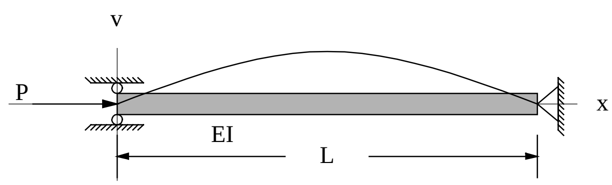
Example 3.4: Buckling load of a simply supported column (p. 202)

The governing differential equation is as follows.

$$\mathrm{EI}\left(\frac{d^2 y}{\mathrm{d}x^2}\right) + P\,y = 0$$

$$y(0) = y(L) = 0$$

The length is $L = 10\,\mathrm{ft}$ and $\mathrm{EI} = 10^6\,\mathrm{lb.in}^2$.



Since the buckling load P is unknown, the element matrices $\boldsymbol{k}_k$ and $\boldsymbol{k}_p$ are computed separately. The matrix $\boldsymbol{k}_p$ is multiplied by the unknown load P. Each of the two matrices are assembled in the usual manner to form global matrices. The final global equations are written in the following form.

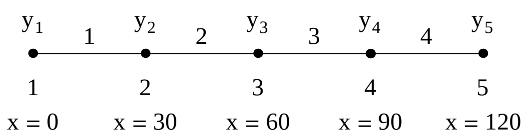
$$\boldsymbol{k}_k\,\boldsymbol{d} + P\,\boldsymbol{k}_p\,\boldsymbol{d} = \mathbf{0} \implies (\boldsymbol{k}_k + P\,\boldsymbol{k}_p)\,\boldsymbol{d} = \mathbf{0}$$

This is a homogeneous system of equations and can be recognized as a generalized eigenvalue problem. A non-zero solution for the nodal degrees of freedom $\boldsymbol{d}$ is possible only if the coefficient matrix cannot be inverted. Thus a necessary condition for a non-trivial solution of the equations is that the determinant of the coefficient matrix be zero.

$$\mathrm{Det}(\boldsymbol{k}_k + P\,\boldsymbol{k}_p) = 0$$

For an n degree of freedom system, evaluating this determinant gives an equation called the *characteristic equation* that is a polynomial of degree n in terms of P. The roots of the characteristic equation represent different buckling loads (eigenvalues). Substituting each buckling load in the global equations one can compute the corresponding nodal displacements. These represent the buckling modes (eigenvectors). Generally one is interested in the lowest buckling load and the corresponding mode shape.

(i) Solution using four linear elements



4 element solution

Nodal locations: {0, 30., 60., 90., 120.}

Element 1

Element nodes: $\{x_1 \to 0,\ x_2 \to 30.\}$

$$\begin{pmatrix} 33333.3 & -33333.3 \\ -33333.3 & 33333.3 \end{pmatrix} \begin{pmatrix} \mathrm{v}_1 \\ \mathrm{v}_2 \end{pmatrix} - \mathrm{P} \begin{pmatrix} 10. & 5. \\ 5. & 10. \end{pmatrix} \begin{pmatrix} \mathrm{v}_1 \\ \mathrm{v}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Element 2

Element nodes: $\{x_2 \to 30.,\ x_3 \to 60.\}$

$$\begin{pmatrix} 33333.3 & -33333.3 \\ -33333.3 & 33333.3 \end{pmatrix} \begin{pmatrix} \mathrm{v}_2 \\ \mathrm{v}_3 \end{pmatrix} - \mathrm{P} \begin{pmatrix} 10. & 5. \\ 5. & 10. \end{pmatrix} \begin{pmatrix} \mathrm{v}_2 \\ \mathrm{v}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Element 3

