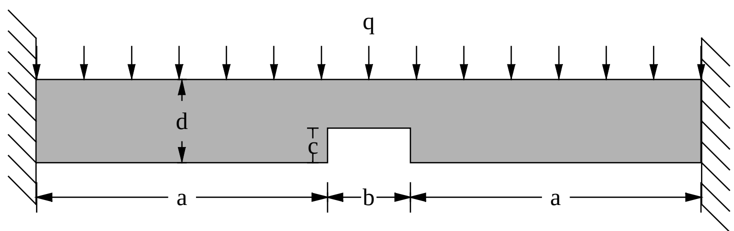
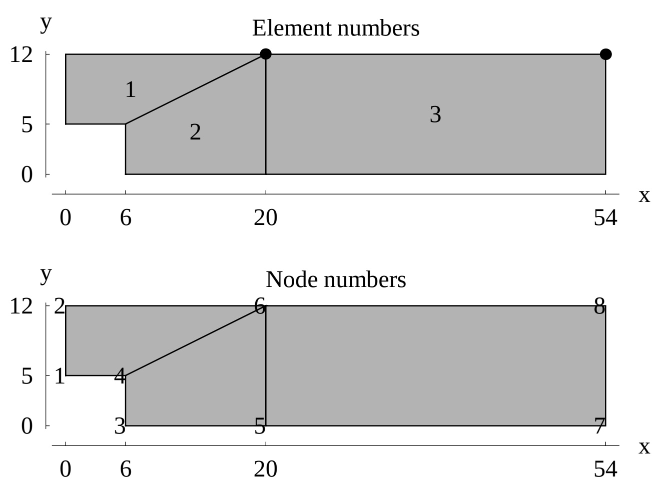
Example 7.7: Notched beam (p. 510)

Find stresses in a notched beam of rectangular cross-section shown in Figure. The following numerical values are used.

$a = 48\,\text{in}; \quad b = 12\,\text{in}; \quad c = 5\,\text{in}; \quad d = 12\,\text{in}; \quad \text{Thickness} = 4\,\text{in}; \quad E = 3\times 10^{6}\,\text{lb}/\text{in}^2; \quad \nu = 0.2; \quad q = 50\,\text{lb}/\text{in}^2$



Since the thickness of the beam is much smaller than the other dimensions, and there are no out of plane loads, the problem can be treated as a plane stress situation. Using symmetry half of the beam is modeled. To show all calculations, a very coarse model involving only three elements is used.



Global equations at start of the element assembly process

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} u_1\\ v_1\\ u_2\\ v_2\\ u_3\\ v_3\\ u_4\\ v_4\\ u_5\\ v_5\\ u_6\\ v_6\\ u_7\\ v_7\\ u_8\\ v_8 \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}$$

Equations for element 1

$E = 3000000;\qquad \nu = 0.2;\qquad h = 4$

Nodal coordinates

Element node	Global node number	x	y
1	1	0.	5.
2	4	6.	5.
3	6	20.	12.
4	2	0.	12.

Interpolation functions and their derivatives

$$\{N_1, N_2, N_3, N_4\} = \left\{\frac{1}{4}(s-1)(t-1), -\frac{1}{4}(s+1)(t-1), \frac{1}{4}(s+1)(t+1), -\frac{1}{4}(s-1)(t+1)\right\}$$

$$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{\frac{t-1}{4}, \frac{1-t}{4}, \frac{t+1}{4}, \frac{1}{4}(-t-1)\right\}$$

$$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \left\{\frac{s-1}{4}, \frac{1}{4}(-s-1), \frac{s+1}{4}, \frac{1-s}{4}\right\}$$

Mapping to the master element

$$x(s,t) = \boldsymbol{N}^T \boldsymbol{x}_n = 3.5\,t\,s + 6.5\,s + 3.5\,t + 6.5$$

$$y(s,t) = \boldsymbol{N}^T \boldsymbol{y}_n = 3.5\,t + 8.5$$

$$\boldsymbol{J} = \begin{pmatrix} 3.5\,t + 6.5 & 3.5\,s + 3.5 \\ 0 & 3.5 \end{pmatrix}; \qquad \det J = 12.25\,t + 22.75$$

$$\text{Plane stress } \boldsymbol{C} = \begin{pmatrix} 3.125\times10^{6} & 625000. & 0 \\ 625000. & 3.125\times10^{6} & 0 \\ 0 & 0 & 1.25\times10^{6} \end{pmatrix}$$

For numerical integration the Gauss quadrature points and weights are

	s	t	Weight
1	−0.57735	−0.57735	1.
2	−0.57735	0.57735	1.
3	0.57735	−0.57735	1.
4	0.57735	0.57735	1.

Computation of element matrices at {−0.57735, −0.57735} with weight = 1.

$$\boldsymbol{J} = \begin{pmatrix} 4.47927 & 1.47927 \\ 0 & 3.5 \end{pmatrix} \qquad \det J = 15.6775$$

$$\{N_1, N_2, N_3, N_4\} = \{0.622008, 0.166667, 0.0446582, 0.166667\}$$

$$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \{-0.394338, 0.394338, 0.105662, -0.105662\}$$

$$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \{-0.394338, -0.105662, 0.105662, 0.394338\}$$

$$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{-0.088036, 0.088036, 0.0235892, -0.0235892\}$$

$$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{-0.0754595, -0.0673977, 0.0202193, 0.122638\}$$

$$\boldsymbol{B}^{T} = \begin{pmatrix} -0.088036 & 0 & 0.088036 & 0 & 0.0235892 & 0 & -0.0235892 & 0 \\ 0 & -0.0754595 & 0 & -0.0673977 & 0 & 0.0202193 & 0 & 0. \\ -0.0754595 & -0.088036 & -0.0673977 & 0.088036 & 0.0202193 & 0.0235892 & 0.122638 & -0. \end{pmatrix}$$

$$\boldsymbol{k} = 10^{6}\begin{pmatrix} 1.96517 & 0.781108 & -1.12016 & -0.288186 & -0.526565 & -0.209297 & -0.318444 \\ 0.781108 & 1.7234 & 0.204736 & 0.389125 & -0.209297 & -0.461783 & -0.776547 \\ -1.12016 & 0.204736 & 1.87489 & -0.697658 & 0.300146 & -0.0548588 & -1.05488 \\ -0.288186 & 0.389125 & -0.697658 & 1.4977 & 0.0772193 & -0.104266 & 0.908625 \\ -0.526565 & -0.209297 & 0.300146 & 0.0772193 & 0.141093 & 0.056081 & 0.0853267 \\ -0.209297 & -0.461783 & -0.0548588 & -0.104266 & 0.056081 & 0.123734 & 0.208075 \\ -0.318444 & -0.776547 & -1.05488 & 0.908625 & 0.0853267 & 0.208075 & 1.28799 \\ -0.283625 & -1.65074 & 0.547781 & -1.78256 & 0.075997 & 0.442314 & -0.340153 \end{pmatrix}$$

Computation of element matrices at {−0.57735, 0.57735} with weight = 1.

$$\boldsymbol{J} = \begin{pmatrix} 8.52073 & 1.47927 \\ 0 & 3.5 \end{pmatrix} \qquad \det J = 29.8225$$

$\{N_1, N_2, N_3, N_4\} = \{0.166667, 0.0446582, 0.166667, 0.622008\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \{-0.105662, 0.105662, 0.394338, -0.394338\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \{-0.394338, -0.105662, 0.105662, 0.394338\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{-0.0124006, 0.0124006, 0.0462798, -0.0462798\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{-0.107427, -0.0354304, 0.0106291, 0.132228\}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -0.0124006 & 0 & 0.0124006 & 0 & 0.0462798 & 0 & -0.0462798 & 0 \\ 0 & -0.107427 & 0 & -0.0354304 & 0 & 0.0106291 & 0 & 0. \\ -0.107427 & -0.0124006 & -0.0354304 & 0.0124006 & 0.0106291 & 0.0462798 & 0.132228 & -0. \end{pmatrix}$$

$\boldsymbol{k} = 10^6$

$$\begin{pmatrix} 1.77816 & 0.297963 & 0.510224 & -0.165885 & -0.384204 & -0.751169 & -1.90418 & 0.619091 \\ 0.297963 & 4.32502 & -0.0338069 & 1.39594 & -0.390325 & -0.511237 & 0.126169 & -5.20972 \\ 0.510224 & -0.0338069 & 0.244508 & -0.0982711 & 0.157784 & -0.234675 & -0.912516 & 0.366753 \\ -0.165885 & 1.39594 & -0.0982711 & 0.490888 & -0.102597 & -0.0548117 & 0.366753 & -1.83202 \\ -0.384204 & -0.390325 & 0.157784 & -0.102597 & 0.815278 & 0.110026 & -0.588859 & 0.382896 \\ -0.751169 & -0.511237 & -0.234675 & -0.0548117 & 0.110026 & 0.361489 & 0.875818 & 0.20456 \\ -1.90418 & 0.126169 & -0.912516 & 0.366753 & -0.588859 & 0.875818 & 3.40556 & -1.36874 \\ 0.619091 & -5.20972 & 0.366753 & -1.83202 & 0.382896 & 0.20456 & -1.36874 & 6.83718 \end{pmatrix}$$

Computation of element matrices at $\{0.57735, -0.57735\}$ with weight = 1.

$$\boldsymbol{J} = \begin{pmatrix} 4.47927 & 5.52073 \\ 0 & 3.5 \end{pmatrix} \qquad \det J = 15.6775$$

$\{N_1, N_2, N_3, N_4\} = \{0.166667, 0.622008, 0.166667, 0.0446582\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \{-0.394338, 0.394338, 0.105662, -0.105662\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \{-0.105662, -0.394338, 0.394338, 0.105662\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{-0.088036, 0.088036, 0.0235892, -0.0235892\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{0.108674, -0.251532, 0.0754595, 0.0673977\}$

$\boldsymbol{B}^{\mathrm{T}} =$

$$\begin{pmatrix} -0.088036 & 0 & 0.088036 & 0 & 0.0235892 & 0 & -0.0235892 & 0 \\ 0 & 0.108674 & 0 & -0.251532 & 0 & 0.0754595 & 0 & 0.0673977 \\ 0.108674 & -0.088036 & -0.251532 & 0.088036 & 0.0754595 & 0.0235892 & 0.0673977 & -0.0235892 \end{pmatrix}$$

$$\boldsymbol{k} = 10^6 \begin{pmatrix} 2.44459 & -1.12493 & -3.66154 & 1.61785 & 0.235849 & -0.0594203 & 0.981107 & - \\ -1.12493 & 2.92194 & 2.11077 & -5.96433 & -0.420264 & 1.44425 & -0.56558 & \\ -3.66154 & 2.11077 & 6.47824 & -2.60369 & -1.08086 & -0.204736 & -1.73584 & \\ 1.61785 & -5.96433 & -2.60369 & 13.0061 & 0.288186 & -3.55678 & 0.697658 & - \\ 0.235849 & -0.420264 & -1.08086 & 0.288186 & 0.555394 & 0.209297 & 0.289615 & - \\ -0.0594203 & 1.44425 & -0.204736 & -3.55678 & 0.209297 & 1.15949 & 0.0548588 & \\ 0.981107 & -0.56558 & -1.73584 & 0.697658 & 0.289615 & 0.0548588 & 0.465117 & - \\ -0.433502 & 1.59814 & 0.697658 & -3.48497 & -0.0772193 & 0.953035 & -0.186937 & \end{pmatrix}$$

Computation of element matrices at {0.57735, 0.57735} with weight = 1.

