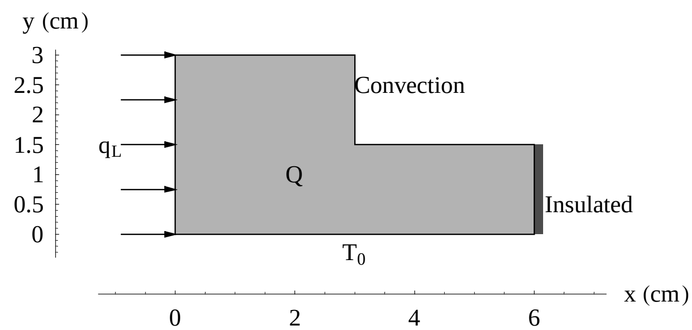
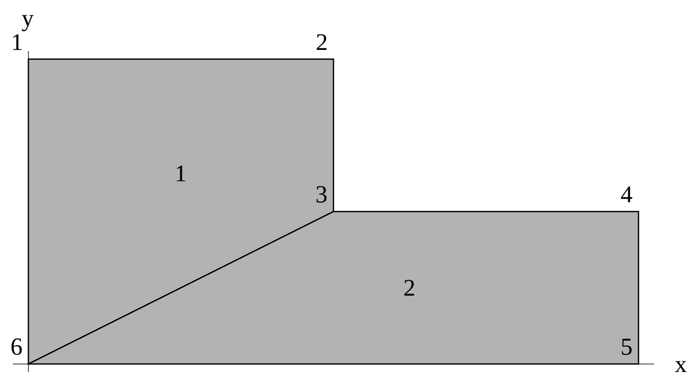
Example 6.21: Heat flow in an L-shaped body using Quad4 elements (p. 445)

Consider two dimensional heat flow over an L-shaped body with thermal conductivity $k = 45\ W/m.°C$ shown in Figure. The bottom is maintained at $T_0 = 110\ °C$. Convection heat loss takes place on the top where the ambient air temperature is $20\ °C$ and the convection heat transfer coefficient is $h = 55\ W/m^2.°C$. The right side is insulated. The left side is subjected to heat flux at a uniform rate of $q_L = 8000\ W/m^2$. Heat is generated in the body at a rate of $Q = 5 \times 10^6\ W/m^3$. Determine temperature distribution in the body.





Global equations at start of the element assembly process

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 1

$k_x = 45;$ $k_y = 45;$ $p = 0;$ $q = 5000000$

$$\boldsymbol{C} = \begin{pmatrix} 45 & 0 \\ 0 & 45 \end{pmatrix}$$

Element thickness = 1

Nodal coordinates

Element node	Global node number	x	y
1	6	0	0
2	3	0.03	0.015
3	2	0.03	0.03
4	1	0	0.03

Interpolation functions and their derivatives

$$\boldsymbol{N}^{\mathrm{T}} = \left\{ \frac{1}{4}(s-1)(t-1),\ -\frac{1}{4}(s+1)(t-1),\ \frac{1}{4}(s+1)(t+1),\ -\frac{1}{4}(s-1)(t+1) \right\}$$

$$\partial \boldsymbol{N}^{\mathrm{T}}/\partial s = \left\{ \frac{t-1}{4},\ \frac{1-t}{4},\ \frac{t+1}{4},\ \frac{1}{4}(-t-1) \right\}$$

$$\partial \boldsymbol{N}^{\mathrm{T}}/\partial t = \left\{ \frac{s-1}{4},\ \frac{1}{4}(-s-1),\ \frac{s+1}{4},\ \frac{1-s}{4} \right\}$$

Mapping to the master element

$$x(s,t) = 0.015\, s + 0.015$$

$$y(s,t) = -0.00375\, t\, s + 0.00375\, s + 0.01125\, t + 0.01875$$

Jacobian matrix, $\boldsymbol{J}$ =

$$\begin{pmatrix} 0.015 & 0 \\ 0.00375 - 0.00375\, t & 0.01125 - 0.00375\, s \end{pmatrix}; \qquad \det J = 0.00016875 - 0.00005625\, s$$

Gauss quadrature points and weights

	Point	Weight
1	$s \to -0.57735$ $t \to -0.57735$	1.
2	$s \to -0.57735$ $t \to 0.57735$	1.
3	$s \to 0.57735$ $t \to -0.57735$	1.
4	$s \to 0.57735$ $t \to 0.57735$	1.

Computation of element matrices at {−0.57735, −0.57735} with weight = 1.

$$\boldsymbol{N}^{\mathrm{T}} = (\,0.622008 \quad 0.166667 \quad 0.0446582 \quad 0.166667\,)$$

$$\partial \boldsymbol{N}^{\mathrm{T}}/\partial s = (\,-0.394338 \quad 0.394338 \quad 0.105662 \quad -0.105662\,)$$

$$\partial \boldsymbol{N}^{\mathrm{T}}/\partial t = (\,-0.394338 \quad -0.105662 \quad 0.105662 \quad 0.394338\,)$$

$$\text{Jacobian matrix, } \boldsymbol{J} = \begin{pmatrix} 0.015 & 0 \\ 0.00591506 & 0.0134151 \end{pmatrix}; \qquad \det J = 0.000201226$$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -14.6976 & 29.3951 & 3.9382 & -18.6358 \\ -29.3951 & -7.8764 & 7.8764 & 29.3951 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 9.78042 & -1.81564 & -2.62065 & -5.34412 \\ -1.81564 & 8.3861 & 0.4865 & -7.05695 \\ -2.62065 & 0.4865 & 0.702202 & 1.43195 \\ -5.34412 & -7.05695 & 1.43195 & 10.9691 \end{pmatrix}$$

$$\boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 625.821 \\ 167.688 \\ 44.9319 \\ 167.688 \end{pmatrix}$$

