. & 0 & 0 & 0 & 30. & -7.5 & 0 \\ 0 & 0 & 0 & -15. & -15. & -15. & 0 & 0 & -7.5 & 60. & -7.5 \\ 0 & 0 & 0 & 0 & -15. & -15. & -15. & 0 & 0 & -7.5 & 60. \\ 0 & 0 & 0 & 0 & 0 & -15. & -15. & -15. & 0 & 0 & -7.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & -15. & -7.5 & 0 & 0 & 0 \end{matrix}\right.$$

Essential boundary conditions

Node	dof	Value
9	T_9	110
10	T_{10}	110
11	T_{11}	110
12	T_{12}	110
13	T_{13}	110

Delete equations {9, 10, 11, 12, 13}.

$$\begin{pmatrix} 30.275 & -7.3625 & 0 & -7.5 & -15. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ -7.3625 & 60.55 & -7.3625 & -15. & -15. & -15. & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & -7.3625 & 30.55 & 0 & -15. & -7.3625 & 0 & 0 & 0 & 0 & 0 & 0 & \\ -7.5 & -15. & 0 & 60. & -15. & 0 & 0 & 0 & -7.5 & -15. & 0 & 0 & \\ -15. & -15. & -15. & -15. & 120. & -15. & 0 & 0 & -15. & -15. & -15. & 0 & \\ 0 & -15. & -7.3625 & 0 & -15. & 90.55 & -7.3625 & 0 & 0 & -15. & -15. & -15. & \\ 0 & 0 & 0 & 0 & 0 & -7.3625 & 60.55 & -7.3625 & 0 & 0 & -15. & -15. & - \\ 0 & 0 & 0 & 0 & 0 & 0 & -7.3625 & 30.275 & 0 & 0 & 0 & -15. & - \end{pmatrix}$$

Extract columns {9, 10, 11, 12, 13}.

Multiply each column by its respective known value {110, 110, 110, 110, 110}.

Move all resulting vectors to the rhs.

After adjusting for essential boundary conditions we have

$$\begin{pmatrix} 30.275 & -7.3625 & 0 & -7.5 & -15. & 0 & 0 & 0 \\ -7.3625 & 60.55 & -7.3625 & -15. & -15. & -15. & 0 & 0 \\ 0 & -7.3625 & 30.55 & 0 & -15. & -7.3625 & 0 & 0 \\ -7.5 & -15. & 0 & 60. & -15. & 0 & 0 & 0 \\ -15. & -15. & -15. & -15. & 120. & -15. & 0 & 0 \\ 0 & -15. & -7.3625 & 0 & -15. & 90.55 & -7.3625 & 0 \\ 0 & 0 & 0 & 0 & 0 & -7.3625 & 60.55 & -7.3625 \\ 0 & 0 & 0 & 0 & 0 & 0 & -7.3625 & 30.275 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \end{pmatrix} = \begin{pmatrix} 349.5 \\ 579. \\ 297.75 \\ 3157.5 \\ 6075. \\ 5810.25 \\ 5529. \\ 2764.5 \end{pmatrix}$$

Solving the final system of global equations we get

$$\{T_1 = 154.962,\ T_2 = 151.228,\ T_3 = 148.673,$$
$$T_4 = 145.433,\ T_5 = 142.521,\ T_6 = 134.871,\ T_7 = 122.436,\ T_8 = 121.088\}$$

Complete table of nodal values

	T
1	154.962
2	151.228
3	148.673
4	145.433
5	142.521
6	134.871
7	122.436
8	121.088
9	110
10	110
11	110
12	110
13	110

Solution for element 1

Coordinates of element center

$x_c = 0.0075; \qquad y_c = 0.0075$

Element dimensions: $a = 0.0075; \qquad b = 0.0075$

Interpolation functions in local element coordinates

$$\boldsymbol{N}^T = \{4444.44\,t\,s - 33.3333\,s - 33.3333\,t + 0.25,\ -4444.44\,t\,s + 33.3333\,s - 33.3333\,t + 0.25,\ 4444.44\,t\,s + 33.3333\,s + 33.3333\,t + 0.25,\ -4444.44\,t\,s - 33.3333\,s + 33.3333\,t + 0.25\}$$

Shift for global coordinates: $s = x - 0.0075; \qquad t = y - 0.0075$

Interpolation functions in global coordinates

$$\boldsymbol{N}^T = \{4444.44\,y\,x - 66.6667\,x - 66.6667\,y + 1.,\ 66.6667\,x - 4444.44\,x\,y,\ 4444.44\,x\,y,\ 66.6667\,y - 4444.44\,x\,y\}$$

Nodal values, $\boldsymbol{d}^T = \{110,\ 110,\ 142.521,\ 145.433\}$

$T(x, y) = \boldsymbol{N}^T\boldsymbol{d} = -12940.9\,x\,y + 2362.17\,y + 110.$

$\partial T/\partial x = -12940.9\,y; \qquad \partial T/\partial y = 2362.17 - 12940.9\,x$

Solution for element 2

Coordinates of element center

$x_c = 0.0075; \qquad y_c = 0.0225$

Element dimensions: $a = 0.0075; \qquad b = 0.0075$

Interpolation functions in local element coordinates

$$\boldsymbol{N}^{T} = \{4444.44\, t\, s - 33.3333\, s - 33.3333\, t + 0.25,\ -4444.44\, t\, s + 33.3333\, s - 33.3333\, t + 0.25,\ 4444.44\, t\, s + 33.3333\, s + 33.3333\, t + 0.25,\ -4444.44\, t\, s - 33.3333\, s + 33.3333\, t + 0.25\}$$

Shift for global coordinates: $s = x - 0.0075; \qquad t = y - 0.0225$

Interpolation functions in global coordinates

$$\boldsymbol{N}^{T} = \{4444.44\, y\, x - 133.333\, x - 66.6667\, y + 2.,\ 133.333\, x - 4444.44\, x\, y,\ 4444.44\, x\, y - 66.6667\, x,\ -4444.44\, y\, x + 66.6667\, x + 66.6667\, y - 1.\}$$

Nodal values, $\boldsymbol{d}^{T} = \{145.433,\ 142.521,\ 151.228,\ 154.962\}$

$T(x, y) = \boldsymbol{N}^{T}\boldsymbol{d} = -3653.37\, y\, x - 139.313\, x + 635.298\, y + 135.903$

$\partial T/\partial x = -3653.37\, y - 139.313; \qquad \partial T/\partial y = 635.298 - 3653.37\, x$

Solution for element 3

Coordinates of element center

$x_c = 0.0225; \qquad y_c = 0.0075$

Element dimensions: $a = 0.0075; \qquad b = 0.0075$

Interpolation functions in local element coordinates

$$\boldsymbol{N}^{T} = \{4444.44\, t\, s - 33.3333\, s - 33.3333\, t + 0.25,\ -4444.44\, t\, s + 33.3333\, s - 33.3333\, t + 0.25,\ 4444.44\, t\, s + 33.3333\, s + 33.3333\, t + 0.25,\ -4444.44\, t\, s - 33.3333\, s + 33.3333\, t + 0.25\}$$

Shift for global coordinates: $s = x - 0.0225; \qquad t = y - 0.0075$

Interpolation functions in global coordinates

$$\boldsymbol{N}^{T} = \{4444.44\, y\, x - 66.6667\, x - 133.333\, y + 2.,\ -4444.44\, y\, x + 66.6667\, x + 66.6667\, y - 1.,\ 4444.44\, x\, y - 66.6667\, y,\ 133.333\, y - 4444.44\, x\, y\}$$

Nodal values, $\boldsymbol{d}^{T} = \{110,\ 110,\ 134.871,\ 142.521\}$

$T(x, y) = \boldsymbol{N}^{T}\boldsymbol{d} = -34001.2\, x\, y + 2678.07\, y + 110.$

$\partial T/\partial x = -34001.2\, y; \qquad \partial T/\partial y = 2678.07 - 34001.2\, x$

Solution for element 4

Coordinates of element center

$x_c = 0.0225; \qquad y_c = 0.0225$

Element dimensions: $a = 0.0075; \qquad b = 0.0075$

Interpolation functions in local element coordinates

$$\boldsymbol{N}^{\mathrm{T}} = \{4444.44\,t\,s - 33.3333\,s - 33.3333\,t + 0.25,\ -4444.44\,t\,s + 33.3333\,s - 33.3333\,t + 0.25,\ 4444.44\,t\,s + 33.3333\,s + 33.3333\,t + 0.25,\ -4444.44\,t\,s - 33.3333\,s + 33.3333\,t + 0.25\}$$

Shift for global coordinates: $s = x - 0.0225; \qquad t = y - 0.0225$

Interpolation functions in global coordinates

$$\boldsymbol{N}^{\mathrm{T}} = \{4444.44\,y\,x - 133.333\,x - 133.333\,y + 4.,\ -4444.44\,y\,x + 133.333\,x + 66.6667\,y - 2.,\ 4444.44\,y\,x - 66.6667\,x - 66.6667\,y + 1.,\ -4444.44\,y\,x + 66.6667\,x + 133.333\,y - 2.\}$$

Nodal values, $\boldsymbol{d}^{\mathrm{T}} = \{142.521,\ 134.871,\ 148.673,\ 151.228\}$

$T(x, y) = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{d} = 22645.1\,y\,x - 849.695\,x + 240.82\,y + 146.559$

$\partial T/\partial x = 22645.1\,y - 849.695; \qquad \partial T/\partial y = 22645.1\,x + 240.82$

Solution for element 5

Coordinates of element center

$x_c = 0.0375; \qquad y_c = 0.0075$

Element dimensions: $a = 0.0075; \qquad b = 0.0075$

Interpolation functions in local element coordinates

$$\boldsymbol{N}^{\mathrm{T}} = \{4444.44\,t\,s - 33.3333\,s - 33.3333\,t + 0.25,\ -4444.44\,t\,s + 33.3333\,s - 33.3333\,t + 0.25,\ 4444.44\,t\,s + 33.3333\,s + 33.3333\,t + 0.25,\ -4444.44\,t\,s - 33.3333\,s + 33.3333\,t + 0.25\}$$

Shift for global coordinates: $s = x - 0.0375; \qquad t = y - 0.0075$

Interpolation functions in global coordinates

$$\boldsymbol{N}^{\mathrm{T}} = \{4444.44\,y\,x - 66.6667\,x - 200.\,y + 3.,\ -4444.44\,y\,x + 66.6667\,x + 133.333\,y - 2.,\ 4444.44\,x\,y - 133.333\,y,\ 200.\,y - 4444.44\,x\,y\}$$

Nodal values, $\boldsymbol{d}^{\mathrm{T}} = \{110,\ 110,\ 122.436,\ 134.871\}$

$T(x, y) = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{d} = -55265.\,x\,y + 3315.99\,y + 110.$

$\partial T/\partial x = -55265.\,y; \qquad \partial T/\partial y = 3315.99 - 55265.\,x$

Solution for element 6

Coordinates of element center

$x_c = 0.0525;\qquad y_c = 0.0075$

Element dimensions: $a = 0.0075;\qquad b = 0.0075$

Interpolation functions in local element coordinates

$$\boldsymbol{N}^T = \{4444.44\,t\,s - 33.3333\,s - 33.3333\,t + 0.25,\ -4444.44\,t\,s + 33.3333\,s - 33.3333\,t + 0.25,\ 4444.44\,t\,s + 33.3333\,s + 33.3333\,t + 0.25,\ -4444.44\,t\,s - 33.3333\,s + 33.3333\,t + 0.25\}$$

Shift for global coordinates: $s = x - 0.0525;\qquad t = y - 0.0075$

Interpolation functions in global coordinates

$$\boldsymbol{N}^T = \{4444.44\,y\,x - 66.6667\,x - 266.667\,y + 4.,\ -4444.44\,y\,x + 66.6667\,x + 200.\,y - 3.,\ 4444.44\,x\,y - 200.\,y,\ 266.667\,y - 4444.44\,x\,y\}$$

Nodal values, $\boldsymbol{d}^T = \{110, 110, 121.088, 122.436\}$

$T(x, y) = \boldsymbol{N}^T\boldsymbol{d} = -5991.37\,x\,y + 1098.67\,y + 110.$

$\partial T/\partial x = -5991.37\,y;\qquad \partial T/\partial y = 1098.67 - 5991.37\,x$

Solution summary

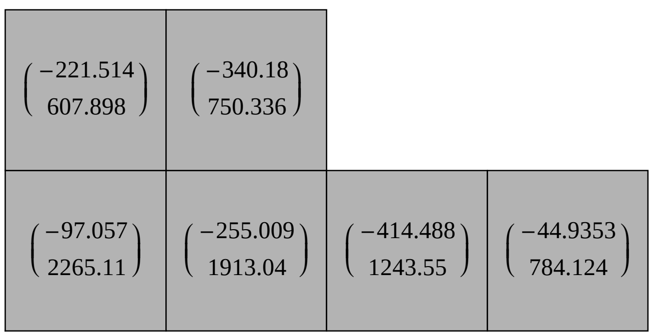
Nodal solution

	x−coord	y−coord	T
1	0	0.03	154.962
2	0.015	0.03	151.228
3	0.03	0.03	148.673
4	0	0.015	145.433
5	0.015	0.015	142.521
6	0.03	0.015	134.871
7	0.045	0.015	122.436
8	0.06	0.015	121.088
9	0	0	110
10	0.015	0	110
11	0.03	0	110
12	0.045	0	110
13	0.06	0	110

Solution at element centroids

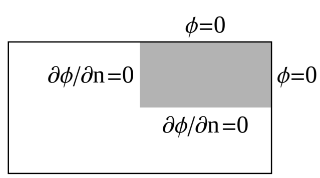
	x−coord	y−coord	T	∂T/∂x	∂T/∂y
1	0.0075	0.0075	126.988	−97.057	2265.11
2	0.0075	0.0225	148.536	−221.514	607.898
3	0.0225	0.0075	124.348	−255.009	1913.04
4	0.0225	0.0225	144.323	−340.18	750.336
5	0.0375	0.0075	119.327	−414.488	1243.55
6	0.0525	0.0075	115.881	−44.9353	784.124

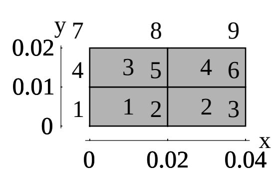
Solution derivatives



Example 5.4: Torsion of a rectangular shaft (p. 342)

Find stresses developed in a 4 cm × 8 cm rectangular shaft when it is subjected to a torque of 500 N-m. The shaft is 1 m long and G = 76.9 GPa. A quarter of the domain needs to be modeled because of symmetry.