$$\boldsymbol{J} = \begin{pmatrix} 8.52073 & 5.52073 \\ 0 & 3.5 \end{pmatrix} \qquad \det J = 29.8225$$

$\{N_1, N_2, N_3, N_4\} = \{0.0446582, 0.166667, 0.622008, 0.166667\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \{-0.105662, 0.105662, 0.394338, -0.394338\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \{-0.105662, -0.394338, 0.394338, 0.105662\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{-0.0124006, 0.0124006, 0.0462798, -0.0462798\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{-0.0106291, -0.132228, 0.0396684, 0.103189\}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -0.0124006 & 0 & 0.0124006 & 0 & 0.0462798 & 0 & -0.0462798 & 0 \\ 0 & -0.0106291 & 0 & -0.132228 & 0 & 0.0396684 & 0 & 0. \\ -0.0106291 & -0.0124006 & -0.132228 & 0.0124006 & 0.0396684 & 0.0462798 & 0.103189 & -0. \end{pmatrix}$$

$$\boldsymbol{k} = 10^6 \begin{pmatrix} 0.0741713 & 0.0294813 & 0.152248 & 0.102597 & -0.276811 & -0.110026 & 0.0503915 & - \\ 0.0294813 & 0.0650461 & 0.234675 & 0.501003 & -0.110026 & -0.242755 & -0.15413 & - \\ 0.152248 & 0.234675 & 2.66445 & -0.366753 & -0.568198 & -0.875818 & -2.2485 & \\ 0.102597 & 0.501003 & -0.366753 & 6.54074 & -0.382896 & -1.86977 & 0.647052 & - \\ -0.276811 & -0.110026 & -0.568198 & -0.382896 & 1.03307 & 0.410622 & -0.188064 & \\ -0.110026 & -0.242755 & -0.875818 & -1.86977 & 0.410622 & 0.905976 & 0.575222 & \\ 0.0503915 & -0.15413 & -2.2485 & 0.647052 & -0.188064 & 0.575222 & 2.38617 & - \\ -0.0220522 & -0.323293 & 1.0079 & -5.17198 & 0.0822999 & 1.20655 & -1.06814 & \end{pmatrix}$$

Summing contributions from all points we get

$\boldsymbol{k} = 10^6$

$$\begin{pmatrix} 6.26209 & -0.0163755 & -4.11923 & 1.26638 & -0.951731 & -1.12991 & -1.19113 & -0.120087 \\ -0.0163755 & 9.0354 & 2.51638 & -3.67826 & -1.12991 & 0.228478 & -1.37009 & -5.58562 \\ -4.11923 & 2.51638 & 11.2621 & -3.76638 & -1.19113 & -1.37009 & -5.95173 & 2.62009 \\ 1.26638 & -3.67826 & -3.76638 & 21.5354 & -0.120087 & -5.58562 & 2.62009 & -12.2715 \\ -0.951731 & -1.12991 & -1.19113 & -0.120087 & 2.54484 & 0.786026 & -0.401981 & 0.463974 \\ -1.12991 & 0.228478 & -1.37009 & -5.58562 & 0.786026 & 2.55069 & 1.71397 & 2.80646 \\ -1.19113 & -1.37009 & -5.95173 & 2.62009 & -0.401981 & 1.71397 & 7.54484 & -2.96397 \\ -0.120087 & -5.58562 & 2.62009 & -12.2715 & 0.463974 & 2.80646 & -2.96397 & 15.0507 \end{pmatrix}$$

$\boldsymbol{r}^T = (\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix})$

Computation of element matrices resulting from NBC

NBC on side 3 with $\{q_n, q_t\} = \{-50, 0\}$

$\{N_1, N_2, N_3, N_4\}_c = \left\{0, 0, \frac{1-a}{2}, \frac{a+1}{2}\right\}$

$x(a) = 10. - 10.\,a;$ $\quad$ $y(a) = 12.$

$dx/da = -10.;$ $\quad$ $dy/da = 0.$

$J_c = 10.$

Gauss point $= -0.57735;$ $\quad$ Weight $= 1.;$ $\quad$ $J_c = 10.$

$\{N_1, N_2, N_3, N_4\}_c = \{0, 0, 0.788675, 0.211325\}$

$\boldsymbol{r}_q^T = (\begin{matrix} 0 & 0 & 0 & 0 & 0 & -1577.35 & 0 & -422.65 \end{matrix})$

Gauss point $= 0.57735;$ $\quad$ Weight $= 1.;$ $\quad$ $J_c = 10.$

$\{N_1, N_2, N_3, N_4\}_c = \{0, 0, 0.211325, 0.788675\}$

$\boldsymbol{r}_q^T = (\begin{matrix} 0 & 0 & 0 & 0 & 0 & -422.65 & 0 & -1577.35 \end{matrix})$

Summing contributions from all Gauss points

$\boldsymbol{r}_q^T = (\begin{matrix} 0 & 0 & 0 & 0 & 0 & -2000. & 0 & -2000. \end{matrix})$

Complete element equations for element 1

$$10^6\begin{pmatrix} 6.26209 & -0.0163755 & -4.11923 & 1.26638 & -0.951731 & -1.12991 & -1.19113 & -0.120087 \\ -0.0163755 & 9.0354 & 2.51638 & -3.67826 & -1.12991 & 0.228478 & -1.37009 & -5.58562 \\ -4.11923 & 2.51638 & 11.2621 & -3.76638 & -1.19113 & -1.37009 & -5.95173 & 2.62009 \\ 1.26638 & -3.67826 & -3.76638 & 21.5354 & -0.120087 & -5.58562 & 2.62009 & -12.2715 \\ -0.951731 & -1.12991 & -1.19113 & -0.120087 & 2.54484 & 0.786026 & -0.401981 & 0.463974 \\ -1.12991 & 0.228478 & -1.37009 & -5.58562 & 0.786026 & 2.55069 & 1.71397 & 2.80646 \\ -1.19113 & -1.37009 & -5.95173 & 2.62009 & -0.401981 & 1.71397 & 7.54484 & -2.96397 \\ -0.120087 & -5.58562 & 2.62009 & -12.2715 & 0.463974 & 2.80646 & -2.96397 & 15.0507 \end{pmatrix}$$

$$\begin{pmatrix} u_1 \\ v_1 \\ u_4 \\ v_4 \\ u_6 \\ v_6 \\ u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ -2000. \\ 0. \\ -2000. \end{pmatrix}$$

The element contributes to {1, 2, 7, 8, 11, 12, 3, 4} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 1 \\ 2 \\ 7 \\ 8 \\ 11 \\ 12 \\ 3 \\ 4 \end{pmatrix}$

and to a global matrix: $\begin{pmatrix} [1,1] & [1,2] & [1,7] & [1,8] & [1,11] & [1,12] & [1,3] & [1,4] \\ [2,1] & [2,2] & [2,7] & [2,8] & [2,11] & [2,12] & [2,3] & [2,4] \\ [7,1] & [7,2] & [7,7] & [7,8] & [7,11] & [7,12] & [7,3] & [7,4] \\ [8,1] & [8,2] & [8,7] & [8,8] & [8,11] & [8,12] & [8,3] & [8,4] \\ [11,1] & [11,2] & [11,7] & [11,8] & [11,11] & [11,12] & [11,3] & [11,4] \\ [12,1] & [12,2] & [12,7] & [12,8] & [12,11] & [12,12] & [12,3] & [12,4] \\ [3,1] & [3,2] & [3,7] & [3,8] & [3,11] & [3,12] & [3,3] & [3,4] \\ [4,1] & [4,2] & [4,7] & [4,8] & [4,11] & [4,12] & [4,3] & [4,4] \end{pmatrix}$

Adding element equations into appropriate locations we have

$$10^6 \left(\begin{array}{llllllllllll}
6.26209 & -0.0163755 & -1.19113 & -0.120087 & 0 & 0 & -4.11923 & 1.26638 & 0 & 0 & -0.951731 & -1 \\
-0.0163755 & 9.0354 & -1.37009 & -5.58562 & 0 & 0 & 2.51638 & -3.67826 & 0 & 0 & -1.12991 & 0 \\
-1.19113 & -1.37009 & 7.54484 & -2.96397 & 0 & 0 & -5.95173 & 2.62009 & 0 & 0 & -0.401981 & 1 \\
-0.120087 & -5.58562 & -2.96397 & 15.0507 & 0 & 0 & 2.62009 & -12.2715 & 0 & 0 & 0.463974 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-4.11923 & 2.51638 & -5.95173 & 2.62009 & 0 & 0 & 11.2621 & -3.76638 & 0 & 0 & -1.19113 & -1 \\
1.26638 & -3.67826 & 2.62009 & -12.2715 & 0 & 0 & -3.76638 & 21.5354 & 0 & 0 & -0.120087 & -5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.951731 & -1.12991 & -0.401981 & 0.463974 & 0 & 0 & -1.19113 & -0.120087 & 0 & 0 & 2.54484 & 0 \\
-1.12991 & 0.228478 & 1.71397 & 2.80646 & 0 & 0 & -1.37009 & -5.58562 & 0 & 0 & 0.786026 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array} \right.$$

Equations for element 2

$E = 3000000;\qquad \nu = 0.2;\qquad h = 4$

Nodal coordinates

Element node	Global node number	x	y
1	3	6.	0.
2	5	20.	0.
3	6	20.	12.
4	4	6.	5.

Interpolation functions and their derivatives

$$\{N_1,\ N_2,\ N_3,\ N_4\} = \left\{\frac{1}{4}(s-1)(t-1),\ -\frac{1}{4}(s+1)(t-1),\ \frac{1}{4}(s+1)(t+1),\ -\frac{1}{4}(s-1)(t+1)\right\}$$

$$\{\partial N_1/\partial s,\ \partial N_2/\partial s,\ \partial N_3/\partial s,\ \partial N_4/\partial s\} = \left\{\frac{t-1}{4},\ \frac{1-t}{4},\ \frac{t+1}{4},\ \frac{1}{4}(-t-1)\right\}$$

$$\{\partial N_1/\partial t,\ \partial N_2/\partial t,\ \partial N_3/\partial t,\ \partial N_4/\partial t\} = \left\{\frac{s-1}{4},\ \frac{1}{4}(-s-1),\ \frac{s+1}{4},\ \frac{1-s}{4}\right\}$$

Mapping to the master element

$$x(s,t) = \boldsymbol{N}^T \boldsymbol{x}_n = 7.\,s + 13.$$

$$y(s,t) = \boldsymbol{N}^T \boldsymbol{y}_n = 1.75\,t\,s + 1.75\,s + 4.25\,t + 4.25$$

$$\boldsymbol{J} = \begin{pmatrix} 7. & 0 \\ 1.75\,t + 1.75 & 1.75\,s + 4.25 \end{pmatrix}; \qquad \det J = 12.25\,s + 29.75$$

$$\text{Plane stress } \boldsymbol{C} = \begin{pmatrix} 3.125\times 10^{6} & 625000. & 0 \\ 625000. & 3.125\times 10^{6} & 0 \\ 0 & 0 & 1.25\times 10^{6} \end{pmatrix}$$

For numerical integration the Gauss quadrature points and weights are

	s	t	Weight
1	−0.57735	−0.57735	1.
2	−0.57735	0.57735	1.
3	0.57735	−0.57735	1.
4	0.57735	0.57735	1.

