CHAPTER SEVEN

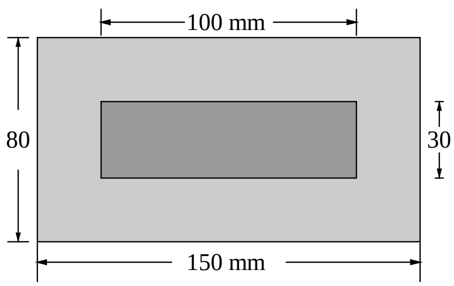
Analysis of Elastic Solids

Example 7.6: Thermal stresses (p. 502)

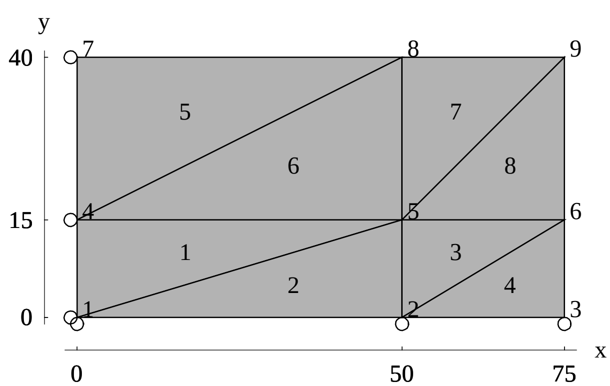
A 5 mm thick symmetric assembly of steel and aluminum plates, shown in Figure, is created at room temperature. Determine stresses and deformed shape if the temperature of the assembly is increased by 70°C above the room temperature. Assume a perfect bond between the two materials. Use the following data.

Steel plate: $80 \times 150\,\text{mm} \quad E = 200\,\text{GPa} \quad \nu = 0.3 \quad \alpha = 12 \times 10^{-6} / °\text{C}$

Aluminum plate: $30 \times 100\,\text{mm} \quad E = 70\,\text{GPa} \quad \nu = 0.33 \quad \alpha = 23 \times 10^{-6} / °\text{C}$



Since the thickness of the assembly is much smaller than the other dimensions, and there are no out of plane loads, the problem can be treated as a plane stress situation. Using symmetry a quarter of the assembly is modeled as shown in Figure. The first two elements are in the aluminum plate and the remaining 6 in the steel plate. A coarse mesh is used to show all calculations. Due to symmetry nodes 2 and 3 can displace in the x direction only while nodes 4 and 7 can displace in the y direction alone. Node 1, being on both the axes of symmetry, cannot displace in either direction. Note that in addition to reducing the model size, the use of symmetry provides enough boundary conditions so that there is no rigid body motion in the model. Since no support conditions are given for the assembly, analysis of a full finite element model would not be possible without introducing artificial supports.



The complete finite element calculations are as follows. The numerical values are in the N-mm units. The displacements will be in mm and the stresses in MPa.

Global equations at start of the element assembly process

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} u_1\\ v_1\\ u_2\\ v_2\\ u_3\\ v_3\\ u_4\\ v_4\\ u_5\\ v_5\\ u_6\\ v_6\\ u_7\\ v_7\\ u_8\\ v_8\\ u_9\\ v_9 \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}$$

Equations for element 1

$h = 5;$ $E = 70000;$ $\nu = 0.33$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 78554.6 & 25923. & 0 \\ 25923. & 78554.6 & 0 \\ 0 & 0 & 26315.8 \end{pmatrix}$

Nodal coordinates

Element node	Global node number	x	y
1	1	0	0
2	5	50	15
3	4	0	15

$x_1 = 0$ $x_2 = 50$ $x_3 = 0$

$y_1 = 0$ $y_2 = 15$ $y_3 = 15$

Using these values we get

$b_1 = 0$ $b_2 = 15$ $b_3 = -15$

$c_1 = -50$ $c_2 = 0$ $c_3 = 50$

$f_1 = 750$ $f_2 = 0$ $f_3 = 0$

Element area, A = 375

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & \frac{1}{50} & 0 & -\frac{1}{50} & 0 \\ 0 & -\frac{1}{15} & 0 & 0 & 0 & \frac{1}{15} \\ -\frac{1}{15} & 0 & 0 & \frac{1}{50} & \frac{1}{15} & -\frac{1}{50} \end{pmatrix}$$

Thus the element stiffness matrix is

$$\boldsymbol{k} = \mathrm{hA}\boldsymbol{BCB}^{\mathrm{T}} = 10^{6}\begin{pmatrix} 0.219298 & 0 & 0 & -0.0657895 & -0.219298 & 0.0657895 \\ 0 & 0.654622 & -0.0648075 & 0 & 0.0648075 & -0.654622 \\ 0 & -0.0648075 & 0.0589159 & 0 & -0.0589159 & 0.0648075 \\ -0.0657895 & 0 & 0 & 0.0197368 & 0.0657895 & -0.0197368 \\ -0.219298 & 0.0648075 & -0.0589159 & 0.0657895 & 0.278214 & -0.130597 \\ 0.0657895 & -0.654622 & 0.0648075 & -0.0197368 & -0.130597 & 0.674358 \end{pmatrix}$$

Load vector due to temperature change

$$\alpha = \frac{23}{1000000}; \qquad \Delta\mathrm{T} = 70; \qquad \boldsymbol{\epsilon}_0^{\mathrm{T}} = \begin{pmatrix} \frac{161}{100000} & \frac{161}{100000} & 0 \end{pmatrix}$$

$$\boldsymbol{r}_{\epsilon}^{\mathrm{T}} = \begin{pmatrix} 0. & -21026.1 & 6307.84 & 0. & -6307.84 & 21026.1 \end{pmatrix}$$

Complete equations for element 1

$$10^{6}\begin{pmatrix} 0.219298 & 0 & 0 & -0.0657895 & -0.219298 & 0.0657895 \\ 0 & 0.654622 & -0.0648075 & 0 & 0.0648075 & -0.654622 \\ 0 & -0.0648075 & 0.0589159 & 0 & -0.0589159 & 0.0648075 \\ -0.0657895 & 0 & 0 & 0.0197368 & 0.0657895 & -0.0197368 \\ -0.219298 & 0.0648075 & -0.0589159 & 0.0657895 & 0.278214 & -0.130597 \\ 0.0657895 & -0.654622 & 0.0648075 & -0.0197368 & -0.130597 & 0.674358 \end{pmatrix}\begin{pmatrix} u_1 \\ v_1 \\ u_5 \\ v_5 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0. \\ -21026.1 \\ 6307.84 \\ 0. \\ -6307.84 \\ 21026.1 \end{pmatrix}$$

The element contributes to {1, 2, 9, 10, 7, 8} global degrees of freedom.

Adding element equations into appropriate locations we have

$$
10^6 \begin{pmatrix}
0.219298 & 0 & 0 & 0 & 0 & 0 & -0.219298 & 0.0657895 & 0 & -0.0657895 & 0 & 0 & 0 & 0 \\
0 & 0.654622 & 0 & 0 & 0 & 0 & 0.0648075 & -0.654622 & -0.0648075 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.219298 & 0.0648075 & 0 & 0 & 0 & 0 & 0.278214 & -0.130597 & -0.0589159 & 0.0657895 & 0 & 0 & 0 & 0 \\
0.0657895 & -0.654622 & 0 & 0 & 0 & 0 & -0.130597 & 0.674358 & 0.0648075 & -0.0197368 & 0 & 0 & 0 & 0 \\
0 & -0.0648075 & 0 & 0 & 0 & 0 & -0.0589159 & 0.0648075 & 0.0589159 & 0 & 0 & 0 & 0 & 0 \\
-0.0657895 & 0 & 0 & 0 & 0 & 0 & 0.0657895 & -0.0197368 & 0 & 0.0197368 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

Equations for element 2

$h = 5;$ $\quad E = 70000;$ $\quad \nu = 0.33$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 78554.6 & 25923. & 0 \\ 25923. & 78554.6 & 0 \\ 0 & 0 & 26315.8 \end{pmatrix}$

Nodal coordinates

Element node	Global node number	x	y
1	1	0	0
2	2	50	0
3	5	50	15

$x_1 = 0 \quad x_2 = 50 \quad x_3 = 50$

$y_1 = 0 \quad y_2 = 0 \quad y_3 = 15$

Using these values we get

$b_1 = -15 \quad b_2 = 15 \quad b_3 = 0$

$c_1 = 0 \quad c_2 = -50 \quad c_3 = 50$

$f_1 = 750 \quad f_2 = 0 \quad f_3 = 0$

Element area, A = 375

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -\frac{1}{50} & 0 & \frac{1}{50} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{15} & 0 & \frac{1}{15} \\ 0 & -\frac{1}{50} & -\frac{1}{15} & \frac{1}{50} & \frac{1}{15} & 0 \end{pmatrix}$$

Thus the element stiffness matrix is

$$\boldsymbol{k} = \mathrm{hA}\boldsymbol{BCB}^{\mathrm{T}} = 10^6 \begin{pmatrix} 0.0589159 & 0 & -0.0589159 & 0.0648075 & 0 & -0.0648075 \\ 0 & 0.0197368 & 0.0657895 & -0.0197368 & -0.0657895 & 0 \\ -0.0589159 & 0.0657895 & 0.278214 & -0.130597 & -0.219298 & 0.0648075 \\ 0.0648075 & -0.0197368 & -0.130597 & 0.674358 & 0.0657895 & -0.654622 \\ 0 & -0.0657895 & -0.219298 & 0.0657895 & 0.219298 & 0 \\ -0.0648075 & 0 & 0.0648075 & -0.654622 & 0 & 0.654622 \end{pmatrix}$$

Load vector due to temperature change

$$\alpha = \frac{23}{1000000}; \qquad \Delta\mathrm{T} = 70; \qquad \boldsymbol{\epsilon}_0^{\mathrm{T}} = \begin{pmatrix} \frac{161}{100000} & \frac{161}{100000} & 0 \end{pmatrix}$$

$$\boldsymbol{r}_\epsilon^{\mathrm{T}} = \begin{pmatrix} -6307.84 & 0. & 6307.84 & -21026.1 & 0. & 21026.1 \end{pmatrix}$$

Complete equations for element 2

$$10^6 \begin{pmatrix} 0.0589159 & 0 & -0.0589159 & 0.0648075 & 0 & -0.0648075 \\ 0 & 0.0197368 & 0.0657895 & -0.0197368 & -0.0657895 & 0 \\ -0.0589159 & 0.0657895 & 0.278214 & -0.130597 & -0.219298 & 0.0648075 \\ 0.0648075 & -0.0197368 & -0.130597 & 0.674358 & 0.0657895 & -0.654622 \\ 0 & -0.0657895 & -0.219298 & 0.0657895 & 0.219298 & 0 \\ -0.0648075 & 0 & 0.0648075 & -0.654622 & 0 & 0.654622 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_5 \\ v_5 \end{pmatrix} = \begin{pmatrix} -6307.84 \\ 0. \\ 6307.84 \\ -21026.1 \\ 0. \\ 21026.1 \end{pmatrix}$$

The element contributes to {1, 2, 3, 4, 9, 10} global degrees of freedom.

Adding element equations into appropriate locations we have

$$10^6\begin{pmatrix}
0.278214 & 0 & -0.0589159 & 0.0648075 & 0 & 0 & -0.219298 & 0.0657895 & 0 & -0. \\
0 & 0.674358 & 0.0657895 & -0.0197368 & 0 & 0 & 0.0648075 & -0.654622 & -0.130597 & 0 \\
-0.0589159 & 0.0657895 & 0.278214 & -0.130597 & 0 & 0 & 0 & 0 & -0.219298 & 0. \\
0.0648075 & -0.0197368 & -0.130597 & 0.674358 & 0 & 0 & 0 & 0 & 0.0657895 & -0. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.219298 & 0.0648075 & 0 & 0 & 0 & 0 & 0.278214 & -0.130597 & -0.0589159 & 0. \\
0.0657895 & -0.654622 & 0 & 0 & 0 & 0 & -0.130597 & 0.674358 & 0.0648075 & -0. \\
0 & -0.130597 & -0.219298 & 0.0657895 & 0 & 0 & -0.0589159 & 0.0648075 & 0.278214 & 0 \\
-0.130597 & 0 & 0.0648075 & -0.654622 & 0 & 0 & 0.0657895 & -0.0197368 & 0 & 0. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Equations for element 3

$h = 5;$ $E = 200000;$ $\nu = 0.3$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 219780. & 65934.1 & 0 \\ 65934.1 & 219780. & 0 \\ 0 & 0 & 76923.1 \end{pmatrix}$

Nodal coordinates

Element node	Global node number	x	y
1	2	50	0
2	6	75	15
3	5	50	15

$x_1 = 50$ $x_2 = 75$ $x_3 = 50$

$y_1 = 0$ $y_2 = 15$ $y_3 = 15$

Using these values we get

$b_1 = 0$ $b_2 = 15$ $b_3 = -15$

$c_1 = -25$ $c_2 = 0$ $c_3 = 25$

$f_1 = 375$ $f_2 = -750$ $f_3 = 750$

Element area, $A = \frac{375}{2}$

$$\boldsymbol{B}^T = \begin{pmatrix} 0 & 0 & \frac{1}{25} & 0 & -\frac{1}{25} & 0 \\ 0 & -\frac{1}{15} & 0 & 0 & 0 & \frac{1}{15} \\ -\frac{1}{15} & 0 & 0 & \frac{1}{25} & \frac{1}{15} & -\frac{1}{25} \end{pmatrix}$$

Thus the element stiffness matrix is

$$\boldsymbol{k} = \mathrm{h}A\boldsymbol{B}\boldsymbol{C}\boldsymbol{B}^T = 10^6 \begin{pmatrix} 0.320513 & 0 & 0 & -0.192308 & -0.320513 & 0.192308 \\ 0 & 0.915751 & -0.164835 & 0 & 0.164835 & -0.915751 \\ 0 & -0.164835 & 0.32967 & 0 & -0.32967 & 0.164835 \\ -0.192308 & 0 & 0 & 0.115385 & 0.192308 & -0.115385 \\ -0.320513 & 0.164835 & -0.32967 & 0.192308 & 0.650183 & -0.357143 \\ 0.192308 & -0.915751 & 0.164835 & -0.115385 & -0.357143 & 1.03114 \end{pmatrix}$$

Load vector due to temperature change

$$\alpha = \frac{3}{250000}; \qquad \Delta T = 70; \qquad \boldsymbol{\epsilon}_0^T = \begin{pmatrix} \frac{21}{25000} & \frac{21}{25000} & 0 \end{pmatrix}$$

$$\boldsymbol{r}_\epsilon^T = (\,0. \quad -15000. \quad 9000. \quad 0. \quad -9000. \quad 15000.\,)$$

Complete equations for element 3

$$10^6 \begin{pmatrix} 0.320513 & 0 & 0 & -0.192308 & -0.320513 & 0.192308 \\ 0 & 0.915751 & -0.164835 & 0 & 0.164835 & -0.915751 \\ 0 & -0.164835 & 0.32967 & 0 & -0.32967 & 0.164835 \\ -0.192308 & 0 & 0 & 0.115385 & 0.192308 & -0.115385 \\ -0.320513 & 0.164835 & -0.32967 & 0.192308 & 0.650183 & -0.357143 \\ 0.192308 & -0.915751 & 0.164835 & -0.115385 & -0.357143 & 1.03114 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ u_6 \\ v_6 \\ u_5 \\ v_5 \end{pmatrix} = \begin{pmatrix} 0. \\ -15000. \\ 9000. \\ 0. \\ -9000. \\ 15000. \end{pmatrix}$$

The element contributes to {3, 4, 11, 12, 9, 10} global degrees of freedom.

Adding element equations into appropriate locations we have

$$10^{6}\begin{pmatrix}
0.278214 & 0 & -0.0589159 & 0.0648075 & 0 & 0 & -0.219298 & 0.0657895 & 0 & -0.1\ldots \\
0 & 0.674358 & 0.0657895 & -0.0197368 & 0 & 0 & 0.0648075 & -0.654622 & -0.130597 & 0 \\
-0.0589159 & 0.0657895 & 0.598727 & -0.130597 & 0 & 0 & 0 & 0 & -0.539811 & 0.2\ldots \\
0.0648075 & -0.0197368 & -0.130597 & 1.59011 & 0 & 0 & 0 & 0 & 0.230625 & -1.5\ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.219298 & 0.0648075 & 0 & 0 & 0 & 0 & 0.278214 & -0.130597 & -0.0589159 & 0.0\ldots \\
0.0657895 & -0.654622 & 0 & 0 & 0 & 0 & -0.130597 & 0.674358 & 0.0648075 & -0.0\ldots \\
0 & -0.130597 & -0.539811 & 0.230625 & 0 & 0 & -0.0589159 & 0.0648075 & 0.928397 & -0.3\ldots \\
-0.130597 & 0 & 0.257115 & -1.57037 & 0 & 0 & 0.0657895 & -0.0197368 & -0.357143 & 1.7\ldots \\
0 & 0 & 0 & -0.164835 & 0 & 0 & 0 & 0 & -0.32967 & 0.1\ldots \\
0 & 0 & -0.192308 & 0 & 0 & 0 & 0 & 0 & 0.192308 & -0.1\ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Equations for element 4

$h = 5;$ $E = 200000;$ $\nu = 0.3$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 219780. & 65934.1 & 0 \\ 65934.1 & 219780. & 0 \\ 0 & 0 & 76923.1 \end{pmatrix}$

Nodal coordinates

Element node	Global node number	x	y
1	2	50	0
2	3	75	0
3	6	75	15

$x_1 = 50$ $x_2 = 75$ $x_3 = 75$

$y_1 = 0$ $y_2 = 0$ $y_3 = 15$

Using these values we get

$b_1 = -15$ $b_2 = 15$ $b_3 = 0$

$c_1 = 0$ $c_2 = -25$ $c_3 = 25$

$f_1 = 1125$ $f_2 = -750$ $f_3 = 0$

Element area, $A = \frac{375}{2}$

$$\boldsymbol{B}^{T} = \begin{pmatrix} -\frac{1}{25} & 0 & \frac{1}{25} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{15} & 0 & \frac{1}{15} \\ 0 & -\frac{1}{25} & -\frac{1}{15} & \frac{1}{25} & \frac{1}{15} & 0 \end{pmatrix}$$

Thus the element stiffness matrix is

$$\boldsymbol{k} = \mathrm{hA}\boldsymbol{BCB}^{T} = 10^{6}\begin{pmatrix} 0.32967 & 0 & -0.32967 & 0.164835 & 0 & -0.164835 \\ 0 & 0.115385 & 0.192308 & -0.115385 & -0.192308 & 0 \\ -0.32967 & 0.192308 & 0.650183 & -0.357143 & -0.320513 & 0.164835 \\ 0.164835 & -0.115385 & -0.357143 & 1.03114 & 0.192308 & -0.915751 \\ 0 & -0.192308 & -0.320513 & 0.192308 & 0.320513 & 0 \\ -0.164835 & 0 & 0.164835 & -0.915751 & 0 & 0.915751 \end{pmatrix}$$

Load vector due to temperature change

$$\alpha = \frac{3}{250000}; \qquad \Delta T = 70; \qquad \boldsymbol{\epsilon}_0^{T} = \begin{pmatrix} \frac{21}{25000} & \frac{21}{25000} & 0 \end{pmatrix}$$

$$\boldsymbol{r}_{\epsilon}^{T} = \begin{pmatrix} -9000. & 0. & 9000. & -15000. & 0. & 15000. \end{pmatrix}$$

Complete equations for element 4

$$10^{6}\begin{pmatrix} 0.32967 & 0 & -0.32967 & 0.164835 & 0 & -0.164835 \\ 0 & 0.115385 & 0.192308 & -0.115385 & -0.192308 & 0 \\ -0.32967 & 0.192308 & 0.650183 & -0.357143 & -0.320513 & 0.164835 \\ 0.164835 & -0.115385 & -0.357143 & 1.03114 & 0.192308 & -0.915751 \\ 0 & -0.192308 & -0.320513 & 0.192308 & 0.320513 & 0 \\ -0.164835 & 0 & 0.164835 & -0.915751 & 0 & 0.915751 \end{pmatrix}\begin{pmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_6 \\ v_6 \end{pmatrix} = \begin{pmatrix} -9000. \\ 0. \\ 9000. \\ -15000. \\ 0. \\ 15000. \end{pmatrix}$$

The element contributes to {3, 4, 5, 6, 11, 12} global degrees of freedom.

Adding element equations into appropriate locations we have

$$10^6 \left(\begin{array}{cccccccc}
0.278214 & 0 & -0.0589159 & 0.0648075 & 0 & 0 & -0.219298 & 0.0657895 \\
0 & 0.674358 & 0.0657895 & -0.0197368 & 0 & 0 & 0.0648075 & -0.654622 \\
-0.0589159 & 0.0657895 & 0.928397 & -0.130597 & -0.32967 & 0.164835 & 0 & 0 \\
0.0648075 & -0.0197368 & -0.130597 & 1.70549 & 0.192308 & -0.115385 & 0 & 0 \\
0 & 0 & -0.32967 & 0.192308 & 0.650183 & -0.357143 & 0 & 0 \\
0 & 0 & 0.164835 & -0.115385 & -0.357143 & 1.03114 & 0 & 0 \\
-0.219298 & 0.0648075 & 0 & 0 & 0 & 0 & 0.278214 & -0.130597 \\
0.0657895 & -0.654622 & 0 & 0 & 0 & 0 & -0.130597 & 0.674358 \\
0 & -0.130597 & -0.539811 & 0.230625 & 0 & 0 & -0.0589159 & 0.0648075 \\
-0.130597 & 0 & 0.257115 & -1.57037 & 0 & 0 & 0.0657895 & -0.0197368 \\
0 & 0 & 0 & -0.357143 & -0.320513 & 0.192308 & 0 & 0 \\
0 & 0 & -0.357143 & 0 & 0.164835 & -0.915751 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array} \right.$$

Equations for element 5

$h = 5;$ $E = 200000;$ $\nu = 0.3$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 219780. & 65934.1 & 0 \\ 65934.1 & 219780. & 0 \\ 0 & 0 & 76923.1 \end{pmatrix}$

Nodal coordinates

Element node	Global node number	x	y
1	4	0	15
2	8	50	40
3	7	0	40

$x_1 = 0$ $x_2 = 50$ $x_3 = 0$

$y_1 = 15$ $y_2 = 40$ $y_3 = 40$

Using these values we get

$b_1 = 0$ $b_2 = 25$ $b_3 = -25$

$c_1 = -50$ $c_2 = 0$ $c_3 = 50$

$f_1 = 2000$ $f_2 = 0$ $f_3 = -750$

Element area, A = 625

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & \frac{1}{50} & 0 & -\frac{1}{50} & 0 \\ 0 & -\frac{1}{25} & 0 & 0 & 0 & \frac{1}{25} \\ -\frac{1}{25} & 0 & 0 & \frac{1}{50} & \frac{1}{25} & -\frac{1}{50} \end{pmatrix}$$

Thus the element stiffness matrix is

$$\boldsymbol{k} = \mathrm{hA}\boldsymbol{BCB}^{\mathrm{T}} = 10^{6}\begin{pmatrix} 0.384615 & 0 & 0 & -0.192308 & -0.384615 & 0.192308 \\ 0 & 1.0989 & -0.164835 & 0 & 0.164835 & -1.0989 \\ 0 & -0.164835 & 0.274725 & 0 & -0.274725 & 0.164835 \\ -0.192308 & 0 & 0 & 0.0961538 & 0.192308 & -0.0961538 \\ -0.384615 & 0.164835 & -0.274725 & 0.192308 & 0.659341 & -0.357143 \\ 0.192308 & -1.0989 & 0.164835 & -0.0961538 & -0.357143 & 1.19505 \end{pmatrix}$$

Load vector due to temperature change

$$\alpha = \frac{3}{250000}; \qquad \Delta\mathrm{T} = 70; \qquad \boldsymbol{\epsilon}_0^{\mathrm{T}} = \begin{pmatrix} \frac{21}{25000} & \frac{21}{25000} & 0 \end{pmatrix}$$

$$\boldsymbol{r}_{\epsilon}^{\mathrm{T}} = \begin{pmatrix} 0. & -30000. & 15000. & 0. & -15000. & 30000. \end{pmatrix}$$

Complete equations for element 5

$$10^{6}\begin{pmatrix} 0.384615 & 0 & 0 & -0.192308 & -0.384615 & 0.192308 \\ 0 & 1.0989 & -0.164835 & 0 & 0.164835 & -1.0989 \\ 0 & -0.164835 & 0.274725 & 0 & -0.274725 & 0.164835 \\ -0.192308 & 0 & 0 & 0.0961538 & 0.192308 & -0.0961538 \\ -0.384615 & 0.164835 & -0.274725 & 0.192308 & 0.659341 & -0.357143 \\ 0.192308 & -1.0989 & 0.164835 & -0.0961538 & -0.357143 & 1.19505 \end{pmatrix}\begin{pmatrix} u_4 \\ v_4 \\ u_8 \\ v_8 \\ u_7 \\ v_7 \end{pmatrix} = \begin{pmatrix} 0. \\ -30000. \\ 15000. \\ 0. \\ -15000. \\ 30000. \end{pmatrix}$$

The element contributes to {7, 8, 15, 16, 13, 14} global degrees of freedom.

Adding element equations into appropriate locations we have

$$
10^{6}\begin{pmatrix}
0.278214 & 0 & -0.0589159 & 0.0648075 & 0 & 0 & -0.219298 & 0.0657895 \\
0 & 0.674358 & 0.0657895 & -0.0197368 & 0 & 0 & 0.0648075 & -0.654622 \\
-0.0589159 & 0.0657895 & 0.928397 & -0.130597 & -0.32967 & 0.164835 & 0 & 0 \\
0.0648075 & -0.0197368 & -0.130597 & 1.70549 & 0.192308 & -0.115385 & 0 & 0 \\
0 & 0 & -0.32967 & 0.192308 & 0.650183 & -0.357143 & 0 & 0 \\
0 & 0 & 0.164835 & -0.115385 & -0.357143 & 1.03114 & 0 & 0 \\
-0.219298 & 0.0648075 & 0 & 0 & 0 & 0 & 0.66283 & -0.130597 \\
0.0657895 & -0.654622 & 0 & 0 & 0 & 0 & -0.130597 & 1.77326 \\
0 & -0.130597 & -0.539811 & 0.230625 & 0 & 0 & -0.0589159 & 0.0648075 \\
-0.130597 & 0 & 0.257115 & -1.57037 & 0 & 0 & 0.0657895 & -0.0197368 \\
0 & 0 & 0 & -0.357143 & -0.320513 & 0.192308 & 0 & 0 \\
0 & 0 & -0.357143 & 0 & 0.164835 & -0.915751 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.384615 & 0.164835 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.192308 & -1.0989 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.164835 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.192308 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

Equations for element 6

$h = 5$; $E = 200000$; $\nu = 0.3$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 219780. & 65934.1 & 0 \\ 65934.1 & 219780. & 0 \\ 0 & 0 & 76923.1 \end{pmatrix}$

Nodal coordinates

Element node	Global node number	x	y
1	4	0	15
2	5	50	15
3	8	50	40

$x_1 = 0$ $x_2 = 50$ $x_3 = 50$

$y_1 = 15$ $y_2 = 15$ $y_3 = 40$

Using these values we get

$b_1 = -25$ $b_2 = 25$ $b_3 = 0$

$c_1 = 0$ $c_2 = -50$ $c_3 = 50$

$f_1 = 1250$ $f_2 = 750$ $f_3 = -750$

Element area, A = 625

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -\frac{1}{50} & 0 & \frac{1}{50} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{25} & 0 & \frac{1}{25} \\ 0 & -\frac{1}{50} & -\frac{1}{25} & \frac{1}{50} & \frac{1}{25} & 0 \end{pmatrix}$$

Thus the element stiffness matrix is

$$\boldsymbol{k} = \mathrm{hA}\boldsymbol{BCB}^{\mathrm{T}} = 10^{6}\begin{pmatrix} 0.274725 & 0 & -0.274725 & 0.164835 & 0 & -0.164835 \\ 0 & 0.0961538 & 0.192308 & -0.0961538 & -0.192308 & 0 \\ -0.274725 & 0.192308 & 0.659341 & -0.357143 & -0.384615 & 0.164835 \\ 0.164835 & -0.0961538 & -0.357143 & 1.19505 & 0.192308 & -1.0989 \\ 0 & -0.192308 & -0.384615 & 0.192308 & 0.384615 & 0 \\ -0.164835 & 0 & 0.164835 & -1.0989 & 0 & 1.0989 \end{pmatrix}$$

Load vector due to temperature change

$$\alpha = \frac{3}{250000}; \qquad \Delta\mathrm{T} = 70; \qquad \boldsymbol{\epsilon}_0^{\mathrm{T}} = \begin{pmatrix} \frac{21}{25000} & \frac{21}{25000} & 0 \end{pmatrix}$$

$$\boldsymbol{r}_{\epsilon}^{\mathrm{T}} = (\,-15000. \quad 0. \quad 15000. \quad -30000. \quad 0. \quad 30000.\,)$$

Complete equations for element 6

$$10^{6}\begin{pmatrix} 0.274725 & 0 & -0.274725 & 0.164835 & 0 & -0.164835 \\ 0 & 0.0961538 & 0.192308 & -0.0961538 & -0.192308 & 0 \\ -0.274725 & 0.192308 & 0.659341 & -0.357143 & -0.384615 & 0.164835 \\ 0.164835 & -0.0961538 & -0.357143 & 1.19505 & 0.192308 & -1.0989 \\ 0 & -0.192308 & -0.384615 & 0.192308 & 0.384615 & 0 \\ -0.164835 & 0 & 0.164835 & -1.0989 & 0 & 1.0989 \end{pmatrix}\begin{pmatrix} u_4 \\ v_4 \\ u_5 \\ v_5 \\ u_8 \\ v_8 \end{pmatrix} = \begin{pmatrix} -15000. \\ 0. \\ 15000. \\ -30000. \\ 0. \\ 30000. \end{pmatrix}$$

The element contributes to {7, 8, 9, 10, 15, 16} global degrees of freedom.

Adding element equations into appropriate locations we have

$$10^6\begin{pmatrix}
0.278214 & 0 & -0.0589159 & 0.0648075 & 0 & 0 & -0.219298 & 0.0657895 \\
0 & 0.674358 & 0.0657895 & -0.0197368 & 0 & 0 & 0.0648075 & -0.654622 \\
-0.0589159 & 0.0657895 & 0.928397 & -0.130597 & -0.32967 & 0.164835 & 0 & 0 \\
0.0648075 & -0.0197368 & -0.130597 & 1.70549 & 0.192308 & -0.115385 & 0 & 0 \\
0 & 0 & -0.32967 & 0.192308 & 0.650183 & -0.357143 & 0 & 0 \\
0 & 0 & 0.164835 & -0.115385 & -0.357143 & 1.03114 & 0 & 0 \\
-0.219298 & 0.0648075 & 0 & 0 & 0 & 0 & 0.937555 & -0.130597 \\
0.0657895 & -0.654622 & 0 & 0 & 0 & 0 & -0.130597 & 1.86941 \\
0 & -0.130597 & -0.539811 & 0.230625 & 0 & 0 & -0.333641 & 0.257115 \\
-0.130597 & 0 & 0.257115 & -1.57037 & 0 & 0 & 0.230625 & -0.115891 \\
0 & 0 & 0 & -0.357143 & -0.320513 & 0.192308 & 0 & 0 \\
0 & 0 & -0.357143 & 0 & 0.164835 & -0.915751 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.384615 & 0.164835 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.192308 & -1.0989 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.357143 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.357143 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Equations for element 7

$h = 5;$ $E = 200000;$ $\nu = 0.3$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 219780. & 65934.1 & 0 \\ 65934.1 & 219780. & 0 \\ 0 & 0 & 76923.1 \end{pmatrix}$

Nodal coordinates

Element node	Global node number	x	y
1	5	50	15
2	9	75	40
3	8	50	40

$x_1 = 50$ $x_2 = 75$ $x_3 = 50$

$y_1 = 15$ $y_2 = 40$ $y_3 = 40$

Using these values we get

$b_1 = 0$ $b_2 = 25$ $b_3 = -25$

$c_1 = -25$ $c_2 = 0$ $c_3 = 25$

$f_1 = 1000$ $f_2 = -1250$ $f_3 = 875$

Element area, $A = \frac{625}{2}$

$$\boldsymbol{B}^T = \begin{pmatrix} 0 & 0 & \frac{1}{25} & 0 & -\frac{1}{25} & 0 \\ 0 & -\frac{1}{25} & 0 & 0 & 0 & \frac{1}{25} \\ -\frac{1}{25} & 0 & 0 & \frac{1}{25} & \frac{1}{25} & -\frac{1}{25} \end{pmatrix}$$

Thus the element stiffness matrix is

$$\boldsymbol{k} = hA\boldsymbol{BCB}^T = 10^6 \begin{pmatrix} 0.192308 & 0 & 0 & -0.192308 & -0.192308 & 0.192308 \\ 0 & 0.549451 & -0.164835 & 0 & 0.164835 & -0.549451 \\ 0 & -0.164835 & 0.549451 & 0 & -0.549451 & 0.164835 \\ -0.192308 & 0 & 0 & 0.192308 & 0.192308 & -0.192308 \\ -0.192308 & 0.164835 & -0.549451 & 0.192308 & 0.741758 & -0.357143 \\ 0.192308 & -0.549451 & 0.164835 & -0.192308 & -0.357143 & 0.741758 \end{pmatrix}$$

Load vector due to temperature change

$$\alpha = \frac{3}{250000}; \qquad \Delta T = 70; \qquad \boldsymbol{\epsilon}_0^T = \begin{pmatrix} \frac{21}{25000} & \frac{21}{25000} & 0 \end{pmatrix}$$

$$\boldsymbol{r}_\epsilon^T = \begin{pmatrix} 0. & -15000. & 15000. & 0. & -15000. & 15000. \end{pmatrix}$$

Complete equations for element 7

$$10^6 \begin{pmatrix} 0.192308 & 0 & 0 & -0.192308 & -0.192308 & 0.192308 \\ 0 & 0.549451 & -0.164835 & 0 & 0.164835 & -0.549451 \\ 0 & -0.164835 & 0.549451 & 0 & -0.549451 & 0.164835 \\ -0.192308 & 0 & 0 & 0.192308 & 0.192308 & -0.192308 \\ -0.192308 & 0.164835 & -0.549451 & 0.192308 & 0.741758 & -0.357143 \\ 0.192308 & -0.549451 & 0.164835 & -0.192308 & -0.357143 & 0.741758 \end{pmatrix} \begin{pmatrix} u_5 \\ v_5 \\ u_9 \\ v_9 \\ u_8 \\ v_8 \end{pmatrix} = \begin{pmatrix} 0. \\ -15000. \\ 15000. \\ 0. \\ -15000. \\ 15000. \end{pmatrix}$$

The element contributes to {9, 10, 17, 18, 15, 16} global degrees of freedom.

Adding element equations into appropriate locations we have

$$10^6\left(\begin{array}{llllllll}
0.278214 & 0 & -0.0589159 & 0.0648075 & 0 & 0 & -0.219298 & 0.0657895 \\
0 & 0.674358 & 0.0657895 & -0.0197368 & 0 & 0 & 0.0648075 & -0.654622 \\
-0.0589159 & 0.0657895 & 0.928397 & -0.130597 & -0.32967 & 0.164835 & 0 & 0 \\
0.0648075 & -0.0197368 & -0.130597 & 1.70549 & 0.192308 & -0.115385 & 0 & 0 \\
0 & 0 & -0.32967 & 0.192308 & 0.650183 & -0.357143 & 0 & 0 \\
0 & 0 & 0.164835 & -0.115385 & -0.357143 & 1.03114 & 0 & 0 \\
-0.219298 & 0.0648075 & 0 & 0 & 0 & 0 & 0.937555 & -0.130597 \\
0.0657895 & -0.654622 & 0 & 0 & 0 & 0 & -0.130597 & 1.86941 \\
0 & -0.130597 & -0.539811 & 0.230625 & 0 & 0 & -0.333641 & 0.257115 \\
-0.130597 & 0 & 0.257115 & -1.57037 & 0 & 0 & 0.230625 & -0.115891 \\
0 & 0 & 0 & -0.357143 & -0.320513 & 0.192308 & 0 & 0 \\
0 & 0 & -0.357143 & 0 & 0.164835 & -0.915751 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.384615 & 0.164835 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.192308 & -1.0989 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.357143 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.357143 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right.$$

Equations for element 8

$h = 5;\qquad E = 200000;\qquad \nu = 0.3$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 219780. & 65934.1 & 0 \\ 65934.1 & 219780. & 0 \\ 0 & 0 & 76923.1 \end{pmatrix}$

Nodal coordinates

Element node	Global node number	x	y
1	5	50	15
2	6	75	15
3	9	75	40

$x_1 = 50\qquad x_2 = 75\qquad x_3 = 75$

$y_1 = 15\qquad y_2 = 15\qquad y_3 = 40$

Using these values we get

$b_1 = -25\qquad b_2 = 25\qquad b_3 = 0$

$c_1 = 0\qquad c_2 = -25\qquad c_3 = 25$

$f_1 = 1875\qquad f_2 = -875\qquad f_3 = -375$

Element area, $A = \frac{625}{2}$

$$\boldsymbol{B}^{T} = \begin{pmatrix} -\frac{1}{25} & 0 & \frac{1}{25} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{25} & 0 & \frac{1}{25} \\ 0 & -\frac{1}{25} & -\frac{1}{25} & \frac{1}{25} & \frac{1}{25} & 0 \end{pmatrix}$$

Thus the element stiffness matrix is

$$\boldsymbol{k} = hA\boldsymbol{B}\boldsymbol{C}\boldsymbol{B}^{T} = 10^{6}\begin{pmatrix} 0.549451 & 0 & -0.549451 & 0.164835 & 0 & -0.164835 \\ 0 & 0.192308 & 0.192308 & -0.192308 & -0.192308 & 0 \\ -0.549451 & 0.192308 & 0.741758 & -0.357143 & -0.192308 & 0.164835 \\ 0.164835 & -0.192308 & -0.357143 & 0.741758 & 0.192308 & -0.549451 \\ 0 & -0.192308 & -0.192308 & 0.192308 & 0.192308 & 0 \\ -0.164835 & 0 & 0.164835 & -0.549451 & 0 & 0.549451 \end{pmatrix}$$

Load vector due to temperature change

$$\alpha = \frac{3}{250000}; \qquad \Delta T = 70; \qquad \boldsymbol{\epsilon}_0^{T} = \begin{pmatrix} \frac{21}{25000} & \frac{21}{25000} & 0 \end{pmatrix}$$

$$\boldsymbol{r}_{\epsilon}^{T} = \begin{pmatrix} -15000. & 0. & 15000. & -15000. & 0. & 15000. \end{pmatrix}$$

Complete equations for element 8

$$10^{6}\begin{pmatrix} 0.549451 & 0 & -0.549451 & 0.164835 & 0 & -0.164835 \\ 0 & 0.192308 & 0.192308 & -0.192308 & -0.192308 & 0 \\ -0.549451 & 0.192308 & 0.741758 & -0.357143 & -0.192308 & 0.164835 \\ 0.164835 & -0.192308 & -0.357143 & 0.741758 & 0.192308 & -0.549451 \\ 0 & -0.192308 & -0.192308 & 0.192308 & 0.192308 & 0 \\ -0.164835 & 0 & 0.164835 & -0.549451 & 0 & 0.549451 \end{pmatrix}\begin{pmatrix} u_5 \\ v_5 \\ u_6 \\ v_6 \\ u_9 \\ v_9 \end{pmatrix} = \begin{pmatrix} -15000. \\ 0. \\ 15000. \\ -15000. \\ 0. \\ 15000. \end{pmatrix}$$

The element contributes to {9, 10, 11, 12, 17, 18} global degrees of freedom.