Computation of element matrices at {−0.57735, 0.57735} with weight = 1.

$$\boldsymbol{N}^{\mathrm{T}} = (\,0.166667 \quad 0.0446582 \quad 0.166667 \quad 0.622008\,)$$

$$\partial \boldsymbol{N}^{\mathrm{T}}/\partial s = (\,-0.105662 \quad 0.105662 \quad 0.394338 \quad -0.394338\,)$$

$$\partial \boldsymbol{N}^{\mathrm{T}}/\partial t = (\,-0.394338 \quad -0.105662 \quad 0.105662 \quad 0.394338\,)$$

$$\text{Jacobian matrix, } \boldsymbol{J} = \begin{pmatrix} 0.015 & 0 \\ 0.00158494 & 0.0134151 \end{pmatrix}; \qquad \det J = 0.000201226$$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -3.9382 & 7.8764 & 25.4569 & -29.3951 \\ -29.3951 & -7.8764 & 7.8764 & 29.3951 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 7.96477 & 1.81564 & -3.00435 & -6.77607 \\ 1.81564 & 1.12352 & 1.25388 & -4.19305 \\ -3.00435 & 1.25388 & 6.43001 & -4.67955 \\ -6.77607 & -4.19305 & -4.67955 & 15.6487 \end{pmatrix}$$

$$\boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 167.688 \\ 44.9319 \\ 167.688 \\ 625.821 \end{pmatrix}$$

Computation of element matrices at {0.57735, −0.57735} with weight = 1.

$$\boldsymbol{N}^{\mathrm{T}} = (\, 0.166667 \quad 0.622008 \quad 0.166667 \quad 0.0446582 \,)$$

$$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{s} = (\, -0.394338 \quad 0.394338 \quad 0.105662 \quad -0.105662 \,)$$

$$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{t} = (\, -0.105662 \quad -0.394338 \quad 0.394338 \quad 0.105662 \,)$$

Jacobian matrix, $\boldsymbol{J} = \begin{pmatrix} 0.015 & 0 \\ 0.00591506 & 0.00908494 \end{pmatrix}$; detJ = 0.000136274

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -21.7028 & 43.4056 & -10.0723 & -11.6305 \\ -11.6305 & -43.4056 & 43.4056 & 11.6305 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 3.71792 & -2.68103 & -1.75527 & 0.718379 \\ -2.68103 & 23.1072 & -14.2347 & -6.19157 \\ -1.75527 & -14.2347 & 12.1758 & 3.81416 \\ 0.718379 & -6.19157 & 3.81416 & 1.65903 \end{pmatrix}$$

$$\boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 113.562 \\ 423.818 \\ 113.562 \\ 30.4288 \end{pmatrix}$$

Computation of element matrices at {0.57735, 0.57735} with weight = 1.

$$\boldsymbol{N}^{\mathrm{T}} = (\, 0.0446582 \quad 0.166667 \quad 0.622008 \quad 0.166667 \,)$$

$$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{s} = (\, -0.105662 \quad 0.105662 \quad 0.394338 \quad -0.394338 \,)$$

$$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{t} = (\, -0.105662 \quad -0.394338 \quad 0.394338 \quad 0.105662 \,)$$

Jacobian matrix, $\boldsymbol{J} = \begin{pmatrix} 0.015 & 0 \\ 0.00158494 & 0.00908494 \end{pmatrix}$; detJ = 0.000136274

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -5.81525 & 11.6305 & 21.7028 & -27.5181 \\ -11.6305 & -43.4056 & 43.4056 & 11.6305 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 1.03689 & 2.68103 & -3.86973 & 0.151811 \\ 2.68103 & 12.3831 & -10.0057 & -5.05843 \\ -3.86973 & -10.0057 & 14.442 & -0.566568 \\ 0.151811 & -5.05843 & -0.566568 & 5.47319 \end{pmatrix}$$

$$\boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 30.4288 \\ 113.562 \\ 423.818 \\ 113.562 \end{pmatrix}$$

Summing contributions from all points we get

$$\boldsymbol{k} = \begin{pmatrix} 22.5 & 0 & -11.25 & -11.25 \\ 0 & 45. & -22.5 & -22.5 \\ -11.25 & -22.5 & 33.75 & 0 \\ -11.25 & -22.5 & 0 & 33.75 \end{pmatrix}$$

$$\boldsymbol{r}^{\mathrm{T}} = (\ 937.5 \quad 750. \quad 750. \quad 937.5\ )$$

Computation of element matrices resulting from NBC

NBC on side 2 with $\alpha = -55$ and $\beta = 1100$

$$\boldsymbol{N}_{\mathrm{c}}^{\mathrm{T}} = \left(0 \quad \tfrac{1-\mathrm{a}}{2} \quad \tfrac{\mathrm{a}+1}{2} \quad 0\right)$$

$\mathrm{x(a)} = 0.03;$ $\qquad$ $\mathrm{y(a)} = 0.0075\,\mathrm{a} + 0.0225$

$\mathrm{dx/da} = 0.;$ $\qquad$ $\mathrm{dy/da} = 0.0075;$ $\qquad$ $\mathrm{J_c} = 0.0075$

