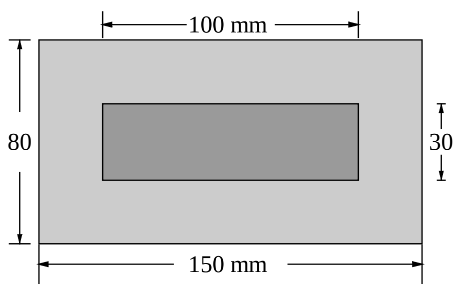
Example 7.6: Thermal stresses (p. 502)

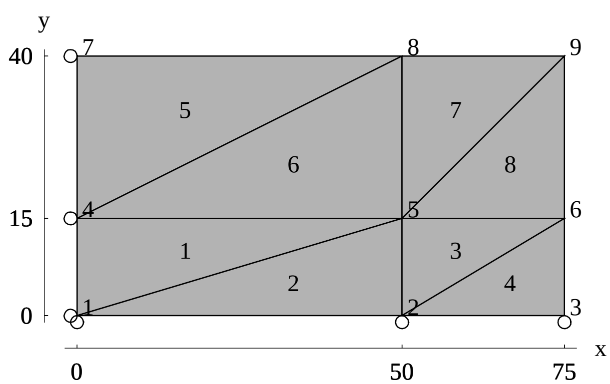
A 5 mm thick symmetric assembly of steel and aluminum plates, shown in Figure, is created at room temperature. Determine stresses and deformed shape if the temperature of the assembly is increased by 70°C above the room temperature. Assume a perfect bond between the two materials. Use the following data.

Steel plate: $80 \times 150\ \text{mm} \quad E = 200\ \text{GPa} \quad \nu = 0.3 \quad \alpha = 12 \times 10^{-6} / {}^\circ\text{C}$

Aluminum plate: $30 \times 100\ \text{mm} \quad E = 70\ \text{GPa} \quad \nu = 0.33 \quad \alpha = 23 \times 10^{-6} / {}^\circ\text{C}$



Since the thickness of the assembly is much smaller than the other dimensions, and there are no out of plane loads, the problem can be treated as a plane stress situation. Using symmetry a quarter of the assembly is modeled as shown in Figure. The first two elements are in the aluminum plate and the remaining 6 in the steel plate. A coarse mesh is used to show all calculations. Due to symmetry nodes 2 and 3 can displace in the x direction only while nodes 4 and 7 can displace in the y direction alone. Node 1, being on both the axes of symmetry, cannot displace in either direction. Note that in addition to reducing the model size, the use of symmetry provides enough boundary conditions so that there is no rigid body motion in the model. Since no support conditions are given for the assembly, analysis of a full finite element model would not be possible without introducing artificial supports.



The complete finite element calculations are as follows. The numerical values are in the N-mm units. The displacements will be in mm and the stresses in MPa.

Global equations at start of the element assembly process

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \\ u_7 \\ v_7 \\ u_8 \\ v_8 \\ u_9 \\ v_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 1

$h = 5;$ $E = 70000;$ $\nu = 0.33$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 78554.6 & 25923. & 0 \\ 25923. & 78554.6 & 0 \\ 0 & 0 & 26315.8 \end{pmatrix}$

Nodal coordinates

Element node	Global node number	x	y
1	1	0	0
2	5	50	15
3	4	0	15

$x_1 = 0$ $x_2 = 50$ $x_3 = 0$

$y_1 = 0$ $y_2 = 15$ $y_3 = 15$

Using these values we get

$b_1 = 0$ $b_2 = 15$ $b_3 = -15$

$c_1 = -50$ $c_2 = 0$ $c_3 = 50$

$f_1 = 750$ $f_2 = 0$ $f_3 = 0$

Element area, A = 375

$$\boldsymbol{B}^T = \begin{pmatrix} 0 & 0 & \frac{1}{50} & 0 & -\frac{1}{50} & 0 \\ 0 & -\frac{1}{15} & 0 & 0 & 0 & \frac{1}{15} \\ -\frac{1}{15} & 0 & 0 & \frac{1}{50} & \frac{1}{15} & -\frac{1}{50} \end{pmatrix}$$

Thus the element stiffness matrix is

$$\boldsymbol{k} = \mathrm{hA}\boldsymbol{BCB}^T = 10^6 \begin{pmatrix} 0.219298 & 0 & 0 & -0.0657895 & -0.219298 & 0.0657895 \\ 0 & 0.654622 & -0.0648075 & 0 & 0.0648075 & -0.654622 \\ 0 & -0.0648075 & 0.0589159 & 0 & -0.0589159 & 0.0648075 \\ -0.0657895 & 0 & 0 & 0.0197368 & 0.0657895 & -0.0197368 \\ -0.219298 & 0.0648075 & -0.0589159 & 0.0657895 & 0.278214 & -0.130597 \\ 0.0657895 & -0.654622 & 0.0648075 & -0.0197368 & -0.130597 & 0.674358 \end{pmatrix}$$

Load vector due to temperature change

$$\alpha = \frac{23}{1000000}; \qquad \Delta T = 70; \qquad \boldsymbol{\epsilon}_0^T = \begin{pmatrix} \frac{161}{100000} & \frac{161}{100000} & 0 \end{pmatrix}$$

$$\boldsymbol{r}_\epsilon^T = \begin{pmatrix} 0. & -21026.1 & 6307.84 & 0. & -6307.84 & 21026.1 \end{pmatrix}$$

Complete equations for element 1

$$10^6 \begin{pmatrix} 0.219298 & 0 & 0 & -0.0657895 & -0.219298 & 0.0657895 \\ 0 & 0.654622 & -0.0648075 & 0 & 0.0648075 & -0.654622 \\ 0 & -0.0648075 & 0.0589159 & 0 & -0.0589159 & 0.0648075 \\ -0.0657895 & 0 & 0 & 0.0197368 & 0.0657895 & -0.0197368 \\ -0.219298 & 0.0648075 & -0.0589159 & 0.0657895 & 0.278214 & -0.130597 \\ 0.0657895 & -0.654622 & 0.0648075 & -0.0197368 & -0.130597 & 0.674358 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_5 \\ v_5 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0. \\ -21026.1 \\ 6307.84 \\ 0. \\ -6307.84 \\ 21026.1 \end{pmatrix}$$

The element contributes to {1, 2, 9, 10, 7, 8} global degrees of freedom.

Adding element equations into appropriate locations we have

$$10^6 \left(\begin{matrix} 0.219298 & 0 & 0 & 0 & 0 & 0 & -0.219298 & 0.0657895 & 0 & -0.0657895 & 0 & 0 & 0 & 0 \\ 0 & 0.654622 & 0 & 0 & 0 & 0 & 0.0648075 & -0.654622 & -0.0648075 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.219298 & 0.0648075 & 0 & 0 & 0 & 0 & 0.278214 & -0.130597 & -0.0589159 & 0.0657895 & 0 & 0 & 0 & 0 \\ 0.0657895 & -0.654622 & 0 & 0 & 0 & 0 & -0.130597 & 0.674358 & 0.0648075 & -0.0197368 & 0 & 0 & 0 & 0 \\ 0 & -0.0648075 & 0 & 0 & 0 & 0 & -0.0589159 & 0.0648075 & 0.0589159 & 0 & 0 & 0 & 0 & 0 \\ -0.0657895 & 0 & 0 & 0 & 0 & 0 & 0.0657895 & -0.0197368 & 0 & 0.0197368 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \right.$$

Equations for element 2

h = 5; E = 70000; $\nu = 0.33$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 78554.6 & 25923. & 0 \\ 25923. & 78554.6 & 0 \\ 0 & 0 & 26315.8 \end{pmatrix}$

Nodal coordinates

Element node	Global node number	x	y
1	1	0	0
2	2	50	0
3	5	50	15

$x_1 = 0 \qquad x_2 = 50 \qquad x_3 = 50$

$y_1 = 0 \qquad y_2 = 0 \qquad y_3 = 15$

Using these values we get

$b_1 = -15 \qquad b_2 = 15 \qquad b_3 = 0$

$c_1 = 0 \qquad c_2 = -50 \qquad c_3 = 50$

$f_1 = 750 \qquad f_2 = 0 \qquad f_3 = 0$

Element area, $A = 375$

$$\boldsymbol{B}^T = \begin{pmatrix} -\frac{1}{50} & 0 & \frac{1}{50} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{15} & 0 & \frac{1}{15} \\ 0 & -\frac{1}{50} & -\frac{1}{15} & \frac{1}{50} & \frac{1}{15} & 0 \end{pmatrix}$$

Thus the element stiffness matrix is

$$\boldsymbol{k} = hA\boldsymbol{BCB}^T = 10^6 \begin{pmatrix} 0.0589159 & 0 & -0.0589159 & 0.0648075 & 0 & -0.0648075 \\ 0 & 0.0197368 & 0.0657895 & -0.0197368 & -0.0657895 & 0 \\ -0.0589159 & 0.0657895 & 0.278214 & -0.130597 & -0.219298 & 0.0648075 \\ 0.0648075 & -0.0197368 & -0.130597 & 0.674358 & 0.0657895 & -0.654622 \\ 0 & -0.0657895 & -0.219298 & 0.0657895 & 0.219298 & 0 \\ -0.0648075 & 0 & 0.0648075 & -0.654622 & 0 & 0.654622 \end{pmatrix}$$

