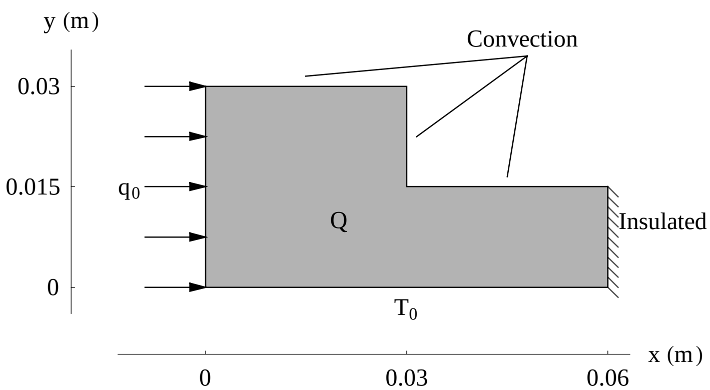
Example 5.3: Heat flow in an L-shaped body (p. 337)

Consider two dimensional heat flow over an L-shaped body with thermal conductivity $k = 45\ W/m.°C$ shown in Figure. The bottom is maintained at $T_0 = 110\,°C$. Convection heat loss takes place on the top where the ambient air temperature is $20\,°C$ and the convection heat transfer coefficient is $h = 55\ W/m^2.°C$. The right side is insulated. The left side is subjected to heat flux at a uniform rate of $q_L = 8000\ W/m^2$. Heat is generated in the body at a rate of $Q = 5\times10^6\ W/m^3$. Determine temperature distribution in the body.



As shown earlier the governing differential equation for a heat flow problem is a special case of the general form. With the numerical values given for this example

$$k_x = k_y = 45; \quad p = 0; \quad q = 5\times10^6$$

The boundary conditions are as follows.

For all nodes on the bottom side, $T = 110$

On the left side ($n_x = -1,\ n_y = 0$):

$$-k\ \frac{\partial T}{\partial n} = k\ \frac{\partial T}{\partial x} = q_L \quad \Longrightarrow \quad \alpha = 0; \beta = 8000$$

On the right side, $\alpha = 0; \beta = 0$

For convection on horizontal portions of the top side ($n_x = 0,\ n_y = 1$):

$$-k\ \frac{\partial T}{\partial n} = -k\ \frac{\partial T}{\partial y} = h(T - T_\infty)$$

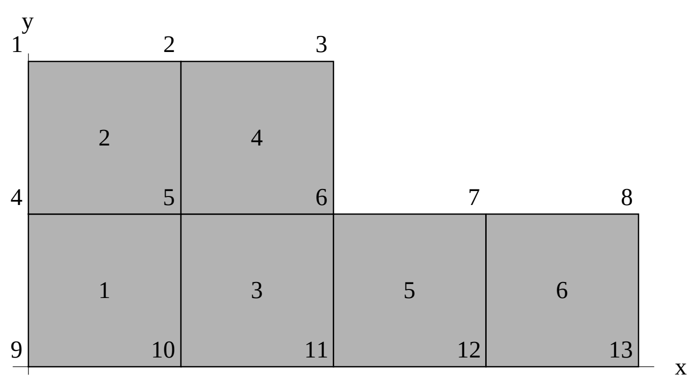
$$\Longrightarrow \quad \alpha = -h = -55; \beta = h\, T_\infty = 55 \times 20 = 1100$$

For convection on vertical portion of the top side ($n_x = 1$, $n_y = 0$):

$$-k\, \frac{\partial T}{\partial n} = -k\, \frac{\partial T}{\partial x} = h(T - T_\infty)$$

$$\Longrightarrow \quad \alpha = -h = -55; \beta = h\, T_\infty = 55 \times 20 = 1100$$

The complete finite element solution is as follows.



