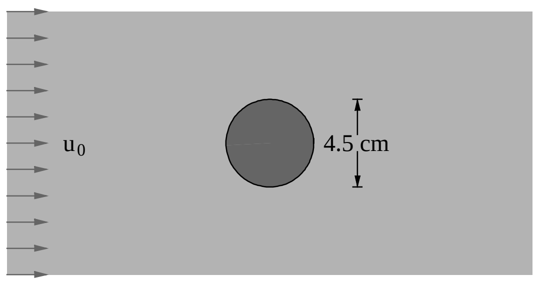
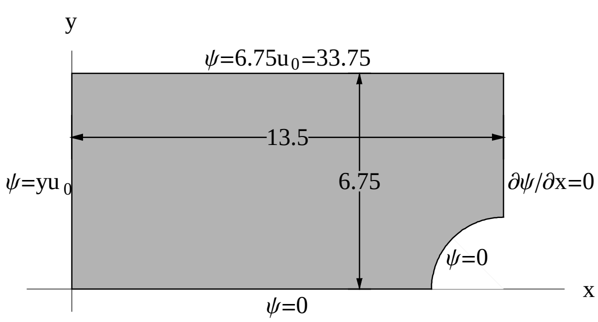
Example 5.6: Stream function formulation for fluid flow around a cylinder (p. 363)

Consider fluid flow in the direction perpendicular to a long cylinder as shown in Figure. The cylinder diameter is 4.5 cm. At a distance of about 3 times the diameter of the cylinder, both the upstream and the downstream, the flow can be considered uniform with a velocity of $u_0 = 5\,\mathrm{cm}/s$ in the x direction. Determine the flow velocity near the cylinder.



We choose a computational domain that extends 3 times the cylinder diameter upstream and downstream and 1.5 times diameter above and below the cylinder. Taking advantage of symmetry we need to model only a quarter of the solution domain.

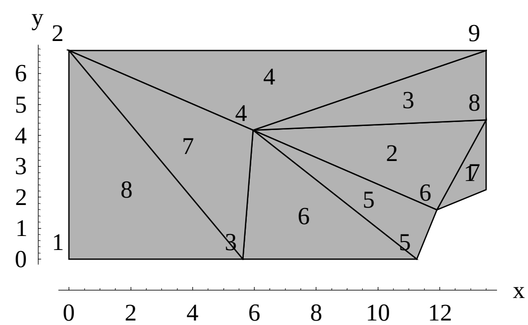


The governing differential equation in terms of stream function $\psi(x, y)$ is as follows.

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$$

Compared to the general form $k_x = k_y = 1$ and $p = q = 0$. The fluid velocity is related to stream function as follows.

$u = \frac{\partial \psi}{\partial y}; \quad v = -\frac{\partial \psi}{\partial x}$



The complete finite element solution is as follows.

Global equations at start of the element assembly process

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ \psi_7 \\ \psi_8 \\ \psi_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 1

$k_x = 1; \quad k_y = 1; \quad p = 0; \quad q = 0$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	6	11.909	1.59099
2	7	13.5	2.25
3	8	13.5	4.5

$x_1 = 11.909$ $\quad x_2 = 13.5$ $\quad x_3 = 13.5$
$y_1 = 1.59099$ $\quad y_2 = 2.25$ $\quad y_3 = 4.5$

Using these values we get

$b_1 = -2.25$ $\quad b_2 = 2.90901$ $\quad b_3 = -0.65901$

$c_1 = 0.$ $\quad c_2 = -1.59099$ $\quad c_3 = 1.59099$

$f_1 = 30.375$ $\quad f_2 = -32.1122$ $\quad f_3 = 5.3169$

Element area, $A = 1.78986$

$$\boldsymbol{B}^{T} = \begin{pmatrix} -2.25 & 2.90901 & -0.65901 \\ 0. & -1.59099 & 1.59099 \end{pmatrix}$$

$$\boldsymbol{k}_k = \begin{pmatrix} 0.707107 & -0.914214 & 0.207107 \\ -0.914214 & 1.53553 & -0.62132 \\ 0.207107 & -0.62132 & 0.414214 \end{pmatrix}; \qquad \boldsymbol{k}_p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_q = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 0.707107 & -0.914214 & 0.207107 \\ -0.914214 & 1.53553 & -0.62132 \\ 0.207107 & -0.62132 & 0.414214 \end{pmatrix} \begin{pmatrix} \psi_6 \\ \psi_7 \\ \psi_8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to {6, 7, 8} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.707107 & -0.914214 & 0.207107 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.914214 & 1.53553 & -0.62132 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.207107 & -0.62132 & 0.414214 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ \psi_7 \\ \psi_8 \\ \psi_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 2

