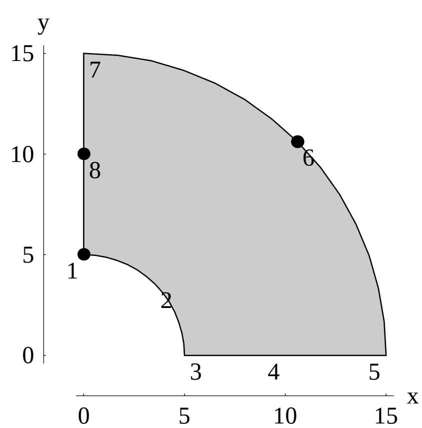
Example 7.9: Pressure Vessel, One element solution (p. 518)

To show all calculations only a single eight node element is used as shown in Figure. Because of symmetry the following boundary conditions are imposed.

At nodes 3, 4, 5: y displacement = 0

At nodes 7, 8, 1: x displacement = 0



Using k – in units the calculations are as follows.

Global equations at start of the element assembly process

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} u_1\\ v_1\\ u_2\\ v_2\\ u_3\\ v_3\\ u_4\\ v_4\\ u_5\\ v_5\\ u_6\\ v_6\\ u_7\\ v_7\\ u_8\\ v_8 \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}$$

Equations for element 1

$E = 30000;\qquad \nu = 0.3;\qquad h = 1$

Nodal coordinates

Element node	Global node number	x	y
1	1	0.	5.
2	2	3.53553	3.53553
3	3	5.	0.
4	4	10.	0.
5	5	15.	0.
6	6	10.6066	10.6066
7	7	0.	15.
8	8	0.	10.

Interpolation functions and their derivatives

$$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\} =$$

$$\left\{-\frac{1}{4}(s-1)(t-1)(s+t+1),\ \frac{1}{2}\left(s^2-1\right)(t-1),\ \frac{1}{4}(t-1)\left(-s^2+t\,s+t+1\right),\ -\frac{1}{2}(s+1)\left(t^2-1\right),\right.$$
$$\left.\frac{1}{4}(s+1)(t+1)(s+t-1),\ -\frac{1}{2}\left(s^2-1\right)(t+1),\ \frac{1}{4}(s-1)(s-t+1)(t+1),\ \frac{1}{2}(s-1)\left(t^2-1\right)\right\}$$

$$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s, \partial N_5/\partial s, \partial N_6/\partial s, \partial N_7/\partial s, \partial N_8/\partial s\} = \left\{-\frac{1}{4}(t-1)(2s+t), s(t-1), -\frac{1}{4}(2s-t)(t-1), \frac{1}{2}\left(1-t^2\right), \frac{1}{4}(t+1)(2s+t), -s(t+1), \frac{1}{4}(2s-t)(t+1), \frac{1}{2}\left(t^2-1\right)\right\}$$

$$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t, \partial N_5/\partial t, \partial N_6/\partial t, \partial N_7/\partial t, \partial N_8/\partial t\} = \left\{-\frac{1}{4}(s-1)(s+2t), \frac{1}{2}\left(s^2-1\right), -\frac{1}{4}(s+1)(s-2t), -(s+1)t, \frac{1}{4}(s+1)(s+2t), \frac{1}{2}\left(1-s^2\right), \frac{1}{4}(s-1)(s-2t), (s-1)t\right\}$$

Mapping to the master element

$$x(s,t) = \boldsymbol{N}^T \boldsymbol{x}_n = -1.03553\, t\, s^2 - 2.07107\, s^2 + 2.5\, t\, s + 5.\, s + 3.53553\, t + 7.07107$$

$$y(s,t) = \boldsymbol{N}^T \boldsymbol{y}_n = -1.03553\, t\, s^2 - 2.07107\, s^2 - 2.5\, t\, s - 5.\, s + 3.53553\, t + 7.07107$$

$$\boldsymbol{J} = \begin{pmatrix} -2.07107\, t\, s - 4.14214\, s + 2.5\, t + 5. & -1.03553\, s^2 + 2.5\, s + 3.53553 \\ -2.07107\, t\, s - 4.14214\, s - 2.5\, t - 5. & -1.03553\, s^2 - 2.5\, s + 3.53553 \end{pmatrix};$$

$$\det J = 5.17767\, t\, s^2 + 10.3553\, s^2 + 17.6777\, t + 35.3553$$

Plane strain $\boldsymbol{C} = \begin{pmatrix} 40384.6 & 17307.7 & 0 \\ 17307.7 & 40384.6 & 0 \\ 0 & 0 & 11538.5 \end{pmatrix}$

For numerical integration the Gauss quadrature points and weights are

	s	t	Weight
1	−0.774597	−0.774597	0.308642
2	−0.774597	0.	0.493827
3	−0.774597	0.774597	0.308642
4	0.	−0.774597	0.493827
5	0.	0.	0.790123
6	0.	0.774597	0.493827
7	0.774597	−0.774597	0.308642
8	0.774597	0.	0.493827
9	0.774597	0.774597	0.308642

Computation of element matrices at {−0.774597, −0.774597} with weight = 0.308642

$$\boldsymbol{J} = \begin{pmatrix} 5.02935 & 0.977722 \\ -1.09766 & 4.85071 \end{pmatrix} \qquad \det J = 25.4691$$

$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\} =$
$\{0.432379, 0.354919, -0.1, 0.0450807, -0.032379, 0.0450807, -0.1, 0.354919\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s, \partial N_5/\partial s, \partial N_6/\partial s, \partial N_7/\partial s, \partial N_8/\partial s\} =$
$\{-1.03095, 1.3746, -0.343649, 0.2, -0.130948, 0.174597, -0.0436492, -0.2\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t, \partial N_5/\partial t, \partial N_6/\partial t, \partial N_7/\partial t, \partial N_8/\partial t\} =$
$\{-1.03095, -0.2, -0.0436492, 0.174597, -0.130948, 0.2, -0.343649, 1.3746\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x, \partial N_5/\partial x, \partial N_6/\partial x, \partial N_7/\partial x, \partial N_8/\partial x\} =$
$\{-0.24078, 0.253178, -0.0673307, 0.0456156, -0.0305831, 0.0418723, -0.0231237, 0.0211513\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y, \partial N_5/\partial y, \partial N_6/\partial y, \partial N_7/\partial y, \partial N_8/\partial y\} =$
$\{-0.164003, -0.0922625, 0.00457284, 0.0267997, -0.0208311, 0.0327912, -0.0661843, 0.279117\}$

$$\boldsymbol{B} = \begin{pmatrix} -0.24078 & 0 & -0.164003 \\ 0 & -0.164003 & -0.24078 \\ 0.253178 & 0 & -0.0922625 \\ 0 & -0.0922625 & 0.253178 \\ -0.0673307 & 0 & 0.00457284 \\ 0 & 0.00457284 & -0.0673307 \\ 0.0456156 & 0 & 0.0267997 \\ 0 & 0.0267997 & 0.0456156 \\ -0.0305831 & 0 & -0.0208311 \\ 0 & -0.0208311 & -0.0305831 \\ 0.0418723 & 0 & 0.0327912 \\ 0 & 0.0327912 & 0.0418723 \\ -0.0231237 & 0 & -0.0661843 \\ 0 & -0.0661843 & -0.0231237 \\ 0.0211513 & 0 & 0.279117 \\ 0 & 0.279117 & 0.0211513 \end{pmatrix}$$

$$\boldsymbol{k} = \left(\begin{array}{ccccccc}
20844.2 & 8954.26 & -17979.8 & -743.723 & 5078.55 & 851.771 & -3885.39 \\
8954.26 & 13797.1 & -3634.26 & -725.672 & 1402.49 & 1232.37 & -1603.11 \\
-17979.8 & -3634.26 & 21120.9 & -5296.74 & -5449.85 & 720.964 & 3442.01 \\
-743.723 & -725.672 & -5296.74 & 8516.24 & 950.184 & -1680.1 & 42.8277 \\
5078.55 & 1402.49 & -5449.85 & 950.184 & 1441.06 & -69.8162 & -963.9 \\
851.771 & 1232.37 & 720.964 & -1680.1 & -69.8162 & 417.829 & -135.287 \\
-3885.39 & -1603.11 & 3442.01 & 42.8277 & -963.9 & -135.287 & 725.704 \\
-1556.48 & -2391.51 & 541.403 & 262.562 & -226.58 & -239.671 & 277.204 \\
2647.56 & 1137.34 & -2283.74 & -94.4652 & 645.061 & 108.189 & -493.51 \\
1137.34 & 1752.46 & -461.612 & -92.1724 & 178.14 & 156.532 & -203.622 \\
-3688.39 & -1650.44 & 3091. & 227.405 & -881.402 & -174.206 & 686.061 \\
-1697.07 & -2621.7 & 779.112 & 1.11193 & -283.018 & -208.113 & 305.289 \\
2752.03 & 1961.38 & -1304.67 & -1229.58 & 466.809 & 389.803 & -495.734 \\
2512.1 & 3950.82 & -2086.26 & 1407.49 & 596.693 & 45.1387 & -466.958 \\
-5768.74 & -6567.66 & -635.765 & 6144.09 & -336.332 & -1691.42 & 984.766 \\
-9458.2 & -14993.9 & 9437.39 & -7689.46 & -2548.09 & 276.018 & 1783.66
\end{array}\right.$$

Computation of element matrices at {−0.774597, 0.} with weight = 0.493827

$$\boldsymbol{J} = \begin{pmatrix} 8.20848 & 0.977722 \\ -1.79152 & 4.85071 \end{pmatrix} \qquad \det J = 41.5685$$