Gauss point = -0.57735; $\qquad$ Weight = 1.; $\qquad$ $\mathrm{J_c} = 0.0075$

$$\boldsymbol{N}_{\mathrm{c}}^{\mathrm{T}} = (\ 0 \quad 0.788675 \quad 0.211325 \quad 0\ )$$

$$\boldsymbol{k}_{\alpha} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.256578 & 0.06875 & 0 \\ 0 & 0.06875 & 0.0184215 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{r}_{\beta}^{\mathrm{T}} = (\ 0 \quad 6.50657 \quad 1.74343 \quad 0\ )$$

Gauss point = 0.57735; $\qquad$ Weight = 1.; $\qquad$ $\mathrm{J_c} = 0.0075$

$$\boldsymbol{N}_{\mathrm{c}}^{\mathrm{T}} = (\ 0 \quad 0.211325 \quad 0.788675 \quad 0\ )$$

$$\boldsymbol{k}_{\alpha} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.0184215 & 0.06875 & 0 \\ 0 & 0.06875 & 0.256578 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{r}_\beta^{\mathrm{T}} = (\,0 \quad 1.74343 \quad 6.50657 \quad 0\,)$$

Summing contributions from all Gauss points

$$\boldsymbol{k}_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.275 & 0.1375 & 0 \\ 0 & 0.1375 & 0.275 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{r}_\beta^{\mathrm{T}} = (\,0 \quad 8.25 \quad 8.25 \quad 0\,)$$

Computation of element matrices resulting from NBC

NBC on side 3 with $\alpha = -55$ and $\beta = 1100$

$$\boldsymbol{N}_c^{\mathrm{T}} = \left(0 \quad 0 \quad \tfrac{1-a}{2} \quad \tfrac{a+1}{2}\right)$$

$x(a) = 0.015 - 0.015\,a$; $\quad y(a) = 0.03$

$dx/da = -0.015$; $\quad dy/da = 0.$; $\quad J_c = 0.015$

Gauss point $= -0.57735$; $\quad$ Weight $= 1.$; $\quad J_c = 0.015$

$$\boldsymbol{N}_c^{\mathrm{T}} = (\,0 \quad 0 \quad 0.788675 \quad 0.211325\,)$$

$$\boldsymbol{k}_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.513157 & 0.1375 \\ 0 & 0 & 0.1375 & 0.036843 \end{pmatrix}$$

$$\boldsymbol{r}_\beta^{\mathrm{T}} = (\,0 \quad 0 \quad 13.0131 \quad 3.48686\,)$$

Gauss point $= 0.57735$; $\quad$ Weight $= 1.$; $\quad J_c = 0.015$

$$\boldsymbol{N}_c^{\mathrm{T}} = (\,0 \quad 0 \quad 0.211325 \quad 0.788675\,)$$

$$\boldsymbol{k}_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.036843 & 0.1375 \\ 0 & 0 & 0.1375 & 0.513157 \end{pmatrix}$$

$$\boldsymbol{r}_\beta^{\mathrm{T}} = (\,0 \quad 0 \quad 3.48686 \quad 13.0131\,)$$

Summing contributions from all Gauss points

$$\boldsymbol{k}_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.55 & 0.275 \\ 0 & 0 & 0.275 & 0.55 \end{pmatrix}$$

$\boldsymbol{r}_\beta^{\mathrm{T}} = (\,0 \quad 0 \quad 16.5 \quad 16.5\,)$

Computation of element matrices resulting from NBC

NBC on side 4 with $\alpha = 0$ and $\beta = 8000$

$\boldsymbol{N}_c^{\mathrm{T}} = \left(\,\frac{a+1}{2} \quad 0 \quad 0 \quad \frac{1-a}{2}\,\right)$

$x(a) = 0;\qquad y(a) = 0.015 - 0.015\,a$

$dx/da = 0.;\qquad dy/da = -0.015;\qquad J_c = 0.015$

Gauss point $= -0.57735$; Weight $= 1.$; $J_c = 0.015$

$\boldsymbol{N}_c^{\mathrm{T}} = (\,0.211325 \quad 0 \quad 0 \quad 0.788675\,)$

$$\boldsymbol{k}_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\boldsymbol{r}_\beta^{\mathrm{T}} = (\,25.359 \quad 0 \quad 0 \quad 94.641\,)$

Gauss point $= 0.57735$; Weight $= 1.$; $J_c = 0.015$

$\boldsymbol{N}_c^{\mathrm{T}} = (\,0.788675 \quad 0 \quad 0 \quad 0.211325\,)$

$$\boldsymbol{k}_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\boldsymbol{r}_\beta^{\mathrm{T}} = (\,94.641 \quad 0 \quad 0 \quad 25.359\,)$

Summing contributions from all Gauss points

$$\boldsymbol{k}_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\boldsymbol{r}_\beta^{\mathrm{T}} = (\,120. \quad 0 \quad 0 \quad 120.\,)$

Complete element equations for element 1

$$\begin{pmatrix} 22.5 & 0 & -11.25 & -11.25 \\ 0 & 45.275 & -22.3625 & -22.5 \\ -11.25 & -22.3625 & 34.575 & 0.275 \\ -11.25 & -22.5 & 0.275 & 34.3 \end{pmatrix} \begin{pmatrix} T_6 \\ T_3 \\ T_2 \\ T_1 \end{pmatrix} = \begin{pmatrix} 1057.5 \\ 758.25 \\ 774.75 \\ 1074. \end{pmatrix}$$

The element contributes to {6, 3, 2, 1} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 6 \\ 3 \\ 2 \\ 1 \end{pmatrix}$

and to a global matrix: $\begin{pmatrix} [6,6] & [6,3] & [6,2] & [6,1] \\ [3,6] & [3,3] & [3,2] & [3,1] \\ [2,6] & [2,3] & [2,2] & [2,1] \\ [1,6] & [1,3] & [1,2] & [1,1] \end{pmatrix}$