The governing differential equation for the problem is

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + 2\,G\,\theta = 0$$

where G is shear modulus and θ is angle of twist per unit length. The boundary condition is $\phi = 0$ on the boundary. As a result of essential boundary condition $\phi = 0$ at nodes {3, 6, 9, 8, 7}. There are no nonzero natural boundary conditions.

Since θ is unknown, we start by arbitrarily assuming Gθ =1. After performing the analysis, we compute the total torque T_a. This torque corresponds to the assumed value of Gθ. Since the relationship between the torque and the angle of twist is linear, the actual value of θ can then be computed using the given value of torque T as follows.

$$\theta = T/(G\,T_a)$$

The actual ϕ values are obtained by multiplying the computed values by the actual Gθ value. The complete finite element solution, using N and m units, is as follows.

Global equations at start of the element assembly process

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\\ \phi_5\\ \phi_6\\ \phi_7\\ \phi_8\\ \phi_9\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}$$

Equations for element 1

Element dimensions: a = 0.01; b = 0.005

$k_x = 1$; $k_y = 1$; p = 0; q = 2

$$\boldsymbol{k}_\mathrm{k} = \begin{pmatrix} 0.833333 & 0.166667 & -0.416667 & -0.583333\\ 0.166667 & 0.833333 & -0.583333 & -0.416667\\ -0.416667 & -0.583333 & 0.833333 & 0.166667\\ -0.583333 & -0.416667 & 0.166667 & 0.833333 \end{pmatrix}; \quad \boldsymbol{k}_\mathrm{p} = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}; \quad \boldsymbol{r}_\mathrm{q} = \begin{pmatrix}0.0001\\0.0001\\0.0001\\0.0001\end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 0.833333 & 0.166667 & -0.416667 & -0.583333\\ 0.166667 & 0.833333 & -0.583333 & -0.416667\\ -0.416667 & -0.583333 & 0.833333 & 0.166667\\ -0.583333 & -0.416667 & 0.166667 & 0.833333 \end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_5\\ \phi_4\end{pmatrix} = \begin{pmatrix}0.0001\\0.0001\\0.0001\\0.0001\end{pmatrix}$$

The element contributes to {1, 2, 5, 4} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 0.833333 & 0.166667 & 0 & -0.583333 & -0.416667 & 0 & 0 & 0 & 0 \\ 0.166667 & 0.833333 & 0 & -0.416667 & -0.583333 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.583333 & -0.416667 & 0 & 0.833333 & 0.166667 & 0 & 0 & 0 & 0 \\ -0.416667 & -0.583333 & 0 & 0.166667 & 0.833333 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \\ \phi_9 \end{pmatrix} = \begin{pmatrix} 0.0001 \\ 0.0001 \\ 0 \\ 0.0001 \\ 0.0001 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 2

Element dimensions: $a = 0.01$; $b = 0.005$

$k_x = 1$; $k_y = 1$; $p = 0$; $q = 2$

$$\boldsymbol{k}_k = \begin{pmatrix} 0.833333 & 0.166667 & -0.416667 & -0.583333 \\ 0.166667 & 0.833333 & -0.583333 & -0.416667 \\ -0.416667 & -0.583333 & 0.833333 & 0.166667 \\ -0.583333 & -0.416667 & 0.166667 & 0.833333 \end{pmatrix}; \quad \boldsymbol{k}_p = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \boldsymbol{r}_q = \begin{pmatrix} 0.0001 \\ 0.0001 \\ 0.0001 \\ 0.0001 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 0.833333 & 0.166667 & -0.416667 & -0.583333 \\ 0.166667 & 0.833333 & -0.583333 & -0.416667 \\ -0.416667 & -0.583333 & 0.833333 & 0.166667 \\ -0.583333 & -0.416667 & 0.166667 & 0.833333 \end{pmatrix} \begin{pmatrix} \phi_2 \\ \phi_3 \\ \phi_6 \\ \phi_5 \end{pmatrix} = \begin{pmatrix} 0.0001 \\ 0.0001 \\ 0.0001 \\ 0.0001 \end{pmatrix}$$

The element contributes to {2, 3, 6, 5} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 0.833333 & 0.166667 & 0 & -0.583333 & -0.416667 & 0 & 0 & 0 & 0 \\ 0.166667 & 1.66667 & 0.166667 & -0.416667 & -1.16667 & -0.416667 & 0 & 0 & 0 \\ 0 & 0.166667 & 0.833333 & 0 & -0.416667 & -0.583333 & 0 & 0 & 0 \\ -0.583333 & -0.416667 & 0 & 0.833333 & 0.166667 & 0 & 0 & 0 & 0 \\ -0.416667 & -1.16667 & -0.416667 & 0.166667 & 1.66667 & 0.166667 & 0 & 0 & 0 \\ 0 & -0.416667 & -0.583333 & 0 & 0.166667 & 0.833333 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \\ \phi_9 \end{pmatrix} = \begin{pmatrix} 0.0001 \\ 0.0002 \\ 0.0001 \\ 0.0001 \\ 0.0002 \\ 0.0001 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 3

Element dimensions: $a = 0.01$; $b = 0.005$

$k_x = 1$; $k_y = 1$; $p = 0$; $q = 2$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 0.833333 & 0.166667 & -0.416667 & -0.583333 \\ 0.166667 & 0.833333 & -0.583333 & -0.416667 \\ -0.416667 & -0.583333 & 0.833333 & 0.166667 \\ -0.583333 & -0.416667 & 0.166667 & 0.833333 \end{pmatrix}; \quad \boldsymbol{k}_{\mathrm{p}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 0.0001 \\ 0.0001 \\ 0.0001 \\ 0.0001 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 0.833333 & 0.166667 & -0.416667 & -0.583333 \\ 0.166667 & 0.833333 & -0.583333 & -0.416667 \\ -0.416667 & -0.583333 & 0.833333 & 0.166667 \\ -0.583333 & -0.416667 & 0.166667 & 0.833333 \end{pmatrix} \begin{pmatrix} \phi_4 \\ \phi_5 \\ \phi_8 \\ \phi_7 \end{pmatrix} = \begin{pmatrix} 0.0001 \\ 0.0001 \\ 0.0001 \\ 0.0001 \end{pmatrix}$$

The element contributes to {4, 5, 8, 7} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 0.833333 & 0.166667 & 0 & -0.583333 & -0.416667 & 0 & 0 & 0 & 0 \\ 0.166667 & 1.66667 & 0.166667 & -0.416667 & -1.16667 & -0.416667 & 0 & 0 & 0 \\ 0 & 0.166667 & 0.833333 & 0 & -0.416667 & -0.583333 & 0 & 0 & 0 \\ -0.583333 & -0.416667 & 0 & 1.66667 & 0.333333 & 0 & -0.583333 & -0.416667 & 0 \\ -0.416667 & -1.16667 & -0.416667 & 0.333333 & 2.5 & 0.166667 & -0.416667 & -0.583333 & 0 \\ 0 & -0.416667 & -0.583333 & 0 & 0.166667 & 0.833333 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.583333 & -0.416667 & 0 & 0.833333 & 0.166667 & 0 \\ 0 & 0 & 0 & -0.416667 & -0.583333 & 0 & 0.166667 & 0.833333 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \\ \phi_9 \end{pmatrix} = \begin{pmatrix} 0.0001 \\ 0.0002 \\ 0.0001 \\ 0.0002 \\ 0.0003 \\ 0.0001 \\ 0.0001 \\ 0.0001 \\ 0 \end{pmatrix}$$

Equations for element 4

Element dimensions: $a = 0.01$; $b = 0.005$

$k_x = 1$; $k_y = 1$; $p = 0$; $q = 2$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 0.833333 & 0.166667 & -0.416667 & -0.583333 \\ 0.166667 & 0.833333 & -0.583333 & -0.416667 \\ -0.416667 & -0.583333 & 0.833333 & 0.166667 \\ -0.583333 & -0.416667 & 0.166667 & 0.833333 \end{pmatrix}; \qquad \boldsymbol{k}_{\mathrm{p}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 0.0001 \\ 0.0001 \\ 0.0001 \\ 0.0001 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 0.833333 & 0.166667 & -0.416667 & -0.583333 \\ 0.166667 & 0.833333 & -0.583333 & -0.416667 \\ -0.416667 & -0.583333 & 0.833333 & 0.166667 \\ -0.583333 & -0.416667 & 0.166667 & 0.833333 \end{pmatrix} \begin{pmatrix} \phi_5 \\ \phi_6 \\ \phi_9 \\ \phi_8 \end{pmatrix} = \begin{pmatrix} 0.0001 \\ 0.0001 \\ 0.0001 \\ 0.0001 \end{pmatrix}$$

The element contributes to {5, 6, 9, 8} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\left(\begin{array}{ccccccccc} 0.833333 & 0.166667 & 0 & -0.583333 & -0.416667 & 0 & 0 & 0 & 0 \\ 0.166667 & 1.66667 & 0.166667 & -0.416667 & -1.16667 & -0.416667 & 0 & 0 & 0 \\ 0 & 0.166667 & 0.833333 & 0 & -0.416667 & -0.583333 & 0 & 0 & 0 \\ -0.583333 & -0.416667 & 0 & 1.66667 & 0.333333 & 0 & -0.583333 & -0.416667 & 0 \\ -0.416667 & -1.16667 & -0.416667 & 0.333333 & 3.33333 & 0.333333 & -0.416667 & -1.16667 & -0.416667 \\ 0 & -0.416667 & -0.583333 & 0 & 0.333333 & 1.66667 & 0 & -0.416667 & -0.583333 \\ 0 & 0 & 0 & -0.583333 & -0.416667 & 0 & 0.833333 & 0.166667 & 0 \\ 0 & 0 & 0 & -0.416667 & -1.16667 & -0.416667 & 0.166667 & 1.66667 & 0.166667 \\ 0 & 0 & 0 & 0 & -0.416667 & -0.583333 & 0 & 0.166667 & 0.833333 \end{array} \right.$$

Essential boundary conditions

Node	dof	Value
3	ϕ_3	0
6	ϕ_6	0
7	ϕ_7	0
8	ϕ_8	0
9	ϕ_9	0

Remove {3, 6, 7, 8, 9} rows and columns.