$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\} = \{-0.1, 0.2, -0.1, 0.112702, -0.1, 0.2, -0.1, 0.887298\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s, \partial N_5/\partial s, \partial N_6/\partial s, \partial N_7/\partial s, \partial N_8/\partial s\} =$
$\{-0.387298, 0.774597, -0.387298, 0.5, -0.387298, 0.774597, -0.387298, -0.5\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t, \partial N_5/\partial t, \partial N_6/\partial t, \partial N_7/\partial t, \partial N_8/\partial t\} =$
$\{-0.343649, -0.2, 0.0436492, 0., -0.0436492, 0.2, 0.343649, 0.\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x, \partial N_5/\partial x, \partial N_6/\partial x, \partial N_7/\partial x, \partial N_8/\partial x\} =$
$\{-0.0600051, 0.0817695, -0.0433133, 0.0583459, -0.0470757, 0.0990086, -0.030384, -0.0583459\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y, \partial N_5/\partial y, \partial N_6/\partial y, \partial N_7/\partial y, \partial N_8/\partial y\} =$
$\{-0.0587504, -0.0577128, 0.0177289, -0.0117604, 0.000490192, 0.0212747, 0.0769695, 0.0117604\}$

$$\boldsymbol{B} = \begin{pmatrix} -0.0600051 & 0 & -0.0587504 \\ 0 & -0.0587504 & -0.0600051 \\ 0.0817695 & 0 & -0.0577128 \\ 0 & -0.0577128 & 0.0817695 \\ -0.0433133 & 0 & 0.0177289 \\ 0 & 0.0177289 & -0.0433133 \\ 0.0583459 & 0 & -0.0117604 \\ 0 & -0.0117604 & 0.0583459 \\ -0.0470757 & 0 & 0.000490192 \\ 0 & 0.000490192 & -0.0470757 \\ 0.0990086 & 0 & 0.0212747 \\ 0 & 0.0212747 & 0.0990086 \\ -0.030384 & 0 & 0.0769695 \\ 0 & 0.0769695 & -0.030384 \\ -0.0583459 & 0 & 0.0117604 \\ 0 & 0.0117604 & -0.0583459 \end{pmatrix}$$

$$\boldsymbol{k} = \begin{pmatrix} 3802.45 & 2087.5 & -3264.46 & 92.517 & 1907.89 & 224.764 & -2738.72 & -561.19 \\ 2087.5 & 3714.22 & -886.54 & 1648.7 & 652.115 & -247.873 & -1050.72 & -256.47 \\ -3264.46 & -886.54 & 6331.83 & -2794.42 & -3178.43 & 1107.13 & 4115.86 & -1139.23 \\ 92.517 & 1648.7 & -2794.42 & 4344.9 & 1231.49 & -1687.1 & -1424.13 & 1692.69 \\ 1907.89 & 652.115 & -3178.43 & 1231.49 & 1629.69 & -454.706 & -2144.4 & 425.98 \\ 224.764 & -247.873 & 1107.13 & -1687.1 & -454.706 & 704.922 & 488.162 & -771.42 \\ -2738.72 & -1050.72 & 4115.86 & -1424.13 & -2144.4 & 488.162 & 2854.88 & -406.31 \\ -561.192 & -256.471 & -1139.23 & 1692.69 & 425.983 & -771.421 & -406.311 & 920.97 \\ 2334.93 & 975.655 & -3197.82 & 974.762 & 1692.4 & -301.551 & -2278.36 & 203.47 \\ 644.631 & 645.197 & 657.753 & -935.202 & -205.225 & 490.159 & 141.292 & -655.35 \\ -5221.16 & -2369. & 6420.68 & -1618.09 & -3465.75 & 405.38 & 4729.67 & -119.67 \\ -1831.31 & -2443.34 & -735.358 & 899.706 & 88.3708 & -703.059 & 165.222 & 1160.85 \\ 440.362 & -459.73 & -3111.79 & 2113.73 & 1414.2 & -981.021 & -1684.04 & 1190.65 \\ -1218.1 & -3316.9 & 2651.43 & -4271. & -1312.05 & 1442.95 & 1680.18 & -1170.3 \\ 2738.72 & 1050.72 & -4115.86 & 1424.13 & 2144.4 & -488.162 & -2854.88 & 406.31 \\ 561.192 & 256.471 & 1139.23 & -1692.69 & -425.983 & 771.421 & 406.311 & -920.97 \end{pmatrix}$$

Computation of element matrices at {−0.774597, 0.774597} with weight = 0.308642

$$\boldsymbol{J} = \begin{pmatrix} 11.3876 & 0.977722 \\ -2.48537 & 4.85071 \end{pmatrix} \qquad \text{detJ} = 57.668$$

$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\} =$
$\{-0.1, 0.0450807, -0.032379, 0.0450807, -0.1, 0.354919, 0.432379, 0.354919\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s, \partial N_5/\partial s, \partial N_6/\partial s, \partial N_7/\partial s, \partial N_8/\partial s\} =$
$\{-0.0436492, 0.174597, -0.130948, 0.2, -0.343649, 1.3746, -1.03095, -0.2\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t, \partial N_5/\partial t, \partial N_6/\partial t, \partial N_7/\partial t, \partial N_8/\partial t\} =$
$\{0.343649, -0.2, 0.130948, -0.174597, 0.0436492, 0.2, 1.03095, -1.3746\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x, \partial N_5/\partial x, \partial N_6/\partial x, \partial N_7/\partial x, \partial N_8/\partial x\} =$
$\{0.011139, 0.00606652, -0.00537101, 0.00929813, -0.0270246, 0.124243, -0.0422859, -0.0760651\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y, \partial N_5/\partial y, \partial N_6/\partial y, \partial N_7/\partial y, \partial N_8/\partial y\} =$
$\{0.0686, -0.0424539, 0.0280782, -0.0378682, 0.0144457, 0.0161884, 0.221059, -0.268049\}$

$$\boldsymbol{B} = \begin{pmatrix} 0.011139 & 0 & 0.0686 \\ 0 & 0.0686 & 0.011139 \\ 0.00606652 & 0 & -0.0424539 \\ 0 & -0.0424539 & 0.00606652 \\ -0.00537101 & 0 & 0.0280782 \\ 0 & 0.0280782 & -0.00537101 \\ 0.00929813 & 0 & -0.0378682 \\ 0 & -0.0378682 & 0.00929813 \\ -0.0270246 & 0 & 0.0144457 \\ 0 & 0.0144457 & -0.0270246 \\ 0.124243 & 0 & 0.0161884 \\ 0 & 0.0161884 & 0.124243 \\ -0.0422859 & 0 & 0.221059 \\ 0 & 0.221059 & -0.0422859 \\ -0.0760651 & 0 & -0.268049 \\ 0 & -0.268049 & -0.0760651 \end{pmatrix}$$

$$
\boldsymbol{k} = \left(\begin{array}{ccccccc}
1055.65 & 392.327 & -549.535 & -60.2104 & 352.572 & 20.6795 & -459.056 \\
392.327 & 3408.11 & 31.0827 & -2079.5 & -49.2711 & 1372.23 & 109.865 \\
-549.535 & 31.0827 & 396.599 & -132.231 & -268.228 & 99.3017 & 370.71 \\
-60.2104 & -2079.5 & -132.231 & 1303.07 & 105.225 & -863.516 & -168.782 \\
352.572 & -49.2711 & -268.228 & 105.225 & 182.646 & -77.4287 & -254.261 \\
20.6795 & 1372.23 & 99.3017 & -863.516 & -77.4287 & 572.611 & 122.196 \\
-459.056 & 109.865 & 370.71 & -168.782 & -254.261 & 122.196 & 356.645 \\
1.05331 & -1845.99 & -151.837 & 1167.16 & 116.272 & -774.531 & -180.779 \\
-12.8613 & -538.055 & -243.792 & 371.43 & 187.633 & -249.687 & -292.962 \\
-331.165 & 650.486 & 262.618 & -474.49 & -179.737 & 321.359 & 251.548 \\
1222.84 & 2662.61 & 400.629 & -1604.7 & -386.311 & 1056.8 & 704.475 \\
1805.93 & 1082.46 & -1052.99 & -339.207 & 689.652 & 189.675 & -919.869 \\
2775.8 & -387.91 & -2111.75 & 828.434 & 1437.97 & -609.595 & -2001.79 \\
162.809 & 10803.5 & 781.801 & -6798.45 & -609.595 & 4508.16 & 962.045 \\
-4385.41 & -2220.65 & 2005.37 & 660.833 & -1252.02 & -362.264 & 1576.24 \\
-1991.43 & -13391.3 & 162.258 & 8084.93 & 4.88228 & -5325.98 & -176.224
\end{array} \right.
$$

Computation of element matrices at {0., −0.774597} with weight = 0.493827

$$\boldsymbol{J} = \begin{pmatrix} 3.06351 & 3.53553 \\ -3.06351 & 3.53553 \end{pmatrix} \qquad \det J = 21.6623$$

