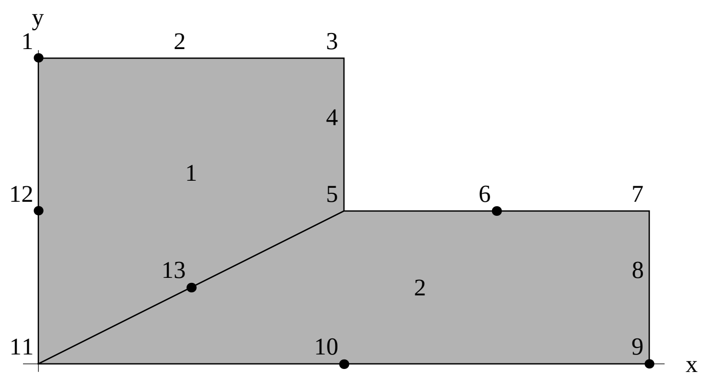
Example 6.22: Heat flow in an L-shaped body using Quad8 elements (p. 449)



Global equations at start of the element assembly process

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} T_1\\ T_2\\ T_3\\ T_4\\ T_5\\ T_6\\ T_7\\ T_8\\ T_9\\ T_{10}\\ T_{11}\\ T_{12}\\ T_{13} \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}$$

Equations for element 1

$k_x = 45$; $k_y = 45$; $p = 0$; $q = 5000000$

$$\boldsymbol{C} = \begin{pmatrix} 45 & 0 \\ 0 & 45 \end{pmatrix}$$

Element thickness = 1

Nodal coordinates

Element node	Global node number	x	y
1	11	0	0
2	13	0.015	0.0075
3	5	0.03	0.015
4	4	0.03	0.0225
5	3	0.03	0.03
6	2	0.015	0.03
7	1	0	0.03
8	12	0	0.015

Interpolation functions and their derivatives

$$\boldsymbol{N}^{\mathrm{T}} = \left\{-\frac{1}{4}(s-1)(t-1)(s+t+1),\ \frac{1}{2}\left(s^2-1\right)(t-1),\ \frac{1}{4}(t-1)\left(-s^2+t\,s+t+1\right),\ -\frac{1}{2}(s+1)\left(t^2-1\right),\right.$$
$$\left.\frac{1}{4}(s+1)(t+1)(s+t-1),\ -\frac{1}{2}\left(s^2-1\right)(t+1),\ \frac{1}{4}(s-1)(s-t+1)(t+1),\ \frac{1}{2}(s-1)\left(t^2-1\right)\right\}$$

$$\partial\boldsymbol{N}^{\mathrm{T}}/\partial s = \left\{-\frac{1}{4}(t-1)(2\,s+t),\ s\,(t-1),\ -\frac{1}{4}(2\,s-t)(t-1),\right.$$
$$\left.\frac{1}{2}\left(1-t^2\right),\ \frac{1}{4}(t+1)(2\,s+t),\ -s\,(t+1),\ \frac{1}{4}(2\,s-t)(t+1),\ \frac{1}{2}\left(t^2-1\right)\right\}$$

$$\partial\boldsymbol{N}^{\mathrm{T}}/\partial t = \left\{-\frac{1}{4}(s-1)(s+2\,t),\ \frac{1}{2}\left(s^2-1\right),\ -\frac{1}{4}(s+1)(s-2\,t),\right.$$
$$\left.-(s+1)\,t,\ \frac{1}{4}(s+1)(s+2\,t),\ \frac{1}{2}\left(1-s^2\right),\ \frac{1}{4}(s-1)(s-2\,t),\ (s-1)\,t\right\}$$

Mapping to the master element

$$x(s,t) = 0.015\,s + 0.015$$

$$y(s,t) = -0.00375\,t\,s + 0.00375\,s + 0.01125\,t + 0.01875$$

Jacobian matrix, $\boldsymbol{J}$ =

$$\begin{pmatrix} 0.015 & 0 \\ 0.00375 - 0.00375\,t & 0.01125 - 0.00375\,s \end{pmatrix}; \qquad \det J = 0.00016875 - 0.00005625\,s$$

Gauss quadrature points and weights

	Point	Weight
1	$s \to -0.774597$ $t \to -0.774597$	0.308642
2	$s \to -0.774597$ $t \to 0.$	0.493827
3	$s \to -0.774597$ $t \to 0.774597$	0.308642
4	$s \to 0.$ $t \to -0.774597$	0.493827
5	$s \to 0.$ $t \to 0.$	0.790123
6	$s \to 0.$ $t \to 0.774597$	0.493827
7	$s \to 0.774597$ $t \to -0.774597$	0.308642
8	$s \to 0.774597$ $t \to 0.$	0.493827
9	$s \to 0.774597$ $t \to 0.774597$	0.308642

Computation of element matrices at {−0.774597, −0.774597} with weight = 0.308642

$$\boldsymbol{N}^{\mathrm{T}} = (\ 0.432379 \quad 0.354919 \quad -0.1 \quad 0.0450807 \quad -0.032379 \quad 0.0450807 \quad -0.1 \quad 0.354919\ )$$

$$\partial \boldsymbol{N}^{\mathrm{T}}/\partial s = (\ -1.03095 \quad 1.3746 \quad -0.343649 \quad 0.2 \quad -0.130948 \quad 0.174597 \quad -0.0436492 \quad -0.2\ )$$

$$\partial \boldsymbol{N}^{\mathrm{T}}/\partial t = (\ -1.03095 \quad -0.2 \quad -0.0436492 \quad 0.174597 \quad -0.130948 \quad 0.2 \quad -0.343649 \quad 1.3746\ )$$

$$\text{Jacobian matrix, } \boldsymbol{J} = \begin{pmatrix} 0.015 & 0 \\ 0.00665474 & 0.0141547 \end{pmatrix}; \qquad \det J = 0.000212321$$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -36.417 & 97.9083 & -21.5419 & 7.86098 & -4.62557 & 5.37122 & 7.86098 & -56.417 \\ -72.8341 & -14.1295 & -3.08371 & 12.3349 & -9.25114 & 14.1295 & -24.278 & 97.1121 \end{pmatrix}$$

$\boldsymbol{k}_{\mathrm{k}}$ =

$$\begin{pmatrix} 19.5542 & -7.47966 & 2.97571 & -3.49348 & 2.48371 & -3.61157 & 4.37026 & -14.7992 \\ -7.47966 & 28.857 & -6.09113 & 1.75569 & -0.950041 & 0.962059 & 3.28123 & -20.3352 \\ 2.97571 & -6.09113 & 1.39649 & -0.611536 & 0.377965 & -0.469694 & -0.278594 & 2.70079 \\ -3.49348 & 1.75569 & -0.611536 & 0.6309 & -0.443731 & 0.638464 & -0.700869 & 2.22457 \\ 2.48371 & -0.950041 & 0.377965 & -0.443731 & 0.315472 & -0.45873 & 0.555096 & -1.87974 \\ -3.61157 & 0.962059 & -0.469694 & 0.638464 & -0.45873 & 0.673807 & -0.887073 & 3.15274 \\ 4.37026 & 3.28123 & -0.278594 & -0.700869 & 0.555096 & -0.887073 & 1.92038 & -8.26042 \\ -14.7992 & -20.3352 & 2.70079 & 2.22457 & -1.87974 & 3.15274 & -8.26042 & 37.1964 \end{pmatrix}$$

$$\boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 141.672 \\ 116.291 \\ -32.7656 \\ 14.7709 \\ -10.6092 \\ 14.7709 \\ -32.7656 \\ 116.291 \end{pmatrix}$$

Computation of element matrices at {−0.774597, 0.} with weight = 0.493827

$$\boldsymbol{N}^{\mathrm{T}} = (\, -0.1 \quad 0.2 \quad -0.1 \quad 0.112702 \quad -0.1 \quad 0.2 \quad -0.1 \quad 0.887298 \,)$$

$$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{s} = (\, -0.387298 \quad 0.774597 \quad -0.387298 \quad 0.5 \quad -0.387298 \quad 0.774597 \quad -0.387298 \quad -0.5 \,)$$

$$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{t} = (\, -0.343649 \quad -0.2 \quad 0.0436492 \quad 0. \quad -0.0436492 \quad 0.2 \quad 0.343649 \quad 0. \,)$$

$$\text{Jacobian matrix, } \boldsymbol{J} = \begin{pmatrix} 0.015 & 0 \\ 0.00375 & 0.0141547 \end{pmatrix}; \qquad \det J = 0.000212321$$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -19.7504 & 55.1722 & -26.5908 & 33.3333 & -25.049 & 48.1074 & -31.8894 & -33.3333 \\ -24.278 & -14.1295 & 3.08371 & 0 & -3.08371 & 14.1295 & 24.278 & 0 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 4.62152 & -3.5228 & 2.12468 & -3.10624 & 2.68748 & -6.10153 & 0.19064 & 3.10624 \\ -3.5228 & 15.3042 & -7.12759 & 8.67719 & -6.31506 & 11.5811 & -9.91985 & -8.67719 \\ 2.12468 & -7.12759 & 3.381 & -4.18207 & 3.09783 & -5.83007 & 4.35415 & 4.18207 \\ -3.10624 & 8.67719 & -4.18207 & 5.2425 & -3.93957 & 7.56608 & -5.0154 & -5.2425 \\ 2.68748 & -6.31506 & 3.09783 & -3.93957 & 3.00533 & -5.89126 & 3.41568 & 3.93957 \\ -6.10153 & 11.5811 & -5.83007 & 7.56608 & -5.89126 & 11.8615 & -5.6198 & -7.56608 \\ 0.19064 & -9.91985 & 4.35415 & -5.0154 & 3.41568 & -5.6198 & 7.57918 & 5.0154 \\ 3.10624 & -8.67719 & 4.18207 & -5.2425 & 3.93957 & -7.56608 & 5.0154 & 5.2425 \end{pmatrix}$$

$$\boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} -52.425 \\ 104.85 \\ -52.425 \\ 59.0838 \\ -52.425 \\ 104.85 \\ -52.425 \\ 465.166 \end{pmatrix}$$