After adjusting for essential boundary conditions we have

$$\begin{pmatrix} 0.833333 & 0.166667 & -0.583333 & -0.416667 \\ 0.166667 & 1.66667 & -0.416667 & -1.16667 \\ -0.583333 & -0.416667 & 1.66667 & 0.333333 \\ -0.416667 & -1.16667 & 0.333333 & 3.33333 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_4 \\ \phi_5 \end{pmatrix} = \begin{pmatrix} 0.0001 \\ 0.0002 \\ 0.0002 \\ 0.0004 \end{pmatrix}$$

Solving the final system of global equations we get

$$\{\phi_1 = 0.000380919,\ \phi_2 = 0.000331898,\ \phi_4 = 0.000285245,\ \phi_5 = 0.000255255\}$$

Complete table of nodal values

	ϕ
1	0.000380919
2	0.000331898
3	0
4	0.000285245
5	0.000255255
6	0
7	0
8	0
9	0

Solution for element 1

Coordinates of element center

$x_c = 0.01;\qquad y_c = 0.005$

Element dimensions: $a = 0.01;\qquad b = 0.005$

Interpolation functions in local element coordinates

$$\boldsymbol{N}^T = \{5000.\,t\,s - 25.\,s - 50.\,t + 0.25,\ -5000.\,t\,s + 25.\,s - 50.\,t + 0.25,\ 5000.\,t\,s + 25.\,s + 50.\,t + 0.25,\ -5000.\,t\,s - 25.\,s + 50.\,t + 0.25\}$$

Shift for global coordinates: $s = x - 0.01;\qquad t = y - 0.005$

Interpolation functions in global coordinates

$$\boldsymbol{N}^T = \{5000.\,y\,x - 50.\,x - 100.\,y + 1.,\ 50.\,x - 5000.\,x\,y,\ 5000.\,x\,y,\ 100.\,y - 5000.\,x\,y\}$$

Nodal values, $\boldsymbol{d}^T = \{0.000380919,\ 0.000331898,\ 0.000255255,\ 0.000285245\}$

$\phi(x, y) = \boldsymbol{N}^T\boldsymbol{d} = 0.0951558\,y\,x - 0.0024511\,x - 0.00956742\,y + 0.000380919$

$\partial\phi/\partial x = 0.0951558\,y - 0.0024511;\qquad \partial\phi/\partial y = 0.0951558\,x - 0.00956742$

Solution for element 2

Coordinates of element center

$x_c = 0.03;\qquad y_c = 0.005$

Element dimensions: $a = 0.01;\qquad b = 0.005$

Interpolation functions in local element coordinates

$$\boldsymbol{N}^T = \{5000.\, t\, s - 25.\, s - 50.\, t + 0.25,\ -5000.\, t\, s + 25.\, s - 50.\, t + 0.25,\ 5000.\, t\, s + 25.\, s + 50.\, t + 0.25,\ -5000.\, t\, s - 25.\, s + 50.\, t + 0.25\}$$

Shift for global coordinates: $s = x - 0.03;\qquad t = y - 0.005$

Interpolation functions in global coordinates

$$\boldsymbol{N}^T = \{5000.\, y\, x - 50.\, x - 200.\, y + 2.,\ -5000.\, y\, x + 50.\, x + 100.\, y - 1.,\ 5000.\, x\, y - 100.\, y,\ 200.\, y - 5000.\, x\, y\}$$

Nodal values, $\boldsymbol{d}^T = \{0.000331898,\ 0,\ 0,\ 0.000255255\}$

$\phi(x, y) = \boldsymbol{N}^T\boldsymbol{d} = 0.383215\, y\, x - 0.0165949\, x - 0.0153286\, y + 0.000663795$

$\partial\phi/\partial x = 0.383215\, y - 0.0165949;\qquad \partial\phi/\partial y = 0.383215\, x - 0.0153286$

Solution for element 3

Coordinates of element center

$x_c = 0.01;\qquad y_c = 0.015$

Element dimensions: $a = 0.01;\qquad b = 0.005$

Interpolation functions in local element coordinates

$$\boldsymbol{N}^T = \{5000.\, t\, s - 25.\, s - 50.\, t + 0.25,\ -5000.\, t\, s + 25.\, s - 50.\, t + 0.25,\ 5000.\, t\, s + 25.\, s + 50.\, t + 0.25,\ -5000.\, t\, s - 25.\, s + 50.\, t + 0.25\}$$

Shift for global coordinates: $s = x - 0.01;\qquad t = y - 0.015$

Interpolation functions in global coordinates

$$\boldsymbol{N}^T = \{5000.\, y\, x - 100.\, x - 100.\, y + 2.,\ 100.\, x - 5000.\, x\, y,\ 5000.\, x\, y - 50.\, x,\ -5000.\, y\, x + 50.\, x + 100.\, y - 1.\}$$

Nodal values, $\boldsymbol{d}^T = \{0.000285245,\ 0.000255255,\ 0,\ 0\}$

$\phi(x, y) = \boldsymbol{N}^T\boldsymbol{d} = 0.149954\, y\, x - 0.00299907\, x - 0.0285245\, y + 0.000570491$

$\partial\phi/\partial x = 0.149954\, y - 0.00299907;\qquad \partial\phi/\partial y = 0.149954\, x - 0.0285245$

Solution for element 4

Coordinates of element center

$x_c = 0.03; \qquad y_c = 0.015$

Element dimensions: $a = 0.01; \qquad b = 0.005$

Interpolation functions in local element coordinates

$$\boldsymbol{N}^T = \{5000.\, t\, s - 25.\, s - 50.\, t + 0.25, \\ -5000.\, t\, s + 25.\, s - 50.\, t + 0.25,\ 5000.\, t\, s + 25.\, s + 50.\, t + 0.25,\ -5000.\, t\, s - 25.\, s + 50.\, t + 0.25\}$$

Shift for global coordinates: $s = x - 0.03; \qquad t = y - 0.015$

Interpolation functions in global coordinates

$$\boldsymbol{N}^T = \{5000.\, y\, x - 100.\, x - 200.\, y + 4., \\ -5000.\, y\, x + 100.\, x + 100.\, y - 2.,\ 5000.\, y\, x - 50.\, x - 100.\, y + 1.,\ -5000.\, y\, x + 50.\, x + 200.\, y - 2.\}$$

Nodal values, $\boldsymbol{d}^T = \{0.000255255,\ 0,\ 0,\ 0\}$

$\phi(x, y) = \boldsymbol{N}^T\boldsymbol{d} = 1.27627\, y\, x - 0.0255255\, x - 0.0510509\, y + 0.00102102$

$\partial\phi/\partial x = 1.27627\, y - 0.0255255; \qquad \partial\phi/\partial y = 1.27627\, x - 0.0510509$

Solution summary

Nodal solution

	x−coord	y−coord	ϕ
1	0.	0.	0.000380919
2	0.02	0.	0.000331898
3	0.04	0.	0
4	0.	0.01	0.000285245
5	0.02	0.01	0.000255255
6	0.04	0.01	0
7	0.	0.02	0
8	0.02	0.02	0
9	0.04	0.02	0

Solution at element centroids

	x−coord	y−coord	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	0.01	0.005	0.000313329	−0.00197532	−0.00861586
2	0.03	0.005	0.000146788	−0.0146788	−0.00383215
3	0.01	0.015	0.000135125	−0.000749769	−0.027025
4	0.03	0.015	0.0000638136	−0.00638136	−0.0127627

	ϕ_a	$\int\int\phi_a\,\mathrm{dA}$
1	$0.0951558\,y\,x - 0.0024511\,x - 0.00956742\,y + 0.000380919$	6.26658×10^{-8}
2	$0.383215\,y\,x - 0.0165949\,x - 0.0153286\,y + 0.000663795$	2.93576×10^{-8}
3	$0.149954\,y\,x - 0.00299907\,x - 0.0285245\,y + 0.000570491$	2.7025×10^{-8}
4	$1.27627\,y\,x - 0.0255255\,x - 0.0510509\,y + 0.00102102$	1.27627×10^{-8}

The total torque is given by

$$T = 2\int\int_A \phi\,\mathrm{dA}$$

Summing $\int\int\phi\,\mathrm{dA}$ contributions from all elements and multiplying by 2 gives the total torque. Since we are modeling a $1/4^{\text{th}}$ of the shape, the torque for the entire section is

$$T_a = 4\times 2\times\sum(\int\int\phi_a\,\mathrm{dA}) = 1.05449\times 10^{-6}\ \mathrm{N\cdot m}$$

Since the actual torque is 500 N-m, the actual value of the angle of twist is

$$\mathrm{G}\theta = 500/T_a = 4.74163\times 10^{8}; \qquad \theta = 0.00616597\ \mathrm{rad/m}$$

The ϕ values are simply scaled by this value of Gθ and thus the solution corresponding to a torque of 500 N-m is as follows.

$$\phi = (500/T_a)\phi_a = 4.74163\times 10^{8}\,\phi_a$$

	ϕ $(\times 10^6)$	$\tau_{yz}=-\partial\phi/\partial x$ (MPa)	$\tau_{xz}=\partial\phi/\partial y$ (MPa)
1	$45.1194\,y\,x - 1.16222\,x - 4.53652\,y + 0.180618$	$1.16222 - 45.1194\,y$	$45.1194\,x - 4.53652$
2	$181.706\,y\,x - 7.86868\,x - 7.26826\,y + 0.314747$	$7.86868 - 181.706\,y$	$181.706\,x - 7.26826$
3	$71.1025\,y\,x - 1.42205\,x - 13.5253\,y + 0.270506$	$1.42205 - 71.1025\,y$	$71.1025\,x - 13.5253$
4	$605.162\,y\,x - 12.1032\,x - 24.2065\,y + 0.484129$	$12.1032 - 605.162\,y$	$605.162\,x - 24.2065$

Stresses at element centroids

	τ_{yz} (MPa)	τ_{xz} (MPa)
1	0.936622	−4.08532
2	6.96015	−1.81706
3	0.355513	−12.8143
4	3.02581	−6.05162

The maximum shear stress occurs at midpoint of the long side (node 7) which from element 3 is

Stresses at node 7: $\tau_{yz} = 2.22045 \times 10^{-16}$ MPa; $\tau_{xz} = -13.5253$ MPa;
$\tau_{max} = 13.5253$ MPa

An exact solution for the problem is available as follows (Roark's Formulas for Stress and Strain, Seventh Edition, p. 401, McGraw-Hill 2002).

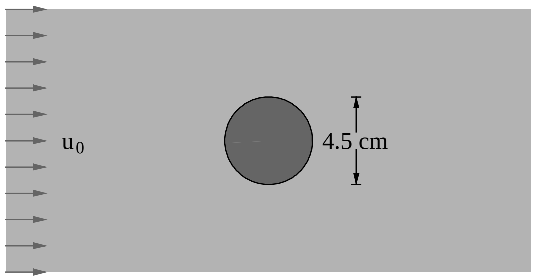
$$\tau_{max} = \frac{3T}{8ab^2}\left(1 + 0.6095\,b/a + 0.8865\,(b/a)^2 - 1.8023\,(b/a)^3 + 0.91\,(b/a)^4\right)$$

where $2a$ is the longer dimension of the section and $2b$ is the shorter dimension.

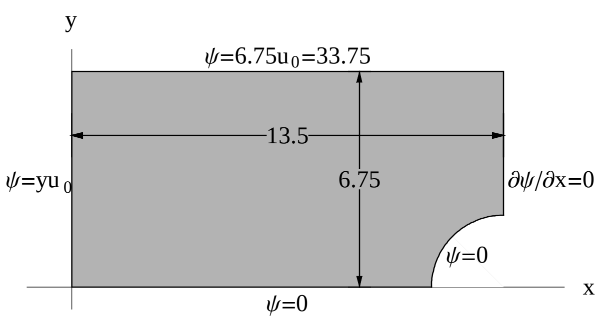
Exact solution: $\tau_{max} = 15.9136$ MPa

Example 5.6: Stream function formulation for fluid flow around a cylinder (p. 363)

Consider fluid flow in the direction perpendicular to a long cylinder as shown in Figure. The cylinder diameter is 4.5 cm. At a distance of about 3 times the diameter of the cylinder, both the upstream and the downstream, the flow can be considered uniform with a velocity of $u_0 = 5\,\mathrm{cm}/s$ in the x direction. Determine the flow velocity near the cylinder.



We choose a computational domain that extends 3 times the cylinder diameter upstream and downstream and 1.5 times diameter above and below the cylinder. Taking advantage of symmetry we need to model only a quarter of the solution domain.

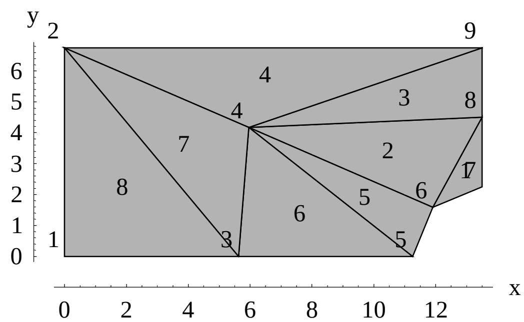


The governing differential equation in terms of stream function $\psi(x, y)$ is as follows.

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$$

Compared to the general form $k_x = k_y = 1$ and $p = q = 0$. The fluid velocity is related to stream function as follows.

$$u = \frac{\partial\psi}{\partial y}; \quad v = -\frac{\partial\psi}{\partial x}$$



The complete finite element solution is as follows.