$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\} = \{-0.1, 0.887298, -0.1, 0.2, -0.1, 0.112702, -0.1, 0.2\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s, \partial N_5/\partial s, \partial N_6/\partial s, \partial N_7/\partial s, \partial N_8/\partial s\} =$
$\{-0.343649, 0., 0.343649, 0.2, -0.0436492, 0., 0.0436492, -0.2\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t, \partial N_5/\partial t, \partial N_6/\partial t, \partial N_7/\partial t, \partial N_8/\partial t\} =$
$\{-0.387298, -0.5, -0.387298, 0.774597, -0.387298, 0.5, -0.387298, 0.774597\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x, \partial N_5/\partial x, \partial N_6/\partial x, \partial N_7/\partial x, \partial N_8/\partial x\} =$
$\{-0.11086, -0.0707107, 0.00131526, 0.142187, -0.0618963, 0.0707107, -0.0476482, 0.0769022\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y, \partial N_5/\partial y, \partial N_6/\partial y, \partial N_7/\partial y, \partial N_8/\partial y\} =$
$\{0.00131526, -0.0707107, -0.11086, 0.0769022, -0.0476482, 0.0707107, -0.0618963, 0.142187\}$

$$\boldsymbol{B} = \begin{pmatrix} -0.11086 & 0 & 0.00131526 \\ 0 & 0.00131526 & -0.11086 \\ -0.0707107 & 0 & -0.0707107 \\ 0 & -0.0707107 & -0.0707107 \\ 0.00131526 & 0 & -0.11086 \\ 0 & -0.11086 & 0.00131526 \\ 0.142187 & 0 & 0.0769022 \\ 0 & 0.0769022 & 0.142187 \\ -0.0618963 & 0 & -0.0476482 \\ 0 & -0.0476482 & -0.0618963 \\ 0.0707107 & 0 & 0.0707107 \\ 0 & 0.0707107 & 0.0707107 \\ -0.0476482 & 0 & -0.0618963 \\ 0 & -0.0618963 & -0.0476482 \\ 0.0769022 & 0 & 0.142187 \\ 0 & 0.142187 & 0.0769022 \end{pmatrix}$$

$$\boldsymbol{k} = \begin{pmatrix} 5309.58 & -44.9939 & 3375.04 & 1439.89 & -80.9891 & 2275.66 & -6797.22 & -155 \\ -44.9939 & 1517.71 & 950.359 & 927.399 & 1517.28 & -80.9891 & -1017.68 & -190 \\ 3375.04 & 950.359 & 2777.21 & 1542.9 & 927.399 & 1439.89 & -5014.69 & -224 \\ 1439.89 & 927.399 & 1542.9 & 2777.21 & 950.359 & 3375.04 & -2532.7 & -359 \\ -80.9891 & 1517.28 & 927.399 & 950.359 & 1517.71 & -44.9939 & -971.508 & -192 \\ 2275.66 & -80.9891 & 1439.89 & 3375.04 & -44.9939 & 5309.58 & -2905.96 & -365 \\ -6797.22 & -1017.68 & -5014.69 & -2532.7 & -971.508 & -2905.96 & 9463.98 & 337 \\ -1555.37 & -1901.93 & -2247.8 & -3590.19 & -1926.9 & -3659.97 & 3374.16 & 505 \\ 2956.64 & 636.927 & 2306.67 & 1226.21 & 616.83 & 1262.71 & -4254.35 & -171 \\ 967.951 & 819.891 & 1164.03 & 1995.78 & 835.362 & 2271.95 & -1841.9 & -266 \\ -3375.04 & -950.359 & -2777.21 & -1542.9 & -927.399 & -1439.89 & 5014.69 & 224 \\ -1439.89 & -927.399 & -1542.9 & -2777.21 & -950.359 & -3375.04 & 2532.7 & 359 \\ 2271.95 & 835.362 & 1995.78 & 1164.03 & 819.891 & 967.951 & -3514.38 & -176 \\ 1262.71 & 616.83 & 1226.21 & 2306.67 & 636.927 & 2956.64 & -2081.74 & -289 \\ -3659.97 & -1926.9 & -3590.19 & -2247.8 & -1901.93 & -1555.37 & 6073.48 & 359 \\ -2905.96 & -971.508 & -2532.7 & -5014.69 & -1017.68 & -6797.22 & 4473.12 & 607 \end{pmatrix}$$

Computation of element matrices at {0., 0.} with weight = 0.790123

$$\boldsymbol{J} = \begin{pmatrix} 5. & 3.53553 \\ -5. & 3.53553 \end{pmatrix} \qquad \text{detJ} = 35.3553$$

$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\} = \{-0.25, 0.5, -0.25, 0.5, -0.25, 0.5, -0.25, 0.5\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s, \partial N_5/\partial s, \partial N_6/\partial s, \partial N_7/\partial s, \partial N_8/\partial s\} = \{0., 0., 0., 0.5, 0., 0., 0., -0.5\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t, \partial N_5/\partial t, \partial N_6/\partial t, \partial N_7/\partial t, \partial N_8/\partial t\} = \{0., -0.5, 0., 0., 0., 0.5, 0., 0.\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x, \partial N_5/\partial x, \partial N_6/\partial x, \partial N_7/\partial x, \partial N_8/\partial x\} = \{0., -0.0707107, 0., 0.05, 0., 0.0707107, 0., -0.05\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y, \partial N_5/\partial y, \partial N_6/\partial y, \partial N_7/\partial y, \partial N_8/\partial y\} = \{0., -0.0707107, 0., -0.05, 0., 0.0707107, 0., 0.05\}$

$$\boldsymbol{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.0707107 & 0 & -0.0707107 \\ 0 & -0.0707107 & -0.0707107 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.05 & 0 & -0.05 \\ 0 & -0.05 & 0.05 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.0707107 & 0 & 0.0707107 \\ 0 & 0.0707107 & 0.0707107 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.05 & 0 & 0.05 \\ 0 & 0.05 & -0.05 \end{pmatrix}$$

$$\boldsymbol{k} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 7252.38 & 4029.1 & 0 & 0 & -2849. & 569.801 & 0 & 0 & -7252.38 & -4029.1 & 0 & 0 \\
0 & 0 & 4029.1 & 7252.38 & 0 & 0 & -569.801 & 2849. & 0 & 0 & -4029.1 & -7252.38 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2849. & -569.801 & 0 & 0 & 3626.19 & -2014.55 & 0 & 0 & 2849. & 569.801 & 0 & 0 \\
0 & 0 & 569.801 & 2849. & 0 & 0 & -2014.55 & 3626.19 & 0 & 0 & -569.801 & -2849. & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -7252.38 & -4029.1 & 0 & 0 & 2849. & -569.801 & 0 & 0 & 7252.38 & 4029.1 & 0 & 0 \\
0 & 0 & -4029.1 & -7252.38 & 0 & 0 & 569.801 & -2849. & 0 & 0 & 4029.1 & 7252.38 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2849. & 569.801 & 0 & 0 & -3626.19 & 2014.55 & 0 & 0 & -2849. & -569.801 & 0 & 0 \\
0 & 0 & -569.801 & -2849. & 0 & 0 & 2014.55 & -3626.19 & 0 & 0 & 569.801 & 2849. & 0 & 0
\end{pmatrix}$$

Computation of element matrices at {0., 0.774597} with weight = 0.493827

$$\boldsymbol{J} = \begin{pmatrix} 6.93649 & 3.53553 \\ -6.93649 & 3.53553 \end{pmatrix} \qquad \det J = 49.0484$$

$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\} = \{-0.1, 0.112702, -0.1, 0.2, -0.1, 0.887298, -0.1, 0.2\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s, \partial N_5/\partial s, \partial N_6/\partial s, \partial N_7/\partial s, \partial N_8/\partial s\} =$
$\{0.0436492, 0., -0.0436492, 0.2, 0.343649, 0., -0.343649, -0.2\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t, \partial N_5/\partial t, \partial N_6/\partial t, \partial N_7/\partial t, \partial N_8/\partial t\} =$
$\{0.387298, -0.5, 0.387298, -0.774597, 0.387298, 0.5, 0.387298, -0.774597\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x, \partial N_5/\partial x, \partial N_6/\partial x, \partial N_7/\partial x, \partial N_8/\partial x\} =$
$\{0.0579186, -0.0707107, 0.0516259, -0.095128, 0.0795434, 0.0707107, 0.0300011, -0.123961\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y, \partial N_5/\partial y, \partial N_6/\partial y, \partial N_7/\partial y, \partial N_8/\partial y\} =$
$\{0.0516259, -0.0707107, 0.0579186, -0.123961, 0.0300011, 0.0707107, 0.0795434, -0.095128\}$

$$
\boldsymbol{B} = \begin{pmatrix}
0.0579186 & 0 & 0.0516259 \\
0 & 0.0516259 & 0.0579186 \\
-0.0707107 & 0 & -0.0707107 \\
0 & -0.0707107 & -0.0707107 \\
0.0516259 & 0 & 0.0579186 \\
0 & 0.0579186 & 0.0516259 \\
-0.095128 & 0 & -0.123961 \\
0 & -0.123961 & -0.095128 \\
0.0795434 & 0 & 0.0300011 \\
0 & 0.0300011 & 0.0795434 \\
0.0707107 & 0 & 0.0707107 \\
0 & 0.0707107 & 0.0707107 \\
0.0300011 & 0 & 0.0795434 \\
0 & 0.0795434 & 0.0300011 \\
-0.123961 & 0 & -0.095128 \\
0 & -0.095128 & -0.123961
\end{pmatrix}
$$

$$
\boldsymbol{k} = \begin{pmatrix}
4026.22 & 2089.17 & -5026.31 & -2737.12 & 3760.5 & 2151.17 & -7177.97 & -4382.37 & 49 \\
2089.17 & 3544.59 & -2674.95 & -4715.42 & 2054.84 & 3760.5 & -4065.36 & -7799.75 & 22 \\
-5026.31 & -2674.95 & 6288.26 & 3493.48 & -4715.42 & -2737.12 & 9029.47 & 5554.52 & -60 \\
-2737.12 & -4715.42 & 3493.48 & 6288.26 & -2674.95 & -5026.31 & 5269.62 & 10454. & -29 \\
3760.5 & 2054.84 & -4715.42 & -2674.95 & 3544.59 & 2089.17 & -6810.43 & -4222.66 & 45 \\
2151.17 & 3760.5 & -2737.12 & -5026.31 & 2089.17 & 4026.22 & -4098.3 & -8395.48 & 23 \\
-7177.97 & -4065.36 & 9029.47 & 5269.62 & -6810.43 & -4098.3 & 13146.4 & 8239.13 & -84 \\
-4382.37 & -7799.75 & 5554.52 & 10454. & -4222.66 & -8395.48 & 8239.13 & 17560. & -49 \\
4939.35 & 2207.14 & -6094.68 & -2950.8 & 4502.5 & 2364.22 & -8441.01 & -4931.21 & 64 \\
1876.12 & 2802.6 & -2461.27 & -3647.04 & 1936.87 & 2847.37 & -3952.16 & -5752.56 & 16 \\
5026.31 & 2674.95 & -6288.26 & -3493.48 & 4715.42 & 2737.12 & -9029.47 & -5554.52 & 60 \\
2737.12 & 4715.42 & -3493.48 & -6288.26 & 2674.95 & 5026.31 & -5269.62 & -10454. & 29 \\
2847.37 & 1936.87 & -3647.04 & -2461.27 & 2802.6 & 1876.12 & -5547.39 & -3673.81 & 30 \\
2364.22 & 4502.5 & -2950.8 & -6094.68 & 2207.14 & 4939.35 & -4211.5 & -10442.7 & 29 \\
-8395.48 & -4222.66 & 10454. & 5554.52 & -7799.75 & -4382.37 & 14830.4 & 8970.92 & -104 \\
-4098.3 & -6810.43 & 5269.62 & 9029.47 & -4065.36 & -7177.97 & 8088.19 & 14830.4 & -42
\end{pmatrix}
$$