Adding element equations into appropriate locations we have

$$10^6 \left(\begin{matrix}
0.278214 & 0 & -0.0589159 & 0.0648075 & 0 & 0 & -0.219298 & 0.0657895 \\
0 & 0.674358 & 0.0657895 & -0.0197368 & 0 & 0 & 0.0648075 & -0.654622 \\
-0.0589159 & 0.0657895 & 0.928397 & -0.130597 & -0.32967 & 0.164835 & 0 & 0 \\
0.0648075 & -0.0197368 & -0.130597 & 1.70549 & 0.192308 & -0.115385 & 0 & 0 \\
0 & 0 & -0.32967 & 0.192308 & 0.650183 & -0.357143 & 0 & 0 \\
0 & 0 & 0.164835 & -0.115385 & -0.357143 & 1.03114 & 0 & 0 \\
-0.219298 & 0.0648075 & 0 & 0 & 0 & 0 & 0.937555 & -0.130597 \\
0.0657895 & -0.654622 & 0 & 0 & 0 & 0 & -0.130597 & 1.86941 \\
0 & -0.130597 & -0.539811 & 0.230625 & 0 & 0 & -0.333641 & 0.257115 \\
-0.130597 & 0 & 0.257115 & -1.57037 & 0 & 0 & 0.230625 & -0.115891 \\
0 & 0 & 0 & -0.357143 & -0.320513 & 0.192308 & 0 & 0 \\
0 & 0 & -0.357143 & 0 & 0.164835 & -0.915751 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.384615 & 0.164835 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.192308 & -1.0989 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.357143 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.357143 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix} \right.$$

Essential boundary conditions

Node	dof	Value
1	u_1 v_1	0 0
2	v_2	0
3	v_3	0
4	u_4	0
7	u_7	0

Remove {1, 2, 4, 6, 7, 13} rows and columns.

After adjusting for essential boundary conditions we have

$$10^6 \left(\begin{array}{ccccccccc}
0.928397 & -0.32967 & 0 & -0.539811 & 0.257115 & 0 & -0.357143 & 0 & 0 \\
-0.32967 & 0.650183 & 0 & 0 & 0 & -0.320513 & 0.164835 & 0 & 0 \\
0 & 0 & 1.86941 & 0.257115 & -0.115891 & 0 & 0 & -1.0989 & -0.35 \\
-0.539811 & 0 & 0.257115 & 2.3295 & -0.714286 & -0.879121 & 0.357143 & 0 & -0.57 \\
0.257115 & 0 & -0.115891 & -0.714286 & 3.64231 & 0.357143 & -0.307692 & 0 & 0.35 \\
0 & -0.320513 & 0 & -0.879121 & 0.357143 & 1.39194 & -0.357143 & 0 & 0 \\
-0.357143 & 0.164835 & 0 & 0.357143 & -0.307692 & -0.357143 & 1.77289 & 0 & 0 \\
0 & 0 & -1.0989 & 0 & 0 & 0 & 0 & 1.19505 & 0.16 \\
0 & 0 & -0.357143 & -0.576923 & 0.357143 & 0 & 0 & 0.164835 & 1.40 \\
0 & 0 & 0 & 0.357143 & -1.64835 & 0 & 0 & -0.0961538 & -0.35 \\
0 & 0 & 0 & 0 & -0.357143 & -0.192308 & 0.192308 & 0 & -0.54 \\
0 & 0 & 0 & -0.357143 & 0 & 0.164835 & -0.549451 & 0 & 0.19
\end{array}\right.$$

Solving the final system of global equations we get

$$\{u_2 = 0.0513447,\ u_3 = 0.0703132,\ v_4 = 0.0252714,\ u_5 = 0.0495551,\ v_5 = 0.0186102,\ u_6 = 0.069253,$$
$$v_6 = 0.0146016,\ v_7 = 0.0445986,\ u_8 = 0.0498168,\ v_8 = 0.0388815,\ u_9 = 0.0716126,\ v_9 = 0.0366734\}$$

Complete table of nodal values

	u	v
1	0	0
2	0.0513447	0
3	0.0703132	0
4	0	0.0252714
5	0.0495551	0.0186102
6	0.069253	0.0146016
7	0	0.0445986
8	0.0498168	0.0388815
9	0.0716126	0.0366734

Computation of reactions

Equation numbers of dof with specified values: {1, 2, 4, 6, 7, 13}

Extracting equations {1, 2, 4, 6, 7, 13} from the global system we have

$$10^6\left(\begin{array}{llllllll}
0.278214 & 0 & -0.0589159 & 0.0648075 & 0 & 0 & -0.219298 & 0.0657895 \\
0 & 0.674358 & 0.0657895 & -0.0197368 & 0 & 0 & 0.0648075 & -0.654622 \\
0.0648075 & -0.0197368 & -0.130597 & 1.70549 & 0.192308 & -0.115385 & 0 & 0 \\
0 & 0 & 0.164835 & -0.115385 & -0.357143 & 1.03114 & 0 & 0 \\
-0.219298 & 0.0648075 & 0 & 0 & 0 & 0 & 0.937555 & -0.130597 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.384615 & 0.164835
\end{array}\right.$$

Substituting the nodal values and re−arranging

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{pmatrix} = 10^6\left(\begin{array}{llllllll}
0.278214 & 0 & -0.0589159 & 0.0648075 & 0 & 0 & -0.219298 & 0.0 \\
0 & 0.674358 & 0.0657895 & -0.0197368 & 0 & 0 & 0.0648075 & -0.6 \\
0.0648075 & -0.0197368 & -0.130597 & 1.70549 & 0.192308 & -0.115385 & 0 & 0 \\
0 & 0 & 0.164835 & -0.115385 & -0.357143 & 1.03114 & 0 & 0 \\
-0.219298 & 0.0648075 & 0 & 0 & 0 & 0 & 0.937555 & -0.1 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.384615 & 0.1
\end{array}\right.$$

Carrying out computations, the reactions are as follows.

Label	dof	Reaction
R_1	u_1	2514.97
R_2	v_1	1389.1
R_3	v_2	312.884
R_4	v_3	−1701.98
R_5	u_4	456.218
R_6	u_7	−2971.19

Sum of Reactions

dof: u	0
dof: v	0

Solution for element 1

$h = 5;$ $E = 70000;$ $\nu = 0.33$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 78554.6 & 25923. & 0 \\ 25923. & 78554.6 & 0 \\ 0 & 0 & 26315.8 \end{pmatrix}$

Element nodes: First node (node # 1): {0, 0}
Second node (node # 5): {50, 15} Third node (node # 4): {0, 15}

$x_1 = 0$ $x_2 = 50$ $x_3 = 0$
$y_1 = 0$ $y_2 = 15$ $y_3 = 15$

Using these values we get

$b_1 = 0$ $b_2 = 15$ $b_3 = -15$

$c_1 = -50$ $c_2 = 0$ $c_3 = 50$

$f_1 = 750$ $f_2 = 0$ $f_3 = 0$

Element area, A = 375

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & \frac{1}{50} & 0 & -\frac{1}{50} & 0 \\ 0 & -\frac{1}{15} & 0 & 0 & 0 & \frac{1}{15} \\ -\frac{1}{15} & 0 & 0 & \frac{1}{50} & \frac{1}{15} & -\frac{1}{50} \end{pmatrix}$$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\left\{1 - \frac{y}{15}, \frac{x}{50}, \frac{y}{15} - \frac{x}{50}\right\}$

$$\boldsymbol{N}^{\mathrm{T}} = \begin{pmatrix} 1-\frac{y}{15} & 0 & \frac{x}{50} & 0 & \frac{y}{15}-\frac{x}{50} & 0 \\ 0 & 1-\frac{y}{15} & 0 & \frac{x}{50} & 0 & \frac{y}{15}-\frac{x}{50} \end{pmatrix}$$

From global solution the displacements at the element nodes are (displacements at nodes {1, 5, 4}):

$\boldsymbol{d}^{\mathrm{T}} = \{0, 0, 0.0495551, 0.0186102, 0, 0.0252714\}$

The displacement distribution over the element is

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{d} = \begin{pmatrix} 0.000991101\,x \\ 0.00168476\,y - 0.000133224\,x \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{d} = (\,0.000991101 \quad 0.00168476 \quad -0.000133224\,)$

Initial strains: $\boldsymbol{\epsilon}_0^{\mathrm{T}} = \left(\,\frac{161}{100000} \quad \frac{161}{100000} \quad 0\,\right)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0) = (\,-46.6793 \quad -10.1709 \quad -3.50591\,)$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{\mathrm{T}} = (\,0.000991101 \quad 0.00168476 \quad 0.00187801 \quad -0.000133224 \quad 0 \quad 0\,)$

$\boldsymbol{\sigma}^{\mathrm{T}} = (\,-46.6793 \quad -10.1709 \quad 0 \quad -3.50591 \quad 0 \quad 0\,)$

Substituting these stress components into appropriate formulas

Principal stresses = $(\,0 \quad -9.83725 \quad -47.0129\,)$

Effective stress (von Mises) = 42.9477

Solution for element 2

$h = 5;$ $E = 70000;$ $\nu = 0.33$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 78554.6 & 25923. & 0 \\ 25923. & 78554.6 & 0 \\ 0 & 0 & 26315.8 \end{pmatrix}$

Element nodes: First node (node # 1): {0, 0}
Second node (node # 2): {50, 0} Third node (node # 5): {50, 15}

$x_1 = 0$ $x_2 = 50$ $x_3 = 50$
$y_1 = 0$ $y_2 = 0$ $y_3 = 15$

Using these values we get

$b_1 = -15$ $b_2 = 15$ $b_3 = 0$

$c_1 = 0$ $c_2 = -50$ $c_3 = 50$

$f_1 = 750 \qquad f_2 = 0 \qquad f_3 = 0$

Element area, A = 375

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -\frac{1}{50} & 0 & \frac{1}{50} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{15} & 0 & \frac{1}{15} \\ 0 & -\frac{1}{50} & -\frac{1}{15} & \frac{1}{50} & \frac{1}{15} & 0 \end{pmatrix}$$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\left\{1 - \frac{x}{50}, \frac{x}{50} - \frac{y}{15}, \frac{y}{15}\right\}$

$$\boldsymbol{N}^{\mathrm{T}} = \begin{pmatrix} 1 - \frac{x}{50} & 0 & \frac{x}{50} - \frac{y}{15} & 0 & \frac{y}{15} & 0 \\ 0 & 1 - \frac{x}{50} & 0 & \frac{x}{50} - \frac{y}{15} & 0 & \frac{y}{15} \end{pmatrix}$$

From global solution the displacements at the element nodes are
(displacements at nodes {1, 2, 5}):
$\boldsymbol{d}^{\mathrm{T}} = \{0, 0, 0.0513447, 0, 0.0495551, 0.0186102\}$

The displacement distribution over the element is

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{d} = \begin{pmatrix} 0.00102689\,x - 0.000119307\,y \\ 0.00124068\,y \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{d} = (\,0.00102689 \quad 0.00124068 \quad -0.000119307\,)$

Initial strains: $\boldsymbol{\epsilon}_0^{\mathrm{T}} = \left(\,\frac{161}{100000} \quad \frac{161}{100000} \quad 0\,\right)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0) = (\,-55.3796 \quad -44.1277 \quad -3.13967\,)$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{\mathrm{T}} = (\,0.00102689 \quad 0.00124068 \quad 0.00207911 \quad -0.000119307 \quad 0 \quad 0\,)$

$\boldsymbol{\sigma}^{\mathrm{T}} = (\,-55.3796 \quad -44.1277 \quad 0 \quad -3.13967 \quad 0 \quad 0\,)$

Substituting these stress components into appropriate formulas

Principal stresses = $(\,0 \quad -43.3109 \quad -56.1964\,)$

Effective stress (von Mises) = 50.9897

Solution for element 3

$h = 5; \qquad E = 200000; \qquad \nu = 0.3$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 219780. & 65934.1 & 0 \\ 65934.1 & 219780. & 0 \\ 0 & 0 & 76923.1 \end{pmatrix}$

Element nodes: First node (node # 2): {50, 0}
Second node (node # 6): {75, 15} Third node (node # 5): {50, 15}

$x_1 = 50 \quad x_2 = 75 \quad x_3 = 50$
$y_1 = 0 \quad y_2 = 15 \quad y_3 = 15$

Using these values we get

$b_1 = 0 \quad b_2 = 15 \quad b_3 = -15$

$c_1 = -25 \quad c_2 = 0 \quad c_3 = 25$

$f_1 = 375 \quad f_2 = -750 \quad f_3 = 750$

Element area, $A = \frac{375}{2}$

$$\boldsymbol{B}^T = \begin{pmatrix} 0 & 0 & \frac{1}{25} & 0 & -\frac{1}{25} & 0 \\ 0 & -\frac{1}{15} & 0 & 0 & 0 & \frac{1}{15} \\ -\frac{1}{15} & 0 & 0 & \frac{1}{25} & \frac{1}{15} & -\frac{1}{25} \end{pmatrix}$$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\left\{1 - \frac{y}{15}, \frac{x}{25} - 2, -\frac{x}{25} + \frac{y}{15} + 2\right\}$

$$\boldsymbol{N}^T = \begin{pmatrix} 1 - \frac{y}{15} & 0 & \frac{x}{25} - 2 & 0 & -\frac{x}{25} + \frac{y}{15} + 2 & 0 \\ 0 & 1 - \frac{y}{15} & 0 & \frac{x}{25} - 2 & 0 & -\frac{x}{25} + \frac{y}{15} + 2 \end{pmatrix}$$

From global solution the displacements at the element nodes are
(displacements at nodes {2, 6, 5}):
$\boldsymbol{d}^T = \{0.0513447, 0, 0.069253, 0.0146016, 0.0495551, 0.0186102\}$

The displacement distribution over the element is

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \boldsymbol{N}^T \boldsymbol{d} = \begin{pmatrix} 0.000787917\,x - 0.000119307\,y + 0.0119489 \\ -0.000160344\,x + 0.00124068\,y + 0.0080172 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^T \boldsymbol{d} = \begin{pmatrix} 0.000787917 & 0.00124068 & -0.000279651 \end{pmatrix}$

Initial strains: $\boldsymbol{\epsilon}_0^T = \begin{pmatrix} \frac{21}{25000} & \frac{21}{25000} & 0 \end{pmatrix}$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0) = \begin{pmatrix} 14.9716 & 84.6275 & -21.5116 \end{pmatrix}$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$$\boldsymbol{\epsilon}^{\mathrm{T}} = (\,0.000787917 \quad 0.00124068 \quad 0.000690601 \quad -0.000279651 \quad 0 \quad 0\,)$$

$$\boldsymbol{\sigma}^{\mathrm{T}} = (\,14.9716 \quad 84.6275 \quad 0 \quad -21.5116 \quad 0 \quad 0\,)$$

Substituting these stress components into appropriate formulas

Principal stresses = (90.7353   8.86375   0)

Effective stress (von Mises) = 86.6441

Solution for element 4

h = 5;   E = 200000; $\nu = 0.3$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 219780. & 65934.1 & 0 \\ 65934.1 & 219780. & 0 \\ 0 & 0 & 76923.1 \end{pmatrix}$

Element nodes: First node (node # 2): {50, 0}
Second node (node # 3): {75, 0}   Third node (node # 6): {75, 15}

$x_1 = 50 \qquad x_2 = 75 \qquad x_3 = 75$
$y_1 = 0 \qquad y_2 = 0 \qquad y_3 = 15$

Using these values we get

$b_1 = -15 \qquad b_2 = 15 \qquad b_3 = 0$

$c_1 = 0 \qquad c_2 = -25 \qquad c_3 = 25$

$f_1 = 1125 \qquad f_2 = -750 \qquad f_3 = 0$

Element area, $A = \dfrac{375}{2}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -\frac{1}{25} & 0 & \frac{1}{25} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{15} & 0 & \frac{1}{15} \\ 0 & -\frac{1}{25} & -\frac{1}{15} & \frac{1}{25} & \frac{1}{15} & 0 \end{pmatrix}$$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\left\{3 - \dfrac{x}{25},\ \dfrac{x}{25} - \dfrac{y}{15} - 2,\ \dfrac{y}{15}\right\}$

$$\boldsymbol{N}^{\mathrm{T}} = \begin{pmatrix} 3 - \frac{x}{25} & 0 & \frac{x}{25} - \frac{y}{15} - 2 & 0 & \frac{y}{15} & 0 \\ 0 & 3 - \frac{x}{25} & 0 & \frac{x}{25} - \frac{y}{15} - 2 & 0 & \frac{y}{15} \end{pmatrix}$$

From global solution the displacements at the element nodes are
(displacements at nodes {2, 3, 6}):

$\boldsymbol{d}^{\mathrm{T}} = \{0.0513447,\ 0,\ 0.0703132,\ 0,\ 0.069253,\ 0.0146016\}$

The displacement distribution over the element is

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \boldsymbol{N}^T \boldsymbol{d} = \begin{pmatrix} 0.000758739\,x - 0.0000706781\,y + 0.0134077 \\ 0.00097344\,y \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^T \boldsymbol{d} = (\,0.000758739 \quad 0.00097344 \quad -0.0000706781\,)$

Initial strains: $\boldsymbol{\epsilon}_0^T = (\,\frac{21}{25000} \quad \frac{21}{25000} \quad 0\,)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0) = (\,-9.06129 \quad 23.9696 \quad -5.43678\,)$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$$\boldsymbol{\epsilon}^T = (\,0.000758739 \quad 0.00097344 \quad 0.000817638 \quad -0.0000706781 \quad 0 \quad 0\,)$$

$$\boldsymbol{\sigma}^T = (\,-9.06129 \quad 23.9696 \quad 0 \quad -5.43678 \quad 0 \quad 0\,)$$

Substituting these stress components into appropriate formulas

Principal stresses = $(\,24.8415 \quad 0 \quad -9.93316\,)$

Effective stress (von Mises) = 31.0245

Solution for element 5

$h = 5;$ $E = 200000;$ $\nu = 0.3$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 219780. & 65934.1 & 0 \\ 65934.1 & 219780. & 0 \\ 0 & 0 & 76923.1 \end{pmatrix}$

Element nodes: First node (node # 4): {0, 15}
Second node (node # 8): {50, 40} Third node (node # 7): {0, 40}

$x_1 = 0 \quad x_2 = 50 \quad x_3 = 0$
$y_1 = 15 \quad y_2 = 40 \quad y_3 = 40$

Using these values we get

$b_1 = 0 \quad b_2 = 25 \quad b_3 = -25$

$c_1 = -50 \quad c_2 = 0 \quad c_3 = 50$

$f_1 = 2000 \quad f_2 = 0 \quad f_3 = -750$

Element area, A = 625

$$\boldsymbol{B}^T = \begin{pmatrix} 0 & 0 & \frac{1}{50} & 0 & -\frac{1}{50} & 0 \\ 0 & -\frac{1}{25} & 0 & 0 & 0 & \frac{1}{25} \\ -\frac{1}{25} & 0 & 0 & \frac{1}{50} & \frac{1}{25} & -\frac{1}{50} \end{pmatrix}$$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\left\{\frac{8}{5}-\frac{y}{25}, \frac{x}{50}, -\frac{x}{50}+\frac{y}{25}-\frac{3}{5}\right\}$

$$\boldsymbol{N}^{T}=\begin{pmatrix} \frac{8}{5}-\frac{y}{25} & 0 & \frac{x}{50} & 0 & -\frac{x}{50}+\frac{y}{25}-\frac{3}{5} & 0 \\ 0 & \frac{8}{5}-\frac{y}{25} & 0 & \frac{x}{50} & 0 & -\frac{x}{50}+\frac{y}{25}-\frac{3}{5} \end{pmatrix}$$

From global solution the displacements at the element nodes are (displacements at nodes {4, 8, 7}):

$\boldsymbol{d}^{T}=\{0, 0.0252714, 0.0498168, 0.0388815, 0, 0.0445986\}$

The displacement distribution over the element is

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}=\boldsymbol{N}^{T}\boldsymbol{d}=\begin{pmatrix} 0.000996336\,x \\ -0.000114342\,x+0.000773089\,y+0.0136751 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon}=\boldsymbol{B}^{T}\boldsymbol{d}=(\,0.000996336 \quad 0.000773089 \quad -0.000114342\,)$

Initial strains: $\boldsymbol{\epsilon}_0^{T}=\left(\,\frac{21}{25000} \quad \frac{21}{25000} \quad 0\,\right)$

In−plane stress components, $\boldsymbol{\sigma}=\boldsymbol{C}(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_0)=(\,29.9479 \quad -4.39778 \quad -8.79555\,)$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{T}=(\,0.000996336 \quad 0.000773089 \quad 0.000801675 \quad -0.000114342 \quad 0 \quad 0\,)$

$\boldsymbol{\sigma}^{T}=(\,29.9479 \quad -4.39778 \quad 0 \quad -8.79555 \quad 0 \quad 0\,)$

Substituting these stress components into appropriate formulas

Principal stresses = (32.0694   0   −6.51919)

Effective stress (von Mises) = 35.7772

Solution for element 6

$h=5;$ $\quad E=200000;$ $\quad \nu=0.3$

Plane stress constitutive matrix, $\boldsymbol{C}=\begin{pmatrix} 219780. & 65934.1 & 0 \\ 65934.1 & 219780. & 0 \\ 0 & 0 & 76923.1 \end{pmatrix}$

Element nodes: First node (node # 4): {0, 15}
Second node (node # 5): {50, 15} Third node (node # 8): {50, 40}

$x_1=0$ $\quad x_2=50$ $\quad x_3=50$
$y_1=15$ $\quad y_2=15$ $\quad y_3=40$

Using these values we get

$b_1 = -25 \quad b_2 = 25 \quad b_3 = 0$

$c_1 = 0 \quad c_2 = -50 \quad c_3 = 50$

$f_1 = 1250 \quad f_2 = 750 \quad f_3 = -750$

Element area, A = 625

$$\boldsymbol{B}^T = \begin{pmatrix} -\frac{1}{50} & 0 & \frac{1}{50} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{25} & 0 & \frac{1}{25} \\ 0 & -\frac{1}{50} & -\frac{1}{25} & \frac{1}{50} & \frac{1}{25} & 0 \end{pmatrix}$$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\left\{1 - \frac{x}{50}, \frac{x}{50} - \frac{y}{25} + \frac{3}{5}, \frac{y}{25} - \frac{3}{5}\right\}$

$$\boldsymbol{N}^T = \begin{pmatrix} 1 - \frac{x}{50} & 0 & \frac{x}{50} - \frac{y}{25} + \frac{3}{5} & 0 & \frac{y}{25} - \frac{3}{5} & 0 \\ 0 & 1 - \frac{x}{50} & 0 & \frac{x}{50} - \frac{y}{25} + \frac{3}{5} & 0 & \frac{y}{25} - \frac{3}{5} \end{pmatrix}$$

From global solution the displacements at the element nodes are (displacements at nodes {4, 5, 8}):

$\boldsymbol{d}^T = \{0, 0.0252714, 0.0495551, 0.0186102, 0.0498168, 0.0388815\}$

The displacement distribution over the element is

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \boldsymbol{N}^T\boldsymbol{d} = \begin{pmatrix} 0.000991101\,x + 0.0000104699\,y - 0.000157049 \\ -0.000133224\,x + 0.000810854\,y + 0.0131086 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^T\boldsymbol{d} = (\,0.000991101 \quad 0.000810854 \quad -0.000122754\,)$

Initial strains: $\boldsymbol{\epsilon}_0^T = \left(\,\frac{21}{25000} \quad \frac{21}{25000} \quad 0\,\right)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0) = (\,31.2874 \quad 3.55695 \quad -9.44265\,)$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^T = (\,0.000991101 \quad 0.000810854 \quad 0.000787734 \quad -0.000122754 \quad 0 \quad 0\,)$

$\boldsymbol{\sigma}^T = (\,31.2874 \quad 3.55695 \quad 0 \quad -9.44265 \quad 0 \quad 0\,)$

Substituting these stress components into appropriate formulas

Principal stresses = (34.1974 0.646946 0)

Effective stress (von Mises) = 33.8785

Solution for element 7

h = 5; E = 200000; $\nu = 0.3$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 219780. & 65934.1 & 0 \\ 65934.1 & 219780. & 0 \\ 0 & 0 & 76923.1 \end{pmatrix}$

Element nodes: First node (node # 5): {50, 15}

Second node (node # 9): {75, 40} Third node (node # 8): {50, 40}

$x_1 = 50$ $x_2 = 75$ $x_3 = 50$

$y_1 = 15$ $y_2 = 40$ $y_3 = 40$

Using these values we get

$b_1 = 0$ $b_2 = 25$ $b_3 = -25$

$c_1 = -25$ $c_2 = 0$ $c_3 = 25$

$f_1 = 1000$ $f_2 = -1250$ $f_3 = 875$

Element area, $A = \dfrac{625}{2}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & \frac{1}{25} & 0 & -\frac{1}{25} & 0 \\ 0 & -\frac{1}{25} & 0 & 0 & 0 & \frac{1}{25} \\ -\frac{1}{25} & 0 & 0 & \frac{1}{25} & \frac{1}{25} & -\frac{1}{25} \end{pmatrix}$$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\left\{\dfrac{8}{5} - \dfrac{y}{25}, \dfrac{x}{25} - 2, -\dfrac{x}{25} + \dfrac{y}{25} + \dfrac{7}{5}\right\}$

$$\boldsymbol{N}^{\mathrm{T}} = \begin{pmatrix} \frac{8}{5} - \frac{y}{25} & 0 & \frac{x}{25} - 2 & 0 & -\frac{x}{25} + \frac{y}{25} + \frac{7}{5} & 0 \\ 0 & \frac{8}{5} - \frac{y}{25} & 0 & \frac{x}{25} - 2 & 0 & -\frac{x}{25} + \frac{y}{25} + \frac{7}{5} \end{pmatrix}$$

From global solution the displacements at the element nodes are

(displacements at nodes {5, 9, 8}):

$\boldsymbol{d}^{\mathrm{T}} = \{0.0495551, 0.0186102, 0.0716126, 0.0366734, 0.0498168, 0.0388815\}$

The displacement distribution over the element is

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{d} = \begin{pmatrix} 0.00087183\,x + 0.0000104699\,y + 0.00580652 \\ -0.0000883238\,x + 0.000810854\,y + 0.0108636 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{d} = \begin{pmatrix} 0.00087183 & 0.000810854 & -0.0000778538 \end{pmatrix}$

Initial strains: $\boldsymbol{\epsilon}_0^{\mathrm{T}} = \begin{pmatrix} \frac{21}{25000} & \frac{21}{25000} & 0 \end{pmatrix}$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0) = \begin{pmatrix} 5.07389 & -4.3071 & -5.98876 \end{pmatrix}$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$$\boldsymbol{\epsilon}^T = (\ 0.00087183 \quad 0.000810854 \quad 0.00083885 \quad -0.0000778538 \quad 0 \quad 0\ )$$

$$\boldsymbol{\sigma}^T = (\ 5.07389 \quad -4.3071 \quad 0 \quad -5.98876 \quad 0 \quad 0\ )$$

Substituting these stress components into appropriate formulas

Principal stresses = (7.99036   0   −7.22357)

Effective stress (von Mises) = 13.1812

Solution for element 8

$h = 5;$ $E = 200000;$ $\nu = 0.3$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 219780. & 65934.1 & 0 \\ 65934.1 & 219780. & 0 \\ 0 & 0 & 76923.1 \end{pmatrix}$

Element nodes: First node (node # 5): {50, 15}
Second node (node # 6): {75, 15} Third node (node # 9): {75, 40}

$x_1 = 50$ $x_2 = 75$ $x_3 = 75$
$y_1 = 15$ $y_2 = 15$ $y_3 = 40$

Using these values we get

$b_1 = -25$ $b_2 = 25$ $b_3 = 0$

$c_1 = 0$ $c_2 = -25$ $c_3 = 25$

$f_1 = 1875$ $f_2 = -875$ $f_3 = -375$

Element area, $A = \frac{625}{2}$

$$\boldsymbol{B}^T = \begin{pmatrix} -\frac{1}{25} & 0 & \frac{1}{25} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{25} & 0 & \frac{1}{25} \\ 0 & -\frac{1}{25} & -\frac{1}{25} & \frac{1}{25} & \frac{1}{25} & 0 \end{pmatrix}$$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\left\{3 - \frac{x}{25}, \frac{x}{25} - \frac{y}{25} - \frac{7}{5}, \frac{y}{25} - \frac{3}{5}\right\}$

$$\boldsymbol{N}^T = \begin{pmatrix} 3 - \frac{x}{25} & 0 & \frac{x}{25} - \frac{y}{25} - \frac{7}{5} & 0 & \frac{y}{25} - \frac{3}{5} & 0 \\ 0 & 3 - \frac{x}{25} & 0 & \frac{x}{25} - \frac{y}{25} - \frac{7}{5} & 0 & \frac{y}{25} - \frac{3}{5} \end{pmatrix}$$

From global solution the displacements at the element nodes are

(displacements at nodes {5, 6, 9}):

$\boldsymbol{d}^{T}$ = {0.0495551, 0.0186102, 0.069253, 0.0146016, 0.0716126, 0.0366734}

The displacement distribution over the element is

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \boldsymbol{N}^{T}\boldsymbol{d} = \begin{pmatrix} 0.000787917\,x + 0.0000943834\,y + 0.00874349 \\ -0.000160344\,x + 0.000882874\,y + 0.0133843 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{T}\boldsymbol{d} = (\,0.000787917 \quad 0.000882874 \quad -0.0000659605\,)$

Initial strains: $\boldsymbol{\epsilon}_0^{T} = \left(\,\frac{21}{25000} \quad \frac{21}{25000} \quad 0\,\right)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0) = (\,-8.62005 \quad 5.98876 \quad -5.07389\,)$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$$\boldsymbol{\epsilon}^{T} = (\,0.000787917 \quad 0.000882874 \quad 0.000843947 \quad -0.0000659605 \quad 0 \quad 0\,)$$

$$\boldsymbol{\sigma}^{T} = (\,-8.62005 \quad 5.98876 \quad 0 \quad -5.07389 \quad 0 \quad 0\,)$$

Substituting these stress components into appropriate formulas

Principal stresses = $(\,7.57809 \quad 0 \quad -10.2094\,)$

Effective stress (von Mises) = 15.4605

Solution summary

Nodal solution

	x	y	u	v
1	0	0	0	0
2	50	0	0.0513447	0
3	75	0	0.0703132	0
4	0	15	0	0.0252714
5	50	15	0.0495551	0.0186102
6	75	15	0.069253	0.0146016
7	0	40	0	0.0445986
8	50	40	0.0498168	0.0388815
9	75	40	0.0716126	0.0366734

Solution at element centers

	Coord	Disp	Stresses	Principal stresses	Effective Stress
1	$\frac{50}{3}$ 10	0.0165184 0.0146272	−46.6793 −10.1709 0 −3.50591 0 0	0 −9.83725 −47.0129	42.9477
2	$\frac{100}{3}$ 5	0.0336333 0.0062034	−55.3796 −44.1277 0 −3.13967 0 0	0 −43.3109 −56.1964	50.9897
3	$\frac{175}{3}$ 10	0.0567176 0.0110706	14.9716 84.6275 0 −21.5116 0 0	90.7353 8.86375 0	86.6441
4	$\frac{200}{3}$ 5	0.0636369 0.0048672	−9.06129 23.9696 0 −5.43678 0 0	24.8415 0 −9.93316	31.0245
5	$\frac{50}{3}$ $\frac{95}{3}$	0.0166056 0.0362505	29.9479 −4.39778 0 −8.79555 0 0	32.0694 0 −6.51919	35.7772
6	$\frac{100}{3}$ $\frac{70}{3}$	0.033124 0.0275877	31.2874 3.55695 0 −9.44265 0 0	34.1974 0.646946 0	33.8785
7	$\frac{175}{3}$ $\frac{95}{3}$	0.0569948 0.0313884	5.07389 −4.3071 0 −5.98876 0 0	7.99036 0 −7.22357	13.1812
8	$\frac{200}{3}$ $\frac{70}{3}$	0.0634735 0.0232951	−8.62005 5.98876 0 −5.07389 0 0	7.57809 0 −10.2094	15.4605

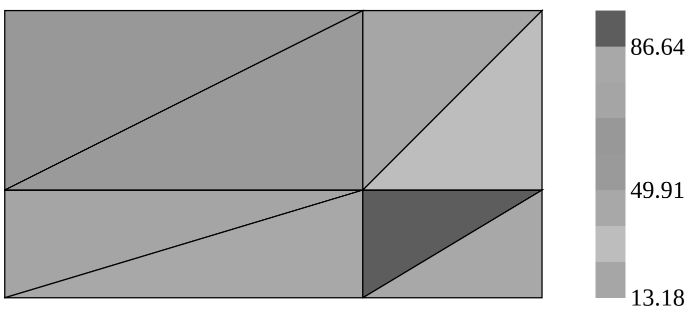
Support reactions

Node	dof	Reaction
1	1	2514.97
1	2	1389.1
2	2	312.884
3	2	−1701.98
4	1	456.218
7	1	−2971.19

Sum of applied loads → (0   0)

Sum of support reactions → (0   0)

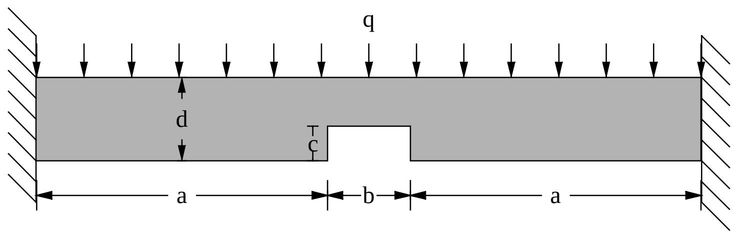
Figure shows equivalent von-Mises stresses in different elements. There are large differences in stresses among neighboring elements, indicating that the solution is not reliable and mesh must be refined.



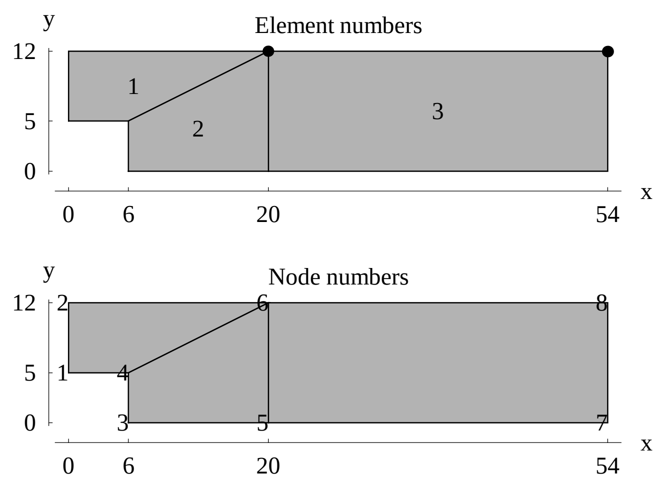
Example 7.7: Notched beam (p. 510)

Find stresses in a notched beam of rectangular cross-section shown in Figure. The following numerical values are used.

$a = 48\,\text{in};\quad b = 12\,\text{in};\quad c = 5\,\text{in};\quad d = 12\,\text{in};\quad \text{Thickness} = 4\,\text{in};\quad E = 3\times 10^6\,\text{lb}/\text{in}^2;\quad \nu = 0.2;\quad q = 50\,\text{lb}/\text{in}^2$



Since the thickness of the beam is much smaller than the other dimensions, and there are no out of plane loads, the problem can be treated as a plane stress situation. Using symmetry half of the beam is modeled. To show all calculations, a very coarse model involving only three elements is used.



Global equations at start of the element assembly process

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} u_1\\ v_1\\ u_2\\ v_2\\ u_3\\ v_3\\ u_4\\ v_4\\ u_5\\ v_5\\ u_6\\ v_6\\ u_7\\ v_7\\ u_8\\ v_8 \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}$$

Equations for element 1

$E = 3000000;$ $\nu = 0.2;$ $h = 4$

Nodal coordinates

Element node	Global node number	x	y
1	1	0.	5.
2	4	6.	5.
3	6	20.	12.
4	2	0.	12.

