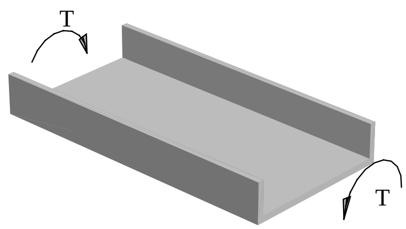
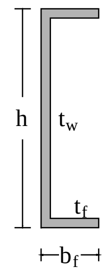
Example 5.7: Torsion constant of a C shape (p. 367)

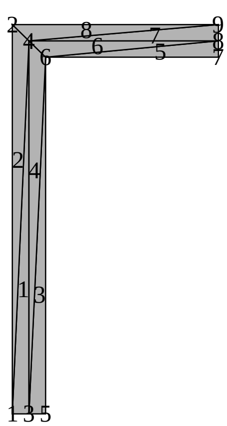
Find torsional constant J for a standard C12×30 section shown in Figure. The section dimensions are as follows.

$$h = 12\,\text{in} \quad t_w = 0.51\,\text{in} \quad b_f = 3.17\,\text{in} \quad t_f = 0.501\,\text{in}$$





Taking advantage of symmetry, we model only half the cross-section. Since ϕ is assigned a zero value on all the outside boundary nodes, the mesh must have some interior nodes. Thus the simplest possible model with triangular elements is an 8 element model shown in Figure. With such a coarse mesh, we don't expect a very accurate solution. The mesh is used to show most computations explicitly.



As a result of essential boundary conditions $\phi = 0$ at nodes {1, 2, 5, 6, 7, 8, 9}. Setting $G\theta = 1$, we can obtain the finite element solution using usual steps.

Global equations at start of the element assembly process

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} \phi_1\\ \phi_2\\ \phi_3\\ \phi_4\\ \phi_5\\ \phi_6\\ \phi_7\\ \phi_8\\ \phi_9 \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}$$

Equations for element 1

$k_x = 1;$ $k_y = 1;$ $p = 0;$ $q = 2$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	1	0.	0.
2	3	0.255	0.
3	4	0.255	5.7495

$x_1 = 0.$ $x_2 = 0.255$ $x_3 = 0.255$

$y_1 = 0.$ $y_2 = 0.$ $y_3 = 5.7495$

Using these values we get

$b_1 = -5.7495$ $b_2 = 5.7495$ $b_3 = 0.$

$c_1 = 0.$ $c_2 = -0.255$ $c_3 = 0.255$

$f_1 = 1.46612$ $f_2 = 0.$ $f_3 = 0.$

Element area, $A = 0.733061$

$$\boldsymbol{B}^T = \begin{pmatrix} -5.7495 & 5.7495 & 0. \\ 0. & -0.255 & 0.255 \end{pmatrix}$$

$$\boldsymbol{k}_k = \begin{pmatrix} 11.2735 & -11.2735 & 0. \\ -11.2735 & 11.2957 & -0.0221758 \\ 0. & -0.0221758 & 0.0221758 \end{pmatrix}; \qquad \boldsymbol{k}_p = \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}; \qquad \boldsymbol{r}_q = \begin{pmatrix} 0.488707\\0.488707\\0.488707 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 11.2735 & -11.2735 & 0 \\ -11.2735 & 11.2957 & -0.0221758 \\ 0 & -0.0221758 & 0.0221758 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} 0.488707 \\ 0.488707 \\ 0.488707 \end{pmatrix}$$

The element contributes to {1, 3, 4} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 11.2735 & 0 & -11.2735 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -11.2735 & 0 & 11.2957 & -0.0221758 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.0221758 & 0.0221758 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \\ \phi_9 \end{pmatrix} = \begin{pmatrix} 0.488707 \\ 0 \\ 0.488707 \\ 0.488707 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 2

$k_x = 1;$ $k_y = 1;$ $p = 0;$ $q = 2$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	4	0.255	5.7495
2	2	0.	6.
3	1	0.	0.