Computation of element matrices at {−0.57735, −0.57735} with weight = 1.

$$\boldsymbol{J} = \begin{pmatrix} 7. & 0 \\ 0.739637 & 3.23964 \end{pmatrix} \qquad \det J = 22.6775$$

$\{N_1, N_2, N_3, N_4\} = \{0.622008, 0.166667, 0.0446582, 0.166667\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \{-0.394338, 0.394338, 0.105662, -0.105662\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \{-0.394338, -0.105662, 0.105662, 0.394338\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{-0.0434724, 0.0597802, 0.0116484, -0.0279562\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{-0.121723, -0.0326155, 0.0326155, 0.121723\}$

$$\boldsymbol{B}^{T} = \begin{pmatrix} -0.0434724 & 0 & 0.0597802 & 0 & 0.0116484 & 0 & -0.0279562 & 0 \\ 0 & -0.121723 & 0 & -0.0326155 & 0 & 0.0326155 & 0 & 0. \\ -0.121723 & -0.0434724 & -0.0326155 & 0.0597802 & 0.0326155 & 0.0116484 & 0.121723 & -0. \end{pmatrix}$$

$$\boldsymbol{k} = 10^{6} \begin{pmatrix} 2.21571 & 0.899997 & -0.286521 & -0.74469 & -0.593697 & -0.241154 & -1.33549 \\ 0.899997 & 4.41427 & -0.251768 & 0.830714 & -0.241154 & -1.1828 & -0.407075 \\ -0.286521 & -0.251768 & 1.13364 & -0.331617 & 0.076773 & 0.0674611 & -0.923892 \\ -0.74469 & 0.830714 & -0.331617 & 0.706754 & 0.199539 & -0.222589 & 0.876768 \\ -0.593697 & -0.241154 & 0.076773 & 0.199539 & 0.159081 & 0.0646169 & 0.357843 \\ -0.241154 & -1.1828 & 0.0674611 & -0.222589 & 0.0646169 & 0.316931 & 0.109076 \\ -1.33549 & -0.407075 & -0.923892 & 0.876768 & 0.357843 & 0.109076 & 1.90154 \\ 0.0858466 & -4.06219 & 0.515924 & -1.31488 & -0.0230025 & 1.08846 & -0.578769 \end{pmatrix}$$

Computation of element matrices at {−0.57735, 0.57735} with weight = 1.

$$\boldsymbol{J} = \begin{pmatrix} 7. & 0 \\ 2.76036 & 3.23964 \end{pmatrix} \qquad \det J = 22.6775$$

$\{N_1, N_2, N_3, N_4\} = \{0.166667, 0.0446582, 0.166667, 0.622008\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \{-0.105662, 0.105662, 0.394338, -0.394338\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \{-0.394338, -0.105662, 0.105662, 0.394338\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{0.0329052, 0.0279562, 0.0434724, -0.104334\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{-0.121723, -0.0326155, 0.0326155, 0.121723\}$

$\boldsymbol{B}^T =$

$$\begin{pmatrix} 0.0329052 & 0 & 0.0279562 & 0 & 0.0434724 & 0 & -0.104334 & 0 \\ 0 & -0.121723 & 0 & -0.0326155 & 0 & 0.0326155 & 0 & 0.121723 \\ -0.121723 & 0.0329052 & -0.0326155 & 0.0279562 & 0.0326155 & 0.0434724 & 0.121723 & -0.104334 \end{pmatrix}$$

$\boldsymbol{k} = 10^6$

$$\begin{pmatrix} 1.98692 & -0.681228 & 0.710917 & -0.446691 & -0.0446606 & -0.539153 & -2.65318 & 1.66707 \\ -0.681228 & 4.32276 & -0.314612 & 1.22969 & -0.178309 & -0.963186 & 1.17415 & -4.58926 \\ 0.710917 & -0.314612 & 0.342162 & -0.155081 & 0.223887 & -0.109076 & -1.27697 & 0.578769 \\ -0.446691 & 1.22969 & -0.155081 & 0.390163 & 0.0230025 & -0.163744 & 0.578769 & -1.45611 \\ -0.0446606 & -0.178309 & 0.223887 & 0.0230025 & 0.656331 & 0.241154 & -0.835557 & -0.0858466 \\ -0.539153 & -0.963186 & -0.109076 & -0.163744 & 0.241154 & 0.515831 & 0.407075 & 0.611099 \\ -2.65318 & 1.17415 & -1.27697 & 0.578769 & -0.835557 & 0.407075 & 4.7657 & -2.15999 \\ 1.66707 & -4.58926 & 0.578769 & -1.45611 & -0.0858466 & 0.611099 & -2.15999 & 5.43427 \end{pmatrix}$$

Computation of element matrices at $\{0.57735, -0.57735\}$ with weight = 1.

$\boldsymbol{J} = \begin{pmatrix} 7. & 0 \\ 0.739637 & 5.26036 \end{pmatrix}$ detJ = 36.8225

$\{N_1, N_2, N_3, N_4\} = \{0.166667, 0.622008, 0.166667, 0.0446582\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \{-0.394338, 0.394338, 0.105662, -0.105662\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \{-0.105662, -0.394338, 0.394338, 0.105662\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{-0.0542115, 0.0642548, 0.00717376, -0.017217\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{-0.0200865, -0.0749639, 0.0749639, 0.0200865\}$

$$\boldsymbol{B}^T = \begin{pmatrix} -0.0542115 & 0 & 0.0642548 & 0 & 0.00717376 & 0 & -0.017217 \\ 0 & -0.0200865 & 0 & -0.0749639 & 0 & 0.0749639 & 0 \\ -0.0200865 & -0.0542115 & -0.0749639 & 0.0642548 & 0.0749639 & 0.00717376 & 0.0200865 \end{pmatrix}$$

$$\boldsymbol{k} = 10^6 \begin{pmatrix} 1.427 & 0.300727 & -1.32609 & 0.136483 & -0.456234 & -0.400639 & 0.355326 \\ 0.300727 & 0.726797 & 0.629405 & 0.0517469 & -0.761483 & -0.764678 & -0.168648 \\ -1.32609 & 0.629405 & 2.93499 & -1.33025 & -0.822472 & 0.344406 & -0.78643 \\ 0.136483 & 0.0517469 & -1.33025 & 3.34674 & 0.837328 & -2.50173 & 0.356439 \\ -0.456234 & -0.761483 & -0.822472 & 0.837328 & 1.05833 & 0.148516 & 0.220381 \\ -0.400639 & -0.764678 & 0.344406 & -2.50173 & 0.148516 & 2.59607 & -0.0922832 \\ 0.355326 & -0.168648 & -0.78643 & 0.356439 & 0.220381 & -0.0922832 & 0.210723 \\ -0.0365704 & -0.0138655 & 0.356439 & -0.896756 & -0.224361 & 0.670336 & -0.0955076 \end{pmatrix}$$

Computation of element matrices at {0.57735, 0.57735} with weight = 1.

$$\boldsymbol{J} = \begin{pmatrix} 7. & 0 \\ 2.76036 & 5.26036 \end{pmatrix} \qquad \det J = 36.8225$$

$\{N_1, N_2, N_3, N_4\} = \{0.0446582, 0.166667, 0.622008, 0.166667\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \{-0.105662, 0.105662, 0.394338, -0.394338\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \{-0.105662, -0.394338, 0.394338, 0.105662\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{-0.00717376, 0.0446557, 0.0267728, -0.0642548\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{-0.0200865, -0.0749639, 0.0749639, 0.0200865\}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -0.00717376 & 0 & 0.0446557 & 0 & 0.0267728 & 0 & -0.0642548 \\ 0 & -0.0200865 & 0 & -0.0749639 & 0 & 0.0749639 & 0 \\ -0.0200865 & -0.00717376 & -0.0749639 & 0.0446557 & 0.0749639 & 0.0267728 & 0.0200865 & - \end{pmatrix}$$

$$\boldsymbol{k} = 10^6 \begin{pmatrix} 0.0979711 & 0.0397948 & 0.12978 & -0.11564 & -0.365633 & -0.148516 & 0.137883 \\ 0.0397948 & 0.195184 & 0.0164383 & 0.634096 & -0.148516 & -0.728437 & 0.0922832 \\ 0.12978 & 0.0164383 & 1.9525 & -0.924495 & -0.484344 & -0.0613487 & -1.59794 \\ -0.11564 & 0.634096 & -0.924495 & 2.95374 & 0.431573 & -2.36648 & 0.608562 \\ -0.365633 & -0.148516 & -0.484344 & 0.431573 & 1.36456 & 0.554271 & -0.514585 \\ -0.148516 & -0.728437 & -0.0613487 & -2.36648 & 0.554271 & 2.71857 & -0.344406 \\ 0.137883 & 0.0922832 & -1.59794 & 0.608562 & -0.514585 & -0.344406 & 1.97464 \\ 0.224361 & -0.100843 & 0.969406 & -1.22136 & -0.837328 & 0.37635 & -0.356439 \end{pmatrix}$$

Summing contributions from all points we get

$$\boldsymbol{k} = 10^{6}\begin{pmatrix} 5.7276 & 0.559291 & -0.771918 & -1.17054 & -1.46023 & -1.32946 & -3.49546 & 1.94071 \\ 0.559291 & 9.65901 & 0.0794621 & 2.74625 & -1.32946 & -3.6391 & 0.690709 & -8.76615 \\ -0.771918 & 0.0794621 & 6.3633 & -2.74144 & -1.00616 & 0.241443 & -4.58523 & 2.42054 \\ -1.17054 & 2.74625 & -2.74144 & 7.3974 & 1.49144 & -5.25454 & 2.42054 & -4.8891 \\ -1.46023 & -1.32946 & -1.00616 & 1.49144 & 3.2383 & 1.00856 & -0.771918 & -1.17054 \\ -1.32946 & -3.6391 & 0.241443 & -5.25454 & 1.00856 & 6.1474 & 0.0794621 & 2.74625 \\ -3.49546 & 0.690709 & -4.58523 & 2.42054 & -0.771918 & 0.0794621 & 8.8526 & -3.19071 \\ 1.94071 & -8.76615 & 2.42054 & -4.8891 & -1.17054 & 2.74625 & -3.19071 & 10.909 \end{pmatrix}$$

$$\boldsymbol{r}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Complete element equations for element 2

$$10^{6}\begin{pmatrix} 5.7276 & 0.559291 & -0.771918 & -1.17054 & -1.46023 & -1.32946 & -3.49546 & 1.94071 \\ 0.559291 & 9.65901 & 0.0794621 & 2.74625 & -1.32946 & -3.6391 & 0.690709 & -8.76615 \\ -0.771918 & 0.0794621 & 6.3633 & -2.74144 & -1.00616 & 0.241443 & -4.58523 & 2.42054 \\ -1.17054 & 2.74625 & -2.74144 & 7.3974 & 1.49144 & -5.25454 & 2.42054 & -4.8891 \\ -1.46023 & -1.32946 & -1.00616 & 1.49144 & 3.2383 & 1.00856 & -0.771918 & -1.17054 \\ -1.32946 & -3.6391 & 0.241443 & -5.25454 & 1.00856 & 6.1474 & 0.0794621 & 2.74625 \\ -3.49546 & 0.690709 & -4.58523 & 2.42054 & -0.771918 & 0.0794621 & 8.8526 & -3.19071 \\ 1.94071 & -8.76615 & 2.42054 & -4.8891 & -1.17054 & 2.74625 & -3.19071 & 10.909 \end{pmatrix}$$

$$\begin{pmatrix} u_3 \\ v_3 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {5, 6, 9, 10, 11, 12, 7, 8} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 5 \\ 6 \\ 9 \\ 10 \\ 11 \\ 12 \\ 7 \\ 8 \end{pmatrix}$

and to a global matrix: $\begin{pmatrix} [5, 5] & [5, 6] & [5, 9] & [5, 10] & [5, 11] & [5, 12] & [5, 7] & [5, 8] \\ [6, 5] & [6, 6] & [6, 9] & [6, 10] & [6, 11] & [6, 12] & [6, 7] & [6, 8] \\ [9, 5] & [9, 6] & [9, 9] & [9, 10] & [9, 11] & [9, 12] & [9, 7] & [9, 8] \\ [10, 5] & [10, 6] & [10, 9] & [10, 10] & [10, 11] & [10, 12] & [10, 7] & [10, 8] \\ [11, 5] & [11, 6] & [11, 9] & [11, 10] & [11, 11] & [11, 12] & [11, 7] & [11, 8] \\ [12, 5] & [12, 6] & [12, 9] & [12, 10] & [12, 11] & [12, 12] & [12, 7] & [12, 8] \\ [7, 5] & [7, 6] & [7, 9] & [7, 10] & [7, 11] & [7, 12] & [7, 7] & [7, 8] \\ [8, 5] & [8, 6] & [8, 9] & [8, 10] & [8, 11] & [8, 12] & [8, 7] & [8, 8] \end{pmatrix}$

