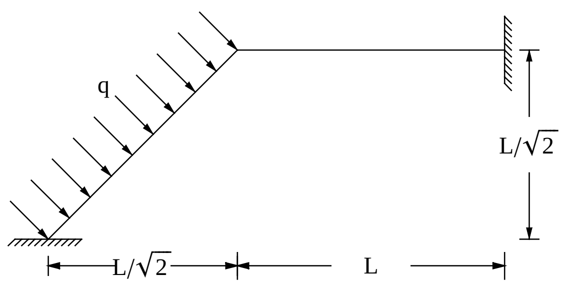
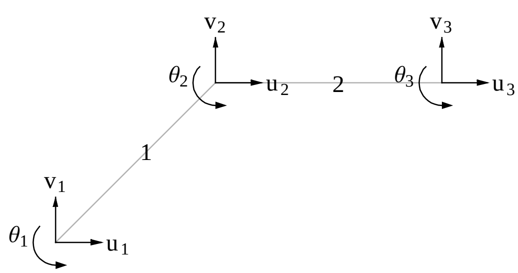
Example 4.11 Frame with distributed load (p. 274)

$$q = 1\,k/\mathrm{ft};\ \ L = 15\,\mathrm{ft};\ \ E = 30\times10^{3}\,k\big/\mathrm{in}^{2};\ \ A = 100\,\mathrm{in}^{2};\ \ I = 1000\,\mathrm{in}^{4}$$





Use k-in units

Global equations at start of the element assembly process

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{pmatrix}\begin{pmatrix} u_1\\ v_1\\ \theta_1\\ u_2\\ v_2\\ \theta_2\\ u_3\\ v_3\\ \theta_3 \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}$$

Equations for element 1

E = 30000; I = 1000; A = 100; q = {0., −0.0833333}

Nodal coordinates

Element node	Global node number	x	y
1	1	0.	0.
2	2	127.279	127.279

Length = 180.; Direction cosines: $\ell_s = 0.707107$ $m_s = 0.707107$

Element equations in local coordinates

$$\begin{pmatrix} 16666.7 & 0 & 0 & -16666.7 & 0 & 0 \\ 0 & 61.7284 & 5555.56 & 0 & -61.7284 & 5555.56 \\ 0 & 5555.56 & 666667. & 0 & -5555.56 & 333333. \\ -16666.7 & 0 & 0 & 16666.7 & 0 & 0 \\ 0 & -61.7284 & -5555.56 & 0 & 61.7284 & -5555.56 \\ 0 & 5555.56 & 333333. & 0 & -5555.56 & 666667. \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{pmatrix} = \begin{pmatrix} 0. \\ -7.5 \\ -225. \\ 0. \\ -7.5 \\ 225. \end{pmatrix}$$

$$\text{Global to local transformation, } \boldsymbol{T} = \begin{pmatrix} 0.707107 & 0.707107 & 0 & 0 & 0 & 0 \\ -0.707107 & 0.707107 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.707107 & 0.707107 & 0 \\ 0 & 0 & 0 & -0.707107 & 0.707107 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Element equations in global coordinates

$$\begin{pmatrix} 8364.2 & 8302.47 & -3928.37 & -8364.2 & -8302.47 & -3928.37 \\ 8302.47 & 8364.2 & 3928.37 & -8302.47 & -8364.2 & 3928.37 \\ -3928.37 & 3928.37 & 666667. & 3928.37 & -3928.37 & 333333. \\ -8364.2 & -8302.47 & 3928.37 & 8364.2 & 8302.47 & 3928.37 \\ -8302.47 & -8364.2 & -3928.37 & 8302.47 & 8364.2 & -3928.37 \\ -3928.37 & 3928.37 & 333333. & 3928.37 & -3928.37 & 666667. \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 5.3033 \\ -5.3033 \\ -225. \\ 5.3033 \\ -5.3033 \\ 225. \end{pmatrix}$$