Element nodes: $\{x_3 \to 60.,\ x_4 \to 90.\}$

$$\begin{pmatrix} 33333.3 & -33333.3 \\ -33333.3 & 33333.3 \end{pmatrix} \begin{pmatrix} \mathrm{v}_3 \\ \mathrm{v}_4 \end{pmatrix} - \mathrm{P} \begin{pmatrix} 10. & 5. \\ 5. & 10. \end{pmatrix} \begin{pmatrix} \mathrm{v}_3 \\ \mathrm{v}_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Element 4

Element nodes: $\{x_4 \to 90.,\ x_5 \to 120.\}$

$$\begin{pmatrix} 33333.3 & -33333.3 \\ -33333.3 & 33333.3 \end{pmatrix} \begin{pmatrix} \mathrm{v}_4 \\ \mathrm{v}_5 \end{pmatrix} - \mathrm{P} \begin{pmatrix} 10. & 5. \\ 5. & 10. \end{pmatrix} \begin{pmatrix} \mathrm{v}_4 \\ \mathrm{v}_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Global equations before boundary conditions

$$\begin{pmatrix} 33333.3 & -33333.3 & 0 & 0 & 0 \\ -33333.3 & 66666.7 & -33333.3 & 0 & 0 \\ 0 & -33333.3 & 66666.7 & -33333.3 & 0 \\ 0 & 0 & -33333.3 & 66666.7 & -33333.3 \\ 0 & 0 & 0 & -33333.3 & 33333.3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix} - P \begin{pmatrix} 10. & 5. & 0 & 0 & 0 \\ 5. & 20. & 5. & 0 & 0 \\ 0 & 5. & 20. & 5. & 0 \\ 0 & 0 & 5. & 20. & 5. \\ 0 & 0 & 0 & 5. & 10. \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Essential boundary conditions

DOF	Value
v_1	0
v_5	0

Incorporating EBC the final system of equations is

$$\begin{pmatrix} 66666.7 & -33333.3 & 0 \\ -33333.3 & 66666.7 & -33333.3 \\ 0 & -33333.3 & 66666.7 \end{pmatrix} \begin{pmatrix} v_2 \\ v_3 \\ v_4 \end{pmatrix} - P \begin{pmatrix} 20. & 5. & 0 \\ 5. & 20. & 5. \\ 0 & 5. & 20. \end{pmatrix} \begin{pmatrix} v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solution of the eigenvalue problem

$$\boldsymbol{k}_k = \begin{pmatrix} 66666.7 & -33333.3 & 0 \\ -33333.3 & 66666.7 & -33333.3 \\ 0 & -33333.3 & 66666.7 \end{pmatrix}$$

$$\boldsymbol{k}_p = \begin{pmatrix} 20. & 5. & 0 \\ 5. & 20. & 5. \\ 0 & 5. & 20. \end{pmatrix}$$

Characteristic equation: $\mathrm{Det}[\boldsymbol{k}_k - P\boldsymbol{k}_p] = 0$

gives $-7000.\,P^3 + 9.\times 10^7\,P^2 - 2.66667\times 10^{11}\,P + 1.48148\times 10^{14} = 0$

Computing roots of the characteristic equation we get the eigenvalues

Eigenvalues: $\{P_1 = 721.295,\ P_2 = 3333.33,\ P_3 = 8802.51\}$

Substituting these eigenvalues into the global equations, the corresponding eigenvectors are

	Eigenvalue	Eigenvector
1	721.295	(0 0.5 0.707107 0.5 0)
2	3333.33	(0 −0.707107 0 0.707107 0)
3	8802.51	(0 0.5 −0.707107 0.5 0)

Using the first eigenvalue, the lowest buckling load is

Buckling load = 721.295

The example corresponds to the classical Euler buckling situation. The buckling load using the Euler's equation is

$$\pi^2 \mathrm{EI}/L^2 = 685.389$$

The four linear element finite element solution is clearly not very accurate. To get a better solution we must either use more linear elements or employ higher-order elements.