Computation of element matrices at {−0.774597, 0.774597} with weight = 0.308642

$$\boldsymbol{N}^{\mathrm{T}} = (\, -0.1 \quad 0.0450807 \quad -0.032379 \quad 0.0450807 \quad -0.1 \quad 0.354919 \quad 0.432379 \quad 0.354919 \,)$$

$$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{s} = (\, -0.0436492 \quad 0.174597 \quad -0.130948 \quad 0.2 \quad -0.343649 \quad 1.3746 \quad -1.03095 \quad -0.2 \,)$$

$$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{t} = (\, 0.343649 \quad -0.2 \quad 0.130948 \quad -0.174597 \quad 0.0436492 \quad 0.2 \quad 1.03095 \quad -1.3746 \,)$$

Jacobian matrix, $\boldsymbol{J} = \begin{pmatrix} 0.015 & 0 \\ 0.000845262 & 0.0141547 \end{pmatrix}$; $\quad$ detJ = 0.000212321

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -4.27803 & 12.436 & -9.25114 & 14.0284 & -23.0837 & 90.8436 & -72.8341 & -7.86098 \\ 24.278 & -14.1295 & 9.25114 & -12.3349 & 3.08371 & 14.1295 & 72.8341 & -97.1121 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} =$$

$$\begin{pmatrix} 1.79212 & -1.16847 & 0.77903 & -1.06007 & 0.511987 & -0.134453 & 6.13329 & -6.85343 \\ -1.16847 & 1.04479 & -0.724727 & 1.02841 & -0.975026 & 2.74273 & -5.70576 & 3.75806 \\ 0.77903 & -0.724727 & 0.504756 & -0.719209 & 0.713867 & -2.09281 & 3.97393 & -2.43484 \\ -1.06007 & 1.02841 & -0.719209 & 1.02901 & -1.06711 & 3.2441 & -5.66232 & 3.20719 \\ 0.511987 & -0.975026 & 0.713867 & -1.06711 & 1.59939 & -6.05538 & 5.62026 & -0.347986 \\ -0.134453 & 2.74273 & -2.09281 & 3.2441 & -6.05538 & 24.9247 & -16.4767 & -6.15221 \\ 6.13329 & -5.70576 & 3.97393 & -5.66232 & 5.62026 & -16.4767 & 31.2867 & -19.1694 \\ -6.85343 & 3.75806 & -2.43484 & 3.20719 & -0.347986 & -6.15221 & -19.1694 & 27.9926 \end{pmatrix}$$

$$\boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} -32.7656 \\ 14.7709 \\ -10.6092 \\ 14.7709 \\ -32.7656 \\ 116.291 \\ 141.672 \\ 116.291 \end{pmatrix}$$

Computation of element matrices at {0., −0.774597} with weight = 0.493827

$$\boldsymbol{N}^{\mathrm{T}} = (\, -0.1 \quad 0.887298 \quad -0.1 \quad 0.2 \quad -0.1 \quad 0.112702 \quad -0.1 \quad 0.2 \,)$$

$$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{s} = (\, -0.343649 \quad 0. \quad 0.343649 \quad 0.2 \quad -0.0436492 \quad 0. \quad 0.0436492 \quad -0.2 \,)$$

$$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{t} = (\, -0.387298 \quad -0.5 \quad -0.387298 \quad 0.774597 \quad -0.387298 \quad 0.5 \quad -0.387298 \quad 0.774597 \,)$$

Jacobian matrix, $\boldsymbol{J} = \begin{pmatrix} 0.015 & 0 \\ 0.00665474 & 0.01125 \end{pmatrix}$; $\quad$ detJ = 0.00016875

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -7.63665 & 19.7177 & 38.1832 & -17.2133 & 12.3634 & -19.7177 & 18.1832 & -43.8799 \\ -34.4265 & -44.4444 & -34.4265 & 68.853 & -34.4265 & 44.4444 & -34.4265 & 68.853 \end{pmatrix}$$

$\boldsymbol{k}_{\rm k} =$

$$\begin{pmatrix} 4.66314 & 5.17309 & 3.35097 & -8.39595 & 4.09039 & -5.17309 & 3.92372 & -7.63228 \\ 5.17309 & 8.86537 & 8.56108 & -12.7483 & 6.65192 & -8.86537 & 7.08225 & -14.7201 \\ 3.35097 & 8.56108 & 9.91179 & -11.3536 & 6.21472 & -8.56108 & 7.04805 & -15.1719 \\ -8.39595 & -12.7483 & -11.3536 & 18.8889 & -9.68694 & 12.7483 & -10.0626 & 20.6102 \\ 4.09039 & 6.65192 & 6.21472 & -9.68694 & 5.01764 & -6.65192 & 5.28747 & -10.9233 \\ -5.17309 & -8.86537 & -8.56108 & 12.7483 & -6.65192 & 8.86537 & -7.08225 & 14.7201 \\ 3.92372 & 7.08225 & 7.04805 & -10.0626 & 5.28747 & -7.08225 & 5.68431 & -11.8809 \\ -7.63228 & -14.7201 & -15.1719 & 20.6102 & -10.9233 & 14.7201 & -11.8809 & 24.9982 \end{pmatrix}$$

$$\boldsymbol{r}_{\rm q} = \begin{pmatrix} -41.6667 \\ 369.708 \\ -41.6667 \\ 83.3333 \\ -41.6667 \\ 46.959 \\ -41.6667 \\ 83.3333 \end{pmatrix}$$

Computation of element matrices at {0., 0.} with weight = 0.790123

$\boldsymbol{N}^{\rm T} = (\ -0.25 \quad 0.5 \quad -0.25 \quad 0.5 \quad -0.25 \quad 0.5 \quad -0.25 \quad 0.5\ )$

$\partial \boldsymbol{N}^{\rm T}/\partial {\rm s} = (\ 0. \quad 0. \quad 0. \quad 0.5 \quad 0. \quad 0. \quad 0. \quad -0.5\ )$

$\partial \boldsymbol{N}^{\rm T}/\partial {\rm t} = (\ 0. \quad -0.5 \quad 0. \quad 0. \quad 0. \quad 0.5 \quad 0. \quad 0.\ )$

Jacobian matrix, $\boldsymbol{J} = \begin{pmatrix} 0.015 & 0 \\ 0.00375 & 0.01125 \end{pmatrix}$; detJ = 0.00016875

$$\boldsymbol{B}^{\rm T} = \begin{pmatrix} 0 & 11.1111 & 0 & 33.3333 & 0 & -11.1111 & 0 & -33.3333 \\ 0 & -44.4444 & 0 & 0 & 0 & 44.4444 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{k}_{\rm k} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12.5926 & 0 & 2.22222 & 0 & -12.5926 & 0 & -2.22222 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2.22222 & 0 & 6.66667 & 0 & -2.22222 & 0 & -6.66667 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -12.5926 & 0 & -2.22222 & 0 & 12.5926 & 0 & 2.22222 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2.22222 & 0 & -6.66667 & 0 & 2.22222 & 0 & 6.66667 \end{pmatrix}$$

$$\boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} -166.667 \\ 333.333 \\ -166.667 \\ 333.333 \\ -166.667 \\ 333.333 \\ -166.667 \\ 333.333 \end{pmatrix}$$

Computation of element matrices at {0., 0.774597} with weight = 0.493827

$$\boldsymbol{N}^{\mathrm{T}} = (\, -0.1 \quad 0.112702 \quad -0.1 \quad 0.2 \quad -0.1 \quad 0.887298 \quad -0.1 \quad 0.2 \,)$$

$$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{s} = (\, 0.0436492 \quad 0. \quad -0.0436492 \quad 0.2 \quad 0.343649 \quad 0. \quad -0.343649 \quad -0.2 \,)$$

$$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{t} = (\, 0.387298 \quad -0.5 \quad 0.387298 \quad -0.774597 \quad 0.387298 \quad 0.5 \quad 0.387298 \quad -0.774597 \,)$$

$$\text{Jacobian matrix, } \boldsymbol{J} = \begin{pmatrix} 0.015 & 0 \\ 0.000845262 & 0.01125 \end{pmatrix}; \qquad \det J = 0.00016875$$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 0.969981 & 2.50448 & -4.84991 & 17.2133 & 20.97 & -2.50448 & -24.8499 & -9.45341 \\ 34.4265 & -44.4444 & 34.4265 & -68.853 & 34.4265 & 44.4444 & 34.4265 & -68.853 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 4.44797 & -5.72864 & 4.4268 & -8.82628 & 4.52072 & 5.72864 & 4.35405 & -8.92328 \\ -5.72864 & 7.43093 & -5.7833 & 11.6372 & -5.54081 & -7.43093 & -5.97114 & 11.3867 \\ 4.4268 & -5.7833 & 4.53265 & -9.20195 & 4.06306 & 5.7833 & 4.89639 & -8.71696 \\ -8.82628 & 11.6372 & -9.20195 & 18.8889 & -7.53528 & -11.6372 & -10.4929 & 17.1676 \\ 4.52072 & -5.54081 & 4.06306 & -7.53528 & 6.09347 & 5.54081 & 2.49031 & -9.63228 \\ 5.72864 & -7.43093 & 5.7833 & -11.6372 & 5.54081 & 7.43093 & 5.97114 & -11.3867 \\ 4.35405 & -5.97114 & 4.89639 & -10.4929 & 2.49031 & 5.97114 & 6.76014 & -8.00795 \\ -8.92328 & 11.3867 & -8.71696 & 17.1676 & -9.63228 & -11.3867 & -8.00795 & 18.1129 \end{pmatrix}$$

$$\boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} -41.6667 \\ 46.959 \\ -41.6667 \\ 83.3333 \\ -41.6667 \\ 369.708 \\ -41.6667 \\ 83.3333 \end{pmatrix}$$