Interpolation functions and their derivatives

$$\{N_1, N_2, N_3, N_4\} = \left\{\frac{1}{4}(s-1)(t-1), -\frac{1}{4}(s+1)(t-1), \frac{1}{4}(s+1)(t+1), -\frac{1}{4}(s-1)(t+1)\right\}$$

$$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{\frac{t-1}{4}, \frac{1-t}{4}, \frac{t+1}{4}, \frac{1}{4}(-t-1)\right\}$$

$$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \left\{\frac{s-1}{4}, \frac{1}{4}(-s-1), \frac{s+1}{4}, \frac{1-s}{4}\right\}$$

Mapping to the master element

$$x(s,t) = \boldsymbol{N}^T \boldsymbol{x}_n = 3.5\,t\,s + 6.5\,s + 3.5\,t + 6.5$$

$$y(s,t) = \boldsymbol{N}^T \boldsymbol{y}_n = 3.5\,t + 8.5$$

$$\boldsymbol{J} = \begin{pmatrix} 3.5\,t + 6.5 & 3.5\,s + 3.5 \\ 0 & 3.5 \end{pmatrix}; \qquad \text{detJ} = 12.25\,t + 22.75$$

$$\text{Plane stress } \boldsymbol{C} = \begin{pmatrix} 3.125\times10^{6} & 625000. & 0 \\ 625000. & 3.125\times10^{6} & 0 \\ 0 & 0 & 1.25\times10^{6} \end{pmatrix}$$

For numerical integration the Gauss quadrature points and weights are

	s	t	Weight
1	−0.57735	−0.57735	1.
2	−0.57735	0.57735	1.
3	0.57735	−0.57735	1.
4	0.57735	0.57735	1.

Computation of element matrices at {−0.57735, −0.57735} with weight = 1.

$$\boldsymbol{J} = \begin{pmatrix} 4.47927 & 1.47927 \\ 0 & 3.5 \end{pmatrix} \qquad \text{detJ} = 15.6775$$

$\{N_1, N_2, N_3, N_4\} = \{0.622008, 0.166667, 0.0446582, 0.166667\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \{-0.394338, 0.394338, 0.105662, -0.105662\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \{-0.394338, -0.105662, 0.105662, 0.394338\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{-0.088036, 0.088036, 0.0235892, -0.0235892\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{-0.0754595, -0.0673977, 0.0202193, 0.122638\}$

$$\boldsymbol{B}^{T} = \begin{pmatrix} -0.088036 & 0 & 0.088036 & 0 & 0.0235892 & 0 & -0.0235892 & 0 \\ 0 & -0.0754595 & 0 & -0.0673977 & 0 & 0.0202193 & 0 & 0. \\ -0.0754595 & -0.088036 & -0.0673977 & 0.088036 & 0.0202193 & 0.0235892 & 0.122638 & -0. \end{pmatrix}$$

$$\boldsymbol{k} = 10^{6}\begin{pmatrix} 1.96517 & 0.781108 & -1.12016 & -0.288186 & -0.526565 & -0.209297 & -0.318444 \\ 0.781108 & 1.7234 & 0.204736 & 0.389125 & -0.209297 & -0.461783 & -0.776547 \\ -1.12016 & 0.204736 & 1.87489 & -0.697658 & 0.300146 & -0.0548588 & -1.05488 \\ -0.288186 & 0.389125 & -0.697658 & 1.4977 & 0.0772193 & -0.104266 & 0.908625 \\ -0.526565 & -0.209297 & 0.300146 & 0.0772193 & 0.141093 & 0.056081 & 0.0853267 \\ -0.209297 & -0.461783 & -0.0548588 & -0.104266 & 0.056081 & 0.123734 & 0.208075 \\ -0.318444 & -0.776547 & -1.05488 & 0.908625 & 0.0853267 & 0.208075 & 1.28799 \\ -0.283625 & -1.65074 & 0.547781 & -1.78256 & 0.075997 & 0.442314 & -0.340153 \end{pmatrix}$$

Computation of element matrices at {−0.57735, 0.57735} with weight = 1.

$$\boldsymbol{J} = \begin{pmatrix} 8.52073 & 1.47927 \\ 0 & 3.5 \end{pmatrix} \qquad \text{detJ} = 29.8225$$

$\{N_1, N_2, N_3, N_4\} = \{0.166667, 0.0446582, 0.166667, 0.622008\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \{-0.105662, 0.105662, 0.394338, -0.394338\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \{-0.394338, -0.105662, 0.105662, 0.394338\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{-0.0124006, 0.0124006, 0.0462798, -0.0462798\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{-0.107427, -0.0354304, 0.0106291, 0.132228\}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -0.0124006 & 0 & 0.0124006 & 0 & 0.0462798 & 0 & -0.0462798 & 0 \\ 0 & -0.107427 & 0 & -0.0354304 & 0 & 0.0106291 & 0 & 0. \\ -0.107427 & -0.0124006 & -0.0354304 & 0.0124006 & 0.0106291 & 0.0462798 & 0.132228 & -0. \end{pmatrix}$$

$\boldsymbol{k} = 10^6$

$$\begin{pmatrix} 1.77816 & 0.297963 & 0.510224 & -0.165885 & -0.384204 & -0.751169 & -1.90418 & 0.619091 \\ 0.297963 & 4.32502 & -0.0338069 & 1.39594 & -0.390325 & -0.511237 & 0.126169 & -5.20972 \\ 0.510224 & -0.0338069 & 0.244508 & -0.0982711 & 0.157784 & -0.234675 & -0.912516 & 0.366753 \\ -0.165885 & 1.39594 & -0.0982711 & 0.490888 & -0.102597 & -0.0548117 & 0.366753 & -1.83202 \\ -0.384204 & -0.390325 & 0.157784 & -0.102597 & 0.815278 & 0.110026 & -0.588859 & 0.382896 \\ -0.751169 & -0.511237 & -0.234675 & -0.0548117 & 0.110026 & 0.361489 & 0.875818 & 0.20456 \\ -1.90418 & 0.126169 & -0.912516 & 0.366753 & -0.588859 & 0.875818 & 3.40556 & -1.36874 \\ 0.619091 & -5.20972 & 0.366753 & -1.83202 & 0.382896 & 0.20456 & -1.36874 & 6.83718 \end{pmatrix}$$

Computation of element matrices at $\{0.57735, -0.57735\}$ with weight = 1.

$$\boldsymbol{J} = \begin{pmatrix} 4.47927 & 5.52073 \\ 0 & 3.5 \end{pmatrix} \qquad \det J = 15.6775$$

$\{N_1, N_2, N_3, N_4\} = \{0.166667, 0.622008, 0.166667, 0.0446582\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \{-0.394338, 0.394338, 0.105662, -0.105662\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \{-0.105662, -0.394338, 0.394338, 0.105662\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{-0.088036, 0.088036, 0.0235892, -0.0235892\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{0.108674, -0.251532, 0.0754595, 0.0673977\}$

$\boldsymbol{B}^{\mathrm{T}} =$

$$\begin{pmatrix} -0.088036 & 0 & 0.088036 & 0 & 0.0235892 & 0 & -0.0235892 & 0 \\ 0 & 0.108674 & 0 & -0.251532 & 0 & 0.0754595 & 0 & 0.0673977 \\ 0.108674 & -0.088036 & -0.251532 & 0.088036 & 0.0754595 & 0.0235892 & 0.0673977 & -0.0235892 \end{pmatrix}$$

$$\boldsymbol{k} = 10^6 \begin{pmatrix} 2.44459 & -1.12493 & -3.66154 & 1.61785 & 0.235849 & -0.0594203 & 0.981107 & {-} \\ -1.12493 & 2.92194 & 2.11077 & -5.96433 & -0.420264 & 1.44425 & -0.56558 & \\ -3.66154 & 2.11077 & 6.47824 & -2.60369 & -1.08086 & -0.204736 & -1.73584 & \\ 1.61785 & -5.96433 & -2.60369 & 13.0061 & 0.288186 & -3.55678 & 0.697658 & {-} \\ 0.235849 & -0.420264 & -1.08086 & 0.288186 & 0.555394 & 0.209297 & 0.289615 & {-} \\ -0.0594203 & 1.44425 & -0.204736 & -3.55678 & 0.209297 & 1.15949 & 0.0548588 & \\ 0.981107 & -0.56558 & -1.73584 & 0.697658 & 0.289615 & 0.0548588 & 0.465117 & {-} \\ -0.433502 & 1.59814 & 0.697658 & -3.48497 & -0.0772193 & 0.953035 & -0.186937 & \end{pmatrix}$$

Computation of element matrices at {0.57735, 0.57735} with weight = 1.

$$\boldsymbol{J} = \begin{pmatrix} 8.52073 & 5.52073 \\ 0 & 3.5 \end{pmatrix} \qquad \det J = 29.8225$$

$\{N_1, N_2, N_3, N_4\} = \{0.0446582, 0.166667, 0.622008, 0.166667\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \{-0.105662, 0.105662, 0.394338, -0.394338\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \{-0.105662, -0.394338, 0.394338, 0.105662\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{-0.0124006, 0.0124006, 0.0462798, -0.0462798\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{-0.0106291, -0.132228, 0.0396684, 0.103189\}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -0.0124006 & 0 & 0.0124006 & 0 & 0.0462798 & 0 & -0.0462798 & 0 \\ 0 & -0.0106291 & 0 & -0.132228 & 0 & 0.0396684 & 0 & 0. \\ -0.0106291 & -0.0124006 & -0.132228 & 0.0124006 & 0.0396684 & 0.0462798 & 0.103189 & -0. \end{pmatrix}$$

$$\boldsymbol{k} = 10^6 \begin{pmatrix} 0.0741713 & 0.0294813 & 0.152248 & 0.102597 & -0.276811 & -0.110026 & 0.0503915 & {-} \\ 0.0294813 & 0.0650461 & 0.234675 & 0.501003 & -0.110026 & -0.242755 & -0.15413 & {-} \\ 0.152248 & 0.234675 & 2.66445 & -0.366753 & -0.568198 & -0.875818 & -2.2485 & \\ 0.102597 & 0.501003 & -0.366753 & 6.54074 & -0.382896 & -1.86977 & 0.647052 & {-} \\ -0.276811 & -0.110026 & -0.568198 & -0.382896 & 1.03307 & 0.410622 & -0.188064 & \\ -0.110026 & -0.242755 & -0.875818 & -1.86977 & 0.410622 & 0.905976 & 0.575222 & \\ 0.0503915 & -0.15413 & -2.2485 & 0.647052 & -0.188064 & 0.575222 & 2.38617 & {-} \\ -0.0220522 & -0.323293 & 1.0079 & -5.17198 & 0.0822999 & 1.20655 & -1.06814 & \end{pmatrix}$$

Summing contributions from all points we get

$$\boldsymbol{k} = 10^6 \begin{pmatrix} 6.26209 & -0.0163755 & -4.11923 & 1.26638 & -0.951731 & -1.12991 & -1.19113 & -0.120087 \\ -0.0163755 & 9.0354 & 2.51638 & -3.67826 & -1.12991 & 0.228478 & -1.37009 & -5.58562 \\ -4.11923 & 2.51638 & 11.2621 & -3.76638 & -1.19113 & -1.37009 & -5.95173 & 2.62009 \\ 1.26638 & -3.67826 & -3.76638 & 21.5354 & -0.120087 & -5.58562 & 2.62009 & -12.2715 \\ -0.951731 & -1.12991 & -1.19113 & -0.120087 & 2.54484 & 0.786026 & -0.401981 & 0.463974 \\ -1.12991 & 0.228478 & -1.37009 & -5.58562 & 0.786026 & 2.55069 & 1.71397 & 2.80646 \\ -1.19113 & -1.37009 & -5.95173 & 2.62009 & -0.401981 & 1.71397 & 7.54484 & -2.96397 \\ -0.120087 & -5.58562 & 2.62009 & -12.2715 & 0.463974 & 2.80646 & -2.96397 & 15.0507 \end{pmatrix}$$

$\boldsymbol{r}^T = (\,0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\,)$

Computation of element matrices resulting from NBC

NBC on side 3 with $\{q_n, q_t\} = \{-50, 0\}$

$\{N_1, N_2, N_3, N_4\}_c = \left\{0, 0, \frac{1-a}{2}, \frac{a+1}{2}\right\}$

$x(a) = 10. - 10.\,a;$ $\qquad y(a) = 12.$

$dx/da = -10.;$ $\qquad dy/da = 0.$

$J_c = 10.$

Gauss point = -0.57735; Weight = 1.; $J_c = 10.$

$\{N_1, N_2, N_3, N_4\}_c = \{0, 0, 0.788675, 0.211325\}$

$\boldsymbol{r}_q^T = (\,0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -1577.35 \quad 0 \quad -422.65\,)$

Gauss point = 0.57735; Weight = 1.; $J_c = 10.$

$\{N_1, N_2, N_3, N_4\}_c = \{0, 0, 0.211325, 0.788675\}$

$\boldsymbol{r}_q^T = (\,0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -422.65 \quad 0 \quad -1577.35\,)$

Summing contributions from all Gauss points

$\boldsymbol{r}_q^T = (\,0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -2000. \quad 0 \quad -2000.\,)$

Complete element equations for element 1

$$10^6\begin{pmatrix} 6.26209 & -0.0163755 & -4.11923 & 1.26638 & -0.951731 & -1.12991 & -1.19113 & -0.120087 \\ -0.0163755 & 9.0354 & 2.51638 & -3.67826 & -1.12991 & 0.228478 & -1.37009 & -5.58562 \\ -4.11923 & 2.51638 & 11.2621 & -3.76638 & -1.19113 & -1.37009 & -5.95173 & 2.62009 \\ 1.26638 & -3.67826 & -3.76638 & 21.5354 & -0.120087 & -5.58562 & 2.62009 & -12.2715 \\ -0.951731 & -1.12991 & -1.19113 & -0.120087 & 2.54484 & 0.786026 & -0.401981 & 0.463974 \\ -1.12991 & 0.228478 & -1.37009 & -5.58562 & 0.786026 & 2.55069 & 1.71397 & 2.80646 \\ -1.19113 & -1.37009 & -5.95173 & 2.62009 & -0.401981 & 1.71397 & 7.54484 & -2.96397 \\ -0.120087 & -5.58562 & 2.62009 & -12.2715 & 0.463974 & 2.80646 & -2.96397 & 15.0507 \end{pmatrix}$$

$$\begin{pmatrix} u_1 \\ v_1 \\ u_4 \\ v_4 \\ u_6 \\ v_6 \\ u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ -2000. \\ 0. \\ -2000. \end{pmatrix}$$

The element contributes to {1, 2, 7, 8, 11, 12, 3, 4} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 1 \\ 2 \\ 7 \\ 8 \\ 11 \\ 12 \\ 3 \\ 4 \end{pmatrix}$

and to a global matrix: $\begin{pmatrix} [1,1] & [1,2] & [1,7] & [1,8] & [1,11] & [1,12] & [1,3] & [1,4] \\ [2,1] & [2,2] & [2,7] & [2,8] & [2,11] & [2,12] & [2,3] & [2,4] \\ [7,1] & [7,2] & [7,7] & [7,8] & [7,11] & [7,12] & [7,3] & [7,4] \\ [8,1] & [8,2] & [8,7] & [8,8] & [8,11] & [8,12] & [8,3] & [8,4] \\ [11,1] & [11,2] & [11,7] & [11,8] & [11,11] & [11,12] & [11,3] & [11,4] \\ [12,1] & [12,2] & [12,7] & [12,8] & [12,11] & [12,12] & [12,3] & [12,4] \\ [3,1] & [3,2] & [3,7] & [3,8] & [3,11] & [3,12] & [3,3] & [3,4] \\ [4,1] & [4,2] & [4,7] & [4,8] & [4,11] & [4,12] & [4,3] & [4,4] \end{pmatrix}$

Adding element equations into appropriate locations we have

$$10^{6}\left(\begin{array}{cccccccccccc}
6.26209 & -0.0163755 & -1.19113 & -0.120087 & 0 & 0 & -4.11923 & 1.26638 & 0 & 0 & -0.951731 & -1 \\
-0.0163755 & 9.0354 & -1.37009 & -5.58562 & 0 & 0 & 2.51638 & -3.67826 & 0 & 0 & -1.12991 & 0 \\
-1.19113 & -1.37009 & 7.54484 & -2.96397 & 0 & 0 & -5.95173 & 2.62009 & 0 & 0 & -0.401981 & 1 \\
-0.120087 & -5.58562 & -2.96397 & 15.0507 & 0 & 0 & 2.62009 & -12.2715 & 0 & 0 & 0.463974 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-4.11923 & 2.51638 & -5.95173 & 2.62009 & 0 & 0 & 11.2621 & -3.76638 & 0 & 0 & -1.19113 & -1 \\
1.26638 & -3.67826 & 2.62009 & -12.2715 & 0 & 0 & -3.76638 & 21.5354 & 0 & 0 & -0.120087 & -5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.951731 & -1.12991 & -0.401981 & 0.463974 & 0 & 0 & -1.19113 & -0.120087 & 0 & 0 & 2.54484 & 0 \\
-1.12991 & 0.228478 & 1.71397 & 2.80646 & 0 & 0 & -1.37009 & -5.58562 & 0 & 0 & 0.786026 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right.$$

Equations for element 2

$E = 3000000;$ $\nu = 0.2;$ $h = 4$

Nodal coordinates

Element node	Global node number	x	y
1	3	6.	0.
2	5	20.	0.
3	6	20.	12.
4	4	6.	5.

Interpolation functions and their derivatives

$$\{N_1, N_2, N_3, N_4\} = \left\{\frac{1}{4}(s-1)(t-1), -\frac{1}{4}(s+1)(t-1), \frac{1}{4}(s+1)(t+1), -\frac{1}{4}(s-1)(t+1)\right\}$$

$$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{\frac{t-1}{4}, \frac{1-t}{4}, \frac{t+1}{4}, \frac{1}{4}(-t-1)\right\}$$

$$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \left\{\frac{s-1}{4}, \frac{1}{4}(-s-1), \frac{s+1}{4}, \frac{1-s}{4}\right\}$$

Mapping to the master element

$$x(s,t) = \boldsymbol{N}^{T}\boldsymbol{x}_n = 7.\,s + 13.$$

$$y(s,t) = \boldsymbol{N}^{T}\boldsymbol{y}_n = 1.75\,t\,s + 1.75\,s + 4.25\,t + 4.25$$

$$\boldsymbol{J} = \begin{pmatrix} 7. & 0 \\ 1.75\,t + 1.75 & 1.75\,s + 4.25 \end{pmatrix}; \qquad \det J = 12.25\,s + 29.75$$

$$\text{Plane stress } \boldsymbol{C} = \begin{pmatrix} 3.125\times 10^{6} & 625000. & 0 \\ 625000. & 3.125\times 10^{6} & 0 \\ 0 & 0 & 1.25\times 10^{6} \end{pmatrix}$$

For numerical integration the Gauss quadrature points and weights are

	s	t	Weight
1	−0.57735	−0.57735	1.
2	−0.57735	0.57735	1.
3	0.57735	−0.57735	1.
4	0.57735	0.57735	1.

Computation of element matrices at {−0.57735, −0.57735} with weight = 1.

$$\boldsymbol{J} = \begin{pmatrix} 7. & 0 \\ 0.739637 & 3.23964 \end{pmatrix} \qquad \det J = 22.6775$$

$\{N_1, N_2, N_3, N_4\} = \{0.622008, 0.166667, 0.0446582, 0.166667\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \{-0.394338, 0.394338, 0.105662, -0.105662\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \{-0.394338, -0.105662, 0.105662, 0.394338\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{-0.0434724, 0.0597802, 0.0116484, -0.0279562\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{-0.121723, -0.0326155, 0.0326155, 0.121723\}$

$$\boldsymbol{B}^{T} = \begin{pmatrix} -0.0434724 & 0 & 0.0597802 & 0 & 0.0116484 & 0 & -0.0279562 & 0 \\ 0 & -0.121723 & 0 & -0.0326155 & 0 & 0.0326155 & 0 & 0. \\ -0.121723 & -0.0434724 & -0.0326155 & 0.0597802 & 0.0326155 & 0.0116484 & 0.121723 & -0. \end{pmatrix}$$

$$\boldsymbol{k} = 10^{6}\begin{pmatrix} 2.21571 & 0.899997 & -0.286521 & -0.74469 & -0.593697 & -0.241154 & -1.33549 \\ 0.899997 & 4.41427 & -0.251768 & 0.830714 & -0.241154 & -1.1828 & -0.407075 \\ -0.286521 & -0.251768 & 1.13364 & -0.331617 & 0.076773 & 0.0674611 & -0.923892 \\ -0.74469 & 0.830714 & -0.331617 & 0.706754 & 0.199539 & -0.222589 & 0.876768 \\ -0.593697 & -0.241154 & 0.076773 & 0.199539 & 0.159081 & 0.0646169 & 0.357843 \\ -0.241154 & -1.1828 & 0.0674611 & -0.222589 & 0.0646169 & 0.316931 & 0.109076 \\ -1.33549 & -0.407075 & -0.923892 & 0.876768 & 0.357843 & 0.109076 & 1.90154 \\ 0.0858466 & -4.06219 & 0.515924 & -1.31488 & -0.0230025 & 1.08846 & -0.578769 \end{pmatrix}$$

Computation of element matrices at {−0.57735, 0.57735} with weight = 1.

$$\boldsymbol{J} = \begin{pmatrix} 7. & 0 \\ 2.76036 & 3.23964 \end{pmatrix} \qquad \det J = 22.6775$$

$\{N_1, N_2, N_3, N_4\} = \{0.166667, 0.0446582, 0.166667, 0.622008\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \{-0.105662, 0.105662, 0.394338, -0.394338\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \{-0.394338, -0.105662, 0.105662, 0.394338\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{0.0329052, 0.0279562, 0.0434724, -0.104334\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{-0.121723, -0.0326155, 0.0326155, 0.121723\}$

$\boldsymbol{B}^{\mathrm{T}} =$

$$\begin{pmatrix} 0.0329052 & 0 & 0.0279562 & 0 & 0.0434724 & 0 & -0.104334 & 0 \\ 0 & -0.121723 & 0 & -0.0326155 & 0 & 0.0326155 & 0 & 0.121723 \\ -0.121723 & 0.0329052 & -0.0326155 & 0.0279562 & 0.0326155 & 0.0434724 & 0.121723 & -0.104334 \end{pmatrix}$$

$\boldsymbol{k} = 10^6$

$$\begin{pmatrix} 1.98692 & -0.681228 & 0.710917 & -0.446691 & -0.0446606 & -0.539153 & -2.65318 & 1.66707 \\ -0.681228 & 4.32276 & -0.314612 & 1.22969 & -0.178309 & -0.963186 & 1.17415 & -4.58926 \\ 0.710917 & -0.314612 & 0.342162 & -0.155081 & 0.223887 & -0.109076 & -1.27697 & 0.578769 \\ -0.446691 & 1.22969 & -0.155081 & 0.390163 & 0.0230025 & -0.163744 & 0.578769 & -1.45611 \\ -0.0446606 & -0.178309 & 0.223887 & 0.0230025 & 0.656331 & 0.241154 & -0.835557 & -0.0858466 \\ -0.539153 & -0.963186 & -0.109076 & -0.163744 & 0.241154 & 0.515831 & 0.407075 & 0.611099 \\ -2.65318 & 1.17415 & -1.27697 & 0.578769 & -0.835557 & 0.407075 & 4.7657 & -2.15999 \\ 1.66707 & -4.58926 & 0.578769 & -1.45611 & -0.0858466 & 0.611099 & -2.15999 & 5.43427 \end{pmatrix}$$

Computation of element matrices at $\{0.57735, -0.57735\}$ with weight = 1.

$$\boldsymbol{J} = \begin{pmatrix} 7. & 0 \\ 0.739637 & 5.26036 \end{pmatrix} \qquad \det J = 36.8225$$

$\{N_1, N_2, N_3, N_4\} = \{0.166667, 0.622008, 0.166667, 0.0446582\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \{-0.394338, 0.394338, 0.105662, -0.105662\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \{-0.105662, -0.394338, 0.394338, 0.105662\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{-0.0542115, 0.0642548, 0.00717376, -0.017217\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{-0.0200865, -0.0749639, 0.0749639, 0.0200865\}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -0.0542115 & 0 & 0.0642548 & 0 & 0.00717376 & 0 & -0.017217 \\ 0 & -0.0200865 & 0 & -0.0749639 & 0 & 0.0749639 & 0 \\ -0.0200865 & -0.0542115 & -0.0749639 & 0.0642548 & 0.0749639 & 0.00717376 & 0.0200865 \end{pmatrix}$$

$$\boldsymbol{k} = 10^6 \begin{pmatrix} 1.427 & 0.300727 & -1.32609 & 0.136483 & -0.456234 & -0.400639 & 0.355326 \\ 0.300727 & 0.726797 & 0.629405 & 0.0517469 & -0.761483 & -0.764678 & -0.168648 \\ -1.32609 & 0.629405 & 2.93499 & -1.33025 & -0.822472 & 0.344406 & -0.78643 \\ 0.136483 & 0.0517469 & -1.33025 & 3.34674 & 0.837328 & -2.50173 & 0.356439 \\ -0.456234 & -0.761483 & -0.822472 & 0.837328 & 1.05833 & 0.148516 & 0.220381 \\ -0.400639 & -0.764678 & 0.344406 & -2.50173 & 0.148516 & 2.59607 & -0.0922832 \\ 0.355326 & -0.168648 & -0.78643 & 0.356439 & 0.220381 & -0.0922832 & 0.210723 \\ -0.0365704 & -0.0138655 & 0.356439 & -0.896756 & -0.224361 & 0.670336 & -0.0955076 \end{pmatrix}$$

Computation of element matrices at {0.57735, 0.57735} with weight = 1.

$$\boldsymbol{J} = \begin{pmatrix} 7. & 0 \\ 2.76036 & 5.26036 \end{pmatrix} \qquad \det J = 36.8225$$

$\{N_1, N_2, N_3, N_4\} = \{0.0446582, 0.166667, 0.622008, 0.166667\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \{-0.105662, 0.105662, 0.394338, -0.394338\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \{-0.105662, -0.394338, 0.394338, 0.105662\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{-0.00717376, 0.0446557, 0.0267728, -0.0642548\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{-0.0200865, -0.0749639, 0.0749639, 0.0200865\}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -0.00717376 & 0 & 0.0446557 & 0 & 0.0267728 & 0 & -0.0642548 \\ 0 & -0.0200865 & 0 & -0.0749639 & 0 & 0.0749639 & 0 \\ -0.0200865 & -0.00717376 & -0.0749639 & 0.0446557 & 0.0749639 & 0.0267728 & 0.0200865 & - \end{pmatrix}$$

$$\boldsymbol{k} = 10^6 \begin{pmatrix} 0.0979711 & 0.0397948 & 0.12978 & -0.11564 & -0.365633 & -0.148516 & 0.137883 \\ 0.0397948 & 0.195184 & 0.0164383 & 0.634096 & -0.148516 & -0.728437 & 0.0922832 \\ 0.12978 & 0.0164383 & 1.9525 & -0.924495 & -0.484344 & -0.0613487 & -1.59794 \\ -0.11564 & 0.634096 & -0.924495 & 2.95374 & 0.431573 & -2.36648 & 0.608562 \\ -0.365633 & -0.148516 & -0.484344 & 0.431573 & 1.36456 & 0.554271 & -0.514585 \\ -0.148516 & -0.728437 & -0.0613487 & -2.36648 & 0.554271 & 2.71857 & -0.344406 \\ 0.137883 & 0.0922832 & -1.59794 & 0.608562 & -0.514585 & -0.344406 & 1.97464 \\ 0.224361 & -0.100843 & 0.969406 & -1.22136 & -0.837328 & 0.37635 & -0.356439 \end{pmatrix}$$

Summing contributions from all points we get

$$\boldsymbol{k} = 10^6 \begin{pmatrix} 5.7276 & 0.559291 & -0.771918 & -1.17054 & -1.46023 & -1.32946 & -3.49546 & 1.94071 \\ 0.559291 & 9.65901 & 0.0794621 & 2.74625 & -1.32946 & -3.6391 & 0.690709 & -8.76615 \\ -0.771918 & 0.0794621 & 6.3633 & -2.74144 & -1.00616 & 0.241443 & -4.58523 & 2.42054 \\ -1.17054 & 2.74625 & -2.74144 & 7.3974 & 1.49144 & -5.25454 & 2.42054 & -4.8891 \\ -1.46023 & -1.32946 & -1.00616 & 1.49144 & 3.2383 & 1.00856 & -0.771918 & -1.17054 \\ -1.32946 & -3.6391 & 0.241443 & -5.25454 & 1.00856 & 6.1474 & 0.0794621 & 2.74625 \\ -3.49546 & 0.690709 & -4.58523 & 2.42054 & -0.771918 & 0.0794621 & 8.8526 & -3.19071 \\ 1.94071 & -8.76615 & 2.42054 & -4.8891 & -1.17054 & 2.74625 & -3.19071 & 10.909 \end{pmatrix}$$

$$\boldsymbol{r}^{\mathrm{T}} = (\,0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\,)$$

Complete element equations for element 2

$$10^6 \begin{pmatrix} 5.7276 & 0.559291 & -0.771918 & -1.17054 & -1.46023 & -1.32946 & -3.49546 & 1.94071 \\ 0.559291 & 9.65901 & 0.0794621 & 2.74625 & -1.32946 & -3.6391 & 0.690709 & -8.76615 \\ -0.771918 & 0.0794621 & 6.3633 & -2.74144 & -1.00616 & 0.241443 & -4.58523 & 2.42054 \\ -1.17054 & 2.74625 & -2.74144 & 7.3974 & 1.49144 & -5.25454 & 2.42054 & -4.8891 \\ -1.46023 & -1.32946 & -1.00616 & 1.49144 & 3.2383 & 1.00856 & -0.771918 & -1.17054 \\ -1.32946 & -3.6391 & 0.241443 & -5.25454 & 1.00856 & 6.1474 & 0.0794621 & 2.74625 \\ -3.49546 & 0.690709 & -4.58523 & 2.42054 & -0.771918 & 0.0794621 & 8.8526 & -3.19071 \\ 1.94071 & -8.76615 & 2.42054 & -4.8891 & -1.17054 & 2.74625 & -3.19071 & 10.909 \end{pmatrix}$$

$$\begin{pmatrix} u_3 \\ v_3 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {5, 6, 9, 10, 11, 12, 7, 8} global degrees of freedom.

$$\text{Locations for element contributions to a global vector:} \begin{pmatrix} 5 \\ 6 \\ 9 \\ 10 \\ 11 \\ 12 \\ 7 \\ 8 \end{pmatrix}$$

$$\text{and to a global matrix:}\begin{pmatrix} [5,5] & [5,6] & [5,9] & [5,10] & [5,11] & [5,12] & [5,7] & [5,8] \\ [6,5] & [6,6] & [6,9] & [6,10] & [6,11] & [6,12] & [6,7] & [6,8] \\ [9,5] & [9,6] & [9,9] & [9,10] & [9,11] & [9,12] & [9,7] & [9,8] \\ [10,5] & [10,6] & [10,9] & [10,10] & [10,11] & [10,12] & [10,7] & [10,8] \\ [11,5] & [11,6] & [11,9] & [11,10] & [11,11] & [11,12] & [11,7] & [11,8] \\ [12,5] & [12,6] & [12,9] & [12,10] & [12,11] & [12,12] & [12,7] & [12,8] \\ [7,5] & [7,6] & [7,9] & [7,10] & [7,11] & [7,12] & [7,7] & [7,8] \\ [8,5] & [8,6] & [8,9] & [8,10] & [8,11] & [8,12] & [8,7] & [8,8] \end{pmatrix}$$

Adding element equations into appropriate locations we have

$$10^6\left(\begin{array}{llllllll} 6.26209 & -0.0163755 & -1.19113 & -0.120087 & 0 & 0 & -4.11923 & 1.26638 \\ -0.0163755 & 9.0354 & -1.37009 & -5.58562 & 0 & 0 & 2.51638 & -3.67826 \\ -1.19113 & -1.37009 & 7.54484 & -2.96397 & 0 & 0 & -5.95173 & 2.62009 \\ -0.120087 & -5.58562 & -2.96397 & 15.0507 & 0 & 0 & 2.62009 & -12.2715 \\ 0 & 0 & 0 & 0 & 5.7276 & 0.559291 & -3.49546 & 1.94071 \\ 0 & 0 & 0 & 0 & 0.559291 & 9.65901 & 0.690709 & -8.76615 \\ -4.11923 & 2.51638 & -5.95173 & 2.62009 & -3.49546 & 0.690709 & 20.1147 & -6.95708 \\ 1.26638 & -3.67826 & 2.62009 & -12.2715 & 1.94071 & -8.76615 & -6.95708 & 32.4444 \\ 0 & 0 & 0 & 0 & -0.771918 & 0.0794621 & -4.58523 & 2.42054 \\ 0 & 0 & 0 & 0 & -1.17054 & 2.74625 & 2.42054 & -4.8891 \\ -0.951731 & -1.12991 & -0.401981 & 0.463974 & -1.46023 & -1.32946 & -1.96304 & -1.29063 \\ -1.12991 & 0.228478 & 1.71397 & 2.80646 & -1.32946 & -3.6391 & -1.29063 & -2.83938 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right.$$

Equations for element 3

$E = 3000000;$ $\nu = 0.2;$ $h = 4$

Nodal coordinates

Element node	Global node number	x	y
1	5	20.	0.
2	7	54.	0.
3	8	54.	12.
4	6	20.	12.

Interpolation functions and their derivatives

$$\{N_1, N_2, N_3, N_4\} = \left\{\frac{1}{4}(s-1)(t-1), -\frac{1}{4}(s+1)(t-1), \frac{1}{4}(s+1)(t+1), -\frac{1}{4}(s-1)(t+1)\right\}$$

$$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{\frac{t-1}{4}, \frac{1-t}{4}, \frac{t+1}{4}, \frac{1}{4}(-t-1)\right\}$$

$$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \left\{\frac{s-1}{4}, \frac{1}{4}(-s-1), \frac{s+1}{4}, \frac{1-s}{4}\right\}$$

Mapping to the master element

$$x(s,t) = \boldsymbol{N}^T \boldsymbol{x}_n = 17.\, s + 37.$$

$$y(s,t) = \boldsymbol{N}^T \boldsymbol{y}_n = 6.\, t + 6.$$

$$\boldsymbol{J} = \begin{pmatrix} 17. & 0 \\ 0 & 6. \end{pmatrix}; \qquad \det J = 102.$$

Plane stress $\boldsymbol{C} = \begin{pmatrix} 3.125\times 10^6 & 625000. & 0 \\ 625000. & 3.125\times 10^6 & 0 \\ 0 & 0 & 1.25\times 10^6 \end{pmatrix}$

For numerical integration the Gauss quadrature points and weights are

	s	t	Weight
1	−0.57735	−0.57735	1.
2	−0.57735	0.57735	1.
3	0.57735	−0.57735	1.
4	0.57735	0.57735	1.

Computation of element matrices at {−0.57735, −0.57735} with weight = 1.

$$\boldsymbol{J} = \begin{pmatrix} 17. & 0 \\ 0 & 6. \end{pmatrix} \qquad \det J = 102.$$

$$\{N_1, N_2, N_3, N_4\} = \{0.622008, 0.166667, 0.0446582, 0.166667\}$$

$$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \{-0.394338, 0.394338, 0.105662, -0.105662\}$$

$$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \{-0.394338, -0.105662, 0.105662, 0.394338\}$$

$$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{-0.0231963, 0.0231963, 0.00621544, -0.00621544\}$$

$$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{-0.0657229, -0.0176104, 0.0176104, 0.0657229\}$$

$$\boldsymbol{B}^T = \begin{pmatrix} -0.0231963 & 0 & 0.0231963 & 0 & 0.00621544 & 0 & -0.00621544 \\ 0 & -0.0657229 & 0 & -0.0176104 & 0 & 0.0176104 & 0 \\ -0.0657229 & -0.0231963 & -0.0176104 & 0.0231963 & 0.0176104 & 0.00621544 & 0.0657229 \end{pmatrix}$$

$\boldsymbol{k} = 10^6$

$$\begin{pmatrix} 2.88899 & 1.16627 & -0.095761 & -0.673344 & -0.774101 & -0.3125 & -2.01912 & -0.180422 \\ 1.16627 & 5.78178 & -0.180422 & 1.20128 & -0.3125 & -1.54922 & -0.673344 & -5.43384 \\ -0.095761 & -0.180422 & 0.844203 & -0.3125 & 0.0256591 & 0.0483439 & -0.774101 & 0.444578 \\ -0.673344 & 1.20128 & -0.3125 & 0.669827 & 0.180422 & -0.321882 & 0.805422 & -1.54922 \\ -0.774101 & -0.3125 & 0.0256591 & 0.180422 & 0.20742 & 0.0837341 & 0.541022 & 0.0483439 \\ -0.3125 & -1.54922 & 0.0483439 & -0.321882 & 0.0837341 & 0.415113 & 0.180422 & 1.45599 \\ -2.01912 & -0.673344 & -0.774101 & 0.805422 & 0.541022 & 0.180422 & 2.2522 & -0.3125 \\ -0.180422 & -5.43384 & 0.444578 & -1.54922 & 0.0483439 & 1.45599 & -0.3125 & 5.52707 \end{pmatrix}$$

Computation of element matrices at {−0.57735, 0.57735} with weight = 1.

$\boldsymbol{J} = \begin{pmatrix} 17. & 0 \\ 0 & 6. \end{pmatrix}$ detJ = 102.

