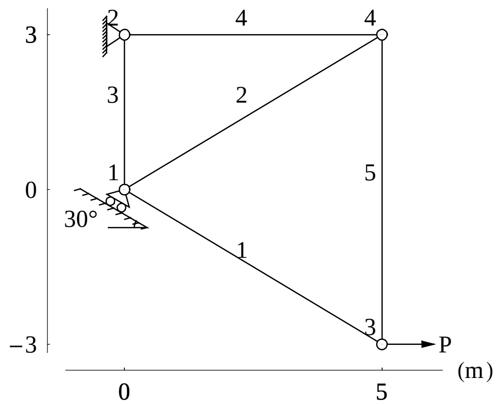
Example 1.17: Five bar truss with inclined support (p. 76)

Consider a five bar pin-jointed structure shown in Figure. All members have the same cross-sectional area and are of the same material, $E = 70$ GPa and $A = 10^{-3}\ m^2$. The load $P = 20$ kN.



For numerical calculations use the N – mm units are convenient. The displacements will be in mm and the stresses in MPa. The complete computations are as follows.

Specified nodal loads

Node	dof	Value
3	u_3	20000.

Global equations at start of the element assembly process

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 20000. \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 1

$E = 70000 \qquad A = 1000$

Element node	Global node number	x	y
1	1	0	0
2	3	5000.	−3000.

$x_1 = 0 \qquad y_1 = 0 \qquad x_2 = 5000. \qquad y_2 = -3000.$

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 5830.95$$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 0.857493$ $\quad m_s = \dfrac{y_2 - y_1}{L} = -0.514496$

Substituting into the truss element equations we get

$$\begin{pmatrix} 8827.13 & -5296.28 & -8827.13 & 5296.28 \\ -5296.28 & 3177.77 & 5296.28 & -3177.77 \\ -8827.13 & 5296.28 & 8827.13 & -5296.28 \\ 5296.28 & -3177.77 & -5296.28 & 3177.77 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {1, 2, 5, 6} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 8827.13 & -5296.28 & 0 & 0 & -8827.13 & 5296.28 & 0 & 0 \\ -5296.28 & 3177.77 & 0 & 0 & 5296.28 & -3177.77 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -8827.13 & 5296.28 & 0 & 0 & 8827.13 & -5296.28 & 0 & 0 \\ 5296.28 & -3177.77 & 0 & 0 & -5296.28 & 3177.77 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 20000. \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 2

$E = 70000$ $\quad A = 1000$

Element node	Global node number	x	y
1	1	0	0
2	4	5000.	3000.

$x_1 = 0$ $\quad y_1 = 0$ $\quad x_2 = 5000.$ $\quad y_2 = 3000.$

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 5830.95$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 0.857493$ $\quad m_s = \dfrac{y_2 - y_1}{L} = 0.514496$

Substituting into the truss element equations we get

$$\begin{pmatrix} 8827.13 & 5296.28 & -8827.13 & -5296.28 \\ 5296.28 & 3177.77 & -5296.28 & -3177.77 \\ -8827.13 & -5296.28 & 8827.13 & 5296.28 \\ -5296.28 & -3177.77 & 5296.28 & 3177.77 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {1, 2, 7, 8} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 17654.3 & 0 & 0 & 0 & -8827.13 & 5296.28 & -8827.13 & -5296.28 \\ 0 & 6355.54 & 0 & 0 & 5296.28 & -3177.77 & -5296.28 & -3177.77 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -8827.13 & 5296.28 & 0 & 0 & 8827.13 & -5296.28 & 0 & 0 \\ 5296.28 & -3177.77 & 0 & 0 & -5296.28 & 3177.77 & 0 & 0 \\ -8827.13 & -5296.28 & 0 & 0 & 0 & 0 & 8827.13 & 5296.28 \\ -5296.28 & -3177.77 & 0 & 0 & 0 & 0 & 5296.28 & 3177.77 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 20000. \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 3

$E = 70000 \qquad A = 1000$

Element node	Global node number	x	y
1	1	0	0
2	2	0	3000.