$x_1 = 0.255$ $x_2 = 0.$ $x_3 = 0.$

$y_1 = 5.7495$ $y_2 = 6.$ $y_3 = 0.$

Using these values we get

$b_1 = 6.$ $b_2 = -5.7495$ $b_3 = -0.2505$

$c_1 = 0.$ $c_2 = 0.255$ $c_3 = -0.255$

$f_1 = 0.$ $f_2 = 0.$ $f_3 = 1.53$

Element area, $A = 0.765$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 6. & -5.7495 & -0.2505 \\ 0. & 0.255 & -0.255 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 11.7647 & -11.2735 & -0.491176 \\ -11.2735 & 10.8241 & 0.44942 \\ -0.491176 & 0.44942 & 0.0417566 \end{pmatrix}; \qquad \boldsymbol{k}_{\mathrm{p}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 0.51 \\ 0.51 \\ 0.51 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 11.7647 & -11.2735 & -0.491176 \\ -11.2735 & 10.8241 & 0.44942 \\ -0.491176 & 0.44942 & 0.0417566 \end{pmatrix} \begin{pmatrix} \phi_4 \\ \phi_2 \\ \phi_1 \end{pmatrix} = \begin{pmatrix} 0.51 \\ 0.51 \\ 0.51 \end{pmatrix}$$

The element contributes to {4, 2, 1} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 11.3153 & 0.44942 & -11.2735 & -0.491176 & 0 & 0 & 0 & 0 & 0 \\ 0.44942 & 10.8241 & 0 & -11.2735 & 0 & 0 & 0 & 0 & 0 \\ -11.2735 & 0 & 11.2957 & -0.0221758 & 0 & 0 & 0 & 0 & 0 \\ -0.491176 & -11.2735 & -0.0221758 & 11.7869 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \\ \phi_9 \end{pmatrix} = \begin{pmatrix} 0.998707 \\ 0.51 \\ 0.488707 \\ 0.998707 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 3

$k_x = 1$; $\quad k_y = 1$; $\quad p = 0$; $\quad q = 2$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	3	0.255	0.
2	5	0.51	0.
3	6	0.51	5.499

$x_1 = 0.255 \quad x_2 = 0.51 \quad x_3 = 0.51$

$y_1 = 0. \quad y_2 = 0. \quad y_3 = 5.499$

Using these values we get

$b_1 = -5.499 \qquad b_2 = 5.499 \qquad b_3 = 0.$

$c_1 = 0. \qquad c_2 = -0.255 \qquad c_3 = 0.255$

$f_1 = 2.80449 \qquad f_2 = -1.40225 \qquad f_3 = 0.$

Element area, A = 0.701123

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -5.499 & 5.499 & 0. \\ 0. & -0.255 & 0.255 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 10.7824 & -10.7824 & 0. \\ -10.7824 & 10.8055 & -0.023186 \\ 0. & -0.023186 & 0.023186 \end{pmatrix}; \qquad \boldsymbol{k}_{\mathrm{p}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 0.467415 \\ 0.467415 \\ 0.467415 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 10.7824 & -10.7824 & 0 \\ -10.7824 & 10.8055 & -0.023186 \\ 0 & -0.023186 & 0.023186 \end{pmatrix} \begin{pmatrix} \phi_3 \\ \phi_5 \\ \phi_6 \end{pmatrix} = \begin{pmatrix} 0.467415 \\ 0.467415 \\ 0.467415 \end{pmatrix}$$

The element contributes to {3, 5, 6} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 11.3153 & 0.44942 & -11.2735 & -0.491176 & 0 & 0 & 0 & 0 & 0 \\ 0.44942 & 10.8241 & 0 & -11.2735 & 0 & 0 & 0 & 0 & 0 \\ -11.2735 & 0 & 22.0781 & -0.0221758 & -10.7824 & 0 & 0 & 0 & 0 \\ -0.491176 & -11.2735 & -0.0221758 & 11.7869 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -10.7824 & 0 & 10.8055 & -0.023186 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.023186 & 0.023186 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \\ \phi_9 \end{pmatrix} = \begin{pmatrix} 0.998707 \\ 0.51 \\ 0.956122 \\ 0.998707 \\ 0.467415 \\ 0.467415 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 4

$k_x = 1;$ $k_y = 1;$ $p = 0;$ $q = 2$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	6	0.51	5.499
2	4	0.255	5.7495
3	3	0.255	0.