$\{N_1, N_2, N_3, N_4\} = \{0.166667, 0.0446582, 0.166667, 0.622008\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \{-0.105662, 0.105662, 0.394338, -0.394338\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \{-0.394338, -0.105662, 0.105662, 0.394338\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{-0.00621544, 0.00621544, 0.0231963, -0.0231963\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{-0.0657229, -0.0176104, 0.0176104, 0.0657229\}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -0.00621544 & 0 & 0.00621544 & 0 & 0.0231963 & 0 & -0.0231963 & 0 \\ 0 & -0.0657229 & 0 & -0.0176104 & 0 & 0.0176104 & 0 & 0.0657229 \\ -0.0657229 & -0.00621544 & -0.0176104 & 0.00621544 & 0.0176104 & 0.0231963 & 0.0657229 & -0.0231963 \end{pmatrix}$$

$\boldsymbol{k} = 10^6$

$$\begin{pmatrix} 2.2522 & 0.3125 & 0.541022 & -0.180422 & -0.774101 & -0.805422 & -2.01912 & 0.673344 \\ 0.3125 & 5.52707 & -0.0483439 & 1.45599 & -0.444578 & -1.54922 & 0.180422 & -5.43384 \\ 0.541022 & -0.0483439 & 0.20742 & -0.0837341 & 0.0256591 & -0.180422 & -0.774101 & 0.3125 \\ -0.180422 & 1.45599 & -0.0837341 & 0.415113 & -0.0483439 & -0.321882 & 0.3125 & -1.54922 \\ -0.774101 & -0.444578 & 0.0256591 & -0.0483439 & 0.844203 & 0.3125 & -0.095761 & 0.180422 \\ -0.805422 & -1.54922 & -0.180422 & -0.321882 & 0.3125 & 0.669827 & 0.673344 & 1.20128 \\ -2.01912 & 0.180422 & -0.774101 & 0.3125 & -0.095761 & 0.673344 & 2.88899 & -1.16627 \\ 0.673344 & -5.43384 & 0.3125 & -1.54922 & 0.180422 & 1.20128 & -1.16627 & 5.78178 \end{pmatrix}$$

Computation of element matrices at {0.57735, −0.57735} with weight = 1.

$\boldsymbol{J} = \begin{pmatrix} 17. & 0 \\ 0 & 6. \end{pmatrix}$ detJ = 102.

$\{N_1, N_2, N_3, N_4\} = \{0.166667, 0.622008, 0.166667, 0.0446582\}$

$\{\partial N_1/\partial s,\ \partial N_2/\partial s,\ \partial N_3/\partial s,\ \partial N_4/\partial s\} = \{-0.394338,\ 0.394338,\ 0.105662,\ -0.105662\}$

$\{\partial N_1/\partial t,\ \partial N_2/\partial t,\ \partial N_3/\partial t,\ \partial N_4/\partial t\} = \{-0.105662,\ -0.394338,\ 0.394338,\ 0.105662\}$

$\{\partial N_1/\partial x,\ \partial N_2/\partial x,\ \partial N_3/\partial x,\ \partial N_4/\partial x\} = \{-0.0231963,\ 0.0231963,\ 0.00621544,\ -0.00621544\}$

$\{\partial N_1/\partial y,\ \partial N_2/\partial y,\ \partial N_3/\partial y,\ \partial N_4/\partial y\} = \{-0.0176104,\ -0.0657229,\ 0.0657229,\ 0.0176104\}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -0.0231963 & 0 & 0.0231963 & 0 & 0.00621544 & 0 & -0.00621544 \\ 0 & -0.0176104 & 0 & -0.0657229 & 0 & 0.0657229 & 0 \\ -0.0176104 & -0.0231963 & -0.0657229 & 0.0231963 & 0.0657229 & 0.00621544 & 0.0176104 \end{pmatrix}$$

$\boldsymbol{k} = 10^6$

$$\begin{pmatrix} 0.844203 & 0.3125 & -0.095761 & 0.180422 & -0.774101 & -0.444578 & 0.0256591 & -0.0483439 \\ 0.3125 & 0.669827 & 0.673344 & 1.20128 & -0.805422 & -1.54922 & -0.180422 & -0.321882 \\ -0.095761 & 0.673344 & 2.88899 & -1.16627 & -2.01912 & 0.180422 & -0.774101 & 0.3125 \\ 0.180422 & 1.20128 & -1.16627 & 5.78178 & 0.673344 & -5.43384 & 0.3125 & -1.54922 \\ -0.774101 & -0.805422 & -2.01912 & 0.673344 & 2.2522 & 0.3125 & 0.541022 & -0.180422 \\ -0.444578 & -1.54922 & 0.180422 & -5.43384 & 0.3125 & 5.52707 & -0.0483439 & 1.45599 \\ 0.0256591 & -0.180422 & -0.774101 & 0.3125 & 0.541022 & -0.0483439 & 0.20742 & -0.0837341 \\ -0.0483439 & -0.321882 & 0.3125 & -1.54922 & -0.180422 & 1.45599 & -0.0837341 & 0.415113 \end{pmatrix}$$

Computation of element matrices at {0.57735, 0.57735} with weight = 1.

$$\boldsymbol{J} = \begin{pmatrix} 17. & 0 \\ 0 & 6. \end{pmatrix} \qquad \mathrm{detJ} = 102.$$

$\{N_1,\ N_2,\ N_3,\ N_4\} = \{0.0446582,\ 0.166667,\ 0.622008,\ 0.166667\}$

$\{\partial N_1/\partial s,\ \partial N_2/\partial s,\ \partial N_3/\partial s,\ \partial N_4/\partial s\} = \{-0.105662,\ 0.105662,\ 0.394338,\ -0.394338\}$

$\{\partial N_1/\partial t,\ \partial N_2/\partial t,\ \partial N_3/\partial t,\ \partial N_4/\partial t\} = \{-0.105662,\ -0.394338,\ 0.394338,\ 0.105662\}$

$\{\partial N_1/\partial x,\ \partial N_2/\partial x,\ \partial N_3/\partial x,\ \partial N_4/\partial x\} = \{-0.00621544,\ 0.00621544,\ 0.0231963,\ -0.0231963\}$

$\{\partial N_1/\partial y,\ \partial N_2/\partial y,\ \partial N_3/\partial y,\ \partial N_4/\partial y\} = \{-0.0176104,\ -0.0657229,\ 0.0657229,\ 0.0176104\}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -0.00621544 & 0 & 0.00621544 & 0 & 0.0231963 & 0 & -0.0231963 \\ 0 & -0.0176104 & 0 & -0.0657229 & 0 & 0.0657229 & 0 \\ -0.0176104 & -0.00621544 & -0.0657229 & 0.00621544 & 0.0657229 & 0.0231963 & 0.0176104 \end{pmatrix}$$

$$\boldsymbol{k} = 10^6$$

$$\begin{pmatrix} 0.20742 & 0.0837341 & 0.541022 & 0.0483439 & -0.774101 & -0.3125 & 0.0256591 & 0.180422 \\ 0.0837341 & 0.415113 & 0.180422 & 1.45599 & -0.3125 & -1.54922 & 0.0483439 & -0.321882 \\ 0.541022 & 0.180422 & 2.2522 & -0.3125 & -2.01912 & -0.673344 & -0.774101 & 0.805422 \\ 0.0483439 & 1.45599 & -0.3125 & 5.52707 & -0.180422 & -5.43384 & 0.444578 & -1.54922 \\ -0.774101 & -0.3125 & -2.01912 & -0.180422 & 2.88899 & 1.16627 & -0.095761 & -0.673344 \\ -0.3125 & -1.54922 & -0.673344 & -5.43384 & 1.16627 & 5.78178 & -0.180422 & 1.20128 \\ 0.0256591 & 0.0483439 & -0.774101 & 0.444578 & -0.095761 & -0.180422 & 0.844203 & -0.3125 \\ 0.180422 & -0.321882 & 0.805422 & -1.54922 & -0.673344 & 1.20128 & -0.3125 & 0.669827 \end{pmatrix}$$

Summing contributions from all points we get

$$\boldsymbol{k} = 10^6 \begin{pmatrix} 6.19281 & 1.875 & 0.890523 & -0.625 & -3.09641 & -1.875 & -3.98693 & 0.625 \\ 1.875 & 12.3938 & 0.625 & 5.31454 & -1.875 & -6.1969 & -0.625 & -11.5114 \\ 0.890523 & 0.625 & 6.19281 & -1.875 & -3.98693 & -0.625 & -3.09641 & 1.875 \\ -0.625 & 5.31454 & -1.875 & 12.3938 & 0.625 & -11.5114 & 1.875 & -6.1969 \\ -3.09641 & -1.875 & -3.98693 & 0.625 & 6.19281 & 1.875 & 0.890523 & -0.625 \\ -1.875 & -6.1969 & -0.625 & -11.5114 & 1.875 & 12.3938 & 0.625 & 5.31454 \\ -3.98693 & -0.625 & -3.09641 & 1.875 & 0.890523 & 0.625 & 6.19281 & -1.875 \\ 0.625 & -11.5114 & 1.875 & -6.1969 & -0.625 & 5.31454 & -1.875 & 12.3938 \end{pmatrix}$$

$$\boldsymbol{r}^{\mathrm{T}} = (\,0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\,)$$

Computation of element matrices resulting from NBC

NBC on side 3 with $\{q_n, q_t\} = \{-50, 0\}$

$\{N_1, N_2, N_3, N_4\}_c = \left\{0, 0, \frac{1-a}{2}, \frac{a+1}{2}\right\}$

$x(a) = 37. - 17.\,a;$ $\qquad$ $y(a) = 12.$

$dx/da = -17.;$ $\qquad$ $dy/da = 0.$

$J_c = 17.$

Gauss point = -0.57735; $\qquad$ Weight = 1.; $\qquad$ $J_c = 17.$

$\{N_1, N_2, N_3, N_4\}_c = \{0, 0, 0.788675, 0.211325\}$

$\boldsymbol{r}_q^{\mathrm{T}} = (\,0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -2681.5 \quad 0 \quad -718.505\,)$

Gauss point = 0.57735; $\qquad$ Weight = 1.; $\qquad$ $J_c = 17.$

$\{N_1, N_2, N_3, N_4\}_c = \{0, 0, 0.211325, 0.788675\}$

$\boldsymbol{r}_q^{\mathrm{T}} = (\,0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -718.505 \quad 0 \quad -2681.5\,)$

Summing contributions from all Gauss points

$$\boldsymbol{r}_q^T = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -3400. \quad 0 \quad -3400.)$$

Complete element equations for element 3

$$10^6 \begin{pmatrix} 6.19281 & 1.875 & 0.890523 & -0.625 & -3.09641 & -1.875 & -3.98693 & 0.625 \\ 1.875 & 12.3938 & 0.625 & 5.31454 & -1.875 & -6.1969 & -0.625 & -11.5114 \\ 0.890523 & 0.625 & 6.19281 & -1.875 & -3.98693 & -0.625 & -3.09641 & 1.875 \\ -0.625 & 5.31454 & -1.875 & 12.3938 & 0.625 & -11.5114 & 1.875 & -6.1969 \\ -3.09641 & -1.875 & -3.98693 & 0.625 & 6.19281 & 1.875 & 0.890523 & -0.625 \\ -1.875 & -6.1969 & -0.625 & -11.5114 & 1.875 & 12.3938 & 0.625 & 5.31454 \\ -3.98693 & -0.625 & -3.09641 & 1.875 & 0.890523 & 0.625 & 6.19281 & -1.875 \\ 0.625 & -11.5114 & 1.875 & -6.1969 & -0.625 & 5.31454 & -1.875 & 12.3938 \end{pmatrix}$$

$$\begin{pmatrix} u_5 \\ v_5 \\ u_7 \\ v_7 \\ u_8 \\ v_8 \\ u_6 \\ v_6 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ -3400. \\ 0. \\ -3400. \end{pmatrix}$$

The element contributes to {9, 10, 13, 14, 15, 16, 11, 12} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 9 \\ 10 \\ 13 \\ 14 \\ 15 \\ 16 \\ 11 \\ 12 \end{pmatrix}$

and to a global matrix:

$$\begin{pmatrix}
[9,9] & [9,10] & [9,13] & [9,14] & [9,15] & [9,16] & [9,11] & [9,12] \\
[10,9] & [10,10] & [10,13] & [10,14] & [10,15] & [10,16] & [10,11] & [10,12] \\
[13,9] & [13,10] & [13,13] & [13,14] & [13,15] & [13,16] & [13,11] & [13,12] \\
[14,9] & [14,10] & [14,13] & [14,14] & [14,15] & [14,16] & [14,11] & [14,12] \\
[15,9] & [15,10] & [15,13] & [15,14] & [15,15] & [15,16] & [15,11] & [15,12] \\
[16,9] & [16,10] & [16,13] & [16,14] & [16,15] & [16,16] & [16,11] & [16,12] \\
[11,9] & [11,10] & [11,13] & [11,14] & [11,15] & [11,16] & [11,11] & [11,12] \\
[12,9] & [12,10] & [12,13] & [12,14] & [12,15] & [12,16] & [12,11] & [12,12]
\end{pmatrix}$$

Adding element equations into appropriate locations we have

$$10^6 \begin{pmatrix}
6.26209 & -0.0163755 & -1.19113 & -0.120087 & 0 & 0 & -4.11923 & 1.26638 \\
-0.0163755 & 9.0354 & -1.37009 & -5.58562 & 0 & 0 & 2.51638 & -3.67826 \\
-1.19113 & -1.37009 & 7.54484 & -2.96397 & 0 & 0 & -5.95173 & 2.62009 \\
-0.120087 & -5.58562 & -2.96397 & 15.0507 & 0 & 0 & 2.62009 & -12.2715 \\
0 & 0 & 0 & 0 & 5.7276 & 0.559291 & -3.49546 & 1.94071 \\
0 & 0 & 0 & 0 & 0.559291 & 9.65901 & 0.690709 & -8.76615 \\
-4.11923 & 2.51638 & -5.95173 & 2.62009 & -3.49546 & 0.690709 & 20.1147 & -6.95708 \\
1.26638 & -3.67826 & 2.62009 & -12.2715 & 1.94071 & -8.76615 & -6.95708 & 32.4444 \\
0 & 0 & 0 & 0 & -0.771918 & 0.0794621 & -4.58523 & 2.42054 \\
0 & 0 & 0 & 0 & -1.17054 & 2.74625 & 2.42054 & -4.8891 \\
-0.951731 & -1.12991 & -0.401981 & 0.463974 & -1.46023 & -1.32946 & -1.96304 & -1.29063 \\
-1.12991 & 0.228478 & 1.71397 & 2.80646 & -1.32946 & -3.6391 & -1.29063 & -2.83938 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Essential boundary conditions

Node	dof	Value
1	u_1	0
2	u_2	0
7	u_7	0
	v_7	0
8	u_8	0
	v_8	0

Remove {1, 3, 13, 14, 15, 16} rows and columns.

After adjusting for essential boundary conditions we have

$$10^6 \begin{pmatrix} 9.0354 & -5.58562 & 0 & 0 & 2.51638 & -3.67826 & 0 & 0 \\ -5.58562 & 15.0507 & 0 & 0 & 2.62009 & -12.2715 & 0 & 0 \\ 0 & 0 & 5.7276 & 0.559291 & -3.49546 & 1.94071 & -0.771918 & -1.17054 \\ 0 & 0 & 0.559291 & 9.65901 & 0.690709 & -8.76615 & 0.0794621 & 2.74625 \\ 2.51638 & 2.62009 & -3.49546 & 0.690709 & 20.1147 & -6.95708 & -4.58523 & 2.42054 \\ -3.67826 & -12.2715 & 1.94071 & -8.76615 & -6.95708 & 32.4444 & 2.42054 & -4.8891 \\ 0 & 0 & -0.771918 & 0.0794621 & -4.58523 & 2.42054 & 12.5561 & -0.866443 \\ 0 & 0 & -1.17054 & 2.74625 & 2.42054 & -4.8891 & -0.866443 & 19.7912 \\ -1.12991 & 0.463974 & -1.46023 & -1.32946 & -1.96304 & -1.29063 & -4.99308 & 0.866443 \\ 0.228478 & 2.80646 & -1.32946 & -3.6391 & -1.29063 & -2.83938 & 0.866443 & -16.766 \end{pmatrix}$$

Solving the final system of global equations we get

$\{v_1 = -0.0183155,\ v_2 = -0.0183204,\ u_3 = 0.00275915,\ v_3 = -0.0166486,\ u_4 = 0.00114552,$
$v_4 = -0.0164634,\ u_5 = 0.00305003,\ v_5 = -0.0113566,\ u_6 = -0.00210128,\ v_6 = -0.0116254\}$

Complete table of nodal values

	u	v
1	0	−0.0183155
2	0	−0.0183204
3	0.00275915	−0.0166486
4	0.00114552	−0.0164634
5	0.00305003	−0.0113566
6	−0.00210128	−0.0116254
7	0	0
8	0	0

Computation of reactions

Equation numbers of dof with specified values: {1, 3, 13, 14, 15, 16}

Extracting equations {1, 3, 13, 14, 15, 16} from the global system we have

$$10^6\begin{pmatrix} 6.26209 & -0.0163755 & -1.19113 & -0.120087 & 0 & 0 & -4.11923 & 1.26638 & 0 & 0 & -0.951 \\ -1.19113 & -1.37009 & 7.54484 & -2.96397 & 0 & 0 & -5.95173 & 2.62009 & 0 & 0 & -0.401 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.890523 & 0.625 & -3.096 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.625 & 5.31454 & 1.875 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3.09641 & -1.875 & 0.890 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.875 & -6.1969 & 0.625 \end{pmatrix}$$

Substituting the nodal values and re−arranging

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{pmatrix} = 10^6\begin{pmatrix} 6.26209 & -0.0163755 & -1.19113 & -0.120087 & 0 & 0 & -4.11923 & 1.26638 & 0 & 0 \\ -1.19113 & -1.37009 & 7.54484 & -2.96397 & 0 & 0 & -5.95173 & 2.62009 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.890523 & 0.625 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.625 & 5.31454 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3.09641 & -1.875 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.875 & -6.1969 \end{pmatrix}$$

Carrying out computations, the reactions are as follows.

Label	dof	Reaction
R_1	u_1	-7931.98
R_2	u_2	10360.8
R_3	u_7	$-19673.$
R_4	v_7	5839.98
R_5	u_8	17244.2
R_6	v_8	4960.02

Sum of Reactions

dof: u	0
dof: v	10800.

Solution for element 1

Element nodal displacements

Element node	Global node number	u	v
1	1	0	-0.0183155
2	4	0.00114552	-0.0164634
3	6	-0.00210128	-0.0116254
4	2	0	-0.0183204

$$\boldsymbol{d}^T = (0 \quad -0.0183155 \quad 0.00114552 \quad -0.0164634 \quad -0.00210128 \quad -0.0116254 \quad 0 \quad -0.0183204)$$

$E = 3000000;$ $\nu = 0.2;$ $h = 4$

$$\text{Plane stress } \boldsymbol{C} = \begin{pmatrix} 3.125\times 10^6 & 625000. & 0 \\ 625000. & 3.125\times 10^6 & 0 \\ 0 & 0 & 1.25\times 10^6 \end{pmatrix}$$

Interpolation functions and their derivatives

$$\{N_1, N_2, N_3, N_4\} = \left\{\frac{1}{4}(s-1)(t-1), -\frac{1}{4}(s+1)(t-1), \frac{1}{4}(s+1)(t+1), -\frac{1}{4}(s-1)(t+1)\right\}$$

$$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{\frac{t-1}{4}, \frac{1-t}{4}, \frac{t+1}{4}, \frac{1}{4}(-t-1)\right\}$$

$$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{\frac{s-1}{4}, \frac{1}{4}(-s-1), \frac{s+1}{4}, \frac{1-s}{4}\right\}$$

Nodal coordinates

Element node	Global node number	x	y
1	1	0.	5.
2	4	6.	5.
3	6	20.	12.
4	2	0.	12.

Mapping to the master element

$$x(s,t) = 1.5\,(s+1)\,(1-t) + 5.\,(s+1)\,(t+1)$$

$$y(s,t) = 1.25\,(1-s)\,(1-t) + 1.25\,(s+1)\,(1-t) + 3.\,(1-s)\,(t+1) + 3.\,(s+1)\,(t+1)$$

$$\boldsymbol{J} = \begin{pmatrix} 1.5\,(1-t) + 5.\,(t+1) & 3.5\,(s+1) \\ 0 & 1.75\,(1-s) + 1.75\,(s+1) \end{pmatrix}; \qquad \text{detJ} = 12.25\,t + 22.75$$

Element solution at $\{s, t\} = \{0, 0\} \Longrightarrow \{x, y\} = \{6.5, 8.5\}$

$$\{N_1, N_2, N_3, N_4\} = \left\{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right\}$$

$$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{-\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}\right\}$$

$$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \left\{-\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right\}$$

$$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{-0.0384615, 0.0384615, 0.0384615, -0.0384615\}$$

$$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{-0.032967, -0.10989, 0.032967, 0.10989\}$$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -0.0384615 & 0 & 0.0384615 & 0 & 0.0384615 & 0 & -0.0384615 & 0 \\ 0 & -0.032967 & 0 & -0.10989 & 0 & 0.032967 & 0 & 0. \\ -0.032967 & -0.0384615 & -0.10989 & 0.0384615 & 0.032967 & 0.0384615 & 0.10989 & -0. \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{d} = (\,-0.00003676 \quad 0.0000164861 \quad 0.000133582\,)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\,-104.571 \quad 28.544 \quad 166.978\,)$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$$\boldsymbol{\epsilon}^{\mathrm{T}} = (\,-0.00003676 \quad 0.0000164861 \quad 5.06848\times10^{-6} \quad 0.000133582 \quad 0 \quad 0\,)$$

$$\boldsymbol{\sigma}^{\mathrm{T}} = (\,-104.571 \quad 28.544 \quad 0 \quad 166.978 \quad 0 \quad 0\,)$$

Substituting these stress components into appropriate formulas

Principal stresses = (141.741   0.   −217.768)

Effective stress (von Mises) = 313.656

Element solution at $\{s, t\} = \{-1, -1\} \Longrightarrow \{x, y\} = \{0., 5.\}$

$\{N_1, N_2, N_3, N_4\} = \{1, 0, 0, 0\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{-\frac{1}{2}, \frac{1}{2}, 0, 0\right\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \left\{-\frac{1}{2}, 0, 0, \frac{1}{2}\right\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{-0.166667, 0.166667, 0, 0.\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{-0.142857, 0., 0, 0.142857\}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -0.166667 & 0 & 0.166667 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.142857 & 0 & 0 & 0 & 0 & 0 & 0.142857 \\ -0.142857 & -0.166667 & 0 & 0.166667 & 0 & 0 & 0.142857 & 0 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{d} = (\, 0.00019092 \quad -6.97447\times 10^{-7} \quad 0.000308689\,)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\, 596.188 \quad 117.145 \quad 385.861\,)$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{\mathrm{T}} = (\, 0.00019092 \quad -6.97447\times 10^{-7} \quad -0.0000475555 \quad 0.000308689 \quad 0 \quad 0\,)$

$\boldsymbol{\sigma}^{\mathrm{T}} = (\, 596.188 \quad 117.145 \quad 0 \quad 385.861 \quad 0 \quad 0\,)$

Substituting these stress components into appropriate formulas

Principal stresses = $(\, 810.824 \quad 0. \quad -97.4913\,)$

Effective stress (von Mises) = 863.706

Element solution at $\{s, t\} = \{-1, 1\} \Longrightarrow \{x, y\} = \{0., 12.\}$

$\{N_1, N_2, N_3, N_4\} = \{0, 0, 0, 1\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{0, 0, \frac{1}{2}, -\frac{1}{2}\right\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \left\{-\frac{1}{2}, 0, 0, \frac{1}{2}\right\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{0., 0, 0.05, -0.05\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{-0.142857, 0, 0., 0.142857\}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0.05 & 0 & -0.05 & 0 \\ 0 & -0.142857 & 0 & 0 & 0 & 0 & 0 & 0.142857 \\ -0.142857 & 0 & 0 & 0 & 0 & 0.05 & 0.142857 & -0.05 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{d} = (\, -0.000105064 \quad -6.97447\times 10^{-7} \quad 0.000334751\,)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\,-328.76 \quad -67.8444 \quad 418.438\,)$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{\mathrm{T}} = (\,-0.000105064 \quad -6.97447\times 10^{-7} \quad 0.0000264403 \quad 0.000334751 \quad 0 \quad 0\,)$

$\boldsymbol{\sigma}^{\mathrm{T}} = (\,-328.76 \quad -67.8444 \quad 0 \quad 418.438 \quad 0 \quad 0\,)$

Substituting these stress components into appropriate formulas

Principal stresses = $(\,240.001 \quad 0. \quad -636.606\,)$

Effective stress (von Mises) = 784.636

Element solution at $\{s, t\} = \{1, -1\} \Longrightarrow \{x, y\} = \{6., 5.\}$

$\{N_1, N_2, N_3, N_4\} = \{0, 1, 0, 0\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{-\frac{1}{2}, \frac{1}{2}, 0, 0\right\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \left\{0, -\frac{1}{2}, \frac{1}{2}, 0\right\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{-0.166667, 0.166667, 0., 0\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{0.333333, -0.47619, 0.142857, 0\}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -0.166667 & 0 & 0.166667 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.333333 & 0 & -0.47619 & 0 & 0.142857 & 0 & 0 \\ 0.333333 & -0.166667 & -0.47619 & 0.166667 & 0.142857 & 0 & 0 & 0 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{d} = (\,0.00019092 \quad 0.0000737645 \quad -0.000536978\,)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\,642.726 \quad 349.839 \quad -671.222\,)$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{\mathrm{T}} = (\,0.00019092 \quad 0.0000737645 \quad -0.000066171 \quad -0.000536978 \quad 0 \quad 0\,)$

$\boldsymbol{\sigma}^{\mathrm{T}} = (\,642.726 \quad 349.839 \quad 0 \quad -671.222 \quad 0 \quad 0\,)$

Substituting these stress components into appropriate formulas

Principal stresses = $(\,1183.29 \quad 0. \quad -190.729\,)$

Effective stress (von Mises) = 1289.28

Element solution at $\{s, t\} = \{1, 1\} \Longrightarrow \{x, y\} = \{20., 12.\}$

$\{N_1, N_2, N_3, N_4\} = \{0, 0, 1, 0\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{0, 0, \frac{1}{2}, -\frac{1}{2}\right\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \left\{0, -\frac{1}{2}, \frac{1}{2}, 0\right\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{0, 0., 0.05, -0.05\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{0, -0.142857, 0.0428571, 0.1\}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0.05 & 0 & -0.05 & 0 \\ 0 & 0 & 0 & -0.142857 & 0 & 0.0428571 & 0 & 0.1 \\ 0 & 0 & -0.142857 & 0 & 0.0428571 & 0.05 & 0.1 & -0.05 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{d} = (\ -0.000105064 \quad 0.0000216411 \quad 0.0000810505\ )$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\ -314.799 \quad 1.96363 \quad 101.313\ )$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{\mathrm{T}} = (\ -0.000105064 \quad 0.0000216411 \quad 0.0000208557 \quad 0.0000810505 \quad 0 \quad 0\ )$

$\boldsymbol{\sigma}^{\mathrm{T}} = (\ -314.799 \quad 1.96363 \quad 0 \quad 101.313 \quad 0 \quad 0\ )$

Substituting these stress components into appropriate formulas

Principal stresses = $(\ 31.5956 \quad 0. \quad -344.431\ )$

Effective stress (von Mises) = 361.266

Solution for element 2

Element nodal displacements

Element node	Global node number	u	v
1	3	0.00275915	−0.0166486
2	5	0.00305003	−0.0113566
3	6	−0.00210128	−0.0116254
4	4	0.00114552	−0.0164634

$\boldsymbol{d}^{\mathrm{T}} = (\ 0.00275915 \quad -0.0166486 \quad 0.00305003 \quad -0.0113566 \quad -0.00210128 \quad -0.0116254 \quad 0.00114552 \quad -0.0164$

E = 3000000; $\nu = 0.2$; h = 4

$$\text{Plane stress } \boldsymbol{C} = \begin{pmatrix} 3.125\times10^{6} & 625000. & 0 \\ 625000. & 3.125\times10^{6} & 0 \\ 0 & 0 & 1.25\times10^{6} \end{pmatrix}$$

Interpolation functions and their derivatives

$$\{N_1,\ N_2,\ N_3,\ N_4\} = \left\{\frac{1}{4}(s-1)(t-1),\ -\frac{1}{4}(s+1)(t-1),\ \frac{1}{4}(s+1)(t+1),\ -\frac{1}{4}(s-1)(t+1)\right\}$$

$$\{\partial N_1/\partial s,\ \partial N_2/\partial s,\ \partial N_3/\partial s,\ \partial N_4/\partial s\} = \left\{\frac{t-1}{4},\ \frac{1-t}{4},\ \frac{t+1}{4},\ \frac{1}{4}(-t-1)\right\}$$

$$\{\partial N_1/\partial s,\ \partial N_2/\partial s,\ \partial N_3/\partial s,\ \partial N_4/\partial s\} = \left\{\frac{s-1}{4},\ \frac{1}{4}(-s-1),\ \frac{s+1}{4},\ \frac{1-s}{4}\right\}$$

Nodal coordinates

Element node	Global node number	x	y
1	3	6.	0.
2	5	20.	0.
3	6	20.	12.
4	4	6.	5.

Mapping to the master element

$$x(s,t) = 1.5\,(1-s)(1-t) + 5.\,(s+1)(1-t) + 1.5\,(1-s)(t+1) + 5.\,(s+1)(t+1)$$

$$y(s,t) = 1.25\,(1-s)(t+1) + 3.\,(s+1)(t+1)$$

$$\boldsymbol{J} = \begin{pmatrix} 3.5\,(1-t) + 3.5\,(t+1) & 0 \\ 1.75\,(t+1) & 1.25\,(1-s) + 3.\,(s+1) \end{pmatrix}; \qquad \det J = 12.25\,s + 29.75$$

Element solution at $\{s, t\} = \{0, 0\} \Longrightarrow \{x, y\} = \{13., 4.25\}$

$$\{N_1,\ N_2,\ N_3,\ N_4\} = \left\{\frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4}\right\}$$

$$\{\partial N_1/\partial s,\ \partial N_2/\partial s,\ \partial N_3/\partial s,\ \partial N_4/\partial s\} = \left\{-\frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ -\frac{1}{4}\right\}$$

$$\{\partial N_1/\partial t,\ \partial N_2/\partial t,\ \partial N_3/\partial t,\ \partial N_4/\partial t\} = \left\{-\frac{1}{4},\ -\frac{1}{4},\ \frac{1}{4},\ \frac{1}{4}\right\}$$

$$\{\partial N_1/\partial x,\ \partial N_2/\partial x,\ \partial N_3/\partial x,\ \partial N_4/\partial x\} = \{-0.0210084,\ 0.0504202,\ 0.0210084,\ -0.0504202\}$$

$$\{\partial N_1/\partial y,\ \partial N_2/\partial y,\ \partial N_3/\partial y,\ \partial N_4/\partial y\} = \{-0.0588235,\ -0.0588235,\ 0.0588235,\ 0.0588235\}$$

$$\boldsymbol{B}^T = \begin{pmatrix} -0.0210084 & 0 & 0.0504202 & 0 & 0.0210084 & 0 & -0.0504202 & 0 \\ 0 & -0.0588235 & 0 & -0.0588235 & 0 & 0.0588235 & 0 & 0. \\ -0.0588235 & -0.0210084 & -0.0588235 & 0.0504202 & 0.0588235 & 0.0210084 & 0.0588235 & -0. \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^T\boldsymbol{d} = (\ -6.08383\times10^{-6} \quad -4.91655\times10^{-6} \quad -0.0000349244\ )$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\ -22.0848 \quad -19.1666 \quad -43.6555\ )$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$$\boldsymbol{\epsilon}^T = (\ -6.08383\times10^{-6} \quad -4.91655\times10^{-6} \quad 2.7501\times10^{-6} \quad -0.0000349244 \quad 0 \quad 0\ )$$

$\boldsymbol{\sigma}^{T} = (\ -22.0848 \quad -19.1666 \quad 0 \quad -43.6555 \quad 0 \quad 0\ )$

Substituting these stress components into appropriate formulas

Principal stresses = $(\ 23.0542 \quad 0. \quad -64.3056\ )$

Effective stress (von Mises) = 78.4169

Element solution at $\{s, t\} = \{-1, -1\} \Longrightarrow \{x, y\} = \{6., 0.\}$

$\{N_1, N_2, N_3, N_4\} = \{1, 0, 0, 0\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{-\frac{1}{2}, \frac{1}{2}, 0, 0\right\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \left\{-\frac{1}{2}, 0, 0, \frac{1}{2}\right\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{-0.0714286, 0.0714286, 0, 0.\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{-0.2, 0., 0, 0.2\}$

$$\boldsymbol{B}^{T} = \begin{pmatrix} -0.0714286 & 0 & 0.0714286 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.2 & 0 & 0 & 0 & 0 & 0 & 0.2 \\ -0.2 & -0.0714286 & 0 & 0.0714286 & 0 & 0 & 0.2 & 0 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{T}\boldsymbol{d} = (\ 0.0000207772 \quad 0.0000370391 \quad 0.0000552707\ )$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\ 88.0783 \quad 128.733 \quad 69.0884\ )$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{T} = (\ 0.0000207772 \quad 0.0000370391 \quad -0.0000144541 \quad 0.0000552707 \quad 0 \quad 0\ )$

$\boldsymbol{\sigma}^{T} = (\ 88.0783 \quad 128.733 \quad 0 \quad 69.0884 \quad 0 \quad 0\ )$

Substituting these stress components into appropriate formulas

Principal stresses = $(\ 180.422 \quad 36.3889 \quad 0.\ )$

Effective stress (von Mises) = 165.26

Element solution at $\{s, t\} = \{-1, 1\} \Longrightarrow \{x, y\} = \{6., 5.\}$

$\{N_1, N_2, N_3, N_4\} = \{0, 0, 0, 1\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{0, 0, \frac{1}{2}, -\frac{1}{2}\right\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \left\{-\frac{1}{2}, 0, 0, \frac{1}{2}\right\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{0.1, 0, 0.0714286, -0.171429\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{-0.2, 0, 0., 0.2\}$

$$\boldsymbol{B}^{T} = \begin{pmatrix} 0.1 & 0 & 0 & 0 & 0.0714286 & 0 & -0.171429 & 0 \\ 0 & -0.2 & 0 & 0 & 0 & 0 & 0 & 0.2 \\ -0.2 & 0.1 & 0 & 0 & 0 & 0.0714286 & 0.2 & -0.171429 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{T}\boldsymbol{d} = (\,-0.0000705504 \quad 0.0000370391 \quad 4.32456\times 10^{-6}\,)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\,-197.32 \quad 71.6533 \quad 5.4057\,)$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{T} = (\,-0.0000705504 \quad 0.0000370391 \quad 8.37781\times 10^{-6} \quad 4.32456\times 10^{-6} \quad 0 \quad 0\,)$

$\boldsymbol{\sigma}^{T} = (\,-197.32 \quad 71.6533 \quad 0 \quad 5.4057 \quad 0 \quad 0\,)$

Substituting these stress components into appropriate formulas

Principal stresses = $(\,71.7619 \quad 0. \quad -197.429\,)$

Effective stress (von Mises) = 241.445

Element solution at $\{s, t\} = \{1, -1\} \Longrightarrow \{x, y\} = \{20., 0.\}$

$\{N_1, N_2, N_3, N_4\} = \{0, 1, 0, 0\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{-\frac{1}{2}, \frac{1}{2}, 0, 0\right\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \left\{0, -\frac{1}{2}, \frac{1}{2}, 0\right\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{-0.0714286, 0.0714286, 0., 0\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{0., -0.0833333, 0.0833333, 0\}$

$$\boldsymbol{B}^{T} = \begin{pmatrix} -0.0714286 & 0 & 0.0714286 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.0833333 & 0 & 0.0833333 & 0 & 0 \\ 0 & -0.0714286 & -0.0833333 & 0.0714286 & 0.0833333 & 0 & 0 & 0 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{T}\boldsymbol{d} = (\,0.0000207772 \quad -0.0000223981 \quad -0.0000512781\,)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\,50.93 \quad -57.0082 \quad -64.0977\,)$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{T} = (\,0.0000207772 \quad -0.0000223981 \quad 4.05212\times 10^{-7} \quad -0.0000512781 \quad 0 \quad 0\,)$

$\boldsymbol{\sigma}^{T} = (\,50.93 \quad -57.0082 \quad 0 \quad -64.0977 \quad 0 \quad 0\,)$

Substituting these stress components into appropriate formulas

Principal stresses = (80.7534   0.   −86.8316)

Effective stress (von Mises) = 145.165

Element solution at {s, t} = {1, 1} ⟹ {x, y} = {20., 12.}

$\{N_1, N_2, N_3, N_4\} = \{0, 0, 1, 0\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{0, 0, \frac{1}{2}, -\frac{1}{2}\right\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \left\{0, -\frac{1}{2}, \frac{1}{2}, 0\right\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{0, 0.0416667, 0.0297619, -0.0714286\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{0, -0.0833333, 0.0833333, 0.\}$

$$\boldsymbol{B}^T = \begin{pmatrix} 0 & 0 & 0.0416667 & 0 & 0.0297619 & 0 & -0.0714286 & 0 \\ 0 & 0 & 0 & -0.0833333 & 0 & 0.0833333 & 0 & 0 \\ 0 & 0 & -0.0833333 & 0.0416667 & 0.0833333 & 0.0297619 & 0 & -0.0714286 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^T \boldsymbol{d}$ = (−0.0000172759   −0.0000223981   −0.0000725057)

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon}$ = (−67.9861   −80.7914   −90.6321)

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^T = (-0.0000172759 \quad -0.0000223981 \quad 9.9185\times10^{-6} \quad -0.0000725057 \quad 0 \quad 0)$

$\boldsymbol{\sigma}^T = (-67.9861 \quad -80.7914 \quad 0 \quad -90.6321 \quad 0 \quad 0)$

Substituting these stress components into appropriate formulas

Principal stresses = (16.4692   0.   −165.247)

Effective stress (von Mises) = 174.067

Solution for element 3

Element nodal displacements

Element node	Global node number	u	v
1	5	0.00305003	−0.0113566
2	7	0	0
3	8	0	0
4	6	−0.00210128	−0.0116254

$\boldsymbol{d}^T = (0.00305003 \quad -0.0113566 \quad 0 \quad 0 \quad 0 \quad 0 \quad -0.00210128 \quad -0.0116254)$

E = 3000000;   ν = 0.2;   h = 4

$$\text{Plane stress } \boldsymbol{C} = \begin{pmatrix} 3.125\times 10^6 & 625000. & 0 \\ 625000. & 3.125\times 10^6 & 0 \\ 0 & 0 & 1.25\times 10^6 \end{pmatrix}$$

Interpolation functions and their derivatives

$$\{N_1,\ N_2,\ N_3,\ N_4\} = \left\{\frac{1}{4}(s-1)(t-1),\ -\frac{1}{4}(s+1)(t-1),\ \frac{1}{4}(s+1)(t+1),\ -\frac{1}{4}(s-1)(t+1)\right\}$$

$$\{\partial N_1/\partial s,\ \partial N_2/\partial s,\ \partial N_3/\partial s,\ \partial N_4/\partial s\} = \left\{\frac{t-1}{4},\ \frac{1-t}{4},\ \frac{t+1}{4},\ \frac{1}{4}(-t-1)\right\}$$

$$\{\partial N_1/\partial s,\ \partial N_2/\partial s,\ \partial N_3/\partial s,\ \partial N_4/\partial s\} = \left\{\frac{s-1}{4},\ \frac{1}{4}(-s-1),\ \frac{s+1}{4},\ \frac{1-s}{4}\right\}$$

Nodal coordinates

Element node	Global node number	x	y
1	5	20.	0.
2	7	54.	0.
3	8	54.	12.
4	6	20.	12.