Computation of element matrices at {0.774597, −0.774597} with weight = 0.308642

$$\boldsymbol{N}^{\mathrm{T}} = (\, -0.1 \quad 0.354919 \quad 0.432379 \quad 0.354919 \quad -0.1 \quad 0.0450807 \quad -0.032379 \quad 0.0450807 \,)$$

$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{s} = (\,0.343649 \quad -1.3746 \quad 1.03095 \quad 0.2 \quad 0.0436492 \quad -0.174597 \quad 0.130948 \quad -0.2\,)$

$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{t} = (\,-0.0436492 \quad -0.2 \quad -1.03095 \quad 1.3746 \quad -0.343649 \quad 0.2 \quad -0.130948 \quad 0.174597\,)$

Jacobian matrix, $\boldsymbol{J} = \begin{pmatrix} 0.015 & 0 \\ 0.00665474 & 0.00834526 \end{pmatrix}$; detJ = 0.000125179

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 25.2304 & -81.0074 & 123.537 & -59.7427 & 21.179 & -22.2721 & 15.6912 & -22.6152 \\ -5.23041 & -23.9657 & -123.537 & 164.716 & -41.179 & 23.9657 & -15.6912 & 20.9217 \end{pmatrix}$$

$\boldsymbol{k}_{\mathrm{k}} =$

$$\begin{pmatrix} 1.15431 & -3.3355 & 6.5424 & -4.1185 & 1.30349 & -1.19491 & 0.830994 & -1.18228 \\ -3.3355 & 12.4076 & -12.2515 & 1.55096 & -1.26704 & 2.13822 & -1.55614 & 2.31337 \\ 6.5424 & -12.2515 & 53.0667 & -48.2094 & 13.3933 & -9.931 & 6.74035 & -9.35088 \\ -4.1185 & 1.55096 & -48.2094 & 53.3758 & -13.9924 & 9.17653 & -6.12339 & 8.34043 \\ 1.30349 & -1.26704 & 13.3933 & -13.9924 & 3.72799 & -2.53589 & 1.70117 & -2.33059 \\ -1.19491 & 2.13822 & -9.931 & 9.17653 & -2.53589 & 1.861 & -1.2614 & 1.74745 \\ 0.830994 & -1.55614 & 6.74035 & -6.12339 & 1.70117 & -1.2614 & 0.856137 & -1.18772 \\ -1.18228 & 2.31337 & -9.35088 & 8.34043 & -2.33059 & 1.74745 & -1.18772 & 1.65021 \end{pmatrix}$$

$$\boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} -19.3177 \\ 68.5624 \\ 83.5258 \\ 68.5624 \\ -19.3177 \\ 8.70856 \\ -6.25489 \\ 8.70856 \end{pmatrix}$$

Computation of element matrices at {0.774597, 0.} with weight = 0.493827

$\boldsymbol{N}^{\mathrm{T}} = (\,-0.1 \quad 0.2 \quad -0.1 \quad 0.887298 \quad -0.1 \quad 0.2 \quad -0.1 \quad 0.112702\,)$

$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{s} = (\,0.387298 \quad -0.774597 \quad 0.387298 \quad 0.5 \quad 0.387298 \quad -0.774597 \quad 0.387298 \quad -0.5\,)$

$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{t} = (\,0.0436492 \quad -0.2 \quad -0.343649 \quad 0. \quad 0.343649 \quad 0.2 \quad -0.0436492 \quad 0.\,)$

Jacobian matrix, $\boldsymbol{J} = \begin{pmatrix} 0.015 & 0 \\ 0.00375 & 0.00834526 \end{pmatrix}$; detJ = 0.000125179

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 24.5123 & -45.6484 & 36.1146 & 33.3333 & 15.5252 & -57.6312 & 27.1275 & -33.3333 \\ 5.23041 & -23.9657 & -41.179 & 0 & 41.179 & 23.9657 & -5.23041 & 0 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 1.74752 & -3.46133 & 1.86341 & 2.27291 & 1.65776 & -3.58101 & 1.77365 & -2.27291 \\ -3.46133 & 7.39426 & -1.84066 & -4.23275 & -4.71669 & 5.72044 & -3.09602 & 4.23275 \\ 1.86341 & -1.84066 & 8.34519 & 3.34873 & -3.15735 & -8.53501 & 3.32442 & -3.34873 \\ 2.27291 & -4.23275 & 3.34873 & 3.09084 & 1.43957 & -5.34386 & 2.5154 & -3.09084 \\ 1.65776 & -4.71669 & -3.15735 & 1.43957 & 5.38752 & 0.256336 & 0.572417 & -1.43957 \\ -3.58101 & 5.72044 & -8.53501 & -5.34386 & 0.256336 & 10.8369 & -4.69766 & 5.34386 \\ 1.77365 & -3.09602 & 3.32442 & 2.5154 & 0.572417 & -4.69766 & 2.1232 & -2.5154 \\ -2.27291 & 4.23275 & -3.34873 & -3.09084 & -1.43957 & 5.34386 & -2.5154 & 3.09084 \end{pmatrix}$$

$$\boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} -30.9084 \\ 61.8168 \\ -30.9084 \\ 274.25 \\ -30.9084 \\ 61.8168 \\ -30.9084 \\ 34.8343 \end{pmatrix}$$

Computation of element matrices at {0.774597, 0.774597} with weight = 0.308642

$$\boldsymbol{N}^{\mathrm{T}} = (\,-0.032379 \quad 0.0450807 \quad -0.1 \quad 0.354919 \quad 0.432379 \quad 0.354919 \quad -0.1 \quad 0.0450807\,)$$

$$\partial\boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{s} = (\,0.130948 \quad -0.174597 \quad 0.0436492 \quad 0.2 \quad 1.03095 \quad -1.3746 \quad 0.343649 \quad -0.2\,)$$

$$\partial\boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{t} = (\,0.130948 \quad -0.2 \quad 0.343649 \quad -1.3746 \quad 1.03095 \quad 0.2 \quad 0.0436492 \quad -0.174597\,)$$

Jacobian matrix, $\boldsymbol{J} = \begin{pmatrix} 0.015 & 0 \\ 0.000845262 & 0.00834526 \end{pmatrix}$; detJ = 0.000125179

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 7.84562 & -10.2893 & 0.589476 & 22.6152 & 61.7684 & -92.9903 & 22.6152 & -12.1544 \\ 15.6912 & -23.9657 & 41.179 & -164.716 & 123.537 & 23.9657 & 5.23041 & -20.9217 \end{pmatrix}$$

$\boldsymbol{k}_{\mathrm{k}} =$

$$\begin{pmatrix} 0.535086 & -0.794151 & 1.13143 & -4.18509 & 4.21272 & -0.61462 & 0.451169 & -0.736548 \\ -0.794151 & 1.18263 & -1.72633 & 6.4586 & -6.25234 & 0.664925 & -0.622496 & 1.08916 \\ 1.13143 & -1.72633 & 2.94875 & -11.7694 & 8.90775 & 1.62049 & 0.397641 & -1.51031 \\ -4.18509 & 6.4586 & -11.7694 & 48.0596 & -32.9491 & -10.5194 & -0.608655 & 5.51353 \\ 4.21272 & -6.25234 & 8.90775 & -32.9491 & 33.1667 & -4.83889 & 3.55205 & -5.79883 \\ -0.61462 & 0.664925 & 1.62049 & -10.5194 & -4.83889 & 16.0325 & -3.43832 & 1.09329 \\ 0.451169 & -0.622496 & 0.397641 & -0.608655 & 3.55205 & -3.43832 & 0.936764 & -0.668147 \\ -0.736548 & 1.08916 & -1.51031 & 5.51353 & -5.79883 & 1.09329 & -0.668147 & 1.01785 \end{pmatrix}$$

$$\boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} -6.25489 \\ 8.70856 \\ -19.3177 \\ 68.5624 \\ 83.5258 \\ 68.5624 \\ -19.3177 \\ 8.70856 \end{pmatrix}$$

Summing contributions from all points we get

$$\boldsymbol{k} = \begin{pmatrix} 38.5159 & -20.3175 & 23.1944 & -30.9127 & 21.4683 & -14.6825 & 22.0278 & -39.2937 \\ -20.3175 & 95.0794 & -26.9841 & 16.3492 & -19.3651 & -5.07937 & -16.5079 & -23.1746 \\ 23.1944 & -26.9841 & 84.0873 & -82.6984 & 33.6111 & -28.0159 & 30.4563 & -33.6508 \\ -30.9127 & 16.3492 & -82.6984 & 155.873 & -68.1746 & 3.65079 & -36.1508 & 42.0635 \\ 21.4683 & -19.3651 & 33.6111 & -68.1746 & 58.3135 & -20.6349 & 23.1944 & -28.4127 \\ -14.6825 & -5.07937 & -28.0159 & 3.65079 & -20.6349 & 95.0794 & -33.4921 & 3.1746 \\ 22.0278 & -16.5079 & 30.4563 & -36.1508 & 23.1944 & -33.4921 & 57.1468 & -46.6746 \\ -39.2937 & -23.1746 & -33.6508 & 42.0635 & -28.4127 & 3.1746 & -46.6746 & 125.968 \end{pmatrix}$$

$$\boldsymbol{r}^{\mathrm{T}} = (\,-250. \quad 1125. \quad -312.5 \quad 1000. \quad -312.5 \quad 1125. \quad -250. \quad 1250.\,)$$