$x_1 = 0.51$ $x_2 = 0.255$ $x_3 = 0.255$

$y_1 = 5.499$ $y_2 = 5.7495$ $y_3 = 0.$

Using these values we get

$b_1 = 5.7495$ $b_2 = -5.499$ $b_3 = -0.2505$

$c_1 = 0.$ $c_2 = 0.255$ $c_3 = -0.255$

$f_1 = -1.46612$ $\quad f_2 = 1.40225$ $\quad f_3 = 1.53$

Element area, A = 0.733061

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 5.7495 & -5.499 & -0.2505 \\ 0. & 0.255 & -0.255 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 11.2735 & -10.7824 & -0.491176 \\ -10.7824 & 10.3348 & 0.447601 \\ -0.491176 & 0.447601 & 0.0435759 \end{pmatrix}; \qquad \boldsymbol{k}_{\mathrm{p}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 0.488707 \\ 0.488707 \\ 0.488707 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 11.2735 & -10.7824 & -0.491176 \\ -10.7824 & 10.3348 & 0.447601 \\ -0.491176 & 0.447601 & 0.0435759 \end{pmatrix} \begin{pmatrix} \phi_6 \\ \phi_4 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} 0.488707 \\ 0.488707 \\ 0.488707 \end{pmatrix}$$

The element contributes to {6, 4, 3} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 11.3153 & 0.44942 & -11.2735 & -0.491176 & 0 & 0 & 0 & 0 & 0 \\ 0.44942 & 10.8241 & 0 & -11.2735 & 0 & 0 & 0 & 0 & 0 \\ -11.2735 & 0 & 22.1216 & 0.425425 & -10.7824 & -0.491176 & 0 & 0 & 0 \\ -0.491176 & -11.2735 & 0.425425 & 22.1216 & 0 & -10.7824 & 0 & 0 & 0 \\ 0 & 0 & -10.7824 & 0 & 10.8055 & -0.023186 & 0 & 0 & 0 \\ 0 & 0 & -0.491176 & -10.7824 & -0.023186 & 11.2967 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \\ \phi_9 \end{pmatrix} = \begin{pmatrix} 0.998707 \\ 0.51 \\ 1.44483 \\ 1.48741 \\ 0.467415 \\ 0.956122 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 5

$k_x = 1;$ $k_y = 1;$ $p = 0;$ $q = 2$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	6	0.51	5.499
2	7	3.17	5.499
3	8	3.17	5.7495

$x_1 = 0.51$ $x_2 = 3.17$ $x_3 = 3.17$
$y_1 = 5.499$ $y_2 = 5.499$ $y_3 = 5.7495$

Using these values we get

$b_1 = -0.2505$ $b_2 = 0.2505$ $b_3 = 0.$

$c_1 = 0.$ $c_2 = -2.66$ $c_3 = 2.66$

$f_1 = 0.794085$ $f_2 = 14.4996$ $f_3 = -14.6273$

Element area, $A = 0.333165$

$$\boldsymbol{B}^T = \begin{pmatrix} -0.2505 & 0.2505 & 0. \\ 0. & -2.66 & 2.66 \end{pmatrix}$$

$$\boldsymbol{k}_k = \begin{pmatrix} 0.0470865 & -0.0470865 & 0. \\ -0.0470865 & 5.35647 & -5.30938 \\ 0. & -5.30938 & 5.30938 \end{pmatrix}; \qquad \boldsymbol{k}_p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_q = \begin{pmatrix} 0.22211 \\ 0.22211 \\ 0.22211 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 0.0470865 & -0.0470865 & 0 \\ -0.0470865 & 5.35647 & -5.30938 \\ 0 & -5.30938 & 5.30938 \end{pmatrix} \begin{pmatrix} \phi_6 \\ \phi_7 \\ \phi_8 \end{pmatrix} = \begin{pmatrix} 0.22211 \\ 0.22211 \\ 0.22211 \end{pmatrix}$$

The element contributes to {6, 7, 8} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 11.3153 & 0.44942 & -11.2735 & -0.491176 & 0 & 0 & 0 & 0 & 0 \\ 0.44942 & 10.8241 & 0 & -11.2735 & 0 & 0 & 0 & 0 & 0 \\ -11.2735 & 0 & 22.1216 & 0.425425 & -10.7824 & -0.491176 & 0 & 0 & 0 \\ -0.491176 & -11.2735 & 0.425425 & 22.1216 & 0 & -10.7824 & 0 & 0 & 0 \\ 0 & 0 & -10.7824 & 0 & 10.8055 & -0.023186 & 0 & 0 & 0 \\ 0 & 0 & -0.491176 & -10.7824 & -0.023186 & 11.3438 & -0.0470865 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.0470865 & 5.35647 & -5.30938 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -5.30938 & 5.30938 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Equations for element 6