Mapping to the master element

$$x(s,t) = 5.(1-s)(1-t) + 13.5(s+1)(1-t) + 5.(1-s)(t+1) + 13.5(s+1)(t+1)$$

$$y(s,t) = 3.(1-s)(t+1) + 3.(s+1)(t+1)$$

$$\boldsymbol{J} = \begin{pmatrix} 8.5(1-t) + 8.5(t+1) & 0 \\ 0 & 3.(1-s) + 3.(s+1) \end{pmatrix}; \qquad \det J = 102.$$

Element solution at $\{s, t\} = \{0, 0\} \Longrightarrow \{x, y\} = \{37., 6.\}$

$$\{N_1,\ N_2,\ N_3,\ N_4\} = \left\{\frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4}\right\}$$

$$\{\partial N_1/\partial s,\ \partial N_2/\partial s,\ \partial N_3/\partial s,\ \partial N_4/\partial s\} = \left\{-\frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ -\frac{1}{4}\right\}$$

$$\{\partial N_1/\partial t,\ \partial N_2/\partial t,\ \partial N_3/\partial t,\ \partial N_4/\partial t\} = \left\{-\frac{1}{4},\ -\frac{1}{4},\ \frac{1}{4},\ \frac{1}{4}\right\}$$

$$\{\partial N_1/\partial x,\ \partial N_2/\partial x,\ \partial N_3/\partial x,\ \partial N_4/\partial x\} = \{-0.0147059,\ 0.0147059,\ 0.0147059,\ -0.0147059\}$$

$$\{\partial N_1/\partial y,\ \partial N_2/\partial y,\ \partial N_3/\partial y,\ \partial N_4/\partial y\} = \{-0.0416667,\ -0.0416667,\ 0.0416667,\ 0.0416667\}$$

$$\boldsymbol{B}^T = \begin{pmatrix} -0.0147059 & 0 & 0.0147059 & 0 & 0.0147059 & 0 & -0.0147059 & 0 \\ 0 & -0.0416667 & 0 & -0.0416667 & 0 & 0.0416667 & 0 & 0. \\ -0.0416667 & -0.0147059 & -0.0416667 & 0.0147059 & 0.0416667 & 0.0147059 & 0.0416667 & -0. \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^T \boldsymbol{d} = (\ -0.0000139523 \quad -0.000011199 \quad 0.000123333\ )$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C\epsilon} = (\,-50.6003 \quad -43.7172 \quad 154.167\,)$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{\mathrm{T}} = (\,-0.0000139523 \quad -0.000011199 \quad 6.28783\times10^{-6} \quad 0.000123333 \quad 0 \quad 0\,)$

$\boldsymbol{\sigma}^{\mathrm{T}} = (\,-50.6003 \quad -43.7172 \quad 0 \quad 154.167 \quad 0 \quad 0\,)$

Substituting these stress components into appropriate formulas

Principal stresses = $(\,107.046 \quad 0. \quad -201.364\,)$

Effective stress (von Mises) = 271.222

Element solution at $\{s, t\} = \{-1, -1\} \Longrightarrow \{x, y\} = \{20., 0.\}$

$\{N_1, N_2, N_3, N_4\} = \{1, 0, 0, 0\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{-\frac{1}{2}, \frac{1}{2}, 0, 0\right\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \left\{-\frac{1}{2}, 0, 0, \frac{1}{2}\right\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{-0.0294118, 0.0294118, 0, 0.\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{-0.0833333, 0., 0, 0.0833333\}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -0.0294118 & 0 & 0.0294118 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.0833333 & 0 & 0 & 0 & 0 & 0 & 0.0833333 \\ -0.0833333 & -0.0294118 & 0 & 0.0294118 & 0 & 0 & 0.0833333 & 0 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{d} = (\,-0.0000897069 \quad -0.0000223981 \quad -0.0000952572\,)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C\epsilon} = (\,-294.333 \quad -126.061 \quad -119.072\,)$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{\mathrm{T}} = (\,-0.0000897069 \quad -0.0000223981 \quad 0.0000280262 \quad -0.0000952572 \quad 0 \quad 0\,)$

$\boldsymbol{\sigma}^{\mathrm{T}} = (\,-294.333 \quad -126.061 \quad 0 \quad -119.072 \quad 0 \quad 0\,)$

Substituting these stress components into appropriate formulas

Principal stresses = $(\,0. \quad -64.3993 \quad -355.994\,)$

Effective stress (von Mises) = 328.563

Element solution at $\{s, t\} = \{-1, 1\} \Longrightarrow \{x, y\} = \{20., 12.\}$

$\{N_1, N_2, N_3, N_4\} = \{0, 0, 0, 1\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{0, 0, \frac{1}{2}, -\frac{1}{2}\right\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \left\{-\frac{1}{2}, 0, 0, \frac{1}{2}\right\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{0., 0, 0.0294118, -0.0294118\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{-0.0833333, 0, 0., 0.0833333\}$

$$\boldsymbol{B}^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0.0294118 & 0 & -0.0294118 & 0 \\ 0 & -0.0833333 & 0 & 0 & 0 & 0 & 0 & 0.0833333 \\ -0.0833333 & 0 & 0 & 0 & 0 & 0.0294118 & 0.0833333 & -0.0294118 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^T \boldsymbol{d} = (\ 0.0000618023 \quad -0.0000223981 \quad -0.000087352\ )$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C\epsilon} = (\ 179.133 \quad -31.3676 \quad -109.19\ )$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^T = (\ 0.0000618023 \quad -0.0000223981 \quad -9.85105 \times 10^{-6} \quad -0.000087352 \quad 0 \quad 0\ )$

$\boldsymbol{\sigma}^T = (\ 179.133 \quad -31.3676 \quad 0 \quad -109.19 \quad 0 \quad 0\ )$

Substituting these stress components into appropriate formulas

Principal stresses = $(\ 225.541 \quad 0. \quad -77.775\ )$

Effective stress (von Mises) = 272.872

Element solution at $\{s, t\} = \{1, -1\} \Longrightarrow \{x, y\} = \{54., 0.\}$

$\{N_1, N_2, N_3, N_4\} = \{0, 1, 0, 0\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{-\frac{1}{2}, \frac{1}{2}, 0, 0\right\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \left\{0, -\frac{1}{2}, \frac{1}{2}, 0\right\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{-0.0294118, 0.0294118, 0., 0\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{0., -0.0833333, 0.0833333, 0\}$

$$\boldsymbol{B}^T = \begin{pmatrix} -0.0294118 & 0 & 0.0294118 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.0833333 & 0 & 0.0833333 & 0 & 0 \\ 0 & -0.0294118 & -0.0833333 & 0.0294118 & 0.0833333 & 0 & 0 & 0 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^T \boldsymbol{d} = (\ -0.0000897069 \quad 0. \quad 0.000334019\ )$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C\epsilon} = (\ -280.334 \quad -56.0668 \quad 417.523\ )$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{T} = (\ -0.0000897069 \quad 0. \quad 0.0000224267 \quad 0.000334019 \quad 0 \quad 0\ )$

$\boldsymbol{\sigma}^{T} = (\ -280.334 \quad -56.0668 \quad 0 \quad 417.523 \quad 0 \quad 0\ )$

Substituting these stress components into appropriate formulas

Principal stresses = $(\ 264.119 \quad 0. \quad -600.519\ )$

Effective stress (von Mises) = 767.457

Element solution at $\{s, t\} = \{1, 1\} \Longrightarrow \{x, y\} = \{54., 12.\}$

$\{N_1, N_2, N_3, N_4\} = \{0, 0, 1, 0\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{0, 0, \frac{1}{2}, -\frac{1}{2}\right\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \left\{0, -\frac{1}{2}, \frac{1}{2}, 0\right\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{0, 0., 0.0294118, -0.0294118\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{0, -0.0833333, 0.0833333, 0.\}$

$$\boldsymbol{B}^{T} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0.0294118 & 0 & -0.0294118 & 0 \\ 0 & 0 & 0 & -0.0833333 & 0 & 0.0833333 & 0 & 0 \\ 0 & 0 & -0.0833333 & 0 & 0.0833333 & 0.0294118 & 0 & -0.0294118 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{T}\boldsymbol{d} = (\ 0.0000618023 \quad 0. \quad 0.000341924\ )$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\ 193.132 \quad 38.6264 \quad 427.405\ )$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{T} = (\ 0.0000618023 \quad 0. \quad -0.0000154506 \quad 0.000341924 \quad 0 \quad 0\ )$

$\boldsymbol{\sigma}^{T} = (\ 193.132 \quad 38.6264 \quad 0 \quad 427.405 \quad 0 \quad 0\ )$

Substituting these stress components into appropriate formulas

Principal stresses = $(\ 550.21 \quad 0. \quad -318.451\ )$

Effective stress (von Mises) = 761.155

Solution summary

Nodal solution

	x	y	u	v
1	0.	5.	0	−0.0183155
2	0.	12.	0	−0.0183204
3	6.	0.	0.00275915	−0.0166486
4	6.	5.	0.00114552	−0.0164634
5	20.	0.	0.00305003	−0.0113566
6	20.	12.	−0.00210128	−0.0116254
7	54.	0.	0	0
8	54.	12.	0	0

Solution at selected points on elements

	Coord	Disp	Stresses	Principal stresses	Effective Stress
1	6.5 8.5	−0.00023894 −0.0161812	−104.571 28.544 0 166.978 0 0	141.741 0. −217.768	313.656
2	13. 4.25	0.00121336 −0.0140235	−22.0848 −19.1666 0 −43.6555 0 0	23.0542 0. −64.3056	78.4169
3	37. 6.	0.000237189 −0.00574551	−50.6003 −43.7172 0 154.167 0 0	107.046 0. −201.364	271.222

Support reactions

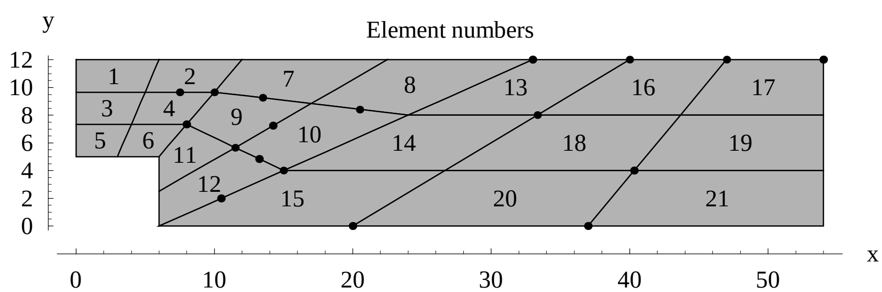
Node	dof	Reaction
1	1	−7931.98
2	1	10360.8
7	1	−19673.
7	2	5839.98
8	1	17244.2
8	2	4960.02

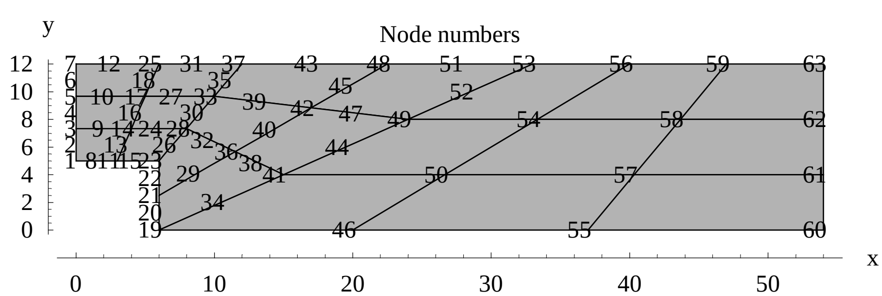
Sum of applied loads → (0 −10800.)

Sum of support reactions → (0 10800.)

Example 7.8: Notched beam using transition from 8 to 4 node elements (p. 516)

Many practical problems can be analyzed efficiently by using higher order elements in the region of high stress gradients and low order elements elsewhere. Appropriate order elements must be used in the transition region between the high to low-order elements. To demonstrate this, consider analysis of the notched beam of Figure. To capture stress concentration in the vicinity of the notch. we employ 8 node elements. Away from the notch the stresses do not change that rapidly and thus we could use 4 node elements. To maintain the compatibility of the displacement field over the entire mesh it is necessary to use 5 node elements in the transition region from 4 to 8 noded elements. Taking advantage of symmetry the right half of the beam is modelled as shown in Figure. The first 12 elements are the 8 noded elements. The elements 16 through 21 are 4 node elements. The elements 13, 14 and 15 in the transition region must have quadratic displacement field along their left edges and linear along the right sides. Thus 5 node elements are used. Due to symmetry nodes 1 through 7 are restrained in the *x* direction. Both horizontal and vertical displacements are zero at nodes 60 through 63 because of the fixed support.





Equations for element 1

$E = 3000000;\qquad \nu = 0.2;\qquad h = 4$

Nodal coordinates

Element node	Global node number	x	y
1	7	0.	12.
2	6	0.	10.8333
3	5	0.	9.66667
4	10	2.5	9.66667
5	17	5.	9.66667
6	18	5.5	10.8333
7	25	6.	12.
8	12	3.	12.

Complete element equations for element 1

$$\left(\begin{array}{ccccc}
1.039\times 10^{7} & -3.7758\times 10^{6} & -1.0664\times 10^{7} & 940803. & 4.69977\times \\
-3.7758\times 10^{6} & 1.98446\times 10^{7} & 2.60747\times 10^{6} & -2.73384\times 10^{7} & -337260. \\
-1.0664\times 10^{7} & 2.60747\times 10^{6} & 2.38523\times 10^{7} & -363938. & -9.64706\times \\
940803. & -2.73384\times 10^{7} & -363938. & 5.36865\times 10^{7} & -728632. \\
4.69977\times 10^{6} & -337260. & -9.64706\times 10^{6} & -728632. & 9.4665\times 1 \\
79407. & 9.64362\times 10^{6} & -2.3953\times 10^{6} & -2.49316\times 10^{7} & 3.28954\times \\
-3.57384\times 10^{6} & 1.13964\times 10^{6} & 88305.9 & 3.02426\times 10^{6} & -4.32916\times \\
1.13964\times 10^{6} & -3.9769\times 10^{6} & 3.02426\times 10^{6} & 220765. & -2.02758\times \\
4.82646\times 10^{6} & -1.63571\times 10^{6} & -6.15553\times 10^{6} & 1.0377\times 10^{6} & 4.26742\times \\
-1.63571\times 10^{6} & 9.21423\times 10^{6} & 1.0377\times 10^{6} & -1.45749\times 10^{7} & -11955.1 \\
-5.52643\times 10^{6} & 725863. & 7.57624\times 10^{6} & 363938. & -5.59103\times \\
725863. & -1.31378\times 10^{7} & 363938. & 2.48849\times 10^{7} & -938035. \\
3.59742\times 10^{6} & 3271.92 & -5.13538\times 10^{6} & -642807. & 3.99154\times \\
-413395. & 5.64421\times 10^{6} & -642807. & -1.21602\times 10^{7} & 1.28649\times \\
-3.74931\times 10^{6} & 1.27253\times 10^{6} & 85141.9 & -3.63133\times 10^{6} & -2.85798\times \\
2.93919\times 10^{6} & 106407. & -3.63133\times 10^{6} & 212855. & -532568.
\end{array}\right.$$

Equations for element 2

$E = 3000000;$ $\nu = 0.2;$ $h = 4$

Nodal coordinates

Element node	Global node number	x	y
1	25	6.	12.
2	18	5.5	10.8333
3	17	5.	9.66667
4	27	7.5	9.66667
5	33	10.	9.66667
6	35	11.	10.8333
7	37	12.	12.
8	31	9.	12.

Complete element equations for element 2

$$\left(\begin{array}{ccccc}
1.27621\times 10^{7} & -4.52645\times 10^{6} & -1.16541\times 10^{7} & 857747. & 4.84711\times 1 \\
-4.52645\times 10^{6} & 2.57748\times 10^{7} & 2.52441\times 10^{6} & -2.98136\times 10^{7} & -595112. \\
-1.16541\times 10^{7} & 2.52441\times 10^{6} & 2.42683\times 10^{7} & -1.09182\times 10^{6} & -8.72604\times 1 \\
857747. & -2.98136\times 10^{7} & -1.09182\times 10^{6} & 5.47263\times 10^{7} & -828300. \\
4.84711\times 10^{6} & -595112. & -8.72604\times 10^{6} & -828300. & 7.82417\times 1 \\
-178446. & 1.0012\times 10^{7} & -2.49497\times 10^{6} & -2.2629\times 10^{7} & 2.4586\times 1 \\
-4.39851\times 10^{6} & 1.7467\times 10^{6} & -1.63984\times 10^{6} & 3.02426\times 10^{6} & -4.01002\times 1 \\
1.7467\times 10^{6} & -6.03857\times 10^{6} & 3.02426\times 10^{6} & -4.09961\times 10^{6} & -756066. \\
5.86093\times 10^{6} & -1.98492\times 10^{6} & -6.77698\times 10^{6} & 1.13737\times 10^{6} & 4.51918\times 1 \\
-1.98492\times 10^{6} & 1.18004\times 10^{7} & 1.13737\times 10^{6} & -1.61285\times 10^{7} & -452532. \\
-5.96494\times 10^{6} & 808920. & 7.16031\times 10^{6} & 1.09182\times 10^{6} & -5.08349\times 1 \\
808920. & -1.42341\times 10^{7} & 1.09182\times 10^{6} & 2.38451\times 10^{7} & -838367. \\
3.83177\times 10^{6} & -406851. & -4.7918\times 10^{6} & -559750. & 3.35618\times 1 \\
-823518. & 6.2301\times 10^{6} & -559750. & -1.13012\times 10^{7} & 937280. \\
-5.28431\times 10^{6} & 2.4333\times 10^{6} & 2.16019\times 10^{6} & -3.63133\times 10^{6} & -2.7271\times 1 \\
4.09997\times 10^{6} & -3.73109\times 10^{6} & -3.63133\times 10^{6} & 5.40047\times 10^{6} & 74499.1
\end{array}\right.$$

Equations for element 3

E = 3000000; ν = 0.2; h = 4

Nodal coordinates

Element node	Global node number	x	y
1	5	0.	9.66667
2	4	0.	8.5
3	3	0.	7.33333
4	9	2.	7.33333
5	14	4.	7.33333
6	16	4.5	8.5
7	17	5.	9.66667
8	10	2.5	9.66667

Complete element equations for element 3

$$\left(\begin{array}{ccccc}
9.77365\times10^{6} & -3.82578\times10^{6} & -8.70305\times10^{6} & 965082. & 4.26335\times \\
-3.82578\times10^{6} & 1.70172\times10^{7} & 2.63175\times10^{6} & -2.25702\times10^{7} & -365994. \\
-8.70305\times10^{6} & 2.63175\times10^{6} & 2.06887\times10^{7} & -444997. & -7.65393\times \\
965082. & -2.25702\times10^{7} & -444997. & 4.44535\times10^{7} & -705731. \\
4.26335\times10^{6} & -365994. & -7.65393\times10^{6} & -705731. & 9.00667\times \\
50672.6 & 8.08326\times10^{6} & -2.3724\times10^{6} & -2.01506\times10^{7} & 3.23061\times \\
-3.97388\times10^{6} & 1.20877\times10^{6} & 108452. & 2.95375\times10^{6} & -5.6626\times1 \\
1.20877\times10^{6} & -3.87026\times10^{6} & 2.95375\times10^{6} & 271131. & -1.91727\times \\
4.59147\times10^{6} & -1.67611\times10^{6} & -5.31167\times10^{6} & 1.08531\times10^{6} & 4.27498\times \\
-1.67611\times10^{6} & 7.98938\times10^{6} & 1.08531\times10^{6} & -1.22634\times10^{7} & -63340.7 \\
-4.63029\times10^{6} & 701585. & 5.02558\times10^{6} & 444997. & -4.72703\times \\
701585. & -1.07631\times10^{7} & 444997. & 1.98322\times10^{7} & -960936. \\
3.54408\times10^{6} & -40538.1 & -4.25781\times10^{6} & -602082. & 3.75577\times \\
-457205. & 4.7953\times10^{6} & -602082. & -9.83193\times10^{6} & 1.24885\times \\
-4.86534\times10^{6} & 1.36632\times10^{6} & 103714. & -3.69633\times10^{6} & -3.25722\times \\
3.03298\times10^{6} & -681572. & -3.69633\times10^{6} & 259286. & -466188.
\end{array}\right.$$

Equations for element 4

E = 3000000; ν = 0.2; h = 4

Nodal coordinates

Element node	Global node number	x	y
1	17	5.	9.66667
2	16	4.5	8.5
3	14	4.	7.33333
4	24	6.	7.33333
5	28	8.	7.33333
6	30	9.	8.5
7	33	10.	9.66667
8	27	7.5	9.66667

Complete element equations for element 4

$$\left(\begin{array}{ccccc}
1.22193\times10^{7} & -4.73397\times10^{6} & -9.70228\times10^{6} & 865580. & 4.44354\times \\
-4.73397\times10^{6} & 2.31313\times10^{7} & 2.53225\times10^{6} & -2.50683\times10^{7} & -681316. \\
-9.70228\times10^{6} & 2.53225\times10^{6} & 2.11973\times10^{7} & -1.33499\times10^{6} & -6.73892\times \\
865580. & -2.50683\times10^{7} & -1.33499\times10^{6} & 4.57249\times10^{7} & -830109. \\
4.44354\times10^{6} & -681316. & -6.73892\times10^{6} & -830109. & 7.45445\times \\
-264649. & 8.53372\times10^{6} & -2.49678\times10^{6} & -1.78631\times10^{7} & 2.20218\times \\
-4.87677\times10^{6} & 1.95135\times10^{6} & -1.5794\times10^{6} & 2.95375\times10^{6} & -5.49229\times \\
1.95135\times10^{6} & -6.12749\times10^{6} & 2.95375\times10^{6} & -3.94851\times10^{6} & -345495. \\
5.67133\times10^{6} & -2.10337\times10^{6} & -5.96739\times10^{6} & 1.20969\times10^{6} & 4.58546\times \\
-2.10337\times10^{6} & 1.0689\times10^{7} & 1.20969\times10^{6} & -1.39027\times10^{7} & -606689. \\
-5.05962\times10^{6} & 801087. & 4.51701\times10^{6} & 1.33499\times10^{6} & -4.21346\times \\
801087. & -1.18365\times10^{7} & 1.33499\times10^{6} & 1.85608\times10^{7} & -836558. \\
3.8285\times10^{6} & -538281. & -3.94219\times10^{6} & -502580. & 3.16422\times \\
-954947. & 5.50636\times10^{6} & -502580. & -9.04288\times10^{6} & 821587. \\
-6.52398\times10^{6} & 2.77225\times10^{6} & 2.2159\times10^{6} & -3.69633\times10^{6} & -3.20299\times \\
4.43892\times10^{6} & -4.82817\times10^{6} & -3.69633\times10^{6} & 5.53976\times10^{6} & 276396.
\end{array}\right.$$

Equations for element 5

$E = 3000000;$ $\nu = 0.2;$ $h = 4$

Nodal coordinates

Element node	Global node number	x	y
1	3	0.	7.33333
2	2	0.	6.16667
3	1	0.	5.
4	8	1.5	5.
5	11	3.	5.
6	13	3.5	6.16667
7	14	4.	7.33333
8	9	2.	7.33333

Complete element equations for element 5

$$\left(\begin{array}{ccccc}
9.48869\times 10^{6} & -3.90299\times 10^{6} & -6.7153\times 10^{6} & 1.00354\times 10^{6} & 3.95938\times \\
-3.90299\times 10^{6} & 1.4333\times 10^{7} & 2.67021\times 10^{6} & -1.78006\times 10^{7} & -411255. \\
-6.7153\times 10^{6} & 2.67021\times 10^{6} & 1.7896\times 10^{7} & -572609. & -5.60222\times \\
1.00354\times 10^{6} & -1.78006\times 10^{7} & -572609. & 3.53875\times 10^{7} & -670012. \\
3.95938\times 10^{6} & -411255. & -5.60222\times 10^{6} & -670012. & 9.03321\times \\
5411.26 & 6.58405\times 10^{6} & -2.33668\times 10^{6} & -1.53554\times 10^{7} & 3.13558\times \\
-4.68741\times 10^{6} & 1.31838\times 10^{6} & 140552. & 2.8414\times 10^{6} & -7.7342\times 1 \\
1.31838\times 10^{6} & -3.90861\times 10^{6} & 2.8414\times 10^{6} & 351380. & -1.73947\times \\
4.53756\times 10^{6} & -1.74046\times 10^{6} & -4.53833\times 10^{6} & 1.16194\times 10^{6} & 4.53432\times \\
-1.74046\times 10^{6} & 6.84838\times 10^{6} & 1.16194\times 10^{6} & -9.99597\times 10^{6} & -146104. \\
-3.76089\times 10^{6} & 663125. & 2.10397\times 10^{6} & 572609. & -3.92159\times \\
663125. & -8.38984\times 10^{6} & 572609. & 1.46125\times 10^{7} & -996655. \\
3.68225\times 10^{6} & -108225. & -3.41738\times 10^{6} & -539158. & 3.70029\times \\
-524892. & 4.03518\times 10^{6} & -539158. & -7.53106\times 10^{6} & 1.18998\times \\
-6.50428\times 10^{6} & 1.51122\times 10^{6} & 132681. & -3.79772\times 10^{6} & -3.96919\times \\
3.17788\times 10^{6} & -1.70148\times 10^{6} & -3.79772\times 10^{6} & 331703. & -362062.
\end{array}\right.$$

Equations for element 6

$E = 3000000;$ $\nu = 0.2;$ $h = 4$

Nodal coordinates

Element node	Global node number	x	y
1	14	4.	7.33333
2	13	3.5	6.16667
3	11	3.	5.
4	15	4.5	5.
5	23	6.	5.
6	26	7.	6.16667
7	28	8.	7.33333
8	24	6.	7.33333

Complete element equations for element 6

$$\left(\begin{array}{ccccc}
1.20474\times 10^{7} & -5.05264\times 10^{6} & -7.72952\times 10^{6} & 879575. & 4.19129\times 1 \\
-5.05264\times 10^{6} & 2.07298\times 10^{7} & 2.54624\times 10^{6} & -2.03362\times 10^{7} & -817100. \\
-7.72952\times 10^{6} & 2.54624\times 10^{6} & 1.85504\times 10^{7} & -1.71783\times 10^{6} & -4.69594\times 1 \\
879575. & -2.03362\times 10^{7} & -1.71783\times 10^{6} & 3.70235\times 10^{7} & -835301. \\
4.19129\times 10^{6} & -817100. & -4.69594\times 10^{6} & -835301. & 7.62699\times 1 \\
-400433. & 7.16382\times 10^{6} & -2.50197\times 10^{6} & -1.30897\times 10^{7} & 1.78621\times 1 \\
-5.714\times 10^{6} & 2.2747\times 10^{6} & -1.48311\times 10^{6} & 2.8414\times 10^{6} & -7.80448\times 1 \\
2.2747\times 10^{6} & -6.47509\times 10^{6} & 2.8414\times 10^{6} & -3.70776\times 10^{6} & 318772. \\
5.68938\times 10^{6} & -2.29093\times 10^{6} & -5.24953\times 10^{6} & 1.32723\times 10^{6} & 4.9394\times 1 \\
-2.29093\times 10^{6} & 9.72794\times 10^{6} & 1.32723\times 10^{6} & -1.1774\times 10^{7} & -854978. \\
-4.17524\times 10^{6} & 787092. & 1.44956\times 10^{6} & 1.71783\times 10^{6} & -3.3993\times 1 \\
787092. & -9.4257\times 10^{6} & 1.71783\times 10^{6} & 1.29765\times 10^{7} & -831366. \\
4.04403\times 10^{6} & -741342. & -3.14471\times 10^{6} & -415191. & 3.17758\times 1 \\
-1.15801\times 10^{6} & 4.93963\times 10^{6} & -415191. & -6.84938\times 10^{6} & 639512. \\
-8.35339\times 10^{6} & 3.29398\times 10^{6} & 2.30281\times 10^{6} & -3.79772\times 10^{6} & -4.03553\times 1 \\
4.96065\times 10^{6} & -6.32423\times 10^{6} & -3.79772\times 10^{6} & 5.75701\times 10^{6} & 594254.
\end{array}\right.$$

Equations for element 7

$E = 3000000;$ $\nu = 0.2;$ $h = 4$

Nodal coordinates

Element node	Global node number	x	y
1	37	12.	12.
2	35	11.	10.8333
3	33	10.	9.66667
4	39	13.5	9.25
5	42	17.	8.83333
6	45	19.75	10.4167
7	48	22.5	12.
8	43	17.25	12.

Complete element equations for element 7

$$\left(\begin{array}{ccccc}
1.83404\times 10^{7} & -4.35842\times 10^{6} & -1.81952\times 10^{7} & 134448. & 6.56745\times \\
-4.35842\times 10^{6} & 4.22068\times 10^{7} & 1.80111\times 10^{6} & -4.56941\times 10^{7} & -543792. \\
-1.81952\times 10^{7} & 1.80111\times 10^{6} & 3.2475\times 10^{7} & 635354. & -1.21258\times \\
134448. & -4.56941\times 10^{7} & 635354. & 7.71512\times 10^{7} & -1.71559\times \\
6.56745\times 10^{6} & -543792. & -1.21258\times 10^{7} & -1.71559\times 10^{6} & 8.65201\times \\
-127125. & 1.44838\times 10^{7} & -3.38226\times 10^{6} & -2.92744\times 10^{7} & 2.30227\times \\
-6.26209\times 10^{6} & 2.32135\times 10^{6} & -874506. & 1.74831\times 10^{6} & -3.60781\times \\
2.32135\times 10^{6} & -1.22346\times 10^{7} & 1.74831\times 10^{6} & -4.1997\times 10^{6} & 588829. \\
7.7732\times 10^{6} & -1.77674\times 10^{6} & -9.38062\times 10^{6} & 696646. & 5.49223\times \\
-1.77674\times 10^{6} & 1.78443\times 10^{7} & 696646. & -2.2906\times 10^{7} & -524631. \\
-7.52043\times 10^{6} & 223969. & 1.00103\times 10^{7} & 2.54253\times 10^{6} & -5.90223\times \\
223969. & -1.85616\times 10^{7} & 2.54253\times 10^{6} & 2.88728\times 10^{7} & -1.26881\times \\
4.30118\times 10^{6} & -506274. & -5.82154\times 10^{6} & -772169. & 3.3185\times 1 \\
-922941. & 8.38971\times 10^{6} & -772169. & -1.3595\times 10^{7} & 799313. \\
-5.00445\times 10^{6} & 2.83879\times 10^{6} & 3.91247\times 10^{6} & -3.26953\times 10^{6} & -2.39436\times \\
4.50546\times 10^{6} & -6.43433\times 10^{6} & -3.26953\times 10^{6} & 9.64515\times 10^{6} & 362408.
\end{array}\right.$$

Equations for element 8

$E = 3000000;$ $\nu = 0.2;$ $h = 4$

Nodal coordinates

Element node	Global node number	x	y
1	48	22.5	12.
2	45	19.75	10.4167
3	42	17.	8.83333
4	47	20.5	8.41667
5	49	24.	8.
6	52	28.5	10.
7	53	33.	12.
8	51	27.75	12.

Complete element equations for element 8

$$\left(\begin{array}{ccccc}
2.22103\times 10^{7} & -5.87167\times 10^{6} & -1.54514\times 10^{7} & 140088. & 6.43023\times 1 \\
-5.87167\times 10^{6} & 5.06997\times 10^{7} & 1.80675\times 10^{6} & -3.90268\times 10^{7} & -1.21061\times 1 \\
-1.54514\times 10^{7} & 1.80675\times 10^{6} & 2.73231\times 10^{7} & -1.23767\times 10^{6} & -6.65328\times 1 \\
140088. & -3.90268\times 10^{7} & -1.23767\times 10^{6} & 6.33113\times 10^{7} & -1.35486\times 1 \\
6.43023\times 10^{6} & -1.21061\times 10^{6} & -6.65328\times 10^{6} & -1.35486\times 10^{6} & 6.4807\times 10 \\
-793945. & 1.38134\times 10^{7} & -3.02153\times 10^{6} & -1.5914\times 10^{7} & 27783.7 \\
-9.20805\times 10^{6} & 3.46648\times 10^{6} & -3.62774\times 10^{6} & 1.56405\times 10^{6} & -7.51542\times 1 \\
3.46648\times 10^{6} & -1.86811\times 10^{7} & 1.56405\times 10^{6} & -1.10325\times 10^{7} & 3.13164\times 1 \\
9.39696\times 10^{6} & -2.44962\times 10^{6} & -8.61461\times 10^{6} & 951272. & 6.38939\times 1 \\
-2.44962\times 10^{6} & 2.13828\times 10^{7} & 951272. & -2.08821\times 10^{7} & -1.45251\times 1 \\
-6.91164\times 10^{6} & 494365. & 3.46578\times 10^{6} & 3.698\times 10^{6} & -3.68259\times 1 \\
494365. & -1.69369\times 10^{7} & 3.698\times 10^{6} & 1.36898\times 10^{7} & -843880. \\
5.55894\times 10^{6} & -1.36086\times 10^{6} & -4.12415\times 10^{6} & -463024. & 2.6861\times 10 \\
-1.77752\times 10^{6} & 1.09074\times 10^{7} & -463024. & -9.2557\times 10^{6} & -48683.2 \\
-1.20253\times 10^{7} & 5.12516\times 10^{6} & 7.68229\times 10^{6} & -3.29785\times 10^{6} & -4.13514\times 1 \\
6.79183\times 10^{6} & -2.21585\times 10^{7} & -3.29785\times 10^{6} & 1.911\times 10^{7} & 1.75113\times 1
\end{array}\right.$$

Equations for element 9

E = 3000000; ν = 0.2; h = 4

Nodal coordinates

Element node	Global node number	x	y
1	33	10.	9.66667
2	30	9.	8.5
3	28	8.	7.33333
4	32	9.75	6.5
5	36	11.5	5.66667
6	40	14.25	7.25
7	42	17.	8.83333
8	39	13.5	9.25

Complete element equations for element 9

$$\left(\begin{array}{ccccc}
1.29673\times 10^{7} & -2.79914\times 10^{6} & -1.07172\times 10^{7} & -1.92152\times 10^{6} & 5.09418\times 1 \\
-2.79914\times 10^{6} & 2.93466\times 10^{7} & -254849. & -2.67301\times 10^{7} & -335148. \\
-1.07172\times 10^{7} & -254849. & 2.08165\times 10^{7} & 3.29751\times 10^{6} & -7.6598\times 1(\\
-1.92152\times 10^{6} & -2.67301\times 10^{7} & 3.29751\times 10^{6} & 4.27671\times 10^{7} & -2.44916\times 1 \\
5.09418\times 10^{6} & -335148. & -7.6598\times 10^{6} & -2.44916\times 10^{6} & 9.48832\times 1 \\
81518.5 & 9.16828\times 10^{6} & -4.11583\times 10^{6} & -1.35721\times 10^{7} & 1.16752\times 1 \\
-6.08691\times 10^{6} & 2.33529\times 10^{6} & 1.64368\times 10^{6} & 619133. & -7.47384\times 1 \\
2.33529\times 10^{6} & -1.12968\times 10^{7} & 619133. & -1.18712\times 10^{6} & 2.18657\times 1 \\
5.47278\times 10^{6} & -1.12164\times 10^{6} & -5.83714\times 10^{6} & -235426. & 4.90524\times 1 \\
-1.12164\times 10^{6} & 1.22482\times 10^{7} & -235426. & -1.3698\times 10^{7} & -522089. \\
-4.13746\times 10^{6} & -620348. & 2.94811\times 10^{6} & 4.37522\times 10^{6} & -3.74916\times 1 \\
-620348. & -9.68526\times 10^{6} & 4.37522\times 10^{6} & 1.06313\times 10^{7} & -1.58679\times 1 \\
3.79255\times 10^{6} & -587934. & -3.75163\times 10^{6} & -986883. & 3.37383\times 1 \\
-1.0046\times 10^{6} & 6.00011\times 10^{6} & -986883. & -6.67113\times 10^{6} & 377908. \\
-6.38524\times 10^{6} & 3.38376\times 10^{6} & 2.55745\times 10^{6} & -2.69888\times 10^{6} & -3.97876\times 1 \\
5.05042\times 10^{6} & -9.05101\times 10^{6} & -2.69888\times 10^{6} & 8.46011\times 10^{6} & 1.16118\times 1
\end{array}\right.$$

Equations for element 10

$E = 3000000;$ $\nu = 0.2;$ $h = 4$

Nodal coordinates

Element node	Global node number	x	y
1	42	17.	8.83333
2	40	14.25	7.25
3	36	11.5	5.66667
4	38	13.25	4.83333
5	41	15.	4.
6	44	19.5	6.
7	49	24.	8.
8	47	20.5	8.41667