Adding element equations into appropriate locations we have

$$10^6 \begin{pmatrix} 6.26209 & -0.0163755 & -1.19113 & -0.120087 & 0 & 0 & -4.11923 & 1.26638 \\ -0.0163755 & 9.0354 & -1.37009 & -5.58562 & 0 & 0 & 2.51638 & -3.67826 \\ -1.19113 & -1.37009 & 7.54484 & -2.96397 & 0 & 0 & -5.95173 & 2.62009 \\ -0.120087 & -5.58562 & -2.96397 & 15.0507 & 0 & 0 & 2.62009 & -12.2715 \\ 0 & 0 & 0 & 0 & 5.7276 & 0.559291 & -3.49546 & 1.94071 \\ 0 & 0 & 0 & 0 & 0.559291 & 9.65901 & 0.690709 & -8.76615 \\ -4.11923 & 2.51638 & -5.95173 & 2.62009 & -3.49546 & 0.690709 & 20.1147 & -6.95708 \\ 1.26638 & -3.67826 & 2.62009 & -12.2715 & 1.94071 & -8.76615 & -6.95708 & 32.4444 \\ 0 & 0 & 0 & 0 & -0.771918 & 0.0794621 & -4.58523 & 2.42054 \\ 0 & 0 & 0 & 0 & -1.17054 & 2.74625 & 2.42054 & -4.8891 \\ -0.951731 & -1.12991 & -0.401981 & 0.463974 & -1.46023 & -1.32946 & -1.96304 & -1.29063 \\ -1.12991 & 0.228478 & 1.71397 & 2.80646 & -1.32946 & -3.6391 & -1.29063 & -2.83938 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Equations for element 3

$E = 3000000;$ $\nu = 0.2;$ $h = 4$

Nodal coordinates

Element node	Global node number	x	y
1	5	20.	0.
2	7	54.	0.
3	8	54.	12.
4	6	20.	12.

Interpolation functions and their derivatives

$$\{N_1,\ N_2,\ N_3,\ N_4\} = \left\{\frac{1}{4}(s-1)(t-1),\ -\frac{1}{4}(s+1)(t-1),\ \frac{1}{4}(s+1)(t+1),\ -\frac{1}{4}(s-1)(t+1)\right\}$$

$$\{\partial N_1/\partial s,\ \partial N_2/\partial s,\ \partial N_3/\partial s,\ \partial N_4/\partial s\} = \left\{\frac{t-1}{4},\ \frac{1-t}{4},\ \frac{t+1}{4},\ \frac{1}{4}(-t-1)\right\}$$

$$\{\partial N_1/\partial t,\ \partial N_2/\partial t,\ \partial N_3/\partial t,\ \partial N_4/\partial t\} = \left\{\frac{s-1}{4},\ \frac{1}{4}(-s-1),\ \frac{s+1}{4},\ \frac{1-s}{4}\right\}$$

Mapping to the master element

$$x(s,t) = \boldsymbol{N}^T \boldsymbol{x}_n = 17.\,s + 37.$$

$$y(s,t) = \boldsymbol{N}^T \boldsymbol{y}_n = 6.\,t + 6.$$

$$\boldsymbol{J} = \begin{pmatrix} 17. & 0 \\ 0 & 6. \end{pmatrix}; \qquad \det J = 102.$$

Plane stress $\boldsymbol{C} = \begin{pmatrix} 3.125\times10^6 & 625000. & 0 \\ 625000. & 3.125\times10^6 & 0 \\ 0 & 0 & 1.25\times10^6 \end{pmatrix}$

For numerical integration the Gauss quadrature points and weights are

	s	t	Weight
1	−0.57735	−0.57735	1.
2	−0.57735	0.57735	1.
3	0.57735	−0.57735	1.
4	0.57735	0.57735	1.

Computation of element matrices at {−0.57735, −0.57735} with weight = 1.

$$\boldsymbol{J} = \begin{pmatrix} 17. & 0 \\ 0 & 6. \end{pmatrix} \qquad \det J = 102.$$

$$\{N_1,\ N_2,\ N_3,\ N_4\} = \{0.622008,\ 0.166667,\ 0.0446582,\ 0.166667\}$$

$$\{\partial N_1/\partial s,\ \partial N_2/\partial s,\ \partial N_3/\partial s,\ \partial N_4/\partial s\} = \{-0.394338,\ 0.394338,\ 0.105662,\ -0.105662\}$$

$$\{\partial N_1/\partial t,\ \partial N_2/\partial t,\ \partial N_3/\partial t,\ \partial N_4/\partial t\} = \{-0.394338,\ -0.105662,\ 0.105662,\ 0.394338\}$$

$$\{\partial N_1/\partial x,\ \partial N_2/\partial x,\ \partial N_3/\partial x,\ \partial N_4/\partial x\} = \{-0.0231963,\ 0.0231963,\ 0.00621544,\ -0.00621544\}$$

$$\{\partial N_1/\partial y,\ \partial N_2/\partial y,\ \partial N_3/\partial y,\ \partial N_4/\partial y\} = \{-0.0657229,\ -0.0176104,\ 0.0176104,\ 0.0657229\}$$

$$\boldsymbol{B}^T = \begin{pmatrix} -0.0231963 & 0 & 0.0231963 & 0 & 0.00621544 & 0 & -0.00621544 \\ 0 & -0.0657229 & 0 & -0.0176104 & 0 & 0.0176104 & 0 \\ -0.0657229 & -0.0231963 & -0.0176104 & 0.0231963 & 0.0176104 & 0.00621544 & 0.0657229 \end{pmatrix}$$

$\boldsymbol{k} = 10^6$

$$\begin{pmatrix} 2.88899 & 1.16627 & -0.095761 & -0.673344 & -0.774101 & -0.3125 & -2.01912 & -0.180422 \\ 1.16627 & 5.78178 & -0.180422 & 1.20128 & -0.3125 & -1.54922 & -0.673344 & -5.43384 \\ -0.095761 & -0.180422 & 0.844203 & -0.3125 & 0.0256591 & 0.0483439 & -0.774101 & 0.444578 \\ -0.673344 & 1.20128 & -0.3125 & 0.669827 & 0.180422 & -0.321882 & 0.805422 & -1.54922 \\ -0.774101 & -0.3125 & 0.0256591 & 0.180422 & 0.20742 & 0.0837341 & 0.541022 & 0.0483439 \\ -0.3125 & -1.54922 & 0.0483439 & -0.321882 & 0.0837341 & 0.415113 & 0.180422 & 1.45599 \\ -2.01912 & -0.673344 & -0.774101 & 0.805422 & 0.541022 & 0.180422 & 2.2522 & -0.3125 \\ -0.180422 & -5.43384 & 0.444578 & -1.54922 & 0.0483439 & 1.45599 & -0.3125 & 5.52707 \end{pmatrix}$$

Computation of element matrices at {−0.57735, 0.57735} with weight = 1.

$$\boldsymbol{J} = \begin{pmatrix} 17. & 0 \\ 0 & 6. \end{pmatrix} \qquad \text{detJ} = 102.$$

$\{N_1, N_2, N_3, N_4\} = \{0.166667, 0.0446582, 0.166667, 0.622008\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \{-0.105662, 0.105662, 0.394338, -0.394338\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \{-0.394338, -0.105662, 0.105662, 0.394338\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{-0.00621544, 0.00621544, 0.0231963, -0.0231963\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{-0.0657229, -0.0176104, 0.0176104, 0.0657229\}$

$$\boldsymbol{B}^{\mathrm{T}} = \left(\begin{matrix} -0.00621544 & 0 & 0.00621544 & 0 & 0.0231963 & 0 & -0.0231963 \\ 0 & -0.0657229 & 0 & -0.0176104 & 0 & 0.0176104 & 0 \\ -0.0657229 & -0.00621544 & -0.0176104 & 0.00621544 & 0.0176104 & 0.0231963 & 0.0657229 \end{matrix} \right.$$

$\boldsymbol{k} = 10^6$

$$\begin{pmatrix} 2.2522 & 0.3125 & 0.541022 & -0.180422 & -0.774101 & -0.805422 & -2.01912 & 0.673344 \\ 0.3125 & 5.52707 & -0.0483439 & 1.45599 & -0.444578 & -1.54922 & 0.180422 & -5.43384 \\ 0.541022 & -0.0483439 & 0.20742 & -0.0837341 & 0.0256591 & -0.180422 & -0.774101 & 0.3125 \\ -0.180422 & 1.45599 & -0.0837341 & 0.415113 & -0.0483439 & -0.321882 & 0.3125 & -1.54922 \\ -0.774101 & -0.444578 & 0.0256591 & -0.0483439 & 0.844203 & 0.3125 & -0.095761 & 0.180422 \\ -0.805422 & -1.54922 & -0.180422 & -0.321882 & 0.3125 & 0.669827 & 0.673344 & 1.20128 \\ -2.01912 & 0.180422 & -0.774101 & 0.3125 & -0.095761 & 0.673344 & 2.88899 & -1.16627 \\ 0.673344 & -5.43384 & 0.3125 & -1.54922 & 0.180422 & 1.20128 & -1.16627 & 5.78178 \end{pmatrix}$$

Computation of element matrices at {0.57735, −0.57735} with weight = 1.

$$\boldsymbol{J} = \begin{pmatrix} 17. & 0 \\ 0 & 6. \end{pmatrix} \qquad \text{detJ} = 102.$$

$\{N_1, N_2, N_3, N_4\} = \{0.166667, 0.622008, 0.166667, 0.0446582\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \{-0.394338, 0.394338, 0.105662, -0.105662\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \{-0.105662, -0.394338, 0.394338, 0.105662\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{-0.0231963, 0.0231963, 0.00621544, -0.00621544\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{-0.0176104, -0.0657229, 0.0657229, 0.0176104\}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -0.0231963 & 0 & 0.0231963 & 0 & 0.00621544 & 0 & -0.00621544 \\ 0 & -0.0176104 & 0 & -0.0657229 & 0 & 0.0657229 & 0 \\ -0.0176104 & -0.0231963 & -0.0657229 & 0.0231963 & 0.0657229 & 0.00621544 & 0.0176104 \end{pmatrix}$$

$\boldsymbol{k} = 10^6$

$$\begin{pmatrix} 0.844203 & 0.3125 & -0.095761 & 0.180422 & -0.774101 & -0.444578 & 0.0256591 & -0.0483439 \\ 0.3125 & 0.669827 & 0.673344 & 1.20128 & -0.805422 & -1.54922 & -0.180422 & -0.321882 \\ -0.095761 & 0.673344 & 2.88899 & -1.16627 & -2.01912 & 0.180422 & -0.774101 & 0.3125 \\ 0.180422 & 1.20128 & -1.16627 & 5.78178 & 0.673344 & -5.43384 & 0.3125 & -1.54922 \\ -0.774101 & -0.805422 & -2.01912 & 0.673344 & 2.2522 & 0.3125 & 0.541022 & -0.180422 \\ -0.444578 & -1.54922 & 0.180422 & -5.43384 & 0.3125 & 5.52707 & -0.0483439 & 1.45599 \\ 0.0256591 & -0.180422 & -0.774101 & 0.3125 & 0.541022 & -0.0483439 & 0.20742 & -0.0837341 \\ -0.0483439 & -0.321882 & 0.3125 & -1.54922 & -0.180422 & 1.45599 & -0.0837341 & 0.415113 \end{pmatrix}$$

Computation of element matrices at {0.57735, 0.57735} with weight = 1.