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 34.3 & 0.275 & -22.5 & 0 & 0 & -11.25 \\ 0.275 & 34.575 & -22.3625 & 0 & 0 & -11.25 \\ -22.5 & -22.3625 & 45.275 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -11.25 & -11.25 & 0 & 0 & 0 & 22.5 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{pmatrix} = \begin{pmatrix} 1074. \\ 774.75 \\ 758.25 \\ 0 \\ 0 \\ 1057.5 \end{pmatrix}$$

Equations for element 2

$k_x = 45;$ $k_y = 45;$ $p = 0;$ $q = 5000000$

$$\boldsymbol{C} = \begin{pmatrix} 45 & 0 \\ 0 & 45 \end{pmatrix}$$

Element thickness = 1

Nodal coordinates

Element node	Global node number	x	y
1	6	0	0
2	5	0.06	0
3	4	0.06	0.015
4	3	0.03	0.015

Interpolation functions and their derivatives

$$\boldsymbol{N}^{\mathrm{T}} = \left\{\frac{1}{4}(s-1)(t-1), -\frac{1}{4}(s+1)(t-1), \frac{1}{4}(s+1)(t+1), -\frac{1}{4}(s-1)(t+1)\right\}$$

$$\partial\boldsymbol{N}^{\mathrm{T}}/\partial s = \left\{\frac{t-1}{4}, \frac{1-t}{4}, \frac{t+1}{4}, \frac{1}{4}(-t-1)\right\}$$

$$\partial\boldsymbol{N}^{\mathrm{T}}/\partial t = \left\{\frac{s-1}{4}, \frac{1}{4}(-s-1), \frac{s+1}{4}, \frac{1-s}{4}\right\}$$

Mapping to the master element

$$x(s,t) = -0.0075\,t\,s + 0.0225\,s + 0.0075\,t + 0.0375$$

$$y(s,t) = 0.0075\,t + 0.0075$$

Jacobian matrix, $\boldsymbol{J}$ =

$$\begin{pmatrix} 0.0225 - 0.0075\,t & 0.0075 - 0.0075\,s \\ 0 & 0.0075 \end{pmatrix}; \qquad \det J = 0.00016875 - 0.00005625\,t$$

Gauss quadrature points and weights

	Point	Weight
1	$s \to -0.57735$ $t \to -0.57735$	1.
2	$s \to -0.57735$ $t \to 0.57735$	1.
3	$s \to 0.57735$ $t \to -0.57735$	1.
4	$s \to 0.57735$ $t \to 0.57735$	1.

Computation of element matrices at {−0.57735, −0.57735} with weight = 1.

$$\boldsymbol{N}^{T} = (\,0.622008 \quad 0.166667 \quad 0.0446582 \quad 0.166667\,)$$

$$\partial \boldsymbol{N}^{T}/\partial s = (\,-0.394338 \quad 0.394338 \quad 0.105662 \quad -0.105662\,)$$

$$\partial \boldsymbol{N}^{T}/\partial t = (\,-0.394338 \quad -0.105662 \quad 0.105662 \quad 0.394338\,)$$

$$\text{Jacobian matrix, } \boldsymbol{J} = \begin{pmatrix} 0.0268301 & 0.0118301 \\ 0 & 0.0075 \end{pmatrix}; \qquad \det J = 0.000201226$$

$$\boldsymbol{B}^{T} = \begin{pmatrix} -14.6976 & 14.6976 & 3.9382 & -3.9382 \\ -29.3951 & -37.2715 & 7.8764 & 58.7903 \end{pmatrix}$$

$$\boldsymbol{k}_{k} = \begin{pmatrix} 9.78042 & 7.96477 & -2.62065 & -15.1245 \\ 7.96477 & 14.5352 & -2.13415 & -20.3658 \\ -2.62065 & -2.13415 & 0.702202 & 4.05261 \\ -15.1245 & -20.3658 & 4.05261 & 31.4378 \end{pmatrix}$$

$$\boldsymbol{r}_{q} = \begin{pmatrix} 625.821 \\ 167.688 \\ 44.9319 \\ 167.688 \end{pmatrix}$$

Computation of element matrices at {−0.57735, 0.57735} with weight = 1.

$$\boldsymbol{N}^{T} = (\,0.166667 \quad 0.0446582 \quad 0.166667 \quad 0.622008\,)$$

$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{s} = (\,-0.105662 \quad 0.105662 \quad 0.394338 \quad -0.394338\,)$

$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{t} = (\,-0.394338 \quad -0.105662 \quad 0.105662 \quad 0.394338\,)$

Jacobian matrix, $\boldsymbol{J} = \begin{pmatrix} 0.0181699 & 0.0118301 \\ 0 & 0.0075 \end{pmatrix}$; detJ = 0.000136274

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -5.81525 & 5.81525 & 21.7028 & -21.7028 \\ -43.4056 & -23.261 & -20.1446 & 86.8113 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 11.761 & 5.98419 & 4.58811 & -22.3333 \\ 5.98419 & 3.52543 & 3.64746 & -13.1571 \\ 4.58811 & 3.64746 & 5.37694 & -13.6125 \\ -22.3333 & -13.1571 & -13.6125 & 49.1029 \end{pmatrix}$$

$$\boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 113.562 \\ 30.4288 \\ 113.562 \\ 423.818 \end{pmatrix}$$

Computation of element matrices at {0.57735, −0.57735} with weight = 1.