Global equations at start of the element assembly process

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} T_1\\ T_2\\ T_3\\ T_4\\ T_5\\ T_6\\ T_7\\ T_8\\ T_9\\ T_{10}\\ T_{11}\\ T_{12}\\ T_{13} \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}$$

Equations for element 1

Element dimensions: a = 0.0075; b = 0.0075

$k_x = 45$; $k_y = 45$; $p = 0$; $q = 5000000$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 30. & -7.5 & -15. & -7.5 \\ -7.5 & 30. & -7.5 & -15. \\ -15. & -7.5 & 30. & -7.5 \\ -7.5 & -15. & -7.5 & 30. \end{pmatrix}; \quad \boldsymbol{k}_{\mathrm{p}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 281.25 \\ 281.25 \\ 281.25 \\ 281.25 \end{pmatrix}$$

NBC on side 4

$L = 0.015$; $\alpha = 0$; $\beta = 8000$

$$\boldsymbol{k}_{\alpha} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \boldsymbol{r}_{\beta} = \begin{pmatrix} 60. \\ 0 \\ 0 \\ 60. \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 30. & -7.5 & -15. & -7.5 \\ -7.5 & 30. & -7.5 & -15. \\ -15. & -7.5 & 30. & -7.5 \\ -7.5 & -15. & -7.5 & 30. \end{pmatrix} \begin{pmatrix} T_9 \\ T_{10} \\ T_5 \\ T_4 \end{pmatrix} = \begin{pmatrix} 341.25 \\ 281.25 \\ 281.25 \\ 341.25 \end{pmatrix}$$

The element contributes to {9, 10, 5, 4} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 30. & -7.5 & 0 & 0 & 0 & -7.5 & -15. & 0 & 0 & 0 \\ 0 & 0 & 0 & -7.5 & 30. & 0 & 0 & 0 & -15. & -7.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7.5 & -15. & 0 & 0 & 0 & 30. & -7.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -15. & -7.5 & 0 & 0 & 0 & -7.5 & 30. & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \\ T_{10} \\ T_{11} \\ T_{12} \\ T_{13} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 341.25 \\ 281.25 \\ 0 \\ 0 \\ 0 \\ 341.25 \\ 281.25 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 2

Element dimensions: a = 0.0075; b = 0.0075

$k_x = 45$; $k_y = 45$; p = 0; q = 5000000

$$\boldsymbol{k}_\mathrm{k} = \begin{pmatrix} 30. & -7.5 & -15. & -7.5 \\ -7.5 & 30. & -7.5 & -15. \\ -15. & -7.5 & 30. & -7.5 \\ -7.5 & -15. & -7.5 & 30. \end{pmatrix}; \qquad \boldsymbol{k}_\mathrm{p} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_\mathrm{q} = \begin{pmatrix} 281.25 \\ 281.25 \\ 281.25 \\ 281.25 \end{pmatrix}$$

NBC on side 3

L = 0.015; $\alpha = -55$; $\beta = 1100$

$$\boldsymbol{k}_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.275 & 0.1375 \\ 0 & 0 & 0.1375 & 0.275 \end{pmatrix}; \qquad \boldsymbol{r}_\beta = \begin{pmatrix} 0 \\ 0 \\ 8.25 \\ 8.25 \end{pmatrix}$$

NBC on side 4

L = 0.015; $\alpha = 0$; $\beta = 8000$

$$\boldsymbol{k}_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_\beta = \begin{pmatrix} 60. \\ 0 \\ 0 \\ 60. \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 30. & -7.5 & -15. & -7.5 \\ -7.5 & 30. & -7.5 & -15. \\ -15. & -7.5 & 30.275 & -7.3625 \\ -7.5 & -15. & -7.3625 & 30.275 \end{pmatrix} \begin{pmatrix} T_4 \\ T_5 \\ T_2 \\ T_1 \end{pmatrix} = \begin{pmatrix} 341.25 \\ 281.25 \\ 289.5 \\ 349.5 \end{pmatrix}$$

The element contributes to {4, 5, 2, 1} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 30.275 & -7.3625 & 0 & -7.5 & -15. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -7.3625 & 30.275 & 0 & -15. & -7.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -7.5 & -15. & 0 & 60. & -15. & 0 & 0 & 0 & -7.5 & -15. & 0 & 0 & 0 \\ -15. & -7.5 & 0 & -15. & 60. & 0 & 0 & 0 & -15. & -7.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7.5 & -15. & 0 & 0 & 0 & 30. & -7.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -15. & -7.5 & 0 & 0 & 0 & -7.5 & 30. & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \\ T_{10} \\ T_{11} \\ T_{12} \\ T_{13} \end{pmatrix} = \begin{pmatrix} 349.5 \\ 289.5 \\ 0 \\ 682.5 \\ 562.5 \\ 0 \\ 0 \\ 0 \\ 341.25 \\ 281.25 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 3