(ii) Solution using four quadratic elements

4 element solution

Nodal locations: {0, 15., 30., 45., 60., 75., 90., 105., 120.}

Element 1

Element nodes: $\{x_1 \to 0,\ x_2 \to 15.,\ x_3 \to 30.\}$

$$\begin{pmatrix} 77777.8 & -88888.9 & 11111.1 \\ -88888.9 & 177778. & -88888.9 \\ 11111.1 & -88888.9 & 77777.8 \end{pmatrix}\begin{pmatrix} \mathrm{v}_1 \\ \mathrm{v}_2 \\ \mathrm{v}_3 \end{pmatrix} - \mathrm{P}\begin{pmatrix} 4. & 2. & -1. \\ 2. & 16. & 2. \\ -1. & 2. & 4. \end{pmatrix}\begin{pmatrix} \mathrm{v}_1 \\ \mathrm{v}_2 \\ \mathrm{v}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Element 2

Element nodes: $\{x_3 \to 30.,\ x_4 \to 45.,\ x_5 \to 60.\}$

$$\begin{pmatrix} 77777.8 & -88888.9 & 11111.1 \\ -88888.9 & 177778. & -88888.9 \\ 11111.1 & -88888.9 & 77777.8 \end{pmatrix}\begin{pmatrix} \mathrm{v}_3 \\ \mathrm{v}_4 \\ \mathrm{v}_5 \end{pmatrix} - \mathrm{P}\begin{pmatrix} 4. & 2. & -1. \\ 2. & 16. & 2. \\ -1. & 2. & 4. \end{pmatrix}\begin{pmatrix} \mathrm{v}_3 \\ \mathrm{v}_4 \\ \mathrm{v}_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Element 3

Element nodes: $\{x_5 \to 60.,\ x_6 \to 75.,\ x_7 \to 90.\}$

$$\begin{pmatrix} 77777.8 & -88888.9 & 11111.1 \\ -88888.9 & 177778. & -88888.9 \\ 11111.1 & -88888.9 & 77777.8 \end{pmatrix}\begin{pmatrix} \mathrm{v}_5 \\ \mathrm{v}_6 \\ \mathrm{v}_7 \end{pmatrix} - \mathrm{P}\begin{pmatrix} 4. & 2. & -1. \\ 2. & 16. & 2. \\ -1. & 2. & 4. \end{pmatrix}\begin{pmatrix} \mathrm{v}_5 \\ \mathrm{v}_6 \\ \mathrm{v}_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Element 4

Element nodes: $\{x_7 \to 90., x_8 \to 105., x_9 \to 120.\}$

$$\begin{pmatrix} 77777.8 & -88888.9 & 11111.1 \\ -88888.9 & 177778. & -88888.9 \\ 11111.1 & -88888.9 & 77777.8 \end{pmatrix} \begin{pmatrix} v_7 \\ v_8 \\ v_9 \end{pmatrix} - P \begin{pmatrix} 4. & 2. & -1. \\ 2. & 16. & 2. \\ -1. & 2. & 4. \end{pmatrix} \begin{pmatrix} v_7 \\ v_8 \\ v_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Global equations before boundary conditions

$$\begin{pmatrix} 77777.8 & -88888.9 & 11111.1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -88888.9 & 177778. & -88888.9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 11111.1 & -88888.9 & 155556. & -88888.9 & 11111.1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -88888.9 & 177778. & -88888.9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 11111.1 & -88888.9 & 155556. & -88888.9 & 11111.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -88888.9 & 177778. & -88888.9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 11111.1 & -88888.9 & 155556. & -88888.9 & 11111.1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -88888.9 & 177778. & -88888.9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 11111.1 & -88888.9 & 77777.8 \end{pmatrix}$$