$\boldsymbol{N}^{\mathrm{T}} = (\,0.166667 \quad 0.622008 \quad 0.166667 \quad 0.0446582\,)$

$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{s} = (\,-0.394338 \quad 0.394338 \quad 0.105662 \quad -0.105662\,)$

$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{t} = (\,-0.105662 \quad -0.394338 \quad 0.394338 \quad 0.105662\,)$

Jacobian matrix, $\boldsymbol{J} = \begin{pmatrix} 0.0268301 & 0.00316987 \\ 0 & 0.0075 \end{pmatrix}$; detJ = 0.000201226

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -14.6976 & 14.6976 & 3.9382 & -3.9382 \\ -7.8764 & -58.7903 & 50.9139 & 15.7528 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 2.51785 & 2.23696 & -4.15542 & -0.599393 \\ 2.23696 & 33.2534 & -26.5802 & -8.91023 \\ -4.15542 & -26.5802 & 23.6134 & 7.12213 \\ -0.599393 & -8.91023 & 7.12213 & 2.38749 \end{pmatrix}$$

$$\boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 167.688 \\ 625.821 \\ 167.688 \\ 44.9319 \end{pmatrix}$$

Computation of element matrices at {0.57735, 0.57735} with weight = 1.

$\boldsymbol{N}^{\mathrm{T}} = (\,0.0446582 \quad 0.166667 \quad 0.622008 \quad 0.166667\,)$

$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{s} = (\,-0.105662 \quad 0.105662 \quad 0.394338 \quad -0.394338\,)$

$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{t} = (-0.105662 \quad -0.394338 \quad 0.394338 \quad 0.105662)$

Jacobian matrix, $\boldsymbol{J} = \begin{pmatrix} 0.0181699 & 0.00316987 \\ 0 & 0.0075 \end{pmatrix}$; detJ = 0.000136274

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -5.81525 & 5.81525 & 21.7028 & -21.7028 \\ -11.6305 & -55.0362 & 43.4056 & 23.261 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 1.03689 & 3.71792 & -3.86973 & -0.88508 \\ 3.71792 & 18.7821 & -13.8755 & -8.62454 \\ -3.86973 & -13.8755 & 14.442 & 3.30316 \\ -0.88508 & -8.62454 & 3.30316 & 6.20646 \end{pmatrix}$$

$$\boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 30.4288 \\ 113.562 \\ 423.818 \\ 113.562 \end{pmatrix}$$

Summing contributions from all points we get

$$\boldsymbol{k} = \begin{pmatrix} 25.0962 & 19.9038 & -6.05769 & -38.9423 \\ 19.9038 & 70.0962 & -38.9423 & -51.0577 \\ -6.05769 & -38.9423 & 44.1346 & 0.865385 \\ -38.9423 & -51.0577 & 0.865385 & 89.1346 \end{pmatrix}$$

$\boldsymbol{r}^{\mathrm{T}} = (937.5 \quad 937.5 \quad 750. \quad 750.)$

Computation of element matrices resulting from NBC

NBC on side 3 with $\alpha = -55$ and $\beta = 1100$

$\boldsymbol{N}_{\mathrm{c}}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & \frac{1-a}{2} & \frac{a+1}{2} \end{pmatrix}$

x(a) = 0.045 – 0.015 a; y(a) = 0.015

dx/da = −0.015; dy/da = 0.; $\mathrm{J_c}$ = 0.015

Gauss point = −0.57735; Weight = 1.; $\mathrm{J_c}$ = 0.015

$\boldsymbol{N}_{\mathrm{c}}^{\mathrm{T}} = (0 \quad 0 \quad 0.788675 \quad 0.211325)$

$$\boldsymbol{k}_{\alpha} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.513157 & 0.1375 \\ 0 & 0 & 0.1375 & 0.036843 \end{pmatrix}$$

$\boldsymbol{r}_{\beta}^{\mathrm{T}} = (0 \quad 0 \quad 13.0131 \quad 3.48686)$

Gauss point = 0.57735; Weight = 1.; $\mathrm{J_c}$ = 0.015

$$\boldsymbol{N}_c^{\mathrm{T}} = (\,0 \quad 0 \quad 0.211325 \quad 0.788675\,)$$

$$\boldsymbol{k}_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.036843 & 0.1375 \\ 0 & 0 & 0.1375 & 0.513157 \end{pmatrix}$$

$$\boldsymbol{r}_\beta^{\mathrm{T}} = (\,0 \quad 0 \quad 3.48686 \quad 13.0131\,)$$

Summing contributions from all Gauss points

$$\boldsymbol{k}_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.55 & 0.275 \\ 0 & 0 & 0.275 & 0.55 \end{pmatrix}$$

$$\boldsymbol{r}_\beta^{\mathrm{T}} = (\,0 \quad 0 \quad 16.5 \quad 16.5\,)$$

Complete element equations for element 2

$$\begin{pmatrix} 25.0962 & 19.9038 & -6.05769 & -38.9423 \\ 19.9038 & 70.0962 & -38.9423 & -51.0577 \\ -6.05769 & -38.9423 & 44.6846 & 1.14038 \\ -38.9423 & -51.0577 & 1.14038 & 89.6846 \end{pmatrix} \begin{pmatrix} T_6 \\ T_5 \\ T_4 \\ T_3 \end{pmatrix} = \begin{pmatrix} 937.5 \\ 937.5 \\ 766.5 \\ 766.5 \end{pmatrix}$$