Complete element equations for element 10

$$\left(\begin{array}{ccccc}
1.70633\times10^{7} & -4.41939\times10^{6} & -8.5983\times10^{6} & -1.59487\times10^{6} & 5.43427\times 1 \\
-4.41939\times10^{6} & 3.85866\times10^{7} & 71800.2 & -2.21058\times10^{7} & -1.25443\times 1 \\
-8.5983\times10^{6} & 71800.2 & 1.82312\times10^{7} & 74843.4 & -3.56588\times 1 \\
-1.59487\times10^{6} & -2.21058\times10^{7} & 74843.4 & 3.69539\times10^{7} & -1.3315\times 1 \\
5.43427\times10^{6} & -1.25443\times10^{6} & -3.56588\times10^{6} & -1.3315\times10^{6} & 9.1922\times 1 \\
-837760. & 1.02102\times10^{7} & -2.99816\times10^{6} & -5.11727\times10^{6} & -2.1988\times 1 \\
-9.20691\times10^{6} & 3.57048\times10^{6} & -314397. & 32975.4 & -1.2271\times 1 \\
3.57048\times10^{6} & -1.82838\times10^{7} & 32975.4 & -5.49612\times10^{6} & 5.23176\times 1 \\
7.22123\times10^{6} & -1.85378\times10^{6} & -5.44631\times10^{6} & 201544. & 6.23978\times 1 \\
-1.85378\times10^{6} & 1.61952\times10^{7} & 201544. & -1.28773\times10^{7} & -1.69683\times 1 \\
-3.88838\times10^{6} & -194688. & -2.91085\times10^{6} & 5.22171\times10^{6} & -2.31265\times 1 \\
-194688. & -9.23812\times10^{6} & 5.22171\times10^{6} & -2.32109\times10^{6} & -747393. \\
5.45934\times10^{6} & -1.60145\times10^{6} & -2.79314\times10^{6} & -389086. & 3.55277\times 1 \\
-2.01812\times10^{6} & 9.74782\times10^{6} & -389086. & -4.57489\times10^{6} & -852728. \\
-1.34845\times10^{7} & 5.68145\times10^{6} & 5.39772\times10^{6} & -2.21562\times10^{6} & -6.26946\times 1 \\
7.34812\times10^{6} & -2.51121\times10^{7} & -2.21562\times10^{6} & 1.55385\times10^{7} & 2.84991\times 1
\end{array}\right.$$

Equations for element 11

E = 3000000; ν = 0.2; h = 4

Nodal coordinates

Element node	Global node number	x	y
1	28	8.	7.33333
2	26	7.	6.16667
3	23	6.	5.
4	22	6.	3.75
5	21	6.	2.5
6	29	8.75	4.08333
7	36	11.5	5.66667
8	32	9.75	6.5

Complete element equations for element 11

$$\left(\begin{array}{ccccc}
8.66496\times10^{6} & -1.72313\times10^{6} & -6.04992\times10^{6} & -2.73051\times10^{6} & 6.47312\times \\
-1.72313\times10^{6} & 1.7769\times10^{7} & -1.06384\times10^{6} & -1.09385\times10^{7} & -1.38552\times \\
-6.04992\times10^{6} & -1.06384\times10^{6} & 1.89395\times10^{7} & 1.65324\times10^{6} & -1.12339\times \\
-2.73051\times10^{6} & -1.09385\times10^{7} & 1.65324\times10^{6} & 1.90996\times10^{7} & 334041. \\
6.47312\times10^{6} & -1.38552\times10^{6} & -1.12339\times10^{7} & 334041. & 2.21584\times \\
-968850. & 7.01682\times10^{6} & -1.33263\times10^{6} & -6.51427\times10^{6} & -5.09142\times \\
-6.60142\times10^{6} & 2.63909\times10^{6} & 7.47218\times10^{6} & -1.95601\times10^{6} & -1.61978\times \\
2.63909\times10^{6} & -1.10734\times10^{7} & -1.95601\times10^{6} & 5.44669\times10^{6} & 5.86686\times \\
3.75242\times10^{6} & -726731. & -3.91519\times10^{6} & -442674. & 6.59708\times \\
-726731. & 7.33253\times10^{6} & -442674. & -6.34727\times10^{6} & -1.50819\times \\
-2.42662\times10^{6} & -691724. & -985300. & 4.77381\times10^{6} & -5.50399\times \\
-691724. & -2.93031\times10^{6} & 4.77381\times10^{6} & -3.77413\times10^{6} & -167314. \\
4.74361\times10^{6} & -1.35168\times10^{6} & -4.41258\times10^{6} & 75860.4 & 6.83\times10^{6} \\
-1.76835\times10^{6} & 5.60796\times10^{6} & 75860.4 & -3.41482\times10^{6} & -1.57401\times \\
-8.55615\times10^{6} & 4.30353\times10^{6} & 185258. & -1.70776\times10^{6} & -9.12289\times \\
5.9702\times10^{6} & -1.27841\times10^{7} & -1.70776\times10^{6} & 6.44263\times10^{6} & 3.52555\times
\end{array}\right.$$

Equations for element 12

$E = 3000000;$ $\nu = 0.2;$ $h = 4$

Nodal coordinates

Element node	Global node number	x	y
1	36	11.5	5.66667
2	29	8.75	4.08333
3	21	6.	2.5
4	20	6.	1.25
5	19	6.	0.
6	34	10.5	2.
7	41	15.	4.
8	38	13.25	4.83333

Complete element equations for element 12

$$\left(\begin{array}{ccccc}
1.26253\times 10^{7} & -3.29451\times 10^{6} & -3.60052\times 10^{6} & -2.46779\times 10^{6} & 6.15795\times 1(\\
-3.29451\times 10^{6} & 2.76631\times 10^{7} & -801123. & -8.24386\times 10^{6} & -2.09804\times 1(\\
-3.60052\times 10^{6} & -801123. & 1.58998\times 10^{7} & -1.76369\times 10^{6} & -5.28306\times 1(\\
-2.46779\times 10^{6} & -8.24386\times 10^{6} & -1.76369\times 10^{6} & 2.1648\times 10^{7} & 933051. \\
6.15795\times 10^{6} & -2.09804\times 10^{6} & -5.28306\times 10^{6} & 933051. & 1.88197\times 1(\\
-1.68137\times 10^{6} & 9.43405\times 10^{6} & -733615. & -2.15213\times 10^{6} & -7.66094\times 1(\\
-9.64983\times 10^{6} & 3.87998\times 10^{6} & 5.16841\times 10^{6} & -2.50502\times 10^{6} & -1.99625\times 1(\\
3.87998\times 10^{6} & -1.86171\times 10^{7} & -2.50502\times 10^{6} & 3.61981\times 10^{6} & 8.70113\times 1(\\
5.42307\times 10^{6} & -1.43244\times 10^{6} & -3.34289\times 10^{6} & -54463.3 & 7.48931\times 1(\\
-1.43244\times 10^{6} & 1.16394\times 10^{7} & -54463.3 & -6.64256\times 10^{6} & -2.59605\times 1(\\
-1.99495\times 10^{6} & -364285. & -7.20807\times 10^{6} & 5.78588\times 10^{6} & -3.41976\times 1(\\
-364285. & -3.46269\times 10^{6} & 5.78588\times 10^{6} & -1.4823\times 10^{7} & 505664. \\
6.44681\times 10^{6} & -2.33619\times 10^{6} & -3.44574\times 10^{6} & 589128. & 6.74557\times 1(\\
-2.75285\times 10^{6} & 1.04853\times 10^{7} & 589128. & -3.34137\times 10^{6} & -2.73082\times 1(\\
-1.54078\times 10^{7} & 6.4466\times 10^{6} & 1.81207\times 10^{6} & -517095. & -1.05472\times 1(\\
8.11327\times 10^{6} & -2.88982\times 10^{7} & -517095. & 9.93509\times 10^{6} & 4.94601\times 1(
\end{array}\right.$$

Equations for element 13

$E = 3000000;$ $\nu = 0.2;$ $h = 4$

Nodal coordinates

Element node	Global node number	x	y
1	53	33.	12.
2	52	28.5	10.
3	49	24.	8.
4	54	33.3333	8.
5	56	40.	12.

Complete element equations for element 13

$$\left(\begin{array}{llllll}
1.42985\times 10^{7} & -2.73728\times 10^{6} & -6.43346\times 10^{6} & -1.7531\times 10^{6} & 262557. & \\
-2.73728\times 10^{6} & 3.42051\times 10^{7} & -86431.8 & -1.77275\times 10^{7} & 194064. & \\
-6.43346\times 10^{6} & -86431.8 & 2.51218\times 10^{7} & -3.21918\times 10^{6} & -8.22788\times 10^{6} & \\
-1.7531\times 10^{6} & -1.77275\times 10^{7} & -3.21918\times 10^{6} & 5.7051\times 10^{7} & 143509. & \\
262557. & 194064. & -8.22788\times 10^{6} & 143509. & 5.06795\times 10^{6} & \\
610731. & 1.37557\times 10^{6} & -1.52316\times 10^{6} & -2.18026\times 10^{7} & 1.12769\times 10^{6} & \\
-3.29583\times 10^{6} & 682485. & -1.1833\times 10^{7} & 4.79941\times 10^{6} & 3.00432\times 10^{6} & \\
682485. & -8.13682\times 10^{6} & 4.79941\times 10^{6} & -2.54729\times 10^{7} & -782779. & \\
-4.83176\times 10^{6} & 1.94716\times 10^{6} & 1.37258\times 10^{6} & 29354.2 & -106953. & - \\
3.19716\times 10^{6} & -9.71638\times 10^{6} & 29354.2 & 7.952\times 10^{6} & -682485. & -
\end{array}\right.$$

Equations for element 14

$E = 3000000;$ $\quad \nu = 0.2;$ $\quad h = 4$

Nodal coordinates

Element node	Global node number	x	y
1	49	24.	8.
2	44	19.5	6.
3	41	15.	4.
4	50	26.6667	4.
5	54	33.3333	8.

Complete element equations for element 14

$$\left(\begin{array}{llllllllll}
1.61265\times10^{7} & -2.53394\times10^{6} & -9.51594\times10^{6} & -1.52696\times10^{6} & 782172. \\
-2.53394\times10^{6} & 3.91386\times10^{7} & 139709. & -2.50295\times10^{7} & 103798. \\
-9.51594\times10^{6} & 139709. & 2.87426\times10^{7} & -2.49705\times10^{6} & -1.11128\times10^{7} \\
-1.52696\times10^{6} & -2.50295\times10^{7} & -2.49705\times10^{6} & 6.73937\times10^{7} & 278434. \\
782172. & 103798. & -1.11128\times10^{7} & 278434. & 6.99479\times10^{6} \\
520464. & 2.51328\times10^{6} & -1.38823\times10^{6} & -2.87736\times10^{7} & 1.28542\times10^{6} \\
-3.73254\times10^{6} & 659681. & -1.07586\times10^{7} & 4.30342\times10^{6} & 3.97826\times10^{6} \\
659681. & -9.26936\times10^{6} & 4.30342\times10^{6} & -2.36733\times10^{7} & -1.00797\times10^{6} \\
-3.66022\times10^{6} & 1.63076\times10^{6} & 2.64463\times10^{6} & -557851. & -642464. \\
2.88076\times10^{6} & -7.35304\times10^{6} & -557851. & 1.00826\times10^{7} & -659681.
\end{array}\right.$$

Equations for element 15

E = 3000000; ν = 0.2; h = 4

Nodal coordinates

Element node	Global node number	x	y
1	41	15.	4.
2	34	10.5	2.
3	19	6.	0.
4	46	20.	0.
5	50	26.6667	4.

Complete element equations for element 15

$$\left(\begin{array}{llllllllll}
1.81228\times10^{7} & -2.40825\times10^{6} & -1.24057\times10^{7} & -1.38977\times10^{6} & 1.22994\times10^{6} \\
-2.40825\times10^{6} & 4.43538\times10^{7} & 276901. & -3.20088\times10^{7} & 46698.2 \\
-1.24057\times10^{7} & 276901. & 3.29383\times10^{7} & -2.04025\times10^{6} & -1.39029\times10^{7} \\
-1.38977\times10^{6} & -3.20088\times10^{7} & -2.04025\times10^{6} & 7.86992\times10^{7} & 369640. \\
1.22994\times10^{6} & 46698.2 & -1.39029\times10^{7} & 369640. & 9.04082\times10^{6} \\
463365. & 3.53065\times10^{6} & -1.29703\times10^{6} & -3.5586\times10^{7} & 1.38812\times10^{6} \\
-4.19374\times10^{6} & 648185. & -1.00662\times10^{7} & 3.98382\times10^{6} & 4.78564\times10^{6} \\
648185. & -1.04429\times10^{7} & 3.98382\times10^{6} & -2.25136\times10^{7} & -1.15627\times10^{6} \\
-2.75322\times10^{6} & 1.43646\times10^{6} & 3.4366\times10^{6} & -923441. & -1.15348\times10^{6} \\
2.68646\times10^{6} & -5.43278\times10^{6} & -923441. & 1.14092\times10^{7} & -648185.
\end{array}\right.$$

Equations for element 16

E = 3000000; ν = 0.2; h = 4

Nodal coordinates

Element node	Global node number	x	y
1	56	40.	12.
2	54	33.3333	8.
3	58	43.6667	8.
4	59	47.	12.

Complete element equations for element 16

$$\left(\begin{array}{llllll}
1.0601\times 10^{7} & -3.47572\times 10^{6} & -2.00657\times 10^{6} & -1.35515\times 10^{6} & -6.53509\times 10^{6} & \\
-3.47572\times 10^{6} & 2.20204\times 10^{7} & -105155. & -7.06087\times 10^{6} & 2.60515\times 10^{6} & \\
-2.00657\times 10^{6} & -105155. & 3.0394\times 10^{6} & 555105. & -747733. & 6 \\
-1.35515\times 10^{6} & -7.06087\times 10^{6} & 555105. & 3.90279\times 10^{6} & 1.9449\times 10^{6} & \\
-6.53509\times 10^{6} & 2.60515\times 10^{6} & -747733. & 1.9449\times 10^{6} & 9.2894\times 10^{6} & \\
2.60515\times 10^{6} & -1.42933\times 10^{7} & 694895. & 1.82638\times 10^{6} & -3.1949\times 10^{6} & \\
-2.05931\times 10^{6} & 975724. & -285093. & -1.14485\times 10^{6} & -2.00657\times 10^{6} & -1 \\
2.22572\times 10^{6} & -666240. & -1.14485\times 10^{6} & 1.3317\times 10^{6} & -1.35515\times 10^{6} &
\end{array}\right.$$

Equations for element 17

E = 3000000; ν = 0.2; h = 4

Nodal coordinates

Element node	Global node number	x	y
1	59	47.	12.
2	58	43.6667	8.
3	62	54.	8.
4	63	54.	12.

Complete element equations for element 17

$$\left(\begin{array}{lllll}
7.33192\times 10^{6} & -2.40857\times 10^{6} & -2.54743\times 10^{6} & -868385. & -3.9109\times 10 \\
-2.40857\times 10^{6} & 1.38478\times 10^{7} & 381615. & -8.41301\times 10^{6} & 2.11838\times 1 \\
-2.54743\times 10^{6} & 381615. & 4.14503\times 10^{6} & 1.43503\times 10^{6} & 229968. \\
-868385. & -8.41301\times 10^{6} & 1.43503\times 10^{6} & 6.66687\times 10^{6} & 1.06497\times 1 \\
-3.9109\times 10^{6} & 2.11838\times 10^{6} & 229968. & 1.06497\times 10^{6} & 6.22837\times 1 \\
2.11838\times 10^{6} & -7.73283\times 10^{6} & -185035. & 4.27063\times 10^{6} & -2.31497\times 1 \\
-873585. & -91425.4 & -1.82757\times 10^{6} & -1.63162\times 10^{6} & -2.54743\times 1 \\
1.15857\times 10^{6} & 2.29806\times 10^{6} & -1.63162\times 10^{6} & -2.52449\times 10^{6} & -868385.
\end{array}\right.$$

Equations for element 18

E = 3000000; $\nu = 0.2$; h = 4

Nodal coordinates

Element node	Global node number	x	y
1	54	33.3333	8.
2	50	26.6667	4.
3	57	40.3333	4.
4	58	43.6667	8.

Complete element equations for element 18

$$\left(\begin{array}{cccccc}
1.09361\times10^{7} & -2.99621\times10^{6} & -3.84805\times10^{6} & -1.1492\times10^{6} & -6.77695\times10^{6} & \\
-2.99621\times10^{6} & 2.42009\times10^{7} & 100796. & -1.10879\times10^{7} & 2.3992\times10^{6} & \\
-3.84805\times10^{6} & 100796. & 3.67169\times10^{6} & 899398. & 703307. & 3 \\
-1.1492\times10^{6} & -1.10879\times10^{7} & 899398. & 6.44755\times10^{6} & 1.6006\times10^{6} & \\
-6.77695\times10^{6} & 2.3992\times10^{6} & 703307. & 1.6006\times10^{6} & 9.92169\times10^{6} & \\
2.3992\times10^{6} & -1.54746\times10^{7} & 350602. & 4.48995\times10^{6} & -2.8506\times10^{6} & \\
-311125. & 496214. & -526954. & -1.3508\times10^{6} & -3.84805\times10^{6} & 1 \\
1.74621\times10^{6} & 2.36159\times10^{6} & -1.3508\times10^{6} & 150385. & -1.1492\times10^{6} &
\end{array}\right.$$

Equations for element 19

E = 3000000; $\nu = 0.2$; h = 4

Nodal coordinates

Element node	Global node number	x	y
1	58	43.6667	8.
2	57	40.3333	4.
3	61	54.	4.
4	62	54.	8.

Complete element equations for element 19

$$\left(\begin{array}{lllll}
8.02226\times 10^{6} & -2.24874\times 10^{6} & -4.23634\times 10^{6} & -799735. & -4.30532\times 1 \\
-2.24874\times 10^{6} & 1.69163\times 10^{7} & 450265. & -1.20586\times 10^{7} & 2.04973\times 1 \\
-4.23634\times 10^{6} & 450265. & 5.03236\times 10^{6} & 1.5498\times 10^{6} & 1.42598\times 1 \\
-799735. & -1.20586\times 10^{7} & 1.5498\times 10^{6} & 9.84921\times 10^{6} & 950201. \\
-4.30532\times 10^{6} & 2.04973\times 10^{6} & 1.42598\times 10^{6} & 950201. & 7.11569\times 1 \\
2.04973\times 10^{6} & -9.29553\times 10^{6} & -299799. & 6.29662\times 10^{6} & -2.2002\times 1 \\
519404. & -251262. & -2.22199\times 10^{6} & -1.70027\times 10^{6} & -4.23634\times 1 \\
998738. & 4.43791\times 10^{6} & -1.70027\times 10^{6} & -4.0872\times 10^{6} & -799735.
\end{array}\right.$$

Equations for element 20

E = 3000000; ν = 0.2; h = 4

Nodal coordinates

Element node	Global node number	x	y
1	50	26.6667	4.
2	46	20.	0.
3	55	37.	0.
4	57	40.3333	4.

Complete element equations for element 20

$$\left(\begin{array}{llllll}
1.17574\times 10^{7} & -2.73792\times 10^{6} & -5.48961\times 10^{6} & -1.03422\times 10^{6} & -7.21872\times 10^{6} & 2.284 \\
-2.73792\times 10^{6} & 2.69773\times 10^{7} & 215780. & -1.48699\times 10^{7} & 2.28422\times 10^{6} & -1.69 \\
-5.48961\times 10^{6} & 215780. & 4.61747\times 10^{6} & 1.10104\times 10^{6} & 1.84087\times 10^{6} & 148960. \\
-1.03422\times 10^{6} & -1.48699\times 10^{7} & 1.10104\times 10^{6} & 9.37658\times 10^{6} & 1.39896\times 10^{6} & 6.76 \\
-7.21872\times 10^{6} & 2.28422\times 10^{6} & 1.84087\times 10^{6} & 1.39896\times 10^{6} & 1.08675\times 10^{7} & -2.64 \\
2.28422\times 10^{6} & -1.6901\times 10^{7} & 148960. & 6.76925\times 10^{6} & -2.64896\times 10^{6} & 2.50 \\
950927. & 237921. & -968719. & -1.46578\times 10^{6} & -5.48961\times 10^{6} & 215780. \\
1.48792\times 10^{6} & 4.7935\times 10^{6} & -1.46578\times 10^{6} & -1.27598\times 10^{6} & -1.03422\times 10^{6} & -1.48
\end{array}\right.$$

Equations for element 21

E = 3000000; ν = 0.2; h = 4

Nodal coordinates

Element node	Global node number	x	y
1	57	40.3333	4.
2	55	37.	0.
3	60	54.	0.
4	61	54.	4.

Complete element equations for element 21

$$\left(\begin{array}{ccccc}
9.03487\times10^{6} & -2.16264\times10^{6} & -5.79274\times10^{6} & -761407. & -4.83226\times \\
-2.16264\times10^{6} & 2.0171\times10^{7} & 488593. & -1.56277\times10^{7} & 2.01141\times \\
-5.79274\times10^{6} & 488593. & 6.1275\times10^{6} & 1.61701\times10^{6} & 2.41417\times \\
-761407. & -1.56277\times10^{7} & 1.61701\times10^{6} & 1.31517\times10^{7} & 882987. \\
-4.83226\times10^{6} & 2.01141\times10^{6} & 2.41417\times10^{6} & 882987. & 8.21083\times \\
2.01141\times10^{6} & -1.09348\times10^{7} & -367013. & 8.20251\times10^{6} & -2.13299\times \\
1.59013\times10^{6} & -337360. & -2.74893\times10^{6} & -1.73859\times10^{6} & -5.79274\times \\
912640. & 6.3915\times10^{6} & -1.73859\times10^{6} & -5.7265\times10^{6} & -761407.
\end{array}\right.$$

Essential boundary conditions

Node	dof	Value
1	u_1	0
2	u_2	0
3	u_3	0
4	u_4	0
5	u_5	0
6	u_6	0
7	u_7	0
60	u_{60} v_{60}	0 0
61	u_{61} v_{61}	0 0
62	u_{62} v_{62}	0 0
63	u_{63} v_{63}	0 0

After adjusting for essential boundary conditions we have

$$\left(\begin{array}{ccccc}
1.15632\times10^{7} & -1.53554\times10^{7} & 6.58405\times10^{6} & 0 & 0 \\
-1.53554\times10^{7} & 3.53875\times10^{7} & -1.78006\times10^{7} & 0 & 0 \\
6.58405\times10^{6} & -1.78006\times10^{7} & 2.84509\times10^{7} & -2.01506\times10^{7} & 8.08326\times \\
0 & 0 & -2.01506\times10^{7} & 4.44535\times10^{7} & -2.25702\times
\end{array}\right.$$

0	0	[illegible]	[illegible]	[illegible]
0	0	8.08326×10^{6}	-2.25702×10^{7}	$3.38974\times$
0	0	0	0	$-2.49316\times$
0	0	0	0	$9.64362\times$
$-72806.$	2.8414×10^{6}	1.31838×10^{6}	0	0
-2.5265×10^{6}	$351380.$	-3.90861×10^{6}	0	0
$-362062.$	-3.79772×10^{6}	1.26062×10^{6}	2.95375×10^{6}	$1.20877\times$
-2.11306×10^{6}	$331703.$	-3.02185×10^{6}	$271131.$	$-3.87026\times$
0	0	$-466188.$	-3.69633×10^{6}	1.0054×1
0	0	-2.07862×10^{6}	$259286.$	$-1.12039\times$
$-562771.$	1.16194×10^{6}	-1.74046×10^{6}	0	0
5.54667×10^{6}	-9.99597×10^{6}	6.84838×10^{6}	0	0
0	0	0	0	$-532568.$
0	0	0	0	$-2.18723\times$
$-996655.$	$572609.$	$663125.$	0	0
-8.45411×10^{6}	1.46125×10^{7}	-8.38984×10^{6}	0	0
1.18998×10^{6}	$-539158.$	$-588232.$	1.08531×10^{6}	$-1.67611\times$
4.75521×10^{6}	-7.53106×10^{6}	1.02853×10^{7}	-1.22634×10^{7}	$7.98938\times$
0	0	0	0	0
0	0	0	0	0
0	0	$-960936.$	$444997.$	$701585.$
0	0	-1.08018×10^{7}	1.98322×10^{7}	$-1.07631\times$
0	0	1.24885×10^{6}	$-602082.$	$-469160.$
0	0	5.90013×10^{6}	-9.83193×10^{6}	$1.18658\times$
0	0	0	0	$-938035.$
0	0	0	0	$-1.31636\times$
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0	0	0	0	0

Solving the final system of global equations we get

$\{v_1 = -0.0433512,\ v_2 = -0.0433977,\ v_3 = -0.0434424,\ v_4 = -0.0434478,\ v_5 = -0.0434335,\ v_6 = -0.043391,$
$v_7 = -0.0433234,\ u_8 = 0.000442152,\ v_8 = -0.0432241,\ u_9 = 0.000121906,\ v_9 = -0.043224,$
$u_{10} = -0.000450314,\ v_{10} = -0.0430701,\ u_{11} = 0.000898366,\ v_{11} = -0.0428902,\ u_{12} = -0.00128943,$
$v_{12} = -0.0428053,\ u_{13} = 0.000645599,\ v_{13} = -0.0427767,\ u_{14} = 0.000207697,\ v_{14} = -0.0425616,$
$u_{15} = 0.0014282,\ v_{15} = -0.0423311,\ u_{16} = -0.000289969,\ v_{16} = -0.0423059,\ u_{17} = -0.000891704,$
$v_{17} = -0.0420074,\ u_{18} = -0.00160899,\ v_{18} = -0.0416635,\ u_{19} = 0.00701662,\ v_{19} = -0.0418628,$
$u_{20} = 0.00578302,\ v_{20} = -0.0418207,\ u_{21} = 0.0045341,\ v_{21} = -0.0417719,\ u_{22} = 0.00321919,$
$v_{22} = -0.0416488,\ u_{23} = 0.00192205,\ v_{23} = -0.0414902,\ u_{24} = 0.000271463,\ v_{24} = -0.0414315,$
$u_{25} = -0.00248781,\ v_{25} = -0.0412825,\ u_{26} = 0.00107483,\ v_{26} = -0.0407164,\ u_{27} = -0.00130038,$
$v_{27} = -0.0402653,\ u_{28} = 0.000258265,\ v_{28} = -0.0398727,\ u_{29} = 0.00299974,\ v_{29} = -0.0391301,$
$u_{30} = -0.000631563,\ v_{30} = -0.0390009,\ u_{31} = -0.00338436,\ v_{31} = -0.038948,\ u_{32} = 0.00091481,$
$v_{32} = -0.0382586,\ u_{33} = -0.00163959,\ v_{33} = -0.0380827,\ u_{34} = 0.00507175,\ v_{34} = -0.0374624,$
$u_{35} = -0.00275544,\ v_{35} = -0.037128,\ u_{36} = 0.00165284,\ v_{36} = -0.0365735,\ u_{37} = -0.00399364,$
$v_{37} = -0.0361434,\ u_{38} = 0.00239557,\ v_{38} = -0.0348281,\ u_{39} = -0.00159262,\ v_{39} = -0.0346746,$
$u_{40} = 0.000170754,\ v_{40} = -0.0338948,\ u_{41} = 0.00315312,\ v_{41} = -0.033045,\ u_{42} = -0.00142344,$
$v_{42} = -0.0311427,\ u_{43} = -0.00453029,\ v_{43} = -0.0309132,\ u_{44} = 0.00116341,\ v_{44} = -0.0285447,$
$u_{45} = -0.0030776,\ v_{45} = -0.028365,\ u_{46} = 0.00696755,\ v_{46} = -0.0280676,\ u_{47} = -0.00115034,$
$v_{47} = -0.0275859,\ u_{48} = -0.00473821,\ v_{48} = -0.025587,\ u_{49} = -0.000855872,\ v_{49} = -0.0240683,$
$u_{50} = 0.00286123,\ v_{50} = -0.021379,\ u_{51} = -0.0047055,\ v_{51} = -0.020438,\ u_{52} = -0.00281548,$
$v_{52} = -0.0196773,\ u_{53} = -0.00449338,\ v_{53} = -0.0155429,\ u_{54} = -0.00103931,\ v_{54} = -0.0149814,$
$u_{55} = 0.00603824,\ v_{55} = -0.0112323,\ u_{56} = -0.00408667,\ v_{56} = -0.00921432,\ u_{57} = 0.00202799,$
$v_{57} = -0.00842634,\ u_{58} = -0.00079608,\ v_{58} = -0.00596352,\ u_{59} = -0.002965,\ v_{59} = -0.00390572\}$

Solution summary

Nodal solution

	x	y	u	v
1	0.	5.	0	−0.0433512
2	0.	6.16667	0	−0.0433977
3	0.	7.33333	0	−0.0434424
4	0.	8.5	0	−0.0434478

5	0.	9.66667	0	−0.0434335
6	0.	10.8333	0	−0.043391
7	0.	12.	0	−0.0433234
8	1.5	5.	0.000442152	−0.0432241
9	2.	7.33333	0.000121906	−0.043224
10	2.5	9.66667	−0.000450314	−0.0430701
11	3.	5.	0.000898366	−0.0428902
12	3.	12.	−0.00128943	−0.0428053
13	3.5	6.16667	0.000645599	−0.0427767
14	4.	7.33333	0.000207697	−0.0425616
15	4.5	5.	0.0014282	−0.0423311
16	4.5	8.5	−0.000289969	−0.0423059
17	5.	9.66667	−0.000891704	−0.0420074
18	5.5	10.8333	−0.00160899	−0.0416635
19	6.	0.	0.00701662	−0.0418628
20	6.	1.25	0.00578302	−0.0418207
21	6.	2.5	0.0045341	−0.0417719
22	6.	3.75	0.00321919	−0.0416488
23	6.	5.	0.00192205	−0.0414902
24	6.	7.33333	0.000271463	−0.0414315
25	6.	12.	−0.00248781	−0.0412825
26	7.	6.16667	0.00107483	−0.0407164
27	7.5	9.66667	−0.00130038	−0.0402653
28	8.	7.33333	0.000258265	−0.0398727
29	8.75	4.08333	0.00299974	−0.0391301
30	9.	8.5	−0.000631563	−0.0390009
31	9.	12.	−0.00338436	−0.038948
32	9.75	6.5	0.00091481	−0.0382586
33	10.	9.66667	−0.00163959	−0.0380827
34	10.5	2.	0.00507175	−0.0374624
35	11.	10.8333	−0.00275544	−0.037128
36	11.5	5.66667	0.00165284	−0.0365735
37	12.	12.	−0.00399364	−0.0361434
38	13.25	4.83333	0.00239557	−0.0348281
39	13.5	9.25	−0.00159262	−0.0346746
40	14.25	7.25	0.000170754	−0.0338948
41	15.	4.	0.00315312	−0.033045
42	17.	8.83333	−0.00142344	−0.0311427

43	17.25	12.	−0.00453029	−0.0309132
44	19.5	6.	0.00116341	−0.0285447
45	19.75	10.4167	−0.0030776	−0.028365
46	20.	0.	0.00696755	−0.0280676
47	20.5	8.41667	−0.00115034	−0.0275859
48	22.5	12.	−0.00473821	−0.025587
49	24.	8.	−0.000855872	−0.0240683
50	26.6667	4.	0.00286123	−0.021379
51	27.75	12.	−0.0047055	−0.020438
52	28.5	10.	−0.00281548	−0.0196773
53	33.	12.	−0.00449338	−0.0155429
54	33.3333	8.	−0.00103931	−0.0149814
55	37.	0.	0.00603824	−0.0112323
56	40.	12.	−0.00408667	−0.00921432
57	40.3333	4.	0.00202799	−0.00842634
58	43.6667	8.	−0.00079608	−0.00596352
59	47.	12.	−0.002965	−0.00390572
60	54.	0.	0	0
61	54.	4.	0	0
62	54.	8.	0	0
63	54.	12.	0	0

Solution at selected points on elements

	Coord	Disp	Stresses	Principal stresses	Effective Stress
1	2.75 10.8333	−0.000829492 −0.0429533	−885.347 −38.5711 0 21.4363 0 0	0. −38.0288 −885.89	867.501
2	8.25 10.8333	−0.0022714 −0.0396234	−629.864 −22.6481 0 81.8768 0 0	0. −11.8016 −640.711	634.892
3	2.25 8.5	−0.000138187 −0.0431627	−194.132 −4.10036 0 27.8944 0 0	0. −0.0904196 −198.142	198.097

4	6.75 8.5	−0.000458893 −0.0408707	−219.926 39.0197 0 137.002 0 0	98.0482 0. −278.954	338.792
5	1.75 6.16667	0.000328313 −0.0432499	552.704 −3.33214 0 0.825049 0 0	552.705 0. −3.33336	554.379
6	5.25 6.16667	0.000888449 −0.0419242	387.678 98.8279 0 17.6001 0 0	388.747 97.7594 0.	350.253
7	15.375 10.625	−0.00302925 −0.0328015	−256.073 −45.2711 0 57.0159 0 0	0. −30.8382 −270.506	256.481
8	24.125 10.2083	−0.00299674 −0.0239479	−57.3007 −49.9626 0 40.0476 0 0	0. −13.4163 −93.8469	87.91
9	11.625 7.875	−0.000281328 −0.0364965	−162.463 −84.5421 0 153.548 0 0	34.9111 0. −281.916	300.895
10	16.875 6.625	0.00065803 −0.0312194	−115.316 −105.519 0 108.113 0 0	0. −2.19399 −218.641	217.552
11	7.875 5.125	0.00201247 −0.0399499	121.192 20.5462 0 17.235 0 0	124.062 17.6766 0.	116.236

12	9.625 3.04167	0.00403588 −0.0383074	58.0026 32.1149 0 −23.7665 0 0	72.1215 17.9961 0.	65.0189
13	32.5833 10.	−0.00268924 −0.0158876	77.2584 −77.4671 0 39.9309 0 0	86.9558 0. −87.1645	150.793
14	24.75 6.	0.00103718 −0.0233625	−82.1807 −50.2688 0 56.8608 0 0	0. −7.16757 −125.282	121.856
15	16.9167 2.	0.00499307 −0.0310929	−41.851 −25.3267 0 25.5573 0 0	0. −6.7293 −60.4484	57.3805
16	41. 10.	−0.00222177 −0.00851624	211.669 −122.818 0 95.078 0 0	236.806 0. −147.955	336.161
17	49.6667 10.	−0.00094027 −0.00246731	690.569 198.064 0 259.817 0 0	802.29 86.3428 0.	762.792
18	36. 6.	0.000763459 −0.0126876	−99.7889 −130.189 0 131.99 0 0	17.873 0. −247.851	257.254

19	48. 6.	0.000307979 −0.00359747	−124.137 149.26 0 334.945 0 0	374.327 0. −349.204	626.723
20	31. 2.	0.00447376 −0.0172763	−196.713 −121.448 0 35.7738 0 0	0. −107.158 −211.003	182.742
21	46.3333 2.	0.00201656 −0.00491466	−769.69 96.9889 0 311.696 0 0	197.445 0. −870.146	983.842

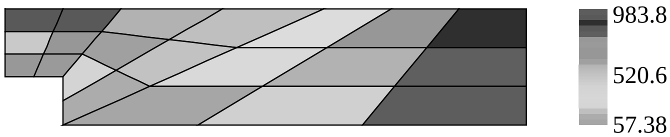
Support reactions

Node	dof	Reaction
1	1	−1492.21
2	1	−3250.09
3	1	−668.521
4	1	1095.16
5	1	1706.8
6	1	5756.08
7	1	2029.83
60	1	−22089.2
60	2	1870.66
61	1	−4913.92
61	2	1954.34
62	1	7693.77
62	2	2332.
63	1	14132.3
63	2	4642.99

Sum of applied loads → (0   −10800.)

Sum of support reactions → (0   10800.)

The effective stresses at element centers are used to create an element stress plot as shown in Figure. There are significant jumps in stresses across element boundaries and thus the model needs to be refined further. Results obtained from refined models using Ansys and Abaqus are presented in Appendix A.

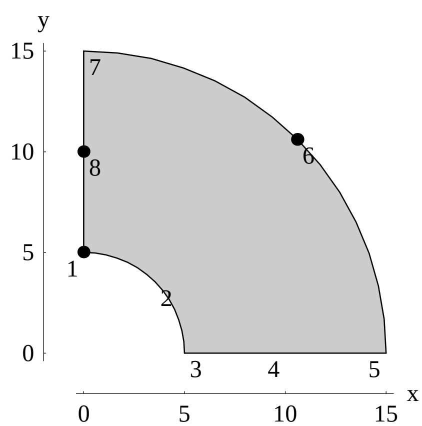


Example 7.9: Pressure Vessel, One element solution (p. 518)

To show all calculations only a single eight node element is used as shown in Figure. Because of symmetry the following boundary conditions are imposed.

At nodes 3, 4, 5: y displacement = 0

At nodes 7, 8, 1: x displacement = 0



Using k – in units the calculations are as follows.

Global equations at start of the element assembly process

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} u_1\\ v_1\\ u_2\\ v_2\\ u_3\\ v_3\\ u_4\\ v_4\\ u_5\\ v_5\\ u_6\\ v_6\\ u_7\\ v_7\\ u_8\\ v_8 \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}$$

Equations for element 1

$E = 30000; \quad \nu = 0.3; \quad h = 1$

Nodal coordinates

Element node	Global node number	x	y
1	1	0.	5.
2	2	3.53553	3.53553
3	3	5.	0.
4	4	10.	0.
5	5	15.	0.
6	6	10.6066	10.6066
7	7	0.	15.
8	8	0.	10.