$k_x = 1;$ $\quad k_y = 1;$ $\quad p = 0;$ $\quad q = 0$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	8	13.5	4.5
2	4	5.9545	4.1705
3	6	11.909	1.59099

$x_1 = 13.5$ $x_2 = 5.9545$ $x_3 = 11.909$
$y_1 = 4.5$ $y_2 = 4.1705$ $y_3 = 1.59099$

Using these values we get

$b_1 = 2.5795$ $b_2 = -2.90901$ $b_3 = 0.329505$

$c_1 = 5.9545$ $c_2 = 1.59099$ $c_3 = -7.5455$

$f_1 = -40.1929$ $f_2 = 32.1122$ $f_3 = 29.5064$

Element area, A = 10.7128

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 2.5795 & -2.90901 & 0.329505 \\ 5.9545 & 1.59099 & -7.5455 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 0.982699 & 0.0459671 & -1.02867 \\ 0.0459671 & 0.256552 & -0.302519 \\ -1.02867 & -0.302519 & 1.33118 \end{pmatrix}; \qquad \boldsymbol{k}_{\mathrm{p}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 0.982699 & 0.0459671 & -1.02867 \\ 0.0459671 & 0.256552 & -0.302519 \\ -1.02867 & -0.302519 & 1.33118 \end{pmatrix} \begin{pmatrix} \psi_8 \\ \psi_4 \\ \psi_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to {8, 4, 6} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.256552 & 0 & -0.302519 & 0 & 0.0459671 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.302519 & 0 & 2.03829 & -0.914214 & -0.821559 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.914214 & 1.53553 & -0.62132 & 0 \\ 0 & 0 & 0 & 0.0459671 & 0 & -0.821559 & -0.62132 & 1.39691 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ \psi_7 \\ \psi_8 \\ \psi_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 3

$k_x = 1;\quad k_y = 1;\quad p = 0;\quad q = 0$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	4	5.9545	4.1705
2	8	13.5	4.5
3	9	13.5	6.75

$x_1 = 5.9545\quad x_2 = 13.5\quad x_3 = 13.5$
$y_1 = 4.1705\quad y_2 = 4.5\quad y_3 = 6.75$

Using these values we get

$b_1 = -2.25\quad b_2 = 2.5795\quad b_3 = -0.329505$

$c_1 = 0.\quad c_2 = -7.5455\quad c_3 = 7.5455$

$f_1 = 30.375\quad f_2 = 16.1088\quad f_3 = -29.5064$

Element area, A = 8.48868

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -2.25 & 2.5795 & -0.329505 \\ 0. & -7.5455 & 7.5455 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 0.149096 & -0.17093 & 0.0218345 \\ -0.17093 & 1.87274 & -1.70181 \\ 0.0218345 & -1.70181 & 1.67997 \end{pmatrix};\quad \boldsymbol{k}_{\mathrm{p}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};\quad \boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 0.149096 & -0.17093 & 0.0218345 \\ -0.17093 & 1.87274 & -1.70181 \\ 0.0218345 & -1.70181 & 1.67997 \end{pmatrix} \begin{pmatrix} \psi_4 \\ \psi_8 \\ \psi_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to {4, 8, 9} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.405647 & 0 & -0.302519 & 0 & -0.124963 & 0.0218345 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.302519 & 0 & 2.03829 & -0.914214 & -0.821559 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.914214 & 1.53553 & -0.62132 & 0 \\ 0 & 0 & 0 & -0.124963 & 0 & -0.821559 & -0.62132 & 3.26965 & -1.70181 \\ 0 & 0 & 0 & 0.0218345 & 0 & 0 & 0 & -1.70181 & 1.67997 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ \psi_7 \\ \psi_8 \\ \psi_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 4

$k_x = 1;$ $k_y = 1;$ $p = 0;$ $q = 0$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	9	13.5	6.75
2	2	0	6.75
3	4	5.9545	4.1705

