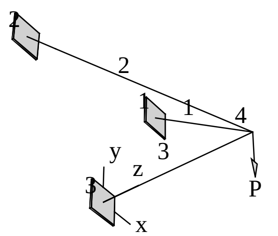
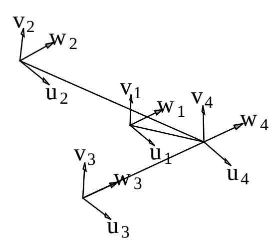
Example 4.2: Three-bar truss (p. 230)

The cross-sectional areas of elements 1 and 2 is 200 mm^2 and that of element 3 is 600 mm^2. All elements are made of the same material with $E = 200$ GPa. The applied load is $P = 20$ kN. The nodal coordinates are as follows.

Node	$x(m)$	$y(m)$	$z(m)$
1	0.96	1.92	0
2	−1.44	1.44	0
3	0	0	0
4	0	0	2





The complete computations are as follows. The numerical values are in the N − mm units. The computed displacements are in mm and the stresses in MPa.

Specified nodal loads

Node	dof	Value
4	u_4	0
	v_4	−20000
	w_4	0

Global equations at start of the element assembly process

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} u_1\\ v_1\\ w_1\\ u_2\\ v_2\\ w_2\\ u_3\\ v_3\\ w_3\\ u_4\\ v_4\\ w_4 \end{pmatrix} = \begin{pmatrix} 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ -20000\\ 0 \end{pmatrix}$$

Equations for element 1

$E = 210000$ $A = 200$

Element node	Global node number	x	y	z
1	1	960.	1920.	0
2	4	0	0	2000.

Direction cosines, $\ell_s = -0.327205$ $m_s = -0.65441$ $n_s = 0.681677$

Substituting into the truss element equations we get

$$\begin{pmatrix} 1532.63 & 3065.27 & -3192.99 & -1532.63 & -3065.27 & 3192.99\\ 3065.27 & 6130.53 & -6385.97 & -3065.27 & -6130.53 & 6385.97\\ -3192.99 & -6385.97 & 6652.06 & 3192.99 & 6385.97 & -6652.06\\ -1532.63 & -3065.27 & 3192.99 & 1532.63 & 3065.27 & -3192.99\\ -3065.27 & -6130.53 & 6385.97 & 3065.27 & 6130.53 & -6385.97\\ 3192.99 & 6385.97 & -6652.06 & -3192.99 & -6385.97 & 6652.06 \end{pmatrix} \begin{pmatrix} u_1\\ v_1\\ w_1\\ u_4\\ v_4\\ w_4 \end{pmatrix} = \begin{pmatrix} 0.\\ 0.\\ 0.\\ 0.\\ 0.\\ 0. \end{pmatrix}$$

The element contributes to {1, 2, 3, 10, 11, 12} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 1532.63 & 3065.27 & -3192.99 & 0 & 0 & 0 & 0 & 0 & 0 & -1532.63 & -3065.27 & 3192.99 \\ 3065.27 & 6130.53 & -6385.97 & 0 & 0 & 0 & 0 & 0 & 0 & -3065.27 & -6130.53 & 6385.97 \\ -3192.99 & -6385.97 & 6652.06 & 0 & 0 & 0 & 0 & 0 & 0 & 3192.99 & 6385.97 & -6652.06 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1532.63 & -3065.27 & 3192.99 & 0 & 0 & 0 & 0 & 0 & 0 & 1532.63 & 3065.27 & -3192.99 \\ -3065.27 & -6130.53 & 6385.97 & 0 & 0 & 0 & 0 & 0 & 0 & 3065.27 & 6130.53 & -6385.97 \\ 3192.99 & 6385.97 & -6652.06 & 0 & 0 & 0 & 0 & 0 & 0 & -3192.99 & -6385.97 & 6652.06 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ u_3 \\ v_3 \\ w_3 \\ u_4 \\ v_4 \\ w_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -20000. \\ 0 \end{pmatrix}$$

Equations for element 2

$E = 210000 \qquad A = 200$

Element node	Global node number	x	y	z
1	2	−1440.	1440.	0
2	4	0	0	2000.