Interpolation functions and their derivatives

$$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\} =$$
$$\left\{-\frac{1}{4}(s-1)(t-1)(s+t+1),\ \frac{1}{2}\left(s^2-1\right)(t-1),\ \frac{1}{4}(t-1)\left(-s^2+t\,s+t+1\right),\ -\frac{1}{2}(s+1)\left(t^2-1\right),\right.$$
$$\left.\frac{1}{4}(s+1)(t+1)(s+t-1),\ -\frac{1}{2}\left(s^2-1\right)(t+1),\ \frac{1}{4}(s-1)(s-t+1)(t+1),\ \frac{1}{2}(s-1)\left(t^2-1\right)\right\}$$

$$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s, \partial N_5/\partial s, \partial N_6/\partial s, \partial N_7/\partial s, \partial N_8/\partial s\} = \Big\{-\frac{1}{4}(t-1)(2s+t),$$
$$s(t-1), -\frac{1}{4}(2s-t)(t-1), \frac{1}{2}\left(1-t^2\right), \frac{1}{4}(t+1)(2s+t), -s(t+1), \frac{1}{4}(2s-t)(t+1), \frac{1}{2}\left(t^2-1\right)\Big\}$$

$$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t, \partial N_5/\partial t, \partial N_6/\partial t, \partial N_7/\partial t, \partial N_8/\partial t\} = \Big\{-\frac{1}{4}(s-1)(s+2t),$$
$$\frac{1}{2}\left(s^2-1\right), -\frac{1}{4}(s+1)(s-2t), -(s+1)t, \frac{1}{4}(s+1)(s+2t), \frac{1}{2}\left(1-s^2\right), \frac{1}{4}(s-1)(s-2t), (s-1)t\Big\}$$

Mapping to the master element

$$x(s,t) = \boldsymbol{N}^T\boldsymbol{x}_n = -1.03553\,t\,s^2 - 2.07107\,s^2 + 2.5\,t\,s + 5.\,s + 3.53553\,t + 7.07107$$

$$y(s,t) = \boldsymbol{N}^T\boldsymbol{y}_n = -1.03553\,t\,s^2 - 2.07107\,s^2 - 2.5\,t\,s - 5.\,s + 3.53553\,t + 7.07107$$

$$\boldsymbol{J} = \begin{pmatrix} -2.07107\,t\,s - 4.14214\,s + 2.5\,t + 5. & -1.03553\,s^2 + 2.5\,s + 3.53553 \\ -2.07107\,t\,s - 4.14214\,s - 2.5\,t - 5. & -1.03553\,s^2 - 2.5\,s + 3.53553 \end{pmatrix};$$

$$\det J = 5.17767\,t\,s^2 + 10.3553\,s^2 + 17.6777\,t + 35.3553$$

$$\text{Plane strain } \boldsymbol{C} = \begin{pmatrix} 40384.6 & 17307.7 & 0 \\ 17307.7 & 40384.6 & 0 \\ 0 & 0 & 11538.5 \end{pmatrix}$$

For numerical integration the Gauss quadrature points and weights are

	s	t	Weight
1	−0.774597	−0.774597	0.308642
2	−0.774597	0.	0.493827
3	−0.774597	0.774597	0.308642
4	0.	−0.774597	0.493827
5	0.	0.	0.790123
6	0.	0.774597	0.493827
7	0.774597	−0.774597	0.308642
8	0.774597	0.	0.493827
9	0.774597	0.774597	0.308642

Computation of element matrices at {−0.774597, −0.774597} with weight = 0.308642

$$\boldsymbol{J} = \begin{pmatrix} 5.02935 & 0.977722 \\ -1.09766 & 4.85071 \end{pmatrix} \qquad \det J = 25.4691$$

$$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\} =$$
$$\{0.432379, 0.354919, -0.1, 0.0450807, -0.032379, 0.0450807, -0.1, 0.354919\}$$

$$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s, \partial N_5/\partial s, \partial N_6/\partial s, \partial N_7/\partial s, \partial N_8/\partial s\} =$$
$$\{-1.03095, 1.3746, -0.343649, 0.2, -0.130948, 0.174597, -0.0436492, -0.2\}$$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t, \partial N_5/\partial t, \partial N_6/\partial t, \partial N_7/\partial t, \partial N_8/\partial t\} =$
$\{-1.03095, -0.2, -0.0436492, 0.174597, -0.130948, 0.2, -0.343649, 1.3746\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x, \partial N_5/\partial x, \partial N_6/\partial x, \partial N_7/\partial x, \partial N_8/\partial x\} =$
$\{-0.24078, 0.253178, -0.0673307, 0.0456156, -0.0305831, 0.0418723, -0.0231237, 0.0211513\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y, \partial N_5/\partial y, \partial N_6/\partial y, \partial N_7/\partial y, \partial N_8/\partial y\} =$
$\{-0.164003, -0.0922625, 0.00457284, 0.0267997, -0.0208311, 0.0327912, -0.0661843, 0.279117\}$

$$\boldsymbol{B} = \begin{pmatrix} -0.24078 & 0 & -0.164003 \\ 0 & -0.164003 & -0.24078 \\ 0.253178 & 0 & -0.0922625 \\ 0 & -0.0922625 & 0.253178 \\ -0.0673307 & 0 & 0.00457284 \\ 0 & 0.00457284 & -0.0673307 \\ 0.0456156 & 0 & 0.0267997 \\ 0 & 0.0267997 & 0.0456156 \\ -0.0305831 & 0 & -0.0208311 \\ 0 & -0.0208311 & -0.0305831 \\ 0.0418723 & 0 & 0.0327912 \\ 0 & 0.0327912 & 0.0418723 \\ -0.0231237 & 0 & -0.0661843 \\ 0 & -0.0661843 & -0.0231237 \\ 0.0211513 & 0 & 0.279117 \\ 0 & 0.279117 & 0.0211513 \end{pmatrix}$$

$$\boldsymbol{k} = \left(\begin{array}{ccccccc}
20844.2 & 8954.26 & -17979.8 & -743.723 & 5078.55 & 851.771 & -3885.39 \\
8954.26 & 13797.1 & -3634.26 & -725.672 & 1402.49 & 1232.37 & -1603.11 \\
-17979.8 & -3634.26 & 21120.9 & -5296.74 & -5449.85 & 720.964 & 3442.01 \\
-743.723 & -725.672 & -5296.74 & 8516.24 & 950.184 & -1680.1 & 42.8277 \\
5078.55 & 1402.49 & -5449.85 & 950.184 & 1441.06 & -69.8162 & -963.9 \\
851.771 & 1232.37 & 720.964 & -1680.1 & -69.8162 & 417.829 & -135.287 \\
-3885.39 & -1603.11 & 3442.01 & 42.8277 & -963.9 & -135.287 & 725.704 \\
-1556.48 & -2391.51 & 541.403 & 262.562 & -226.58 & -239.671 & 277.204 \\
2647.56 & 1137.34 & -2283.74 & -94.4652 & 645.061 & 108.189 & -493.51 \\
1137.34 & 1752.46 & -461.612 & -92.1724 & 178.14 & 156.532 & -203.622 \\
-3688.39 & -1650.44 & 3091. & 227.405 & -881.402 & -174.206 & 686.061 \\
-1697.07 & -2621.7 & 779.112 & 1.11193 & -283.018 & -208.113 & 305.289 \\
2752.03 & 1961.38 & -1304.67 & -1229.58 & 466.809 & 389.803 & -495.734 \\
2512.1 & 3950.82 & -2086.26 & 1407.49 & 596.693 & 45.1387 & -466.958 \\
-5768.74 & -6567.66 & -635.765 & 6144.09 & -336.332 & -1691.42 & 984.766 \\
-9458.2 & -14993.9 & 9437.39 & -7689.46 & -2548.09 & 276.018 & 1783.66
\end{array}\right.$$

Computation of element matrices at {−0.774597, 0.} with weight = 0.493827

$$\boldsymbol{J} = \begin{pmatrix} 8.20848 & 0.977722 \\ -1.79152 & 4.85071 \end{pmatrix} \qquad \det J = 41.5685$$

$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\} = \{-0.1, 0.2, -0.1, 0.112702, -0.1, 0.2, -0.1, 0.887298\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s, \partial N_5/\partial s, \partial N_6/\partial s, \partial N_7/\partial s, \partial N_8/\partial s\} =$
$\{-0.387298, 0.774597, -0.387298, 0.5, -0.387298, 0.774597, -0.387298, -0.5\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t, \partial N_5/\partial t, \partial N_6/\partial t, \partial N_7/\partial t, \partial N_8/\partial t\} =$
$\{-0.343649, -0.2, 0.0436492, 0., -0.0436492, 0.2, 0.343649, 0.\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x, \partial N_5/\partial x, \partial N_6/\partial x, \partial N_7/\partial x, \partial N_8/\partial x\} =$
$\{-0.0600051, 0.0817695, -0.0433133, 0.0583459, -0.0470757, 0.0990086, -0.030384, -0.0583459\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y, \partial N_5/\partial y, \partial N_6/\partial y, \partial N_7/\partial y, \partial N_8/\partial y\} =$
$\{-0.0587504, -0.0577128, 0.0177289, -0.0117604, 0.000490192, 0.0212747, 0.0769695, 0.0117604\}$

$$\boldsymbol{B} = \begin{pmatrix} -0.0600051 & 0 & -0.0587504 \\ 0 & -0.0587504 & -0.0600051 \\ 0.0817695 & 0 & -0.0577128 \\ 0 & -0.0577128 & 0.0817695 \\ -0.0433133 & 0 & 0.0177289 \\ 0 & 0.0177289 & -0.0433133 \\ 0.0583459 & 0 & -0.0117604 \\ 0 & -0.0117604 & 0.0583459 \\ -0.0470757 & 0 & 0.000490192 \\ 0 & 0.000490192 & -0.0470757 \\ 0.0990086 & 0 & 0.0212747 \\ 0 & 0.0212747 & 0.0990086 \\ -0.030384 & 0 & 0.0769695 \\ 0 & 0.0769695 & -0.030384 \\ -0.0583459 & 0 & 0.0117604 \\ 0 & 0.0117604 & -0.0583459 \end{pmatrix}$$

$$\boldsymbol{k} = \begin{pmatrix} 3802.45 & 2087.5 & -3264.46 & 92.517 & 1907.89 & 224.764 & -2738.72 & -561.19 \\ 2087.5 & 3714.22 & -886.54 & 1648.7 & 652.115 & -247.873 & -1050.72 & -256.47 \\ -3264.46 & -886.54 & 6331.83 & -2794.42 & -3178.43 & 1107.13 & 4115.86 & -1139.23 \\ 92.517 & 1648.7 & -2794.42 & 4344.9 & 1231.49 & -1687.1 & -1424.13 & 1692.69 \\ 1907.89 & 652.115 & -3178.43 & 1231.49 & 1629.69 & -454.706 & -2144.4 & 425.98 \\ 224.764 & -247.873 & 1107.13 & -1687.1 & -454.706 & 704.922 & 488.162 & -771.42 \\ -2738.72 & -1050.72 & 4115.86 & -1424.13 & -2144.4 & 488.162 & 2854.88 & -406.31 \\ -561.192 & -256.471 & -1139.23 & 1692.69 & 425.983 & -771.421 & -406.311 & 920.97 \\ 2334.93 & 975.655 & -3197.82 & 974.762 & 1692.4 & -301.551 & -2278.36 & 203.47 \\ 644.631 & 645.197 & 657.753 & -935.202 & -205.225 & 490.159 & 141.292 & -655.35 \\ -5221.16 & -2369. & 6420.68 & -1618.09 & -3465.75 & 405.38 & 4729.67 & -119.67 \\ -1831.31 & -2443.34 & -735.358 & 899.706 & 88.3708 & -703.059 & 165.222 & 1160.85 \\ 440.362 & -459.73 & -3111.79 & 2113.73 & 1414.2 & -981.021 & -1684.04 & 1190.65 \\ -1218.1 & -3316.9 & 2651.43 & -4271. & -1312.05 & 1442.95 & 1680.18 & -1170.3 \\ 2738.72 & 1050.72 & -4115.86 & 1424.13 & 2144.4 & -488.162 & -2854.88 & 406.31 \\ 561.192 & 256.471 & 1139.23 & -1692.69 & -425.983 & 771.421 & 406.311 & -920.97 \end{pmatrix}$$

Computation of element matrices at {−0.774597, 0.774597} with weight = 0.308642

$$\boldsymbol{J} = \begin{pmatrix} 11.3876 & 0.977722 \\ -2.48537 & 4.85071 \end{pmatrix} \qquad \det J = 57.668$$

$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\} =$
$\{-0.1, 0.0450807, -0.032379, 0.0450807, -0.1, 0.354919, 0.432379, 0.354919\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s, \partial N_5/\partial s, \partial N_6/\partial s, \partial N_7/\partial s, \partial N_8/\partial s\}$ =
$\{-0.0436492, 0.174597, -0.130948, 0.2, -0.343649, 1.3746, -1.03095, -0.2\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t, \partial N_5/\partial t, \partial N_6/\partial t, \partial N_7/\partial t, \partial N_8/\partial t\}$ =
$\{0.343649, -0.2, 0.130948, -0.174597, 0.0436492, 0.2, 1.03095, -1.3746\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x, \partial N_5/\partial x, \partial N_6/\partial x, \partial N_7/\partial x, \partial N_8/\partial x\}$ =
$\{0.011139, 0.00606652, -0.00537101, 0.00929813, -0.0270246, 0.124243, -0.0422859, -0.0760651\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y, \partial N_5/\partial y, \partial N_6/\partial y, \partial N_7/\partial y, \partial N_8/\partial y\}$ =
$\{0.0686, -0.0424539, 0.0280782, -0.0378682, 0.0144457, 0.0161884, 0.221059, -0.268049\}$

$$\boldsymbol{B} = \begin{pmatrix} 0.011139 & 0 & 0.0686 \\ 0 & 0.0686 & 0.011139 \\ 0.00606652 & 0 & -0.0424539 \\ 0 & -0.0424539 & 0.00606652 \\ -0.00537101 & 0 & 0.0280782 \\ 0 & 0.0280782 & -0.00537101 \\ 0.00929813 & 0 & -0.0378682 \\ 0 & -0.0378682 & 0.00929813 \\ -0.0270246 & 0 & 0.0144457 \\ 0 & 0.0144457 & -0.0270246 \\ 0.124243 & 0 & 0.0161884 \\ 0 & 0.0161884 & 0.124243 \\ -0.0422859 & 0 & 0.221059 \\ 0 & 0.221059 & -0.0422859 \\ -0.0760651 & 0 & -0.268049 \\ 0 & -0.268049 & -0.0760651 \end{pmatrix}$$

$$
\boldsymbol{k} = \left(\begin{array}{ccccccc}
1055.65 & 392.327 & -549.535 & -60.2104 & 352.572 & 20.6795 & -459.056 \\
392.327 & 3408.11 & 31.0827 & -2079.5 & -49.2711 & 1372.23 & 109.865 \\
-549.535 & 31.0827 & 396.599 & -132.231 & -268.228 & 99.3017 & 370.71 \\
-60.2104 & -2079.5 & -132.231 & 1303.07 & 105.225 & -863.516 & -168.782 \\
352.572 & -49.2711 & -268.228 & 105.225 & 182.646 & -77.4287 & -254.261 \\
20.6795 & 1372.23 & 99.3017 & -863.516 & -77.4287 & 572.611 & 122.196 \\
-459.056 & 109.865 & 370.71 & -168.782 & -254.261 & 122.196 & 356.645 \\
1.05331 & -1845.99 & -151.837 & 1167.16 & 116.272 & -774.531 & -180.779 \\
-12.8613 & -538.055 & -243.792 & 371.43 & 187.633 & -249.687 & -292.962 \\
-331.165 & 650.486 & 262.618 & -474.49 & -179.737 & 321.359 & 251.548 \\
1222.84 & 2662.61 & 400.629 & -1604.7 & -386.311 & 1056.8 & 704.475 \\
1805.93 & 1082.46 & -1052.99 & -339.207 & 689.652 & 189.675 & -919.869 \\
2775.8 & -387.91 & -2111.75 & 828.434 & 1437.97 & -609.595 & -2001.79 \\
162.809 & 10803.5 & 781.801 & -6798.45 & -609.595 & 4508.16 & 962.045 \\
-4385.41 & -2220.65 & 2005.37 & 660.833 & -1252.02 & -362.264 & 1576.24 \\
-1991.43 & -13391.3 & 162.258 & 8084.93 & 4.88228 & -5325.98 & -176.224
\end{array}\right.
$$

Computation of element matrices at {0., −0.774597} with weight = 0.493827

$$\boldsymbol{J} = \begin{pmatrix} 3.06351 & 3.53553 \\ -3.06351 & 3.53553 \end{pmatrix} \qquad \det J = 21.6623$$

$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\} = \{-0.1, 0.887298, -0.1, 0.2, -0.1, 0.112702, -0.1, 0.2\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s, \partial N_5/\partial s, \partial N_6/\partial s, \partial N_7/\partial s, \partial N_8/\partial s\} =$
$\{-0.343649, 0., 0.343649, 0.2, -0.0436492, 0., 0.0436492, -0.2\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t, \partial N_5/\partial t, \partial N_6/\partial t, \partial N_7/\partial t, \partial N_8/\partial t\} =$
$\{-0.387298, -0.5, -0.387298, 0.774597, -0.387298, 0.5, -0.387298, 0.774597\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x, \partial N_5/\partial x, \partial N_6/\partial x, \partial N_7/\partial x, \partial N_8/\partial x\} =$
$\{-0.11086, -0.0707107, 0.00131526, 0.142187, -0.0618963, 0.0707107, -0.0476482, 0.0769022\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y, \partial N_5/\partial y, \partial N_6/\partial y, \partial N_7/\partial y, \partial N_8/\partial y\} =$
$\{0.00131526, -0.0707107, -0.11086, 0.0769022, -0.0476482, 0.0707107, -0.0618963, 0.142187\}$

$$\boldsymbol{B} = \begin{pmatrix} -0.11086 & 0 & 0.00131526 \\ 0 & 0.00131526 & -0.11086 \\ -0.0707107 & 0 & -0.0707107 \\ 0 & -0.0707107 & -0.0707107 \\ 0.00131526 & 0 & -0.11086 \\ 0 & -0.11086 & 0.00131526 \\ 0.142187 & 0 & 0.0769022 \\ 0 & 0.0769022 & 0.142187 \\ -0.0618963 & 0 & -0.0476482 \\ 0 & -0.0476482 & -0.0618963 \\ 0.0707107 & 0 & 0.0707107 \\ 0 & 0.0707107 & 0.0707107 \\ -0.0476482 & 0 & -0.0618963 \\ 0 & -0.0618963 & -0.0476482 \\ 0.0769022 & 0 & 0.142187 \\ 0 & 0.142187 & 0.0769022 \end{pmatrix}$$

$$\boldsymbol{k} = \begin{pmatrix} 5309.58 & -44.9939 & 3375.04 & 1439.89 & -80.9891 & 2275.66 & -6797.22 & -155 \\ -44.9939 & 1517.71 & 950.359 & 927.399 & 1517.28 & -80.9891 & -1017.68 & -190 \\ 3375.04 & 950.359 & 2777.21 & 1542.9 & 927.399 & 1439.89 & -5014.69 & -224 \\ 1439.89 & 927.399 & 1542.9 & 2777.21 & 950.359 & 3375.04 & -2532.7 & -359 \\ -80.9891 & 1517.28 & 927.399 & 950.359 & 1517.71 & -44.9939 & -971.508 & -192 \\ 2275.66 & -80.9891 & 1439.89 & 3375.04 & -44.9939 & 5309.58 & -2905.96 & -365 \\ -6797.22 & -1017.68 & -5014.69 & -2532.7 & -971.508 & -2905.96 & 9463.98 & 337 \\ -1555.37 & -1901.93 & -2247.8 & -3590.19 & -1926.9 & -3659.97 & 3374.16 & 505 \\ 2956.64 & 636.927 & 2306.67 & 1226.21 & 616.83 & 1262.71 & -4254.35 & -171 \\ 967.951 & 819.891 & 1164.03 & 1995.78 & 835.362 & 2271.95 & -1841.9 & -266 \\ -3375.04 & -950.359 & -2777.21 & -1542.9 & -927.399 & -1439.89 & 5014.69 & 224 \\ -1439.89 & -927.399 & -1542.9 & -2777.21 & -950.359 & -3375.04 & 2532.7 & 359 \\ 2271.95 & 835.362 & 1995.78 & 1164.03 & 819.891 & 967.951 & -3514.38 & -176 \\ 1262.71 & 616.83 & 1226.21 & 2306.67 & 636.927 & 2956.64 & -2081.74 & -289 \\ -3659.97 & -1926.9 & -3590.19 & -2247.8 & -1901.93 & -1555.37 & 6073.48 & 359 \\ -2905.96 & -971.508 & -2532.7 & -5014.69 & -1017.68 & -6797.22 & 4473.12 & 607 \end{pmatrix}$$

Computation of element matrices at {0., 0.} with weight = 0.790123

$$\boldsymbol{J} = \begin{pmatrix} 5. & 3.53553 \\ -5. & 3.53553 \end{pmatrix} \qquad \text{detJ} = 35.3553$$

$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\} = \{-0.25, 0.5, -0.25, 0.5, -0.25, 0.5, -0.25, 0.5\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s, \partial N_5/\partial s, \partial N_6/\partial s, \partial N_7/\partial s, \partial N_8/\partial s\} = \{0., 0., 0., 0.5, 0., 0., 0., -0.5\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t, \partial N_5/\partial t, \partial N_6/\partial t, \partial N_7/\partial t, \partial N_8/\partial t\} = \{0., -0.5, 0., 0., 0., 0.5, 0., 0.\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x, \partial N_5/\partial x, \partial N_6/\partial x, \partial N_7/\partial x, \partial N_8/\partial x\} = \{0., -0.0707107, 0., 0.05, 0., 0.0707107, 0., -0.05\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y, \partial N_5/\partial y, \partial N_6/\partial y, \partial N_7/\partial y, \partial N_8/\partial y\} = \{0., -0.0707107, 0., -0.05, 0., 0.0707107, 0., 0.05\}$

$$\boldsymbol{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.0707107 & 0 & -0.0707107 \\ 0 & -0.0707107 & -0.0707107 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.05 & 0 & -0.05 \\ 0 & -0.05 & 0.05 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.0707107 & 0 & 0.0707107 \\ 0 & 0.0707107 & 0.0707107 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.05 & 0 & 0.05 \\ 0 & 0.05 & -0.05 \end{pmatrix}$$

$$\boldsymbol{k} = \left(\begin{array}{cccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 7252.38 & 4029.1 & 0 & 0 & -2849. & 569.801 & 0 & 0 & -7252.38 & -4029.1 & 0 & 0 \\
0 & 0 & 4029.1 & 7252.38 & 0 & 0 & -569.801 & 2849. & 0 & 0 & -4029.1 & -7252.38 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2849. & -569.801 & 0 & 0 & 3626.19 & -2014.55 & 0 & 0 & 2849. & 569.801 & 0 & 0 \\
0 & 0 & 569.801 & 2849. & 0 & 0 & -2014.55 & 3626.19 & 0 & 0 & -569.801 & -2849. & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -7252.38 & -4029.1 & 0 & 0 & 2849. & -569.801 & 0 & 0 & 7252.38 & 4029.1 & 0 & 0 \\
0 & 0 & -4029.1 & -7252.38 & 0 & 0 & 569.801 & -2849. & 0 & 0 & 4029.1 & 7252.38 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2849. & 569.801 & 0 & 0 & -3626.19 & 2014.55 & 0 & 0 & -2849. & -569.801 & 0 & 0 \\
0 & 0 & -569.801 & -2849. & 0 & 0 & 2014.55 & -3626.19 & 0 & 0 & 569.801 & 2849. & 0 & 0
\end{array}\right.$$

Computation of element matrices at {0., 0.774597} with weight = 0.493827

$$\boldsymbol{J} = \begin{pmatrix} 6.93649 & 3.53553 \\ -6.93649 & 3.53553 \end{pmatrix} \qquad \det J = 49.0484$$

$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\} = \{-0.1, 0.112702, -0.1, 0.2, -0.1, 0.887298, -0.1, 0.2\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s, \partial N_5/\partial s, \partial N_6/\partial s, \partial N_7/\partial s, \partial N_8/\partial s\} =$
$\{0.0436492, 0., -0.0436492, 0.2, 0.343649, 0., -0.343649, -0.2\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t, \partial N_5/\partial t, \partial N_6/\partial t, \partial N_7/\partial t, \partial N_8/\partial t\} =$
$\{0.387298, -0.5, 0.387298, -0.774597, 0.387298, 0.5, 0.387298, -0.774597\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x, \partial N_5/\partial x, \partial N_6/\partial x, \partial N_7/\partial x, \partial N_8/\partial x\} =$
$\{0.0579186, -0.0707107, 0.0516259, -0.095128, 0.0795434, 0.0707107, 0.0300011, -0.123961\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y, \partial N_5/\partial y, \partial N_6/\partial y, \partial N_7/\partial y, \partial N_8/\partial y\} =$
$\{0.0516259, -0.0707107, 0.0579186, -0.123961, 0.0300011, 0.0707107, 0.0795434, -0.095128\}$

$$
\boldsymbol{B} = \begin{pmatrix}
0.0579186 & 0 & 0.0516259 \\
0 & 0.0516259 & 0.0579186 \\
-0.0707107 & 0 & -0.0707107 \\
0 & -0.0707107 & -0.0707107 \\
0.0516259 & 0 & 0.0579186 \\
0 & 0.0579186 & 0.0516259 \\
-0.095128 & 0 & -0.123961 \\
0 & -0.123961 & -0.095128 \\
0.0795434 & 0 & 0.0300011 \\
0 & 0.0300011 & 0.0795434 \\
0.0707107 & 0 & 0.0707107 \\
0 & 0.0707107 & 0.0707107 \\
0.0300011 & 0 & 0.0795434 \\
0 & 0.0795434 & 0.0300011 \\
-0.123961 & 0 & -0.095128 \\
0 & -0.095128 & -0.123961
\end{pmatrix}
$$

$$
\boldsymbol{k} = \begin{pmatrix}
4026.22 & 2089.17 & -5026.31 & -2737.12 & 3760.5 & 2151.17 & -7177.97 & -4382.37 & 49 \\
2089.17 & 3544.59 & -2674.95 & -4715.42 & 2054.84 & 3760.5 & -4065.36 & -7799.75 & 22 \\
-5026.31 & -2674.95 & 6288.26 & 3493.48 & -4715.42 & -2737.12 & 9029.47 & 5554.52 & -60 \\
-2737.12 & -4715.42 & 3493.48 & 6288.26 & -2674.95 & -5026.31 & 5269.62 & 10454. & -29 \\
3760.5 & 2054.84 & -4715.42 & -2674.95 & 3544.59 & 2089.17 & -6810.43 & -4222.66 & 45 \\
2151.17 & 3760.5 & -2737.12 & -5026.31 & 2089.17 & 4026.22 & -4098.3 & -8395.48 & 23 \\
-7177.97 & -4065.36 & 9029.47 & 5269.62 & -6810.43 & -4098.3 & 13146.4 & 8239.13 & -84 \\
-4382.37 & -7799.75 & 5554.52 & 10454. & -4222.66 & -8395.48 & 8239.13 & 17560. & -49 \\
4939.35 & 2207.14 & -6094.68 & -2950.8 & 4502.5 & 2364.22 & -8441.01 & -4931.21 & 64 \\
1876.12 & 2802.6 & -2461.27 & -3647.04 & 1936.87 & 2847.37 & -3952.16 & -5752.56 & 16 \\
5026.31 & 2674.95 & -6288.26 & -3493.48 & 4715.42 & 2737.12 & -9029.47 & -5554.52 & 60 \\
2737.12 & 4715.42 & -3493.48 & -6288.26 & 2674.95 & 5026.31 & -5269.62 & -10454. & 29 \\
2847.37 & 1936.87 & -3647.04 & -2461.27 & 2802.6 & 1876.12 & -5547.39 & -3673.81 & 30 \\
2364.22 & 4502.5 & -2950.8 & -6094.68 & 2207.14 & 4939.35 & -4211.5 & -10442.7 & 29 \\
-8395.48 & -4222.66 & 10454. & 5554.52 & -7799.75 & -4382.37 & 14830.4 & 8970.92 & -104 \\
-4098.3 & -6810.43 & 5269.62 & 9029.47 & -4065.36 & -7177.97 & 8088.19 & 14830.4 & -42
\end{pmatrix}
$$

Computation of element matrices at {0.774597, −0.774597} with weight = 0.308642

$$
\boldsymbol{J} = \begin{pmatrix} 1.09766 & 4.85071 \\ -5.02935 & 0.977722 \end{pmatrix} \qquad \det J = 25.4691
$$

$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\} =$
$\{-0.1, 0.354919, 0.432379, 0.354919, -0.1, 0.0450807, -0.032379, 0.0450807\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s, \partial N_5/\partial s, \partial N_6/\partial s, \partial N_7/\partial s, \partial N_8/\partial s\} =$
$\{0.343649, -1.3746, 1.03095, 0.2, 0.0436492, -0.174597, 0.130948, -0.2\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t, \partial N_5/\partial t, \partial N_6/\partial t, \partial N_7/\partial t, \partial N_8/\partial t\} =$
$\{-0.0436492, -0.2, -1.03095, 1.3746, -0.343649, 0.2, -0.130948, 0.174597\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x, \partial N_5/\partial x, \partial N_6/\partial x, \partial N_7/\partial x, \partial N_8/\partial x\} =$
$\{0.00457284, -0.0922625, -0.164003, 0.279117, -0.0661843, 0.0327912, -0.0208311, 0.0267997\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y, \partial N_5/\partial y, \partial N_6/\partial y, \partial N_7/\partial y, \partial N_8/\partial y\} =$
$\{-0.0673307, 0.253178, -0.24078, 0.0211513, -0.0231237, 0.0418723, -0.0305831, 0.0456156\}$

$$\boldsymbol{B} = \begin{pmatrix} 0.00457284 & 0 & -0.0673307 \\ 0 & -0.0673307 & 0.00457284 \\ -0.0922625 & 0 & 0.253178 \\ 0 & 0.253178 & -0.0922625 \\ -0.164003 & 0 & -0.24078 \\ 0 & -0.24078 & -0.164003 \\ 0.279117 & 0 & 0.0211513 \\ 0 & 0.0211513 & 0.279117 \\ -0.0661843 & 0 & -0.0231237 \\ 0 & -0.0231237 & -0.0661843 \\ 0.0327912 & 0 & 0.0418723 \\ 0 & 0.0418723 & 0.0327912 \\ -0.0208311 & 0 & -0.0305831 \\ 0 & -0.0305831 & -0.0208311 \\ 0.0267997 & 0 & 0.0456156 \\ 0 & 0.0456156 & 0.0267997 \end{pmatrix}$$

$$\boldsymbol{k} = \left(\begin{array}{ccccccc}
417.829 & -69.8162 & -1680.1 & 720.964 & 1232.37 & 851.771 & 276.018 \\
-69.8162 & 1441.06 & 950.184 & -5449.85 & 1402.49 & 5078.55 & -2548.09 \\
-1680.1 & 950.184 & 8516.24 & -5296.74 & -725.672 & -743.723 & -7689.46 \\
720.964 & -5449.85 & -5296.74 & 21120.9 & -3634.26 & -17979.8 & 9437.39 \\
1232.37 & 1402.49 & -725.672 & -3634.26 & 13797.1 & 8954.26 & -14993.9 \\
851.771 & 5078.55 & -743.723 & -17979.8 & 8954.26 & 20844.2 & -9458.2 \\
276.018 & -2548.09 & -7689.46 & 9437.39 & -14993.9 & -9458.2 & 24772.6 \\
-1691.42 & -336.332 & 6144.09 & -635.765 & -6567.66 & -5768.74 & 1338.69 \\
45.1387 & 596.693 & 1407.49 & -2086.26 & 3950.82 & 2512.1 & -5908.81 \\
389.803 & 466.809 & -1229.58 & -1304.67 & 1961.38 & 2752.03 & -1005.09 \\
-208.113 & -283.018 & 1.11193 & 779.112 & -2621.7 & -1697.07 & 2985.89 \\
-174.206 & -881.402 & 227.405 & 3091. & -1650.44 & -3688.39 & 1653. \\
156.532 & 178.14 & -92.1724 & -461.612 & 1752.46 & 1137.34 & -1904.47 \\
108.189 & 645.061 & -94.4652 & -2283.74 & 1137.34 & 2647.56 & -1201.35 \\
-239.671 & -226.58 & 262.562 & 541.403 & -2391.51 & -1556.48 & 2462.17 \\
-135.287 & -963.9 & 42.8277 & 3442.01 & -1603.11 & -3885.39 & 1783.66
\end{array}\right.$$

Computation of element matrices at {0.774597, 0.} with weight = 0.493827

$$\boldsymbol{J} = \begin{pmatrix} 1.79152 & 4.85071 \\ -8.20848 & 0.977722 \end{pmatrix} \qquad \det J = 41.5685$$

$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\} = \{-0.1, 0.2, -0.1, 0.887298, -0.1, 0.2, -0.1, 0.112702\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s, \partial N_5/\partial s, \partial N_6/\partial s, \partial N_7/\partial s, \partial N_8/\partial s\} =$
$\{0.387298, -0.774597, 0.387298, 0.5, 0.387298, -0.774597, 0.387298, -0.5\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t, \partial N_5/\partial t, \partial N_6/\partial t, \partial N_7/\partial t, \partial N_8/\partial t\} =$
$\{0.0436492, -0.2, -0.343649, 0., 0.343649, 0.2, -0.0436492, 0.\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x, \partial N_5/\partial x, \partial N_6/\partial x, \partial N_7/\partial x, \partial N_8/\partial x\} =$
$\{0.0177289, -0.0577128, -0.0587504, 0.0117604, 0.0769695, 0.0212747, 0.000490192, -0.0117604\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y, \partial N_5/\partial y, \partial N_6/\partial y, \partial N_7/\partial y, \partial N_8/\partial y\} =$
$\{-0.0433133, 0.0817695, -0.0600051, -0.0583459, -0.030384, 0.0990086, -0.0470757, 0.0583459\}$

$$\boldsymbol{B} = \begin{pmatrix} 0.0177289 & 0 & -0.0433133 \\ 0 & -0.0433133 & 0.0177289 \\ -0.0577128 & 0 & 0.0817695 \\ 0 & 0.0817695 & -0.0577128 \\ -0.0587504 & 0 & -0.0600051 \\ 0 & -0.0600051 & -0.0587504 \\ 0.0117604 & 0 & -0.0583459 \\ 0 & -0.0583459 & 0.0117604 \\ 0.0769695 & 0 & -0.030384 \\ 0 & -0.030384 & 0.0769695 \\ 0.0212747 & 0 & 0.0990086 \\ 0 & 0.0990086 & 0.0212747 \\ 0.000490192 & 0 & -0.0470757 \\ 0 & -0.0470757 & 0.000490192 \\ -0.0117604 & 0 & 0.0583459 \\ 0 & 0.0583459 & -0.0117604 \end{pmatrix}$$

$$\boldsymbol{k} = \begin{pmatrix} 704.922 & -454.706 & -1687.1 & 1107.13 & -247.873 & 224.764 & 771.421 & -488.16 \\ -454.706 & 1629.69 & 1231.49 & -3178.43 & 652.115 & 1907.89 & -425.983 & 2144.4 \\ -1687.1 & 1231.49 & 4344.9 & -2794.42 & 1648.7 & 92.517 & -1692.69 & 1424.13 \\ 1107.13 & -3178.43 & -2794.42 & 6331.83 & -886.54 & -3264.46 & 1139.23 & -4115.86 \\ -247.873 & 652.115 & 1648.7 & -886.54 & 3714.22 & 2087.5 & 256.471 & 1050.72 \\ 224.764 & 1907.89 & 92.517 & -3264.46 & 2087.5 & 3802.45 & 561.192 & 2738.72 \\ 771.421 & -425.983 & -1692.69 & 1139.23 & 256.471 & 561.192 & 920.977 & -406.31 \\ -488.162 & 2144.4 & 1424.13 & -4115.86 & 1050.72 & 2738.72 & -406.311 & 2854.88 \\ 1442.95 & -1312.05 & -4271. & 2651.43 & -3316.9 & -1218.1 & 1170.3 & -1680.18 \\ -981.021 & 1414.2 & 2113.73 & -3111.79 & -459.73 & 440.362 & -1190.65 & 1684.04 \\ -703.059 & 88.3708 & 899.706 & -735.358 & -2443.34 & -1831.31 & -1160.85 & -165.22 \\ 405.38 & -3465.75 & -1618.09 & 6420.68 & -2369. & -5221.16 & 119.679 & -4729.67 \\ 490.159 & -205.225 & -935.202 & 657.753 & 645.197 & 644.631 & 655.35 & -141.29 \\ -301.551 & 1692.4 & 974.762 & -3197.82 & 975.655 & 2334.93 & -203.471 & 2278.36 \\ -771.421 & 425.983 & 1692.69 & -1139.23 & -256.471 & -561.192 & -920.977 & 406.31 \\ 488.162 & -2144.4 & -1424.13 & 4115.86 & -1050.72 & -2738.72 & 406.311 & -2854.88 \end{pmatrix}$$

Computation of element matrices at {0.774597, 0.774597} with weight = 0.308642

$$\boldsymbol{J} = \begin{pmatrix} 2.48537 & 4.85071 \\ -11.3876 & 0.977722 \end{pmatrix} \qquad \det J = 57.668$$