Load vector due to temperature change

$$\alpha = \frac{23}{1000000}; \qquad \Delta T = 70; \qquad \boldsymbol{\epsilon}_0^T = \left(\frac{161}{100000} \quad \frac{161}{100000} \quad 0 \right)$$

$$\boldsymbol{r}_\epsilon^T = (-6307.84 \quad 0. \quad 6307.84 \quad -21026.1 \quad 0. \quad 21026.1)$$

Complete equations for element 2

$$10^6 \begin{pmatrix} 0.0589159 & 0 & -0.0589159 & 0.0648075 & 0 & -0.0648075 \\ 0 & 0.0197368 & 0.0657895 & -0.0197368 & -0.0657895 & 0 \\ -0.0589159 & 0.0657895 & 0.278214 & -0.130597 & -0.219298 & 0.0648075 \\ 0.0648075 & -0.0197368 & -0.130597 & 0.674358 & 0.0657895 & -0.654622 \\ 0 & -0.0657895 & -0.219298 & 0.0657895 & 0.219298 & 0 \\ -0.0648075 & 0 & 0.0648075 & -0.654622 & 0 & 0.654622 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_5 \\ v_5 \end{pmatrix} = \begin{pmatrix} -6307.84 \\ 0. \\ 6307.84 \\ -21026.1 \\ 0. \\ 21026.1 \end{pmatrix}$$

The element contributes to {1, 2, 3, 4, 9, 10} global degrees of freedom.

Adding element equations into appropriate locations we have

$$10^6\begin{pmatrix} 0.278214 & 0 & -0.0589159 & 0.0648075 & 0 & 0 & -0.219298 & 0.0657895 & 0 & -0. \\ 0 & 0.674358 & 0.0657895 & -0.0197368 & 0 & 0 & 0.0648075 & -0.654622 & -0.130597 & 0 \\ -0.0589159 & 0.0657895 & 0.278214 & -0.130597 & 0 & 0 & 0 & 0 & -0.219298 & 0. \\ 0.0648075 & -0.0197368 & -0.130597 & 0.674358 & 0 & 0 & 0 & 0 & 0.0657895 & -0. \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.219298 & 0.0648075 & 0 & 0 & 0 & 0 & 0.278214 & -0.130597 & -0.0589159 & 0. \\ 0.0657895 & -0.654622 & 0 & 0 & 0 & 0 & -0.130597 & 0.674358 & 0.0648075 & -0. \\ 0 & -0.130597 & -0.219298 & 0.0657895 & 0 & 0 & -0.0589159 & 0.0648075 & 0.278214 & 0 \\ -0.130597 & 0 & 0.0648075 & -0.654622 & 0 & 0 & 0.0657895 & -0.0197368 & 0 & 0. \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Equations for element 3

$h = 5;$ $E = 200000;$ $\nu = 0.3$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 219780. & 65934.1 & 0 \\ 65934.1 & 219780. & 0 \\ 0 & 0 & 76923.1 \end{pmatrix}$

Nodal coordinates

Element node	Global node number	x	y
1	2	50	0
2	6	75	15
3	5	50	15

$x_1 = 50$ $x_2 = 75$ $x_3 = 50$

$y_1 = 0$ $y_2 = 15$ $y_3 = 15$

Using these values we get

$b_1 = 0$ $b_2 = 15$ $b_3 = -15$

$c_1 = -25 \qquad c_2 = 0 \qquad c_3 = 25$

$f_1 = 375 \qquad f_2 = -750 \qquad f_3 = 750$

Element area, $A = \frac{375}{2}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & \frac{1}{25} & 0 & -\frac{1}{25} & 0 \\ 0 & -\frac{1}{15} & 0 & 0 & 0 & \frac{1}{15} \\ -\frac{1}{15} & 0 & 0 & \frac{1}{25} & \frac{1}{15} & -\frac{1}{25} \end{pmatrix}$$

Thus the element stiffness matrix is

$$\boldsymbol{k} = \mathrm{h}A\boldsymbol{B}\boldsymbol{C}\boldsymbol{B}^{\mathrm{T}} = 10^6 \begin{pmatrix} 0.320513 & 0 & 0 & -0.192308 & -0.320513 & 0.192308 \\ 0 & 0.915751 & -0.164835 & 0 & 0.164835 & -0.915751 \\ 0 & -0.164835 & 0.32967 & 0 & -0.32967 & 0.164835 \\ -0.192308 & 0 & 0 & 0.115385 & 0.192308 & -0.115385 \\ -0.320513 & 0.164835 & -0.32967 & 0.192308 & 0.650183 & -0.357143 \\ 0.192308 & -0.915751 & 0.164835 & -0.115385 & -0.357143 & 1.03114 \end{pmatrix}$$

Load vector due to temperature change

$$\alpha = \frac{3}{250000}; \qquad \Delta T = 70; \qquad \boldsymbol{\epsilon}_0^{\mathrm{T}} = \begin{pmatrix} \frac{21}{25000} & \frac{21}{25000} & 0 \end{pmatrix}$$

$$\boldsymbol{r}_\epsilon^{\mathrm{T}} = (\,0. \quad -15000. \quad 9000. \quad 0. \quad -9000. \quad 15000.\,)$$

Complete equations for element 3

$$10^6 \begin{pmatrix} 0.320513 & 0 & 0 & -0.192308 & -0.320513 & 0.192308 \\ 0 & 0.915751 & -0.164835 & 0 & 0.164835 & -0.915751 \\ 0 & -0.164835 & 0.32967 & 0 & -0.32967 & 0.164835 \\ -0.192308 & 0 & 0 & 0.115385 & 0.192308 & -0.115385 \\ -0.320513 & 0.164835 & -0.32967 & 0.192308 & 0.650183 & -0.357143 \\ 0.192308 & -0.915751 & 0.164835 & -0.115385 & -0.357143 & 1.03114 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ u_6 \\ v_6 \\ u_5 \\ v_5 \end{pmatrix} = \begin{pmatrix} 0. \\ -15000. \\ 9000. \\ 0. \\ -9000. \\ 15000. \end{pmatrix}$$

The element contributes to {3, 4, 11, 12, 9, 10} global degrees of freedom.

Adding element equations into appropriate locations we have

$$10^6 \begin{pmatrix}
0.278214 & 0 & -0.0589159 & 0.0648075 & 0 & 0 & -0.219298 & 0.0657895 & 0 & -0.1 \\
0 & 0.674358 & 0.0657895 & -0.0197368 & 0 & 0 & 0.0648075 & -0.654622 & -0.130597 & 0 \\
-0.0589159 & 0.0657895 & 0.598727 & -0.130597 & 0 & 0 & 0 & 0 & -0.539811 & 0.2 \\
0.0648075 & -0.0197368 & -0.130597 & 1.59011 & 0 & 0 & 0 & 0 & 0.230625 & -1.5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.219298 & 0.0648075 & 0 & 0 & 0 & 0 & 0.278214 & -0.130597 & -0.0589159 & 0.0 \\
0.0657895 & -0.654622 & 0 & 0 & 0 & 0 & -0.130597 & 0.674358 & 0.0648075 & -0.0 \\
0 & -0.130597 & -0.539811 & 0.230625 & 0 & 0 & -0.0589159 & 0.0648075 & 0.928397 & -0.3 \\
-0.130597 & 0 & 0.257115 & -1.57037 & 0 & 0 & 0.0657895 & -0.0197368 & -0.357143 & 1.7 \\
0 & 0 & 0 & -0.164835 & 0 & 0 & 0 & 0 & -0.32967 & 0.1 \\
0 & 0 & -0.192308 & 0 & 0 & 0 & 0 & 0 & 0.192308 & -0.1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Equations for element 4

$h = 5;$ $E = 200000;$ $\nu = 0.3$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 219780. & 65934.1 & 0 \\ 65934.1 & 219780. & 0 \\ 0 & 0 & 76923.1 \end{pmatrix}$

Nodal coordinates

Element node	Global node number	x	y
1	2	50	0
2	3	75	0
3	6	75	15

$x_1 = 50$ $x_2 = 75$ $x_3 = 75$

$y_1 = 0$ $y_2 = 0$ $y_3 = 15$

Using these values we get

$b_1 = -15$ $b_2 = 15$ $b_3 = 0$

$c_1 = 0$ $c_2 = -25$ $c_3 = 25$

$f_1 = 1125$ $f_2 = -750$ $f_3 = 0$

Element area, $A = \frac{375}{2}$

$$\boldsymbol{B}^{T} = \begin{pmatrix} -\frac{1}{25} & 0 & \frac{1}{25} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{15} & 0 & \frac{1}{15} \\ 0 & -\frac{1}{25} & -\frac{1}{15} & \frac{1}{25} & \frac{1}{15} & 0 \end{pmatrix}$$

Thus the element stiffness matrix is

$$\boldsymbol{k} = hA\boldsymbol{BCB}^{T} = 10^{6}\begin{pmatrix} 0.32967 & 0 & -0.32967 & 0.164835 & 0 & -0.164835 \\ 0 & 0.115385 & 0.192308 & -0.115385 & -0.192308 & 0 \\ -0.32967 & 0.192308 & 0.650183 & -0.357143 & -0.320513 & 0.164835 \\ 0.164835 & -0.115385 & -0.357143 & 1.03114 & 0.192308 & -0.915751 \\ 0 & -0.192308 & -0.320513 & 0.192308 & 0.320513 & 0 \\ -0.164835 & 0 & 0.164835 & -0.915751 & 0 & 0.915751 \end{pmatrix}$$