Global equations at start of the element assembly process

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ \psi_7 \\ \psi_8 \\ \psi_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 1

$k_x = 1;$ $k_y = 1;$ $p = 0;$ $q = 0$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	6	11.909	1.59099
2	7	13.5	2.25
3	8	13.5	4.5

$x_1 = 11.909$ $x_2 = 13.5$ $x_3 = 13.5$
$y_1 = 1.59099$ $y_2 = 2.25$ $y_3 = 4.5$

Using these values we get

$b_1 = -2.25$ $b_2 = 2.90901$ $b_3 = -0.65901$

$c_1 = 0.$ $c_2 = -1.59099$ $c_3 = 1.59099$

$f_1 = 30.375$ $f_2 = -32.1122$ $f_3 = 5.3169$

Element area, $A = 1.78986$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -2.25 & 2.90901 & -0.65901 \\ 0. & -1.59099 & 1.59099 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 0.707107 & -0.914214 & 0.207107 \\ -0.914214 & 1.53553 & -0.62132 \\ 0.207107 & -0.62132 & 0.414214 \end{pmatrix}; \qquad \boldsymbol{k}_{\mathrm{p}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 0.707107 & -0.914214 & 0.207107 \\ -0.914214 & 1.53553 & -0.62132 \\ 0.207107 & -0.62132 & 0.414214 \end{pmatrix} \begin{pmatrix} \psi_6 \\ \psi_7 \\ \psi_8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to {6, 7, 8} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.707107 & -0.914214 & 0.207107 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.914214 & 1.53553 & -0.62132 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.207107 & -0.62132 & 0.414214 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ \psi_7 \\ \psi_8 \\ \psi_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 2

$k_x = 1;$ $k_y = 1;$ $p = 0;$ $q = 0$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	8	13.5	4.5
2	4	5.9545	4.1705
3	6	11.909	1.59099

$x_1 = 13.5 \quad x_2 = 5.9545 \quad x_3 = 11.909$

$y_1 = 4.5 \quad y_2 = 4.1705 \quad y_3 = 1.59099$

Using these values we get

$b_1 = 2.5795 \quad b_2 = -2.90901 \quad b_3 = 0.329505$

$c_1 = 5.9545 \quad c_2 = 1.59099 \quad c_3 = -7.5455$

$f_1 = -40.1929 \quad f_2 = 32.1122 \quad f_3 = 29.5064$

Element area, A = 10.7128

$$\boldsymbol{B}^T = \begin{pmatrix} 2.5795 & -2.90901 & 0.329505 \\ 5.9545 & 1.59099 & -7.5455 \end{pmatrix}$$

$$\boldsymbol{k}_k = \begin{pmatrix} 0.982699 & 0.0459671 & -1.02867 \\ 0.0459671 & 0.256552 & -0.302519 \\ -1.02867 & -0.302519 & 1.33118 \end{pmatrix}; \qquad \boldsymbol{k}_p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_q = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 0.982699 & 0.0459671 & -1.02867 \\ 0.0459671 & 0.256552 & -0.302519 \\ -1.02867 & -0.302519 & 1.33118 \end{pmatrix} \begin{pmatrix} \psi_8 \\ \psi_4 \\ \psi_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to {8, 4, 6} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.256552 & 0 & -0.302519 & 0 & 0.0459671 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.302519 & 0 & 2.03829 & -0.914214 & -0.821559 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.914214 & 1.53553 & -0.62132 & 0 \\ 0 & 0 & 0 & 0.0459671 & 0 & -0.821559 & -0.62132 & 1.39691 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ \psi_7 \\ \psi_8 \\ \psi_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 3

$k_x = 1;$ $k_y = 1;$ $p = 0;$ $q = 0$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	4	5.9545	4.1705
2	8	13.5	4.5
3	9	13.5	6.75

$x_1 = 5.9545$ $x_2 = 13.5$ $x_3 = 13.5$
$y_1 = 4.1705$ $y_2 = 4.5$ $y_3 = 6.75$

Using these values we get

$b_1 = -2.25$ $b_2 = 2.5795$ $b_3 = -0.329505$

$c_1 = 0.$ $c_2 = -7.5455$ $c_3 = 7.5455$

$f_1 = 30.375$ $f_2 = 16.1088$ $f_3 = -29.5064$

Element area, $A = 8.48868$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -2.25 & 2.5795 & -0.329505 \\ 0. & -7.5455 & 7.5455 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 0.149096 & -0.17093 & 0.0218345 \\ -0.17093 & 1.87274 & -1.70181 \\ 0.0218345 & -1.70181 & 1.67997 \end{pmatrix}; \qquad \boldsymbol{k}_{\mathrm{p}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 0.149096 & -0.17093 & 0.0218345 \\ -0.17093 & 1.87274 & -1.70181 \\ 0.0218345 & -1.70181 & 1.67997 \end{pmatrix} \begin{pmatrix} \psi_4 \\ \psi_8 \\ \psi_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to {4, 8, 9} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.405647 & 0 & -0.302519 & 0 & -0.124963 & 0.0218345 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.302519 & 0 & 2.03829 & -0.914214 & -0.821559 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.914214 & 1.53553 & -0.62132 & 0 \\ 0 & 0 & 0 & -0.124963 & 0 & -0.821559 & -0.62132 & 3.26965 & -1.70181 \\ 0 & 0 & 0 & 0.0218345 & 0 & 0 & 0 & -1.70181 & 1.67997 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ \psi_7 \\ \psi_8 \\ \psi_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 4

$k_x = 1$; $k_y = 1$; $p = 0$; $q = 0$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	9	13.5	6.75
2	2	0	6.75
3	4	5.9545	4.1705

$x_1 = 13.5$ $x_2 = 0$ $x_3 = 5.9545$

$y_1 = 6.75$ $y_2 = 6.75$ $y_3 = 4.1705$

Using these values we get

$b_1 = 2.5795$ $b_2 = -2.5795$ $b_3 = 0.$

$c_1 = 5.9545$ $c_2 = 7.5455$ $c_3 = -13.5$

$f_1 = -40.1929$ $f_2 = -16.1088$ $f_3 = 91.125$

Element area, $A = 17.4117$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 2.5795 & -2.5795 & 0. \\ 5.9545 & 7.5455 & -13.5 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 0.604623 & 0.549572 & -1.1542 \\ 0.549572 & 0.913014 & -1.46259 \\ -1.1542 & -1.46259 & 2.61678 \end{pmatrix}; \qquad \boldsymbol{k}_{\mathrm{p}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 0.604623 & 0.549572 & -1.1542 \\ 0.549572 & 0.913014 & -1.46259 \\ -1.1542 & -1.46259 & 2.61678 \end{pmatrix} \begin{pmatrix} \psi_9 \\ \psi_2 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to {9, 2, 4} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.913014 & 0 & -1.46259 & 0 & 0 & 0 & 0 & 0.549572 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.46259 & 0 & 3.02243 & 0 & -0.302519 & 0 & -0.124963 & -1.13236 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.302519 & 0 & 2.03829 & -0.914214 & -0.821559 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.914214 & 1.53553 & -0.62132 & 0 \\ 0 & 0 & 0 & -0.124963 & 0 & -0.821559 & -0.62132 & 3.26965 & -1.70181 \\ 0 & 0.549572 & 0 & -1.13236 & 0 & 0 & 0 & -1.70181 & 2.2846 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ \psi_7 \\ \psi_8 \\ \psi_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 5

$k_x = 1$; $k_y = 1$; $p = 0$; $q = 0$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	5	11.25	0
2	6	11.909	1.59099
3	4	5.9545	4.1705

$x_1 = 11.25$ $x_2 = 11.909$ $x_3 = 5.9545$

$y_1 = 0$ $y_2 = 1.59099$ $y_3 = 4.1705$

Using these values we get

$b_1 = -2.5795$ $b_2 = 4.1705$ $b_3 = -1.59099$

$c_1 = -5.9545$ $c_2 = 5.2955$ $c_3 = 0.65901$

$f_1 = 40.1929$ $f_2 = -46.9181$ $f_3 = 17.8986$

Element area, A = 5.58674

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -2.5795 & 4.1705 & -1.59099 \\ -5.9545 & 5.2955 & 0.65901 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 1.88437 & -1.89242 & 0.00804988 \\ -1.89242 & 2.03318 & -0.140754 \\ 0.00804988 & -0.140754 & 0.132705 \end{pmatrix}; \qquad \boldsymbol{k}_{\mathrm{p}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 1.88437 & -1.89242 & 0.00804988 \\ -1.89242 & 2.03318 & -0.140754 \\ 0.00804988 & -0.140754 & 0.132705 \end{pmatrix} \begin{pmatrix} \psi_5 \\ \psi_6 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to {5, 6, 4} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.913014 & 0 & -1.46259 & 0 & 0 & 0 & 0 & 0.549572 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.46259 & 0 & 3.15513 & 0.00804988 & -0.443273 & 0 & -0.124963 & -1.13236 \\ 0 & 0 & 0 & 0.00804988 & 1.88437 & -1.89242 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.443273 & -1.89242 & 4.07147 & -0.914214 & -0.821559 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.914214 & 1.53553 & -0.62132 & 0 \\ 0 & 0 & 0 & -0.124963 & 0 & -0.821559 & -0.62132 & 3.26965 & -1.70181 \\ 0 & 0.549572 & 0 & -1.13236 & 0 & 0 & 0 & -1.70181 & 2.2846 \end{pmatrix}$$

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ \psi_7 \\ \psi_8 \\ \psi_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 6

$k_x = 1; \qquad k_y = 1; \qquad p = 0; \qquad q = 0$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	4	5.9545	4.1705
2	3	5.625	0
3	5	11.25	0

$x_1 = 5.9545 \qquad x_2 = 5.625 \qquad x_3 = 11.25$

$y_1 = 4.1705 \qquad y_2 = 0 \qquad y_3 = 0$

Using these values we get

$b_1 = 0 \qquad b_2 = -4.1705 \qquad b_3 = 4.1705$

$c_1 = 5.625 \qquad c_2 = -5.2955 \qquad c_3 = -0.329505$

$f_1 = 0 \qquad f_2 = 46.9181 \qquad f_3 = -23.459$

Element area, A = 11.7295

$$\boldsymbol{B}^T = \begin{pmatrix} 0 & -4.1705 & 4.1705 \\ 5.625 & -5.2955 & -0.329505 \end{pmatrix}$$

$$\boldsymbol{k}_k = \begin{pmatrix} 0.67438 & -0.634876 & -0.0395043 \\ -0.634876 & 0.968397 & -0.33352 \\ -0.0395043 & -0.33352 & 0.373025 \end{pmatrix}; \qquad \boldsymbol{k}_p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_q = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 0.67438 & -0.634876 & -0.0395043 \\ -0.634876 & 0.968397 & -0.33352 \\ -0.0395043 & -0.33352 & 0.373025 \end{pmatrix} \begin{pmatrix} \psi_4 \\ \psi_3 \\ \psi_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to {4, 3, 5} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.913014 & 0 & -1.46259 & 0 & 0 & 0 & 0 & 0.549572 \\ 0 & 0 & 0.968397 & -0.634876 & -0.33352 & 0 & 0 & 0 & 0 \\ 0 & -1.46259 & -0.634876 & 3.82951 & -0.0314544 & -0.443273 & 0 & -0.124963 & -1.13236 \\ 0 & 0 & -0.33352 & -0.0314544 & 2.2574 & -1.89242 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.443273 & -1.89242 & 4.07147 & -0.914214 & -0.821559 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.914214 & 1.53553 & -0.62132 & 0 \\ 0 & 0 & 0 & -0.124963 & 0 & -0.821559 & -0.62132 & 3.26965 & -1.70181 \\ 0 & 0.549572 & 0 & -1.13236 & 0 & 0 & 0 & -1.70181 & 2.2846 \end{pmatrix}$$

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ \psi_7 \\ \psi_8 \\ \psi_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 7

$k_x = 1;$ $k_y = 1;$ $p = 0;$ $q = 0$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	3	5.625	0
2	4	5.9545	4.1705
3	2	0	6.75

$x_1 = 5.625$ $x_2 = 5.9545$ $x_3 = 0$

$y_1 = 0$ $y_2 = 4.1705$ $y_3 = 6.75$

Using these values we get

$b_1 = -2.5795$ $b_2 = 6.75$ $b_3 = -4.1705$

$c_1 = -5.9545$ $c_2 = 5.625$ $c_3 = 0.329505$

$f_1 = 40.1929 \qquad f_2 = -37.9688 \qquad f_3 = 23.459$

Element area, A = 12.8416

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -2.5795 & 6.75 & -4.1705 \\ -5.9545 & 5.625 & 0.329505 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 0.819796 & -0.991032 & 0.171236 \\ -0.991032 & 1.50299 & -0.511957 \\ 0.171236 & -0.511957 & 0.340721 \end{pmatrix}; \qquad \boldsymbol{k}_{\mathrm{p}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 0.819796 & -0.991032 & 0.171236 \\ -0.991032 & 1.50299 & -0.511957 \\ 0.171236 & -0.511957 & 0.340721 \end{pmatrix} \begin{pmatrix} \psi_3 \\ \psi_4 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to {3, 4, 2} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.25373 & 0.171236 & -1.97454 & 0 & 0 & 0 & 0 & 0.549572 \\ 0 & 0.171236 & 1.78819 & -1.62591 & -0.33352 & 0 & 0 & 0 & 0 \\ 0 & -1.97454 & -1.62591 & 5.3325 & -0.0314544 & -0.443273 & 0 & -0.124963 & -1.13236 \\ 0 & 0 & -0.33352 & -0.0314544 & 2.2574 & -1.89242 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.443273 & -1.89242 & 4.07147 & -0.914214 & -0.821559 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.914214 & 1.53553 & -0.62132 & 0 \\ 0 & 0 & 0 & -0.124963 & 0 & -0.821559 & -0.62132 & 3.26965 & -1.70181 \\ 0 & 0.549572 & 0 & -1.13236 & 0 & 0 & 0 & -1.70181 & 2.2846 \end{pmatrix}$$