Element dimensions: $a = 0.0075$; $b = 0.0075$

$k_x = 45$; $k_y = 45$; $p = 0$; $q = 5000000$

$$\boldsymbol{k}_k = \begin{pmatrix} 30. & -7.5 & -15. & -7.5 \\ -7.5 & 30. & -7.5 & -15. \\ -15. & -7.5 & 30. & -7.5 \\ -7.5 & -15. & -7.5 & 30. \end{pmatrix}; \quad \boldsymbol{k}_p = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \boldsymbol{r}_q = \begin{pmatrix} 281.25 \\ 281.25 \\ 281.25 \\ 281.25 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 30. & -7.5 & -15. & -7.5 \\ -7.5 & 30. & -7.5 & -15. \\ -15. & -7.5 & 30. & -7.5 \\ -7.5 & -15. & -7.5 & 30. \end{pmatrix} \begin{pmatrix} T_{10} \\ T_{11} \\ T_6 \\ T_5 \end{pmatrix} = \begin{pmatrix} 281.25 \\ 281.25 \\ 281.25 \\ 281.25 \end{pmatrix}$$

The element contributes to {10, 11, 6, 5} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 30.275 & -7.3625 & 0 & -7.5 & -15. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -7.3625 & 30.275 & 0 & -15. & -7.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -7.5 & -15. & 0 & 60. & -15. & 0 & 0 & 0 & -7.5 & -15. & 0 & 0 & 0 \\ -15. & -7.5 & 0 & -15. & 90. & -7.5 & 0 & 0 & -15. & -15. & -15. & 0 & 0 \\ 0 & 0 & 0 & 0 & -7.5 & 30. & 0 & 0 & 0 & -15. & -7.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7.5 & -15. & 0 & 0 & 0 & 30. & -7.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -15. & -15. & -15. & 0 & 0 & -7.5 & 60. & -7.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & -15. & -7.5 & 0 & 0 & 0 & -7.5 & 30. & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \\ T_{10} \\ T_{11} \\ T_{12} \\ T_{13} \end{pmatrix} = \begin{pmatrix} 349.5 \\ 289.5 \\ 0 \\ 682.5 \\ 843.75 \\ 281.25 \\ 0 \\ 0 \\ 341.25 \\ 562.5 \\ 281.25 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 4

Element dimensions: a = 0.0075; b = 0.0075

$k_x = 45$; $k_y = 45$; $p = 0$; $q = 5000000$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 30. & -7.5 & -15. & -7.5 \\ -7.5 & 30. & -7.5 & -15. \\ -15. & -7.5 & 30. & -7.5 \\ -7.5 & -15. & -7.5 & 30. \end{pmatrix}; \quad \boldsymbol{k}_{\mathrm{p}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 281.25 \\ 281.25 \\ 281.25 \\ 281.25 \end{pmatrix}$$

NBC on side 2

$L = 0.015$; $\alpha = -55$; $\beta = 1100$

$$\boldsymbol{k}_{\alpha} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.275 & 0.1375 & 0 \\ 0 & 0.1375 & 0.275 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \boldsymbol{r}_{\beta} = \begin{pmatrix} 0 \\ 8.25 \\ 8.25 \\ 0 \end{pmatrix}$$

NBC on side 3

$L = 0.015$; $\alpha = -55$; $\beta = 1100$

$$\boldsymbol{k}_{\alpha} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.275 & 0.1375 \\ 0 & 0 & 0.1375 & 0.275 \end{pmatrix}; \quad \boldsymbol{r}_{\beta} = \begin{pmatrix} 0 \\ 0 \\ 8.25 \\ 8.25 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 30. & -7.5 & -15. & -7.5 \\ -7.5 & 30.275 & -7.3625 & -15. \\ -15. & -7.3625 & 30.55 & -7.3625 \\ -7.5 & -15. & -7.3625 & 30.275 \end{pmatrix} \begin{pmatrix} T_5 \\ T_6 \\ T_3 \\ T_2 \end{pmatrix} = \begin{pmatrix} 281.25 \\ 289.5 \\ 297.75 \\ 289.5 \end{pmatrix}$$