Load vector due to temperature change

$$\alpha = \frac{3}{250000}; \qquad \Delta T = 70; \qquad \boldsymbol{\epsilon}_0^{T} = \begin{pmatrix} \frac{21}{25000} & \frac{21}{25000} & 0 \end{pmatrix}$$

$$\boldsymbol{r}_{\epsilon}^{T} = (\ -9000. \quad 0. \quad 9000. \quad -15000. \quad 0. \quad 15000.\ )$$

Complete equations for element 4

$$10^{6}\begin{pmatrix} 0.32967 & 0 & -0.32967 & 0.164835 & 0 & -0.164835 \\ 0 & 0.115385 & 0.192308 & -0.115385 & -0.192308 & 0 \\ -0.32967 & 0.192308 & 0.650183 & -0.357143 & -0.320513 & 0.164835 \\ 0.164835 & -0.115385 & -0.357143 & 1.03114 & 0.192308 & -0.915751 \\ 0 & -0.192308 & -0.320513 & 0.192308 & 0.320513 & 0 \\ -0.164835 & 0 & 0.164835 & -0.915751 & 0 & 0.915751 \end{pmatrix}\begin{pmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_6 \\ v_6 \end{pmatrix} = \begin{pmatrix} -9000. \\ 0. \\ 9000. \\ -15000. \\ 0. \\ 15000. \end{pmatrix}$$

The element contributes to {3, 4, 5, 6, 11, 12} global degrees of freedom.

Adding element equations into appropriate locations we have

$$10^6 \left(\begin{array}{cccccccc}
0.278214 & 0 & -0.0589159 & 0.0648075 & 0 & 0 & -0.219298 & 0.0657895 \\
0 & 0.674358 & 0.0657895 & -0.0197368 & 0 & 0 & 0.0648075 & -0.654622 \\
-0.0589159 & 0.0657895 & 0.928397 & -0.130597 & -0.32967 & 0.164835 & 0 & 0 \\
0.0648075 & -0.0197368 & -0.130597 & 1.70549 & 0.192308 & -0.115385 & 0 & 0 \\
0 & 0 & -0.32967 & 0.192308 & 0.650183 & -0.357143 & 0 & 0 \\
0 & 0 & 0.164835 & -0.115385 & -0.357143 & 1.03114 & 0 & 0 \\
-0.219298 & 0.0648075 & 0 & 0 & 0 & 0 & 0.278214 & -0.130597 \\
0.0657895 & -0.654622 & 0 & 0 & 0 & 0 & -0.130597 & 0.674358 \\
0 & -0.130597 & -0.539811 & 0.230625 & 0 & 0 & -0.0589159 & 0.0648075 \\
-0.130597 & 0 & 0.257115 & -1.57037 & 0 & 0 & 0.0657895 & -0.0197368 \\
0 & 0 & 0 & -0.357143 & -0.320513 & 0.192308 & 0 & 0 \\
0 & 0 & -0.357143 & 0 & 0.164835 & -0.915751 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array} \right.$$

Equations for element 5

$h = 5;$ $E = 200000;$ $\nu = 0.3$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 219780. & 65934.1 & 0 \\ 65934.1 & 219780. & 0 \\ 0 & 0 & 76923.1 \end{pmatrix}$

Nodal coordinates

Element node	Global node number	x	y
1	4	0	15
2	8	50	40
3	7	0	40

$x_1 = 0$ $x_2 = 50$ $x_3 = 0$

$y_1 = 15$ $y_2 = 40$ $y_3 = 40$

Using these values we get

$b_1 = 0$ $b_2 = 25$ $b_3 = -25$

$c_1 = -50$ $c_2 = 0$ $c_3 = 50$

$f_1 = 2000$ $f_2 = 0$ $f_3 = -750$

Element area, A = 625

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & \frac{1}{50} & 0 & -\frac{1}{50} & 0 \\ 0 & -\frac{1}{25} & 0 & 0 & 0 & \frac{1}{25} \\ -\frac{1}{25} & 0 & 0 & \frac{1}{50} & \frac{1}{25} & -\frac{1}{50} \end{pmatrix}$$

Thus the element stiffness matrix is

$$\boldsymbol{k} = \mathrm{hA}\boldsymbol{BCB}^{\mathrm{T}} = 10^6 \begin{pmatrix} 0.384615 & 0 & 0 & -0.192308 & -0.384615 & 0.192308 \\ 0 & 1.0989 & -0.164835 & 0 & 0.164835 & -1.0989 \\ 0 & -0.164835 & 0.274725 & 0 & -0.274725 & 0.164835 \\ -0.192308 & 0 & 0 & 0.0961538 & 0.192308 & -0.0961538 \\ -0.384615 & 0.164835 & -0.274725 & 0.192308 & 0.659341 & -0.357143 \\ 0.192308 & -1.0989 & 0.164835 & -0.0961538 & -0.357143 & 1.19505 \end{pmatrix}$$

Load vector due to temperature change

$$\alpha = \frac{3}{250000}; \qquad \Delta\mathrm{T} = 70; \qquad \boldsymbol{\epsilon}_0^{\mathrm{T}} = \begin{pmatrix} \frac{21}{25000} & \frac{21}{25000} & 0 \end{pmatrix}$$

$$\boldsymbol{r}_\epsilon^{\mathrm{T}} = (\, 0. \quad -30000. \quad 15000. \quad 0. \quad -15000. \quad 30000.\,)$$

Complete equations for element 5

$$10^6 \begin{pmatrix} 0.384615 & 0 & 0 & -0.192308 & -0.384615 & 0.192308 \\ 0 & 1.0989 & -0.164835 & 0 & 0.164835 & -1.0989 \\ 0 & -0.164835 & 0.274725 & 0 & -0.274725 & 0.164835 \\ -0.192308 & 0 & 0 & 0.0961538 & 0.192308 & -0.0961538 \\ -0.384615 & 0.164835 & -0.274725 & 0.192308 & 0.659341 & -0.357143 \\ 0.192308 & -1.0989 & 0.164835 & -0.0961538 & -0.357143 & 1.19505 \end{pmatrix} \begin{pmatrix} \mathrm{u}_4 \\ \mathrm{v}_4 \\ \mathrm{u}_8 \\ \mathrm{v}_8 \\ \mathrm{u}_7 \\ \mathrm{v}_7 \end{pmatrix} = \begin{pmatrix} 0. \\ -30000. \\ 15000. \\ 0. \\ -15000. \\ 30000. \end{pmatrix}$$

The element contributes to {7, 8, 15, 16, 13, 14} global degrees of freedom.

Adding element equations into appropriate locations we have

$$10^6 \begin{pmatrix} 0.278214 & 0 & -0.0589159 & 0.0648075 & 0 & 0 & -0.219298 & 0.0657895 \\ 0 & 0.674358 & 0.0657895 & -0.0197368 & 0 & 0 & 0.0648075 & -0.654622 \\ -0.0589159 & 0.0657895 & 0.928397 & -0.130597 & -0.32967 & 0.164835 & 0 & 0 \\ 0.0648075 & -0.0197368 & -0.130597 & 1.70549 & 0.192308 & -0.115385 & 0 & 0 \\ 0 & 0 & -0.32967 & 0.192308 & 0.650183 & -0.357143 & 0 & 0 \\ 0 & 0 & 0.164835 & -0.115385 & -0.357143 & 1.03114 & 0 & 0 \\ -0.219298 & 0.0648075 & 0 & 0 & 0 & 0 & 0.66283 & -0.130597 \\ 0.0657895 & -0.654622 & 0 & 0 & 0 & 0 & -0.130597 & 1.77326 \\ 0 & -0.130597 & -0.539811 & 0.230625 & 0 & 0 & -0.0589159 & 0.0648075 \\ -0.130597 & 0 & 0.257115 & -1.57037 & 0 & 0 & 0.0657895 & -0.0197368 \\ 0 & 0 & 0 & -0.357143 & -0.320513 & 0.192308 & 0 & 0 \\ 0 & 0 & -0.357143 & 0 & 0.164835 & -0.915751 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.384615 & 0.164835 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.192308 & -1.0989 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.164835 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.192308 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Equations for element 6

$h = 5;$ $E = 200000;$ $\nu = 0.3$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 219780. & 65934.1 & 0 \\ 65934.1 & 219780. & 0 \\ 0 & 0 & 76923.1 \end{pmatrix}$

Nodal coordinates

Element node	Global node number	x	y
1	4	0	15
2	5	50	15
3	8	50	40

$x_1 = 0$ $x_2 = 50$ $x_3 = 50$

$y_1 = 15$ $y_2 = 15$ $y_3 = 40$

Using these values we get

$b_1 = -25$ $b_2 = 25$ $b_3 = 0$

$c_1 = 0$ $c_2 = -50$ $c_3 = 50$

$f_1 = 1250$ $f_2 = 750$ $f_3 = -750$

Element area, A = 625

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -\frac{1}{50} & 0 & \frac{1}{50} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{25} & 0 & \frac{1}{25} \\ 0 & -\frac{1}{50} & -\frac{1}{25} & \frac{1}{50} & \frac{1}{25} & 0 \end{pmatrix}$$