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ \psi_7 \\ \psi_8 \\ \psi_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 8

$k_x = 1$; $k_y = 1$; $p = 0$; $q = 0$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	2	0	6.75
2	1	0	0
3	3	5.625	0

$x_1 = 0$ $x_2 = 0$ $x_3 = 5.625$
$y_1 = 6.75$ $y_2 = 0$ $y_3 = 0$

Using these values we get

$b_1 = 0$ $b_2 = -6.75$ $b_3 = 6.75$

$c_1 = 5.625$ $c_2 = -5.625$ $c_3 = 0$

$f_1 = 0$ $f_2 = 37.9688$ $f_3 = 0$

Element area, $A = 18.9844$

$$\boldsymbol{B}^T = \begin{pmatrix} 0 & -6.75 & 6.75 \\ 5.625 & -5.625 & 0 \end{pmatrix}$$

$$\boldsymbol{k}_k = \begin{pmatrix} 0.416667 & -0.416667 & 0 \\ -0.416667 & 1.01667 & -0.6 \\ 0 & -0.6 & 0.6 \end{pmatrix}; \qquad \boldsymbol{k}_p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_q = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 0.416667 & -0.416667 & 0 \\ -0.416667 & 1.01667 & -0.6 \\ 0 & -0.6 & 0.6 \end{pmatrix} \begin{pmatrix} \psi_2 \\ \psi_1 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to {2, 1, 3} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 1.01667 & -0.416667 & -0.6 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.416667 & 1.6704 & 0.171236 & -1.97454 & 0 & 0 & 0 & 0 & 0.5495 \\ -0.6 & 0.171236 & 2.38819 & -1.62591 & -0.33352 & 0 & 0 & 0 & 0 \\ 0 & -1.97454 & -1.62591 & 5.3325 & -0.0314544 & -0.443273 & 0 & -0.124963 & -1.1323 \\ 0 & 0 & -0.33352 & -0.0314544 & 2.2574 & -1.89242 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.443273 & -1.89242 & 4.07147 & -0.914214 & -0.821559 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.914214 & 1.53553 & -0.62132 & 0 \\ 0 & 0 & 0 & -0.124963 & 0 & -0.821559 & -0.62132 & 3.26965 & -1.7018 \\ 0 & 0.549572 & 0 & -1.13236 & 0 & 0 & 0 & -1.70181 & 2.2846 \end{pmatrix}$$

Essential boundary conditions

Node	dof	Value
1	ψ_1	0
2	ψ_2	33.75
3	ψ_3	0
5	ψ_5	0
6	ψ_6	0
7	ψ_7	0
9	ψ_9	33.75

Delete equations {1, 2, 3, 5, 6, 7, 9}.

$$\begin{pmatrix} 0 & -1.97454 & -1.62591 & 5.3325 & -0.0314544 & -0.443273 & 0 & -0.124963 & -1.13236 \\ 0 & 0 & 0 & -0.124963 & 0 & -0.821559 & -0.62132 & 3.26965 & -1.70181 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 33.75 \\ 0 \\ \psi_4 \\ 0 \\ 0 \\ 0 \\ \psi_8 \\ 33.75 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Extract columns {1, 2, 3, 5, 6, 7, 9}.

Multiply each column by its respective known value {0, 33.75, 0, 0, 0, 0, 33.75}.

Move all resulting vectors to the rhs.

After adjusting for essential boundary conditions we have

$$\begin{pmatrix} 5.3325 & -0.124963 \\ -0.124963 & 3.26965 \end{pmatrix} \begin{pmatrix} \psi_4 \\ \psi_8 \end{pmatrix} = \begin{pmatrix} 104.858 \\ 57.436 \end{pmatrix}$$

Solving the final system of global equations we get

$$\{\psi_4 = 20.0936,\ \psi_8 = 18.3344\}$$

Complete table of nodal values

	ψ
1	0
2	33.75
3	0
4	20.0936
5	0
6	0
7	0
8	18.3344
9	33.75

Solution for element 1

Nodal coordinates

Element node	Global node number	x	y
1	6	11.909	1.59099
2	7	13.5	2.25
3	8	13.5	4.5

$x_1 = 11.909 \qquad x_2 = 13.5 \qquad x_3 = 13.5$
$y_1 = 1.59099 \qquad y_2 = 2.25 \qquad y_3 = 4.5$

$b_1 = -2.25 \qquad b_2 = 2.90901 \qquad b_3 = -0.65901$

$c_1 = 0. \qquad c_2 = -1.59099 \qquad c_3 = 1.59099$

$f_1 = 30.375 \qquad f_2 = -32.1122 \qquad f_3 = 5.3169$

Element area, A = 1.78986

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^T =$

$$\{8.48528 - 0.628539\,x,\ 0.812634\,x - 0.444444\,y - 8.97056,\ -0.184095\,x + 0.444444\,y + 1.48528\}$$

Nodal values, $\boldsymbol{d}^T = \{0, 0, 18.3344\}$

$\psi(x, y) = \boldsymbol{N}^T\boldsymbol{d} = -3.37526\,x + 8.14861\,y + 27.2317$

$\partial\psi/\partial x = -3.37526;\qquad \partial\psi/\partial y = 8.14861$

Solution for element 2

Nodal coordinates

Element node	Global node number	x	y
1	8	13.5	4.5
2	4	5.9545	4.1705
3	6	11.909	1.59099

$x_1 = 13.5 \qquad x_2 = 5.9545 \qquad x_3 = 11.909$
$y_1 = 4.5 \qquad y_2 = 4.1705 \qquad y_3 = 1.59099$

$b_1 = 2.5795 \qquad b_2 = -2.90901 \qquad b_3 = 0.329505$

$c_1 = 5.9545 \qquad c_2 = 1.59099 \qquad c_3 = -7.5455$

$f_1 = -40.1929 \qquad f_2 = 32.1122 \qquad f_3 = 29.5064$

Element area, $A = 10.7128$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^T = \{0.120393\,x + 0.277914\,y - 1.87592,$
$-0.135772\,x + 0.0742562\,y + 1.49877,\; 0.015379\,x - 0.352171\,y + 1.37715\}$

Nodal values, $\boldsymbol{d}^T = \{18.3344, 20.0936, 0\}$

$\psi(x, y) = \boldsymbol{N}^T\boldsymbol{d} = -0.520816\,x + 6.58746\,y - 4.27818$

$\partial\psi/\partial x = -0.520816;\qquad \partial\psi/\partial y = 6.58746$

Solution for element 3

Nodal coordinates

Element node	Global node number	x	y
1	4	5.9545	4.1705
2	8	13.5	4.5
3	9	13.5	6.75

$x_1 = 5.9545 \qquad x_2 = 13.5 \qquad x_3 = 13.5$
$y_1 = 4.1705 \qquad y_2 = 4.5 \qquad y_3 = 6.75$

$b_1 = -2.25 \qquad b_2 = 2.5795 \qquad b_3 = -0.329505$

$c_1 = 0.$ $\quad c_2 = -7.5455$ $\quad c_3 = 7.5455$

$f_1 = 30.375$ $\quad f_2 = 16.1088$ $\quad f_3 = -29.5064$

Element area, A = 8.48868

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^T =$

$$\{1.78915 - 0.132529\,x,\ 0.151938\,x - 0.444444\,y + 0.948838,\ -0.0194085\,x + 0.444444\,y - 1.73799\}$$

Nodal values, $\boldsymbol{d}^T = \{20.0936, 18.3344, 33.75\}$

$\psi(x, y) = \boldsymbol{N}^T\boldsymbol{d} = -0.532342\,x + 6.85139\,y - 5.31027$

$\partial\psi/\partial x = -0.532342;$ $\quad \partial\psi/\partial y = 6.85139$

Solution for element 4

Nodal coordinates

Element node	Global node number	x	y
1	9	13.5	6.75
2	2	0	6.75
3	4	5.9545	4.1705

$x_1 = 13.5$ $\quad x_2 = 0$ $\quad x_3 = 5.9545$
$y_1 = 6.75$ $\quad y_2 = 6.75$ $\quad y_3 = 4.1705$

$b_1 = 2.5795$ $\quad b_2 = -2.5795$ $\quad b_3 = 0.$

$c_1 = 5.9545$ $\quad c_2 = 7.5455$ $\quad c_3 = -13.5$

$f_1 = -40.1929$ $\quad f_2 = -16.1088$ $\quad f_3 = 91.125$

Element area, A = 17.4117

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^T =$

$$\{0.0740741\,x + 0.170992\,y - 1.1542,\ -0.0740741\,x + 0.216679\,y - 0.462586,\ 2.61678 - 0.387671\,y\}$$

Nodal values, $\boldsymbol{d}^T = \{33.75, 33.75, 20.0936\}$

$\psi(x, y) = \boldsymbol{N}^T\boldsymbol{d} = 5.2942\,y - 1.98584$

$\partial\psi/\partial x = 0;$ $\quad \partial\psi/\partial y = 5.2942$

Solution for element 5

Nodal coordinates

Element node	Global node number	x	y
1	5	11.25	0
2	6	11.909	1.59099
3	4	5.9545	4.1705

$x_1 = 11.25$ $x_2 = 11.909$ $x_3 = 5.9545$
$y_1 = 0$ $y_2 = 1.59099$ $y_3 = 4.1705$

$b_1 = -2.5795$ $b_2 = 4.1705$ $b_3 = -1.59099$

$c_1 = -5.9545$ $c_2 = 5.2955$ $c_3 = 0.65901$

$f_1 = 40.1929$ $f_2 = -46.9181$ $f_3 = 17.8986$

Element area, $A = 5.58674$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^T = \{-0.23086\,x - 0.532914\,y + 3.59717,$
$0.37325\,x + 0.473934\,y - 4.19906,\ -0.14239\,x + 0.0589798\,y + 1.60189\}$

Nodal values, $\boldsymbol{d}^T = \{0,\ 0,\ 20.0936\}$

$\psi(x, y) = \boldsymbol{N}^T\boldsymbol{d} = -2.86112\,x + 1.18512\,y + 32.1876$

$\partial\psi/\partial x = -2.86112;$ $\partial\psi/\partial y = 1.18512$

Solution for element 6

Nodal coordinates

Element node	Global node number	x	y
1	4	5.9545	4.1705
2	3	5.625	0
3	5	11.25	0

$x_1 = 5.9545$ $x_2 = 5.625$ $x_3 = 11.25$
$y_1 = 4.1705$ $y_2 = 0$ $y_3 = 0$

$b_1 = 0$ $b_2 = -4.1705$ $b_3 = 4.1705$

$c_1 = 5.625$ $c_2 = -5.2955$ $c_3 = -0.329505$

$f_1 = 0$ $f_2 = 46.9181$ $f_3 = -23.459$

Element area, $A = 11.7295$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^T = \{0.23978\,y,\ -0.177778\,x - 0.225734\,y + 2.,\ 0.177778\,x - 0.014046\,y - 1.\}$

Nodal values, $\boldsymbol{d}^T = \{20.0936, 0, 0\}$

$\psi(x, y) = \boldsymbol{N}^T\boldsymbol{d} = 4.81803\, y$

$\partial\psi/\partial x = 0;$ $\partial\psi/\partial y = 4.81803$

Solution for element 7

Nodal coordinates

Element node	Global node number	x	y
1	3	5.625	0
2	4	5.9545	4.1705
3	2	0	6.75

$x_1 = 5.625$ $x_2 = 5.9545$ $x_3 = 0$
$y_1 = 0$ $y_2 = 4.1705$ $y_3 = 6.75$

$b_1 = -2.5795$ $b_2 = 6.75$ $b_3 = -4.1705$

$c_1 = -5.9545$ $c_2 = 5.625$ $c_3 = 0.329505$

$f_1 = 40.1929$ $f_2 = -37.9688$ $f_3 = 23.459$

Element area, $A = 12.8416$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^T = \{-0.100436\, x - 0.231844\, y + 1.56495,$
$0.262818\, x + 0.219015\, y - 1.47835,\ -0.162382\, x + 0.0128296\, y + 0.9134\}$

Nodal values, $\boldsymbol{d}^T = \{0, 20.0936, 33.75\}$

$\psi(x, y) = \boldsymbol{N}^T\boldsymbol{d} = -0.199449\, x + 4.83379\, y + 1.1219$