The element contributes to {1, 2, 3, 4, 5, 6} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 8364.2 & 8302.47 & -3928.37 & -8364.2 & -8302.47 & -3928.37 & 0 & 0 & 0 \\ 8302.47 & 8364.2 & 3928.37 & -8302.47 & -8364.2 & 3928.37 & 0 & 0 & 0 \\ -3928.37 & 3928.37 & 666667. & 3928.37 & -3928.37 & 333333. & 0 & 0 & 0 \\ -8364.2 & -8302.47 & 3928.37 & 8364.2 & 8302.47 & 3928.37 & 0 & 0 & 0 \\ -8302.47 & -8364.2 & -3928.37 & 8302.47 & 8364.2 & -3928.37 & 0 & 0 & 0 \\ -3928.37 & 3928.37 & 333333. & 3928.37 & -3928.37 & 666667. & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \\ u_3 \\ v_3 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} 5.3033 \\ -5.3033 \\ -225. \\ 5.3033 \\ -5.3033 \\ 225. \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 2

$E = 30000;$ $I = 1000;$ $A = 100;$ $q = \{0, 0\}$

Nodal coordinates

Element node	Global node number	x	y
1	2	127.279	127.279
2	3	307.279	127.279

Length = 180.; Direction cosines: $\ell_s = 1.$ $m_s = 0.$

Element equations in local coordinates

$$\begin{pmatrix} 16666.7 & 0 & 0 & -16666.7 & 0 & 0 \\ 0 & 61.7284 & 5555.56 & 0 & -61.7284 & 5555.56 \\ 0 & 5555.56 & 666667. & 0 & -5555.56 & 333333. \\ -16666.7 & 0 & 0 & 16666.7 & 0 & 0 \\ 0 & -61.7284 & -5555.56 & 0 & 61.7284 & -5555.56 \\ 0 & 5555.56 & 333333. & 0 & -5555.56 & 666667. \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Global to local transformation, $\boldsymbol{T} = \begin{pmatrix} 1. & 0. & 0 & 0 & 0 & 0 \\ 0. & 1. & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1. & 0. & 0 \\ 0 & 0 & 0 & 0. & 1. & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Element equations in global coordinates

$$\begin{pmatrix} 16666.7 & 0 & 0 & -16666.7 & 0 & 0 \\ 0 & 61.7284 & 5555.56 & 0 & -61.7284 & 5555.56 \\ 0 & 5555.56 & 666667. & 0 & -5555.56 & 333333. \\ -16666.7 & 0 & 0 & 16666.7 & 0 & 0 \\ 0 & -61.7284 & -5555.56 & 0 & 61.7284 & -5555.56 \\ 0 & 5555.56 & 333333. & 0 & -5555.56 & 666667. \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ \theta_2 \\ u_3 \\ v_3 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to {4, 5, 6, 7, 8, 9} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 8364.2 & 8302.47 & -3928.37 & -8364.2 & -8302.47 & -3928.37 & 0 & 0 & \\ 8302.47 & 8364.2 & 3928.37 & -8302.47 & -8364.2 & 3928.37 & 0 & 0 & \\ -3928.37 & 3928.37 & 666667. & 3928.37 & -3928.37 & 333333. & 0 & 0 & \\ -8364.2 & -8302.47 & 3928.37 & 25030.9 & 8302.47 & 3928.37 & -16666.7 & 0 & \\ -8302.47 & -8364.2 & -3928.37 & 8302.47 & 8425.93 & 1627.18 & 0 & -61.7284 & 5 \\ -3928.37 & 3928.37 & 333333. & 3928.37 & 1627.18 & 1.33333\times 10^{6} & 0 & -5555.56 & 33 \\ 0 & 0 & 0 & -16666.7 & 0 & 0 & 16666.7 & 0 & \\ 0 & 0 & 0 & 0 & -61.7284 & -5555.56 & 0 & 61.7284 & -5 \\ 0 & 0 & 0 & 0 & 5555.56 & 333333. & 0 & -5555.56 & 66 \end{pmatrix}$$

Essential boundary conditions

Node	dof	Value
1	u_1	0
	v_1	0
	θ_1	0
3	u_3	0
	v_3	0
	θ_3	0

Remove {1, 2, 3, 7, 8, 9} rows and columns.