Thus the element stiffness matrix is

$$\boldsymbol{k} = \mathrm{hA}\boldsymbol{BCB}^{\mathrm{T}} = 10^6 \begin{pmatrix} 0.274725 & 0 & -0.274725 & 0.164835 & 0 & -0.164835 \\ 0 & 0.0961538 & 0.192308 & -0.0961538 & -0.192308 & 0 \\ -0.274725 & 0.192308 & 0.659341 & -0.357143 & -0.384615 & 0.164835 \\ 0.164835 & -0.0961538 & -0.357143 & 1.19505 & 0.192308 & -1.0989 \\ 0 & -0.192308 & -0.384615 & 0.192308 & 0.384615 & 0 \\ -0.164835 & 0 & 0.164835 & -1.0989 & 0 & 1.0989 \end{pmatrix}$$

Load vector due to temperature change

$$\alpha = \frac{3}{250000}; \qquad \Delta\mathrm{T} = 70; \qquad \boldsymbol{\epsilon}_0^{\mathrm{T}} = \begin{pmatrix} \frac{21}{25000} & \frac{21}{25000} & 0 \end{pmatrix}$$

$$\boldsymbol{r}_\epsilon^{\mathrm{T}} = \begin{pmatrix} -15000. & 0. & 15000. & -30000. & 0. & 30000. \end{pmatrix}$$

Complete equations for element 6

$$10^6 \begin{pmatrix} 0.274725 & 0 & -0.274725 & 0.164835 & 0 & -0.164835 \\ 0 & 0.0961538 & 0.192308 & -0.0961538 & -0.192308 & 0 \\ -0.274725 & 0.192308 & 0.659341 & -0.357143 & -0.384615 & 0.164835 \\ 0.164835 & -0.0961538 & -0.357143 & 1.19505 & 0.192308 & -1.0989 \\ 0 & -0.192308 & -0.384615 & 0.192308 & 0.384615 & 0 \\ -0.164835 & 0 & 0.164835 & -1.0989 & 0 & 1.0989 \end{pmatrix} \begin{pmatrix} u_4 \\ v_4 \\ u_5 \\ v_5 \\ u_8 \\ v_8 \end{pmatrix} = \begin{pmatrix} -15000. \\ 0. \\ 15000. \\ -30000. \\ 0. \\ 30000. \end{pmatrix}$$

The element contributes to {7, 8, 9, 10, 15, 16} global degrees of freedom.

Adding element equations into appropriate locations we have

$$10^6\left(\begin{array}{cccccccc}
0.278214 & 0 & -0.0589159 & 0.0648075 & 0 & 0 & -0.219298 & 0.0657895 \\
0 & 0.674358 & 0.0657895 & -0.0197368 & 0 & 0 & 0.0648075 & -0.654622 \\
-0.0589159 & 0.0657895 & 0.928397 & -0.130597 & -0.32967 & 0.164835 & 0 & 0 \\
0.0648075 & -0.0197368 & -0.130597 & 1.70549 & 0.192308 & -0.115385 & 0 & 0 \\
0 & 0 & -0.32967 & 0.192308 & 0.650183 & -0.357143 & 0 & 0 \\
0 & 0 & 0.164835 & -0.115385 & -0.357143 & 1.03114 & 0 & 0 \\
-0.219298 & 0.0648075 & 0 & 0 & 0 & 0 & 0.937555 & -0.130597 \\
0.0657895 & -0.654622 & 0 & 0 & 0 & 0 & -0.130597 & 1.86941 \\
0 & -0.130597 & -0.539811 & 0.230625 & 0 & 0 & -0.333641 & 0.257115 \\
-0.130597 & 0 & 0.257115 & -1.57037 & 0 & 0 & 0.230625 & -0.115891 \\
0 & 0 & 0 & -0.357143 & -0.320513 & 0.192308 & 0 & 0 \\
0 & 0 & -0.357143 & 0 & 0.164835 & -0.915751 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.384615 & 0.164835 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.192308 & -1.0989 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.357143 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.357143 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right.$$

Equations for element 7

$h = 5;$ $E = 200000;$ $\nu = 0.3$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 219780. & 65934.1 & 0 \\ 65934.1 & 219780. & 0 \\ 0 & 0 & 76923.1 \end{pmatrix}$

Nodal coordinates

Element node	Global node number	x	y
1	5	50	15
2	9	75	40
3	8	50	40

$x_1 = 50$ $x_2 = 75$ $x_3 = 50$

$y_1 = 15$ $y_2 = 40$ $y_3 = 40$

Using these values we get

$b_1 = 0$ $b_2 = 25$ $b_3 = -25$

$c_1 = -25$ $c_2 = 0$ $c_3 = 25$

$f_1 = 1000$ $f_2 = -1250$ $f_3 = 875$

Element area, $A = \frac{625}{2}$

$$\boldsymbol{B}^{T} = \begin{pmatrix} 0 & 0 & \frac{1}{25} & 0 & -\frac{1}{25} & 0 \\ 0 & -\frac{1}{25} & 0 & 0 & 0 & \frac{1}{25} \\ -\frac{1}{25} & 0 & 0 & \frac{1}{25} & \frac{1}{25} & -\frac{1}{25} \end{pmatrix}$$

Thus the element stiffness matrix is

$$\boldsymbol{k} = hA\boldsymbol{B}\boldsymbol{C}\boldsymbol{B}^{T} = 10^{6}\begin{pmatrix} 0.192308 & 0 & 0 & -0.192308 & -0.192308 & 0.192308 \\ 0 & 0.549451 & -0.164835 & 0 & 0.164835 & -0.549451 \\ 0 & -0.164835 & 0.549451 & 0 & -0.549451 & 0.164835 \\ -0.192308 & 0 & 0 & 0.192308 & 0.192308 & -0.192308 \\ -0.192308 & 0.164835 & -0.549451 & 0.192308 & 0.741758 & -0.357143 \\ 0.192308 & -0.549451 & 0.164835 & -0.192308 & -0.357143 & 0.741758 \end{pmatrix}$$

Load vector due to temperature change

$$\alpha = \frac{3}{250000}; \qquad \Delta T = 70; \qquad \boldsymbol{\epsilon}_0^{T} = \begin{pmatrix} \frac{21}{25000} & \frac{21}{25000} & 0 \end{pmatrix}$$

$$\boldsymbol{r}_{\epsilon}^{T} = \begin{pmatrix} 0. & -15000. & 15000. & 0. & -15000. & 15000. \end{pmatrix}$$

Complete equations for element 7

$$10^{6}\begin{pmatrix} 0.192308 & 0 & 0 & -0.192308 & -0.192308 & 0.192308 \\ 0 & 0.549451 & -0.164835 & 0 & 0.164835 & -0.549451 \\ 0 & -0.164835 & 0.549451 & 0 & -0.549451 & 0.164835 \\ -0.192308 & 0 & 0 & 0.192308 & 0.192308 & -0.192308 \\ -0.192308 & 0.164835 & -0.549451 & 0.192308 & 0.741758 & -0.357143 \\ 0.192308 & -0.549451 & 0.164835 & -0.192308 & -0.357143 & 0.741758 \end{pmatrix}\begin{pmatrix} u_5 \\ v_5 \\ u_9 \\ v_9 \\ u_8 \\ v_8 \end{pmatrix} = \begin{pmatrix} 0. \\ -15000. \\ 15000. \\ 0. \\ -15000. \\ 15000. \end{pmatrix}$$

The element contributes to {9, 10, 17, 18, 15, 16} global degrees of freedom.

Adding element equations into appropriate locations we have

$$10^6\begin{pmatrix}
0.278214 & 0 & -0.0589159 & 0.0648075 & 0 & 0 & -0.219298 & 0.0657895 \\
0 & 0.674358 & 0.0657895 & -0.0197368 & 0 & 0 & 0.0648075 & -0.654622 \\
-0.0589159 & 0.0657895 & 0.928397 & -0.130597 & -0.32967 & 0.164835 & 0 & 0 \\
0.0648075 & -0.0197368 & -0.130597 & 1.70549 & 0.192308 & -0.115385 & 0 & 0 \\
0 & 0 & -0.32967 & 0.192308 & 0.650183 & -0.357143 & 0 & 0 \\
0 & 0 & 0.164835 & -0.115385 & -0.357143 & 1.03114 & 0 & 0 \\
-0.219298 & 0.0648075 & 0 & 0 & 0 & 0 & 0.937555 & -0.130597 \\
0.0657895 & -0.654622 & 0 & 0 & 0 & 0 & -0.130597 & 1.86941 \\
0 & -0.130597 & -0.539811 & 0.230625 & 0 & 0 & -0.333641 & 0.257115 \\
-0.130597 & 0 & 0.257115 & -1.57037 & 0 & 0 & 0.230625 & -0.115891 \\
0 & 0 & 0 & -0.357143 & -0.320513 & 0.192308 & 0 & 0 \\
0 & 0 & -0.357143 & 0 & 0.164835 & -0.915751 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.384615 & 0.164835 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.192308 & -1.0989 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.357143 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.357143 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Equations for element 8