Direction cosines, $\ell_s = 0.504497 \qquad m_s = -0.504497 \qquad n_s = 0.70069$

Substituting into the truss element equations we get

$$\begin{pmatrix} 3745.09 & -3745.09 & 5201.51 & -3745.09 & 3745.09 & -5201.51 \\ -3745.09 & 3745.09 & -5201.51 & 3745.09 & -3745.09 & 5201.51 \\ 5201.51 & -5201.51 & 7224.32 & -5201.51 & 5201.51 & -7224.32 \\ -3745.09 & 3745.09 & -5201.51 & 3745.09 & -3745.09 & 5201.51 \\ 3745.09 & -3745.09 & 5201.51 & -3745.09 & 3745.09 & -5201.51 \\ -5201.51 & 5201.51 & -7224.32 & 5201.51 & -5201.51 & 7224.32 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ w_2 \\ u_4 \\ v_4 \\ w_4 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {4, 5, 6, 10, 11, 12} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 1532.63 & 3065.27 & -3192.99 & 0 & 0 & 0 & 0 & 0 & 0 & -1532.63 & -3065.27 & 319 \\ 3065.27 & 6130.53 & -6385.97 & 0 & 0 & 0 & 0 & 0 & 0 & -3065.27 & -6130.53 & 638 \\ -3192.99 & -6385.97 & 6652.06 & 0 & 0 & 0 & 0 & 0 & 0 & 3192.99 & 6385.97 & -665 \\ 0 & 0 & 0 & 3745.09 & -3745.09 & 5201.51 & 0 & 0 & 0 & -3745.09 & 3745.09 & -520 \\ 0 & 0 & 0 & -3745.09 & 3745.09 & -5201.51 & 0 & 0 & 0 & 3745.09 & -3745.09 & 520 \\ 0 & 0 & 0 & 5201.51 & -5201.51 & 7224.32 & 0 & 0 & 0 & -5201.51 & 5201.51 & -722 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ -1532.63 & -3065.27 & 3192.99 & -3745.09 & 3745.09 & -5201.51 & 0 & 0 & 0 & 5277.72 & -679.818 & 200 \\ -3065.27 & -6130.53 & 6385.97 & 3745.09 & -3745.09 & 5201.51 & 0 & 0 & 0 & -679.818 & 9875.62 & -1158 \\ 3192.99 & 6385.97 & -6652.06 & -5201.51 & 5201.51 & -7224.32 & 0 & 0 & 0 & 2008.52 & -11587.5 & 1387 \end{pmatrix}$$

Equations for element 3

$E = 210000$ $\quad$ $A = 600$

Element node	Global node number	x	y	z
1	3	0	0	0
2	4	0	0	2000.

Direction cosines, $\ell_s = 0$ $\quad m_s = 0$ $\quad n_s = 1.$

Substituting into the truss element equations we get

$$\begin{pmatrix} 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 63000. & 0. & 0. & -63000. \\ 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & -63000. & 0. & 0. & 63000. \end{pmatrix} \begin{pmatrix} u_3 \\ v_3 \\ w_3 \\ u_4 \\ v_4 \\ w_4 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {7, 8, 9, 10, 11, 12} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\left(\begin{array}{ccccccccccc}
1532.63 & 3065.27 & -3192.99 & 0 & 0 & 0 & 0 & 0 & 0 & -1532.63 & -3065.27 \\
3065.27 & 6130.53 & -6385.97 & 0 & 0 & 0 & 0 & 0 & 0 & -3065.27 & -6130.53 \\
-3192.99 & -6385.97 & 6652.06 & 0 & 0 & 0 & 0 & 0 & 0 & 3192.99 & 6385.97 \\
0 & 0 & 0 & 3745.09 & -3745.09 & 5201.51 & 0 & 0 & 0 & -3745.09 & 3745.09 \\
0 & 0 & 0 & -3745.09 & 3745.09 & -5201.51 & 0 & 0 & 0 & 3745.09 & -3745.09 \\
0 & 0 & 0 & 5201.51 & -5201.51 & 7224.32 & 0 & 0 & 0 & -5201.51 & 5201.51 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 63000. & 0 & 0 \\
-1532.63 & -3065.27 & 3192.99 & -3745.09 & 3745.09 & -5201.51 & 0 & 0 & 0 & 5277.72 & -679.818 \\
-3065.27 & -6130.53 & 6385.97 & 3745.09 & -3745.09 & 5201.51 & 0 & 0 & 0 & -679.818 & 9875.62 \\
3192.99 & 6385.97 & -6652.06 & -5201.51 & 5201.51 & -7224.32 & 0 & 0 & -63000. & 2008.52 & -11587.5
\end{array}\right.$$

Essential boundary conditions

Node	dof	Value
1	u_1	0
	v_1	0
	w_1	0
2	u_2	0
	v_2	0
	w_2	0
3	u_3	0
	v_3	0
	w_3	0

Remove {1, 2, 3, 4, 5, 6, 7, 8, 9} rows and columns.