$x_1 = 0 \qquad y_1 = 0 \qquad x_2 = 0 \qquad y_2 = 3000.$

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 3000.$$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 0 \qquad m_s = \dfrac{y_2 - y_1}{L} = 1.$

Substituting into the truss element equations we get

$$\begin{pmatrix} 0. & 0. & 0. & 0. \\ 0. & 23333.3 & 0. & -23333.3 \\ 0. & 0. & 0. & 0. \\ 0. & -23333.3 & 0. & 23333.3 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {1, 2, 3, 4} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 17654.3 & 0 & 0 & 0 & -8827.13 & 5296.28 & -8827.13 & -5296.28 \\ 0 & 29688.9 & 0 & -23333.3 & 5296.28 & -3177.77 & -5296.28 & -3177.77 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -23333.3 & 0 & 23333.3 & 0 & 0 & 0 & 0 \\ -8827.13 & 5296.28 & 0 & 0 & 8827.13 & -5296.28 & 0 & 0 \\ 5296.28 & -3177.77 & 0 & 0 & -5296.28 & 3177.77 & 0 & 0 \\ -8827.13 & -5296.28 & 0 & 0 & 0 & 0 & 8827.13 & 5296.28 \\ -5296.28 & -3177.77 & 0 & 0 & 0 & 0 & 5296.28 & 3177.77 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 20000. \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 4

$E = 70000 \qquad A = 1000$

Element node	Global node number	x	y
1	2	0	3000.
2	4	5000.	3000.

$x_1 = 0 \qquad y_1 = 3000. \qquad x_2 = 5000. \qquad y_2 = 3000.$

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 5000.$$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 1. \qquad m_s = \dfrac{y_2 - y_1}{L} = 0.$

Substituting into the truss element equations we get

$$\begin{pmatrix} 14000. & 0. & -14000. & 0. \\ 0. & 0. & 0. & 0. \\ -14000. & 0. & 14000. & 0. \\ 0. & 0. & 0. & 0. \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {3, 4, 7, 8} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 17654.3 & 0 & 0 & 0 & -8827.13 & 5296.28 & -8827.13 & -5296.28 \\ 0 & 29688.9 & 0 & -23333.3 & 5296.28 & -3177.77 & -5296.28 & -3177.77 \\ 0 & 0 & 14000. & 0 & 0 & 0 & -14000. & 0 \\ 0 & -23333.3 & 0 & 23333.3 & 0 & 0 & 0 & 0 \\ -8827.13 & 5296.28 & 0 & 0 & 8827.13 & -5296.28 & 0 & 0 \\ 5296.28 & -3177.77 & 0 & 0 & -5296.28 & 3177.77 & 0 & 0 \\ -8827.13 & -5296.28 & -14000. & 0 & 0 & 0 & 22827.1 & 5296.28 \\ -5296.28 & -3177.77 & 0 & 0 & 0 & 0 & 5296.28 & 3177.77 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 20000. \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 5

$E = 70000 \qquad A = 1000$

Element node	Global node number	x	y
1	3	5000.	−3000.
2	4	5000.	3000.

$x_1 = 5000. \qquad y_1 = -3000. \qquad x_2 = 5000. \qquad y_2 = 3000.$

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 6000.$$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 0. \qquad m_s = \dfrac{y_2 - y_1}{L} = 1.$

Substituting into the truss element equations we get

$$\begin{pmatrix} 0. & 0. & 0. & 0. \\ 0. & 11666.7 & 0. & -11666.7 \\ 0. & 0. & 0. & 0. \\ 0. & -11666.7 & 0. & 11666.7 \end{pmatrix} \begin{pmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {5, 6, 7, 8} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 17654.3 & 0 & 0 & 0 & -8827.13 & 5296.28 & -8827.13 & -5296.28 \\ 0 & 29688.9 & 0 & -23333.3 & 5296.28 & -3177.77 & -5296.28 & -3177.77 \\ 0 & 0 & 14000. & 0 & 0 & 0 & -14000. & 0 \\ 0 & -23333.3 & 0 & 23333.3 & 0 & 0 & 0 & 0 \\ -8827.13 & 5296.28 & 0 & 0 & 8827.13 & -5296.28 & 0 & 0 \\ 5296.28 & -3177.77 & 0 & 0 & -5296.28 & 14844.4 & 0 & -11666.7 \\ -8827.13 & -5296.28 & -14000. & 0 & 0 & 0 & 22827.1 & 5296.28 \\ -5296.28 & -3177.77 & 0 & 0 & 0 & -11666.7 & 5296.28 & 14844.4 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 20000. \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Essential boundary conditions

Node	dof	Value
2	u_2	0
	v_2	0

Remove {3, 4} rows and columns.