$h = 5;$ $E = 200000;$ $\nu = 0.3$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 219780. & 65934.1 & 0 \\ 65934.1 & 219780. & 0 \\ 0 & 0 & 76923.1 \end{pmatrix}$

Nodal coordinates

Element node	Global node number	x	y
1	5	50	15
2	6	75	15
3	9	75	40

$x_1 = 50$ $x_2 = 75$ $x_3 = 75$

$y_1 = 15$ $y_2 = 15$ $y_3 = 40$

Using these values we get

$b_1 = -25$ $b_2 = 25$ $b_3 = 0$

$c_1 = 0$ $c_2 = -25$ $c_3 = 25$

$f_1 = 1875$ $f_2 = -875$ $f_3 = -375$

Element area, $A = \frac{625}{2}$

$$\boldsymbol{B}^{T} = \begin{pmatrix} -\frac{1}{25} & 0 & \frac{1}{25} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{25} & 0 & \frac{1}{25} \\ 0 & -\frac{1}{25} & -\frac{1}{25} & \frac{1}{25} & \frac{1}{25} & 0 \end{pmatrix}$$

Thus the element stiffness matrix is

$$\boldsymbol{k} = hA\boldsymbol{BCB}^{T} = 10^{6}\begin{pmatrix} 0.549451 & 0 & -0.549451 & 0.164835 & 0 & -0.164835 \\ 0 & 0.192308 & 0.192308 & -0.192308 & -0.192308 & 0 \\ -0.549451 & 0.192308 & 0.741758 & -0.357143 & -0.192308 & 0.164835 \\ 0.164835 & -0.192308 & -0.357143 & 0.741758 & 0.192308 & -0.549451 \\ 0 & -0.192308 & -0.192308 & 0.192308 & 0.192308 & 0 \\ -0.164835 & 0 & 0.164835 & -0.549451 & 0 & 0.549451 \end{pmatrix}$$

Load vector due to temperature change

$$\alpha = \frac{3}{250000}; \qquad \Delta T = 70; \qquad \boldsymbol{\epsilon}_0^{T} = \begin{pmatrix} \frac{21}{25000} & \frac{21}{25000} & 0 \end{pmatrix}$$

$$\boldsymbol{r}_{\epsilon}^{T} = (\,-15000. \quad 0. \quad 15000. \quad -15000. \quad 0. \quad 15000.\,)$$

Complete equations for element 8

$$10^{6}\begin{pmatrix} 0.549451 & 0 & -0.549451 & 0.164835 & 0 & -0.164835 \\ 0 & 0.192308 & 0.192308 & -0.192308 & -0.192308 & 0 \\ -0.549451 & 0.192308 & 0.741758 & -0.357143 & -0.192308 & 0.164835 \\ 0.164835 & -0.192308 & -0.357143 & 0.741758 & 0.192308 & -0.549451 \\ 0 & -0.192308 & -0.192308 & 0.192308 & 0.192308 & 0 \\ -0.164835 & 0 & 0.164835 & -0.549451 & 0 & 0.549451 \end{pmatrix}\begin{pmatrix} u_5 \\ v_5 \\ u_6 \\ v_6 \\ u_9 \\ v_9 \end{pmatrix} = \begin{pmatrix} -15000. \\ 0. \\ 15000. \\ -15000. \\ 0. \\ 15000. \end{pmatrix}$$

The element contributes to {9, 10, 11, 12, 17, 18} global degrees of freedom.

Adding element equations into appropriate locations we have

$$10^6 \left(\begin{array}{llllllll}
0.278214 & 0 & -0.0589159 & 0.0648075 & 0 & 0 & -0.219298 & 0.0657895 \\
0 & 0.674358 & 0.0657895 & -0.0197368 & 0 & 0 & 0.0648075 & -0.654622 \\
-0.0589159 & 0.0657895 & 0.928397 & -0.130597 & -0.32967 & 0.164835 & 0 & 0 \\
0.0648075 & -0.0197368 & -0.130597 & 1.70549 & 0.192308 & -0.115385 & 0 & 0 \\
0 & 0 & -0.32967 & 0.192308 & 0.650183 & -0.357143 & 0 & 0 \\
0 & 0 & 0.164835 & -0.115385 & -0.357143 & 1.03114 & 0 & 0 \\
-0.219298 & 0.0648075 & 0 & 0 & 0 & 0 & 0.937555 & -0.130597 \\
0.0657895 & -0.654622 & 0 & 0 & 0 & 0 & -0.130597 & 1.86941 \\
0 & -0.130597 & -0.539811 & 0.230625 & 0 & 0 & -0.333641 & 0.257115 \\
-0.130597 & 0 & 0.257115 & -1.57037 & 0 & 0 & 0.230625 & -0.115891 \\
0 & 0 & 0 & -0.357143 & -0.320513 & 0.192308 & 0 & 0 \\
0 & 0 & -0.357143 & 0 & 0.164835 & -0.915751 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.384615 & 0.164835 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.192308 & -1.0989 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.357143 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.357143 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right.$$

Essential boundary conditions

Node	dof	Value
1	u_1 v_1	0 0
2	v_2	0
3	v_3	0
4	u_4	0
7	u_7	0

Remove {1, 2, 4, 6, 7, 13} rows and columns.

After adjusting for essential boundary conditions we have

$$10^{6}\left(\begin{array}{lllllllll}
0.928397 & -0.32967 & 0 & -0.539811 & 0.257115 & 0 & -0.357143 & 0 & 0 \\
-0.32967 & 0.650183 & 0 & 0 & 0 & -0.320513 & 0.164835 & 0 & 0 \\
0 & 0 & 1.86941 & 0.257115 & -0.115891 & 0 & 0 & -1.0989 & -0.35 \\
-0.539811 & 0 & 0.257115 & 2.3295 & -0.714286 & -0.879121 & 0.357143 & 0 & -0.57 \\
0.257115 & 0 & -0.115891 & -0.714286 & 3.64231 & 0.357143 & -0.307692 & 0 & 0.35 \\
0 & -0.320513 & 0 & -0.879121 & 0.357143 & 1.39194 & -0.357143 & 0 & 0 \\
-0.357143 & 0.164835 & 0 & 0.357143 & -0.307692 & -0.357143 & 1.77289 & 0 & 0 \\
0 & 0 & -1.0989 & 0 & 0 & 0 & 0 & 1.19505 & 0.16 \\
0 & 0 & -0.357143 & -0.576923 & 0.357143 & 0 & 0 & 0.164835 & 1.40 \\
0 & 0 & 0 & 0.357143 & -1.64835 & 0 & 0 & -0.0961538 & -0.35 \\
0 & 0 & 0 & 0 & -0.357143 & -0.192308 & 0.192308 & 0 & -0.54 \\
0 & 0 & 0 & -0.357143 & 0 & 0.164835 & -0.549451 & 0 & 0.19
\end{array}\right.$$

Solving the final system of global equations we get

$$\{u_2 = 0.0513447,\ u_3 = 0.0703132,\ v_4 = 0.0252714,\ u_5 = 0.0495551,\ v_5 = 0.0186102,\ u_6 = 0.069253,$$
$$v_6 = 0.0146016,\ v_7 = 0.0445986,\ u_8 = 0.0498168,\ v_8 = 0.0388815,\ u_9 = 0.0716126,\ v_9 = 0.0366734\}$$

Complete table of nodal values

	u	v
1	0	0
2	0.0513447	0
3	0.0703132	0
4	0	0.0252714
5	0.0495551	0.0186102
6	0.069253	0.0146016
7	0	0.0445986
8	0.0498168	0.0388815
9	0.0716126	0.0366734

Computation of reactions

Equation numbers of dof with specified values: {1, 2, 4, 6, 7, 13}

Extracting equations {1, 2, 4, 6, 7, 13} from the global system we have

$$10^6\begin{pmatrix} 0.278214 & 0 & -0.0589159 & 0.0648075 & 0 & 0 & -0.219298 & 0.0657895 \\ 0 & 0.674358 & 0.0657895 & -0.0197368 & 0 & 0 & 0.0648075 & -0.654622 \\ 0.0648075 & -0.0197368 & -0.130597 & 1.70549 & 0.192308 & -0.115385 & 0 & 0 \\ 0 & 0 & 0.164835 & -0.115385 & -0.357143 & 1.03114 & 0 & 0 \\ -0.219298 & 0.0648075 & 0 & 0 & 0 & 0 & 0.937555 & -0.130597 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.384615 & 0.164835 \end{pmatrix}$$

Substituting the nodal values and re−arranging

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{pmatrix} = 10^6\begin{pmatrix} 0.278214 & 0 & -0.0589159 & 0.0648075 & 0 & 0 & -0.219298 & 0.0 \\ 0 & 0.674358 & 0.0657895 & -0.0197368 & 0 & 0 & 0.0648075 & -0.6 \\ 0.0648075 & -0.0197368 & -0.130597 & 1.70549 & 0.192308 & -0.115385 & 0 & 0 \\ 0 & 0 & 0.164835 & -0.115385 & -0.357143 & 1.03114 & 0 & 0 \\ -0.219298 & 0.0648075 & 0 & 0 & 0 & 0 & 0.937555 & -0.1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.384615 & 0.1 \end{pmatrix}$$

Carrying out computations, the reactions are as follows.