Computation of element matrices resulting from NBC

NBC on side 2 with $\alpha = -55$ and $\beta = 1100$

$$\boldsymbol{N}_{\mathrm{c}}^{\mathrm{T}} = \left(0 \quad 0 \quad \tfrac{1-\mathrm{a}}{2} + \tfrac{1}{2}(\mathrm{a}^2 - 1) \quad 1 - \mathrm{a}^2 \quad \tfrac{\mathrm{a}+1}{2} + \tfrac{1}{2}(\mathrm{a}^2 - 1) \quad 0 \quad 0 \quad 0\right)$$

$x(\mathrm{a}) = 0.03;$ $\quad$ $y(\mathrm{a}) = 0.0075\,\mathrm{a} + 0.0225$

$dx/d\mathrm{a} = 0.\,\mathrm{a} + 0.;$ $\quad$ $dy/d\mathrm{a} = 0.\,\mathrm{a} + 0.0075;$ $\quad$ $J_{\mathrm{c}} = \sqrt{(0.\,\mathrm{a} + 0.)^2 + (0.\,\mathrm{a} + 0.0075)^2}$

Gauss point $= -0.774597;$ $\quad$ Weight $= 0.555556;$ $\quad$ $J_{\mathrm{c}} = 0.0075$

$$\boldsymbol{N}_{\mathrm{c}}^{\mathrm{T}} = (\,0 \quad 0 \quad 0.687298 \quad 0.4 \quad -0.0872983 \quad 0 \quad 0 \quad 0\,)$$

$$\boldsymbol{k}_{\alpha} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.108254 & 0.0630023 & -0.01375 & 0 & 0 & 0 \\ 0 & 0 & 0.0630023 & 0.0366667 & -0.00800235 & 0 & 0 & 0 \\ 0 & 0 & -0.01375 & -0.00800235 & 0.00174648 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\boldsymbol{r}_\beta^{\mathrm{T}} = (\,0 \quad 0 \quad 3.15012 \quad 1.83333 \quad -0.400117 \quad 0 \quad 0 \quad 0\,)$

Gauss point = 0.; Weight = 0.888889; $\mathrm{J_c} = 0.0075$

$\boldsymbol{N}_{\mathrm{c}}^{\mathrm{T}} = (\,0 \quad 0 \quad 0. \quad 1. \quad 0. \quad 0 \quad 0 \quad 0\,)$

$$\boldsymbol{k}_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.366667 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\boldsymbol{r}_\beta^{\mathrm{T}} = (\,0 \quad 0 \quad 0 \quad 7.33333 \quad 0 \quad 0 \quad 0 \quad 0\,)$

Gauss point = 0.774597; Weight = 0.555556; $\mathrm{J_c} = 0.0075$

$\boldsymbol{N}_{\mathrm{c}}^{\mathrm{T}} = (\,0 \quad 0 \quad -0.0872983 \quad 0.4 \quad 0.687298 \quad 0 \quad 0 \quad 0\,)$

$$\boldsymbol{k}_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.00174648 & -0.00800235 & -0.01375 & 0 & 0 & 0 \\ 0 & 0 & -0.00800235 & 0.0366667 & 0.0630023 & 0 & 0 & 0 \\ 0 & 0 & -0.01375 & 0.0630023 & 0.108254 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\boldsymbol{r}_\beta^{\mathrm{T}} = (\,0 \quad 0 \quad -0.400117 \quad 1.83333 \quad 3.15012 \quad 0 \quad 0 \quad 0\,)$

Summing contributions from all Gauss points

$$\boldsymbol{k}_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.11 & 0.055 & -0.0275 & 0 & 0 & 0 \\ 0 & 0 & 0.055 & 0.44 & 0.055 & 0 & 0 & 0 \\ 0 & 0 & -0.0275 & 0.055 & 0.11 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\boldsymbol{r}_\beta^{\mathrm{T}} = (\,0 \quad 0 \quad 2.75 \quad 11. \quad 2.75 \quad 0 \quad 0 \quad 0\,)$

Computation of element matrices resulting from NBC

NBC on side 3 with $\alpha = -55$ and $\beta = 1100$

$$\boldsymbol{N}_c^T = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{1-a}{2} + \frac{1}{2}(a^2-1) & 1-a^2 & \frac{a+1}{2} + \frac{1}{2}(a^2-1) & 0 \end{pmatrix}$$

$x(a) = 0.015 - 0.015\,a$; $y(a) = 0.03$

$dx/da = 0.\,a - 0.015$; $dy/da = 0.\,a + 0.$; $J_c = \sqrt{(0.\,a + 0.)^2 + (0.\,a + 0.015)^2}$

Gauss point = −0.774597; Weight = 0.555556; $J_c = 0.015$

$$\boldsymbol{N}_c^T = (\, 0 \quad 0 \quad 0 \quad 0 \quad 0.687298 \quad 0.4 \quad -0.0872983 \quad 0 \,)$$

$$\boldsymbol{k}_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.216507 & 0.126005 & -0.0275 & 0 \\ 0 & 0 & 0 & 0 & 0.126005 & 0.0733333 & -0.0160047 & 0 \\ 0 & 0 & 0 & 0 & -0.0275 & -0.0160047 & 0.00349296 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{r}_\beta^T = (\, 0 \quad 0 \quad 0 \quad 0 \quad 6.30023 \quad 3.66667 \quad -0.800235 \quad 0 \,)$$

Gauss point = 0.; Weight = 0.888889; $J_c = 0.015$

$$\boldsymbol{N}_c^T = (\, 0 \quad 0 \quad 0 \quad 0 \quad 0. \quad 1. \quad 0. \quad 0 \,)$$

$$\boldsymbol{k}_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.733333 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{r}_\beta^T = (\, 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 14.6667 \quad 0 \quad 0 \,)$$

Gauss point = 0.774597; Weight = 0.555556; $J_c = 0.015$

$$\boldsymbol{N}_c^T = (\, 0 \quad 0 \quad 0 \quad 0 \quad -0.0872983 \quad 0.4 \quad 0.687298 \quad 0 \,)$$

$$\boldsymbol{k}_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.00349296 & -0.0160047 & -0.0275 & 0 \\ 0 & 0 & 0 & 0 & -0.0160047 & 0.0733333 & 0.126005 & 0 \\ 0 & 0 & 0 & 0 & -0.0275 & 0.126005 & 0.216507 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\boldsymbol{r}_\beta^{\mathrm{T}} = (\,0 \quad 0 \quad 0 \quad 0 \quad -0.800235 \quad 3.66667 \quad 6.30023 \quad 0\,)$

Summing contributions from all Gauss points

$$\boldsymbol{k}_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.22 & 0.11 & -0.055 & 0 \\ 0 & 0 & 0 & 0 & 0.11 & 0.88 & 0.11 & 0 \\ 0 & 0 & 0 & 0 & -0.055 & 0.11 & 0.22 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\boldsymbol{r}_\beta^{\mathrm{T}} = (\,0 \quad 0 \quad 0 \quad 0 \quad 5.5 \quad 22. \quad 5.5 \quad 0\,)$

Computation of element matrices resulting from NBC

NBC on side 4 with $\alpha = 0$ and $\beta = 8000$

$\boldsymbol{N}_{\mathrm{c}}^{\mathrm{T}} = \left(\,\frac{a+1}{2} + \frac{1}{2}(a^2-1) \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \frac{1-a}{2} + \frac{1}{2}(a^2-1) \quad 1-a^2\,\right)$

$x(a) = 0;$ $\qquad$ $y(a) = 0.015 - 0.015\,a$

$dx/da = 0.\,a + 0.;$ $\qquad$ $dy/da = 0.\,a - 0.015;$ $\qquad$ $J_c = \sqrt{(0.\,a - 0.015)^2 + (0.\,a + 0.)^2}$

Gauss point $= -0.774597;$ $\qquad$ Weight $= 0.555556;$ $\qquad$ $J_c = 0.015$

$\boldsymbol{N}_{\mathrm{c}}^{\mathrm{T}} = (\,-0.0872983 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.687298 \quad 0.4\,)$

$$\boldsymbol{k}_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\boldsymbol{r}_\beta^{\mathrm{T}} = (-5.81989 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 45.8199 \quad 26.6667)$

Gauss point = 0.; Weight = 0.888889; $\mathrm{J}_{\mathrm{c}} = 0.015$

$\boldsymbol{N}_{\mathrm{c}}^{\mathrm{T}} = (0. \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0. \quad 1.)$

$$\boldsymbol{k}_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\boldsymbol{r}_\beta^{\mathrm{T}} = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 106.667)$

Gauss point = 0.774597; Weight = 0.555556; $\mathrm{J}_{\mathrm{c}} = 0.015$

$\boldsymbol{N}_{\mathrm{c}}^{\mathrm{T}} = (0.687298 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -0.0872983 \quad 0.4)$

$$\boldsymbol{k}_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\boldsymbol{r}_\beta^{\mathrm{T}} = (45.8199 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -5.81989 \quad 26.6667)$

Summing contributions from all Gauss points

$$\boldsymbol{k}_\alpha = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

$$\boldsymbol{r}_\beta^{\mathrm{T}} = (\,40. \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 40. \quad 160.\,)$$