The element contributes to {5, 6, 3, 2} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 30.275 & -7.3625 & 0 & -7.5 & -15. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -7.3625 & 60.55 & -7.3625 & -15. & -15. & -15. & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -7.3625 & 30.55 & 0 & -15. & -7.3625 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -7.5 & -15. & 0 & 60. & -15. & 0 & 0 & 0 & -7.5 & -15. & 0 & 0 & 0 \\ -15. & -15. & -15. & -15. & 120. & -15. & 0 & 0 & -15. & -15. & -15. & 0 & 0 \\ 0 & -15. & -7.3625 & 0 & -15. & 60.275 & 0 & 0 & 0 & -15. & -7.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7.5 & -15. & 0 & 0 & 0 & 30. & -7.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -15. & -15. & -15. & 0 & 0 & -7.5 & 60. & -7.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & -15. & -7.5 & 0 & 0 & 0 & -7.5 & 30. & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \\ T_{10} \\ T_{11} \\ T_{12} \\ T_{13} \end{pmatrix} = \begin{pmatrix} 349.5 \\ 579. \\ 297.75 \\ 682.5 \\ 1125. \\ 570.75 \\ 0 \\ 0 \\ 341.25 \\ 562.5 \\ 281.25 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 5

Element dimensions: $a = 0.0075$; $b = 0.0075$

$k_x = 45$; $k_y = 45$; $p = 0$; $q = 5000000$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 30. & -7.5 & -15. & -7.5 \\ -7.5 & 30. & -7.5 & -15. \\ -15. & -7.5 & 30. & -7.5 \\ -7.5 & -15. & -7.5 & 30. \end{pmatrix}; \qquad \boldsymbol{k}_{\mathrm{p}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 281.25 \\ 281.25 \\ 281.25 \\ 281.25 \end{pmatrix}$$

NBC on side 3

$L = 0.015$; $\alpha = -55$; $\beta = 1100$

$$\boldsymbol{k}_{\alpha} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.275 & 0.1375 \\ 0 & 0 & 0.1375 & 0.275 \end{pmatrix}; \qquad \boldsymbol{r}_{\beta} = \begin{pmatrix} 0 \\ 0 \\ 8.25 \\ 8.25 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 30. & -7.5 & -15. & -7.5 \\ -7.5 & 30. & -7.5 & -15. \\ -15. & -7.5 & 30.275 & -7.3625 \\ -7.5 & -15. & -7.3625 & 30.275 \end{pmatrix} \begin{pmatrix} T_{11} \\ T_{12} \\ T_7 \\ T_6 \end{pmatrix} = \begin{pmatrix} 281.25 \\ 281.25 \\ 289.5 \\ 289.5 \end{pmatrix}$$

The element contributes to {11, 12, 7, 6} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 30.275 & -7.3625 & 0 & -7.5 & -15. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -7.3625 & 60.55 & -7.3625 & -15. & -15. & -15. & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -7.3625 & 30.55 & 0 & -15. & -7.3625 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -7.5 & -15. & 0 & 60. & -15. & 0 & 0 & 0 & -7.5 & -15. & 0 & 0 & 0 \\ -15. & -15. & -15. & -15. & 120. & -15. & 0 & 0 & -15. & -15. & -15. & 0 & 0 \\ 0 & -15. & -7.3625 & 0 & -15. & 90.55 & -7.3625 & 0 & 0 & -15. & -15. & -15. & 0 \\ 0 & 0 & 0 & 0 & 0 & -7.3625 & 30.275 & 0 & 0 & 0 & -15. & -7.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7.5 & -15. & 0 & 0 & 0 & 30. & -7.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -15. & -15. & -15. & 0 & 0 & -7.5 & 60. & -7.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & -15. & -15. & -15. & 0 & 0 & -7.5 & 60. & -7.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & -15. & -7.5 & 0 & 0 & 0 & -7.5 & 30. & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Equations for element 6