$x_1 = 13.5$ $x_2 = 0$ $x_3 = 5.9545$

$y_1 = 6.75$ $y_2 = 6.75$ $y_3 = 4.1705$

Using these values we get

$b_1 = 2.5795$ $b_2 = -2.5795$ $b_3 = 0.$

$c_1 = 5.9545$ $c_2 = 7.5455$ $c_3 = -13.5$

$f_1 = -40.1929$ $f_2 = -16.1088$ $f_3 = 91.125$

Element area, A = 17.4117

$$\boldsymbol{B}^T = \begin{pmatrix} 2.5795 & -2.5795 & 0. \\ 5.9545 & 7.5455 & -13.5 \end{pmatrix}$$

$$\boldsymbol{k}_k = \begin{pmatrix} 0.604623 & 0.549572 & -1.1542 \\ 0.549572 & 0.913014 & -1.46259 \\ -1.1542 & -1.46259 & 2.61678 \end{pmatrix}; \quad \boldsymbol{k}_p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \boldsymbol{r}_q = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 0.604623 & 0.549572 & -1.1542 \\ 0.549572 & 0.913014 & -1.46259 \\ -1.1542 & -1.46259 & 2.61678 \end{pmatrix} \begin{pmatrix} \psi_9 \\ \psi_2 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to {9, 2, 4} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.913014 & 0 & -1.46259 & 0 & 0 & 0 & 0 & 0.549572 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.46259 & 0 & 3.02243 & 0 & -0.302519 & 0 & -0.124963 & -1.13236 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.302519 & 0 & 2.03829 & -0.914214 & -0.821559 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.914214 & 1.53553 & -0.62132 & 0 \\ 0 & 0 & 0 & -0.124963 & 0 & -0.821559 & -0.62132 & 3.26965 & -1.70181 \\ 0 & 0.549572 & 0 & -1.13236 & 0 & 0 & 0 & -1.70181 & 2.2846 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ \psi_7 \\ \psi_8 \\ \psi_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 5

$k_x = 1;$ $k_y = 1;$ $p = 0;$ $q = 0$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	5	11.25	0
2	6	11.909	1.59099
3	4	5.9545	4.1705

$x_1 = 11.25$ $x_2 = 11.909$ $x_3 = 5.9545$

$y_1 = 0$ $y_2 = 1.59099$ $y_3 = 4.1705$

Using these values we get

$b_1 = -2.5795$ $b_2 = 4.1705$ $b_3 = -1.59099$

$c_1 = -5.9545$ $c_2 = 5.2955$ $c_3 = 0.65901$

$f_1 = 40.1929$ $f_2 = -46.9181$ $f_3 = 17.8986$

Element area, $A = 5.58674$

$$\boldsymbol{B}^T = \begin{pmatrix} -2.5795 & 4.1705 & -1.59099 \\ -5.9545 & 5.2955 & 0.65901 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 1.88437 & -1.89242 & 0.00804988 \\ -1.89242 & 2.03318 & -0.140754 \\ 0.00804988 & -0.140754 & 0.132705 \end{pmatrix}; \qquad \boldsymbol{k}_{\mathrm{p}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 1.88437 & -1.89242 & 0.00804988 \\ -1.89242 & 2.03318 & -0.140754 \\ 0.00804988 & -0.140754 & 0.132705 \end{pmatrix} \begin{pmatrix} \psi_5 \\ \psi_6 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to {5, 6, 4} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.913014 & 0 & -1.46259 & 0 & 0 & 0 & 0 & 0.549572 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.46259 & 0 & 3.15513 & 0.00804988 & -0.443273 & 0 & -0.124963 & -1.13236 \\ 0 & 0 & 0 & 0.00804988 & 1.88437 & -1.89242 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.443273 & -1.89242 & 4.07147 & -0.914214 & -0.821559 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.914214 & 1.53553 & -0.62132 & 0 \\ 0 & 0 & 0 & -0.124963 & 0 & -0.821559 & -0.62132 & 3.26965 & -1.70181 \\ 0 & 0.549572 & 0 & -1.13236 & 0 & 0 & 0 & -1.70181 & 2.2846 \end{pmatrix}$$

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ \psi_7 \\ \psi_8 \\ \psi_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 6

$k_x = 1$; $\qquad k_y = 1$; $\qquad p = 0$; $\qquad q = 0$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	4	5.9545	4.1705
2	3	5.625	0
3	5	11.25	0