Complete element equations for element 1

$$\begin{pmatrix} 38.5159 & -20.3175 & 23.1944 & -30.9127 & 21.4683 & -14.6825 & 22.0278 & -39.2937\\ -20.3175 & 95.0794 & -26.9841 & 16.3492 & -19.3651 & -5.07937 & -16.5079 & -23.1746\\ 23.1944 & -26.9841 & 84.1973 & -82.6434 & 33.5836 & -28.0159 & 30.4563 & -33.6508\\ -30.9127 & 16.3492 & -82.6434 & 156.313 & -68.1196 & 3.65079 & -36.1508 & 42.0635\\ 21.4683 & -19.3651 & 33.5836 & -68.1196 & 58.6435 & -20.5249 & 23.1394 & -28.4127\\ -14.6825 & -5.07937 & -28.0159 & 3.65079 & -20.5249 & 95.9594 & -33.3821 & 3.1746\\ 22.0278 & -16.5079 & 30.4563 & -36.1508 & 23.1394 & -33.3821 & 57.3668 & -46.6746\\ -39.2937 & -23.1746 & -33.6508 & 42.0635 & -28.4127 & 3.1746 & -46.6746 & 125.968 \end{pmatrix} \begin{pmatrix} T_{11}\\ T_{13}\\ T_5\\ T_4\\ T_3\\ T_2\\ T_1\\ T_{12} \end{pmatrix} =$$

$$\begin{pmatrix} -210.\\ 1125.\\ -309.75\\ 1011.\\ -304.25\\ 1147.\\ -204.5\\ 1410. \end{pmatrix}$$

The element contributes to {11, 13, 5, 4, 3, 2, 1, 12} global degrees of freedom.

$$\text{Locations for element contributions to a global vector:} \begin{pmatrix} 11\\ 13\\ 5\\ 4\\ 3\\ 2\\ 1\\ 12 \end{pmatrix}$$

and to a global matrix: $\begin{pmatrix} [11, 11] & [11, 13] & [11, 5] & [11, 4] & [11, 3] & [11, 2] & [11, 1] & [11, 12] \\ [13, 11] & [13, 13] & [13, 5] & [13, 4] & [13, 3] & [13, 2] & [13, 1] & [13, 12] \\ [5, 11] & [5, 13] & [5, 5] & [5, 4] & [5, 3] & [5, 2] & [5, 1] & [5, 12] \\ [4, 11] & [4, 13] & [4, 5] & [4, 4] & [4, 3] & [4, 2] & [4, 1] & [4, 12] \\ [3, 11] & [3, 13] & [3, 5] & [3, 4] & [3, 3] & [3, 2] & [3, 1] & [3, 12] \\ [2, 11] & [2, 13] & [2, 5] & [2, 4] & [2, 3] & [2, 2] & [2, 1] & [2, 12] \\ [1, 11] & [1, 13] & [1, 5] & [1, 4] & [1, 3] & [1, 2] & [1, 1] & [1, 12] \\ [12, 11] & [12, 13] & [12, 5] & [12, 4] & [12, 3] & [12, 2] & [12, 1] & [12, 12] \end{pmatrix}$

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 57.3668 & -33.3821 & 23.1394 & -36.1508 & 30.4563 & 0 & 0 & 0 & 0 & 0 & 22.0278 & -46.6746 & -16.5079 \\ -33.3821 & 95.9594 & -20.5249 & 3.65079 & -28.0159 & 0 & 0 & 0 & 0 & 0 & -14.6825 & 3.1746 & -5.07937 \\ 23.1394 & -20.5249 & 58.6435 & -68.1196 & 33.5836 & 0 & 0 & 0 & 0 & 0 & 21.4683 & -28.4127 & -19.3651 \\ -36.1508 & 3.65079 & -68.1196 & 156.313 & -82.6434 & 0 & 0 & 0 & 0 & 0 & -30.9127 & 42.0635 & 16.3492 \\ 30.4563 & -28.0159 & 33.5836 & -82.6434 & 84.1973 & 0 & 0 & 0 & 0 & 0 & 23.1944 & -33.6508 & -26.9841 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 22.0278 & -14.6825 & 21.4683 & -30.9127 & 23.1944 & 0 & 0 & 0 & 0 & 0 & 38.5159 & -39.2937 & -20.3175 \\ -46.6746 & 3.1746 & -28.4127 & 42.0635 & -33.6508 & 0 & 0 & 0 & 0 & 0 & -39.2937 & 125.968 & -23.1746 \\ -16.5079 & -5.07937 & -19.3651 & 16.3492 & -26.9841 & 0 & 0 & 0 & 0 & 0 & -20.3175 & -23.1746 & 95.0794 \end{pmatrix}$$

Equations for element 2

$k_x = 45;$ $k_y = 45;$ $p = 0;$ $q = 5000000$

$\boldsymbol{C} = \begin{pmatrix} 45 & 0 \\ 0 & 45 \end{pmatrix}$

Element thickness = 1

Nodal coordinates

Element node	Global node number	x	y
1	11	0	0
2	10	0.03	0
3	9	0.06	0
4	8	0.06	0.0075
5	7	0.06	0.015
6	6	0.045	0.015
7	5	0.03	0.015
8	13	0.015	0.0075

Interpolation functions and their derivatives

$$\boldsymbol{N}^{T} = \left\{-\frac{1}{4}(s-1)(t-1)(s+t+1),\ \frac{1}{2}\left(s^2-1\right)(t-1),\ \frac{1}{4}(t-1)\left(-s^2+t\,s+t+1\right),\ -\frac{1}{2}(s+1)\left(t^2-1\right),\right.$$
$$\left.\frac{1}{4}(s+1)(t+1)(s+t-1),\ -\frac{1}{2}\left(s^2-1\right)(t+1),\ \frac{1}{4}(s-1)(s-t+1)(t+1),\ \frac{1}{2}(s-1)\left(t^2-1\right)\right\}$$

$$\partial\boldsymbol{N}^{T}/\partial s = \left\{-\frac{1}{4}(t-1)(2\,s+t),\ s\,(t-1),\ -\frac{1}{4}(2\,s-t)(t-1),\right.$$
$$\left.\frac{1}{2}\left(1-t^2\right),\ \frac{1}{4}(t+1)(2\,s+t),\ -s\,(t+1),\ \frac{1}{4}(2\,s-t)(t+1),\ \frac{1}{2}\left(t^2-1\right)\right\}$$

$$\partial\boldsymbol{N}^{T}/\partial t = \left\{-\frac{1}{4}(s-1)(s+2\,t),\ \frac{1}{2}\left(s^2-1\right),\ -\frac{1}{4}(s+1)(s-2\,t),\right.$$
$$\left.-(s+1)\,t,\ \frac{1}{4}(s+1)(s+2\,t),\ \frac{1}{2}\left(1-s^2\right),\ \frac{1}{4}(s-1)(s-2\,t),\ (s-1)\,t\right\}$$

Mapping to the master element

$$x(s,t) = -0.0075\,t\,s + 0.0225\,s + 0.0075\,t + 0.0375$$

$$y(s,t) = 0.0075\,t + 0.0075$$

Jacobian matrix, $\boldsymbol{J}$ =

$$\begin{pmatrix} 0.0225 - 0.0075\,t & 0.0075 - 0.0075\,s \\ 0 & 0.0075 \end{pmatrix}; \qquad \det J = 0.00016875 - 0.00005625\,t$$

Gauss quadrature points and weights

	Point	Weight
1	$s \to -0.774597$ $t \to -0.774597$	0.308642
2	$s \to -0.774597$ $t \to 0.$	0.493827
3	$s \to -0.774597$ $t \to 0.774597$	0.308642
4	$s \to 0.$ $t \to -0.774597$	0.493827
5	$s \to 0.$ $t \to 0.$	0.790123
6	$s \to 0.$ $t \to 0.774597$	0.493827
7	$s \to 0.774597$ $t \to -0.774597$	0.308642
8	$s \to 0.774597$ $t \to 0.$	0.493827
9	$s \to 0.774597$ $t \to 0.774597$	0.308642

Computation of element matrices at $\{-0.774597, -0.774597\}$ with weight = 0.308642

$$\boldsymbol{N}^T = (\,0.432379 \quad 0.354919 \quad -0.1 \quad 0.0450807 \quad -0.032379 \quad 0.0450807 \quad -0.1 \quad 0.354919\,)$$

$$\partial \boldsymbol{N}^T/\partial s = (\,-1.03095 \quad 1.3746 \quad -0.343649 \quad 0.2 \quad -0.130948 \quad 0.174597 \quad -0.0436492 \quad -0.2\,)$$

$$\partial \boldsymbol{N}^T/\partial t = (\,-1.03095 \quad -0.2 \quad -0.0436492 \quad 0.174597 \quad -0.130948 \quad 0.2 \quad -0.343649 \quad 1.3746\,)$$

Jacobian matrix, $\boldsymbol{J} = \begin{pmatrix} 0.0283095 & 0.0133095 \\ 0 & 0.0075 \end{pmatrix}$; $\quad$ $\det J = 0.000212321$

$$\boldsymbol{B}^T = \begin{pmatrix} -36.417 & 48.5561 & -12.139 & 7.06477 & -4.62557 & 6.16743 & -1.54186 & -7.06477 \\ -72.8341 & -112.834 & 15.722 & 10.7424 & -9.25114 & 15.722 & -43.0837 & 195.817 \end{pmatrix}$$

$\boldsymbol{k}_k =$

$$\begin{pmatrix} 19.5542 & 19.0201 & -2.07316 & -3.06596 & 2.48371 & -4.0391 & 9.41913 & -41.299 \\ 19.0201 & 44.4967 & -6.96943 & -2.56282 & 2.41587 & -4.34818 & 14.1148 & -66.167 \\ -2.07316 & -6.96943 & 1.16345 & 0.245151 & -0.263326 & 0.508137 & -1.94228 & 9.33146 \\ -3.06596 & -2.56282 & 0.245151 & 0.487486 & -0.389428 & 0.626535 & -1.39695 & 6.05598 \\ 2.48371 & 2.41587 & -0.263326 & -0.389428 & 0.315472 & -0.513033 & 1.19639 & -5.24566 \\ -4.0391 & -4.34818 & 0.508137 & 0.626535 & -0.513033 & 0.841079 & -2.02551 & 8.95008 \\ 9.41913 & 14.1148 & -1.94228 & -1.39695 & 1.19639 & -2.02551 & 5.48078 & -24.8463 \\ -41.299 & -66.167 & 9.33146 & 6.05598 & -5.24566 & 8.95008 & -24.8463 & 113.22 \end{pmatrix}$$

$$\boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} 141.672 \\ 116.291 \\ -32.7656 \\ 14.7709 \\ -10.6092 \\ 14.7709 \\ -32.7656 \\ 116.291 \end{pmatrix}$$