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \end{pmatrix} - P \begin{pmatrix} 4. & 2. & -1. & 0 & 0 & 0 & 0 & 0 & 0 \\ 2. & 16. & 2. & 0 & 0 & 0 & 0 & 0 & 0 \\ -1. & 2. & 8. & 2. & -1. & 0 & 0 & 0 & 0 \\ 0 & 0 & 2. & 16. & 2. & 0 & 0 & 0 & 0 \\ 0 & 0 & -1. & 2. & 8. & 2. & -1. & 0 & 0 \\ 0 & 0 & 0 & 0 & 2. & 16. & 2. & 0 & 0 \\ 0 & 0 & 0 & 0 & -1. & 2. & 8. & 2. & -1. \\ 0 & 0 & 0 & 0 & 0 & 0 & 2. & 16. & 2. \\ 0 & 0 & 0 & 0 & 0 & 0 & -1. & 2. & 4. \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Essential boundary conditions

DOF	Value
v_1	0
v_9	0

Incorporating EBC the final system of equations is

$$\begin{pmatrix} 177778. & -88888.9 & 0 & 0 & 0 & 0 & 0 \\ -88888.9 & 155556. & -88888.9 & 11111.1 & 0 & 0 & 0 \\ 0 & -88888.9 & 177778. & -88888.9 & 0 & 0 & 0 \\ 0 & 11111.1 & -88888.9 & 155556. & -88888.9 & 11111.1 & 0 \\ 0 & 0 & 0 & -88888.9 & 177778. & -88888.9 & 0 \\ 0 & 0 & 0 & 11111.1 & -88888.9 & 155556. & -88888.9 \\ 0 & 0 & 0 & 0 & 0 & -88888.9 & 177778. \end{pmatrix}$$

$$\begin{pmatrix} v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{pmatrix} - P \begin{pmatrix} 16. & 2. & 0 & 0 & 0 & 0 & 0 \\ 2. & 8. & 2. & -1. & 0 & 0 & 0 \\ 0 & 2. & 16. & 2. & 0 & 0 & 0 \\ 0 & -1. & 2. & 8. & 2. & -1. & 0 \\ 0 & 0 & 0 & 2. & 16. & 2. & 0 \\ 0 & 0 & 0 & -1. & 2. & 8. & 2. \\ 0 & 0 & 0 & 0 & 0 & 2. & 16. \end{pmatrix} \begin{pmatrix} v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solution of the eigenvalue problem

$$\boldsymbol{k}_k = \begin{pmatrix} 177778. & -88888.9 & 0 & 0 & 0 & 0 & 0 \\ -88888.9 & 155556. & -88888.9 & 11111.1 & 0 & 0 & 0 \\ 0 & -88888.9 & 177778. & -88888.9 & 0 & 0 & 0 \\ 0 & 11111.1 & -88888.9 & 155556. & -88888.9 & 11111.1 & 0 \\ 0 & 0 & 0 & -88888.9 & 177778. & -88888.9 & 0 \\ 0 & 0 & 0 & 11111.1 & -88888.9 & 155556. & -88888.9 \\ 0 & 0 & 0 & 0 & 0 & -88888.9 & 177778. \end{pmatrix}$$

$$\boldsymbol{k}_p = \begin{pmatrix} 16. & 2. & 0 & 0 & 0 & 0 & 0 \\ 2. & 8. & 2. & -1. & 0 & 0 & 0 \\ 0 & 2. & 16. & 2. & 0 & 0 & 0 \\ 0 & -1. & 2. & 8. & 2. & -1. & 0 \\ 0 & 0 & 0 & 2. & 16. & 2. & 0 \\ 0 & 0 & 0 & -1. & 2. & 8. & 2. \\ 0 & 0 & 0 & 0 & 0 & 2. & 16. \end{pmatrix}$$

Characteristic equation: $\text{Det}[\boldsymbol{k}_k - P\boldsymbol{k}_p] = 0$ gives $-2.6112\times 10^{7}\,P^7 + 3.48046\times 10^{12}\,P^6 - 1.67258\times 10^{17}\,P^5 + 3.64352\times 10^{21}\,P^4 - 3.75246\times 10^{25}\,P^3 + 1.75502\times 10^{29}\,P^2 - 3.19639\times 10^{32}\,P + 1.47981\times 10^{35} = 0$