After adjusting for essential boundary conditions we have

$$\begin{pmatrix} 25030.9 & 8302.47 & 3928.37 \\ 8302.47 & 8425.93 & 1627.18 \\ 3928.37 & 1627.18 & 1.33333\times 10^{6} \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 5.3033 \\ -5.3033 \\ 225. \end{pmatrix}$$

Solving the final system of global equations we get

$$\{u_2 = 0.000601607,\ v_2 = -0.00125474,\ \theta_2 = 0.000168509\}$$

Complete table of nodal values

	u	v	θ
1	0	0	0
2	0.000601607	−0.00125474	0.000168509
3	0	0	0

Computation of reactions

Equation numbers of dof with specified values: {1, 2, 3, 7, 8, 9}

Extracting equations {1, 2, 3, 7, 8, 9} from the global system we have

$$\begin{pmatrix} 8364.2 & 8302.47 & -3928.37 & -8364.2 & -8302.47 & -3928.37 & 0 & 0 & 0 \\ 8302.47 & 8364.2 & 3928.37 & -8302.47 & -8364.2 & 3928.37 & 0 & 0 & 0 \\ -3928.37 & 3928.37 & 666667. & 3928.37 & -3928.37 & 333333. & 0 & 0 & 0 \\ 0 & 0 & 0 & -16666.7 & 0 & 0 & 16666.7 & 0 & 0 \\ 0 & 0 & 0 & 0 & -61.7284 & -5555.56 & 0 & 61.7284 & -5555.56 \\ 0 & 0 & 0 & 0 & 5555.56 & 333333. & 0 & -5555.56 & 666667. \end{pmatrix}$$

$$\begin{pmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \\ u_3 \\ v_3 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} R_1 + 5.3033 \\ R_2 - 5.3033 \\ R_3 - 225. \\ R_4 \\ R_5 \\ R_6 \end{pmatrix}$$

Substituting the nodal values and re−arranging

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{pmatrix} = \begin{pmatrix} 8364.2 & 8302.47 & -3928.37 & -8364.2 & -8302.47 & -3928.37 & 0 & 0 & 0 \\ 8302.47 & 8364.2 & 3928.37 & -8302.47 & -8364.2 & 3928.37 & 0 & 0 & 0 \\ -3928.37 & 3928.37 & 666667. & 3928.37 & -3928.37 & 333333. & 0 & 0 & 0 \\ 0 & 0 & 0 & -16666.7 & 0 & 0 & 16666.7 & 0 & 0 \\ 0 & 0 & 0 & 0 & -61.7284 & -5555.56 & 0 & 61.7284 & -5555 \\ 0 & 0 & 0 & 0 & 5555.56 & 333333. & 0 & -5555.56 & 66666 \end{pmatrix}$$

Carrying out computations, the reactions are as follows.

Label	dof	Reaction
R_1	u_1	-0.579812
R_2	v_1	11.4653
R_3	θ_1	288.462
R_4	u_3	-10.0268
R_5	v_3	-0.858707
R_6	θ_3	49.1988

Sum of Reactions

dof: u	-10.6066
dof: v	10.6066
dof: θ	337.661

Solution for element 1

$E = 30000;$ $I = 1000;$ $A = 100;$ $q = \{0., -0.0833333\}$

Length = 180.; Direction cosines: $\ell_s = 0.707107$ $m_s = 0.707107$

Nodal values in global coordinates, $\boldsymbol{d}^T = (0 \quad 0 \quad 0 \quad 0.000601607 \quad -0.00125474 \quad 0.000168509)$

Global to local transformation, $\boldsymbol{T} = \begin{pmatrix} 0.707107 & 0.707107 & 0 & 0 & 0 & 0 \\ -0.707107 & 0.707107 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.707107 & 0.707107 & 0 \\ 0 & 0 & 0 & -0.707107 & 0.707107 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Nodal values in local coordinates, $\boldsymbol{d}_\ell^T = \boldsymbol{T}\boldsymbol{d} = (0 \quad 0 \quad 0 \quad -0.000461833 \quad -0.00131263 \quad 0.000168509)$

Axial displacement interpolation functions, $\boldsymbol{N}_u^T = \{1. - 0.00555556\,s,\ 0.00555556\,s\}$