$\partial\psi/\partial x = -0.199449;$ $\partial\psi/\partial y = 4.83379$

Solution for element 8

Nodal coordinates

Element node	Global node number	x	y
1	2	0	6.75
2	1	0	0
3	3	5.625	0

$x_1 = 0$ $x_2 = 0$ $x_3 = 5.625$
$y_1 = 6.75$ $y_2 = 0$ $y_3 = 0$

$b_1 = 0$ $b_2 = -6.75$ $b_3 = 6.75$

$c_1 = 5.625$ $c_2 = -5.625$ $c_3 = 0$

$f_1 = 0$ $f_2 = 37.9688$ $f_3 = 0$

Element area, A = 18.9844

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^T = \{0.148148\,y,\ -0.177778\,x - 0.148148\,y + 1.,\ 0.177778\,x\}$

Nodal values, $\boldsymbol{d}^T = \{33.75,\ 0,\ 0\}$

$\psi(x, y) = \boldsymbol{N}^T\boldsymbol{d} = 5.\,y$

$\partial\psi/\partial x = 0;$ $\partial\psi/\partial y = 5.$

Solution summary

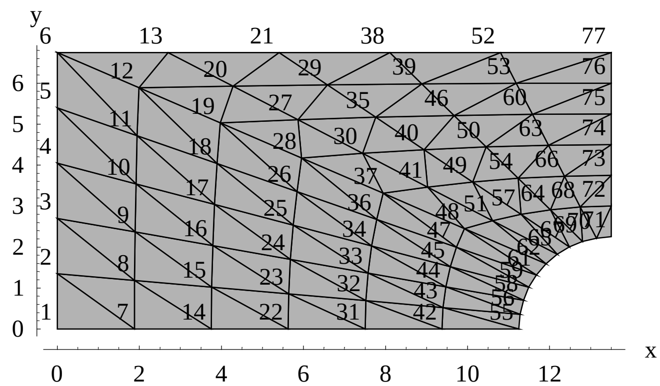
Nodal solution

	x−coord	y−coord	ψ
1	0	0	0
2	0	6.75	33.75
3	5.625	0	0
4	5.9545	4.1705	20.0936
5	11.25	0	0
6	11.909	1.59099	0
7	13.5	2.25	0
8	13.5	4.5	18.3344
9	13.5	6.75	33.75

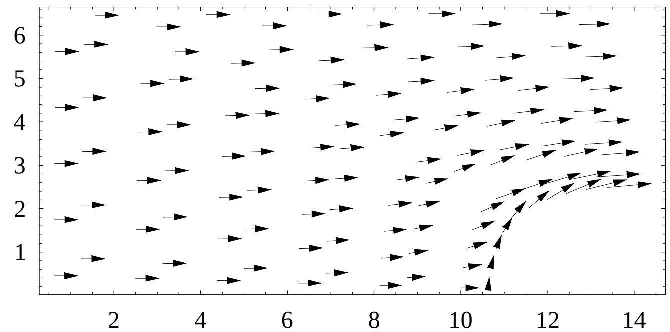
Solution at element centroids

	x−coord	y−coord	ψ	$\partial\psi/\partial x$	$\partial\psi/\partial y$
1	12.9697	2.78033	6.11146	−3.37526	8.14861
2	10.4545	3.4205	12.8093	−0.520816	6.58746
3	10.9848	5.14017	24.0593	−0.532342	6.85139
4	6.48483	5.89017	29.1979	0	5.2942
5	9.7045	1.9205	6.69786	−2.86112	1.18512
6	7.60983	1.39017	6.69786	0	4.81803
7	3.85983	3.64017	17.9479	−0.199449	4.83379
8	1.875	2.25	11.25	0	5.

In order to get a better solution we use a 120 element model as shown in Figure. The following table shows partial results for the stream function values and the velocities in the x and y direction obtained at the centroids of the elements. Using the *u* and *v* values the velocity vectors shown in Figure 5.XXX. are obtained. The velocity vectors are tangent to the stream lines and show that the finite element solution is reasonable.



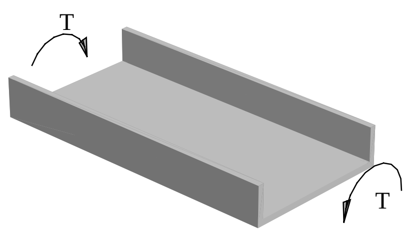
x	y	ψ	$u=\partial\psi/\partial y$	$v=-\partial\psi/\partial x$
11.5662	2.02003	3.12636	4.35271	3.67236
10.8125	2.22896	6.34974	6.18323	2.19341
9.85069	2.85397	11.5733	4.41314	1.68798
8.96234	3.07584	13.6316	5.32864	0.65605
8.13516	3.68791	17.319	4.99405	0.608097
7.11216	3.92273	18.9254	5.17981	0.24295
6.41963	4.52185	22.17	5.09408	0.236236
5.26198	4.76962	23.6071	5.10947	0.0807255
4.7041	5.35579	26.6377	5.07742	0.0794096
3.41179	5.6165	28.0187	5.05675	0.0197334
2.98857	6.18973	30.9208	5.03959	0.0194431
1.56161	6.46339	32.3102	5.02342	0
11.9899	2.20921	2.9261	5.79379	3.55624
11.3639	2.42207	6.05247	6.87944	2.36017
10.676	3.00993	11.1313	5.19344	1.99278
9.91652	3.23292	13.4288	5.6888	1.0255
9.36214	3.81065	17.1543	5.26698	0.978779
8.46913	4.04377	18.9862	5.31546	0.441183
8.04823	4.61137	22.1837	5.29723	0.440073
7.02174	4.85463	23.761	5.20322	0.179558

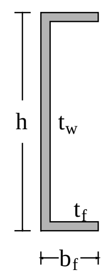


Example 5.7: Torsion constant of a C shape (p. 367)

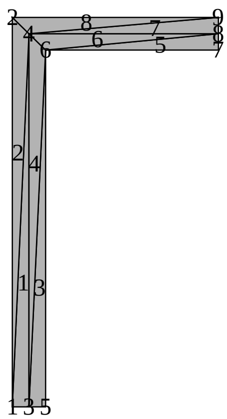
Find torsional constant J for a standard C12×30 section shown in Figure. The section dimensions are as follows.

$$h = 12\,\text{in} \quad t_w = 0.51\,\text{in} \quad b_f = 3.17\,\text{in} \quad t_f = 0.501\,\text{in}$$





Taking advantage of symmetry, we model only half the cross-section. Since ϕ is assigned a zero value on all the outside boundary nodes, the mesh must have some interior nodes. Thus the simplest possible model with triangular elements is an 8 element model shown in Figure. With such a coarse mesh, we don't expect a very accurate solution. The mesh is used to show most computations explicitly.



As a result of essential boundary conditions $\phi = 0$ at nodes {1, 2, 5, 6, 7, 8, 9}. Setting $G\,\theta = 1$, we can obtain the finite element solution using usual steps.

Global equations at start of the element assembly process

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \\ \phi_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 1

$k_x = 1;$ $\quad k_y = 1;$ $\quad p = 0;$ $\quad q = 2$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	1	0.	0.
2	3	0.255	0.
3	4	0.255	5.7495

$x_1 = 0.$ $\quad x_2 = 0.255$ $\quad x_3 = 0.255$

$y_1 = 0.$ $\quad y_2 = 0.$ $\quad y_3 = 5.7495$

Using these values we get

$b_1 = -5.7495 \qquad b_2 = 5.7495 \qquad b_3 = 0.$

$c_1 = 0. \qquad c_2 = -0.255 \qquad c_3 = 0.255$

$f_1 = 1.46612 \qquad f_2 = 0. \qquad f_3 = 0.$

Element area, A = 0.733061

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -5.7495 & 5.7495 & 0. \\ 0. & -0.255 & 0.255 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 11.2735 & -11.2735 & 0. \\ -11.2735 & 11.2957 & -0.0221758 \\ 0. & -0.0221758 & 0.0221758 \end{pmatrix}; \qquad \boldsymbol{k}_{\mathrm{p}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 0.488707 \\ 0.488707 \\ 0.488707 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 11.2735 & -11.2735 & 0 \\ -11.2735 & 11.2957 & -0.0221758 \\ 0 & -0.0221758 & 0.0221758 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} 0.488707 \\ 0.488707 \\ 0.488707 \end{pmatrix}$$

The element contributes to {1, 3, 4} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 11.2735 & 0 & -11.2735 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -11.2735 & 0 & 11.2957 & -0.0221758 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.0221758 & 0.0221758 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \\ \phi_9 \end{pmatrix} = \begin{pmatrix} 0.488707 \\ 0 \\ 0.488707 \\ 0.488707 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 2

$k_x = 1;$ $k_y = 1;$ $p = 0;$ $q = 2$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	4	0.255	5.7495
2	2	0.	6.
3	1	0.	0.

$x_1 = 0.255$ $x_2 = 0.$ $x_3 = 0.$

$y_1 = 5.7495$ $y_2 = 6.$ $y_3 = 0.$

Using these values we get

$b_1 = 6.$ $b_2 = -5.7495$ $b_3 = -0.2505$

$c_1 = 0.$ $c_2 = 0.255$ $c_3 = -0.255$

$f_1 = 0.$ $f_2 = 0.$ $f_3 = 1.53$

Element area, $A = 0.765$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 6. & -5.7495 & -0.2505 \\ 0. & 0.255 & -0.255 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 11.7647 & -11.2735 & -0.491176 \\ -11.2735 & 10.8241 & 0.44942 \\ -0.491176 & 0.44942 & 0.0417566 \end{pmatrix}; \quad \boldsymbol{k}_{\mathrm{p}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 0.51 \\ 0.51 \\ 0.51 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 11.7647 & -11.2735 & -0.491176 \\ -11.2735 & 10.8241 & 0.44942 \\ -0.491176 & 0.44942 & 0.0417566 \end{pmatrix} \begin{pmatrix} \phi_4 \\ \phi_2 \\ \phi_1 \end{pmatrix} = \begin{pmatrix} 0.51 \\ 0.51 \\ 0.51 \end{pmatrix}$$

The element contributes to {4, 2, 1} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 11.3153 & 0.44942 & -11.2735 & -0.491176 & 0 & 0 & 0 & 0 & 0 \\ 0.44942 & 10.8241 & 0 & -11.2735 & 0 & 0 & 0 & 0 & 0 \\ -11.2735 & 0 & 11.2957 & -0.0221758 & 0 & 0 & 0 & 0 & 0 \\ -0.491176 & -11.2735 & -0.0221758 & 11.7869 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \\ \phi_9 \end{pmatrix} = \begin{pmatrix} 0.998707 \\ 0.51 \\ 0.488707 \\ 0.998707 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 3

$k_x = 1;$ $k_y = 1;$ $p = 0;$ $q = 2$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	3	0.255	0.
2	5	0.51	0.
3	6	0.51	5.499

$x_1 = 0.255$ $x_2 = 0.51$ $x_3 = 0.51$

$y_1 = 0.$ $y_2 = 0.$ $y_3 = 5.499$

Using these values we get

$b_1 = -5.499$ $b_2 = 5.499$ $b_3 = 0.$

$c_1 = 0.$ $c_2 = -0.255$ $c_3 = 0.255$

$f_1 = 2.80449$ $f_2 = -1.40225$ $f_3 = 0.$

Element area, $A = 0.701123$

$$\boldsymbol{B}^T = \begin{pmatrix} -5.499 & 5.499 & 0. \\ 0. & -0.255 & 0.255 \end{pmatrix}$$

$$\boldsymbol{k}_k = \begin{pmatrix} 10.7824 & -10.7824 & 0. \\ -10.7824 & 10.8055 & -0.023186 \\ 0. & -0.023186 & 0.023186 \end{pmatrix}; \quad \boldsymbol{k}_p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \boldsymbol{r}_q = \begin{pmatrix} 0.467415 \\ 0.467415 \\ 0.467415 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 10.7824 & -10.7824 & 0 \\ -10.7824 & 10.8055 & -0.023186 \\ 0 & -0.023186 & 0.023186 \end{pmatrix} \begin{pmatrix} \phi_3 \\ \phi_5 \\ \phi_6 \end{pmatrix} = \begin{pmatrix} 0.467415 \\ 0.467415 \\ 0.467415 \end{pmatrix}$$

The element contributes to {3, 5, 6} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 11.3153 & 0.44942 & -11.2735 & -0.491176 & 0 & 0 & 0 & 0 & 0 \\ 0.44942 & 10.8241 & 0 & -11.2735 & 0 & 0 & 0 & 0 & 0 \\ -11.2735 & 0 & 22.0781 & -0.0221758 & -10.7824 & 0 & 0 & 0 & 0 \\ -0.491176 & -11.2735 & -0.0221758 & 11.7869 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -10.7824 & 0 & 10.8055 & -0.023186 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.023186 & 0.023186 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \\ \phi_9 \end{pmatrix} = \begin{pmatrix} 0.998707 \\ 0.51 \\ 0.956122 \\ 0.998707 \\ 0.467415 \\ 0.467415 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 4

$k_x = 1;$ $k_y = 1;$ $p = 0;$ $q = 2$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	6	0.51	5.499
2	4	0.255	5.7495
3	3	0.255	0.