After adjusting for essential boundary conditions we have

$$\begin{pmatrix} 5277.72 & -679.818 & 2008.52 \\ -679.818 & 9875.62 & -11587.5 \\ 2008.52 & -11587.5 & 76876.4 \end{pmatrix} \begin{pmatrix} u_4 \\ v_4 \\ w_4 \end{pmatrix} = \begin{pmatrix} 0 \\ -20000. \\ 0 \end{pmatrix}$$

Solving the final system of global equations we get

$$\{u_4 = -0.178143,\ v_4 = -2.46857,\ w_4 = -0.367431\}$$

Complete table of nodal values

	u	v	w
1	0	0	0
2	0	0	0
3	0	0	0
4	−0.178143	−2.46857	−0.367431

Computation of reactions

Equation numbers of dof with specified values: {1, 2, 3, 4, 5, 6, 7, 8, 9}

Extracting equations {1, 2, 3, 4, 5, 6, 7, 8, 9} from the global system we have

$$
\left(\begin{array}{cccccccccccc}
1532.63 & 3065.27 & -3192.99 & 0 & 0 & 0 & 0 & 0 & 0 & -1532.63 & -3065.27 & 3 \\
3065.27 & 6130.53 & -6385.97 & 0 & 0 & 0 & 0 & 0 & 0 & -3065.27 & -6130.53 & 6 \\
-3192.99 & -6385.97 & 6652.06 & 0 & 0 & 0 & 0 & 0 & 0 & 3192.99 & 6385.97 & -6 \\
0 & 0 & 0 & 3745.09 & -3745.09 & 5201.51 & 0 & 0 & 0 & -3745.09 & 3745.09 & -5 \\
0 & 0 & 0 & -3745.09 & 3745.09 & -5201.51 & 0 & 0 & 0 & 3745.09 & -3745.09 & 5 \\
0 & 0 & 0 & 5201.51 & -5201.51 & 7224.32 & 0 & 0 & 0 & -5201.51 & 5201.51 & -7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 63000. & 0 & 0 & -63
\end{array}\right.
$$

Substituting the nodal values and re−arranging

$$
\begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \\ R_9 \end{pmatrix} =
\left(\begin{array}{ccccccccccc}
1532.63 & 3065.27 & -3192.99 & 0 & 0 & 0 & 0 & 0 & 0 & -1532.63 & -3065. \\
3065.27 & 6130.53 & -6385.97 & 0 & 0 & 0 & 0 & 0 & 0 & -3065.27 & -6130. \\
-3192.99 & -6385.97 & 6652.06 & 0 & 0 & 0 & 0 & 0 & 0 & 3192.99 & 6385. \\
0 & 0 & 0 & 3745.09 & -3745.09 & 5201.51 & 0 & 0 & 0 & -3745.09 & 3745. \\
0 & 0 & 0 & -3745.09 & 3745.09 & -5201.51 & 0 & 0 & 0 & 3745.09 & -3745. \\
0 & 0 & 0 & 5201.51 & -5201.51 & 7224.32 & 0 & 0 & 0 & -5201.51 & 5201. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 63000. & 0 & 0
\end{array}\right.
$$

Carrying out computations, the reactions are as follows.

Label	dof	Reaction
R_1	u_1	6666.67
R_2	v_1	13333.3
R_3	w_1	−13888.9
R_4	u_2	−6666.67
R_5	v_2	6666.67
R_6	w_2	−9259.26
R_7	u_3	0
R_8	v_3	0
R_9	w_3	23148.1

Sum of Reactions

dof: u	0
dof: v	20000.
dof: w	0

Solution for element 1

Nodal coordinates

Element node	Global node number	x	y	z
1	1	960.	1920.	0
2	4	0	0	2000.