Computing roots of the characteristic equation we get the eigenvalues

Eigenvalues:

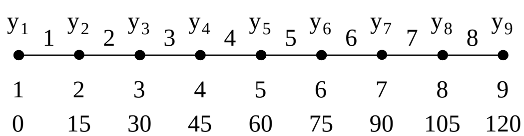
$\{P_1 = 685.74,\ P_2 = 2762.18,\ P_3 = 6373.94,\ P_4 = 11111.1,\ P_5 = 21406.4,\ P_6 = 35756.3,\ P_7 = 55194.\}$

Substituting these eigenvalues into the global equations, the corresponding eigenvectors are

	Eigenvalue	Eigenvector
1	685.74	$(0 \quad 0.191327 \quad 0.353581 \quad 0.461903 \quad 0.50004 \quad 0.461903 \quad 0.353581 \quad 0.191327 \quad 0$
2	2762.18	$(0 \quad -0.353471 \quad -0.500117 \quad -0.353471 \quad 0 \quad 0.353471 \quad 0.500117 \quad 0.353471 \quad 0)$
3	6373.94	$(0 \quad 0.467899 \quad 0.348932 \quad -0.19381 \quad -0.493465 \quad -0.19381 \quad 0.348932 \quad 0.467899$
4	11111.1	$(0 \quad 0.5 \quad 0 \quad -0.5 \quad 0 \quad 0.5 \quad 0 \quad -0.5 \quad 0)$
5	21406.4	$(0 \quad 0.340986 \quad -0.426486 \quad -0.141241 \quad 0.603142 \quad -0.141241 \quad -0.426486 \quad 0.34098$
6	35756.3	$(0 \quad -0.249323 \quad 0.612924 \quad -0.249323 \quad 0 \quad 0.249323 \quad -0.612924 \quad 0.249323 \quad 0)$
7	55194.	$(0 \quad 0.125228 \quad -0.443236 \quad 0.302327 \quad -0.62683 \quad 0.302327 \quad -0.443236 \quad 0.125228$

Buckling load using 4 quadratic elements = 685.74

(iii) Solution using eight linear elements



8 element solution

Nodal locations: {0, 15., 30., 45., 60., 75., 90., 105., 120.}

Element 1

Element nodes: $\{x_1 \to 0, x_2 \to 15.\}$

$$\begin{pmatrix} 66666.7 & -66666.7 \\ -66666.7 & 66666.7 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - P \begin{pmatrix} 5. & 2.5 \\ 2.5 & 5. \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Element 2

Element nodes: $\{x_2 \to 15., x_3 \to 30.\}$

$$\begin{pmatrix} 66666.7 & -66666.7 \\ -66666.7 & 66666.7 \end{pmatrix} \begin{pmatrix} v_2 \\ v_3 \end{pmatrix} - P \begin{pmatrix} 5. & 2.5 \\ 2.5 & 5. \end{pmatrix} \begin{pmatrix} v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Element 3

Element nodes: $\{x_3 \to 30., x_4 \to 45.\}$

$$\begin{pmatrix} 66666.7 & -66666.7 \\ -66666.7 & 66666.7 \end{pmatrix} \begin{pmatrix} v_3 \\ v_4 \end{pmatrix} - P \begin{pmatrix} 5. & 2.5 \\ 2.5 & 5. \end{pmatrix} \begin{pmatrix} v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Element 4

Element nodes: $\{x_4 \to 45., x_5 \to 60.\}$

$$\begin{pmatrix} 66666.7 & -66666.7 \\ -66666.7 & 66666.7 \end{pmatrix} \begin{pmatrix} v_4 \\ v_5 \end{pmatrix} - P \begin{pmatrix} 5. & 2.5 \\ 2.5 & 5. \end{pmatrix} \begin{pmatrix} v_4 \\ v_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Element 5