$x_1 = 5.9545$ $\quad x_2 = 5.625$ $\quad x_3 = 11.25$

$y_1 = 4.1705$ $\quad y_2 = 0$ $\quad y_3 = 0$

Using these values we get

$b_1 = 0$ $\quad b_2 = -4.1705$ $\quad b_3 = 4.1705$

$c_1 = 5.625$ $\quad c_2 = -5.2955$ $\quad c_3 = -0.329505$

$f_1 = 0$ $\quad f_2 = 46.9181$ $\quad f_3 = -23.459$

Element area, A = 11.7295

$$\boldsymbol{B}^T = \begin{pmatrix} 0 & -4.1705 & 4.1705 \\ 5.625 & -5.2955 & -0.329505 \end{pmatrix}$$

$$\boldsymbol{k}_k = \begin{pmatrix} 0.67438 & -0.634876 & -0.0395043 \\ -0.634876 & 0.968397 & -0.33352 \\ -0.0395043 & -0.33352 & 0.373025 \end{pmatrix}; \qquad \boldsymbol{k}_p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_q = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 0.67438 & -0.634876 & -0.0395043 \\ -0.634876 & 0.968397 & -0.33352 \\ -0.0395043 & -0.33352 & 0.373025 \end{pmatrix} \begin{pmatrix} \psi_4 \\ \psi_3 \\ \psi_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to {4, 3, 5} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.913014 & 0 & -1.46259 & 0 & 0 & 0 & 0 & 0.549572 \\ 0 & 0 & 0.968397 & -0.634876 & -0.33352 & 0 & 0 & 0 & 0 \\ 0 & -1.46259 & -0.634876 & 3.82951 & -0.0314544 & -0.443273 & 0 & -0.124963 & -1.13236 \\ 0 & 0 & -0.33352 & -0.0314544 & 2.2574 & -1.89242 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.443273 & -1.89242 & 4.07147 & -0.914214 & -0.821559 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.914214 & 1.53553 & -0.62132 & 0 \\ 0 & 0 & 0 & -0.124963 & 0 & -0.821559 & -0.62132 & 3.26965 & -1.70181 \\ 0 & 0.549572 & 0 & -1.13236 & 0 & 0 & 0 & -1.70181 & 2.2846 \end{pmatrix}$$

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ \psi_7 \\ \psi_8 \\ \psi_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 7

$k_x = 1;$ $k_y = 1;$ $p = 0;$ $q = 0$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	3	5.625	0
2	4	5.9545	4.1705
3	2	0	6.75

$x_1 = 5.625$ $x_2 = 5.9545$ $x_3 = 0$

$y_1 = 0$ $y_2 = 4.1705$ $y_3 = 6.75$

Using these values we get

$b_1 = -2.5795$ $b_2 = 6.75$ $b_3 = -4.1705$

$c_1 = -5.9545$ $c_2 = 5.625$ $c_3 = 0.329505$

$f_1 = 40.1929 \qquad f_2 = -37.9688 \qquad f_3 = 23.459$

Element area, A = 12.8416

$$\boldsymbol{B}^T = \begin{pmatrix} -2.5795 & 6.75 & -4.1705 \\ -5.9545 & 5.625 & 0.329505 \end{pmatrix}$$

$$\boldsymbol{k}_k = \begin{pmatrix} 0.819796 & -0.991032 & 0.171236 \\ -0.991032 & 1.50299 & -0.511957 \\ 0.171236 & -0.511957 & 0.340721 \end{pmatrix}; \qquad \boldsymbol{k}_p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_q = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 0.819796 & -0.991032 & 0.171236 \\ -0.991032 & 1.50299 & -0.511957 \\ 0.171236 & -0.511957 & 0.340721 \end{pmatrix} \begin{pmatrix} \psi_3 \\ \psi_4 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to {3, 4, 2} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.25373 & 0.171236 & -1.97454 & 0 & 0 & 0 & 0 & 0.549572 \\ 0 & 0.171236 & 1.78819 & -1.62591 & -0.33352 & 0 & 0 & 0 & 0 \\ 0 & -1.97454 & -1.62591 & 5.3325 & -0.0314544 & -0.443273 & 0 & -0.124963 & -1.13236 \\ 0 & 0 & -0.33352 & -0.0314544 & 2.2574 & -1.89242 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.443273 & -1.89242 & 4.07147 & -0.914214 & -0.821559 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.914214 & 1.53553 & -0.62132 & 0 \\ 0 & 0 & 0 & -0.124963 & 0 & -0.821559 & -0.62132 & 3.26965 & -1.70181 \\ 0 & 0.549572 & 0 & -1.13236 & 0 & 0 & 0 & -1.70181 & 2.2846 \end{pmatrix}$$

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ \psi_7 \\ \psi_8 \\ \psi_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 8