Computation of element matrices at {−0.774597, 0.} with weight = 0.493827

$$\boldsymbol{N}^{\mathrm{T}} = (\,-0.1 \quad 0.2 \quad -0.1 \quad 0.112702 \quad -0.1 \quad 0.2 \quad -0.1 \quad 0.887298\,)$$

$$\partial\boldsymbol{N}^{\mathrm{T}}/\partial\mathrm{s} = (\,-0.387298 \quad 0.774597 \quad -0.387298 \quad 0.5 \quad -0.387298 \quad 0.774597 \quad -0.387298 \quad -0.5\,)$$

$$\partial\boldsymbol{N}^{\mathrm{T}}/\partial\mathrm{t} = (\,-0.343649 \quad -0.2 \quad 0.0436492 \quad 0. \quad -0.0436492 \quad 0.2 \quad 0.343649 \quad 0.\,)$$

Jacobian matrix, $\boldsymbol{J} = \begin{pmatrix} 0.0225 & 0.0133095 \\ 0 & 0.0075 \end{pmatrix}$; detJ = 0.00016875

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -17.2133 & 34.4265 & -17.2133 & 22.2222 & -17.2133 & 34.4265 & -17.2133 & -22.2222 \\ -15.2733 & -87.7599 & 36.3665 & -39.4355 & 24.7267 & -34.4265 & 76.3665 & 39.4355 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 1.98589 & 2.80421 & -0.971774 & 0.824223 & -0.305107 & -0.250448 & -3.26277 & -0 \\ 2.80421 & 33.3262 & -14.1904 & 15.8471 & -10.3598 & 15.7742 & -27.3544 & -15 \\ -0.971774 & -14.1904 & 6.07056 & -6.81242 & 4.4832 & -6.91711 & 11.5255 & 6 \\ 0.824223 & 15.8471 & -6.81242 & 7.68369 & -5.0911 & 7.95998 & -12.7277 & -7 \\ -0.305107 & -10.3598 & 4.4832 & -5.0911 & 3.4039 & -5.41443 & 8.1922 & 5 \\ -0.250448 & 15.7742 & -6.91711 & 7.95998 & -5.41443 & 8.88889 & -12.0811 & -7 \\ -3.26277 & -27.3544 & 11.5255 & -12.7277 & 8.1922 & -12.0811 & 22.9805 & 12 \\ -0.824223 & -15.8471 & 6.81242 & -7.68369 & 5.0911 & -7.95998 & 12.7277 & 7 \end{pmatrix}$$

$$\boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} -41.6667 \\ 83.3333 \\ -41.6667 \\ 46.959 \\ -41.6667 \\ 83.3333 \\ -41.6667 \\ 369.708 \end{pmatrix}$$

Computation of element matrices at {−0.774597, 0.774597} with weight = 0.308642

$$\boldsymbol{N}^{\mathrm{T}} = (\,-0.1 \quad 0.0450807 \quad -0.032379 \quad 0.0450807 \quad -0.1 \quad 0.354919 \quad 0.432379 \quad 0.354919\,)$$

$$\partial \boldsymbol{N}^{\mathrm{T}}/\partial s = (\,-0.0436492 \quad 0.174597 \quad -0.130948 \quad 0.2 \quad -0.343649 \quad 1.3746 \quad -1.03095 \quad -0.2\,)$$

$$\partial \boldsymbol{N}^{\mathrm{T}}/\partial t = (\,0.343649 \quad -0.2 \quad 0.130948 \quad -0.174597 \quad 0.0436492 \quad 0.2 \quad 1.03095 \quad -1.3746\,)$$

$$\text{Jacobian matrix, } \boldsymbol{J} = \begin{pmatrix} 0.0166905 & 0.0133095 \\ 0 & 0.0075 \end{pmatrix}; \qquad \det J = 0.000125179$$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -2.61521 & 10.4608 & -7.84562 & 11.9828 & -20.5895 & 82.3579 & -61.7684 & -11.9828 \\ 50.4608 & -45.2304 & 31.3825 & -44.5443 & 42.3579 & -119.485 & 247.074 & -162.015 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} =$$

$$\begin{pmatrix} 4.43887 & -4.01567 & 2.78889 & -3.9624 & 3.80972 & -10.857 & 21.9569 & -14.1592 \\ -4.01567 & 3.74706 & -2.61053 & 3.72078 & -3.70538 & 10.8939 & -20.5526 & 12.5225 \\ 2.78889 & -2.61053 & 1.81929 & -2.59385 & 2.59196 & -7.64269 & 14.3232 & -8.67632 \\ -3.9624 & 3.72078 & -2.59385 & 3.69935 & -3.70933 & 10.9693 & -20.4213 & 12.2975 \\ 3.80972 & -3.70538 & 2.59196 & -3.70933 & 3.85641 & -11.7474 & 20.4064 & -11.5024 \\ -10.857 & 10.8939 & -7.64269 & 10.9693 & -11.7474 & 36.6141 & -60.1708 & 31.9407 \\ 21.9569 & -20.5526 & 14.3232 & -20.4213 & 20.4064 & -60.1708 & 112.767 & -68.3085 \\ -14.1592 & 12.5225 & -8.67632 & 12.2975 & -11.5024 & 31.9407 & -68.3085 & 45.8857 \end{pmatrix}$$

$$\boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} -19.3177 \\ 8.70856 \\ -6.25489 \\ 8.70856 \\ -19.3177 \\ 68.5624 \\ 83.5258 \\ 68.5624 \end{pmatrix}$$

Computation of element matrices at $\{0., -0.774597\}$ with weight = 0.493827

$$\boldsymbol{N}^{\mathrm{T}} = (\,-0.1 \quad 0.887298 \quad -0.1 \quad 0.2 \quad -0.1 \quad 0.112702 \quad -0.1 \quad 0.2\,)$$

$$\partial \boldsymbol{N}^{\mathrm{T}}/\partial s = (\,-0.343649 \quad 0. \quad 0.343649 \quad 0.2 \quad -0.0436492 \quad 0. \quad 0.0436492 \quad -0.2\,)$$

$$\partial \boldsymbol{N}^{\mathrm{T}}/\partial t = (\,-0.387298 \quad -0.5 \quad -0.387298 \quad 0.774597 \quad -0.387298 \quad 0.5 \quad -0.387298 \quad 0.774597\,)$$

$$\text{Jacobian matrix, } \boldsymbol{J} = \begin{pmatrix} 0.0283095 & 0.0075 \\ 0 & 0.0075 \end{pmatrix}; \qquad \det J = 0.000212321$$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -12.139 & 0 & 12.139 & 7.06477 & -1.54186 & 0 & 1.54186 & -7.06477 \\ -39.5008 & -66.6667 & -63.7788 & 96.2148 & -50.0979 & 66.6667 & -53.1816 & 110.344 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 8.05719 & 12.425 & 11.1915 & -18.3366 & 9.42527 & -12.425 & 9.82338 & -20.1607 \\ 12.425 & 20.97 & 20.0616 & -30.2643 & 15.7583 & -20.97 & 16.7283 & -34.7088 \\ 11.1915 & 20.0616 & 19.8878 & -28.5487 & 14.9874 & -20.0616 & 16.0919 & -33.6099 \\ -18.3366 & -30.2643 & -28.5487 & 43.9136 & -22.7941 & 30.2643 & -24.0912 & 49.857 \\ 9.42527 & 15.7583 & 14.9874 & -22.7941 & 11.8531 & -15.7583 & 12.5596 & -26.0312 \\ -12.425 & -20.97 & -20.0616 & 30.2643 & -15.7583 & 20.97 & -16.7283 & 34.7088 \\ 9.82338 & 16.7283 & 16.0919 & -24.0912 & 12.5596 & -16.7283 & 13.3558 & -27.7394 \\ -20.1607 & -34.7088 & -33.6099 & 49.857 & -26.0312 & 34.7088 & -27.7394 & 57.6842 \end{pmatrix}$$

$$\boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} -52.425 \\ 465.166 \\ -52.425 \\ 104.85 \\ -52.425 \\ 59.0838 \\ -52.425 \\ 104.85 \end{pmatrix}$$

Computation of element matrices at {0., 0.} with weight = 0.790123

$$\boldsymbol{N}^{\mathrm{T}} = \begin{pmatrix} -0.25 & 0.5 & -0.25 & 0.5 & -0.25 & 0.5 & -0.25 & 0.5 \end{pmatrix}$$

$$\partial \boldsymbol{N}^{\mathrm{T}}/\partial s = \begin{pmatrix} 0. & 0. & 0. & 0.5 & 0. & 0. & 0. & -0.5 \end{pmatrix}$$

$$\partial \boldsymbol{N}^{\mathrm{T}}/\partial t = \begin{pmatrix} 0. & -0.5 & 0. & 0. & 0. & 0.5 & 0. & 0. \end{pmatrix}$$

Jacobian matrix, $\boldsymbol{J} = \begin{pmatrix} 0.0225 & 0.0075 \\ 0 & 0.0075 \end{pmatrix}$; detJ = 0.00016875

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 22.2222 & 0 & 0 & 0 & -22.2222 \\ 0 & -66.6667 & 0 & -22.2222 & 0 & 66.6667 & 0 & 22.2222 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 26.6667 & 0 & 8.88889 & 0 & -26.6667 & 0 & -8.88889 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8.88889 & 0 & 5.92593 & 0 & -8.88889 & 0 & -5.92593 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -26.6667 & 0 & -8.88889 & 0 & 26.6667 & 0 & 8.88889 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -8.88889 & 0 & -5.92593 & 0 & 8.88889 & 0 & 5.92593 \end{pmatrix}$$

$$\boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} -166.667 \\ 333.333 \\ -166.667 \\ 333.333 \\ -166.667 \\ 333.333 \\ -166.667 \\ 333.333 \end{pmatrix}$$