$\boldsymbol{J} = \begin{pmatrix} 17. & 0 \\ 0 & 6. \end{pmatrix}$ detJ = 102.

$\{N_1, N_2, N_3, N_4\} = \{0.0446582, 0.166667, 0.622008, 0.166667\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \{-0.105662, 0.105662, 0.394338, -0.394338\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \{-0.105662, -0.394338, 0.394338, 0.105662\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{-0.00621544, 0.00621544, 0.0231963, -0.0231963\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{-0.0176104, -0.0657229, 0.0657229, 0.0176104\}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -0.00621544 & 0 & 0.00621544 & 0 & 0.0231963 & 0 & -0.0231963 \\ 0 & -0.0176104 & 0 & -0.0657229 & 0 & 0.0657229 & 0 \\ -0.0176104 & -0.00621544 & -0.0657229 & 0.00621544 & 0.0657229 & 0.0231963 & 0.0176104 \end{pmatrix}$$

$$\boldsymbol{k} = 10^6$$

$$\begin{pmatrix} 0.20742 & 0.0837341 & 0.541022 & 0.0483439 & -0.774101 & -0.3125 & 0.0256591 & 0.180422 \\ 0.0837341 & 0.415113 & 0.180422 & 1.45599 & -0.3125 & -1.54922 & 0.0483439 & -0.321882 \\ 0.541022 & 0.180422 & 2.2522 & -0.3125 & -2.01912 & -0.673344 & -0.774101 & 0.805422 \\ 0.0483439 & 1.45599 & -0.3125 & 5.52707 & -0.180422 & -5.43384 & 0.444578 & -1.54922 \\ -0.774101 & -0.3125 & -2.01912 & -0.180422 & 2.88899 & 1.16627 & -0.095761 & -0.673344 \\ -0.3125 & -1.54922 & -0.673344 & -5.43384 & 1.16627 & 5.78178 & -0.180422 & 1.20128 \\ 0.0256591 & 0.0483439 & -0.774101 & 0.444578 & -0.095761 & -0.180422 & 0.844203 & -0.3125 \\ 0.180422 & -0.321882 & 0.805422 & -1.54922 & -0.673344 & 1.20128 & -0.3125 & 0.669827 \end{pmatrix}$$

Summing contributions from all points we get

$$\boldsymbol{k} = 10^6 \begin{pmatrix} 6.19281 & 1.875 & 0.890523 & -0.625 & -3.09641 & -1.875 & -3.98693 & 0.625 \\ 1.875 & 12.3938 & 0.625 & 5.31454 & -1.875 & -6.1969 & -0.625 & -11.5114 \\ 0.890523 & 0.625 & 6.19281 & -1.875 & -3.98693 & -0.625 & -3.09641 & 1.875 \\ -0.625 & 5.31454 & -1.875 & 12.3938 & 0.625 & -11.5114 & 1.875 & -6.1969 \\ -3.09641 & -1.875 & -3.98693 & 0.625 & 6.19281 & 1.875 & 0.890523 & -0.625 \\ -1.875 & -6.1969 & -0.625 & -11.5114 & 1.875 & 12.3938 & 0.625 & 5.31454 \\ -3.98693 & -0.625 & -3.09641 & 1.875 & 0.890523 & 0.625 & 6.19281 & -1.875 \\ 0.625 & -11.5114 & 1.875 & -6.1969 & -0.625 & 5.31454 & -1.875 & 12.3938 \end{pmatrix}$$

$$\boldsymbol{r}^{\mathrm{T}} = (\, 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \,)$$

Computation of element matrices resulting from NBC

NBC on side 3 with $\{q_n, q_t\} = \{-50, 0\}$

$$\{N_1, N_2, N_3, N_4\}_c = \left\{0, 0, \frac{1 - a}{2}, \frac{a + 1}{2}\right\}$$

$x(a) = 37. - 17.\,a;$ $\qquad$ $y(a) = 12.$

$dx/da = -17.;$ $\qquad$ $dy/da = 0.$

$J_c = 17.$

Gauss point = -0.57735; $\qquad$ Weight = 1.; $\qquad$ $J_c = 17.$

$\{N_1, N_2, N_3, N_4\}_c = \{0, 0, 0.788675, 0.211325\}$

$$\boldsymbol{r}_q^{\mathrm{T}} = (\, 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -2681.5 \quad 0 \quad -718.505 \,)$$

Gauss point = 0.57735; $\qquad$ Weight = 1.; $\qquad$ $J_c = 17.$

$\{N_1, N_2, N_3, N_4\}_c = \{0, 0, 0.211325, 0.788675\}$

$$\boldsymbol{r}_q^{\mathrm{T}} = (\, 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -718.505 \quad 0 \quad -2681.5 \,)$$

Summing contributions from all Gauss points

$\boldsymbol{r}_q^T = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -3400. \quad 0 \quad -3400.)$

Complete element equations for element 3

$$10^6 \begin{pmatrix} 6.19281 & 1.875 & 0.890523 & -0.625 & -3.09641 & -1.875 & -3.98693 & 0.625 \\ 1.875 & 12.3938 & 0.625 & 5.31454 & -1.875 & -6.1969 & -0.625 & -11.5114 \\ 0.890523 & 0.625 & 6.19281 & -1.875 & -3.98693 & -0.625 & -3.09641 & 1.875 \\ -0.625 & 5.31454 & -1.875 & 12.3938 & 0.625 & -11.5114 & 1.875 & -6.1969 \\ -3.09641 & -1.875 & -3.98693 & 0.625 & 6.19281 & 1.875 & 0.890523 & -0.625 \\ -1.875 & -6.1969 & -0.625 & -11.5114 & 1.875 & 12.3938 & 0.625 & 5.31454 \\ -3.98693 & -0.625 & -3.09641 & 1.875 & 0.890523 & 0.625 & 6.19281 & -1.875 \\ 0.625 & -11.5114 & 1.875 & -6.1969 & -0.625 & 5.31454 & -1.875 & 12.3938 \end{pmatrix}$$

$$\begin{pmatrix} u_5 \\ v_5 \\ u_7 \\ v_7 \\ u_8 \\ v_8 \\ u_6 \\ v_6 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ -3400. \\ 0. \\ -3400. \end{pmatrix}$$

The element contributes to {9, 10, 13, 14, 15, 16, 11, 12} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 9 \\ 10 \\ 13 \\ 14 \\ 15 \\ 16 \\ 11 \\ 12 \end{pmatrix}$

and to a global matrix:

$$\begin{pmatrix}
[9, 9] & [9, 10] & [9, 13] & [9, 14] & [9, 15] & [9, 16] & [9, 11] & [9, 12] \\
[10, 9] & [10, 10] & [10, 13] & [10, 14] & [10, 15] & [10, 16] & [10, 11] & [10, 12] \\
[13, 9] & [13, 10] & [13, 13] & [13, 14] & [13, 15] & [13, 16] & [13, 11] & [13, 12] \\
[14, 9] & [14, 10] & [14, 13] & [14, 14] & [14, 15] & [14, 16] & [14, 11] & [14, 12] \\
[15, 9] & [15, 10] & [15, 13] & [15, 14] & [15, 15] & [15, 16] & [15, 11] & [15, 12] \\
[16, 9] & [16, 10] & [16, 13] & [16, 14] & [16, 15] & [16, 16] & [16, 11] & [16, 12] \\
[11, 9] & [11, 10] & [11, 13] & [11, 14] & [11, 15] & [11, 16] & [11, 11] & [11, 12] \\
[12, 9] & [12, 10] & [12, 13] & [12, 14] & [12, 15] & [12, 16] & [12, 11] & [12, 12]
\end{pmatrix}$$

Adding element equations into appropriate locations we have

$$10^6 \begin{pmatrix}
6.26209 & -0.0163755 & -1.19113 & -0.120087 & 0 & 0 & -4.11923 & 1.26638 \\
-0.0163755 & 9.0354 & -1.37009 & -5.58562 & 0 & 0 & 2.51638 & -3.67826 \\
-1.19113 & -1.37009 & 7.54484 & -2.96397 & 0 & 0 & -5.95173 & 2.62009 \\
-0.120087 & -5.58562 & -2.96397 & 15.0507 & 0 & 0 & 2.62009 & -12.2715 \\
0 & 0 & 0 & 0 & 5.7276 & 0.559291 & -3.49546 & 1.94071 \\
0 & 0 & 0 & 0 & 0.559291 & 9.65901 & 0.690709 & -8.76615 \\
-4.11923 & 2.51638 & -5.95173 & 2.62009 & -3.49546 & 0.690709 & 20.1147 & -6.95708 \\
1.26638 & -3.67826 & 2.62009 & -12.2715 & 1.94071 & -8.76615 & -6.95708 & 32.4444 \\
0 & 0 & 0 & 0 & -0.771918 & 0.0794621 & -4.58523 & 2.42054 \\
0 & 0 & 0 & 0 & -1.17054 & 2.74625 & 2.42054 & -4.8891 \\
-0.951731 & -1.12991 & -0.401981 & 0.463974 & -1.46023 & -1.32946 & -1.96304 & -1.29063 \\
-1.12991 & 0.228478 & 1.71397 & 2.80646 & -1.32946 & -3.6391 & -1.29063 & -2.83938 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Essential boundary conditions

Node	dof	Value
1	u_1	0
2	u_2	0
7	u_7 v_7	0 0
8	u_8 v_8	0 0

Remove {1, 3, 13, 14, 15, 16} rows and columns.