$k_x = 1;\quad k_y = 1;\quad p = 0;\quad q = 0$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	2	0	6.75
2	1	0	0
3	3	5.625	0

$x_1 = 0\quad x_2 = 0\quad x_3 = 5.625$
$y_1 = 6.75\quad y_2 = 0\quad y_3 = 0$

Using these values we get

$b_1 = 0\quad b_2 = -6.75\quad b_3 = 6.75$

$c_1 = 5.625\quad c_2 = -5.625\quad c_3 = 0$

$f_1 = 0\quad f_2 = 37.9688\quad f_3 = 0$

Element area, A = 18.9844

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 0 & -6.75 & 6.75 \\ 5.625 & -5.625 & 0 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 0.416667 & -0.416667 & 0 \\ -0.416667 & 1.01667 & -0.6 \\ 0 & -0.6 & 0.6 \end{pmatrix};\qquad \boldsymbol{k}_{\mathrm{p}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};\qquad \boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 0.416667 & -0.416667 & 0 \\ -0.416667 & 1.01667 & -0.6 \\ 0 & -0.6 & 0.6 \end{pmatrix}\begin{pmatrix} \psi_2 \\ \psi_1 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to {2, 1, 3} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 1.01667 & -0.416667 & -0.6 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.416667 & 1.6704 & 0.171236 & -1.97454 & 0 & 0 & 0 & 0 & 0.5495 \\ -0.6 & 0.171236 & 2.38819 & -1.62591 & -0.33352 & 0 & 0 & 0 & 0 \\ 0 & -1.97454 & -1.62591 & 5.3325 & -0.0314544 & -0.443273 & 0 & -0.124963 & -1.1323 \\ 0 & 0 & -0.33352 & -0.0314544 & 2.2574 & -1.89242 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.443273 & -1.89242 & 4.07147 & -0.914214 & -0.821559 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.914214 & 1.53553 & -0.62132 & 0 \\ 0 & 0 & 0 & -0.124963 & 0 & -0.821559 & -0.62132 & 3.26965 & -1.7018 \\ 0 & 0.549572 & 0 & -1.13236 & 0 & 0 & 0 & -1.70181 & 2.2846 \end{pmatrix}$$

Essential boundary conditions

Node	dof	Value
1	ψ_1	0
2	ψ_2	33.75
3	ψ_3	0
5	ψ_5	0
6	ψ_6	0
7	ψ_7	0
9	ψ_9	33.75

Delete equations {1, 2, 3, 5, 6, 7, 9}.

$$\begin{pmatrix} 0 & -1.97454 & -1.62591 & 5.3325 & -0.0314544 & -0.443273 & 0 & -0.124963 & -1.13236 \\ 0 & 0 & 0 & -0.124963 & 0 & -0.821559 & -0.62132 & 3.26965 & -1.70181 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 33.75 \\ 0 \\ \psi_4 \\ 0 \\ 0 \\ 0 \\ \psi_8 \\ 33.75 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Extract columns {1, 2, 3, 5, 6, 7, 9}.

Multiply each column by its respective known value {0, 33.75, 0, 0, 0, 0, 33.75}.

Move all resulting vectors to the rhs.

After adjusting for essential boundary conditions we have

$$\begin{pmatrix} 5.3325 & -0.124963 \\ -0.124963 & 3.26965 \end{pmatrix} \begin{pmatrix} \psi_4 \\ \psi_8 \end{pmatrix} = \begin{pmatrix} 104.858 \\ 57.436 \end{pmatrix}$$

Solving the final system of global equations we get

$$\{\psi_4 = 20.0936, \psi_8 = 18.3344\}$$

Complete table of nodal values

	ψ
1	0
2	33.75
3	0
4	20.0936
5	0
6	0
7	0
8	18.3344
9	33.75

Solution for element 1

Nodal coordinates

Element node	Global node number	x	y
1	6	11.909	1.59099
2	7	13.5	2.25
3	8	13.5	4.5

$x_1 = 11.909$ $\quad x_2 = 13.5$ $\quad x_3 = 13.5$
$y_1 = 1.59099$ $\quad y_2 = 2.25$ $\quad y_3 = 4.5$