$k_x = 1;$ $k_y = 1;$ $p = 0;$ $q = 2$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	8	3.17	5.7495
2	4	0.255	5.7495
3	6	0.51	5.499

$x_1 = 3.17$ $x_2 = 0.255$ $x_3 = 0.51$

$y_1 = 5.7495$ $y_2 = 5.7495$ $y_3 = 5.499$

Using these values we get

$b_1 = 0.2505$ $b_2 = -0.2505$ $b_3 = 0.$

$c_1 = 0.255$ $c_2 = 2.66$ $c_3 = -2.915$

$f_1 = -1.53$ $f_2 = -14.4996$ $f_3 = 16.7598$

Element area, $A = 0.365104$

$$\boldsymbol{B}^T = \begin{pmatrix} 0.2505 & -0.2505 & 0. \\ 0.255 & 2.66 & -2.915 \end{pmatrix}$$

$$\boldsymbol{k}_k = \begin{pmatrix} 0.0874924 & 0.42149 & -0.508982 \\ 0.42149 & 4.88789 & -5.30938 \\ -0.508982 & -5.30938 & 5.81836 \end{pmatrix}; \qquad \boldsymbol{k}_p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_q = \begin{pmatrix} 0.243403 \\ 0.243403 \\ 0.243403 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 0.0874924 & 0.42149 & -0.508982 \\ 0.42149 & 4.88789 & -5.30938 \\ -0.508982 & -5.30938 & 5.81836 \end{pmatrix} \begin{pmatrix} \phi_8 \\ \phi_4 \\ \phi_6 \end{pmatrix} = \begin{pmatrix} 0.243403 \\ 0.243403 \\ 0.243403 \end{pmatrix}$$

The element contributes to {8, 4, 6} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 11.3153 & 0.44942 & -11.2735 & -0.491176 & 0 & 0 & 0 & 0 & 0 \\ 0.44942 & 10.8241 & 0 & -11.2735 & 0 & 0 & 0 & 0 & 0 \\ -11.2735 & 0 & 22.1216 & 0.425425 & -10.7824 & -0.491176 & 0 & 0 & 0 \\ -0.491176 & -11.2735 & 0.425425 & 27.0095 & 0 & -16.0917 & 0 & 0.42149 & 0 \\ 0 & 0 & -10.7824 & 0 & 10.8055 & -0.023186 & 0 & 0 & 0 \\ 0 & 0 & -0.491176 & -16.0917 & -0.023186 & 17.1622 & -0.0470865 & -0.508982 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.0470865 & 5.35647 & -5.30938 & 0 \\ 0 & 0 & 0 & 0.42149 & 0 & -0.508982 & -5.30938 & 5.39687 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Equations for element 7

$k_x = 1$; $k_y = 1$; $p = 0$; $q = 2$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	4	0.255	5.7495
2	8	3.17	5.7495
3	9	3.17	6.

$x_1 = 0.255$ $x_2 = 3.17$ $x_3 = 3.17$
$y_1 = 5.7495$ $y_2 = 5.7495$ $y_3 = 6.$

Using these values we get

$b_1 = -0.2505$ $b_2 = 0.2505$ $b_3 = 0.$

$c_1 = 0.$ $c_2 = -2.915$ $c_3 = 2.915$

$f_1 = 0.794085$ $f_2 = 16.6959$ $f_3 = -16.7598$

Element area, $A = 0.365104$

$$\boldsymbol{B}^T = \begin{pmatrix} -0.2505 & 0.2505 & 0. \\ 0. & -2.915 & 2.915 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 0.0429674 & -0.0429674 & 0. \\ -0.0429674 & 5.86133 & -5.81836 \\ 0. & -5.81836 & 5.81836 \end{pmatrix}; \qquad \boldsymbol{k}_{\mathrm{p}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 0.243403 \\ 0.243403 \\ 0.243403 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 0.0429674 & -0.0429674 & 0 \\ -0.0429674 & 5.86133 & -5.81836 \\ 0 & -5.81836 & 5.81836 \end{pmatrix} \begin{pmatrix} \phi_4 \\ \phi_8 \\ \phi_9 \end{pmatrix} = \begin{pmatrix} 0.243403 \\ 0.243403 \\ 0.243403 \end{pmatrix}$$