Label	dof	Reaction
R_1	u_1	2514.97
R_2	v_1	1389.1
R_3	v_2	312.884
R_4	v_3	−1701.98
R_5	u_4	456.218
R_6	u_7	−2971.19

Sum of Reactions

dof: u	0
dof: v	0

Solution for element 1

$h = 5;$ $E = 70000;$ $\nu = 0.33$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 78554.6 & 25923. & 0 \\ 25923. & 78554.6 & 0 \\ 0 & 0 & 26315.8 \end{pmatrix}$

Element nodes: First node (node # 1): {0, 0}
Second node (node # 5): {50, 15} Third node (node # 4): {0, 15}

$x_1 = 0$ $x_2 = 50$ $x_3 = 0$
$y_1 = 0$ $y_2 = 15$ $y_3 = 15$

Using these values we get

$b_1 = 0$ $b_2 = 15$ $b_3 = -15$

$c_1 = -50$ $c_2 = 0$ $c_3 = 50$

$f_1 = 750$ $f_2 = 0$ $f_3 = 0$

Element area, A = 375

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & \frac{1}{50} & 0 & -\frac{1}{50} & 0 \\ 0 & -\frac{1}{15} & 0 & 0 & 0 & \frac{1}{15} \\ -\frac{1}{15} & 0 & 0 & \frac{1}{50} & \frac{1}{15} & -\frac{1}{50} \end{pmatrix}$$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\left\{1 - \frac{y}{15}, \frac{x}{50}, \frac{y}{15} - \frac{x}{50}\right\}$

$$\boldsymbol{N}^{\mathrm{T}} = \begin{pmatrix} 1-\frac{y}{15} & 0 & \frac{x}{50} & 0 & \frac{y}{15}-\frac{x}{50} & 0 \\ 0 & 1-\frac{y}{15} & 0 & \frac{x}{50} & 0 & \frac{y}{15}-\frac{x}{50} \end{pmatrix}$$

From global solution the displacements at the element nodes are

(displacements at nodes {1, 5, 4}):

$\boldsymbol{d}^{\mathrm{T}} = \{0, 0, 0.0495551, 0.0186102, 0, 0.0252714\}$

The displacement distribution over the element is

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{d} = \begin{pmatrix} 0.000991101\,x \\ 0.00168476\,y - 0.000133224\,x \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{d} = (\,0.000991101 \quad 0.00168476 \quad -0.000133224\,)$

Initial strains: $\boldsymbol{\epsilon}_0^{\mathrm{T}} = \left(\,\frac{161}{100000} \quad \frac{161}{100000} \quad 0\,\right)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0) = (\,-46.6793 \quad -10.1709 \quad -3.50591\,)$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{\mathrm{T}} = (\,0.000991101 \quad 0.00168476 \quad 0.00187801 \quad -0.000133224 \quad 0 \quad 0\,)$

$\boldsymbol{\sigma}^{\mathrm{T}} = (\,-46.6793 \quad -10.1709 \quad 0 \quad -3.50591 \quad 0 \quad 0\,)$

Substituting these stress components into appropriate formulas

Principal stresses = $(\,0 \quad -9.83725 \quad -47.0129\,)$

Effective stress (von Mises) = 42.9477

Solution for element 2

$h = 5;\qquad E = 70000;\qquad \nu = 0.33$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 78554.6 & 25923. & 0 \\ 25923. & 78554.6 & 0 \\ 0 & 0 & 26315.8 \end{pmatrix}$

Element nodes: First node (node # 1): {0, 0}

Second node (node # 2): {50, 0} Third node (node # 5): {50, 15}

$x_1 = 0 \qquad x_2 = 50 \qquad x_3 = 50$

$y_1 = 0 \qquad y_2 = 0 \qquad y_3 = 15$

Using these values we get

$b_1 = -15 \qquad b_2 = 15 \qquad b_3 = 0$

$c_1 = 0 \qquad c_2 = -50 \qquad c_3 = 50$

$f_1 = 750$ $f_2 = 0$ $f_3 = 0$

Element area, A = 375

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -\frac{1}{50} & 0 & \frac{1}{50} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{15} & 0 & \frac{1}{15} \\ 0 & -\frac{1}{50} & -\frac{1}{15} & \frac{1}{50} & \frac{1}{15} & 0 \end{pmatrix}$$

Substituting these into the formulas for triangle interpolation functions we get

$$\text{Interpolation functions, } \left\{1 - \frac{x}{50},\ \frac{x}{50} - \frac{y}{15},\ \frac{y}{15}\right\}$$

$$\boldsymbol{N}^{\mathrm{T}} = \begin{pmatrix} 1 - \frac{x}{50} & 0 & \frac{x}{50} - \frac{y}{15} & 0 & \frac{y}{15} & 0 \\ 0 & 1 - \frac{x}{50} & 0 & \frac{x}{50} - \frac{y}{15} & 0 & \frac{y}{15} \end{pmatrix}$$

From global solution the displacements at the element nodes are (displacements at nodes {1, 2, 5}):

$$\boldsymbol{d}^{\mathrm{T}} = \{0, 0, 0.0513447, 0, 0.0495551, 0.0186102\}$$

The displacement distribution over the element is

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{d} = \begin{pmatrix} 0.00102689\,x - 0.000119307\,y \\ 0.00124068\,y \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{d} = (\, 0.00102689 \quad 0.00124068 \quad -0.000119307 \,)$

Initial strains: $\boldsymbol{\epsilon}_0^{\mathrm{T}} = \left(\, \frac{161}{100000} \quad \frac{161}{100000} \quad 0 \,\right)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0) = (\, -55.3796 \quad -44.1277 \quad -3.13967 \,)$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$$\boldsymbol{\epsilon}^{\mathrm{T}} = (\, 0.00102689 \quad 0.00124068 \quad 0.00207911 \quad -0.000119307 \quad 0 \quad 0 \,)$$

$$\boldsymbol{\sigma}^{\mathrm{T}} = (\, -55.3796 \quad -44.1277 \quad 0 \quad -3.13967 \quad 0 \quad 0 \,)$$

Substituting these stress components into appropriate formulas

Principal stresses = $(\, 0 \quad -43.3109 \quad -56.1964 \,)$

Effective stress (von Mises) = 50.9897

Solution for element 3

$h = 5;$ $E = 200000;$ $\nu = 0.3$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 219780. & 65934.1 & 0 \\ 65934.1 & 219780. & 0 \\ 0 & 0 & 76923.1 \end{pmatrix}$

Element nodes: First node (node # 2): {50, 0}
Second node (node # 6): {75, 15} Third node (node # 5): {50, 15}

$x_1 = 50$ $x_2 = 75$ $x_3 = 50$
$y_1 = 0$ $y_2 = 15$ $y_3 = 15$

Using these values we get

$b_1 = 0$ $b_2 = 15$ $b_3 = -15$

$c_1 = -25$ $c_2 = 0$ $c_3 = 25$

$f_1 = 375$ $f_2 = -750$ $f_3 = 750$

Element area, $A = \frac{375}{2}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & \frac{1}{25} & 0 & -\frac{1}{25} & 0 \\ 0 & -\frac{1}{15} & 0 & 0 & 0 & \frac{1}{15} \\ -\frac{1}{15} & 0 & 0 & \frac{1}{25} & \frac{1}{15} & -\frac{1}{25} \end{pmatrix}$$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\left\{1 - \frac{y}{15}, \frac{x}{25} - 2, -\frac{x}{25} + \frac{y}{15} + 2\right\}$

$$\boldsymbol{N}^{\mathrm{T}} = \begin{pmatrix} 1 - \frac{y}{15} & 0 & \frac{x}{25} - 2 & 0 & -\frac{x}{25} + \frac{y}{15} + 2 & 0 \\ 0 & 1 - \frac{y}{15} & 0 & \frac{x}{25} - 2 & 0 & -\frac{x}{25} + \frac{y}{15} + 2 \end{pmatrix}$$

From global solution the displacements at the element nodes are
(displacements at nodes {2, 6, 5}):
$\boldsymbol{d}^{\mathrm{T}} = \{0.0513447, 0, 0.069253, 0.0146016, 0.0495551, 0.0186102\}$

The displacement distribution over the element is

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{d} = \begin{pmatrix} 0.000787917\,x - 0.000119307\,y + 0.0119489 \\ -0.000160344\,x + 0.00124068\,y + 0.0080172 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{d} = \begin{pmatrix} 0.000787917 & 0.00124068 & -0.000279651 \end{pmatrix}$

Initial strains: $\boldsymbol{\epsilon}_0^{\mathrm{T}} = \begin{pmatrix} \frac{21}{25000} & \frac{21}{25000} & 0 \end{pmatrix}$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0) = \begin{pmatrix} 14.9716 & 84.6275 & -21.5116 \end{pmatrix}$

Computing out−of−plane strain and stress components using
appropriate formulas, the complete strain and stress vectors are as follows.

$$\boldsymbol{\epsilon}^{T} = (\,0.000787917 \quad 0.00124068 \quad 0.000690601 \quad -0.000279651 \quad 0 \quad 0\,)$$

$$\boldsymbol{\sigma}^{T} = (\,14.9716 \quad 84.6275 \quad 0 \quad -21.5116 \quad 0 \quad 0\,)$$