$b_1 = -2.25$ $\quad b_2 = 2.90901$ $\quad b_3 = -0.65901$

$c_1 = 0.$ $\quad c_2 = -1.59099$ $\quad c_3 = 1.59099$

$f_1 = 30.375$ $\quad f_2 = -32.1122$ $\quad f_3 = 5.3169$

Element area, $A = 1.78986$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^T =$

$$\{8.48528 - 0.628539\,x,\ 0.812634\,x - 0.444444\,y - 8.97056,\ -0.184095\,x + 0.444444\,y + 1.48528\}$$

Nodal values, $\boldsymbol{d}^T = \{0, 0, 18.3344\}$

$\psi(x, y) = \boldsymbol{N}^T\boldsymbol{d} = -3.37526\,x + 8.14861\,y + 27.2317$

$\partial\psi/\partial x = -3.37526;\qquad \partial\psi/\partial y = 8.14861$

Solution for element 2

Nodal coordinates

Element node	Global node number	x	y
1	8	13.5	4.5
2	4	5.9545	4.1705
3	6	11.909	1.59099

$x_1 = 13.5 \quad x_2 = 5.9545 \quad x_3 = 11.909$
$y_1 = 4.5 \quad y_2 = 4.1705 \quad y_3 = 1.59099$

$b_1 = 2.5795 \quad b_2 = -2.90901 \quad b_3 = 0.329505$

$c_1 = 5.9545 \quad c_2 = 1.59099 \quad c_3 = -7.5455$

$f_1 = -40.1929 \quad f_2 = 32.1122 \quad f_3 = 29.5064$

Element area, $A = 10.7128$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^T = \{0.120393\,x + 0.277914\,y - 1.87592,$
$-0.135772\,x + 0.0742562\,y + 1.49877,\ 0.015379\,x - 0.352171\,y + 1.37715\}$

Nodal values, $\boldsymbol{d}^T = \{18.3344, 20.0936, 0\}$

$\psi(x, y) = \boldsymbol{N}^T\boldsymbol{d} = -0.520816\,x + 6.58746\,y - 4.27818$

$\partial\psi/\partial x = -0.520816;\qquad \partial\psi/\partial y = 6.58746$

Solution for element 3

Nodal coordinates

Element node	Global node number	x	y
1	4	5.9545	4.1705
2	8	13.5	4.5
3	9	13.5	6.75

$x_1 = 5.9545 \quad x_2 = 13.5 \quad x_3 = 13.5$
$y_1 = 4.1705 \quad y_2 = 4.5 \quad y_3 = 6.75$

$b_1 = -2.25 \quad b_2 = 2.5795 \quad b_3 = -0.329505$

$c_1 = 0.$ $\quad c_2 = -7.5455$ $\quad c_3 = 7.5455$

$f_1 = 30.375$ $\quad f_2 = 16.1088$ $\quad f_3 = -29.5064$

Element area, A = 8.48868

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^T =$

$$\{1.78915 - 0.132529\,x,\ 0.151938\,x - 0.444444\,y + 0.948838,\ -0.0194085\,x + 0.444444\,y - 1.73799\}$$

Nodal values, $\boldsymbol{d}^T = \{20.0936,\ 18.3344,\ 33.75\}$

$\psi(x, y) = \boldsymbol{N}^T\boldsymbol{d} = -0.532342\,x + 6.85139\,y - 5.31027$

$\partial\psi/\partial x = -0.532342;$ $\quad \partial\psi/\partial y = 6.85139$

Solution for element 4

Nodal coordinates

Element node	Global node number	x	y
1	9	13.5	6.75
2	2	0	6.75
3	4	5.9545	4.1705

$x_1 = 13.5$ $\quad x_2 = 0$ $\quad x_3 = 5.9545$
$y_1 = 6.75$ $\quad y_2 = 6.75$ $\quad y_3 = 4.1705$

$b_1 = 2.5795$ $\quad b_2 = -2.5795$ $\quad b_3 = 0.$

$c_1 = 5.9545$ $\quad c_2 = 7.5455$ $\quad c_3 = -13.5$

$f_1 = -40.1929$ $\quad f_2 = -16.1088$ $\quad f_3 = 91.125$

Element area, A = 17.4117

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^T =$

$$\{0.0740741\,x + 0.170992\,y - 1.1542,\ -0.0740741\,x + 0.216679\,y - 0.462586,\ 2.61678 - 0.387671\,y\}$$

Nodal values, $\boldsymbol{d}^T = \{33.75,\ 33.75,\ 20.0936\}$

$\psi(x, y) = \boldsymbol{N}^T\boldsymbol{d} = 5.2942\,y - 1.98584$

$\partial\psi/\partial x = 0;$ $\quad \partial\psi/\partial y = 5.2942$

Solution for element 5

Nodal coordinates

Element node	Global node number	x	y
1	5	11.25	0
2	6	11.909	1.59099
3	4	5.9545	4.1705