After adjusting for essential boundary conditions we have

$$10^6\left(\begin{array}{ccccccccc}
9.0354 & -5.58562 & 0 & 0 & 2.51638 & -3.67826 & 0 & 0 & -1 \\
-5.58562 & 15.0507 & 0 & 0 & 2.62009 & -12.2715 & 0 & 0 & 0 \\
0 & 0 & 5.7276 & 0.559291 & -3.49546 & 1.94071 & -0.771918 & -1.17054 & -1 \\
0 & 0 & 0.559291 & 9.65901 & 0.690709 & -8.76615 & 0.0794621 & 2.74625 & -1 \\
2.51638 & 2.62009 & -3.49546 & 0.690709 & 20.1147 & -6.95708 & -4.58523 & 2.42054 & -1 \\
-3.67826 & -12.2715 & 1.94071 & -8.76615 & -6.95708 & 32.4444 & 2.42054 & -4.8891 & -1 \\
0 & 0 & -0.771918 & 0.0794621 & -4.58523 & 2.42054 & 12.5561 & -0.866443 & -4 \\
0 & 0 & -1.17054 & 2.74625 & 2.42054 & -4.8891 & -0.866443 & 19.7912 & 0 \\
-1.12991 & 0.463974 & -1.46023 & -1.32946 & -1.96304 & -1.29063 & -4.99308 & 0.866443 & 1 \\
0.228478 & 2.80646 & -1.32946 & -3.6391 & -1.29063 & -2.83938 & 0.866443 & -16.766 & -0
\end{array}\right.$$

Solving the final system of global equations we get

$\{v_1 = -0.0183155,\ v_2 = -0.0183204,\ u_3 = 0.00275915,\ v_3 = -0.0166486,\ u_4 = 0.00114552,$
$v_4 = -0.0164634,\ u_5 = 0.00305003,\ v_5 = -0.0113566,\ u_6 = -0.00210128,\ v_6 = -0.0116254\}$

Complete table of nodal values

	u	v
1	0	−0.0183155
2	0	−0.0183204
3	0.00275915	−0.0166486
4	0.00114552	−0.0164634
5	0.00305003	−0.0113566
6	−0.00210128	−0.0116254
7	0	0
8	0	0

Computation of reactions

Equation numbers of dof with specified values: {1, 3, 13, 14, 15, 16}

Extracting equations {1, 3, 13, 14, 15, 16} from the global system we have

$$10^{6}\begin{pmatrix} 6.26209 & -0.0163755 & -1.19113 & -0.120087 & 0 & 0 & -4.11923 & 1.26638 & 0 & 0 & -0.951 \\ -1.19113 & -1.37009 & 7.54484 & -2.96397 & 0 & 0 & -5.95173 & 2.62009 & 0 & 0 & -0.401 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.890523 & 0.625 & -3.096 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.625 & 5.31454 & 1.875 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3.09641 & -1.875 & 0.890 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.875 & -6.1969 & 0.625 \end{pmatrix}$$

Substituting the nodal values and re−arranging

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{pmatrix} = 10^{6}\begin{pmatrix} 6.26209 & -0.0163755 & -1.19113 & -0.120087 & 0 & 0 & -4.11923 & 1.26638 & 0 & 0 \\ -1.19113 & -1.37009 & 7.54484 & -2.96397 & 0 & 0 & -5.95173 & 2.62009 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.890523 & 0.625 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.625 & 5.31454 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3.09641 & -1.875 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.875 & -6.1969 \end{pmatrix}$$

Carrying out computations, the reactions are as follows.

Label	dof	Reaction
R_1	u_1	−7931.98
R_2	u_2	10360.8
R_3	u_7	−19673.
R_4	v_7	5839.98
R_5	u_8	17244.2
R_6	v_8	4960.02

Sum of Reactions

dof: u	0
dof: v	10800.

Solution for element 1

Element nodal displacements

Element node	Global node number	u	v
1	1	0	−0.0183155
2	4	0.00114552	−0.0164634
3	6	−0.00210128	−0.0116254
4	2	0	−0.0183204

$$\boldsymbol{d}^T = (0 \quad -0.0183155 \quad 0.00114552 \quad -0.0164634 \quad -0.00210128 \quad -0.0116254 \quad 0 \quad -0.0183204)$$

$E = 3000000;$ $\nu = 0.2;$ $h = 4$

$$\text{Plane stress } \boldsymbol{C} = \begin{pmatrix} 3.125\times10^6 & 625000. & 0 \\ 625000. & 3.125\times10^6 & 0 \\ 0 & 0 & 1.25\times10^6 \end{pmatrix}$$

Interpolation functions and their derivatives

$$\{N_1, N_2, N_3, N_4\} = \left\{\frac{1}{4}(s-1)(t-1), -\frac{1}{4}(s+1)(t-1), \frac{1}{4}(s+1)(t+1), -\frac{1}{4}(s-1)(t+1)\right\}$$

$$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{\frac{t-1}{4}, \frac{1-t}{4}, \frac{t+1}{4}, \frac{1}{4}(-t-1)\right\}$$

$$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{\frac{s-1}{4}, \frac{1}{4}(-s-1), \frac{s+1}{4}, \frac{1-s}{4}\right\}$$

Nodal coordinates

Element node	Global node number	x	y
1	1	0.	5.
2	4	6.	5.
3	6	20.	12.
4	2	0.	12.

Mapping to the master element

$$x(s,t) = 1.5\,(s+1)\,(1-t) + 5.\,(s+1)\,(t+1)$$

$$y(s,t) = 1.25\,(1-s)\,(1-t) + 1.25\,(s+1)\,(1-t) + 3.\,(1-s)\,(t+1) + 3.\,(s+1)\,(t+1)$$

$$\boldsymbol{J} = \begin{pmatrix} 1.5\,(1-t) + 5.\,(t+1) & 3.5\,(s+1) \\ 0 & 1.75\,(1-s) + 1.75\,(s+1) \end{pmatrix}; \qquad \det J = 12.25\,t + 22.75$$

Element solution at $\{s, t\} = \{0, 0\} \Longrightarrow \{x, y\} = \{6.5, 8.5\}$

$$\{N_1, N_2, N_3, N_4\} = \left\{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right\}$$

$$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{-\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}\right\}$$

$$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \left\{-\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right\}$$

$$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{-0.0384615, 0.0384615, 0.0384615, -0.0384615\}$$

$$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{-0.032967, -0.10989, 0.032967, 0.10989\}$$

$$\boldsymbol{B}^T = \begin{pmatrix} -0.0384615 & 0 & 0.0384615 & 0 & 0.0384615 & 0 & -0.0384615 & 0 \\ 0 & -0.032967 & 0 & -0.10989 & 0 & 0.032967 & 0 & 0. \\ -0.032967 & -0.0384615 & -0.10989 & 0.0384615 & 0.032967 & 0.0384615 & 0.10989 & -0. \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^T\boldsymbol{d} = (\,-0.00003676 \quad 0.0000164861 \quad 0.000133582\,)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\,-104.571 \quad 28.544 \quad 166.978\,)$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$$\boldsymbol{\epsilon}^T = (\,-0.00003676 \quad 0.0000164861 \quad 5.06848\times10^{-6} \quad 0.000133582 \quad 0 \quad 0\,)$$

$$\boldsymbol{\sigma}^T = (\,-104.571 \quad 28.544 \quad 0 \quad 166.978 \quad 0 \quad 0\,)$$

Substituting these stress components into appropriate formulas

Principal stresses = $(\,141.741 \quad 0. \quad -217.768\,)$

Effective stress (von Mises) = 313.656

Element solution at $\{s, t\} = \{-1, -1\} \Longrightarrow \{x, y\} = \{0., 5.\}$

$\{N_1, N_2, N_3, N_4\} = \{1, 0, 0, 0\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{-\frac{1}{2}, \frac{1}{2}, 0, 0\right\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \left\{-\frac{1}{2}, 0, 0, \frac{1}{2}\right\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{-0.166667, 0.166667, 0, 0.\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{-0.142857, 0., 0, 0.142857\}$

$$\boldsymbol{B}^T = \begin{pmatrix} -0.166667 & 0 & 0.166667 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.142857 & 0 & 0 & 0 & 0 & 0 & 0.142857 \\ -0.142857 & -0.166667 & 0 & 0.166667 & 0 & 0 & 0.142857 & 0 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^T\boldsymbol{d} = (\ 0.00019092 \quad -6.97447\times10^{-7} \quad 0.000308689\ )$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\ 596.188 \quad 117.145 \quad 385.861\ )$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^T = (\ 0.00019092 \quad -6.97447\times10^{-7} \quad -0.0000475555 \quad 0.000308689 \quad 0 \quad 0\ )$

$\boldsymbol{\sigma}^T = (\ 596.188 \quad 117.145 \quad 0 \quad 385.861 \quad 0 \quad 0\ )$

Substituting these stress components into appropriate formulas

Principal stresses = $(\ 810.824 \quad 0. \quad -97.4913\ )$

Effective stress (von Mises) = 863.706

Element solution at $\{s, t\} = \{-1, 1\} \Longrightarrow \{x, y\} = \{0., 12.\}$

$\{N_1, N_2, N_3, N_4\} = \{0, 0, 0, 1\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{0, 0, \frac{1}{2}, -\frac{1}{2}\right\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \left\{-\frac{1}{2}, 0, 0, \frac{1}{2}\right\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{0., 0, 0.05, -0.05\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{-0.142857, 0, 0., 0.142857\}$

$$\boldsymbol{B}^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0.05 & 0 & -0.05 & 0 \\ 0 & -0.142857 & 0 & 0 & 0 & 0 & 0 & 0.142857 \\ -0.142857 & 0 & 0 & 0 & 0 & 0.05 & 0.142857 & -0.05 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^T\boldsymbol{d} = (\ -0.000105064 \quad -6.97447\times10^{-7} \quad 0.000334751\ )$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\,-328.76 \quad -67.8444 \quad 418.438\,)$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{\mathrm{T}} = (\,-0.000105064 \quad -6.97447\times 10^{-7} \quad 0.0000264403 \quad 0.000334751 \quad 0 \quad 0\,)$

$\boldsymbol{\sigma}^{\mathrm{T}} = (\,-328.76 \quad -67.8444 \quad 0 \quad 418.438 \quad 0 \quad 0\,)$

Substituting these stress components into appropriate formulas

Principal stresses = $(\,240.001 \quad 0. \quad -636.606\,)$

Effective stress (von Mises) = 784.636

Element solution at $\{s, t\} = \{1, -1\} \Longrightarrow \{x, y\} = \{6., 5.\}$

$\{N_1, N_2, N_3, N_4\} = \{0, 1, 0, 0\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{-\frac{1}{2}, \frac{1}{2}, 0, 0\right\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \left\{0, -\frac{1}{2}, \frac{1}{2}, 0\right\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{-0.166667, 0.166667, 0., 0\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{0.333333, -0.47619, 0.142857, 0\}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -0.166667 & 0 & 0.166667 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.333333 & 0 & -0.47619 & 0 & 0.142857 & 0 & 0 \\ 0.333333 & -0.166667 & -0.47619 & 0.166667 & 0.142857 & 0 & 0 & 0 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{d} = (\,0.00019092 \quad 0.0000737645 \quad -0.000536978\,)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\,642.726 \quad 349.839 \quad -671.222\,)$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{\mathrm{T}} = (\,0.00019092 \quad 0.0000737645 \quad -0.000066171 \quad -0.000536978 \quad 0 \quad 0\,)$

$\boldsymbol{\sigma}^{\mathrm{T}} = (\,642.726 \quad 349.839 \quad 0 \quad -671.222 \quad 0 \quad 0\,)$

Substituting these stress components into appropriate formulas

Principal stresses = $(\,1183.29 \quad 0. \quad -190.729\,)$

Effective stress (von Mises) = 1289.28

Element solution at $\{s, t\} = \{1, 1\} \Longrightarrow \{x, y\} = \{20., 12.\}$

$\{N_1, N_2, N_3, N_4\} = \{0, 0, 1, 0\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{0, 0, \frac{1}{2}, -\frac{1}{2}\right\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \left\{0, -\frac{1}{2}, \frac{1}{2}, 0\right\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{0, 0., 0.05, -0.05\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{0, -0.142857, 0.0428571, 0.1\}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0.05 & 0 & -0.05 & 0 \\ 0 & 0 & 0 & -0.142857 & 0 & 0.0428571 & 0 & 0.1 \\ 0 & 0 & -0.142857 & 0 & 0.0428571 & 0.05 & 0.1 & -0.05 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{d} = (\ -0.000105064 \quad 0.0000216411 \quad 0.0000810505\ )$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\ -314.799 \quad 1.96363 \quad 101.313\ )$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{\mathrm{T}} = (\ -0.000105064 \quad 0.0000216411 \quad 0.0000208557 \quad 0.0000810505 \quad 0 \quad 0\ )$