Element dimensions: $a = 0.0075$; $b = 0.0075$

$k_x = 45;$ $\quad k_y = 45;$ $\quad p = 0;$ $\quad q = 5000000$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 30. & -7.5 & -15. & -7.5 \\ -7.5 & 30. & -7.5 & -15. \\ -15. & -7.5 & 30. & -7.5 \\ -7.5 & -15. & -7.5 & 30. \end{pmatrix}; \qquad \boldsymbol{k}_{\mathrm{p}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 281.25 \\ 281.25 \\ 281.25 \\ 281.25 \end{pmatrix}$$

NBC on side 3

$L = 0.015;$ $\quad \alpha = -55;$ $\quad \beta = 1100$

$$\boldsymbol{k}_{\alpha} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.275 & 0.1375 \\ 0 & 0 & 0.1375 & 0.275 \end{pmatrix}; \qquad \boldsymbol{r}_{\beta} = \begin{pmatrix} 0 \\ 0 \\ 8.25 \\ 8.25 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 30. & -7.5 & -15. & -7.5 \\ -7.5 & 30. & -7.5 & -15. \\ -15. & -7.5 & 30.275 & -7.3625 \\ -7.5 & -15. & -7.3625 & 30.275 \end{pmatrix} \begin{pmatrix} T_{12} \\ T_{13} \\ T_8 \\ T_7 \end{pmatrix} = \begin{pmatrix} 281.25 \\ 281.25 \\ 289.5 \\ 289.5 \end{pmatrix}$$

The element contributes to {12, 13, 8, 7} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\left(\begin{array}{cccccccccccccc} 30.275 & -7.3625 & 0 & -7.5 & -15. & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -7.3625 & 60.55 & -7.3625 & -15. & -15. & -15. & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -7.3625 & 30.55 & 0 & -15. & -7.3625 & 0 & 0 & 0 & 0 & 0 & 0 \\ -7.5 & -15. & 0 & 60. & -15. & 0 & 0 & 0 & -7.5 & -15. & 0 & 0 \\ -15. & -15. & -15. & -15. & 120. & -15. & 0 & 0 & -15. & -15. & -15. & 0 \\ 0 & -15. & -7.3625 & 0 & -15. & 90.55 & -7.3625 & 0 & 0 & -15. & -15. & -1 \\ 0 & 0 & 0 & 0 & 0 & -7.3625 & 60.55 & -7.3625 & 0 & 0 & -15. & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -7.3625 & 30.275 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -7.5 & -15. & 0 & 0 & 0 & 30. & -7.5 & 0 & 0 \\ 0 & 0 & 0 & -15. & -15. & -15. & 0 & 0 & -7.5 & 60. & -7.5 & 0 \\ 0 & 0 & 0 & 0 & -15. & -15. & -15. & 0 & 0 & -7.5 & 60. & - \\ 0 & 0 & 0 & 0 & 0 & -15. & -15. & -15. & 0 & 0 & -7.5 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & -15. & -7.5 & 0 & 0 & 0 & - \end{array}\right.$$

Essential boundary conditions

Node	dof	Value
9	T_9	110
10	T_{10}	110
11	T_{11}	110
12	T_{12}	110
13	T_{13}	110

Delete equations {9, 10, 11, 12, 13}.

$$\begin{pmatrix} 30.275 & -7.3625 & 0 & -7.5 & -15. & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -7.3625 & 60.55 & -7.3625 & -15. & -15. & -15. & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -7.3625 & 30.55 & 0 & -15. & -7.3625 & 0 & 0 & 0 & 0 & 0 & 0 \\ -7.5 & -15. & 0 & 60. & -15. & 0 & 0 & 0 & -7.5 & -15. & 0 & 0 \\ -15. & -15. & -15. & -15. & 120. & -15. & 0 & 0 & -15. & -15. & -15. & 0 \\ 0 & -15. & -7.3625 & 0 & -15. & 90.55 & -7.3625 & 0 & 0 & -15. & -15. & -15. \\ 0 & 0 & 0 & 0 & 0 & -7.3625 & 60.55 & -7.3625 & 0 & 0 & -15. & -15. & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -7.3625 & 30.275 & 0 & 0 & 0 & -15. & - \end{pmatrix}$$

Extract columns {9, 10, 11, 12, 13}.