The element contributes to {6, 5, 4, 3} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 6 \\ 5 \\ 4 \\ 3 \end{pmatrix}$

and to a global matrix: $\begin{pmatrix} [6,6] & [6,5] & [6,4] & [6,3] \\ [5,6] & [5,5] & [5,4] & [5,3] \\ [4,6] & [4,5] & [4,4] & [4,3] \\ [3,6] & [3,5] & [3,4] & [3,3] \end{pmatrix}$

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 34.3 & 0.275 & -22.5 & 0 & 0 & -11.25 \\ 0.275 & 34.575 & -22.3625 & 0 & 0 & -11.25 \\ -22.5 & -22.3625 & 134.96 & 1.14038 & -51.0577 & -38.9423 \\ 0 & 0 & 1.14038 & 44.6846 & -38.9423 & -6.05769 \\ 0 & 0 & -51.0577 & -38.9423 & 70.0962 & 19.9038 \\ -11.25 & -11.25 & -38.9423 & -6.05769 & 19.9038 & 47.5962 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{pmatrix} = \begin{pmatrix} 1074. \\ 774.75 \\ 1524.75 \\ 766.5 \\ 937.5 \\ 1995. \end{pmatrix}$$

Essential boundary conditions

Node	dof	Value
5	T_5	110
6	T_6	110

Delete equations {5, 6}.

$$\begin{pmatrix} 34.3 & 0.275 & -22.5 & 0 & 0 & -11.25 \\ 0.275 & 34.575 & -22.3625 & 0 & 0 & -11.25 \\ -22.5 & -22.3625 & 134.96 & 1.14038 & -51.0577 & -38.9423 \\ 0 & 0 & 1.14038 & 44.6846 & -38.9423 & -6.05769 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ 110 \\ 110 \end{pmatrix} = \begin{pmatrix} 1074. \\ 774.75 \\ 1524.75 \\ 766.5 \end{pmatrix}$$

Extract columns {5, 6}.

Multiply each column by its respective known value {110, 110}.

Move all resulting vectors to the rhs.

After adjusting for essential boundary conditions we have

$$\begin{pmatrix} 34.3 & 0.275 & -22.5 & 0 \\ 0.275 & 34.575 & -22.3625 & 0 \\ -22.5 & -22.3625 & 134.96 & 1.14038 \\ 0 & 0 & 1.14038 & 44.6846 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{pmatrix} = \begin{pmatrix} 2311.5 \\ 2012.25 \\ 11424.8 \\ 5716.5 \end{pmatrix}$$

Solving the final system of global equations we get

$\{T_1 = 153.394,\ T_2 = 142.907,\ T_3 = 132.853,\ T_4 = 124.539\}$

Complete table of nodal values

	T
1	153.394
2	142.907
3	132.853
4	124.539
5	110
6	110

Solution for element 1

Element nodal values

Element node	Global node number	T
1	6	110
2	3	132.853
3	2	142.907
4	1	153.394

$\boldsymbol{d}^{\mathrm{T}} = (\,110 \quad 132.853 \quad 142.907 \quad 153.394\,)$

Nodal values = (110 132.853 142.907 153.394)

Interpolation functions and their derivatives

$$\boldsymbol{N}^{\mathrm{T}} = \left\{ \frac{1}{4}(s-1)(t-1),\ -\frac{1}{4}(s+1)(t-1),\ \frac{1}{4}(s+1)(t+1),\ -\frac{1}{4}(s-1)(t+1) \right\}$$

$$\partial \boldsymbol{N}^{\mathrm{T}}/\partial s = \left\{ \frac{t-1}{4},\ \frac{1-t}{4},\ \frac{t+1}{4},\ \frac{1}{4}(-t-1) \right\}$$

$$\partial \boldsymbol{N}^{\mathrm{T}}/\partial t = \left\{ \frac{s-1}{4},\ \frac{1}{4}(-s-1),\ \frac{s+1}{4},\ \frac{1-s}{4} \right\}$$

Nodal coordinates

Element node	Global node number	x	y
1	6	0	0
2	3	0.03	0.015
3	2	0.03	0.03
4	1	0	0.03

Mapping to the master element

$$x(s,t) = 0.0075\,(s+1)(1-t) + 0.0075\,(s+1)(t+1)$$

$$y(s,t) = 0.00375\,(s+1)(1-t) + 0.0075\,(1-s)(t+1) + 0.0075\,(s+1)(t+1)$$

$$\boldsymbol{J} = \begin{pmatrix} 0.0075\,(1-t) + 0.0075\,(t+1) & 0 \\ 0.00375\,(1-t) & 0.0075\,(1-s) + 0.00375\,(s+1) \end{pmatrix};$$

$$\det J = 0.00016875 - 0.00005625\,s$$

Solution at $\{s, t\} = \{0., 0.\} \Longrightarrow \{x, y\} = \{0.015, 0.01875\}$

Interpolation functions & their derivatives

$\boldsymbol{N}^{\mathrm{T}} = \{0.25, 0.25, 0.25, 0.25\}$

$\partial \boldsymbol{N}^{\mathrm{T}}/\partial s = \{-0.25, 0.25, 0.25, -0.25\}$

$\partial \boldsymbol{N}^{\mathrm{T}}/\partial t = \{-0.25, -0.25, 0.25, 0.25\}$

Jacobian matrix, $\boldsymbol{J} = \begin{pmatrix} 0.015 & 0. \\ 0.00375 & 0.01125 \end{pmatrix};$ $\det J = 0.00016875$

$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{x} = \{-11.1111, 22.2222, 11.1111, -22.2222\}$
$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{y} = \{-22.2222, -22.2222, 22.2222, 22.2222\}$

$T = 134.788;\qquad \partial T/\partial \mathrm{x} = -90.8204;\qquad \partial T/\partial \mathrm{y} = 1187.71$