The element contributes to {4, 8, 9} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\left(\begin{array}{ccccccccc} 11.3153 & 0.44942 & -11.2735 & -0.491176 & 0 & 0 & 0 & 0 & \\ 0.44942 & 10.8241 & 0 & -11.2735 & 0 & 0 & 0 & 0 & \\ -11.2735 & 0 & 22.1216 & 0.425425 & -10.7824 & -0.491176 & 0 & 0 & \\ -0.491176 & -11.2735 & 0.425425 & 27.0525 & 0 & -16.0917 & 0 & 0.378522 & \\ 0 & 0 & -10.7824 & 0 & 10.8055 & -0.023186 & 0 & 0 & \\ 0 & 0 & -0.491176 & -16.0917 & -0.023186 & 17.1622 & -0.0470865 & -0.508982 & \\ 0 & 0 & 0 & 0 & 0 & -0.0470865 & 5.35647 & -5.30938 & \\ 0 & 0 & 0 & 0.378522 & 0 & -0.508982 & -5.30938 & 11.2582 & - \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5.81836 & \end{array} \right.$$

Equations for element 8

$k_x = 1;$ $\quad k_y = 1;$ $\quad p = 0;$ $\quad q = 2$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	9	3.17	6.
2	2	0.	6.
3	4	0.255	5.7495

$x_1 = 3.17$ $\quad x_2 = 0.$ $\quad x_3 = 0.255$

$y_1 = 6.$ $\quad y_2 = 6.$ $\quad y_3 = 5.7495$

Using these values we get

$b_1 = 0.2505$ $\quad b_2 = -0.2505$ $\quad b_3 = 0.$

$c_1 = 0.255$ $\quad c_2 = 2.915$ $\quad c_3 = -3.17$

$f_1 = -1.53 \qquad f_2 = -16.6959 \qquad f_3 = 19.02$

Element area, A = 0.397043

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 0.2505 & -0.2505 & 0. \\ 0.255 & 2.915 & -3.17 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 0.0804544 & 0.428528 & -0.508982 \\ 0.428528 & 5.38984 & -5.81836 \\ -0.508982 & -5.81836 & 6.32735 \end{pmatrix}; \qquad \boldsymbol{k}_{\mathrm{p}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 0.264695 \\ 0.264695 \\ 0.264695 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 0.0804544 & 0.428528 & -0.508982 \\ 0.428528 & 5.38984 & -5.81836 \\ -0.508982 & -5.81836 & 6.32735 \end{pmatrix} \begin{pmatrix} \phi_9 \\ \phi_2 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} 0.264695 \\ 0.264695 \\ 0.264695 \end{pmatrix}$$

The element contributes to {9, 2, 4} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\left(\begin{array}{cccccccc} 11.3153 & 0.44942 & -11.2735 & -0.491176 & 0 & 0 & 0 & 0 \\ 0.44942 & 16.2139 & 0 & -17.0919 & 0 & 0 & 0 & 0 \\ -11.2735 & 0 & 22.1216 & 0.425425 & -10.7824 & -0.491176 & 0 & 0 \\ -0.491176 & -17.0919 & 0.425425 & 33.3798 & 0 & -16.0917 & 0 & 0.378522 \\ 0 & 0 & -10.7824 & 0 & 10.8055 & -0.023186 & 0 & 0 \\ 0 & 0 & -0.491176 & -16.0917 & -0.023186 & 17.1622 & -0.0470865 & -0.508982 \\ 0 & 0 & 0 & 0 & 0 & -0.0470865 & 5.35647 & -5.30938 \\ 0 & 0 & 0 & 0.378522 & 0 & -0.508982 & -5.30938 & 11.2582 \\ 0 & 0.428528 & 0 & -0.508982 & 0 & 0 & 0 & -5.81836 \end{array}\right.$$

Essential boundary conditions

Node	dof	Value
1	ϕ_1	0
2	ϕ_2	0
5	ϕ_5	0
6	ϕ_6	0
7	ϕ_7	0
8	ϕ_8	0
9	ϕ_9	0

Remove {1, 2, 5, 6, 7, 8, 9} rows and columns.