Element nodes: $\{x_5 \to 60., x_6 \to 75.\}$

$$\begin{pmatrix} 66666.7 & -66666.7 \\ -66666.7 & 66666.7 \end{pmatrix} \begin{pmatrix} v_5 \\ v_6 \end{pmatrix} - P \begin{pmatrix} 5. & 2.5 \\ 2.5 & 5. \end{pmatrix} \begin{pmatrix} v_5 \\ v_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Element 6

Element nodes: $\{x_6 \to 75., x_7 \to 90.\}$

$$\begin{pmatrix} 66666.7 & -66666.7 \\ -66666.7 & 66666.7 \end{pmatrix} \begin{pmatrix} v_6 \\ v_7 \end{pmatrix} - P \begin{pmatrix} 5. & 2.5 \\ 2.5 & 5. \end{pmatrix} \begin{pmatrix} v_6 \\ v_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Element 7

Element nodes: $\{x_7 \to 90., x_8 \to 105.\}$

$$\begin{pmatrix} 66666.7 & -66666.7 \\ -66666.7 & 66666.7 \end{pmatrix} \begin{pmatrix} v_7 \\ v_8 \end{pmatrix} - P \begin{pmatrix} 5. & 2.5 \\ 2.5 & 5. \end{pmatrix} \begin{pmatrix} v_7 \\ v_8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Element 8

Element nodes: $\{x_8 \to 105., x_9 \to 120.\}$

$$\begin{pmatrix} 66666.7 & -66666.7 \\ -66666.7 & 66666.7 \end{pmatrix} \begin{pmatrix} v_8 \\ v_9 \end{pmatrix} - P \begin{pmatrix} 5. & 2.5 \\ 2.5 & 5. \end{pmatrix} \begin{pmatrix} v_8 \\ v_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Global equations before boundary conditions

$$\begin{pmatrix} 66666.7 & -66666.7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -66666.7 & 133333. & -66666.7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -66666.7 & 133333. & -66666.7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -66666.7 & 133333. & -66666.7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -66666.7 & 133333. & -66666.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -66666.7 & 133333. & -66666.7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -66666.7 & 133333. & -66666.7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -66666.7 & 133333. & -66666.7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -66666.7 & 66666.7 \end{pmatrix}$$

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \end{pmatrix} - P \begin{pmatrix} 5. & 2.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2.5 & 10. & 2.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2.5 & 10. & 2.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2.5 & 10. & 2.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.5 & 10. & 2.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.5 & 10. & 2.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.5 & 10. & 2.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2.5 & 10. & 2.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.5 & 5. \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Essential boundary conditions

DOF	Value
v_1	0
v_9	0

Incorporating EBC the final system of equations is

$$\begin{pmatrix} 133333. & -66666.7 & 0 & 0 & 0 & 0 & 0 \\ -66666.7 & 133333. & -66666.7 & 0 & 0 & 0 & 0 \\ 0 & -66666.7 & 133333. & -66666.7 & 0 & 0 & 0 \\ 0 & 0 & -66666.7 & 133333. & -66666.7 & 0 & 0 \\ 0 & 0 & 0 & -66666.7 & 133333. & -66666.7 & 0 \\ 0 & 0 & 0 & 0 & -66666.7 & 133333. & -66666.7 \\ 0 & 0 & 0 & 0 & 0 & -66666.7 & 133333. \end{pmatrix}$$

$$\begin{pmatrix} v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{pmatrix} - P \begin{pmatrix} 10. & 2.5 & 0 & 0 & 0 & 0 & 0 \\ 2.5 & 10. & 2.5 & 0 & 0 & 0 & 0 \\ 0 & 2.5 & 10. & 2.5 & 0 & 0 & 0 \\ 0 & 0 & 2.5 & 10. & 2.5 & 0 & 0 \\ 0 & 0 & 0 & 2.5 & 10. & 2.5 & 0 \\ 0 & 0 & 0 & 0 & 2.5 & 10. & 2.5 \\ 0 & 0 & 0 & 0 & 0 & 2.5 & 10. \end{pmatrix} \begin{pmatrix} v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solution of the eigenvalue problem