Solution at $\{\mathrm{s}, \mathrm{t}\} = \{-1., -1.\} \Longrightarrow \{\mathrm{x}, \mathrm{y}\} = \{0., 0.\}$

Interpolation functions & their derivatives

$\boldsymbol{N}^{\mathrm{T}} = \{1., 0., 0., 0.\}$
$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{s} = \{-0.5, 0.5, 0., 0.\}$
$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{t} = \{-0.5, 0., 0., 0.5\}$

Jacobian matrix, $\boldsymbol{J} = \begin{pmatrix} 0.015 & 0. \\ 0.0075 & 0.015 \end{pmatrix}$; $\qquad$ $\det\mathrm{J} = 0.000225$

$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{x} = \{-16.6667, 33.3333, 0., -16.6667\}$
$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{y} = \{-33.3333, 0., 0., 33.3333\}$

$T = 110.;\qquad \partial T/\partial \mathrm{x} = 38.551;\qquad \partial T/\partial \mathrm{y} = 1446.45$

Solution at $\{\mathrm{s}, \mathrm{t}\} = \{-1., 1.\} \Longrightarrow \{\mathrm{x}, \mathrm{y}\} = \{0., 0.03\}$

Interpolation functions & their derivatives

$\boldsymbol{N}^{\mathrm{T}} = \{0., 0., 0., 1.\}$
$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{s} = \{0., 0., 0.5, -0.5\}$
$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{t} = \{-0.5, 0., 0., 0.5\}$

Jacobian matrix, $\boldsymbol{J} = \begin{pmatrix} 0.015 & 0. \\ 0. & 0.015 \end{pmatrix}$; $\qquad$ $\det\mathrm{J} = 0.000225$

$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{x} = \{0., 0., 33.3333, -33.3333\}$
$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{y} = \{-33.3333, 0., 0., 33.3333\}$

$T = 153.394;\qquad \partial T/\partial \mathrm{x} = -349.563;\qquad \partial T/\partial \mathrm{y} = 1446.45$

Solution at $\{\mathrm{s}, \mathrm{t}\} = \{1., -1.\} \Longrightarrow \{\mathrm{x}, \mathrm{y}\} = \{0.03, 0.015\}$

Interpolation functions & their derivatives

$\boldsymbol{N}^{\mathrm{T}} = \{0., 1., 0., 0.\}$
$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{s} = \{-0.5, 0.5, 0., 0.\}$
$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{t} = \{0., -0.5, 0.5, 0.\}$

Jacobian matrix, $\boldsymbol{J} = \begin{pmatrix} 0.015 & 0. \\ 0.0075 & 0.0075 \end{pmatrix}$; $\qquad$ $\det\mathrm{J} = 0.0001125$

$\partial \boldsymbol{N}^{T}/\partial x = \{-33.3333, 66.6667, -33.3333, 0.\}$
$\partial \boldsymbol{N}^{T}/\partial y = \{0., -66.6667, 66.6667, 0.\}$

$T = 132.853$; $\partial T/\partial x = 426.665$; $\partial T/\partial y = 670.225$

Solution at $\{s, t\} = \{1., 1.\} \Longrightarrow \{x, y\} = \{0.03, 0.03\}$

Interpolation functions & their derivatives

$\boldsymbol{N}^{T} = \{0., 0., 1., 0.\}$
$\partial \boldsymbol{N}^{T}/\partial s = \{0., 0., 0.5, -0.5\}$
$\partial \boldsymbol{N}^{T}/\partial t = \{0., -0.5, 0.5, 0.\}$

Jacobian matrix, $\boldsymbol{J} = \begin{pmatrix} 0.015 & 0. \\ 0. & 0.0075 \end{pmatrix}$; $\det J = 0.0001125$

$\partial \boldsymbol{N}^{T}/\partial x = \{0., 0., 33.3333, -33.3333\}$
$\partial \boldsymbol{N}^{T}/\partial y = \{0., -66.6667, 66.6667, 0.\}$

$T = 142.907$; $\partial T/\partial x = -349.563$; $\partial T/\partial y = 670.225$

Solution for element 2

Element nodal values

Element node	Global node number	T
1	6	110
2	5	110
3	4	124.539
4	3	132.853

$\boldsymbol{d}^{T} = (110 \quad 110 \quad 124.539 \quad 132.853)$

Nodal values = $(110 \quad 110 \quad 124.539 \quad 132.853)$

Interpolation functions and their derivatives

$$\boldsymbol{N}^{T} = \left\{\frac{1}{4}(s-1)(t-1), -\frac{1}{4}(s+1)(t-1), \frac{1}{4}(s+1)(t+1), -\frac{1}{4}(s-1)(t+1)\right\}$$

$$\partial \boldsymbol{N}^{T}/\partial s = \left\{\frac{t-1}{4}, \frac{1-t}{4}, \frac{t+1}{4}, \frac{1}{4}(-t-1)\right\}$$

$$\partial \boldsymbol{N}^{T}/\partial t = \left\{\frac{s-1}{4}, \frac{1}{4}(-s-1), \frac{s+1}{4}, \frac{1-s}{4}\right\}$$

Nodal coordinates

Element node	Global node number	x	y
1	6	0	0
2	5	0.06	0
3	4	0.06	0.015
4	3	0.03	0.015