Computation of element matrices at $\{0.774597, -0.774597\}$ with weight = 0.308642

$$
\boldsymbol{J} = \begin{pmatrix} 1.09766 & 4.85071 \\ -5.02935 & 0.977722 \end{pmatrix} \qquad \text{detJ} = 25.4691
$$

$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\} =$
$\{-0.1, 0.354919, 0.432379, 0.354919, -0.1, 0.0450807, -0.032379, 0.0450807\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s, \partial N_5/\partial s, \partial N_6/\partial s, \partial N_7/\partial s, \partial N_8/\partial s\} =$
$\{0.343649, -1.3746, 1.03095, 0.2, 0.0436492, -0.174597, 0.130948, -0.2\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t, \partial N_5/\partial t, \partial N_6/\partial t, \partial N_7/\partial t, \partial N_8/\partial t\} =$
$\{-0.0436492, -0.2, -1.03095, 1.3746, -0.343649, 0.2, -0.130948, 0.174597\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x, \partial N_5/\partial x, \partial N_6/\partial x, \partial N_7/\partial x, \partial N_8/\partial x\} =$
$\{0.00457284, -0.0922625, -0.164003, 0.279117, -0.0661843, 0.0327912, -0.0208311, 0.0267997\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y, \partial N_5/\partial y, \partial N_6/\partial y, \partial N_7/\partial y, \partial N_8/\partial y\} =$
$\{-0.0673307, 0.253178, -0.24078, 0.0211513, -0.0231237, 0.0418723, -0.0305831, 0.0456156\}$

$$\boldsymbol{B} = \begin{pmatrix} 0.00457284 & 0 & -0.0673307 \\ 0 & -0.0673307 & 0.00457284 \\ -0.0922625 & 0 & 0.253178 \\ 0 & 0.253178 & -0.0922625 \\ -0.164003 & 0 & -0.24078 \\ 0 & -0.24078 & -0.164003 \\ 0.279117 & 0 & 0.0211513 \\ 0 & 0.0211513 & 0.279117 \\ -0.0661843 & 0 & -0.0231237 \\ 0 & -0.0231237 & -0.0661843 \\ 0.0327912 & 0 & 0.0418723 \\ 0 & 0.0418723 & 0.0327912 \\ -0.0208311 & 0 & -0.0305831 \\ 0 & -0.0305831 & -0.0208311 \\ 0.0267997 & 0 & 0.0456156 \\ 0 & 0.0456156 & 0.0267997 \end{pmatrix}$$

$$\boldsymbol{k} = \left(\begin{array}{ccccccc}
417.829 & -69.8162 & -1680.1 & 720.964 & 1232.37 & 851.771 & 276.018 \\
-69.8162 & 1441.06 & 950.184 & -5449.85 & 1402.49 & 5078.55 & -2548.09 \\
-1680.1 & 950.184 & 8516.24 & -5296.74 & -725.672 & -743.723 & -7689.46 \\
720.964 & -5449.85 & -5296.74 & 21120.9 & -3634.26 & -17979.8 & 9437.39 \\
1232.37 & 1402.49 & -725.672 & -3634.26 & 13797.1 & 8954.26 & -14993.9 \\
851.771 & 5078.55 & -743.723 & -17979.8 & 8954.26 & 20844.2 & -9458.2 \\
276.018 & -2548.09 & -7689.46 & 9437.39 & -14993.9 & -9458.2 & 24772.6 \\
-1691.42 & -336.332 & 6144.09 & -635.765 & -6567.66 & -5768.74 & 1338.69 \\
45.1387 & 596.693 & 1407.49 & -2086.26 & 3950.82 & 2512.1 & -5908.81 \\
389.803 & 466.809 & -1229.58 & -1304.67 & 1961.38 & 2752.03 & -1005.09 \\
-208.113 & -283.018 & 1.11193 & 779.112 & -2621.7 & -1697.07 & 2985.89 \\
-174.206 & -881.402 & 227.405 & 3091. & -1650.44 & -3688.39 & 1653. \\
156.532 & 178.14 & -92.1724 & -461.612 & 1752.46 & 1137.34 & -1904.47 \\
108.189 & 645.061 & -94.4652 & -2283.74 & 1137.34 & 2647.56 & -1201.35 \\
-239.671 & -226.58 & 262.562 & 541.403 & -2391.51 & -1556.48 & 2462.17 \\
-135.287 & -963.9 & 42.8277 & 3442.01 & -1603.11 & -3885.39 & 1783.66
\end{array}\right.$$

Computation of element matrices at {0.774597, 0.} with weight = 0.493827

$$\boldsymbol{J} = \begin{pmatrix} 1.79152 & 4.85071 \\ -8.20848 & 0.977722 \end{pmatrix} \qquad \det J = 41.5685$$

$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\} = \{-0.1, 0.2, -0.1, 0.887298, -0.1, 0.2, -0.1, 0.112702\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s, \partial N_5/\partial s, \partial N_6/\partial s, \partial N_7/\partial s, \partial N_8/\partial s\} =$
$\{0.387298, -0.774597, 0.387298, 0.5, 0.387298, -0.774597, 0.387298, -0.5\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t, \partial N_5/\partial t, \partial N_6/\partial t, \partial N_7/\partial t, \partial N_8/\partial t\} =$
$\{0.0436492, -0.2, -0.343649, 0., 0.343649, 0.2, -0.0436492, 0.\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x, \partial N_5/\partial x, \partial N_6/\partial x, \partial N_7/\partial x, \partial N_8/\partial x\} =$
$\{0.0177289, -0.0577128, -0.0587504, 0.0117604, 0.0769695, 0.0212747, 0.000490192, -0.0117604\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y, \partial N_5/\partial y, \partial N_6/\partial y, \partial N_7/\partial y, \partial N_8/\partial y\} =$
$\{-0.0433133, 0.0817695, -0.0600051, -0.0583459, -0.030384, 0.0990086, -0.0470757, 0.0583459\}$

$$\boldsymbol{B} = \begin{pmatrix} 0.0177289 & 0 & -0.0433133 \\ 0 & -0.0433133 & 0.0177289 \\ -0.0577128 & 0 & 0.0817695 \\ 0 & 0.0817695 & -0.0577128 \\ -0.0587504 & 0 & -0.0600051 \\ 0 & -0.0600051 & -0.0587504 \\ 0.0117604 & 0 & -0.0583459 \\ 0 & -0.0583459 & 0.0117604 \\ 0.0769695 & 0 & -0.030384 \\ 0 & -0.030384 & 0.0769695 \\ 0.0212747 & 0 & 0.0990086 \\ 0 & 0.0990086 & 0.0212747 \\ 0.000490192 & 0 & -0.0470757 \\ 0 & -0.0470757 & 0.000490192 \\ -0.0117604 & 0 & 0.0583459 \\ 0 & 0.0583459 & -0.0117604 \end{pmatrix}$$

$$\boldsymbol{k} = \begin{pmatrix} 704.922 & -454.706 & -1687.1 & 1107.13 & -247.873 & 224.764 & 771.421 & -488.16 \\ -454.706 & 1629.69 & 1231.49 & -3178.43 & 652.115 & 1907.89 & -425.983 & 2144.4 \\ -1687.1 & 1231.49 & 4344.9 & -2794.42 & 1648.7 & 92.517 & -1692.69 & 1424.13 \\ 1107.13 & -3178.43 & -2794.42 & 6331.83 & -886.54 & -3264.46 & 1139.23 & -4115.86 \\ -247.873 & 652.115 & 1648.7 & -886.54 & 3714.22 & 2087.5 & 256.471 & 1050.72 \\ 224.764 & 1907.89 & 92.517 & -3264.46 & 2087.5 & 3802.45 & 561.192 & 2738.72 \\ 771.421 & -425.983 & -1692.69 & 1139.23 & 256.471 & 561.192 & 920.977 & -406.31 \\ -488.162 & 2144.4 & 1424.13 & -4115.86 & 1050.72 & 2738.72 & -406.311 & 2854.8 \\ 1442.95 & -1312.05 & -4271. & 2651.43 & -3316.9 & -1218.1 & 1170.3 & -1680.1 \\ -981.021 & 1414.2 & 2113.73 & -3111.79 & -459.73 & 440.362 & -1190.65 & 1684.04 \\ -703.059 & 88.3708 & 899.706 & -735.358 & -2443.34 & -1831.31 & -1160.85 & -165.22 \\ 405.38 & -3465.75 & -1618.09 & 6420.68 & -2369. & -5221.16 & 119.679 & -4729.67 \\ 490.159 & -205.225 & -935.202 & 657.753 & 645.197 & 644.631 & 655.35 & -141.29 \\ -301.551 & 1692.4 & 974.762 & -3197.82 & 975.655 & 2334.93 & -203.471 & 2278.36 \\ -771.421 & 425.983 & 1692.69 & -1139.23 & -256.471 & -561.192 & -920.977 & 406.31 \\ 488.162 & -2144.4 & -1424.13 & 4115.86 & -1050.72 & -2738.72 & 406.311 & -2854.8 \end{pmatrix}$$