$$\boldsymbol{k}_k = \begin{pmatrix} 133333. & -66666.7 & 0 & 0 & 0 & 0 & 0 \\ -66666.7 & 133333. & -66666.7 & 0 & 0 & 0 & 0 \\ 0 & -66666.7 & 133333. & -66666.7 & 0 & 0 & 0 \\ 0 & 0 & -66666.7 & 133333. & -66666.7 & 0 & 0 \\ 0 & 0 & 0 & -66666.7 & 133333. & -66666.7 & 0 \\ 0 & 0 & 0 & 0 & -66666.7 & 133333. & -66666.7 \\ 0 & 0 & 0 & 0 & 0 & -66666.7 & 133333. \end{pmatrix}$$

$$\boldsymbol{k}_p = \begin{pmatrix} 10. & 2.5 & 0 & 0 & 0 & 0 & 0 \\ 2.5 & 10. & 2.5 & 0 & 0 & 0 & 0 \\ 0 & 2.5 & 10. & 2.5 & 0 & 0 & 0 \\ 0 & 0 & 2.5 & 10. & 2.5 & 0 & 0 \\ 0 & 0 & 0 & 2.5 & 10. & 2.5 & 0 \\ 0 & 0 & 0 & 0 & 2.5 & 10. & 2.5 \\ 0 & 0 & 0 & 0 & 0 & 2.5 & 10. \end{pmatrix}$$

Characteristic equation: $\mathrm{Det}[\boldsymbol{k}_k - P\boldsymbol{k}_p] = 0$ gives $-6.63086 \times 10^{6}\, P^{7} + 8.58724 \times 10^{11}\, P^{6} - 4.14583 \times 10^{16}\, P^{5} + 9.36574 \times 10^{20}\, P^{4} - 1.02469 \times 10^{25}\, P^{3} + 5.11605 \times 10^{28}\, P^{2} - 9.83265 \times 10^{31}\, P + 4.68221 \times 10^{34} = 0$

Computing roots of the characteristic equation we get the eigenvalues

Eigenvalues:

$\{P_1 = 694.242,\ P_2 = 2885.18,\ P_3 = 6908.92,\ P_4 = 13333.3,\ P_5 = 22798.,\ P_6 = 35210.1,\ P_7 = 47674.5\}$

Substituting these eigenvalues into the global equations, the corresponding eigenvectors are

	Eigenvalue	Eigenvector
1	694.242	$(0 \quad 0.191342 \quad 0.353553 \quad 0.46194 \quad 0.5 \quad 0.46194 \quad 0.353553 \quad 0.191342 \quad 0)$
2	2885.18	$(0 \quad 0.353553 \quad 0.5 \quad 0.353553 \quad 0 \quad -0.353553 \quad -0.5 \quad -0.353553 \quad 0)$
3	6908.92	$(0 \quad 0.46194 \quad 0.353553 \quad -0.191342 \quad -0.5 \quad -0.191342 \quad 0.353553 \quad 0.46194 \quad 0)$
4	13333.3	$(0 \quad 0.5 \quad 0 \quad -0.5 \quad 0 \quad 0.5 \quad 0 \quad -0.5 \quad 0)$
5	22798.	$(0 \quad -0.46194 \quad 0.353553 \quad 0.191342 \quad -0.5 \quad 0.191342 \quad 0.353553 \quad -0.46194 \quad 0)$
6	35210.1	$(0 \quad -0.353553 \quad 0.5 \quad -0.353553 \quad 0 \quad 0.353553 \quad -0.5 \quad 0.353553 \quad 0)$
7	47674.5	$(0 \quad -0.191342 \quad 0.353553 \quad -0.46194 \quad 0.5 \quad -0.46194 \quad 0.353553 \quad -0.191342 \quad 0)$

Buckling load using 8 linear elements = 694.242