Substituting these stress components into appropriate formulas

Principal stresses = (90.7353   8.86375   0)

Effective stress (von Mises) = 86.6441

Solution for element 4

$h = 5;$ $E = 200000;$ $\nu = 0.3$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 219780. & 65934.1 & 0 \\ 65934.1 & 219780. & 0 \\ 0 & 0 & 76923.1 \end{pmatrix}$

Element nodes: First node (node # 2): {50, 0}
Second node (node # 3): {75, 0} Third node (node # 6): {75, 15}

$x_1 = 50$ $x_2 = 75$ $x_3 = 75$
$y_1 = 0$ $y_2 = 0$ $y_3 = 15$

Using these values we get

$b_1 = -15$ $b_2 = 15$ $b_3 = 0$

$c_1 = 0$ $c_2 = -25$ $c_3 = 25$

$f_1 = 1125$ $f_2 = -750$ $f_3 = 0$

Element area, $A = \frac{375}{2}$

$$\boldsymbol{B}^{T} = \begin{pmatrix} -\frac{1}{25} & 0 & \frac{1}{25} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{15} & 0 & \frac{1}{15} \\ 0 & -\frac{1}{25} & -\frac{1}{15} & \frac{1}{25} & \frac{1}{15} & 0 \end{pmatrix}$$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\left\{3 - \frac{x}{25},\ \frac{x}{25} - \frac{y}{15} - 2,\ \frac{y}{15}\right\}$

$$\boldsymbol{N}^{T} = \begin{pmatrix} 3 - \frac{x}{25} & 0 & \frac{x}{25} - \frac{y}{15} - 2 & 0 & \frac{y}{15} & 0 \\ 0 & 3 - \frac{x}{25} & 0 & \frac{x}{25} - \frac{y}{15} - 2 & 0 & \frac{y}{15} \end{pmatrix}$$

From global solution the displacements at the element nodes are
(displacements at nodes {2, 3, 6}):
$\boldsymbol{d}^{T} = \{0.0513447, 0, 0.0703132, 0, 0.069253, 0.0146016\}$

The displacement distribution over the element is

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \boldsymbol{N}^T \boldsymbol{d} = \begin{pmatrix} 0.000758739\,x - 0.0000706781\,y + 0.0134077 \\ 0.00097344\,y \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^T \boldsymbol{d} = (\, 0.000758739 \quad 0.00097344 \quad -0.0000706781 \,)$

Initial strains: $\boldsymbol{\epsilon}_0^T = \left(\frac{21}{25000} \quad \frac{21}{25000} \quad 0 \right)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0) = (\, -9.06129 \quad 23.9696 \quad -5.43678 \,)$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$$\boldsymbol{\epsilon}^T = (\, 0.000758739 \quad 0.00097344 \quad 0.000817638 \quad -0.0000706781 \quad 0 \quad 0 \,)$$

$$\boldsymbol{\sigma}^T = (\, -9.06129 \quad 23.9696 \quad 0 \quad -5.43678 \quad 0 \quad 0 \,)$$

Substituting these stress components into appropriate formulas

Principal stresses = (24.8415   0   −9.93316)

Effective stress (von Mises) = 31.0245

Solution for element 5

$h = 5;\qquad E = 200000;\qquad \nu = 0.3$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 219780. & 65934.1 & 0 \\ 65934.1 & 219780. & 0 \\ 0 & 0 & 76923.1 \end{pmatrix}$

Element nodes: First node (node # 4): {0, 15} Second node (node # 8): {50, 40} Third node (node # 7): {0, 40}

$x_1 = 0 \qquad x_2 = 50 \qquad x_3 = 0$
$y_1 = 15 \qquad y_2 = 40 \qquad y_3 = 40$

Using these values we get

$b_1 = 0 \qquad b_2 = 25 \qquad b_3 = -25$

$c_1 = -50 \qquad c_2 = 0 \qquad c_3 = 50$

$f_1 = 2000 \qquad f_2 = 0 \qquad f_3 = -750$

Element area, A = 625

$$\boldsymbol{B}^T = \begin{pmatrix} 0 & 0 & \frac{1}{50} & 0 & -\frac{1}{50} & 0 \\ 0 & -\frac{1}{25} & 0 & 0 & 0 & \frac{1}{25} \\ -\frac{1}{25} & 0 & 0 & \frac{1}{50} & \frac{1}{25} & -\frac{1}{50} \end{pmatrix}$$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\left\{\frac{8}{5}-\frac{y}{25}, \frac{x}{50}, -\frac{x}{50}+\frac{y}{25}-\frac{3}{5}\right\}$

$$\boldsymbol{N}^{T}=\begin{pmatrix} \frac{8}{5}-\frac{y}{25} & 0 & \frac{x}{50} & 0 & -\frac{x}{50}+\frac{y}{25}-\frac{3}{5} & 0 \\ 0 & \frac{8}{5}-\frac{y}{25} & 0 & \frac{x}{50} & 0 & -\frac{x}{50}+\frac{y}{25}-\frac{3}{5} \end{pmatrix}$$

From global solution the displacements at the element nodes are
(displacements at nodes {4, 8, 7}):

$\boldsymbol{d}^{T} = \{0, 0.0252714, 0.0498168, 0.0388815, 0, 0.0445986\}$

The displacement distribution over the element is

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \boldsymbol{N}^{T}\boldsymbol{d} = \begin{pmatrix} 0.000996336\,x \\ -0.000114342\,x + 0.000773089\,y + 0.0136751 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{T}\boldsymbol{d} = (\,0.000996336 \quad 0.000773089 \quad -0.000114342\,)$

Initial strains: $\boldsymbol{\epsilon}_0^{T} = \left(\,\frac{21}{25000} \quad \frac{21}{25000} \quad 0\,\right)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0) = (\,29.9479 \quad -4.39778 \quad -8.79555\,)$

Computing out−of−plane strain and stress components using
appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{T} = (\,0.000996336 \quad 0.000773089 \quad 0.000801675 \quad -0.000114342 \quad 0 \quad 0\,)$

$\boldsymbol{\sigma}^{T} = (\,29.9479 \quad -4.39778 \quad 0 \quad -8.79555 \quad 0 \quad 0\,)$

Substituting these stress components into appropriate formulas

Principal stresses = $(\,32.0694 \quad 0 \quad -6.51919\,)$

Effective stress (von Mises) = 35.7772

Solution for element 6

$h = 5;$ $E = 200000;$ $\nu = 0.3$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 219780. & 65934.1 & 0 \\ 65934.1 & 219780. & 0 \\ 0 & 0 & 76923.1 \end{pmatrix}$

Element nodes: First node (node # 4): {0, 15}
Second node (node # 5): {50, 15} Third node (node # 8): {50, 40}

$x_1 = 0$ $x_2 = 50$ $x_3 = 50$
$y_1 = 15$ $y_2 = 15$ $y_3 = 40$

Using these values we get

$b_1 = -25 \quad b_2 = 25 \quad b_3 = 0$

$c_1 = 0 \quad c_2 = -50 \quad c_3 = 50$

$f_1 = 1250 \quad f_2 = 750 \quad f_3 = -750$

Element area, A = 625

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -\frac{1}{50} & 0 & \frac{1}{50} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{25} & 0 & \frac{1}{25} \\ 0 & -\frac{1}{50} & -\frac{1}{25} & \frac{1}{50} & \frac{1}{25} & 0 \end{pmatrix}$$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\left\{1 - \frac{x}{50}, \frac{x}{50} - \frac{y}{25} + \frac{3}{5}, \frac{y}{25} - \frac{3}{5}\right\}$

$$\boldsymbol{N}^{\mathrm{T}} = \begin{pmatrix} 1 - \frac{x}{50} & 0 & \frac{x}{50} - \frac{y}{25} + \frac{3}{5} & 0 & \frac{y}{25} - \frac{3}{5} & 0 \\ 0 & 1 - \frac{x}{50} & 0 & \frac{x}{50} - \frac{y}{25} + \frac{3}{5} & 0 & \frac{y}{25} - \frac{3}{5} \end{pmatrix}$$

From global solution the displacements at the element nodes are

(displacements at nodes {4, 5, 8}):

$\boldsymbol{d}^{\mathrm{T}} = \{0, 0.0252714, 0.0495551, 0.0186102, 0.0498168, 0.0388815\}$

The displacement distribution over the element is

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{d} = \begin{pmatrix} 0.000991101\,x + 0.0000104699\,y - 0.000157049 \\ -0.000133224\,x + 0.000810854\,y + 0.0131086 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{d} = (\,0.000991101 \quad 0.000810854 \quad -0.000122754\,)$

Initial strains: $\boldsymbol{\epsilon}_0^{\mathrm{T}} = \left(\,\frac{21}{25000} \quad \frac{21}{25000} \quad 0\,\right)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0) = (\,31.2874 \quad 3.55695 \quad -9.44265\,)$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$$\boldsymbol{\epsilon}^{\mathrm{T}} = (\,0.000991101 \quad 0.000810854 \quad 0.000787734 \quad -0.000122754 \quad 0 \quad 0\,)$$

$$\boldsymbol{\sigma}^{\mathrm{T}} = (\,31.2874 \quad 3.55695 \quad 0 \quad -9.44265 \quad 0 \quad 0\,)$$