$x_1 = 11.25$ $x_2 = 11.909$ $x_3 = 5.9545$
$y_1 = 0$ $y_2 = 1.59099$ $y_3 = 4.1705$

$b_1 = -2.5795$ $b_2 = 4.1705$ $b_3 = -1.59099$

$c_1 = -5.9545$ $c_2 = 5.2955$ $c_3 = 0.65901$

$f_1 = 40.1929$ $f_2 = -46.9181$ $f_3 = 17.8986$

Element area, A = 5.58674

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^T = \{-0.23086\,x - 0.532914\,y + 3.59717,$
$0.37325\,x + 0.473934\,y - 4.19906,\ -0.14239\,x + 0.0589798\,y + 1.60189\}$

Nodal values, $\boldsymbol{d}^T = \{0,\ 0,\ 20.0936\}$

$\psi(x, y) = \boldsymbol{N}^T\boldsymbol{d} = -2.86112\,x + 1.18512\,y + 32.1876$

$\partial\psi/\partial x = -2.86112;$ $\partial\psi/\partial y = 1.18512$

Solution for element 6

Nodal coordinates

Element node	Global node number	x	y
1	4	5.9545	4.1705
2	3	5.625	0
3	5	11.25	0

$x_1 = 5.9545$ $x_2 = 5.625$ $x_3 = 11.25$
$y_1 = 4.1705$ $y_2 = 0$ $y_3 = 0$

$b_1 = 0$ $b_2 = -4.1705$ $b_3 = 4.1705$

$c_1 = 5.625$ $c_2 = -5.2955$ $c_3 = -0.329505$

$f_1 = 0$ $f_2 = 46.9181$ $f_3 = -23.459$

Element area, A = 11.7295

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^T = \{0.23978\,y,\ -0.177778\,x - 0.225734\,y + 2.,\ 0.177778\,x - 0.014046\,y - 1.\}$

Nodal values, $\boldsymbol{d}^T = \{20.0936, 0, 0\}$

$\psi(x, y) = \boldsymbol{N}^T\boldsymbol{d} = 4.81803\, y$

$\partial\psi/\partial x = 0;$ $\partial\psi/\partial y = 4.81803$

Solution for element 7

Nodal coordinates

Element node	Global node number	x	y
1	3	5.625	0
2	4	5.9545	4.1705
3	2	0	6.75

$x_1 = 5.625$ $x_2 = 5.9545$ $x_3 = 0$
$y_1 = 0$ $y_2 = 4.1705$ $y_3 = 6.75$

$b_1 = -2.5795$ $b_2 = 6.75$ $b_3 = -4.1705$

$c_1 = -5.9545$ $c_2 = 5.625$ $c_3 = 0.329505$

$f_1 = 40.1929$ $f_2 = -37.9688$ $f_3 = 23.459$

Element area, $A = 12.8416$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^T = \{-0.100436\, x - 0.231844\, y + 1.56495,$
$0.262818\, x + 0.219015\, y - 1.47835,\ -0.162382\, x + 0.0128296\, y + 0.9134\}$

Nodal values, $\boldsymbol{d}^T = \{0, 20.0936, 33.75\}$

$\psi(x, y) = \boldsymbol{N}^T\boldsymbol{d} = -0.199449\, x + 4.83379\, y + 1.1219$

$\partial\psi/\partial x = -0.199449;$ $\partial\psi/\partial y = 4.83379$

Solution for element 8

Nodal coordinates

Element node	Global node number	x	y
1	2	0	6.75
2	1	0	0
3	3	5.625	0

$x_1 = 0$ $x_2 = 0$ $x_3 = 5.625$
$y_1 = 6.75$ $y_2 = 0$ $y_3 = 0$

$b_1 = 0$ $b_2 = -6.75$ $b_3 = 6.75$

$c_1 = 5.625 \qquad c_2 = -5.625 \qquad c_3 = 0$

$f_1 = 0 \qquad f_2 = 37.9688 \qquad f_3 = 0$

Element area, A = 18.9844

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^T = \{0.148148\,y,\ -0.177778\,x - 0.148148\,y + 1.,\ 0.177778\,x\}$