Mapping to the master element

$$x(s,t) = 0.015\,(s+1)\,(1-t) + 0.0075\,(1-s)\,(t+1) + 0.015\,(s+1)\,(t+1)$$

$$y(s,t) = 0.00375\,(1-s)\,(t+1) + 0.00375\,(s+1)\,(t+1)$$

$$\boldsymbol{J} = \begin{pmatrix} 0.015\,(1-t) + 0.0075\,(t+1) & 0.0075\,(1-s) \\ 0 & 0.00375\,(1-s) + 0.00375\,(s+1) \end{pmatrix};$$

$$\det J = 0.00016875 - 0.00005625\,t$$

Solution at $\{s, t\} = \{0., 0.\} \Longrightarrow \{x, y\} = \{0.0375, 0.0075\}$

Interpolation functions & their derivatives

$$\boldsymbol{N}^{T} = \{0.25, 0.25, 0.25, 0.25\}$$
$$\partial \boldsymbol{N}^{T}/\partial s = \{-0.25, 0.25, 0.25, -0.25\}$$
$$\partial \boldsymbol{N}^{T}/\partial t = \{-0.25, -0.25, 0.25, 0.25\}$$

Jacobian matrix, $\boldsymbol{J} = \begin{pmatrix} 0.0225 & 0.0075 \\ 0. & 0.0075 \end{pmatrix}$; $\quad \det J = 0.00016875$

$$\partial \boldsymbol{N}^{T}/\partial x = \{-11.1111, 11.1111, 11.1111, -11.1111\}$$
$$\partial \boldsymbol{N}^{T}/\partial y = \{-22.2222, -44.4444, 22.2222, 44.4444\}$$

$T = 119.348$; $\quad \partial T/\partial x = -92.3768$; $\quad \partial T/\partial y = 1338.8$

Solution at $\{s, t\} = \{-1., -1.\} \Longrightarrow \{x, y\} = \{0., 0.\}$

Interpolation functions & their derivatives

$$\boldsymbol{N}^{T} = \{1., 0., 0., 0.\}$$
$$\partial \boldsymbol{N}^{T}/\partial s = \{-0.5, 0.5, 0., 0.\}$$
$$\partial \boldsymbol{N}^{T}/\partial t = \{-0.5, 0., 0., 0.5\}$$

Jacobian matrix, $\boldsymbol{J} = \begin{pmatrix} 0.03 & 0.015 \\ 0. & 0.0075 \end{pmatrix}$; $\quad \det J = 0.000225$

$$\partial \boldsymbol{N}^{T}/\partial x = \{-16.6667, 16.6667, 0., 0.\}$$
$$\partial \boldsymbol{N}^{T}/\partial y = \{-33.3333, -33.3333, 0., 66.6667\}$$

$T = 110.$; $\quad \partial T/\partial x = 0.$; $\quad \partial T/\partial y = 1523.56$

Solution at $\{s, t\} = \{-1., 1.\} \Longrightarrow \{x, y\} = \{0.03, 0.015\}$

Interpolation functions & their derivatives

$N^T = \{0., 0., 0., 1.\}$
$\partial N^T/\partial s = \{0., 0., 0.5, -0.5\}$
$\partial N^T/\partial t = \{-0.5, 0., 0., 0.5\}$

Jacobian matrix, $J = \begin{pmatrix} 0.015 & 0.015 \\ 0. & 0.0075 \end{pmatrix}$; detJ = 0.0001125

$\partial N^T/\partial x = \{0., 0., 33.3333, -33.3333\}$
$\partial N^T/\partial y = \{-66.6667, 0., -66.6667, 133.333\}$

$T = 132.853$; $\partial T/\partial x = -277.13$; $\partial T/\partial y = 2077.82$

Solution at $\{s, t\} = \{1., -1.\} \Longrightarrow \{x, y\} = \{0.06, 0.\}$

Interpolation functions & their derivatives

$N^T = \{0., 1., 0., 0.\}$
$\partial N^T/\partial s = \{-0.5, 0.5, 0., 0.\}$
$\partial N^T/\partial t = \{0., -0.5, 0.5, 0.\}$

Jacobian matrix, $J = \begin{pmatrix} 0.03 & 0. \\ 0. & 0.0075 \end{pmatrix}$; detJ = 0.000225

$\partial N^T/\partial x = \{-16.6667, 16.6667, 0., 0.\}$
$\partial N^T/\partial y = \{0., -66.6667, 66.6667, 0.\}$

$T = 110.$; $\partial T/\partial x = 0.$; $\partial T/\partial y = 969.295$

Solution at $\{s, t\} = \{1., 1.\} \Longrightarrow \{x, y\} = \{0.06, 0.015\}$

Interpolation functions & their derivatives

$N^T = \{0., 0., 1., 0.\}$
$\partial N^T/\partial s = \{0., 0., 0.5, -0.5\}$
$\partial N^T/\partial t = \{0., -0.5, 0.5, 0.\}$

Jacobian matrix, $J = \begin{pmatrix} 0.015 & 0. \\ 0. & 0.0075 \end{pmatrix}$; detJ = 0.0001125

$\partial N^T/\partial x = \{0., 0., 33.3333, -33.3333\}$
$\partial N^T/\partial y = \{0., -66.6667, 66.6667, 0.\}$

$T = 124.539$; $\partial T/\partial x = -277.13$; $\partial T/\partial y = 969.295$

Solution summary

Nodal solution

	x	y	T
1	0	0.03	153.394
2	0.03	0.03	142.907
3	0.03	0.015	132.853
4	0.06	0.015	124.539
5	0.06	0	110
6	0	0	110

Solution at selected points on the elements

	x	y	T	$\partial T/\partial x$	$\partial T/\partial y$
1	0.015	0.01875	134.788	−90.8204	1187.71
2	0.0375	0.0075	119.348	−92.3768	1338.8