Multiply each column by its respective known value {110, 110, 110, 110, 110}.

Move all resulting vectors to the rhs.

After adjusting for essential boundary conditions we have

$$\begin{pmatrix} 30.275 & -7.3625 & 0 & -7.5 & -15. & 0 & 0 & 0 \\ -7.3625 & 60.55 & -7.3625 & -15. & -15. & -15. & 0 & 0 \\ 0 & -7.3625 & 30.55 & 0 & -15. & -7.3625 & 0 & 0 \\ -7.5 & -15. & 0 & 60. & -15. & 0 & 0 & 0 \\ -15. & -15. & -15. & -15. & 120. & -15. & 0 & 0 \\ 0 & -15. & -7.3625 & 0 & -15. & 90.55 & -7.3625 & 0 \\ 0 & 0 & 0 & 0 & 0 & -7.3625 & 60.55 & -7.3625 \\ 0 & 0 & 0 & 0 & 0 & 0 & -7.3625 & 30.275 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \end{pmatrix} = \begin{pmatrix} 349.5 \\ 579. \\ 297.75 \\ 3157.5 \\ 6075. \\ 5810.25 \\ 5529. \\ 2764.5 \end{pmatrix}$$

Solving the final system of global equations we get

$\{T_1 = 154.962,\ T_2 = 151.228,\ T_3 = 148.673,$
$T_4 = 145.433,\ T_5 = 142.521,\ T_6 = 134.871,\ T_7 = 122.436,\ T_8 = 121.088\}$

Complete table of nodal values

	T
1	154.962
2	151.228
3	148.673
4	145.433
5	142.521
6	134.871
7	122.436
8	121.088
9	110
10	110
11	110
12	110
13	110

Solution for element 1

Coordinates of element center

$x_c = 0.0075; \qquad y_c = 0.0075$

Element dimensions: $a = 0.0075; \qquad b = 0.0075$

Interpolation functions in local element coordinates

$$\boldsymbol{N}^T = \{4444.44\,t\,s - 33.3333\,s - 33.3333\,t + 0.25,\ -4444.44\,t\,s + 33.3333\,s - 33.3333\,t + 0.25,\ 4444.44\,t\,s + 33.3333\,s + 33.3333\,t + 0.25,\ -4444.44\,t\,s - 33.3333\,s + 33.3333\,t + 0.25\}$$

Shift for global coordinates: $s = x - 0.0075; \qquad t = y - 0.0075$

Interpolation functions in global coordinates

$$\boldsymbol{N}^T = \{4444.44\,y\,x - 66.6667\,x - 66.6667\,y + 1.,\ 66.6667\,x - 4444.44\,x\,y,\ 4444.44\,x\,y,\ 66.6667\,y - 4444.44\,x\,y\}$$

Nodal values, $\boldsymbol{d}^T = \{110,\ 110,\ 142.521,\ 145.433\}$

$T(x, y) = \boldsymbol{N}^T\boldsymbol{d} = -12940.9\,x\,y + 2362.17\,y + 110.$

$\partial T/\partial x = -12940.9\,y; \qquad \partial T/\partial y = 2362.17 - 12940.9\,x$

Solution for element 2

Coordinates of element center

$x_c = 0.0075; \qquad y_c = 0.0225$

Element dimensions: $a = 0.0075; \qquad b = 0.0075$

Interpolation functions in local element coordinates

$$\boldsymbol{N}^{T} = \{4444.44\, t\, s - 33.3333\, s - 33.3333\, t + 0.25,\ -4444.44\, t\, s + 33.3333\, s - 33.3333\, t + 0.25,\ 4444.44\, t\, s + 33.3333\, s + 33.3333\, t + 0.25,\ -4444.44\, t\, s - 33.3333\, s + 33.3333\, t + 0.25\}$$