Computation of element matrices at {0.774597, 0.774597} with weight = 0.308642

$$\boldsymbol{J} = \begin{pmatrix} 2.48537 & 4.85071 \\ -11.3876 & 0.977722 \end{pmatrix} \qquad \text{detJ} = 57.668$$

$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\} =$
$\{-0.032379, 0.0450807, -0.1, 0.354919, 0.432379, 0.354919, -0.1, 0.0450807\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s, \partial N_5/\partial s, \partial N_6/\partial s, \partial N_7/\partial s, \partial N_8/\partial s\} =$
$\{0.130948, -0.174597, 0.0436492, 0.2, 1.03095, -1.3746, 0.343649, -0.2\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t, \partial N_5/\partial t, \partial N_6/\partial t, \partial N_7/\partial t, \partial N_8/\partial t\} =$
$\{0.130948, -0.2, 0.343649, -1.3746, 1.03095, 0.2, 0.0436492, -0.174597\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x, \partial N_5/\partial x, \partial N_6/\partial x, \partial N_7/\partial x, \partial N_8/\partial x\} =$
$\{0.0280782, -0.0424539, 0.0686, -0.268049, 0.221059, 0.0161884, 0.0144457, -0.0378682\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y, \partial N_5/\partial y, \partial N_6/\partial y, \partial N_7/\partial y, \partial N_8/\partial y\} =$
$\{-0.00537101, 0.00606652, 0.011139, -0.0760651, -0.0422859, 0.124243, -0.0270246, 0.00929813\}$

$$\boldsymbol{B} = \begin{pmatrix} 0.0280782 & 0 & -0.00537101 \\ 0 & -0.00537101 & 0.0280782 \\ -0.0424539 & 0 & 0.00606652 \\ 0 & 0.00606652 & -0.0424539 \\ 0.0686 & 0 & 0.011139 \\ 0 & 0.011139 & 0.0686 \\ -0.268049 & 0 & -0.0760651 \\ 0 & -0.0760651 & -0.268049 \\ 0.221059 & 0 & -0.0422859 \\ 0 & -0.0422859 & 0.221059 \\ 0.0161884 & 0 & 0.124243 \\ 0 & 0.124243 & 0.0161884 \\ 0.0144457 & 0 & -0.0270246 \\ 0 & -0.0270246 & 0.0144457 \\ -0.0378682 & 0 & 0.00929813 \\ 0 & 0.00929813 & -0.0378682 \end{pmatrix}$$

$$
\boldsymbol{k} = \begin{pmatrix}
572.611 & -77.4287 & -863.516 & 99.3017 & 1372.23 & 20.6795 & -5325.98 \\
-77.4287 & 182.646 & 105.225 & -268.228 & -49.2711 & 352.572 & 4.882 \\
-863.516 & 105.225 & 1303.07 & -132.231 & -2079.5 & -60.2104 & 8084.93 \\
99.3017 & -268.228 & -132.231 & 396.599 & 31.0827 & -549.535 & 162.258 \\
1372.23 & -49.2711 & -2079.5 & 31.0827 & 3408.11 & 392.327 & -13391.3 \\
20.6795 & 352.572 & -60.2104 & -549.535 & 392.327 & 1055.65 & -1991.43 \\
-5325.98 & 4.88228 & 8084.93 & 162.258 & -13391.3 & -1991.43 & 52833.9 \\
-362.264 & -1252.02 & 660.833 & 2005.37 & -2220.65 & -4385.41 & 10468.3 \\
4508.16 & -609.595 & -6798.45 & 781.801 & 10803.5 & 162.809 & -41931.4 \\
-609.595 & 1437.97 & 828.434 & -2111.75 & -387.91 & 2775.8 & 38.438 \\
189.675 & 689.652 & -339.207 & -1052.99 & 1082.46 & 1805.93 & -5059.91 \\
1056.8 & -386.311 & -1604.7 & 400.629 & 2662.61 & 1222.84 & -10512.1 \\
321.359 & -179.737 & -474.49 & 262.618 & 650.486 & -331.165 & -2361.12 \\
-249.687 & 187.633 & 371.43 & -243.792 & -538.055 & -12.8613 & 2005.87 \\
-774.531 & 116.272 & 1167.16 & -151.837 & -1845.99 & 1.05331 & 7150.92 \\
122.196 & -254.261 & -168.782 & 370.71 & 109.865 & -459.056 & -176.224
\end{pmatrix}
$$

Summing contributions from all points we get

$$
\boldsymbol{k} = \begin{pmatrix}
36733.5 & 12876.3 & -27675.8 & -81.2534 & 13375.3 & 6621.25 & -25336.9 & -1059 \\
12876.3 & 29235.1 & -3927.41 & -13841. & 7582.79 & 13375.3 & -10596.2 & -1363 \\
-27675.8 & -3927.41 & 58331.4 & -7381.3 & -13841. & -81.2534 & 7797.13 & 1135 \\
-81.2534 & -13841. & -7381.3 & 58331.4 & -3927.41 & -27675.8 & 11355.9 & 1008 \\
13375.3 & 7582.79 & -13841. & -3927.41 & 29235.1 & 12876.3 & -39273.3 & -1357 \\
6621.25 & 13375.3 & -81.2534 & -27675.8 & 12876.3 & 36733.5 & -17417.6 & -2125 \\
-25336.9 & -10596.2 & 7797.13 & 11355.9 & -39273.3 & -17417.6 & 108701. & 2068 \\
-10596.2 & -13639.6 & 11355.9 & 10088.9 & -13571.5 & -21256.5 & 20689.5 & 5760 \\
18861.9 & 3094.06 & -19175.3 & 874.112 & 19081.9 & 4640.68 & -62430.1 & -1160 \\
3094.06 & 9989.62 & 874.112 & -9681.35 & 3679.14 & 12055.6 & -7762.13 & -1982 \\
-6756.94 & 862.763 & -5843.92 & -13070.1 & -4928.03 & 862.763 & 1719.55 & -1135 \\
862.763 & -4928.03 & -13070.1 & -5843.92 & 862.763 & -6756.94 & -11355.9 & -1960 \\
12055.6 & 3679.14 & -9681.35 & 874.112 & 9989.62 & 3094.06 & -16853.6 & -351 \\
4640.68 & 19081.9 & 874.112 & -19175.3 & 3094.06 & 18861.9 & -3516.93 & -1903 \\
-21256.5 & -13571.5 & 10088.9 & 11355.9 & -13639.6 & -10596.2 & 25676. & 1860 \\
-17417.6 & -39273.3 & 11355.9 & 7797.13 & -10596.2 & -25336.9 & 18603.3 & 2567
\end{pmatrix}
$$

$\boldsymbol{r}^{\mathrm{T}} = (\,0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0\,)$

Computation of element matrices resulting from NBC

NBC on side 1 with $\{q_n, q_t\} = \{-20, 0\}$

$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\}_c = \left\{\frac{1-a}{2} + \frac{1}{2}(a^2-1), 1-a^2, \frac{a+1}{2} + \frac{1}{2}(a^2-1), 0, 0, 0, 0, 0\right\}$

$x(a) = -1.03553\, a^2 + 2.5\, a + 3.53553;$ $\quad y(a) = -1.03553\, a^2 - 2.5\, a + 3.53553$

$dx/da = 2.5 - 2.07107\, a;$ $\quad dy/da = -2.07107\, a - 2.5$

$J_c = \sqrt{(-2.07107\, a - 2.5)^2 + (2.07107\, a - 2.5)^2}$

Gauss point = −0.774597; Weight = 0.555556; $J_c = 4.20086$

$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\}_c = \{0.687298, 0.4, -0.0872983, 0, 0, 0, 0, 0\}$

$\boldsymbol{r}_q^T = (\,6.84059 \quad 31.3427 \quad 3.98115 \quad 18.2411 \quad -0.868868 \quad -3.98104 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\,)$

Gauss point = 0.; Weight = 0.888889; $J_c = 3.53553$

$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\}_c = \{0., 1., 0., 0, 0, 0, 0, 0\}$

$\boldsymbol{r}_q^T = (\,0 \quad 0 \quad 44.4444 \quad 44.4444 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\,)$

Gauss point = 0.774597; Weight = 0.555556; $J_c = 4.20086$

$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\}_c = \{-0.0872983, 0.4, 0.687298, 0, 0, 0, 0, 0\}$

$\boldsymbol{r}_q^T = (\,-3.98104 \quad -0.868868 \quad 18.2411 \quad 3.98115 \quad 31.3427 \quad 6.84059 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\,)$

Summing contributions from all Gauss points

$\boldsymbol{r}_q^T = (\,2.85955 \quad 30.4738 \quad 66.6667 \quad 66.6667 \quad 30.4738 \quad 2.85955 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\,)$