Substituting these stress components into appropriate formulas

Principal stresses = (34.1974   0.646946   0)

Effective stress (von Mises) = 33.8785

Solution for element 7

h = 5; E = 200000; $\nu = 0.3$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 219780. & 65934.1 & 0 \\ 65934.1 & 219780. & 0 \\ 0 & 0 & 76923.1 \end{pmatrix}$

Element nodes: First node (node # 5): {50, 15}

Second node (node # 9): {75, 40} Third node (node # 8): {50, 40}

$x_1 = 50$ $x_2 = 75$ $x_3 = 50$

$y_1 = 15$ $y_2 = 40$ $y_3 = 40$

Using these values we get

$b_1 = 0$ $b_2 = 25$ $b_3 = -25$

$c_1 = -25$ $c_2 = 0$ $c_3 = 25$

$f_1 = 1000$ $f_2 = -1250$ $f_3 = 875$

Element area, $A = \frac{625}{2}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & \frac{1}{25} & 0 & -\frac{1}{25} & 0 \\ 0 & -\frac{1}{25} & 0 & 0 & 0 & \frac{1}{25} \\ -\frac{1}{25} & 0 & 0 & \frac{1}{25} & \frac{1}{25} & -\frac{1}{25} \end{pmatrix}$$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\left\{\frac{8}{5} - \frac{y}{25}, \frac{x}{25} - 2, -\frac{x}{25} + \frac{y}{25} + \frac{7}{5}\right\}$

$$\boldsymbol{N}^{\mathrm{T}} = \begin{pmatrix} \frac{8}{5} - \frac{y}{25} & 0 & \frac{x}{25} - 2 & 0 & -\frac{x}{25} + \frac{y}{25} + \frac{7}{5} & 0 \\ 0 & \frac{8}{5} - \frac{y}{25} & 0 & \frac{x}{25} - 2 & 0 & -\frac{x}{25} + \frac{y}{25} + \frac{7}{5} \end{pmatrix}$$

From global solution the displacements at the element nodes are

(displacements at nodes {5, 9, 8}):

$\boldsymbol{d}^{\mathrm{T}} = \{0.0495551, 0.0186102, 0.0716126, 0.0366734, 0.0498168, 0.0388815\}$

The displacement distribution over the element is

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{d} = \begin{pmatrix} 0.00087183\,x + 0.0000104699\,y + 0.00580652 \\ -0.0000883238\,x + 0.000810854\,y + 0.0108636 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{d} = \begin{pmatrix} 0.00087183 & 0.000810854 & -0.0000778538 \end{pmatrix}$

Initial strains: $\boldsymbol{\epsilon}_0^{\mathrm{T}} = \begin{pmatrix} \frac{21}{25000} & \frac{21}{25000} & 0 \end{pmatrix}$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0) = \begin{pmatrix} 5.07389 & -4.3071 & -5.98876 \end{pmatrix}$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$$\boldsymbol{\epsilon}^{T} = (\,0.00087183 \quad 0.000810854 \quad 0.00083885 \quad -0.0000778538 \quad 0 \quad 0\,)$$

$$\boldsymbol{\sigma}^{T} = (\,5.07389 \quad -4.3071 \quad 0 \quad -5.98876 \quad 0 \quad 0\,)$$

Substituting these stress components into appropriate formulas

Principal stresses = (7.99036   0   −7.22357)

Effective stress (von Mises) = 13.1812

Solution for element 8

$h = 5;$ $E = 200000;$ $\nu = 0.3$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 219780. & 65934.1 & 0 \\ 65934.1 & 219780. & 0 \\ 0 & 0 & 76923.1 \end{pmatrix}$

Element nodes: First node (node # 5): {50, 15}
Second node (node # 6): {75, 15} Third node (node # 9): {75, 40}

$x_1 = 50$ $x_2 = 75$ $x_3 = 75$
$y_1 = 15$ $y_2 = 15$ $y_3 = 40$

Using these values we get

$b_1 = -25$ $b_2 = 25$ $b_3 = 0$

$c_1 = 0$ $c_2 = -25$ $c_3 = 25$

$f_1 = 1875$ $f_2 = -875$ $f_3 = -375$

Element area, $A = \frac{625}{2}$

$$\boldsymbol{B}^{T} = \begin{pmatrix} -\frac{1}{25} & 0 & \frac{1}{25} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{25} & 0 & \frac{1}{25} \\ 0 & -\frac{1}{25} & -\frac{1}{25} & \frac{1}{25} & \frac{1}{25} & 0 \end{pmatrix}$$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\left\{3 - \frac{x}{25}, \frac{x}{25} - \frac{y}{25} - \frac{7}{5}, \frac{y}{25} - \frac{3}{5}\right\}$

$$\boldsymbol{N}^{T} = \begin{pmatrix} 3 - \frac{x}{25} & 0 & \frac{x}{25} - \frac{y}{25} - \frac{7}{5} & 0 & \frac{y}{25} - \frac{3}{5} & 0 \\ 0 & 3 - \frac{x}{25} & 0 & \frac{x}{25} - \frac{y}{25} - \frac{7}{5} & 0 & \frac{y}{25} - \frac{3}{5} \end{pmatrix}$$

From global solution the displacements at the element nodes are
(displacements at nodes {5, 6, 9}):

$$\boldsymbol{d}^{\mathrm{T}} = \{0.0495551, 0.0186102, 0.069253, 0.0146016, 0.0716126, 0.0366734\}$$

The displacement distribution over the element is

$$\begin{pmatrix} \mathrm{u(x,y)} \\ \mathrm{v(x,y)} \end{pmatrix} = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{d} = \begin{pmatrix} 0.000787917\,\mathrm{x} + 0.0000943834\,\mathrm{y} + 0.00874349 \\ -0.000160344\,\mathrm{x} + 0.000882874\,\mathrm{y} + 0.0133843 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{d} = (\ 0.000787917 \quad 0.000882874 \quad -0.0000659605\ )$

Initial strains: $\boldsymbol{\epsilon}_0^{\mathrm{T}} = \left(\ \frac{21}{25000} \quad \frac{21}{25000} \quad 0\ \right)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0) = (\ -8.62005 \quad 5.98876 \quad -5.07389\ )$

Computing out−of−plane strain and stress components using
appropriate formulas, the complete strain and stress vectors are as follows.

$$\boldsymbol{\epsilon}^{\mathrm{T}} = (\ 0.000787917 \quad 0.000882874 \quad 0.000843947 \quad -0.0000659605 \quad 0 \quad 0\ )$$

$$\boldsymbol{\sigma}^{\mathrm{T}} = (\ -8.62005 \quad 5.98876 \quad 0 \quad -5.07389 \quad 0 \quad 0\ )$$

Substituting these stress components into appropriate formulas

Principal stresses = (7.57809   0   −10.2094)

Effective stress (von Mises) = 15.4605

Solution summary

Nodal solution

	x	y	u	v
1	0	0	0	0
2	50	0	0.0513447	0
3	75	0	0.0703132	0
4	0	15	0	0.0252714
5	50	15	0.0495551	0.0186102
6	75	15	0.069253	0.0146016
7	0	40	0	0.0445986
8	50	40	0.0498168	0.0388815
9	75	40	0.0716126	0.0366734

Solution at element centers

	Coord	Disp	Stresses	Principal stresses	Effective Stress
1	$\frac{50}{3}$ 10	0.0165184 0.0146272	−46.6793 −10.1709 0 −3.50591 0 0	0 −9.83725 −47.0129	42.9477
2	$\frac{100}{3}$ 5	0.0336333 0.0062034	−55.3796 −44.1277 0 −3.13967 0 0	0 −43.3109 −56.1964	50.9897
3	$\frac{175}{3}$ 10	0.0567176 0.0110706	14.9716 84.6275 0 −21.5116 0 0	90.7353 8.86375 0	86.6441
4	$\frac{200}{3}$ 5	0.0636369 0.0048672	−9.06129 23.9696 0 −5.43678 0 0	24.8415 0 −9.93316	31.0245
5	$\frac{50}{3}$ $\frac{95}{3}$	0.0166056 0.0362505	29.9479 −4.39778 0 −8.79555 0 0	32.0694 0 −6.51919	35.7772
6	$\frac{100}{3}$ $\frac{70}{3}$	0.033124 0.0275877	31.2874 3.55695 0 −9.44265 0 0	34.1974 0.646946 0	33.8785
7	$\frac{175}{3}$ $\frac{95}{3}$	0.0569948 0.0313884	5.07389 −4.3071 0 −5.98876 0 0	7.99036 0 −7.22357	13.1812
8	$\frac{200}{3}$ $\frac{70}{3}$	0.0634735 0.0232951	−8.62005 5.98876 0 −5.07389 0 0	7.57809 0 −10.2094	15.4605

Support reactions

Node	dof	Reaction
1	1	2514.97
1	2	1389.1
2	2	312.884
3	2	−1701.98
4	1	456.218
7	1	−2971.19

Sum of applied loads → (0   0)

Sum of support reactions → (0   0)

Figure shows equivalent von-Mises stresses in different elements. There are large differences in stresses among neighboring elements, indicating that the solution is not reliable and mesh must be refined.