Computation of element matrices at {0., 0.774597} with weight = 0.493827

$$\boldsymbol{N}^{\mathrm{T}} = (\ -0.1 \quad 0.112702 \quad -0.1 \quad 0.2 \quad -0.1 \quad 0.887298 \quad -0.1 \quad 0.2\ )$$

$$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{s} = (\ 0.0436492 \quad 0. \quad -0.0436492 \quad 0.2 \quad 0.343649 \quad 0. \quad -0.343649 \quad -0.2\ )$$

$$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{t} = (\ 0.387298 \quad -0.5 \quad 0.387298 \quad -0.774597 \quad 0.387298 \quad 0.5 \quad 0.387298 \quad -0.774597\ )$$

$$\text{Jacobian matrix, } \boldsymbol{J} = \begin{pmatrix} 0.0166905 & 0.0075 \\ 0 & 0.0075 \end{pmatrix}; \qquad \det J = 0.000125179$$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 2.61521 & 0 & -2.61521 & 11.9828 & 20.5895 & 0 & -20.5895 & -11.9828 \\ 49.0246 & -66.6667 & 54.255 & -115.262 & 31.0503 & 66.6667 & 72.2293 & -91.2967 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} =$$

$$\begin{pmatrix} 6.70472 & -9.09162 & 7.37996 & -15.6317 & 4.38425 & 9.09162 & 9.70043 & -12.5377 \\ -9.09162 & 12.3634 & -10.0616 & 21.3754 & -5.75829 & -12.3634 & -13.3949 & 16.931 \\ 7.37996 & -10.0616 & 8.20741 & -17.483 & 4.53645 & 10.0616 & 11.0509 & -13.6917 \\ -15.6317 & 21.3754 & -17.483 & 37.3562 & -9.26939 & -21.3754 & -23.8453 & 28.8732 \\ 4.38425 & -5.75829 & 4.53645 & -9.26939 & 3.86121 & 5.75829 & 5.05949 & -8.57201 \\ 9.09162 & -12.3634 & 10.0616 & -21.3754 & 5.75829 & 12.3634 & 13.3949 & -16.931 \\ 9.70043 & -13.3949 & 11.0509 & -23.8453 & 5.05949 & 13.3949 & 15.6919 & -17.6574 \\ -12.5377 & 16.931 & -13.6917 & 28.8732 & -8.57201 & -16.931 & -17.6574 & 23.5856 \end{pmatrix}$$

$$\boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} -30.9084 \\ 34.8343 \\ -30.9084 \\ 61.8168 \\ -30.9084 \\ 274.25 \\ -30.9084 \\ 61.8168 \end{pmatrix}$$

Computation of element matrices at {0.774597, −0.774597} with weight = 0.308642

$\boldsymbol{N}^{\mathrm{T}} = (\,-0.1 \quad 0.354919 \quad 0.432379 \quad 0.354919 \quad -0.1 \quad 0.0450807 \quad -0.032379 \quad 0.0450807\,)$

$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{s} = (\,0.343649 \quad -1.3746 \quad 1.03095 \quad 0.2 \quad 0.0436492 \quad -0.174597 \quad 0.130948 \quad -0.2\,)$

$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{t} = (\,-0.0436492 \quad -0.2 \quad -1.03095 \quad 1.3746 \quad -0.343649 \quad 0.2 \quad -0.130948 \quad 0.174597\,)$

Jacobian matrix, $\boldsymbol{J} = \begin{pmatrix} 0.0283095 & 0.00169052 \\ 0 & 0.0075 \end{pmatrix}$; detJ = 0.000212321

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 12.139 & -48.5561 & 36.417 & 7.06477 & 1.54186 & -6.16743 & 4.62557 & -7.06477 \\ -8.55606 & -15.722 & -145.668 & 181.687 & -46.1674 & 28.0568 & -18.5023 & 24.872 \end{pmatrix}$$

$\boldsymbol{k}_{\mathrm{k}} =$

$$\begin{pmatrix} 0.650416 & -1.34147 & 4.97897 & -4.33125 & 1.22004 & -0.928676 & 0.632412 & -0.880441 \\ -1.34147 & 7.68152 & 1.5391 & -9.43507 & 1.91967 & -0.41769 & 0.195491 & -0.141544 \\ 4.97897 & 1.5391 & 66.4843 & -77.2871 & 19.9973 & -12.7145 & 8.44461 & -11.4427 \\ -4.33125 & -9.43507 & -77.2871 & 97.4911 & -24.7034 & 14.9037 & -9.81675 & 13.1787 \\ 1.22004 & 1.91967 & 19.9973 & -24.7034 & 6.2924 & -3.84779 & 2.53999 & -3.41828 \\ -0.928676 & -0.41769 & -12.7145 & 14.9037 & -3.84779 & 2.4335 & -1.61495 & 2.18632 \\ 0.632412 & 0.195491 & 8.44461 & -9.81675 & 2.53999 & -1.61495 & 1.07261 & -1.45342 \\ -0.880441 & -0.141544 & -11.4427 & 13.1787 & -3.41828 & 2.18632 & -1.45342 & 1.97142 \end{pmatrix}$$

$$\boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} -32.7656 \\ 116.291 \\ 141.672 \\ 116.291 \\ -32.7656 \\ 14.7709 \\ -10.6092 \\ 14.7709 \end{pmatrix}$$

Computation of element matrices at {0.774597, 0.} with weight = 0.493827

$\boldsymbol{N}^{\mathrm{T}} = (\,-0.1 \quad 0.2 \quad -0.1 \quad 0.887298 \quad -0.1 \quad 0.2 \quad -0.1 \quad 0.112702\,)$

$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{s} = (\,0.387298 \quad -0.774597 \quad 0.387298 \quad 0.5 \quad 0.387298 \quad -0.774597 \quad 0.387298 \quad -0.5\,)$

$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{t} = (\,0.0436492 \quad -0.2 \quad -0.343649 \quad 0. \quad 0.343649 \quad 0.2 \quad -0.0436492 \quad 0.\,)$

Jacobian matrix, $\boldsymbol{J} = \begin{pmatrix} 0.0225 & 0.00169052 \\ 0 & 0.0075 \end{pmatrix}$; detJ = 0.00016875

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 17.2133 & -34.4265 & 17.2133 & 22.2222 & 17.2133 & -34.4265 & 17.2133 & -22.2222 \\ 1.93996 & -18.9068 & -49.6998 & -5.00896 & 41.94 & 34.4265 & -9.69981 & 5.00896 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} =$$

$$\begin{pmatrix} 1.12522 & -2.35977 & 0.749552 & 1.398 & 1.41622 & -1.97177 & 1.04055 & -1.398 \\ -2.35977 & 5.78495 & 1.30152 & -2.51374 & -5.19579 & 2.00359 & -1.5345 & 2.51374 \\ 0.749552 & 1.30152 & 10.3739 & 2.36798 & -6.70542 & -8.63844 & 2.91891 & -2.36798 \\ 1.398 & -2.51374 & 2.36798 & 1.94594 & 0.646654 & -3.51553 & 1.61664 & -1.94594 \\ 1.41622 & -5.19579 & -6.70542 & 0.646654 & 7.70721 & 3.1922 & -0.414426 & -0.646654 \\ -1.97177 & 2.00359 & -8.63844 & -3.51553 & 3.1922 & 8.88889 & -3.47446 & 3.51553 \\ 1.04055 & -1.5345 & 2.91891 & 1.61664 & -0.414426 & -3.47446 & 1.46394 & -1.61664 \\ -1.398 & 2.51374 & -2.36798 & -1.94594 & -0.646654 & 3.51553 & -1.61664 & 1.94594 \end{pmatrix}$$

$$\boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} -41.6667 \\ 83.3333 \\ -41.6667 \\ 369.708 \\ -41.6667 \\ 83.3333 \\ -41.6667 \\ 46.959 \end{pmatrix}$$

Computation of element matrices at {0.774597, 0.774597} with weight = 0.308642

$$\boldsymbol{N}^{\mathrm{T}} = (\,-0.032379 \quad 0.0450807 \quad -0.1 \quad 0.354919 \quad 0.432379 \quad 0.354919 \quad -0.1 \quad 0.0450807\,)$$

$$\partial\boldsymbol{N}^{\mathrm{T}}/\partial s = (\,0.130948 \quad -0.174597 \quad 0.0436492 \quad 0.2 \quad 1.03095 \quad -1.3746 \quad 0.343649 \quad -0.2\,)$$

$$\partial\boldsymbol{N}^{\mathrm{T}}/\partial t = (\,0.130948 \quad -0.2 \quad 0.343649 \quad -1.3746 \quad 1.03095 \quad 0.2 \quad 0.0436492 \quad -0.174597\,)$$

Jacobian matrix, $\boldsymbol{J} = \begin{pmatrix} 0.0166905 & 0.00169052 \\ 0 & 0.0075 \end{pmatrix}$; detJ = 0.000125179

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 7.84562 & -10.4608 & 2.61521 & 11.9828 & 61.7684 & -82.3579 & 20.5895 & -11.9828 \\ 15.6912 & -24.3088 & 45.2304 & -185.981 & 123.537 & 45.2304 & 1.17895 & -20.5786 \end{pmatrix}$$