$\boldsymbol{\sigma}^{\mathrm{T}} = (\ -314.799 \quad 1.96363 \quad 0 \quad 101.313 \quad 0 \quad 0\ )$

Substituting these stress components into appropriate formulas

Principal stresses = $(\ 31.5956 \quad 0. \quad -344.431\ )$

Effective stress (von Mises) = 361.266

Solution for element 2

Element nodal displacements

Element node	Global node number	u	v
1	3	0.00275915	−0.0166486
2	5	0.00305003	−0.0113566
3	6	−0.00210128	−0.0116254
4	4	0.00114552	−0.0164634

$\boldsymbol{d}^{\mathrm{T}} = (\ 0.00275915 \quad -0.0166486 \quad 0.00305003 \quad -0.0113566 \quad -0.00210128 \quad -0.0116254 \quad 0.00114552 \quad -0.0164($

$E = 3000000; \qquad \nu = 0.2; \qquad h = 4$

$$\text{Plane stress } \boldsymbol{C} = \begin{pmatrix} 3.125\times10^{6} & 625000. & 0 \\ 625000. & 3.125\times10^{6} & 0 \\ 0 & 0 & 1.25\times10^{6} \end{pmatrix}$$

Interpolation functions and their derivatives

$$\{N_1,\ N_2,\ N_3,\ N_4\} = \left\{\frac{1}{4}\,(s-1)\,(t-1),\ -\frac{1}{4}\,(s+1)\,(t-1),\ \frac{1}{4}\,(s+1)\,(t+1),\ -\frac{1}{4}\,(s-1)\,(t+1)\right\}$$

$$\{\partial N_1/\partial s,\ \partial N_2/\partial s,\ \partial N_3/\partial s,\ \partial N_4/\partial s\} = \left\{\frac{t-1}{4},\ \frac{1-t}{4},\ \frac{t+1}{4},\ \frac{1}{4}\,(-t-1)\right\}$$

$$\{\partial N_1/\partial s,\ \partial N_2/\partial s,\ \partial N_3/\partial s,\ \partial N_4/\partial s\} = \left\{\frac{s-1}{4},\ \frac{1}{4}\,(-s-1),\ \frac{s+1}{4},\ \frac{1-s}{4}\right\}$$

Nodal coordinates

Element node	Global node number	x	y
1	3	6.	0.
2	5	20.	0.
3	6	20.	12.
4	4	6.	5.

Mapping to the master element

$$x(s,t) = 1.5\,(1-s)\,(1-t) + 5.\,(s+1)\,(1-t) + 1.5\,(1-s)\,(t+1) + 5.\,(s+1)\,(t+1)$$

$$y(s,t) = 1.25\,(1-s)\,(t+1) + 3.\,(s+1)\,(t+1)$$

$$\boldsymbol{J} = \begin{pmatrix} 3.5\,(1-t) + 3.5\,(t+1) & 0 \\ 1.75\,(t+1) & 1.25\,(1-s) + 3.\,(s+1) \end{pmatrix}; \qquad \det J = 12.25\,s + 29.75$$

Element solution at $\{s,\ t\} = \{0,\ 0\} \Longrightarrow \{x,\ y\} = \{13.,\ 4.25\}$

$$\{N_1,\ N_2,\ N_3,\ N_4\} = \left\{\frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4}\right\}$$

$$\{\partial N_1/\partial s,\ \partial N_2/\partial s,\ \partial N_3/\partial s,\ \partial N_4/\partial s\} = \left\{-\frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ -\frac{1}{4}\right\}$$

$$\{\partial N_1/\partial t,\ \partial N_2/\partial t,\ \partial N_3/\partial t,\ \partial N_4/\partial t\} = \left\{-\frac{1}{4},\ -\frac{1}{4},\ \frac{1}{4},\ \frac{1}{4}\right\}$$

$$\{\partial N_1/\partial x,\ \partial N_2/\partial x,\ \partial N_3/\partial x,\ \partial N_4/\partial x\} = \{-0.0210084,\ 0.0504202,\ 0.0210084,\ -0.0504202\}$$

$$\{\partial N_1/\partial y,\ \partial N_2/\partial y,\ \partial N_3/\partial y,\ \partial N_4/\partial y\} = \{-0.0588235,\ -0.0588235,\ 0.0588235,\ 0.0588235\}$$

$$\boldsymbol{B}^T = \begin{pmatrix} -0.0210084 & 0 & 0.0504202 & 0 & 0.0210084 & 0 & -0.0504202 & 0 \\ 0 & -0.0588235 & 0 & -0.0588235 & 0 & 0.0588235 & 0 & 0. \\ -0.0588235 & -0.0210084 & -0.0588235 & 0.0504202 & 0.0588235 & 0.0210084 & 0.0588235 & -0. \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^T \boldsymbol{d} = (\ -6.08383\times10^{-6} \quad -4.91655\times10^{-6} \quad -0.0000349244\ )$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\ -22.0848 \quad -19.1666 \quad -43.6555\ )$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$$\boldsymbol{\epsilon}^T = (\ -6.08383\times10^{-6} \quad -4.91655\times10^{-6} \quad 2.7501\times10^{-6} \quad -0.0000349244 \quad 0 \quad 0\ )$$

$\boldsymbol{\sigma}^{\mathrm{T}} = (\,-22.0848 \quad -19.1666 \quad 0 \quad -43.6555 \quad 0 \quad 0\,)$

Substituting these stress components into appropriate formulas

Principal stresses = $(\,23.0542 \quad 0. \quad -64.3056\,)$

Effective stress (von Mises) = 78.4169

Element solution at $\{s, t\} = \{-1, -1\} \Longrightarrow \{x, y\} = \{6., 0.\}$

$\{N_1, N_2, N_3, N_4\} = \{1, 0, 0, 0\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{-\frac{1}{2}, \frac{1}{2}, 0, 0\right\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \left\{-\frac{1}{2}, 0, 0, \frac{1}{2}\right\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{-0.0714286, 0.0714286, 0, 0.\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{-0.2, 0., 0, 0.2\}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -0.0714286 & 0 & 0.0714286 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.2 & 0 & 0 & 0 & 0 & 0 & 0.2 \\ -0.2 & -0.0714286 & 0 & 0.0714286 & 0 & 0 & 0.2 & 0 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{d} = (\,0.0000207772 \quad 0.0000370391 \quad 0.0000552707\,)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\,88.0783 \quad 128.733 \quad 69.0884\,)$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{\mathrm{T}} = (\,0.0000207772 \quad 0.0000370391 \quad -0.0000144541 \quad 0.0000552707 \quad 0 \quad 0\,)$

$\boldsymbol{\sigma}^{\mathrm{T}} = (\,88.0783 \quad 128.733 \quad 0 \quad 69.0884 \quad 0 \quad 0\,)$

Substituting these stress components into appropriate formulas

Principal stresses = $(\,180.422 \quad 36.3889 \quad 0.\,)$

Effective stress (von Mises) = 165.26

Element solution at $\{s, t\} = \{-1, 1\} \Longrightarrow \{x, y\} = \{6., 5.\}$

$\{N_1, N_2, N_3, N_4\} = \{0, 0, 0, 1\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{0, 0, \frac{1}{2}, -\frac{1}{2}\right\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \left\{-\frac{1}{2}, 0, 0, \frac{1}{2}\right\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{0.1, 0, 0.0714286, -0.171429\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{-0.2, 0, 0., 0.2\}$

$$\boldsymbol{B}^T = \begin{pmatrix} 0.1 & 0 & 0 & 0 & 0.0714286 & 0 & -0.171429 & 0 \\ 0 & -0.2 & 0 & 0 & 0 & 0 & 0 & 0.2 \\ -0.2 & 0.1 & 0 & 0 & 0 & 0.0714286 & 0.2 & -0.171429 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^T\boldsymbol{d} = (\,-0.0000705504 \quad 0.0000370391 \quad 4.32456\times 10^{-6}\,)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\,-197.32 \quad 71.6533 \quad 5.4057\,)$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^T = (\,-0.0000705504 \quad 0.0000370391 \quad 8.37781\times 10^{-6} \quad 4.32456\times 10^{-6} \quad 0 \quad 0\,)$

$\boldsymbol{\sigma}^T = (\,-197.32 \quad 71.6533 \quad 0 \quad 5.4057 \quad 0 \quad 0\,)$

Substituting these stress components into appropriate formulas

Principal stresses = $(\,71.7619 \quad 0. \quad -197.429\,)$

Effective stress (von Mises) = 241.445

Element solution at $\{s, t\} = \{1, -1\} \Longrightarrow \{x, y\} = \{20., 0.\}$

$\{N_1, N_2, N_3, N_4\} = \{0, 1, 0, 0\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{-\frac{1}{2}, \frac{1}{2}, 0, 0\right\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \left\{0, -\frac{1}{2}, \frac{1}{2}, 0\right\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{-0.0714286, 0.0714286, 0., 0\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{0., -0.0833333, 0.0833333, 0\}$

$$\boldsymbol{B}^T = \begin{pmatrix} -0.0714286 & 0 & 0.0714286 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.0833333 & 0 & 0.0833333 & 0 & 0 \\ 0 & -0.0714286 & -0.0833333 & 0.0714286 & 0.0833333 & 0 & 0 & 0 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^T\boldsymbol{d} = (\,0.0000207772 \quad -0.0000223981 \quad -0.0000512781\,)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\,50.93 \quad -57.0082 \quad -64.0977\,)$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^T = (\,0.0000207772 \quad -0.0000223981 \quad 4.05212\times 10^{-7} \quad -0.0000512781 \quad 0 \quad 0\,)$

$\boldsymbol{\sigma}^T = (\,50.93 \quad -57.0082 \quad 0 \quad -64.0977 \quad 0 \quad 0\,)$

Substituting these stress components into appropriate formulas

Principal stresses = (80.7534   0.   −86.8316)

Effective stress (von Mises) = 145.165

Element solution at {s, t} = {1, 1} ⟹ {x, y} = {20., 12.}

$\{N_1, N_2, N_3, N_4\} = \{0, 0, 1, 0\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{0, 0, \frac{1}{2}, -\frac{1}{2}\right\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \left\{0, -\frac{1}{2}, \frac{1}{2}, 0\right\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{0, 0.0416667, 0.0297619, -0.0714286\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{0, -0.0833333, 0.0833333, 0.\}$

$$\boldsymbol{B}^{T} = \begin{pmatrix} 0 & 0 & 0.0416667 & 0 & 0.0297619 & 0 & -0.0714286 & 0 \\ 0 & 0 & 0 & -0.0833333 & 0 & 0.0833333 & 0 & 0 \\ 0 & 0 & -0.0833333 & 0.0416667 & 0.0833333 & 0.0297619 & 0 & -0.0714286 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{T}\boldsymbol{d}$ = (−0.0000172759   −0.0000223981   −0.0000725057)

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon}$ = (−67.9861   −80.7914   −90.6321)

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{T} = (\ -0.0000172759 \quad -0.0000223981 \quad 9.9185\times10^{-6} \quad -0.0000725057 \quad 0 \quad 0\ )$