$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\} =$
$\{-0.032379, 0.0450807, -0.1, 0.354919, 0.432379, 0.354919, -0.1, 0.0450807\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s, \partial N_5/\partial s, \partial N_6/\partial s, \partial N_7/\partial s, \partial N_8/\partial s\} =$
$\{0.130948, -0.174597, 0.0436492, 0.2, 1.03095, -1.3746, 0.343649, -0.2\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t, \partial N_5/\partial t, \partial N_6/\partial t, \partial N_7/\partial t, \partial N_8/\partial t\} =$
$\{0.130948, -0.2, 0.343649, -1.3746, 1.03095, 0.2, 0.0436492, -0.174597\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x, \partial N_5/\partial x, \partial N_6/\partial x, \partial N_7/\partial x, \partial N_8/\partial x\} =$
$\{0.0280782, -0.0424539, 0.0686, -0.268049, 0.221059, 0.0161884, 0.0144457, -0.0378682\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y, \partial N_5/\partial y, \partial N_6/\partial y, \partial N_7/\partial y, \partial N_8/\partial y\} =$
$\{-0.00537101, 0.00606652, 0.011139, -0.0760651, -0.0422859, 0.124243, -0.0270246, 0.00929813\}$

$$\boldsymbol{B} = \begin{pmatrix} 0.0280782 & 0 & -0.00537101 \\ 0 & -0.00537101 & 0.0280782 \\ -0.0424539 & 0 & 0.00606652 \\ 0 & 0.00606652 & -0.0424539 \\ 0.0686 & 0 & 0.011139 \\ 0 & 0.011139 & 0.0686 \\ -0.268049 & 0 & -0.0760651 \\ 0 & -0.0760651 & -0.268049 \\ 0.221059 & 0 & -0.0422859 \\ 0 & -0.0422859 & 0.221059 \\ 0.0161884 & 0 & 0.124243 \\ 0 & 0.124243 & 0.0161884 \\ 0.0144457 & 0 & -0.0270246 \\ 0 & -0.0270246 & 0.0144457 \\ -0.0378682 & 0 & 0.00929813 \\ 0 & 0.00929813 & -0.0378682 \end{pmatrix}$$

$$
\boldsymbol{k} = \left(\begin{array}{lllllll}
572.611 & -77.4287 & -863.516 & 99.3017 & 1372.23 & 20.6795 & -5325.98 \\
-77.4287 & 182.646 & 105.225 & -268.228 & -49.2711 & 352.572 & 4.882 \\
-863.516 & 105.225 & 1303.07 & -132.231 & -2079.5 & -60.2104 & 8084.93 \\
99.3017 & -268.228 & -132.231 & 396.599 & 31.0827 & -549.535 & 162.258 \\
1372.23 & -49.2711 & -2079.5 & 31.0827 & 3408.11 & 392.327 & -13391.3 \\
20.6795 & 352.572 & -60.2104 & -549.535 & 392.327 & 1055.65 & -1991.43 \\
-5325.98 & 4.88228 & 8084.93 & 162.258 & -13391.3 & -1991.43 & 52833.9 \\
-362.264 & -1252.02 & 660.833 & 2005.37 & -2220.65 & -4385.41 & 10468.3 \\
4508.16 & -609.595 & -6798.45 & 781.801 & 10803.5 & 162.809 & -41931.4 \\
-609.595 & 1437.97 & 828.434 & -2111.75 & -387.91 & 2775.8 & 38.438 \\
189.675 & 689.652 & -339.207 & -1052.99 & 1082.46 & 1805.93 & -5059.91 \\
1056.8 & -386.311 & -1604.7 & 400.629 & 2662.61 & 1222.84 & -10512.1 \\
321.359 & -179.737 & -474.49 & 262.618 & 650.486 & -331.165 & -2361.12 \\
-249.687 & 187.633 & 371.43 & -243.792 & -538.055 & -12.8613 & 2005.87 \\
-774.531 & 116.272 & 1167.16 & -151.837 & -1845.99 & 1.05331 & 7150.92 \\
122.196 & -254.261 & -168.782 & 370.71 & 109.865 & -459.056 & -176.224
\end{array}\right.
$$

Summing contributions from all points we get

$$
\boldsymbol{k} = \left(\begin{array}{llllllll}
36733.5 & 12876.3 & -27675.8 & -81.2534 & 13375.3 & 6621.25 & -25336.9 & -1059 \\
12876.3 & 29235.1 & -3927.41 & -13841. & 7582.79 & 13375.3 & -10596.2 & -1363 \\
-27675.8 & -3927.41 & 58331.4 & -7381.3 & -13841. & -81.2534 & 7797.13 & 1135 \\
-81.2534 & -13841. & -7381.3 & 58331.4 & -3927.41 & -27675.8 & 11355.9 & 1008 \\
13375.3 & 7582.79 & -13841. & -3927.41 & 29235.1 & 12876.3 & -39273.3 & -1357 \\
6621.25 & 13375.3 & -81.2534 & -27675.8 & 12876.3 & 36733.5 & -17417.6 & -2125 \\
-25336.9 & -10596.2 & 7797.13 & 11355.9 & -39273.3 & -17417.6 & 108701. & 2068 \\
-10596.2 & -13639.6 & 11355.9 & 10088.9 & -13571.5 & -21256.5 & 20689.5 & 5760 \\
18861.9 & 3094.06 & -19175.3 & 874.112 & 19081.9 & 4640.68 & -62430.1 & -1160 \\
3094.06 & 9989.62 & 874.112 & -9681.35 & 3679.14 & 12055.6 & -7762.13 & -1982 \\
-6756.94 & 862.763 & -5843.92 & -13070.1 & -4928.03 & 862.763 & 1719.55 & -1135 \\
862.763 & -4928.03 & -13070.1 & -5843.92 & 862.763 & -6756.94 & -11355.9 & -1960 \\
12055.6 & 3679.14 & -9681.35 & 874.112 & 9989.62 & 3094.06 & -16853.6 & -351 \\
4640.68 & 19081.9 & 874.112 & -19175.3 & 3094.06 & 18861.9 & -3516.93 & -1903 \\
-21256.5 & -13571.5 & 10088.9 & 11355.9 & -13639.6 & -10596.2 & 25676. & 1860 \\
-17417.6 & -39273.3 & 11355.9 & 7797.13 & -10596.2 & -25336.9 & 18603.3 & 2567
\end{array}\right.
$$

$\boldsymbol{r}^{\mathrm{T}} = (\,0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\,)$

Computation of element matrices resulting from NBC

NBC on side 1 with $\{q_n, q_t\} = \{-20, 0\}$

$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\}_c = \left\{\frac{1-a}{2} + \frac{1}{2}\left(a^2-1\right), 1-a^2, \frac{a+1}{2} + \frac{1}{2}\left(a^2-1\right), 0, 0, 0, 0, 0\right\}$

$x(a) = -1.03553\,a^2 + 2.5\,a + 3.53553;$ $\quad y(a) = -1.03553\,a^2 - 2.5\,a + 3.53553$

$dx/da = 2.5 - 2.07107\,a;$ $\quad dy/da = -2.07107\,a - 2.5$

$J_c = \sqrt{(-2.07107\,a - 2.5)^2 + (2.07107\,a - 2.5)^2}$

Gauss point = −0.774597; Weight = 0.555556; $J_c = 4.20086$

$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\}_c = \{0.687298, 0.4, -0.0872983, 0, 0, 0, 0, 0\}$

$\boldsymbol{r}_q^T = (\,6.84059 \quad 31.3427 \quad 3.98115 \quad 18.2411 \quad -0.868868 \quad -3.98104 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\,)$

Gauss point = 0.; Weight = 0.888889; $J_c = 3.53553$

$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\}_c = \{0., 1., 0., 0, 0, 0, 0, 0\}$

$\boldsymbol{r}_q^T = (\,0 \quad 0 \quad 44.4444 \quad 44.4444 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\,)$

Gauss point = 0.774597; Weight = 0.555556; $J_c = 4.20086$

$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\}_c = \{-0.0872983, 0.4, 0.687298, 0, 0, 0, 0, 0\}$

$\boldsymbol{r}_q^T = (\,-3.98104 \quad -0.868868 \quad 18.2411 \quad 3.98115 \quad 31.3427 \quad 6.84059 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\,)$

Summing contributions from all Gauss points

$\boldsymbol{r}_q^T = (\,2.85955 \quad 30.4738 \quad 66.6667 \quad 66.6667 \quad 30.4738 \quad 2.85955 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\,)$

Complete element equations for element 1

$$\left(\begin{array}{cccccccc}
36733.5 & 12876.3 & -27675.8 & -81.2534 & 13375.3 & 6621.25 & -25336.9 & -10596.2 \\
12876.3 & 29235.1 & -3927.41 & -13841. & 7582.79 & 13375.3 & -10596.2 & -13639.6 \\
-27675.8 & -3927.41 & 58331.4 & -7381.3 & -13841. & -81.2534 & 7797.13 & 11355.9 \\
-81.2534 & -13841. & -7381.3 & 58331.4 & -3927.41 & -27675.8 & 11355.9 & 10088.9 \\
13375.3 & 7582.79 & -13841. & -3927.41 & 29235.1 & 12876.3 & -39273.3 & -13571.5 \\
6621.25 & 13375.3 & -81.2534 & -27675.8 & 12876.3 & 36733.5 & -17417.6 & -21256.5 \\
-25336.9 & -10596.2 & 7797.13 & 11355.9 & -39273.3 & -17417.6 & 108701. & 20689.5 \\
-10596.2 & -13639.6 & 11355.9 & 10088.9 & -13571.5 & -21256.5 & 20689.5 & 57600.7 \\
18861.9 & 3094.06 & -19175.3 & 874.112 & 19081.9 & 4640.68 & -62430.1 & -11608.3 \\
3094.06 & 9989.62 & 874.112 & -9681.35 & 3679.14 & 12055.6 & -7762.13 & -19829.7 \\
-6756.94 & 862.763 & -5843.92 & -13070.1 & -4928.03 & 862.763 & 1719.55 & -11355.9 \\
862.763 & -4928.03 & -13070.1 & -5843.92 & 862.763 & -6756.94 & -11355.9 & -19605.6 \\
12055.6 & 3679.14 & -9681.35 & 874.112 & 9989.62 & 3094.06 & -16853.6 & -3516.93 \\
4640.68 & 19081.9 & 874.112 & -19175.3 & 3094.06 & 18861.9 & -3516.93 & -19034.2 \\
-21256.5 & -13571.5 & 10088.9 & 11355.9 & -13639.6 & -10596.2 & 25676. & 18603.3 \\
-17417.6 & -39273.3 & 11355.9 & 7797.13 & -10596.2 & -25336.9 & 18603.3 & 25676.
\end{array}\right.$$

The element contributes to {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \end{pmatrix}$

and to a global matrix: $$\begin{pmatrix}
[1, 1] & [1, 2] & [1, 3] & [1, 4] & [1, 5] & [1, 6] & [1, 7] & [1, 8] & [1, 9] \\
[2, 1] & [2, 2] & [2, 3] & [2, 4] & [2, 5] & [2, 6] & [2, 7] & [2, 8] & [2, 9] \\
[3, 1] & [3, 2] & [3, 3] & [3, 4] & [3, 5] & [3, 6] & [3, 7] & [3, 8] & [3, 9] \\
[4, 1] & [4, 2] & [4, 3] & [4, 4] & [4, 5] & [4, 6] & [4, 7] & [4, 8] & [4, 9] \\
[5, 1] & [5, 2] & [5, 3] & [5, 4] & [5, 5] & [5, 6] & [5, 7] & [5, 8] & [5, 9] \\
[6, 1] & [6, 2] & [6, 3] & [6, 4] & [6, 5] & [6, 6] & [6, 7] & [6, 8] & [6, 9] \\
[7, 1] & [7, 2] & [7, 3] & [7, 4] & [7, 5] & [7, 6] & [7, 7] & [7, 8] & [7, 9] \\
[8, 1] & [8, 2] & [8, 3] & [8, 4] & [8, 5] & [8, 6] & [8, 7] & [8, 8] & [8, 9] \\
[9, 1] & [9, 2] & [9, 3] & [9, 4] & [9, 5] & [9, 6] & [9, 7] & [9, 8] & [9, 9] \\
[10, 1] & [10, 2] & [10, 3] & [10, 4] & [10, 5] & [10, 6] & [10, 7] & [10, 8] & [10, 9] \\
[11, 1] & [11, 2] & [11, 3] & [11, 4] & [11, 5] & [11, 6] & [11, 7] & [11, 8] & [11, 9] \\
[12, 1] & [12, 2] & [12, 3] & [12, 4] & [12, 5] & [12, 6] & [12, 7] & [12, 8] & [12, 9] \\
[13, 1] & [13, 2] & [13, 3] & [13, 4] & [13, 5] & [13, 6] & [13, 7] & [13, 8] & [13, 9] \\
[14, 1] & [14, 2] & [14, 3] & [14, 4] & [14, 5] & [14, 6] & [14, 7] & [14, 8] & [14, 9] \\
[15, 1] & [15, 2] & [15, 3] & [15, 4] & [15, 5] & [15, 6] & [15, 7] & [15, 8] & [15, 9] \\
[16, 1] & [16, 2] & [16, 3] & [16, 4] & [16, 5] & [16, 6] & [16, 7] & [16, 8] & [16, 9]
\end{pmatrix}$$

Adding element equations into appropriate locations we have

$$\begin{pmatrix}
36733.5 & 12876.3 & -27675.8 & -81.2534 & 13375.3 & 6621.25 & -25336.9 & -10596.2 \\
12876.3 & 29235.1 & -3927.41 & -13841. & 7582.79 & 13375.3 & -10596.2 & -13639.6 \\
-27675.8 & -3927.41 & 58331.4 & -7381.3 & -13841. & -81.2534 & 7797.13 & 11355.9 \\
-81.2534 & -13841. & -7381.3 & 58331.4 & -3927.41 & -27675.8 & 11355.9 & 10088.9 \\
13375.3 & 7582.79 & -13841. & -3927.41 & 29235.1 & 12876.3 & -39273.3 & -13571.5 \\
6621.25 & 13375.3 & -81.2534 & -27675.8 & 12876.3 & 36733.5 & -17417.6 & -21256.5 \\
-25336.9 & -10596.2 & 7797.13 & 11355.9 & -39273.3 & -17417.6 & 108701. & 20689.5 \\
-10596.2 & -13639.6 & 11355.9 & 10088.9 & -13571.5 & -21256.5 & 20689.5 & 57600.7 \\
18861.9 & 3094.06 & -19175.3 & 874.112 & 19081.9 & 4640.68 & -62430.1 & -11608.3 \\
3094.06 & 9989.62 & 874.112 & -9681.35 & 3679.14 & 12055.6 & -7762.13 & -19829.7 \\
-6756.94 & 862.763 & -5843.92 & -13070.1 & -4928.03 & 862.763 & 1719.55 & -11355.9 \\
862.763 & -4928.03 & -13070.1 & -5843.92 & 862.763 & -6756.94 & -11355.9 & -19605.6 \\
12055.6 & 3679.14 & -9681.35 & 874.112 & 9989.62 & 3094.06 & -16853.6 & -3516.93 \\
4640.68 & 19081.9 & 874.112 & -19175.3 & 3094.06 & 18861.9 & -3516.93 & -19034.2 \\
-21256.5 & -13571.5 & 10088.9 & 11355.9 & -13639.6 & -10596.2 & 25676. & 18603.3 \\
-17417.6 & -39273.3 & 11355.9 & 7797.13 & -10596.2 & -25336.9 & 18603.3 & 25676.
\end{pmatrix}$$

Essential boundary conditions

Node	dof	Value
1	u_1	0
3	v_3	0
4	v_4	0
5	v_5	0
7	u_7	0
8	u_8	0

Remove {1, 6, 8, 10, 13, 15} rows and columns.

After adjusting for essential boundary conditions we have

$$\left(\begin{array}{ccccccccc}
29235.1 & -3927.41 & -13841. & 7582.79 & -10596.2 & 3094.06 & 862.763 & -4928.03 & \\
-3927.41 & 58331.4 & -7381.3 & -13841. & 7797.13 & -19175.3 & -5843.92 & -13070.1 & \\
-13841. & -7381.3 & 58331.4 & -3927.41 & 11355.9 & 874.112 & -13070.1 & -5843.92 & - \\
7582.79 & -13841. & -3927.41 & 29235.1 & -39273.3 & 19081.9 & -4928.03 & 862.763 & \\
-10596.2 & 7797.13 & 11355.9 & -39273.3 & 108701. & -62430.1 & 1719.55 & -11355.9 & \\
3094.06 & -19175.3 & 874.112 & 19081.9 & -62430.1 & 53178.9 & -1545.83 & 11827.3 & \\
862.763 & -5843.92 & -13070.1 & -4928.03 & 1719.55 & -1545.83 & 42911. & 14248. & \\
-4928.03 & -13070.1 & -5843.92 & 862.763 & -11355.9 & 11827.3 & 14248. & 42911. & \\
19081.9 & 874.112 & -19175.3 & 3094.06 & -3516.93 & -1981.59 & 11827.3 & -1545.83 & \\
-39273.3 & 11355.9 & 7797.13 & -10596.2 & 18603.3 & -3516.93 & -11355.9 & 1719.55 & -
\end{array}\right.$$

Solving the final system of global equations we get

$$\{v_1 = 0.00494694,\ u_2 = 0.0033909,\ v_2 = 0.0033909,\ u_3 = 0.00494694,\ u_4 = 0.00271213,$$
$$u_5 = 0.00225656,\ u_6 = 0.00152523,\ v_6 = 0.00152523,\ v_7 = 0.00225656,\ v_8 = 0.00271213\}$$

Complete table of nodal values

	u	v
1	0	0.00494694
2	0.0033909	0.0033909
3	0.00494694	0
4	0.00271213	0
5	0.00225656	0
6	0.00152523	0.00152523
7	0	0.00225656
8	0	0.00271213

Computation of reactions

Equation numbers of dof with specified values: {1, 6, 8, 10, 13, 15}

Extracting equations {1, 6, 8, 10, 13, 15} from the global system we have

$$\begin{pmatrix} 36733.5 & 12876.3 & -27675.8 & -81.2534 & 13375.3 & 6621.25 & -25336.9 & -10596.2 & 188 \\ 6621.25 & 13375.3 & -81.2534 & -27675.8 & 12876.3 & 36733.5 & -17417.6 & -21256.5 & 46 \\ -10596.2 & -13639.6 & 11355.9 & 10088.9 & -13571.5 & -21256.5 & 20689.5 & 57600.7 & -116 \\ 3094.06 & 9989.62 & 874.112 & -9681.35 & 3679.14 & 12055.6 & -7762.13 & -19829.7 & -33 \\ 12055.6 & 3679.14 & -9681.35 & 874.112 & 9989.62 & 3094.06 & -16853.6 & -3516.93 & 110 \\ -21256.5 & -13571.5 & 10088.9 & 11355.9 & -13639.6 & -10596.2 & 25676. & 18603.3 & -190 \end{pmatrix}$$

Substituting the nodal values and re−arranging

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{pmatrix} = \begin{pmatrix} 36733.5 & 12876.3 & -27675.8 & -81.2534 & 13375.3 & 6621.25 & -25336.9 & -10596.2 \\ 6621.25 & 13375.3 & -81.2534 & -27675.8 & 12876.3 & 36733.5 & -17417.6 & -21256.5 \\ -10596.2 & -13639.6 & 11355.9 & 10088.9 & -13571.5 & -21256.5 & 20689.5 & 57600.7 \\ 3094.06 & 9989.62 & 874.112 & -9681.35 & 3679.14 & 12055.6 & -7762.13 & -19829.7 \\ 12055.6 & 3679.14 & -9681.35 & 874.112 & 9989.62 & 3094.06 & -16853.6 & -3516.9 \\ -21256.5 & -13571.5 & 10088.9 & 11355.9 & -13639.6 & -10596.2 & 25676. & 18603.3 \end{pmatrix}$$

Carrying out computations, the reactions are as follows.

Label	dof	Reaction
R_1	u_1	−39.0268
R_2	v_3	−39.0268
R_3	v_4	−52.5151
R_4	v_5	−8.4581
R_5	u_7	−8.4581
R_6	u_8	−52.5151

Sum of Reactions

dof: u	−100.
dof: v	−100.

Solution for element 1

Element nodal displacements

Element node	Global node number	u	v
1	1	0	0.00494694
2	2	0.0033909	0.0033909
3	3	0.00494694	0
4	4	0.00271213	0
5	5	0.00225656	0
6	6	0.00152523	0.00152523
7	7	0	0.00225656
8	8	0	0.00271213

$\boldsymbol{d}^T = ($ 0 0.00494694 0.0033909 0.0033909 0.00494694 0 0.00271213 0 0.00225656 0 0.00152523 0.(

$E = 30000; \qquad \nu = 0.3; \qquad h = 1$

$$\text{Plane strain } \boldsymbol{C} = \begin{pmatrix} 40384.6 & 17307.7 & 0 \\ 17307.7 & 40384.6 & 0 \\ 0 & 0 & 11538.5 \end{pmatrix}$$

Interpolation functions and their derivatives

$$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\} =$$

$$\left\{-\frac{1}{4}(s-1)(t-1)(s+t+1),\ \frac{1}{2}\left(s^2-1\right)(t-1),\ \frac{1}{4}(t-1)\left(-s^2+t\,s+t+1\right),\ -\frac{1}{2}(s+1)\left(t^2-1\right),\right.$$
$$\left.\frac{1}{4}(s+1)(t+1)(s+t-1),\ -\frac{1}{2}\left(s^2-1\right)(t+1),\ \frac{1}{4}(s-1)(s-t+1)(t+1),\ \frac{1}{2}(s-1)\left(t^2-1\right)\right\}$$

$$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s, \partial N_5/\partial s, \partial N_6/\partial s, \partial N_7/\partial s, \partial N_8/\partial s\} = \Big\{-\frac{1}{4}(t-1)(2s+t),$$
$$s(t-1), -\frac{1}{4}(2s-t)(t-1), \frac{1}{2}\left(1-t^2\right), \frac{1}{4}(t+1)(2s+t), -s(t+1), \frac{1}{4}(2s-t)(t+1), \frac{1}{2}\left(t^2-1\right)\Big\}$$

$$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s, \partial N_5/\partial s, \partial N_6/\partial s, \partial N_7/\partial s, \partial N_8/\partial s\} = \Big\{-\frac{1}{4}(s-1)(s+2t),$$
$$\frac{1}{2}\left(s^2-1\right), -\frac{1}{4}(s+1)(s-2t), -(s+1)t, \frac{1}{4}(s+1)(s+2t), \frac{1}{2}\left(1-s^2\right), \frac{1}{4}(s-1)(s-2t), (s-1)t\Big\}$$

Nodal coordinates

Element node	Global node number	x	y
1	1	0.	5.
2	2	3.53553	3.53553
3	3	5.	0.
4	4	10.	0.
5	5	15.	0.
6	6	10.6066	10.6066
7	7	0.	15.
8	8	0.	10.

Mapping to the master element

$$x(s,t) = 1.76777\left(1-s^2\right)(1-t) + 5.3033\left(1-s^2\right)(t+1) +$$
$$5.(s+1)\left(1-t^2\right) + 5.\left(\frac{1}{4}(s+1)(1-t) - \frac{1}{4}\left(1-s^2\right)(1-t) - \frac{1}{4}(s+1)\left(1-t^2\right)\right) +$$
$$15.\left(\frac{1}{4}(s+1)(t+1) - \frac{1}{4}\left(1-s^2\right)(t+1) - \frac{1}{4}(s+1)\left(1-t^2\right)\right)$$

$$y(s,t) = 1.76777\left(1-s^2\right)(1-t) + 5.3033\left(1-s^2\right)(t+1) +$$
$$5.(1-s)\left(1-t^2\right) + 5.\left(\frac{1}{4}(1-s)(1-t) - \frac{1}{4}\left(1-s^2\right)(1-t) - \frac{1}{4}(1-s)\left(1-t^2\right)\right) +$$
$$15.\left(\frac{1}{4}(1-s)(t+1) - \frac{1}{4}\left(1-s^2\right)(t+1) - \frac{1}{4}(1-s)\left(1-t^2\right)\right)$$

$$\boldsymbol{J} = \left(\begin{array}{c} -3.53553\,s\,(1-t) - 10.6066\,s\,(t+1) + 5.(1-t^2) + 5.\left(\frac{1}{2}s(1-t) + \frac{1-t}{4} + \frac{1}{4}(t^2-1)\right) + 15.\left(\frac{1}{2}s\,(\right. \\ -3.53553\,s\,(1-t) - 10.6066\,s\,(t+1) - 5.(1-t^2) + 5.\left(\frac{1}{2}s(1-t) + \frac{t-1}{4} + \frac{1}{4}(1-t^2)\right) + 15.\left(\frac{1}{4}(-t- \right. \end{array}$$

Element solution at $\{s, t\} = \{0, 0\} \Longrightarrow \{x, y\} = \{7.07107, 7.07107\}$

$$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\} = \left\{-\frac{1}{4}, \frac{1}{2}, -\frac{1}{4}, \frac{1}{2}, -\frac{1}{4}, \frac{1}{2}, -\frac{1}{4}, \frac{1}{2}\right\}$$

$$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s, \partial N_5/\partial s, \partial N_6/\partial s, \partial N_7/\partial s, \partial N_8/\partial s\} = \left\{0, 0, 0, \frac{1}{2}, 0, 0, 0, -\frac{1}{2}\right\}$$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t, \partial N_5/\partial t, \partial N_6/\partial t, \partial N_7/\partial t, \partial N_8/\partial t\} = \left\{0, -\frac{1}{2}, 0, 0, 0, \frac{1}{2}, 0, 0\right\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x, \partial N_5/\partial x, \partial N_6/\partial x, \partial N_7/\partial x, \partial N_8/\partial x\} =$
$\{0, -0.0707107, 0, 0.05, 0, 0.0707107, 0, -0.05\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y, \partial N_5/\partial y, \partial N_6/\partial y, \partial N_7/\partial y, \partial N_8/\partial y\} =$
$\{0, -0.0707107, 0, -0.05, 0, 0.0707107, 0, 0.05\}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & -0.0707107 & 0 & 0 & 0 & 0.05 & 0 & 0 & 0 & 0.0707107 & 0 & 0 & 0 & -0. \\ 0 & 0 & 0 & -0.0707107 & 0 & 0 & 0 & -0.05 & 0 & 0 & 0 & 0.0707107 & 0 & 0 & 0 \\ 0 & 0 & -0.0707107 & -0.0707107 & 0 & 0 & -0.05 & 0.05 & 0 & 0 & 0.0707107 & 0.0707107 & 0 & 0 & 0. \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{d} = (\,3.6836\times10^{-6} \quad 3.6836\times10^{-6} \quad -0.000535058\,)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\,0.212515 \quad 0.212515 \quad -6.17375\,)$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{\mathrm{T}} = (\,3.6836\times10^{-6} \quad 3.6836\times10^{-6} \quad 0 \quad -0.000535058 \quad 0 \quad 0\,)$

$\boldsymbol{\sigma}^{\mathrm{T}} = (\,0.212515 \quad 0.212515 \quad 0.127509 \quad -6.17375 \quad 0 \quad 0\,)$

Substituting these stress components into appropriate formulas

Principal stresses = $(\,6.38626 \quad 0.127509 \quad -5.96123\,)$

Effective stress (von Mises) = 10.6936

Element solution at $\{s, t\} = \{-1, -1\} \Longrightarrow \{x, y\} = \{0., 5.\}$

$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\} = \{1, 0, 0, 0, 0, 0, 0, 0\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s, \partial N_5/\partial s, \partial N_6/\partial s, \partial N_7/\partial s, \partial N_8/\partial s\} = \left\{-\frac{3}{2}, 2, -\frac{1}{2}, 0, 0, 0, 0, 0\right\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t, \partial N_5/\partial t, \partial N_6/\partial t, \partial N_7/\partial t, \partial N_8/\partial t\} = \left\{-\frac{3}{2}, 0, 0, 0, 0, 0, -\frac{1}{2}, 2\right\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x, \partial N_5/\partial x, \partial N_6/\partial x, \partial N_7/\partial x, \partial N_8/\partial x\} =$
$\{-0.356302, 0.437535, -0.109384, 0, 0, 0, -0.00938363, 0.0375345\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y, \partial N_5/\partial y, \partial N_6/\partial y, \partial N_7/\partial y, \partial N_8/\partial y\} =$
$\{-0.3, 0., 0., 0, 0, 0, -0.1, 0.4\}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -0.356302 & 0 & 0.437535 & 0 & -0.109384 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.00 \\ 0 & -0.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.3 & -0.356302 & 0 & 0.437535 & 0 & -0.109384 & 0 & 0 & 0 & 0 & 0 & 0 & -0.1 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{d} = (\,0.00094252 \quad -0.000624887 \quad -0.000198345\,)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C\epsilon} = (\ 27.248 \quad -8.92296 \quad -2.2886\ )$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{\mathrm{T}} = (\ 0.00094252 \quad -0.000624887 \quad 0 \quad -0.000198345 \quad 0 \quad 0\ )$

$\boldsymbol{\sigma}^{\mathrm{T}} = (\ 27.248 \quad -8.92296 \quad 5.4975 \quad -2.2886 \quad 0 \quad 0\ )$

Substituting these stress components into appropriate formulas

Principal stresses = $(\ 27.3922 \quad 5.4975 \quad -9.06718\ )$

Effective stress (von Mises) = 31.7867

Element solution at $\{s, t\} = \{-1, 1\} \Longrightarrow \{x, y\} = \{0., 15.\}$

$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\} = \{0, 0, 0, 0, 0, 0, 1, 0\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s, \partial N_5/\partial s, \partial N_6/\partial s, \partial N_7/\partial s, \partial N_8/\partial s\} = \left\{0, 0, 0, 0, -\frac{1}{2}, 2, -\frac{3}{2}, 0\right\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t, \partial N_5/\partial t, \partial N_6/\partial t, \partial N_7/\partial t, \partial N_8/\partial t\} = \left\{\frac{1}{2}, 0, 0, 0, 0, 0, \frac{3}{2}, -2\right\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x, \partial N_5/\partial x, \partial N_6/\partial x, \partial N_7/\partial x, \partial N_8/\partial x\} =$
$\{0.00938363, 0, 0, 0, -0.0364612, 0.145845, -0.0812327, -0.0375345\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y, \partial N_5/\partial y, \partial N_6/\partial y, \partial N_7/\partial y, \partial N_8/\partial y\} =$
$\{0.1, 0, 0, 0, 0., 0., 0.3, -0.4\}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 0.00938363 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.0364612 & 0 & 0.145845 & 0 & - \\ 0 & 0.1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.1 & 0.00938363 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.0364612 & 0 & 0.145845 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{d} = (\ 0.00014017 \quad 0.0000868105 \quad -0.0000162379\ )$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C\epsilon} = (\ 7.16319 \quad 5.93182 \quad -0.18736\ )$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{\mathrm{T}} = (\ 0.00014017 \quad 0.0000868105 \quad 0 \quad -0.0000162379 \quad 0 \quad 0\ )$

$\boldsymbol{\sigma}^{\mathrm{T}} = (\ 7.16319 \quad 5.93182 \quad 3.9285 \quad -0.18736 \quad 0 \quad 0\ )$

Substituting these stress components into appropriate formulas

Principal stresses = $(\ 7.19107 \quad 5.90395 \quad 3.9285\ )$

Effective stress (von Mises) = 2.84635

Element solution at $\{s, t\} = \{1, -1\} \Longrightarrow \{x, y\} = \{5., 0.\}$

$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\} = \{0, 0, 1, 0, 0, 0, 0, 0\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s, \partial N_5/\partial s, \partial N_6/\partial s, \partial N_7/\partial s, \partial N_8/\partial s\} = \left\{\frac{1}{2}, -2, \frac{3}{2}, 0, 0, 0, 0, 0\right\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t, \partial N_5/\partial t, \partial N_6/\partial t, \partial N_7/\partial t, \partial N_8/\partial t\} = \left\{0, 0, -\frac{3}{2}, 2, -\frac{1}{2}, 0, 0, 0\right\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x, \partial N_5/\partial x, \partial N_6/\partial x, \partial N_7/\partial x, \partial N_8/\partial x\} =$
$\{0., 0., -0.3, 0.4, -0.1, 0, 0, 0\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y, \partial N_5/\partial y, \partial N_6/\partial y, \partial N_7/\partial y, \partial N_8/\partial y\} =$
$\{-0.109384, 0.437535, -0.356302, 0.0375345, -0.00938363, 0, 0, 0\}$

$$\boldsymbol{B}^T = \begin{pmatrix} 0 & 0 & 0 & 0 & -0.3 & 0 & 0.4 & 0 & \cdots \\ 0 & -0.109384 & 0 & 0.437535 & 0 & -0.356302 & 0 & 0.0375345 & \\ -0.109384 & 0 & 0.437535 & 0 & -0.356302 & -0.3 & 0.0375345 & 0.4 & \cdots \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^T\boldsymbol{d} = (\ -0.000624887 \quad 0.00094252 \quad -0.000198345\ )$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\ -8.92296 \quad 27.248 \quad -2.2886\ )$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^T = (\ -0.000624887 \quad 0.00094252 \quad 0 \quad -0.000198345 \quad 0 \quad 0\ )$

$\boldsymbol{\sigma}^T = (\ -8.92296 \quad 27.248 \quad 5.4975 \quad -2.2886 \quad 0 \quad 0\ )$

Substituting these stress components into appropriate formulas

Principal stresses = $(\ 27.3922 \quad 5.4975 \quad -9.06718\ )$

Effective stress (von Mises) = 31.7867

Element solution at $\{s, t\} = \{1, 1\} \Longrightarrow \{x, y\} = \{15., 0.\}$

$\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8\} = \{0, 0, 0, 0, 1, 0, 0, 0\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s, \partial N_5/\partial s, \partial N_6/\partial s, \partial N_7/\partial s, \partial N_8/\partial s\} = \left\{0, 0, 0, 0, \frac{3}{2}, -2, \frac{1}{2}, 0\right\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t, \partial N_5/\partial t, \partial N_6/\partial t, \partial N_7/\partial t, \partial N_8/\partial t\} = \left\{0, 0, \frac{1}{2}, -2, \frac{3}{2}, 0, 0, 0\right\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x, \partial N_5/\partial x, \partial N_6/\partial x, \partial N_7/\partial x, \partial N_8/\partial x\} =$
$\{0, 0, 0.1, -0.4, 0.3, 0., 0., 0\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y, \partial N_5/\partial y, \partial N_6/\partial y, \partial N_7/\partial y, \partial N_8/\partial y\} =$
$\{0, 0, 0.00938363, -0.0375345, -0.0812327, 0.145845, -0.0364612, 0\}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0.1 & 0 & -0.4 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.00938363 & 0 & -0.0375345 & 0 & -0.0812327 & 0 \\ 0 & 0 & 0 & 0 & 0.00938363 & 0.1 & -0.0375345 & -0.4 & -0.0812327 & 0.3 & 0. \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{d} = (\ 0.0000868105 \quad 0.00014017 \quad -0.0000162379\ )$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\ 5.93182 \quad 7.16319 \quad -0.18736\ )$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{\mathrm{T}} = (\ 0.0000868105 \quad 0.00014017 \quad 0 \quad -0.0000162379 \quad 0 \quad 0\ )$

$\boldsymbol{\sigma}^{\mathrm{T}} = (\ 5.93182 \quad 7.16319 \quad 3.9285 \quad -0.18736 \quad 0 \quad 0\ )$

Substituting these stress components into appropriate formulas

Principal stresses = (7.19107 5.90395 3.9285)

Effective stress (von Mises) = 2.84635

Solution summary

Nodal solution

	x	y	u	v
1	0.	5.	0	0.00494694
2	3.53553	3.53553	0.0033909	0.0033909
3	5.	0.	0.00494694	0
4	10.	0.	0.00271213	0
5	15.	0.	0.00225656	0
6	10.6066	10.6066	0.00152523	0.00152523
7	0.	15.	0	0.00225656
8	0.	10.	0	0.00271213

Solution at selected points on elements

	Coord	Disp	Stresses	Principal stresses	Effective Stress
1	7.07107 7.07107	0.00201325 0.00201325	0.212515 0.212515 0.127509 −6.17375 0 0	6.38626 0.127509 −5.96123	10.6936

Support reactions

Node	dof	Reaction
1	1	−39.0268
3	2	−39.0268
4	2	−52.5151
5	2	−8.4581
7	1	−8.4581
8	1	−52.5151

Sum of applied loads → (100. 100.)

Sum of support reactions → (−100. −100.)

■ 7.0.1 Rotating Disks and Flywheels

Disks and flywheels are common components of rotating machines. Stresses are generated in these machine parts due to inertia forces that depend on the angular velocity and mass. Consider a disk of uniform thickness, shown in Figure, that is rotating at a constant angular velocity of ω rad/s. The inertia force is given by $m\,r\,\omega^2$where m is the mass density per unit volume and r is the radial distance from the center of the disk. Treating this centrifugal force as the body force in the radial direction, a plane stress model of the disk can be used to determine stresses in the disk. The body force components in the x and y directions are as follows.

$$b_x = m\,x\,\omega^2 \text{ and } b_y = m\,y\,\omega^2$$

These forces are not constant over an element. For a simplified analysis we can assign constant values for each element by using the values of these forces at element centroids. For a more accurate analysis we can use these expressions directly and carry out integration to get the equivalent body force. This later procedure is used in the numerical example presented in this section.

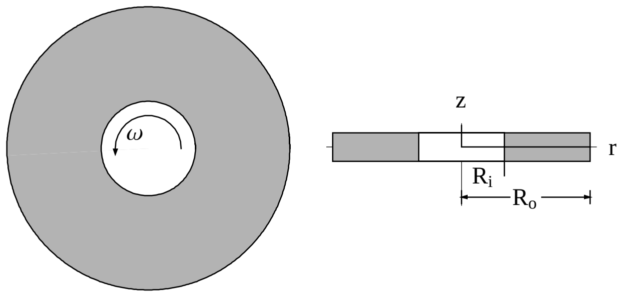
For thin disks, the following analytical solution is available.

$$u(r) = \frac{r}{E}\ \frac{\nu+3}{8}\ m\,\omega^2 R_o^2 \left((\nu+1)\left(\frac{R_i}{r}\right)^2 - \frac{1-\nu^2}{\nu+3}\left(\frac{r}{R_o}\right)^2 + \left(\left(\frac{R_i}{R_o}\right)^2 + 1\right)(1-\nu)\right)$$

$$\sigma_r(r) = \frac{\nu+3}{8}\ m\,\omega^2 R_o^2 \left(-\left(\frac{R_i}{r}\right)^2 - \left(\frac{r}{R_i}\right)^2 + \left(\frac{R_i}{R_o}\right)^2 + 1\right)$$

$$\sigma_t(r) = \frac{\nu+3}{8}\ m\,\omega^2 R_o^2 \left(\left(\frac{R_i}{r}\right)^2 + \left(\frac{R_i}{R_o}\right)^2 - \frac{3\,\nu+1}{\nu+3}\left(\frac{r}{R_o}\right)^2 + 1\right)$$

where u is radial displacement, σ_r is radial stress, σ_t is the tangential stress, R_i is the inner radius (radius of the shaft), R_o is the outer radius, E is modulus of elasticity, and ν is Poisson's ratio.



For disks with variable thickness, and spoked flywheels, analytical solutions are not available. However plane stress finite element models can be used effectively to determine stresses in these situations.