After adjusting for essential boundary conditions we have

$$\begin{pmatrix} 22.1216 & 0.425425 \\ 0.425425 & 33.3798 \end{pmatrix} \begin{pmatrix} \phi_3 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} 1.44483 \\ 2.23892 \end{pmatrix}$$

Solving the final system of global equations we get

$$\{\phi_3 = 0.0640388, \phi_4 = 0.0662577\}$$

Complete table of nodal values

	ϕ
1	0
2	0
3	0.0640388
4	0.0662577
5	0
6	0
7	0
8	0
9	0

Solution for element 1

Nodal coordinates

Element node	Global node number	x	y
1	1	0.	0.
2	3	0.255	0.
3	4	0.255	5.7495

$x_1 = 0.$ $x_2 = 0.255$ $x_3 = 0.255$
$y_1 = 0.$ $y_2 = 0.$ $y_3 = 5.7495$

$b_1 = -5.7495$ $b_2 = 5.7495$ $b_3 = 0.$

$c_1 = 0.$ $c_2 = -0.255$ $c_3 = 0.255$

$f_1 = 1.46612$ $f_2 = 0.$ $f_3 = 0.$

Element area, $A = 0.733061$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^T = \{1. - 3.92157\,x,\ 3.92157\,x - 0.173928\,y,\ 0.173928\,y\}$

Nodal values, $\boldsymbol{d}^T = \{0, 0.0640388, 0.0662577\}$

$\phi(x, y) = \boldsymbol{N}^T \boldsymbol{d} = 0.251132\,x + 0.000385934\,y$

$\partial\phi/\partial x = 0.251132$; $\quad \partial\phi/\partial y = 0.000385934$

Solution for element 2

Nodal coordinates

Element node	Global node number	x	y
1	4	0.255	5.7495
2	2	0.	6.
3	1	0.	0.

$x_1 = 0.255 \quad x_2 = 0. \quad x_3 = 0.$
$y_1 = 5.7495 \quad y_2 = 6. \quad y_3 = 0.$

$b_1 = 6. \quad b_2 = -5.7495 \quad b_3 = -0.2505$

$c_1 = 0. \quad c_2 = 0.255 \quad c_3 = -0.255$

$f_1 = 0. \quad f_2 = 0. \quad f_3 = 1.53$

Element area, $A = 0.765$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^T = \{3.92157\,x,\ 0.166667\,y - 3.75784\,x,\ -0.163725\,x - 0.166667\,y + 1.\}$

Nodal values, $\boldsymbol{d}^T = \{0.0662577,\ 0,\ 0\}$

$\phi(x, y) = \boldsymbol{N}^T \boldsymbol{d} = 0.259834\,x$

$\partial\phi/\partial x = 0.259834$; $\quad \partial\phi/\partial y = 0$

Solution for element 3

Nodal coordinates

Element node	Global node number	x	y
1	3	0.255	0.
2	5	0.51	0.
3	6	0.51	5.499

$x_1 = 0.255 \quad x_2 = 0.51 \quad x_3 = 0.51$
$y_1 = 0. \quad y_2 = 0. \quad y_3 = 5.499$

$b_1 = -5.499 \quad b_2 = 5.499 \quad b_3 = 0.$

$c_1 = 0. \quad c_2 = -0.255 \quad c_3 = 0.255$

$f_1 = 2.80449 \quad f_2 = -1.40225 \quad f_3 = 0.$

Element area, A = 0.701123

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^{T} = \{2. - 3.92157\,x,\ 3.92157\,x - 0.181851\,y - 1.,\ 0.181851\,y\}$

Nodal values, $\boldsymbol{d}^{T} = \{0.0640388,\ 0,\ 0\}$

$\phi(x, y) = \boldsymbol{N}^{T}\boldsymbol{d} = 0.128078 - 0.251132\,x$

$\partial\phi/\partial x = -0.251132;\qquad \partial\phi/\partial y = 0$

Solution for element 4

Nodal coordinates

Element node	Global node number	x	y
1	6	0.51	5.499
2	4	0.255	5.7495
3	3	0.255	0.