Nodal values, $\boldsymbol{d}^T = \{33.75,\ 0,\ 0\}$

$\psi(x, y) = \boldsymbol{N}^T\boldsymbol{d} = 5.\,y$

$\partial\psi/\partial x = 0; \qquad \partial\psi/\partial y = 5.$

Solution summary

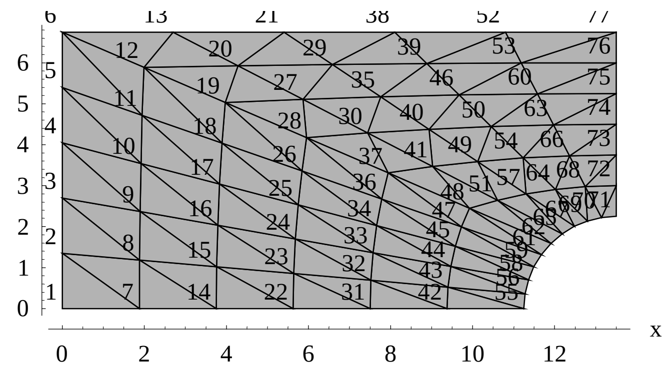
Nodal solution

	x−coord	y−coord	ψ
1	0	0	0
2	0	6.75	33.75
3	5.625	0	0
4	5.9545	4.1705	20.0936
5	11.25	0	0
6	11.909	1.59099	0
7	13.5	2.25	0
8	13.5	4.5	18.3344
9	13.5	6.75	33.75

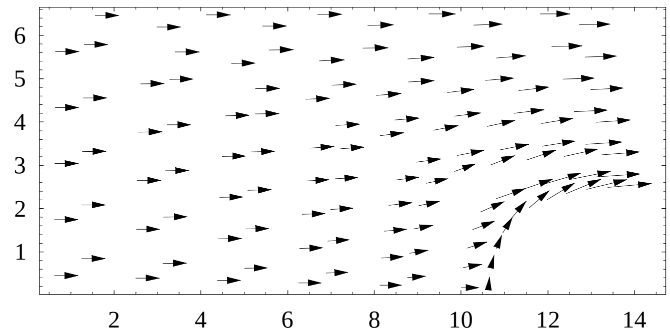
Solution at element centroids

	x−coord	y−coord	ψ	$\partial\psi/\partial x$	$\partial\psi/\partial y$
1	12.9697	2.78033	6.11146	−3.37526	8.14861
2	10.4545	3.4205	12.8093	−0.520816	6.58746
3	10.9848	5.14017	24.0593	−0.532342	6.85139
4	6.48483	5.89017	29.1979	0	5.2942
5	9.7045	1.9205	6.69786	−2.86112	1.18512
6	7.60983	1.39017	6.69786	0	4.81803
7	3.85983	3.64017	17.9479	−0.199449	4.83379
8	1.875	2.25	11.25	0	5.

In order to get a better solution we use a 120 element model as shown in Figure. The following table shows partial results for the stream function values and the velocities in the x and y direction obtained at the centroids of the elements. Using the u and v values the velocity vectors shown in Figure 5.XXX. are obtained. The velocity vectors are tangent to the stream lines and show that the finite element solution is



x	y	ψ	$u=\partial\psi/\partial y$	$v=-\partial\psi/\partial x$
11.5662	2.02003	3.12636	4.35271	3.67236
10.8125	2.22896	6.34974	6.18323	2.19341
9.85069	2.85397	11.5733	4.41314	1.68798
8.96234	3.07584	13.6316	5.32864	0.65605
8.13516	3.68791	17.319	4.99405	0.608097
7.11216	3.92273	18.9254	5.17981	0.24295
6.41963	4.52185	22.17	5.09408	0.236236
5.26198	4.76962	23.6071	5.10947	0.0807255
4.7041	5.35579	26.6377	5.07742	0.0794096
3.41179	5.6165	28.0187	5.05675	0.0197334
2.98857	6.18973	30.9208	5.03959	0.0194431
1.56161	6.46339	32.3102	5.02342	0
11.9899	2.20921	2.9261	5.79379	3.55624
11.3639	2.42207	6.05247	6.87944	2.36017
10.676	3.00993	11.1313	5.19344	1.99278
9.91652	3.23292	13.4288	5.6888	1.0255
9.36214	3.81065	17.1543	5.26698	0.978779
8.46913	4.04377	18.9862	5.31546	0.441183
8.04823	4.61137	22.1837	5.29723	0.440073
7.02174	4.85463	23.761	5.20322	0.179558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