Shift for global coordinates: $s = x - 0.0075; \qquad t = y - 0.0225$

Interpolation functions in global coordinates

$$\boldsymbol{N}^{T} = \{4444.44\, y\, x - 133.333\, x - 66.6667\, y + 2.,\ 133.333\, x - 4444.44\, x\, y,\ 4444.44\, x\, y - 66.6667\, x,\ -4444.44\, y\, x + 66.6667\, x + 66.6667\, y - 1.\}$$

Nodal values, $\boldsymbol{d}^{T} = \{145.433,\ 142.521,\ 151.228,\ 154.962\}$

$T(x, y) = \boldsymbol{N}^{T}\boldsymbol{d} = -3653.37\, y\, x - 139.313\, x + 635.298\, y + 135.903$

$\partial T/\partial x = -3653.37\, y - 139.313; \qquad \partial T/\partial y = 635.298 - 3653.37\, x$

Solution for element 3

Coordinates of element center

$x_c = 0.0225; \qquad y_c = 0.0075$

Element dimensions: $a = 0.0075; \qquad b = 0.0075$

Interpolation functions in local element coordinates

$$\boldsymbol{N}^{T} = \{4444.44\, t\, s - 33.3333\, s - 33.3333\, t + 0.25,\ -4444.44\, t\, s + 33.3333\, s - 33.3333\, t + 0.25,\ 4444.44\, t\, s + 33.3333\, s + 33.3333\, t + 0.25,\ -4444.44\, t\, s - 33.3333\, s + 33.3333\, t + 0.25\}$$

Shift for global coordinates: $s = x - 0.0225; \qquad t = y - 0.0075$

Interpolation functions in global coordinates

$$\boldsymbol{N}^{T} = \{4444.44\, y\, x - 66.6667\, x - 133.333\, y + 2.,\ -4444.44\, y\, x + 66.6667\, x + 66.6667\, y - 1.,\ 4444.44\, x\, y - 66.6667\, y,\ 133.333\, y - 4444.44\, x\, y\}$$

Nodal values, $\boldsymbol{d}^{T} = \{110,\ 110,\ 134.871,\ 142.521\}$

$T(x, y) = \boldsymbol{N}^{T}\boldsymbol{d} = -34001.2\, x\, y + 2678.07\, y + 110.$

$\partial T/\partial x = -34001.2\, y; \qquad \partial T/\partial y = 2678.07 - 34001.2\, x$

Solution for element 4

Coordinates of element center

$x_c = 0.0225;\qquad y_c = 0.0225$

Element dimensions: $a = 0.0075;\qquad b = 0.0075$

Interpolation functions in local element coordinates

$$\boldsymbol{N}^{T} = \{4444.44\, t\, s - 33.3333\, s - 33.3333\, t + 0.25,\ -4444.44\, t\, s + 33.3333\, s - 33.3333\, t + 0.25,\ 4444.44\, t\, s + 33.3333\, s + 33.3333\, t + 0.25,\ -4444.44\, t\, s - 33.3333\, s + 33.3333\, t + 0.25\}$$

Shift for global coordinates: $s = x - 0.0225;\qquad t = y - 0.0225$

Interpolation functions in global coordinates

$$\boldsymbol{N}^{T} = \{4444.44\, y\, x - 133.333\, x - 133.333\, y + 4.,\ -4444.44\, y\, x + 133.333\, x + 66.6667\, y - 2.,\ 4444.44\, y\, x - 66.6667\, x - 66.6667\, y + 1.,\ -4444.44\, y\, x + 66.6667\, x + 133.333\, y - 2.\}$$

Nodal values, $\boldsymbol{d}^{T} = \{142.521,\ 134.871,\ 148.673,\ 151.228\}$

$T(x, y) = \boldsymbol{N}^{T}\boldsymbol{d} = 22645.1\, y\, x - 849.695\, x + 240.82\, y + 146.559$

$\partial T/\partial x = 22645.1\, y - 849.695;\qquad \partial T/\partial y = 22645.1\, x + 240.82$