$x_1 = 0.51 \qquad x_2 = 0.255 \qquad x_3 = 0.255$
$y_1 = 5.499 \qquad y_2 = 5.7495 \qquad y_3 = 0.$

$b_1 = 5.7495 \qquad b_2 = -5.499 \qquad b_3 = -0.2505$

$c_1 = 0. \qquad c_2 = 0.255 \qquad c_3 = -0.255$

$f_1 = -1.46612 \qquad f_2 = 1.40225 \qquad f_3 = 1.53$

Element area, A = 0.733061

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^{T} =$
$\{3.92157\,x - 1.,\ -3.75071\,x + 0.173928\,y + 0.956431,\ -0.170859\,x - 0.173928\,y + 1.04357\}$

Nodal values, $\boldsymbol{d}^{T} = \{0,\ 0.0662577,\ 0.0640388\}$

$\phi(x, y) = \boldsymbol{N}^{T}\boldsymbol{d} = -0.259455\,x + 0.000385934\,y + 0.1302$

$\partial\phi/\partial x = -0.259455;\qquad \partial\phi/\partial y = 0.000385934$

Solution for element 5

Nodal coordinates

Element node	Global node number	x	y
1	6	0.51	5.499
2	7	3.17	5.499
3	8	3.17	5.7495

$x_1 = 0.51$ $x_2 = 3.17$ $x_3 = 3.17$
$y_1 = 5.499$ $y_2 = 5.499$ $y_3 = 5.7495$

$b_1 = -0.2505$ $b_2 = 0.2505$ $b_3 = 0.$

$c_1 = 0.$ $c_2 = -2.66$ $c_3 = 2.66$

$f_1 = 0.794085$ $f_2 = 14.4996$ $f_3 = -14.6273$

Element area, A = 0.333165

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^T = \{1.19173 - 0.37594\,x,\ 0.37594\,x - 3.99202\,y + 21.7604,\ 3.99202\,y - 21.9521\}$

Nodal values, $\boldsymbol{d}^T = \{0, 0, 0\}$

$\phi(x, y) = \boldsymbol{N}^T\boldsymbol{d} = 0$

$\partial\phi/\partial x = 0;$ $\partial\phi/\partial y = 0$

Solution for element 6

Nodal coordinates

Element node	Global node number	x	y
1	8	3.17	5.7495
2	4	0.255	5.7495
3	6	0.51	5.499

$x_1 = 3.17$ $x_2 = 0.255$ $x_3 = 0.51$
$y_1 = 5.7495$ $y_2 = 5.7495$ $y_3 = 5.499$

$b_1 = 0.2505$ $b_2 = -0.2505$ $b_3 = 0.$

$c_1 = 0.255$ $c_2 = 2.66$ $c_3 = -2.915$

$f_1 = -1.53$ $f_2 = -14.4996$ $f_3 = 16.7598$

Element area, A = 0.365104

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^T =$
$\{0.343053\,x + 0.349216\,y - 2.09529,\ -0.343053\,x + 3.6428\,y - 19.8568,\ 22.9521 - 3.99202\,y\}$

Nodal values, $\boldsymbol{d}^T = \{0, 0.0662577, 0\}$

$\phi(x, y) = \boldsymbol{N}^T\boldsymbol{d} = -0.0227299\,x + 0.241364\,y - 1.31567$

$\partial\phi/\partial x = -0.0227299;$ $\partial\phi/\partial y = 0.241364$

Solution for element 7

Nodal coordinates

Element node	Global node number	x	y
1	4	0.255	5.7495
2	8	3.17	5.7495
3	9	3.17	6.

$x_1 = 0.255$ $x_2 = 3.17$ $x_3 = 3.17$
$y_1 = 5.7495$ $y_2 = 5.7495$ $y_3 = 6.$

$b_1 = -0.2505$ $b_2 = 0.2505$ $b_3 = 0.$

$c_1 = 0.$ $c_2 = -2.915$ $c_3 = 2.915$

$f_1 = 0.794085$ $f_2 = 16.6959$ $f_3 = -16.7598$

Element area, A = 0.365104

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^T = \{1.08748 - 0.343053\,x,\ 0.343053\,x - 3.99202\,y + 22.8646,\ 3.99202\,y - 22.9521\}$