$x_1 = 0.51$ $x_2 = 0.255$ $x_3 = 0.255$
$y_1 = 5.499$ $y_2 = 5.7495$ $y_3 = 0.$

Using these values we get

$b_1 = 5.7495$ $b_2 = -5.499$ $b_3 = -0.2505$

$c_1 = 0.$ $c_2 = 0.255$ $c_3 = -0.255$

$f_1 = -1.46612$ $f_2 = 1.40225$ $f_3 = 1.53$

Element area, A = 0.733061

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 5.7495 & -5.499 & -0.2505 \\ 0. & 0.255 & -0.255 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 11.2735 & -10.7824 & -0.491176 \\ -10.7824 & 10.3348 & 0.447601 \\ -0.491176 & 0.447601 & 0.0435759 \end{pmatrix}; \qquad \boldsymbol{k}_{\mathrm{p}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 0.488707 \\ 0.488707 \\ 0.488707 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 11.2735 & -10.7824 & -0.491176 \\ -10.7824 & 10.3348 & 0.447601 \\ -0.491176 & 0.447601 & 0.0435759 \end{pmatrix} \begin{pmatrix} \phi_6 \\ \phi_4 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} 0.488707 \\ 0.488707 \\ 0.488707 \end{pmatrix}$$

The element contributes to {6, 4, 3} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 11.3153 & 0.44942 & -11.2735 & -0.491176 & 0 & 0 & 0 & 0 & 0 \\ 0.44942 & 10.8241 & 0 & -11.2735 & 0 & 0 & 0 & 0 & 0 \\ -11.2735 & 0 & 22.1216 & 0.425425 & -10.7824 & -0.491176 & 0 & 0 & 0 \\ -0.491176 & -11.2735 & 0.425425 & 22.1216 & 0 & -10.7824 & 0 & 0 & 0 \\ 0 & 0 & -10.7824 & 0 & 10.8055 & -0.023186 & 0 & 0 & 0 \\ 0 & 0 & -0.491176 & -10.7824 & -0.023186 & 11.2967 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \\ \phi_9 \end{pmatrix} = \begin{pmatrix} 0.998707 \\ 0.51 \\ 1.44483 \\ 1.48741 \\ 0.467415 \\ 0.956122 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 5

$k_x = 1$; $k_y = 1$; $p = 0$; $q = 2$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	6	0.51	5.499
2	7	3.17	5.499
3	8	3.17	5.7495

$x_1 = 0.51$ $x_2 = 3.17$ $x_3 = 3.17$

$y_1 = 5.499$ $y_2 = 5.499$ $y_3 = 5.7495$

Using these values we get

$b_1 = -0.2505$ $b_2 = 0.2505$ $b_3 = 0.$

$c_1 = 0.$ $c_2 = -2.66$ $c_3 = 2.66$

$f_1 = 0.794085 \qquad f_2 = 14.4996 \qquad f_3 = -14.6273$

Element area, A = 0.333165

$$\boldsymbol{B}^T = \begin{pmatrix} -0.2505 & 0.2505 & 0. \\ 0. & -2.66 & 2.66 \end{pmatrix}$$

$$\boldsymbol{k}_k = \begin{pmatrix} 0.0470865 & -0.0470865 & 0. \\ -0.0470865 & 5.35647 & -5.30938 \\ 0. & -5.30938 & 5.30938 \end{pmatrix}; \qquad \boldsymbol{k}_p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_q = \begin{pmatrix} 0.22211 \\ 0.22211 \\ 0.22211 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 0.0470865 & -0.0470865 & 0 \\ -0.0470865 & 5.35647 & -5.30938 \\ 0 & -5.30938 & 5.30938 \end{pmatrix} \begin{pmatrix} \phi_6 \\ \phi_7 \\ \phi_8 \end{pmatrix} = \begin{pmatrix} 0.22211 \\ 0.22211 \\ 0.22211 \end{pmatrix}$$

The element contributes to {6, 7, 8} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 11.3153 & 0.44942 & -11.2735 & -0.491176 & 0 & 0 & 0 & 0 & 0 \\ 0.44942 & 10.8241 & 0 & -11.2735 & 0 & 0 & 0 & 0 & 0 \\ -11.2735 & 0 & 22.1216 & 0.425425 & -10.7824 & -0.491176 & 0 & 0 & 0 \\ -0.491176 & -11.2735 & 0.425425 & 22.1216 & 0 & -10.7824 & 0 & 0 & 0 \\ 0 & 0 & -10.7824 & 0 & 10.8055 & -0.023186 & 0 & 0 & 0 \\ 0 & 0 & -0.491176 & -10.7824 & -0.023186 & 11.3438 & -0.0470865 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.0470865 & 5.35647 & -5.30938 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -5.30938 & 5.30938 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Equations for element 6

$k_x = 1; \qquad k_y = 1; \qquad p = 0; \qquad q = 2$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	8	3.17	5.7495
2	4	0.255	5.7495
3	6	0.51	5.499

$x_1 = 3.17 \qquad x_2 = 0.255 \qquad x_3 = 0.51$
$y_1 = 5.7495 \qquad y_2 = 5.7495 \qquad y_3 = 5.499$

Using these values we get

$b_1 = 0.2505 \qquad b_2 = -0.2505 \qquad b_3 = 0.$

$c_1 = 0.255 \qquad c_2 = 2.66 \qquad c_3 = -2.915$

$f_1 = -1.53 \qquad f_2 = -14.4996 \qquad f_3 = 16.7598$

Element area, A = 0.365104

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 0.2505 & -0.2505 & 0. \\ 0.255 & 2.66 & -2.915 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 0.0874924 & 0.42149 & -0.508982 \\ 0.42149 & 4.88789 & -5.30938 \\ -0.508982 & -5.30938 & 5.81836 \end{pmatrix}; \qquad \boldsymbol{k}_{\mathrm{p}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 0.243403 \\ 0.243403 \\ 0.243403 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 0.0874924 & 0.42149 & -0.508982 \\ 0.42149 & 4.88789 & -5.30938 \\ -0.508982 & -5.30938 & 5.81836 \end{pmatrix} \begin{pmatrix} \phi_8 \\ \phi_4 \\ \phi_6 \end{pmatrix} = \begin{pmatrix} 0.243403 \\ 0.243403 \\ 0.243403 \end{pmatrix}$$

The element contributes to {8, 4, 6} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 11.3153 & 0.44942 & -11.2735 & -0.491176 & 0 & 0 & 0 & 0 & 0 \\ 0.44942 & 10.8241 & 0 & -11.2735 & 0 & 0 & 0 & 0 & 0 \\ -11.2735 & 0 & 22.1216 & 0.425425 & -10.7824 & -0.491176 & 0 & 0 & 0 \\ -0.491176 & -11.2735 & 0.425425 & 27.0095 & 0 & -16.0917 & 0 & 0.42149 & 0 \\ 0 & 0 & -10.7824 & 0 & 10.8055 & -0.023186 & 0 & 0 & 0 \\ 0 & 0 & -0.491176 & -16.0917 & -0.023186 & 17.1622 & -0.0470865 & -0.508982 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.0470865 & 5.35647 & -5.30938 & 0 \\ 0 & 0 & 0 & 0.42149 & 0 & -0.508982 & -5.30938 & 5.39687 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Equations for element 7

$k_x = 1; \qquad k_y = 1; \qquad p = 0; \qquad q = 2$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	4	0.255	5.7495
2	8	3.17	5.7495
3	9	3.17	6.

$x_1 = 0.255 \qquad x_2 = 3.17 \qquad x_3 = 3.17$

$y_1 = 5.7495 \qquad y_2 = 5.7495 \qquad y_3 = 6.$

Using these values we get

$b_1 = -0.2505 \qquad b_2 = 0.2505 \qquad b_3 = 0.$

$c_1 = 0. \qquad c_2 = -2.915 \qquad c_3 = 2.915$

$f_1 = 0.794085 \qquad f_2 = 16.6959 \qquad f_3 = -16.7598$

Element area, $A = 0.365104$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -0.2505 & 0.2505 & 0. \\ 0. & -2.915 & 2.915 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 0.0429674 & -0.0429674 & 0. \\ -0.0429674 & 5.86133 & -5.81836 \\ 0. & -5.81836 & 5.81836 \end{pmatrix}; \qquad \boldsymbol{k}_{\mathrm{p}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 0.243403 \\ 0.243403 \\ 0.243403 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 0.0429674 & -0.0429674 & 0 \\ -0.0429674 & 5.86133 & -5.81836 \\ 0 & -5.81836 & 5.81836 \end{pmatrix} \begin{pmatrix} \phi_4 \\ \phi_8 \\ \phi_9 \end{pmatrix} = \begin{pmatrix} 0.243403 \\ 0.243403 \\ 0.243403 \end{pmatrix}$$

The element contributes to {4, 8, 9} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\left(\begin{array}{ccccccccc} 11.3153 & 0.44942 & -11.2735 & -0.491176 & 0 & 0 & 0 & 0 & \\ 0.44942 & 10.8241 & 0 & -11.2735 & 0 & 0 & 0 & 0 & \\ -11.2735 & 0 & 22.1216 & 0.425425 & -10.7824 & -0.491176 & 0 & 0 & \\ -0.491176 & -11.2735 & 0.425425 & 27.0525 & 0 & -16.0917 & 0 & 0.378522 & \\ 0 & 0 & -10.7824 & 0 & 10.8055 & -0.023186 & 0 & 0 & \\ 0 & 0 & -0.491176 & -16.0917 & -0.023186 & 17.1622 & -0.0470865 & -0.508982 & \\ 0 & 0 & 0 & 0 & 0 & -0.0470865 & 5.35647 & -5.30938 & \\ 0 & 0 & 0 & 0.378522 & 0 & -0.508982 & -5.30938 & 11.2582 & - \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5.81836 & \end{array} \right.$$

Equations for element 8

$k_x = 1;$ $k_y = 1;$ $p = 0;$ $q = 2$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	9	3.17	6.
2	2	0.	6.
3	4	0.255	5.7495

$x_1 = 3.17$ $x_2 = 0.$ $x_3 = 0.255$
$y_1 = 6.$ $y_2 = 6.$ $y_3 = 5.7495$

Using these values we get

$b_1 = 0.2505$ $b_2 = -0.2505$ $b_3 = 0.$

$c_1 = 0.255$ $c_2 = 2.915$ $c_3 = -3.17$

$f_1 = -1.53$ $f_2 = -16.6959$ $f_3 = 19.02$

Element area, A = 0.397043

$$\boldsymbol{B}^{T} = \begin{pmatrix} 0.2505 & -0.2505 & 0. \\ 0.255 & 2.915 & -3.17 \end{pmatrix}$$

$$\boldsymbol{k}_k = \begin{pmatrix} 0.0804544 & 0.428528 & -0.508982 \\ 0.428528 & 5.38984 & -5.81836 \\ -0.508982 & -5.81836 & 6.32735 \end{pmatrix}; \quad \boldsymbol{k}_p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \boldsymbol{r}_q = \begin{pmatrix} 0.264695 \\ 0.264695 \\ 0.264695 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 0.0804544 & 0.428528 & -0.508982 \\ 0.428528 & 5.38984 & -5.81836 \\ -0.508982 & -5.81836 & 6.32735 \end{pmatrix} \begin{pmatrix} \phi_9 \\ \phi_2 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} 0.264695 \\ 0.264695 \\ 0.264695 \end{pmatrix}$$

The element contributes to {9, 2, 4} global degrees of freedom.

Adding element equations into appropriate locations we have

$$
\left(\begin{array}{cccccccc}
11.3153 & 0.44942 & -11.2735 & -0.491176 & 0 & 0 & 0 & 0 \\
0.44942 & 16.2139 & 0 & -17.0919 & 0 & 0 & 0 & 0 \\
-11.2735 & 0 & 22.1216 & 0.425425 & -10.7824 & -0.491176 & 0 & 0 \\
-0.491176 & -17.0919 & 0.425425 & 33.3798 & 0 & -16.0917 & 0 & 0.378522 \\
0 & 0 & -10.7824 & 0 & 10.8055 & -0.023186 & 0 & 0 \\
0 & 0 & -0.491176 & -16.0917 & -0.023186 & 17.1622 & -0.0470865 & -0.508982 \\
0 & 0 & 0 & 0 & 0 & -0.0470865 & 5.35647 & -5.30938 \\
0 & 0 & 0 & 0.378522 & 0 & -0.508982 & -5.30938 & 11.2582 \\
0 & 0.428528 & 0 & -0.508982 & 0 & 0 & 0 & -5.81836
\end{array}\right.
$$

Essential boundary conditions

Node	dof	Value
1	ϕ_1	0
2	ϕ_2	0
5	ϕ_5	0
6	ϕ_6	0
7	ϕ_7	0
8	ϕ_8	0
9	ϕ_9	0

Remove {1, 2, 5, 6, 7, 8, 9} rows and columns.