Complete element equations for element 1

$$\left(\begin{array}{cccccccc}
36733.5 & 12876.3 & -27675.8 & -81.2534 & 13375.3 & 6621.25 & -25336.9 & -10596.2 \\
12876.3 & 29235.1 & -3927.41 & -13841. & 7582.79 & 13375.3 & -10596.2 & -13639.6 \\
-27675.8 & -3927.41 & 58331.4 & -7381.3 & -13841. & -81.2534 & 7797.13 & 11355.9 \\
-81.2534 & -13841. & -7381.3 & 58331.4 & -3927.41 & -27675.8 & 11355.9 & 10088.9 \\
13375.3 & 7582.79 & -13841. & -3927.41 & 29235.1 & 12876.3 & -39273.3 & -13571.5 \\
6621.25 & 13375.3 & -81.2534 & -27675.8 & 12876.3 & 36733.5 & -17417.6 & -21256.5 \\
-25336.9 & -10596.2 & 7797.13 & 11355.9 & -39273.3 & -17417.6 & 108701. & 20689.5 \\
-10596.2 & -13639.6 & 11355.9 & 10088.9 & -13571.5 & -21256.5 & 20689.5 & 57600.7 \\
18861.9 & 3094.06 & -19175.3 & 874.112 & 19081.9 & 4640.68 & -62430.1 & -11608.3 \\
3094.06 & 9989.62 & 874.112 & -9681.35 & 3679.14 & 12055.6 & -7762.13 & -19829.7 \\
-6756.94 & 862.763 & -5843.92 & -13070.1 & -4928.03 & 862.763 & 1719.55 & -11355.9 \\
862.763 & -4928.03 & -13070.1 & -5843.92 & 862.763 & -6756.94 & -11355.9 & -19605.6 \\
12055.6 & 3679.14 & -9681.35 & 874.112 & 9989.62 & 3094.06 & -16853.6 & -3516.93 \\
4640.68 & 19081.9 & 874.112 & -19175.3 & 3094.06 & 18861.9 & -3516.93 & -19034.2 \\
-21256.5 & -13571.5 & 10088.9 & 11355.9 & -13639.6 & -10596.2 & 25676. & 18603.3 \\
-17417.6 & -39273.3 & 11355.9 & 7797.13 & -10596.2 & -25336.9 & 18603.3 & 25676.
\end{array}\right.$$

The element contributes to {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \end{pmatrix}$

and to a global matrix: $\begin{pmatrix} [1, 1] & [1, 2] & [1, 3] & [1, 4] & [1, 5] & [1, 6] & [1, 7] & [1, 8] & [1, 9] \\ [2, 1] & [2, 2] & [2, 3] & [2, 4] & [2, 5] & [2, 6] & [2, 7] & [2, 8] & [2, 9] \\ [3, 1] & [3, 2] & [3, 3] & [3, 4] & [3, 5] & [3, 6] & [3, 7] & [3, 8] & [3, 9] \\ [4, 1] & [4, 2] & [4, 3] & [4, 4] & [4, 5] & [4, 6] & [4, 7] & [4, 8] & [4, 9] \\ [5, 1] & [5, 2] & [5, 3] & [5, 4] & [5, 5] & [5, 6] & [5, 7] & [5, 8] & [5, 9] \\ [6, 1] & [6, 2] & [6, 3] & [6, 4] & [6, 5] & [6, 6] & [6, 7] & [6, 8] & [6, 9] \\ [7, 1] & [7, 2] & [7, 3] & [7, 4] & [7, 5] & [7, 6] & [7, 7] & [7, 8] & [7, 9] \\ [8, 1] & [8, 2] & [8, 3] & [8, 4] & [8, 5] & [8, 6] & [8, 7] & [8, 8] & [8, 9] \\ [9, 1] & [9, 2] & [9, 3] & [9, 4] & [9, 5] & [9, 6] & [9, 7] & [9, 8] & [9, 9] \\ [10, 1] & [10, 2] & [10, 3] & [10, 4] & [10, 5] & [10, 6] & [10, 7] & [10, 8] & [10, 9] \\ [11, 1] & [11, 2] & [11, 3] & [11, 4] & [11, 5] & [11, 6] & [11, 7] & [11, 8] & [11, 9] \\ [12, 1] & [12, 2] & [12, 3] & [12, 4] & [12, 5] & [12, 6] & [12, 7] & [12, 8] & [12, 9] \\ [13, 1] & [13, 2] & [13, 3] & [13, 4] & [13, 5] & [13, 6] & [13, 7] & [13, 8] & [13, 9] \\ [14, 1] & [14, 2] & [14, 3] & [14, 4] & [14, 5] & [14, 6] & [14, 7] & [14, 8] & [14, 9] \\ [15, 1] & [15, 2] & [15, 3] & [15, 4] & [15, 5] & [15, 6] & [15, 7] & [15, 8] & [15, 9] \\ [16, 1] & [16, 2] & [16, 3] & [16, 4] & [16, 5] & [16, 6] & [16, 7] & [16, 8] & [16, 9] \end{pmatrix}$

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 36733.5 & 12876.3 & -27675.8 & -81.2534 & 13375.3 & 6621.25 & -25336.9 & -10596.2 \\ 12876.3 & 29235.1 & -3927.41 & -13841. & 7582.79 & 13375.3 & -10596.2 & -13639.6 \\ -27675.8 & -3927.41 & 58331.4 & -7381.3 & -13841. & -81.2534 & 7797.13 & 11355.9 \\ -81.2534 & -13841. & -7381.3 & 58331.4 & -3927.41 & -27675.8 & 11355.9 & 10088.9 \\ 13375.3 & 7582.79 & -13841. & -3927.41 & 29235.1 & 12876.3 & -39273.3 & -13571.5 \\ 6621.25 & 13375.3 & -81.2534 & -27675.8 & 12876.3 & 36733.5 & -17417.6 & -21256.5 \\ -25336.9 & -10596.2 & 7797.13 & 11355.9 & -39273.3 & -17417.6 & 108701. & 20689.5 \\ -10596.2 & -13639.6 & 11355.9 & 10088.9 & -13571.5 & -21256.5 & 20689.5 & 57600.7 \\ 18861.9 & 3094.06 & -19175.3 & 874.112 & 19081.9 & 4640.68 & -62430.1 & -11608.3 \\ 3094.06 & 9989.62 & 874.112 & -9681.35 & 3679.14 & 12055.6 & -7762.13 & -19829.7 \\ -6756.94 & 862.763 & -5843.92 & -13070.1 & -4928.03 & 862.763 & 1719.55 & -11355.9 \\ 862.763 & -4928.03 & -13070.1 & -5843.92 & 862.763 & -6756.94 & -11355.9 & -19605.6 \\ 12055.6 & 3679.14 & -9681.35 & 874.112 & 9989.62 & 3094.06 & -16853.6 & -3516.93 \\ 4640.68 & 19081.9 & 874.112 & -19175.3 & 3094.06 & 18861.9 & -3516.93 & -19034.2 \\ -21256.5 & -13571.5 & 10088.9 & 11355.9 & -13639.6 & -10596.2 & 25676. & 18603.3 \\ -17417.6 & -39273.3 & 11355.9 & 7797.13 & -10596.2 & -25336.9 & 18603.3 & 25676. \end{pmatrix}$$

Essential boundary conditions

Node	dof	Value
1	u_1	0
3	v_3	0
4	v_4	0
5	v_5	0
7	u_7	0
8	u_8	0

Remove {1, 6, 8, 10, 13, 15} rows and columns.

After adjusting for essential boundary conditions we have

$$\left(\begin{array}{ccccccccc}
29235.1 & -3927.41 & -13841. & 7582.79 & -10596.2 & 3094.06 & 862.763 & -4928.03 & \\
-3927.41 & 58331.4 & -7381.3 & -13841. & 7797.13 & -19175.3 & -5843.92 & -13070.1 & \\
-13841. & -7381.3 & 58331.4 & -3927.41 & 11355.9 & 874.112 & -13070.1 & -5843.92 & - \\
7582.79 & -13841. & -3927.41 & 29235.1 & -39273.3 & 19081.9 & -4928.03 & 862.763 & \\
-10596.2 & 7797.13 & 11355.9 & -39273.3 & 108701. & -62430.1 & 1719.55 & -11355.9 & \\
3094.06 & -19175.3 & 874.112 & 19081.9 & -62430.1 & 53178.9 & -1545.83 & 11827.3 & \\
862.763 & -5843.92 & -13070.1 & -4928.03 & 1719.55 & -1545.83 & 42911. & 14248. & \\
-4928.03 & -13070.1 & -5843.92 & 862.763 & -11355.9 & 11827.3 & 14248. & 42911. & \\
19081.9 & 874.112 & -19175.3 & 3094.06 & -3516.93 & -1981.59 & 11827.3 & -1545.83 & \\
-39273.3 & 11355.9 & 7797.13 & -10596.2 & 18603.3 & -3516.93 & -11355.9 & 1719.55 & -
\end{array}\right.$$

Solving the final system of global equations we get

$$\{v_1 = 0.00494694,\ u_2 = 0.0033909,\ v_2 = 0.0033909,\ u_3 = 0.00494694,\ u_4 = 0.00271213,$$
$$u_5 = 0.00225656,\ u_6 = 0.00152523,\ v_6 = 0.00152523,\ v_7 = 0.00225656,\ v_8 = 0.00271213\}$$

Complete table of nodal values

	u	v
1	0	0.00494694
2	0.0033909	0.0033909
3	0.00494694	0
4	0.00271213	0
5	0.00225656	0
6	0.00152523	0.00152523
7	0	0.00225656
8	0	0.00271213