$\boldsymbol{\sigma}^{T} = (\ -67.9861 \quad -80.7914 \quad 0 \quad -90.6321 \quad 0 \quad 0\ )$

Substituting these stress components into appropriate formulas

Principal stresses = (16.4692   0.   −165.247)

Effective stress (von Mises) = 174.067

Solution for element 3

Element nodal displacements

Element node	Global node number	u	v
1	5	0.00305003	−0.0113566
2	7	0	0
3	8	0	0
4	6	−0.00210128	−0.0116254

$\boldsymbol{d}^{T} = (\ 0.00305003 \quad -0.0113566 \quad 0 \quad 0 \quad 0 \quad 0 \quad -0.00210128 \quad -0.0116254\ )$

E = 3000000;   ν = 0.2;   h = 4

Plane stress $\boldsymbol{C} = \begin{pmatrix} 3.125\times10^{6} & 625000. & 0 \\ 625000. & 3.125\times10^{6} & 0 \\ 0 & 0 & 1.25\times10^{6} \end{pmatrix}$

Interpolation functions and their derivatives

$$\{N_1, N_2, N_3, N_4\} = \left\{\frac{1}{4}(s-1)(t-1), -\frac{1}{4}(s+1)(t-1), \frac{1}{4}(s+1)(t+1), -\frac{1}{4}(s-1)(t+1)\right\}$$

$$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{\frac{t-1}{4}, \frac{1-t}{4}, \frac{t+1}{4}, \frac{1}{4}(-t-1)\right\}$$

$$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{\frac{s-1}{4}, \frac{1}{4}(-s-1), \frac{s+1}{4}, \frac{1-s}{4}\right\}$$

Nodal coordinates

Element node	Global node number	x	y
1	5	20.	0.
2	7	54.	0.
3	8	54.	12.
4	6	20.	12.

Mapping to the master element

$$x(s,t) = 5.(1-s)(1-t) + 13.5(s+1)(1-t) + 5.(1-s)(t+1) + 13.5(s+1)(t+1)$$

$$y(s,t) = 3.(1-s)(t+1) + 3.(s+1)(t+1)$$

$$\boldsymbol{J} = \begin{pmatrix} 8.5(1-t) + 8.5(t+1) & 0 \\ 0 & 3.(1-s) + 3.(s+1) \end{pmatrix}; \qquad \det J = 102.$$

Element solution at $\{s, t\} = \{0, 0\} \Longrightarrow \{x, y\} = \{37., 6.\}$

$$\{N_1, N_2, N_3, N_4\} = \left\{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right\}$$

$$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{-\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}\right\}$$

$$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \left\{-\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right\}$$

$$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{-0.0147059, 0.0147059, 0.0147059, -0.0147059\}$$

$$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{-0.0416667, -0.0416667, 0.0416667, 0.0416667\}$$

$$\boldsymbol{B}^{T} = \begin{pmatrix} -0.0147059 & 0 & 0.0147059 & 0 & 0.0147059 & 0 & -0.0147059 & 0 \\ 0 & -0.0416667 & 0 & -0.0416667 & 0 & 0.0416667 & 0 & 0. \\ -0.0416667 & -0.0147059 & -0.0416667 & 0.0147059 & 0.0416667 & 0.0147059 & 0.0416667 & -0. \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{T}\boldsymbol{d} = (\,-0.0000139523 \quad -0.000011199 \quad 0.000123333\,)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\,-50.6003 \quad -43.7172 \quad 154.167\,)$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{\mathrm{T}} = (\,-0.0000139523 \quad -0.000011199 \quad 6.28783\times 10^{-6} \quad 0.000123333 \quad 0 \quad 0\,)$

$\boldsymbol{\sigma}^{\mathrm{T}} = (\,-50.6003 \quad -43.7172 \quad 0 \quad 154.167 \quad 0 \quad 0\,)$

Substituting these stress components into appropriate formulas

Principal stresses = $(\,107.046 \quad 0. \quad -201.364\,)$

Effective stress (von Mises) = 271.222

Element solution at $\{s, t\} = \{-1, -1\} \Longrightarrow \{x, y\} = \{20., 0.\}$

$\{N_1, N_2, N_3, N_4\} = \{1, 0, 0, 0\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{-\frac{1}{2}, \frac{1}{2}, 0, 0\right\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \left\{-\frac{1}{2}, 0, 0, \frac{1}{2}\right\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{-0.0294118, 0.0294118, 0, 0.\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{-0.0833333, 0., 0, 0.0833333\}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -0.0294118 & 0 & 0.0294118 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.0833333 & 0 & 0 & 0 & 0 & 0 & 0.0833333 \\ -0.0833333 & -0.0294118 & 0 & 0.0294118 & 0 & 0 & 0.0833333 & 0 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{d} = (\,-0.0000897069 \quad -0.0000223981 \quad -0.0000952572\,)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\,-294.333 \quad -126.061 \quad -119.072\,)$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{\mathrm{T}} = (\,-0.0000897069 \quad -0.0000223981 \quad 0.0000280262 \quad -0.0000952572 \quad 0 \quad 0\,)$

$\boldsymbol{\sigma}^{\mathrm{T}} = (\,-294.333 \quad -126.061 \quad 0 \quad -119.072 \quad 0 \quad 0\,)$

Substituting these stress components into appropriate formulas

Principal stresses = $(\,0. \quad -64.3993 \quad -355.994\,)$

Effective stress (von Mises) = 328.563

Element solution at $\{s, t\} = \{-1, 1\} \Longrightarrow \{x, y\} = \{20., 12.\}$

$\{N_1, N_2, N_3, N_4\} = \{0, 0, 0, 1\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{0, 0, \frac{1}{2}, -\frac{1}{2}\right\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \left\{-\frac{1}{2}, 0, 0, \frac{1}{2}\right\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{0., 0, 0.0294118, -0.0294118\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{-0.0833333, 0, 0., 0.0833333\}$

$$\boldsymbol{B}^{T} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0.0294118 & 0 & -0.0294118 & 0 \\ 0 & -0.0833333 & 0 & 0 & 0 & 0 & 0 & 0.0833333 \\ -0.0833333 & 0 & 0 & 0 & 0 & 0.0294118 & 0.0833333 & -0.0294118 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{T}\boldsymbol{d} = (\, 0.0000618023 \quad -0.0000223981 \quad -0.000087352 \,)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\, 179.133 \quad -31.3676 \quad -109.19 \,)$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{T} = (\, 0.0000618023 \quad -0.0000223981 \quad -9.85105\times 10^{-6} \quad -0.000087352 \quad 0 \quad 0 \,)$

$\boldsymbol{\sigma}^{T} = (\, 179.133 \quad -31.3676 \quad 0 \quad -109.19 \quad 0 \quad 0 \,)$

Substituting these stress components into appropriate formulas

Principal stresses = $(\, 225.541 \quad 0. \quad -77.775 \,)$

Effective stress (von Mises) = 272.872

Element solution at $\{s, t\} = \{1, -1\} \Longrightarrow \{x, y\} = \{54., 0.\}$

$\{N_1, N_2, N_3, N_4\} = \{0, 1, 0, 0\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{-\frac{1}{2}, \frac{1}{2}, 0, 0\right\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \left\{0, -\frac{1}{2}, \frac{1}{2}, 0\right\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{-0.0294118, 0.0294118, 0., 0\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{0., -0.0833333, 0.0833333, 0\}$

$$\boldsymbol{B}^{T} = \begin{pmatrix} -0.0294118 & 0 & 0.0294118 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.0833333 & 0 & 0.0833333 & 0 & 0 \\ 0 & -0.0294118 & -0.0833333 & 0.0294118 & 0.0833333 & 0 & 0 & 0 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{T}\boldsymbol{d} = (\, -0.0000897069 \quad 0. \quad 0.000334019 \,)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\, -280.334 \quad -56.0668 \quad 417.523 \,)$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{\mathrm{T}} = (\, -0.0000897069 \quad 0. \quad 0.0000224267 \quad 0.000334019 \quad 0 \quad 0 \,)$

$\boldsymbol{\sigma}^{\mathrm{T}} = (\, -280.334 \quad -56.0668 \quad 0 \quad 417.523 \quad 0 \quad 0 \,)$

Substituting these stress components into appropriate formulas

Principal stresses = $(\, 264.119 \quad 0. \quad -600.519 \,)$

Effective stress (von Mises) = 767.457

Element solution at $\{s, t\} = \{1, 1\} \Longrightarrow \{x, y\} = \{54., 12.\}$

$\{N_1, N_2, N_3, N_4\} = \{0, 0, 1, 0\}$

$\{\partial N_1/\partial s, \partial N_2/\partial s, \partial N_3/\partial s, \partial N_4/\partial s\} = \left\{0, 0, \frac{1}{2}, -\frac{1}{2}\right\}$

$\{\partial N_1/\partial t, \partial N_2/\partial t, \partial N_3/\partial t, \partial N_4/\partial t\} = \left\{0, -\frac{1}{2}, \frac{1}{2}, 0\right\}$

$\{\partial N_1/\partial x, \partial N_2/\partial x, \partial N_3/\partial x, \partial N_4/\partial x\} = \{0, 0., 0.0294118, -0.0294118\}$

$\{\partial N_1/\partial y, \partial N_2/\partial y, \partial N_3/\partial y, \partial N_4/\partial y\} = \{0, -0.0833333, 0.0833333, 0.\}$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0.0294118 & 0 & -0.0294118 & 0 \\ 0 & 0 & 0 & -0.0833333 & 0 & 0.0833333 & 0 & 0 \\ 0 & 0 & -0.0833333 & 0 & 0.0833333 & 0.0294118 & 0 & -0.0294118 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{d} = (\, 0.0000618023 \quad 0. \quad 0.000341924 \,)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\, 193.132 \quad 38.6264 \quad 427.405 \,)$

Computing out−of−plane strain and stress components using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{\mathrm{T}} = (\, 0.0000618023 \quad 0. \quad -0.0000154506 \quad 0.000341924 \quad 0 \quad 0 \,)$

$\boldsymbol{\sigma}^{\mathrm{T}} = (\, 193.132 \quad 38.6264 \quad 0 \quad 427.405 \quad 0 \quad 0 \,)$

Substituting these stress components into appropriate formulas

Principal stresses = $(\, 550.21 \quad 0. \quad -318.451 \,)$

Effective stress (von Mises) = 761.155

Solution summary

Nodal solution

	x	y	u	v
1	0.	5.	0	−0.0183155
2	0.	12.	0	−0.0183204
3	6.	0.	0.00275915	−0.0166486
4	6.	5.	0.00114552	−0.0164634
5	20.	0.	0.00305003	−0.0113566
6	20.	12.	−0.00210128	−0.0116254
7	54.	0.	0	0
8	54.	12.	0	0

Solution at selected points on elements

	Coord	Disp	Stresses	Principal stresses	Effective Stress
1	6.5 8.5	−0.00023894 −0.0161812	−104.571 28.544 0 166.978 0 0	141.741 0. −217.768	313.656
2	13. 4.25	0.00121336 −0.0140235	−22.0848 −19.1666 0 −43.6555 0 0	23.0542 0. −64.3056	78.4169
3	37. 6.	0.000237189 −0.00574551	−50.6003 −43.7172 0 154.167 0 0	107.046 0. −201.364	271.222

Support reactions

Node	dof	Reaction
1	1	−7931.98
2	1	10360.8
7	1	−19673.
7	2	5839.98
8	1	17244.2
8	2	4960.02

Sum of applied loads → (0   −10800.)

Sum of support reactions → (0   10800.)