Direction cosines, $\ell_s = -0.327205$ $m_s = -0.65441$ $n_s = 0.681677$

Global to local transformation matrix, $\boldsymbol{T}$ =

$$\begin{pmatrix} -0.327205 & -0.65441 & 0.681677 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.327205 & -0.65441 & 0.681677 \end{pmatrix}$$

Element nodal displacements in global coordinates, $\boldsymbol{d} = \begin{pmatrix} u_1 \\ v_1 \\ w_1 \\ u_4 \\ v_4 \\ w_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -0.178143 \\ -2.46857 \\ -0.367431 \end{pmatrix}$

Element nodal displacements in local coordinates, $\boldsymbol{d}_\ell = \boldsymbol{T}\ \boldsymbol{d} = \begin{pmatrix} 0. \\ 1.42328 \end{pmatrix}$

Axial displacements at element ends, $d_1 = 0.$ $d_2 = 1.42328$

$E = 210000$ $A = 200$

Axial strain, $\epsilon = (d_2 - d_1)/L = 0.000485109$

Axial stress, $\sigma = E\epsilon = 101.873$ Axial force $= \sigma A = 20374.6$

Solution for element 2

Nodal coordinates

Element node	Global node number	x	y	z
1	2	−1440.	1440.	0
2	4	0	0	2000.

Direction cosines, $\ell_s = 0.504497$ $m_s = -0.504497$ $n_s = 0.70069$

Global to local transformation matrix, $\boldsymbol{T} =$

$$\begin{pmatrix} 0.504497 & -0.504497 & 0.70069 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.504497 & -0.504497 & 0.70069 \end{pmatrix}$$

Element nodal displacements in global coordinates, $\boldsymbol{d} = \begin{pmatrix} u_2 \\ v_2 \\ w_2 \\ u_4 \\ v_4 \\ w_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -0.178143 \\ -2.46857 \\ -0.367431 \end{pmatrix}$

Element nodal displacements in local coordinates, $\boldsymbol{d}_\ell = \boldsymbol{T}\ \boldsymbol{d} = \begin{pmatrix} 0. \\ 0.89806 \end{pmatrix}$

Axial displacements at element ends, $d_1 = 0.$ $d_2 = 0.89806$

$E = 210000$ $A = 200$

Axial strain, $\epsilon = (d_2 - d_1)/L = 0.000314631$

Axial stress, $\sigma = E\epsilon = 66.0725$ Axial force $= \sigma A = 13214.5$

Solution for element 3

Nodal coordinates

Element node	Global node number	x	y	z
1	3	0	0	0
2	4	0	0	2000.

Direction cosines, $\ell_s = 0$ $m_s = 0$ $n_s = 1.$

Global to local transformation matrix, $\boldsymbol{T} = \begin{pmatrix} 0 & 0 & 1. & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1. \end{pmatrix}$

Element nodal displacements in global coordinates, $\boldsymbol{d} = \begin{pmatrix} u_3 \\ v_3 \\ w_3 \\ u_4 \\ v_4 \\ w_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -0.178143 \\ -2.46857 \\ -0.367431 \end{pmatrix}$

Element nodal displacements in local coordinates, $\boldsymbol{d}_\ell = \boldsymbol{T}\ \boldsymbol{d} = \begin{pmatrix} 0. \\ -0.367431 \end{pmatrix}$

Axial displacements at element ends, $d_1 = 0.$ $d_2 = -0.367431$

$E = 210000$ $A = 600$

Axial strain, $\epsilon = (d_2 - d_1)/L = -0.000183715$

Axial stress, $\sigma = E\epsilon = -38.5802$ Axial force $= \sigma A = -23148.1$

Solution summary

Nodal solution

	x–coord	y–coord	z–coord	u	v	w
1	960.	1920.	0	0	0	0
2	−1440.	1440.	0	0	0	0
3	0	0	0	0	0	0
4	0	0	2000.	−0.178143	−2.46857	−0.367431

Element solution

	Stress	Axial force
1	101.873	20374.6
2	66.0725	13214.5
3	−38.5802	−23148.1

Support reactions

Node	dof	Reaction
1	u_1	6666.67
1	v_1	13333.3
1	w_1	−13888.9
2	u_2	−6666.67
2	v_2	6666.67
2	w_2	−9259.26
3	u_3	0.
3	v_3	0.
3	w_3	23148.1

Sum of applied loads → (0 −20000. 0)

Sum of support reactions → (0 20000. 0)