Computation of reactions

Equation numbers of dof with specified values: {1, 6, 8, 10, 13, 15}

Extracting equations {1, 6, 8, 10, 13, 15} from the global system we have

$$\left(\begin{array}{ccccccccc}
36733.5 & 12876.3 & -27675.8 & -81.2534 & 13375.3 & 6621.25 & -25336.9 & -10596.2 & 188 \\
6621.25 & 13375.3 & -81.2534 & -27675.8 & 12876.3 & 36733.5 & -17417.6 & -21256.5 & 46 \\
-10596.2 & -13639.6 & 11355.9 & 10088.9 & -13571.5 & -21256.5 & 20689.5 & 57600.7 & -116 \\
3094.06 & 9989.62 & 874.112 & -9681.35 & 3679.14 & 12055.6 & -7762.13 & -19829.7 & -33 \\
12055.6 & 3679.14 & -9681.35 & 874.112 & 9989.62 & 3094.06 & -16853.6 & -3516.93 & 110 \\
-21256.5 & -13571.5 & 10088.9 & 11355.9 & -13639.6 & -10596.2 & 25676. & 18603.3 & -190
\end{array}\right.$$

Substituting the nodal values and re−arranging

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{pmatrix} = \left(\begin{array}{cccccccc}
36733.5 & 12876.3 & -27675.8 & -81.2534 & 13375.3 & 6621.25 & -25336.9 & -10596.2 \\
6621.25 & 13375.3 & -81.2534 & -27675.8 & 12876.3 & 36733.5 & -17417.6 & -21256.5 \\
-10596.2 & -13639.6 & 11355.9 & 10088.9 & -13571.5 & -21256.5 & 20689.5 & 57600.7 \\
3094.06 & 9989.62 & 874.112 & -9681.35 & 3679.14 & 12055.6 & -7762.13 & -19829.7 \\
12055.6 & 3679.14 & -9681.35 & 874.112 & 9989.62 & 3094.06 & -16853.6 & -3516.9 \\
-21256.5 & -13571.5 & 10088.9 & 11355.9 & -13639.6 & -10596.2 & 25676. & 18603.3
\end{array}\right.$$

Carrying out computations, the reactions are as follows.

Label	dof	Reaction
R_1	u_1	−39.0268
R_2	v_3	−39.0268
R_3	v_4	−52.5151
R_4	v_5	−8.4581
R_5	u_7	−8.4581
R_6	u_8	−52.5151

Sum of Reactions

dof: u	−100.
dof: v	−100.

Solution for element 1

Element nodal displacements

Element node	Global node number	u	v
1	1	0	0.00494694
2	2	0.0033909	0.0033909
3	3	0.00494694	0
4	4	0.00271213	0
5	5	0.00225656	0
6	6	0.00152523	0.00152523
7	7	0	0.00225656
8	8	0	0.00271213

$\boldsymbol{d}^T = (0 \quad 0.00494694 \quad 0.0033909 \quad 0.0033909 \quad 0.00494694 \quad 0 \quad 0.00271213 \quad 0 \quad 0.00225656 \quad 0 \quad 0.00152523 \quad 0.$

$E = 30000; \qquad \nu = 0.3; \qquad h = 1$

Plane strain $\boldsymbol{C} = \begin{pmatrix} 40384.6 & 17307.7 & 0 \\ 17307.7 & 40384.6 & 0 \\ 0 & 0 & 11538.5 \end{pmatrix}$

Interpolation functions and their derivatives

$$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\} =$$

$$\left\{-\frac{1}{4}(s-1)(t-1)(s+t+1),\ \frac{1}{2}\left(s^2-1\right)(t-1),\ \frac{1}{4}(t-1)\left(-s^2+t\,s+t+1\right),\ -\frac{1}{2}(s+1)\left(t^2-1\right),\right.$$
$$\left.\frac{1}{4}(s+1)(t+1)(s+t-1),\ -\frac{1}{2}\left(s^2-1\right)(t+1),\ \frac{1}{4}(s-1)(s-t+1)(t+1),\ \frac{1}{2}(s-1)\left(t^2-1\right)\right\}$$

$$\{\partial N_1/\partial s,\ \partial N_2/\partial s,\ \partial N_3/\partial s,\ \partial N_4/\partial s,\ \partial N_5/\partial s,\ \partial N_6/\partial s,\ \partial N_7/\partial s,\ \partial N_8/\partial s\} = \Big\{-\frac{1}{4}(t-1)(2s+t),\ s(t-1),\ -\frac{1}{4}(2s-t)(t-1),\ \frac{1}{2}\left(1-t^2\right),\ \frac{1}{4}(t+1)(2s+t),\ -s(t+1),\ \frac{1}{4}(2s-t)(t+1),\ \frac{1}{2}\left(t^2-1\right)\Big\}$$

$$\{\partial N_1/\partial s,\ \partial N_2/\partial s,\ \partial N_3/\partial s,\ \partial N_4/\partial s,\ \partial N_5/\partial s,\ \partial N_6/\partial s,\ \partial N_7/\partial s,\ \partial N_8/\partial s\} = \Big\{-\frac{1}{4}(s-1)(s+2t),\ \frac{1}{2}\left(s^2-1\right),\ -\frac{1}{4}(s+1)(s-2t),\ -(s+1)t,\ \frac{1}{4}(s+1)(s+2t),\ \frac{1}{2}\left(1-s^2\right),\ \frac{1}{4}(s-1)(s-2t),\ (s-1)t\Big\}$$

Nodal coordinates

Element node	Global node number	x	y
1	1	0.	5.
2	2	3.53553	3.53553
3	3	5.	0.
4	4	10.	0.
5	5	15.	0.
6	6	10.6066	10.6066
7	7	0.	15.
8	8	0.	10.

Mapping to the master element

$$x(s,t) = 1.76777\left(1-s^2\right)(1-t) + 5.3033\left(1-s^2\right)(t+1) + 5.(s+1)\left(1-t^2\right) + 5.\left(\frac{1}{4}(s+1)(1-t) - \frac{1}{4}\left(1-s^2\right)(1-t) - \frac{1}{4}(s+1)\left(1-t^2\right)\right) + 15.\left(\frac{1}{4}(s+1)(t+1) - \frac{1}{4}\left(1-s^2\right)(t+1) - \frac{1}{4}(s+1)\left(1-t^2\right)\right)$$

$$y(s,t) = 1.76777\left(1-s^2\right)(1-t) + 5.3033\left(1-s^2\right)(t+1) + 5.(1-s)\left(1-t^2\right) + 5.\left(\frac{1}{4}(1-s)(1-t) - \frac{1}{4}\left(1-s^2\right)(1-t) - \frac{1}{4}(1-s)\left(1-t^2\right)\right) + 15.\left(\frac{1}{4}(1-s)(t+1) - \frac{1}{4}\left(1-s^2\right)(t+1) - \frac{1}{4}(1-s)\left(1-t^2\right)\right)$$

$$\boldsymbol{J} = \left(\begin{matrix} -3.53553\,s\,(1-t) - 10.6066\,s\,(t+1) + 5.(1-t^2) + 5.\left(\frac{1}{2}s(1-t) + \frac{1-t}{4} + \frac{1}{4}(t^2-1)\right) + 15.\left(\frac{1}{2}s\,(\right. \\ -3.53553\,s\,(1-t) - 10.6066\,s\,(t+1) - 5.(1-t^2) + 5.\left(\frac{1}{2}s(1-t) + \frac{t-1}{4} + \frac{1}{4}(1-t^2)\right) + 15.\left(\frac{1}{4}(-t- \right. \end{matrix}\right.$$

Element solution at $\{s, t\} = \{0, 0\} \Longrightarrow \{x, y\} = \{7.07107, 7.07107\}$

$$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\} = \left\{-\frac{1}{4}, \frac{1}{2}, -\frac{1}{4}, \frac{1}{2}, -\frac{1}{4}, \frac{1}{2}, -\frac{1}{4}, \frac{1}{2}\right\}$$

$$\{\partial N_1/\partial s,\ \partial N_2/\partial s,\ \partial N_3/\partial s,\ \partial N_4/\partial s,\ \partial N_5/\partial s,\ \partial N_6/\partial s,\ \partial N_7/\partial s,\ \partial N_8/\partial s\} = \left\{0, 0, 0, \frac{1}{2}, 0, 0, 0, -\frac{1}{2}\right\}$$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t, \partial N_5/\partial t, \partial N_6/\partial t, \partial N_7/\partial t, \partial N_8/\partial t\} = \left\{0, -\frac{1}{2}, 0, 0, 0, \frac{1}{2}, 0, 0\right\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x, \partial N_5/\partial x, \partial N_6/\partial x, \partial N_7/\partial x, \partial N_8/\partial x\} =$
$\{0, -0.0707107, 0, 0.05, 0, 0.0707107, 0, -0.05\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y, \partial N_5/\partial y, \partial N_6/\partial y, \partial N_7/\partial y, \partial N_8/\partial y\} =$
$\{0, -0.0707107, 0, -0.05, 0, 0.0707107, 0, 0.05\}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & -0.0707107 & 0 & 0 & 0 & 0.05 & 0 & 0 & 0 & 0.0707107 & 0 & 0 & 0 & -0. \\ 0 & 0 & 0 & -0.0707107 & 0 & 0 & 0 & -0.05 & 0 & 0 & 0 & 0.0707107 & 0 & 0 & 0 \\ 0 & 0 & -0.0707107 & -0.0707107 & 0 & 0 & -0.05 & 0.05 & 0 & 0 & 0.0707107 & 0.0707107 & 0 & 0 & 0. \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{d} = (\, 3.6836 \times 10^{-6} \quad 3.6836 \times 10^{-6} \quad -0.000535058 \,)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\, 0.212515 \quad 0.212515 \quad -6.17375 \,)$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{\mathrm{T}} = (\, 3.6836 \times 10^{-6} \quad 3.6836 \times 10^{-6} \quad 0 \quad -0.000535058 \quad 0 \quad 0 \,)$

$\boldsymbol{\sigma}^{\mathrm{T}} = (\, 0.212515 \quad 0.212515 \quad 0.127509 \quad -6.17375 \quad 0 \quad 0 \,)$