Axial displacement, $u(s) = \boldsymbol{N}_u^T \begin{pmatrix} d_1 \\ d_4 \end{pmatrix} = -2.56574 \times 10^{-6}\, s$

Axial force, EA du(s)/ds = -7.69721

Beam bending interpolation functions, $\boldsymbol{N}_v^T =$

$$\{3.42936 \times 10^{-7} s^3 - 0.0000925926 s^2 + 1,\ 0.0000308642 s^3 - 0.0111111 s^2 + s,\ 0.0000925926 s^2 - 3.42936 \times 10^{-7} s^3,\ 0.0000308642 s^3 - 0.00555556 s^2\}$$

Transverse displacement, $v(s) = \boldsymbol{N}_v^T \begin{pmatrix} d_2 \\ d_3 \\ d_5 \\ d_6 \end{pmatrix} = 5.65104 \times 10^{-9} s^3 - 1.0577 \times 10^{-6} s^2$

Fixed−end displacement solution, $= -1.15741\times10^{-10}\,(180.-s)^2\,s^2$

Total transverse displacement, $v(s) = -1.15741\times10^{-10}\,s^4 + 4.73177\times10^{-8}\,s^3 - 4.8077\times10^{-6}\,s^2$

Bending moment, $M = EI\,d^2v(s)/ds^2 = -0.0416667\,s^2 + 8.51719\,s - 288.462$

Shear force, $V(s) = dM/ds = 8.51719 - 0.0833333\,s$

Solution for element 2

$E = 30000;$ $I = 1000;$ $A = 100;$ $q = \{0, 0\}$

Length $= 180.;$ Direction cosines: $\ell_s = 1.$ $m_s = 0.$

Nodal values in global coordinates, $\boldsymbol{d}^T = (\,0.000601607 \quad -0.00125474 \quad 0.000168509 \quad 0 \quad 0 \quad 0\,)$

$$\text{Global to local transformation, } \boldsymbol{T} = \begin{pmatrix} 1. & 0. & 0 & 0 & 0 & 0 \\ 0. & 1. & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1. & 0. & 0 \\ 0 & 0 & 0 & 0. & 1. & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Nodal values in local coordinates, $\boldsymbol{d}_\ell^T = \boldsymbol{T}\boldsymbol{d} = (\,0.000601607 \quad -0.00125474 \quad 0.000168509 \quad 0 \quad 0 \quad 0\,)$

Axial displacement interpolation functions, $\boldsymbol{N}_u^T = \{1. - 0.00555556\,s,\ 0.00555556\,s\}$

Axial displacement, $u(s) = \boldsymbol{N}_u^T \begin{pmatrix} d_1 \\ d_4 \end{pmatrix} = 0.000601607 - 3.34226\times10^{-6}\,s$

Axial force, $EA\ du(s)/ds = -10.0268$

Beam bending interpolation functions, $\boldsymbol{N}_v^T =$

$$\{3.42936\times10^{-7}\,s^3 - 0.0000925926\,s^2 + 1,\ 0.0000308642\,s^3 - 0.0111111\,s^2 + s,\ 0.0000925926\,s^2 - 3.42936\times10^{-7}\,s^3,\ 0.0000308642\,s^3 - 0.00555556\,s^2\}$$

Transverse displacement, $v(s) = \boldsymbol{N}_v^T \begin{pmatrix} d_2 \\ d_3 \\ d_5 \\ d_6 \end{pmatrix} =$

$$4.77059\times10^{-9}\,s^3 - 1.75614\times10^{-6}\,s^2 + 0.000168509\,s - 0.00125474$$

Bending moment, $M = EI\,d^2v(s)/ds^2 = 0.858707\,s - 105.368$

Shear force, $V(s) = dM/ds = 0.858707$

Forces at element ends

	x	y	Axial force	Bending moment	Shear force
1	0	0	−7.69721	−288.462	8.51719
	127.279	127.279	−7.69721	−105.368	−6.48281
2	127.279	127.279	−10.0268	−105.368	0.858707
	307.279	127.279	−10.0268	49.1988	0.858707