Nodal values, $\boldsymbol{d}^T = \{0.0662577, 0, 0\}$

$\phi(x, y) = \boldsymbol{N}^T\boldsymbol{d} = 0.0720538 - 0.0227299\,x$

$\partial\phi/\partial x = -0.0227299;$ $\partial\phi/\partial y = 0$

Solution for element 8

Nodal coordinates

Element node	Global node number	x	y
1	9	3.17	6.
2	2	0.	6.
3	4	0.255	5.7495

$x_1 = 3.17$ $x_2 = 0.$ $x_3 = 0.255$
$y_1 = 6.$ $y_2 = 6.$ $y_3 = 5.7495$

$b_1 = 0.2505$ $b_2 = -0.2505$ $b_3 = 0.$

$c_1 = 0.255$ $c_2 = 2.915$ $c_3 = -3.17$

$f_1 = -1.53$ $f_2 = -16.6959$ $f_3 = 19.02$

Element area, A = 0.397043

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^T$ =

$\{0.315457\,x + 0.321124\,y - 1.92675,\ -0.315457\,x + 3.67089\,y - 21.0253,\ 23.9521 - 3.99202\,y\}$

Nodal values, $\boldsymbol{d}^T = \{0, 0, 0.0662577\}$

$\phi(x, y) = \boldsymbol{N}^T\boldsymbol{d} = 1.58701 - 0.264502\,y$

$\partial\phi/\partial x = 0; \qquad \partial\phi/\partial y = -0.264502$

Solution summary

Nodal solution

	x–coord	y–coord	ϕ
1	0.	0.	0
2	0.	6.	0
3	0.255	0.	0.0640388
4	0.255	5.7495	0.0662577
5	0.51	0.	0
6	0.51	5.499	0
7	3.17	5.499	0
8	3.17	5.7495	0
9	3.17	6.	0

Solution at element centroids

	x–coord	y–coord	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	0.17	1.9165	0.0434322	0.251132	0.000385934
2	0.085	3.9165	0.0220859	0.259834	0
3	0.425	1.833	0.0213463	−0.251132	0
4	0.34	3.7495	0.0434322	−0.259455	0.000385934
5	2.28333	5.5825	0	0	0
6	1.31167	5.666	0.0220859	−0.0227299	0.241364
7	2.19833	5.833	0.0220859	−0.0227299	0
8	1.14167	5.9165	0.0220859	0	−0.264502

Solutions over the remaining elements can be determined in exactly the same manner.

The total torque is given by

$$T = 2\iint_A \phi \, dA$$

The integral of ϕ over each element can be evaluated as described earlier. Since ϕ is a linear function over each element, using the procedure for integration over a triangle discussed earlier, it can be shown that the integral over each element is

$$\iint_{A^{(e)}} \phi^{(e)}\, \mathrm{dA} = \frac{A^{(e)}}{3}\,(\phi_1 + \phi_2 + \phi_3)$$

where $A^{(e)}$ is the area of the element and ϕ_1, ϕ_2, ϕ_3 are the values at its nodes. Using this formula the integral of ϕ over each element is evaluated and the results are summarized as follows.

	ϕ	$\iint \phi\, \mathrm{dA}$
1	$0.251132\,x + 0.000385934\,y$	0.0318384
2	$0.259834\,x$	0.0168957
3	$0.128078 - 0.251132\,x$	0.0149663
4	$-0.259455\,x + 0.000385934\,y + 0.1302$	0.0318384
5	0	0
6	$-0.0227299\,x + 0.241364\,y - 1.31567$	0.00806364
7	$0.0720538 - 0.0227299\,x$	0.00806364
8	$1.58701 - 0.264502\,y$	0.00876904

Summing $\iint \phi\, \mathrm{dA}$ contributions from all elements and multiplying by 2 gives the total torque. Since we are modeling half of the C shape, the torque for the entire section is twice this value. The the total torque is

$$\mathrm{T} = 2\times 2\times \sum\left(\iint \phi\, \mathrm{dA}\right) = 0.481741$$

Since $T = J\,G\,\theta$ and we have used $G\,\theta = 1$, the computations show that the torsional constant J for the section is 0.48 in^4. The J value tabulated in the steel design handbook for this section is 0.87 in^4. As expected the computed value has a large error of almost 45%. Solution improves considerably if we use a finer mesh involving 64 triangular elements.

Using 8 element: J = 0.481741; Error = 45.%

Using 64 elements: J = 0.754029; Error = 13.%