Substituting these stress components into appropriate formulas

Principal stresses = $(\, 6.38626 \quad 0.127509 \quad -5.96123 \,)$

Effective stress (von Mises) = 10.6936

Element solution at $\{s, t\} = \{-1, -1\} \Longrightarrow \{x, y\} = \{0., 5.\}$

$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\} = \{1, 0, 0, 0, 0, 0, 0, 0\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s, \partial N_5/\partial s, \partial N_6/\partial s, \partial N_7/\partial s, \partial N_8/\partial s\} = \left\{-\frac{3}{2}, 2, -\frac{1}{2}, 0, 0, 0, 0, 0\right\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t, \partial N_5/\partial t, \partial N_6/\partial t, \partial N_7/\partial t, \partial N_8/\partial t\} = \left\{-\frac{3}{2}, 0, 0, 0, 0, 0, -\frac{1}{2}, 2\right\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x, \partial N_5/\partial x, \partial N_6/\partial x, \partial N_7/\partial x, \partial N_8/\partial x\} =$
$\{-0.356302, 0.437535, -0.109384, 0, 0, 0, -0.00938363, 0.0375345\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y, \partial N_5/\partial y, \partial N_6/\partial y, \partial N_7/\partial y, \partial N_8/\partial y\} =$
$\{-0.3, 0., 0., 0, 0, 0, -0.1, 0.4\}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -0.356302 & 0 & 0.437535 & 0 & -0.109384 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.00 \\ 0 & -0.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.3 & -0.356302 & 0 & 0.437535 & 0 & -0.109384 & 0 & 0 & 0 & 0 & 0 & 0 & -0.1 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{d} = (\, 0.00094252 \quad -0.000624887 \quad -0.000198345 \,)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\,27.248 \quad -8.92296 \quad -2.2886\,)$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{\mathrm{T}} = (\,0.00094252 \quad -0.000624887 \quad 0 \quad -0.000198345 \quad 0 \quad 0\,)$

$\boldsymbol{\sigma}^{\mathrm{T}} = (\,27.248 \quad -8.92296 \quad 5.4975 \quad -2.2886 \quad 0 \quad 0\,)$

Substituting these stress components into appropriate formulas

Principal stresses = $(\,27.3922 \quad 5.4975 \quad -9.06718\,)$

Effective stress (von Mises) = 31.7867

Element solution at $\{s, t\} = \{-1, 1\} \Longrightarrow \{x, y\} = \{0., 15.\}$

$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\} = \{0, 0, 0, 0, 0, 0, 1, 0\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s, \partial N_5/\partial s, \partial N_6/\partial s, \partial N_7/\partial s, \partial N_8/\partial s\} = \left\{0, 0, 0, 0, -\frac{1}{2}, 2, -\frac{3}{2}, 0\right\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t, \partial N_5/\partial t, \partial N_6/\partial t, \partial N_7/\partial t, \partial N_8/\partial t\} = \left\{\frac{1}{2}, 0, 0, 0, 0, 0, \frac{3}{2}, -2\right\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x, \partial N_5/\partial x, \partial N_6/\partial x, \partial N_7/\partial x, \partial N_8/\partial x\} =$
$\{0.00938363, 0, 0, 0, -0.0364612, 0.145845, -0.0812327, -0.0375345\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y, \partial N_5/\partial y, \partial N_6/\partial y, \partial N_7/\partial y, \partial N_8/\partial y\} =$
$\{0.1, 0, 0, 0, 0., 0., 0.3, -0.4\}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 0.00938363 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.0364612 & 0 & 0.145845 & 0 & - \\ 0 & 0.1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.1 & 0.00938363 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.0364612 & 0 & 0.145845 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{d} = (\,0.00014017 \quad 0.0000868105 \quad -0.0000162379\,)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\,7.16319 \quad 5.93182 \quad -0.18736\,)$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{\mathrm{T}} = (\,0.00014017 \quad 0.0000868105 \quad 0 \quad -0.0000162379 \quad 0 \quad 0\,)$

$\boldsymbol{\sigma}^{\mathrm{T}} = (\,7.16319 \quad 5.93182 \quad 3.9285 \quad -0.18736 \quad 0 \quad 0\,)$

Substituting these stress components into appropriate formulas

Principal stresses = $(\,7.19107 \quad 5.90395 \quad 3.9285\,)$

Effective stress (von Mises) = 2.84635

Element solution at $\{s, t\} = \{1, -1\} \Longrightarrow \{x, y\} = \{5., 0.\}$

$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\} = \{0, 0, 1, 0, 0, 0, 0, 0\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s, \partial N_5/\partial s, \partial N_6/\partial s, \partial N_7/\partial s, \partial N_8/\partial s\} = \left\{\frac{1}{2}, -2, \frac{3}{2}, 0, 0, 0, 0, 0\right\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t, \partial N_5/\partial t, \partial N_6/\partial t, \partial N_7/\partial t, \partial N_8/\partial t\} = \left\{0, 0, -\frac{3}{2}, 2, -\frac{1}{2}, 0, 0, 0\right\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x, \partial N_5/\partial x, \partial N_6/\partial x, \partial N_7/\partial x, \partial N_8/\partial x\} =$
$\{0., 0., -0.3, 0.4, -0.1, 0, 0, 0\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y, \partial N_5/\partial y, \partial N_6/\partial y, \partial N_7/\partial y, \partial N_8/\partial y\} =$
$\{-0.109384, 0.437535, -0.356302, 0.0375345, -0.00938363, 0, 0, 0\}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 0 & -0.3 & 0 & 0.4 & 0 & \cdot \\ 0 & -0.109384 & 0 & 0.437535 & 0 & -0.356302 & 0 & 0.0375345 & \\ -0.109384 & 0 & 0.437535 & 0 & -0.356302 & -0.3 & 0.0375345 & 0.4 & \cdot \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{d} = (\ -0.000624887 \quad 0.00094252 \quad -0.000198345\ )$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\ -8.92296 \quad 27.248 \quad -2.2886\ )$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{\mathrm{T}} = (\ -0.000624887 \quad 0.00094252 \quad 0 \quad -0.000198345 \quad 0 \quad 0\ )$

$\boldsymbol{\sigma}^{\mathrm{T}} = (\ -8.92296 \quad 27.248 \quad 5.4975 \quad -2.2886 \quad 0 \quad 0\ )$

Substituting these stress components into appropriate formulas

Principal stresses = $(\ 27.3922 \quad 5.4975 \quad -9.06718\ )$

Effective stress (von Mises) = 31.7867

Element solution at $\{s, t\} = \{1, 1\} \Longrightarrow \{x, y\} = \{15., 0.\}$

$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\} = \{0, 0, 0, 0, 1, 0, 0, 0\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s, \partial N_5/\partial s, \partial N_6/\partial s, \partial N_7/\partial s, \partial N_8/\partial s\} = \left\{0, 0, 0, 0, \frac{3}{2}, -2, \frac{1}{2}, 0\right\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t, \partial N_5/\partial t, \partial N_6/\partial t, \partial N_7/\partial t, \partial N_8/\partial t\} = \left\{0, 0, \frac{1}{2}, -2, \frac{3}{2}, 0, 0, 0\right\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x, \partial N_5/\partial x, \partial N_6/\partial x, \partial N_7/\partial x, \partial N_8/\partial x\} =$
$\{0, 0, 0.1, -0.4, 0.3, 0., 0., 0\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y, \partial N_5/\partial y, \partial N_6/\partial y, \partial N_7/\partial y, \partial N_8/\partial y\} =$
$\{0, 0, 0.00938363, -0.0375345, -0.0812327, 0.145845, -0.0364612, 0\}$

$$B^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0.1 & 0 & -0.4 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.00938363 & 0 & -0.0375345 & 0 & -0.0812327 & 0 \\ 0 & 0 & 0 & 0 & 0.00938363 & 0.1 & -0.0375345 & -0.4 & -0.0812327 & 0.3 & 0. \end{pmatrix}$$

In−plane strain components, $\epsilon = B^{\mathrm{T}} d = (\, 0.0000868105 \quad 0.00014017 \quad -0.0000162379 \,)$

In−plane stress components, $\sigma = C\epsilon = (\, 5.93182 \quad 7.16319 \quad -0.18736 \,)$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\epsilon^{\mathrm{T}} = (\, 0.0000868105 \quad 0.00014017 \quad 0 \quad -0.0000162379 \quad 0 \quad 0 \,)$

$\sigma^{\mathrm{T}} = (\, 5.93182 \quad 7.16319 \quad 3.9285 \quad -0.18736 \quad 0 \quad 0 \,)$

Substituting these stress components into appropriate formulas

Principal stresses = (7.19107   5.90395   3.9285)

Effective stress (von Mises) = 2.84635

Solution summary

Nodal solution

	x	y	u	v
1	0.	5.	0	0.00494694
2	3.53553	3.53553	0.0033909	0.0033909
3	5.	0.	0.00494694	0
4	10.	0.	0.00271213	0
5	15.	0.	0.00225656	0
6	10.6066	10.6066	0.00152523	0.00152523
7	0.	15.	0	0.00225656
8	0.	10.	0	0.00271213

Solution at selected points on elements

	Coord	Disp	Stresses	Principal stresses	Effective Stress
1	7.07107 7.07107	0.00201325 0.00201325	0.212515 0.212515 0.127509 −6.17375 0 0	6.38626 0.127509 −5.96123	10.6936

Support reactions

Node	dof	Reaction
1	1	-39.0268
3	2	-39.0268
4	2	-52.5151
5	2	-8.4581
7	1	-8.4581
8	1	-52.5151

Sum of applied loads $\rightarrow$ $(\,100. \quad 100.\,)$

Sum of support reactions $\rightarrow$ $(\,-100. \quad -100.\,)$