After adjusting for essential boundary conditions we have

$$\begin{pmatrix} 17654.3 & 0 & -8827.13 & 5296.28 & -8827.13 & -5296.28 \\ 0 & 29688.9 & 5296.28 & -3177.77 & -5296.28 & -3177.77 \\ -8827.13 & 5296.28 & 8827.13 & -5296.28 & 0 & 0 \\ 5296.28 & -3177.77 & -5296.28 & 14844.4 & 0 & -11666.7 \\ -8827.13 & -5296.28 & 0 & 0 & 22827.1 & 5296.28 \\ -5296.28 & -3177.77 & 0 & -11666.7 & 5296.28 & 14844.4 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 20000. \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Multipoint constraint due to inclined support at node 1: $u_1 \sin(\pi/6) + v_1 \cos(\pi/6) = 0$. The augmented global equations with the Lagrange multiplier are as follows.

$$\begin{pmatrix} 17654.3 & 0 & -8827.13 & 5296.28 & -8827.13 & -5296.28 & 1/2 \\ 0 & 29688.9 & 5296.28 & -3177.77 & -5296.28 & -3177.77 & \frac{\sqrt{3}}{2} \\ -8827.13 & 5296.28 & 8827.13 & -5296.28 & 0 & 0 & 0 \\ 5296.28 & -3177.77 & -5296.28 & 14844.4 & 0 & -11666.7 & 0 \\ -8827.13 & -5296.28 & 0 & 0 & 22827.1 & 5296.28 & 0 \\ -5296.28 & -3177.77 & 0 & -11666.7 & 5296.28 & 14844.4 & 0 \\ 1/2 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 20000. \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving the final system of global equations we get

$$\{u_1 = 5.14286,\ v_1 = -2.96923,\ u_3 = 16.8629,\ v_3 = 12.788,\ u_4 = -1.42857,\ v_4 = 11.7594,\ \lambda = 80000.\}$$

Solution for element 1

Nodal coordinates

Element node	Global node number	x	y
1	1	0	0
2	3	5000.	−3000.

$x_1 = 0 \qquad y_1 = 0 \qquad x_2 = 5000. \qquad y_2 = -3000.$

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 5830.95$

Direction cosines: $\ell_s = \frac{x_2 - x_1}{L} = 0.857493 \qquad m_s = \frac{y_2 - y_1}{L} = -0.514496$

Global to local transformation matrix, $\boldsymbol{T} = \begin{pmatrix} 0.857493 & -0.514496 & 0 & 0 \\ 0 & 0 & 0.857493 & -0.514496 \end{pmatrix}$

Element nodal displacements in global coordinates, $\boldsymbol{d} = \begin{pmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 5.14286 \\ -2.96923 \\ 16.8629 \\ 12.788 \end{pmatrix}$

Element nodal displacements in local coordinates, $\boldsymbol{d}_\ell = \boldsymbol{T}\,\boldsymbol{d} = \begin{pmatrix} 5.93762 \\ 7.88048 \end{pmatrix}$

$E = 70000 \qquad A = 1000$

Axial strain, $\epsilon = (d_2 - d_1)/L = 0.000333197$

Axial stress, $\sigma = E\epsilon = 23.3238 \qquad$ Axial force $= \sigma A = 23323.8$

In a similar manner we can compute the solutions over the remaining elements.

	Stress	Axial force
1	23.3238	23323.8
2	23.3238	23323.8
3	69.282	69282.
4	$-20.$	$-20000.$
5	$-12.$	$-12000.$