Solution for element 5

Coordinates of element center

$x_c = 0.0375;\qquad y_c = 0.0075$

Element dimensions: $a = 0.0075;\qquad b = 0.0075$

Interpolation functions in local element coordinates

$$\boldsymbol{N}^{T} = \{4444.44\, t\, s - 33.3333\, s - 33.3333\, t + 0.25,\ -4444.44\, t\, s + 33.3333\, s - 33.3333\, t + 0.25,\ 4444.44\, t\, s + 33.3333\, s + 33.3333\, t + 0.25,\ -4444.44\, t\, s - 33.3333\, s + 33.3333\, t + 0.25\}$$

Shift for global coordinates: $s = x - 0.0375;\qquad t = y - 0.0075$

Interpolation functions in global coordinates

$$\boldsymbol{N}^{T} = \{4444.44\, y\, x - 66.6667\, x - 200.\, y + 3.,\ -4444.44\, y\, x + 66.6667\, x + 133.333\, y - 2.,\ 4444.44\, x\, y - 133.333\, y,\ 200.\, y - 4444.44\, x\, y\}$$

Nodal values, $\boldsymbol{d}^{T} = \{110,\ 110,\ 122.436,\ 134.871\}$

$T(x, y) = \boldsymbol{N}^{T}\boldsymbol{d} = -55265.\, x\, y + 3315.99\, y + 110.$

$\partial T/\partial x = -55265.\, y;\qquad \partial T/\partial y = 3315.99 - 55265.\, x$

Solution for element 6

Coordinates of element center

$x_c = 0.0525; \qquad y_c = 0.0075$

Element dimensions: $a = 0.0075; \qquad b = 0.0075$

Interpolation functions in local element coordinates

$$\boldsymbol{N}^T = \{4444.44\, t\, s - 33.3333\, s - 33.3333\, t + 0.25,\ -4444.44\, t\, s + 33.3333\, s - 33.3333\, t + 0.25,\ 4444.44\, t\, s + 33.3333\, s + 33.3333\, t + 0.25,\ -4444.44\, t\, s - 33.3333\, s + 33.3333\, t + 0.25\}$$

Shift for global coordinates: $s = x - 0.0525; \qquad t = y - 0.0075$

Interpolation functions in global coordinates

$$\boldsymbol{N}^T = \{4444.44\, y\, x - 66.6667\, x - 266.667\, y + 4.,\ -4444.44\, y\, x + 66.6667\, x + 200.\, y - 3.,\ 4444.44\, x\, y - 200.\, y,\ 266.667\, y - 4444.44\, x\, y\}$$

Nodal values, $\boldsymbol{d}^T = \{110, 110, 121.088, 122.436\}$

$T(x, y) = \boldsymbol{N}^T\boldsymbol{d} = -5991.37\, x\, y + 1098.67\, y + 110.$

$\partial T/\partial x = -5991.37\, y; \qquad \partial T/\partial y = 1098.67 - 5991.37\, x$

Solution summary

Nodal solution

	x−coord	y−coord	T
1	0	0.03	154.962
2	0.015	0.03	151.228
3	0.03	0.03	148.673
4	0	0.015	145.433
5	0.015	0.015	142.521
6	0.03	0.015	134.871
7	0.045	0.015	122.436
8	0.06	0.015	121.088
9	0	0	110
10	0.015	0	110
11	0.03	0	110
12	0.045	0	110
13	0.06	0	110

Solution at element centroids

	x−coord	y−coord	T	∂T/∂x	∂T/∂y
1	0.0075	0.0075	126.988	−97.057	2265.11
2	0.0075	0.0225	148.536	−221.514	607.898
3	0.0225	0.0075	124.348	−255.009	1913.04
4	0.0225	0.0225	144.323	−340.18	750.336
5	0.0375	0.0075	119.327	−414.488	1243.55
6	0.0525	0.0075	115.881	−44.9353	784.124

Solution derivatives

$\begin{pmatrix} -221.514 \\ 607.898 \end{pmatrix}$	$\begin{pmatrix} -340.18 \\ 750.336 \end{pmatrix}$		
$\begin{pmatrix} -97.057 \\ 2265.11 \end{pmatrix}$	$\begin{pmatrix} -255.009 \\ 1913.04 \end{pmatrix}$	$\begin{pmatrix} -414.488 \\ 1243.55 \end{pmatrix}$	$\begin{pmatrix} -44.9353 \\ 784.124 \end{pmatrix}$