After adjusting for essential boundary conditions we have

$$
\begin{pmatrix} 22.1216 & 0.425425 \\ 0.425425 & 33.3798 \end{pmatrix} \begin{pmatrix} \phi_3 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} 1.44483 \\ 2.23892 \end{pmatrix}
$$

Solving the final system of global equations we get

$$\{\phi_3 = 0.0640388,\ \phi_4 = 0.0662577\}$$

Complete table of nodal values

	ϕ
1	0
2	0
3	0.0640388
4	0.0662577
5	0
6	0
7	0
8	0
9	0

Solution for element 1

Nodal coordinates

Element node	Global node number	x	y
1	1	0.	0.
2	3	0.255	0.
3	4	0.255	5.7495

$x_1 = 0.$ $\quad x_2 = 0.255$ $\quad x_3 = 0.255$
$y_1 = 0.$ $\quad y_2 = 0.$ $\quad y_3 = 5.7495$

$b_1 = -5.7495$ $\quad b_2 = 5.7495$ $\quad b_3 = 0.$

$c_1 = 0.$ $\quad c_2 = -0.255$ $\quad c_3 = 0.255$

$f_1 = 1.46612$ $\quad f_2 = 0.$ $\quad f_3 = 0.$

Element area, $A = 0.733061$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^T = \{1. - 3.92157\,x,\ 3.92157\,x - 0.173928\,y,\ 0.173928\,y\}$

Nodal values, $\boldsymbol{d}^T = \{0,\ 0.0640388,\ 0.0662577\}$

$\phi(x, y) = \boldsymbol{N}^T\boldsymbol{d} = 0.251132\,x + 0.000385934\,y$

$\partial\phi/\partial x = 0.251132;$ $\qquad \partial\phi/\partial y = 0.000385934$

Solution for element 2

Nodal coordinates

Element node	Global node number	x	y
1	4	0.255	5.7495
2	2	0.	6.
3	1	0.	0.

$x_1 = 0.255$ $\quad x_2 = 0.$ $\quad x_3 = 0.$
$y_1 = 5.7495$ $\quad y_2 = 6.$ $\quad y_3 = 0.$

$b_1 = 6.$ $\quad b_2 = -5.7495$ $\quad b_3 = -0.2505$

$c_1 = 0.$ $\quad c_2 = 0.255$ $\quad c_3 = -0.255$

$f_1 = 0.$ $\quad f_2 = 0.$ $\quad f_3 = 1.53$

Element area, $A = 0.765$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^T = \{3.92157\,x,\ 0.166667\,y - 3.75784\,x,\ -0.163725\,x - 0.166667\,y + 1.\}$

Nodal values, $\boldsymbol{d}^T = \{0.0662577, 0, 0\}$

$\phi(x, y) = \boldsymbol{N}^T\boldsymbol{d} = 0.259834\,x$

$\partial\phi/\partial x = 0.259834;$ $\quad \partial\phi/\partial y = 0$

Solution for element 3

Nodal coordinates

Element node	Global node number	x	y
1	3	0.255	0.
2	5	0.51	0.
3	6	0.51	5.499

$x_1 = 0.255$ $\quad x_2 = 0.51$ $\quad x_3 = 0.51$
$y_1 = 0.$ $\quad y_2 = 0.$ $\quad y_3 = 5.499$

$b_1 = -5.499$ $\quad b_2 = 5.499$ $\quad b_3 = 0.$

$c_1 = 0.$ $\quad c_2 = -0.255$ $\quad c_3 = 0.255$

$f_1 = 2.80449$ $\quad f_2 = -1.40225$ $\quad f_3 = 0.$

Element area, $A = 0.701123$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^T = \{2. - 3.92157\,x,\ 3.92157\,x - 0.181851\,y - 1.,\ 0.181851\,y\}$

Nodal values, $\boldsymbol{d}^T = \{0.0640388, 0, 0\}$

$\phi(x, y) = \boldsymbol{N}^{T}\boldsymbol{d} = 0.128078 - 0.251132\,x$

$\partial\phi/\partial x = -0.251132;$ $\partial\phi/\partial y = 0$

Solution for element 4

Nodal coordinates

Element node	Global node number	x	y
1	6	0.51	5.499
2	4	0.255	5.7495
3	3	0.255	0.

$x_1 = 0.51$ $x_2 = 0.255$ $x_3 = 0.255$
$y_1 = 5.499$ $y_2 = 5.7495$ $y_3 = 0.$

$b_1 = 5.7495$ $b_2 = -5.499$ $b_3 = -0.2505$

$c_1 = 0.$ $c_2 = 0.255$ $c_3 = -0.255$

$f_1 = -1.46612$ $f_2 = 1.40225$ $f_3 = 1.53$

Element area, A = 0.733061

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^{T} =$
$\{3.92157\,x - 1.,\ -3.75071\,x + 0.173928\,y + 0.956431,\ -0.170859\,x - 0.173928\,y + 1.04357\}$

Nodal values, $\boldsymbol{d}^{T} = \{0,\ 0.0662577,\ 0.0640388\}$

$\phi(x, y) = \boldsymbol{N}^{T}\boldsymbol{d} = -0.259455\,x + 0.000385934\,y + 0.1302$

$\partial\phi/\partial x = -0.259455;$ $\partial\phi/\partial y = 0.000385934$

Solution for element 5

Nodal coordinates

Element node	Global node number	x	y
1	6	0.51	5.499
2	7	3.17	5.499
3	8	3.17	5.7495

$x_1 = 0.51$ $x_2 = 3.17$ $x_3 = 3.17$
$y_1 = 5.499$ $y_2 = 5.499$ $y_3 = 5.7495$

$b_1 = -0.2505$ $b_2 = 0.2505$ $b_3 = 0.$

$c_1 = 0.$ $c_2 = -2.66$ $c_3 = 2.66$

$f_1 = 0.794085$ $f_2 = 14.4996$ $f_3 = -14.6273$

Element area, A = 0.333165

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^T = \{1.19173 - 0.37594\,x,\ 0.37594\,x - 3.99202\,y + 21.7604,\ 3.99202\,y - 21.9521\}$

Nodal values, $\boldsymbol{d}^T = \{0, 0, 0\}$

$\phi(x, y) = \boldsymbol{N}^T\boldsymbol{d} = 0$

$\partial\phi/\partial x = 0;$ $\partial\phi/\partial y = 0$

Solution for element 6

Nodal coordinates

Element node	Global node number	x	y
1	8	3.17	5.7495
2	4	0.255	5.7495
3	6	0.51	5.499

$x_1 = 3.17$ $x_2 = 0.255$ $x_3 = 0.51$
$y_1 = 5.7495$ $y_2 = 5.7495$ $y_3 = 5.499$

$b_1 = 0.2505$ $b_2 = -0.2505$ $b_3 = 0.$

$c_1 = 0.255$ $c_2 = 2.66$ $c_3 = -2.915$

$f_1 = -1.53$ $f_2 = -14.4996$ $f_3 = 16.7598$

Element area, A = 0.365104

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^T =$
$\{0.343053\,x + 0.349216\,y - 2.09529,\ -0.343053\,x + 3.6428\,y - 19.8568,\ 22.9521 - 3.99202\,y\}$

Nodal values, $\boldsymbol{d}^T = \{0, 0.0662577, 0\}$

$\phi(x, y) = \boldsymbol{N}^T\boldsymbol{d} = -0.0227299\,x + 0.241364\,y - 1.31567$

$\partial\phi/\partial x = -0.0227299;$ $\partial\phi/\partial y = 0.241364$

Solution for element 7

Nodal coordinates

Element node	Global node number	x	y
1	4	0.255	5.7495
2	8	3.17	5.7495
3	9	3.17	6.

$x_1 = 0.255 \quad x_2 = 3.17 \quad x_3 = 3.17$
$y_1 = 5.7495 \quad y_2 = 5.7495 \quad y_3 = 6.$

$b_1 = -0.2505 \quad b_2 = 0.2505 \quad b_3 = 0.$

$c_1 = 0. \quad c_2 = -2.915 \quad c_3 = 2.915$

$f_1 = 0.794085 \quad f_2 = 16.6959 \quad f_3 = -16.7598$

Element area, $A = 0.365104$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^T = \{1.08748 - 0.343053\,x,\ 0.343053\,x - 3.99202\,y + 22.8646,\ 3.99202\,y - 22.9521\}$

Nodal values, $\boldsymbol{d}^T = \{0.0662577, 0, 0\}$

$\phi(x, y) = \boldsymbol{N}^T\boldsymbol{d} = 0.0720538 - 0.0227299\,x$

$\partial\phi/\partial x = -0.0227299; \quad \partial\phi/\partial y = 0$

Solution for element 8

Nodal coordinates

Element node	Global node number	x	y
1	9	3.17	6.
2	2	0.	6.
3	4	0.255	5.7495

$x_1 = 3.17 \quad x_2 = 0. \quad x_3 = 0.255$
$y_1 = 6. \quad y_2 = 6. \quad y_3 = 5.7495$

$b_1 = 0.2505 \quad b_2 = -0.2505 \quad b_3 = 0.$

$c_1 = 0.255 \quad c_2 = 2.915 \quad c_3 = -3.17$

$f_1 = -1.53 \quad f_2 = -16.6959 \quad f_3 = 19.02$

Element area, $A = 0.397043$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^T =$
$\{0.315457\,x + 0.321124\,y - 1.92675,\ -0.315457\,x + 3.67089\,y - 21.0253,\ 23.9521 - 3.99202\,y\}$

Nodal values, $\boldsymbol{d}^{T} = \{0, 0, 0.0662577\}$

$\phi(x, y) = \boldsymbol{N}^{T}\boldsymbol{d} = 1.58701 - 0.264502\, y$

$\partial\phi/\partial x = 0; \qquad \partial\phi/\partial y = -0.264502$

Solution summary

Nodal solution

	x−coord	y−coord	ϕ
1	0.	0.	0
2	0.	6.	0
3	0.255	0.	0.0640388
4	0.255	5.7495	0.0662577
5	0.51	0.	0
6	0.51	5.499	0
7	3.17	5.499	0
8	3.17	5.7495	0
9	3.17	6.	0

Solution at element centroids

	x−coord	y−coord	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	0.17	1.9165	0.0434322	0.251132	0.000385934
2	0.085	3.9165	0.0220859	0.259834	0
3	0.425	1.833	0.0213463	−0.251132	0
4	0.34	3.7495	0.0434322	−0.259455	0.000385934
5	2.28333	5.5825	0	0	0
6	1.31167	5.666	0.0220859	−0.0227299	0.241364
7	2.19833	5.833	0.0220859	−0.0227299	0
8	1.14167	5.9165	0.0220859	0	−0.264502

Solutions over the remaining elements can be determined in exactly the same manner.

The total torque is given by

$$T = 2\iint_A \phi \, dA$$

The integral of ϕ over each element can be evaluated as described earlier. Since ϕ is a linear function over each element, using the procedure for integration over a triangle discussed earlier, it can be shown that the integral over each element is

$$\iint\limits_{A^{(e)}} \phi^{(e)}\, \mathrm{dA} = \frac{A^{(e)}}{3}\,(\phi_1 + \phi_2 + \phi_3)$$

where $A^{(e)}$ is the area of the element and ϕ_1, ϕ_2, ϕ_3 are the values at its nodes. Using this formula the integral of ϕ over each element is evaluated and the results are summarized as follows.

	ϕ	$\iint \phi\, \mathrm{dA}$
1	$0.251132\,x + 0.000385934\,y$	0.0318384
2	$0.259834\,x$	0.0168957
3	$0.128078 - 0.251132\,x$	0.0149663
4	$-0.259455\,x + 0.000385934\,y + 0.1302$	0.0318384
5	0	0
6	$-0.0227299\,x + 0.241364\,y - 1.31567$	0.00806364
7	$0.0720538 - 0.0227299\,x$	0.00806364
8	$1.58701 - 0.264502\,y$	0.00876904

Summing $\iint \phi\, \mathrm{dA}$ contributions from all elements and multiplying by 2 gives the total torque. Since we are modeling half of the C shape, the torque for the entire section is twice this value. The the total torque is

$$\mathrm{T} = 2\times 2\times \sum\left(\iint \phi\, \mathrm{dA}\right) = 0.481741$$

Since $T = J\, G\, \theta$ and we have used $G\, \theta = 1$, the computations show that the torsional constant J for the section is 0.48 in^4. The J value tabulated in the steel design handbook for this section is 0.87 in^4. As expected the computed value has a large error of almost 45%. Solution improves considerably if we use a finer mesh involving 64 triangular elements.

Using 8 element: J = 0.481741; Error = 45.%

Using 64 elements: J = 0.754029; Error = 13.%