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 0.535086 & -0.80585 & 1.26959 & -4.91023 & 4.21272 & 0.110527 & 0.313011 & -0 \\ -0.80585 & 1.21762 & -1.95914 & 7.64218 & -6.34445 & -0.413723 & -0.42429 & 1 \\ 1.26959 & -1.95914 & 3.56869 & -14.5705 & 9.99547 & 3.18234 & 0.186326 & -1 \\ -4.91023 & 7.64218 & -14.5705 & 60.3855 & -38.6582 & -16.3408 & 0.047739 & 6 \\ 4.21272 & -6.34445 & 9.99547 & -38.6582 & 33.1667 & 0.870176 & 2.46433 & -5 \\ 0.110527 & -0.413723 & 3.18234 & -16.3408 & 0.870176 & 15.3494 & -2.85544 & 0 \\ 0.313011 & -0.42429 & 0.186326 & 0.047739 & 2.46433 & -2.85544 & 0.739454 & -0 \\ -0.724849 & 1.08765 & -1.67273 & 6.40434 & -5.70673 & 0.0975428 & -0.471128 & 0 \end{pmatrix}$$

$$\boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} -6.25489 \\ 8.70856 \\ -19.3177 \\ 68.5624 \\ 83.5258 \\ 68.5624 \\ -19.3177 \\ 8.70856 \end{pmatrix}$$

Summing contributions from all points we get

$$\boldsymbol{k} = \begin{pmatrix} 43.0516 & 16.6349 & 25.3135 & -48.0159 & 26.6468 & -21.2698 & 49.623 & -91.9841 \\ 16.6349 & 156.254 & -12.8889 & 12.6984 & -11.2698 & -36.5079 & -32.2222 & -92.6984 \\ 25.3135 & -12.8889 & 117.575 & -144.683 & 49.623 & -42.2222 & 62.5992 & -55.3175 \\ -48.0159 & 12.6984 & -144.683 & 258.889 & -103.968 & 14.6032 & -90.6349 & 101.111 \\ 26.6468 & -11.2698 & 49.623 & -103.968 & 70.4563 & -27.4603 & 52.004 & -56.0317 \\ -21.2698 & -36.5079 & -42.2222 & 14.6032 & -27.4603 & 133.016 & -85.5556 & 65.3968 \\ 49.623 & -32.2222 & 62.5992 & -90.6349 & 52.004 & -85.5556 & 173.552 & -129.365 \\ -91.9841 & -92.6984 & -55.3175 & 101.111 & -56.0317 & 65.3968 & -129.365 & 258.889 \end{pmatrix}$$

$\boldsymbol{r}^{\mathrm{T}} = (-250. \quad 1250. \quad -250. \quad 1125. \quad -312.5 \quad 1000. \quad -312.5 \quad 1125.)$

Computation of element matrices resulting from NBC

NBC on side 3 with $\alpha = -55$ and $\beta = 1100$

$\boldsymbol{N}_{\mathrm{c}}^{\mathrm{T}} = \left(0 \quad 0 \quad 0 \quad 0 \quad \frac{1-a}{2} + \frac{1}{2}(a^2 - 1) \quad 1 - a^2 \quad \frac{a+1}{2} + \frac{1}{2}(a^2 - 1) \quad 0\right)$

$x(a) = 0.045 - 0.015\,a;$ $\qquad y(a) = 0.015$

$dx/da = 0.\,a - 0.015;$ $\qquad dy/da = 0.\,a + 0.;$ $\qquad J_c = \sqrt{(0.\,a + 0.)^2 + (0.\,a + 0.015)^2}$

Gauss point $= -0.774597;$ $\qquad$ Weight $= 0.555556;$ $\qquad J_c = 0.015$

$\boldsymbol{N}_{\mathrm{c}}^{\mathrm{T}} = (0 \quad 0 \quad 0 \quad 0 \quad 0.687298 \quad 0.4 \quad -0.0872983 \quad 0)$

$$\boldsymbol{k}_{\alpha} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.216507 & 0.126005 & -0.0275 & 0 \\ 0 & 0 & 0 & 0 & 0.126005 & 0.0733333 & -0.0160047 & 0 \\ 0 & 0 & 0 & 0 & -0.0275 & -0.0160047 & 0.00349296 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\boldsymbol{r}_\beta^{\mathrm{T}} = (\,0 \quad 0 \quad 0 \quad 0 \quad 6.30023 \quad 3.66667 \quad -0.800235 \quad 0\,)$

Gauss point = 0.;   Weight = 0.888889; $\mathrm{J_c} = 0.015$

$\boldsymbol{N}_\mathrm{c}^{\mathrm{T}} = (\,0 \quad 0 \quad 0 \quad 0 \quad 0. \quad 1. \quad 0. \quad 0\,)$

$$\boldsymbol{k}_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.733333 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\boldsymbol{r}_\beta^{\mathrm{T}} = (\,0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 14.6667 \quad 0 \quad 0\,)$

Gauss point = 0.774597;   Weight = 0.555556; $\mathrm{J_c} = 0.015$

$\boldsymbol{N}_\mathrm{c}^{\mathrm{T}} = (\,0 \quad 0 \quad 0 \quad 0 \quad -0.0872983 \quad 0.4 \quad 0.687298 \quad 0\,)$

$$\boldsymbol{k}_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.00349296 & -0.0160047 & -0.0275 & 0 \\ 0 & 0 & 0 & 0 & -0.0160047 & 0.0733333 & 0.126005 & 0 \\ 0 & 0 & 0 & 0 & -0.0275 & 0.126005 & 0.216507 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\boldsymbol{r}_\beta^{\mathrm{T}} = (\,0 \quad 0 \quad 0 \quad 0 \quad -0.800235 \quad 3.66667 \quad 6.30023 \quad 0\,)$

Summing contributions from all Gauss points

$$\boldsymbol{k}_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.22 & 0.11 & -0.055 & 0 \\ 0 & 0 & 0 & 0 & 0.11 & 0.88 & 0.11 & 0 \\ 0 & 0 & 0 & 0 & -0.055 & 0.11 & 0.22 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\boldsymbol{r}_\beta^{\mathrm{T}} = (\,0 \quad 0 \quad 0 \quad 0 \quad 5.5 \quad 22. \quad 5.5 \quad 0\,)$

Complete element equations for element 2

$$\begin{pmatrix} 43.0516 & 16.6349 & 25.3135 & -48.0159 & 26.6468 & -21.2698 & 49.623 & -91.9841 \\ 16.6349 & 156.254 & -12.8889 & 12.6984 & -11.2698 & -36.5079 & -32.2222 & -92.6984 \\ 25.3135 & -12.8889 & 117.575 & -144.683 & 49.623 & -42.2222 & 62.5992 & -55.3175 \\ -48.0159 & 12.6984 & -144.683 & 258.889 & -103.968 & 14.6032 & -90.6349 & 101.111 \\ 26.6468 & -11.2698 & 49.623 & -103.968 & 70.6763 & -27.3503 & 51.949 & -56.0317 \\ -21.2698 & -36.5079 & -42.2222 & 14.6032 & -27.3503 & 133.896 & -85.4456 & 65.3968 \\ 49.623 & -32.2222 & 62.5992 & -90.6349 & 51.949 & -85.4456 & 173.772 & -129.365 \\ -91.9841 & -92.6984 & -55.3175 & 101.111 & -56.0317 & 65.3968 & -129.365 & 258.889 \end{pmatrix} \begin{pmatrix} T_{11} \\ T_{10} \\ T_9 \\ T_8 \\ T_7 \\ T_6 \\ T_5 \\ T_{13} \end{pmatrix} =$$

$$\begin{pmatrix} -250. \\ 1250. \\ -250. \\ 1125. \\ -307. \\ 1022. \\ -307. \\ 1125. \end{pmatrix}$$

The element contributes to {11, 10, 9, 8, 7, 6, 5, 13} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 11 \\ 10 \\ 9 \\ 8 \\ 7 \\ 6 \\ 5 \\ 13 \end{pmatrix}$

and to a global matrix: $\begin{pmatrix} [11, 11] & [11, 10] & [11, 9] & [11, 8] & [11, 7] & [11, 6] & [11, 5] & [11, 13] \\ [10, 11] & [10, 10] & [10, 9] & [10, 8] & [10, 7] & [10, 6] & [10, 5] & [10, 13] \\ [9, 11] & [9, 10] & [9, 9] & [9, 8] & [9, 7] & [9, 6] & [9, 5] & [9, 13] \\ [8, 11] & [8, 10] & [8, 9] & [8, 8] & [8, 7] & [8, 6] & [8, 5] & [8, 13] \\ [7, 11] & [7, 10] & [7, 9] & [7, 8] & [7, 7] & [7, 6] & [7, 5] & [7, 13] \\ [6, 11] & [6, 10] & [6, 9] & [6, 8] & [6, 7] & [6, 6] & [6, 5] & [6, 13] \\ [5, 11] & [5, 10] & [5, 9] & [5, 8] & [5, 7] & [5, 6] & [5, 5] & [5, 13] \\ [13, 11] & [13, 10] & [13, 9] & [13, 8] & [13, 7] & [13, 6] & [13, 5] & [13, 13] \end{pmatrix}$

Adding element equations into appropriate locations we have

$$
\begin{pmatrix}
57.3668 & -33.3821 & 23.1394 & -36.1508 & 30.4563 & 0 & 0 & 0 & 0 & \\
-33.3821 & 95.9594 & -20.5249 & 3.65079 & -28.0159 & 0 & 0 & 0 & 0 & \\
23.1394 & -20.5249 & 58.6435 & -68.1196 & 33.5836 & 0 & 0 & 0 & 0 & \\
-36.1508 & 3.65079 & -68.1196 & 156.313 & -82.6434 & 0 & 0 & 0 & 0 & \\
30.4563 & -28.0159 & 33.5836 & -82.6434 & 257.969 & -85.4456 & 51.949 & -90.6349 & 62.5992 & - \\
0 & 0 & 0 & 0 & -85.4456 & 133.896 & -27.3503 & 14.6032 & -42.2222 & - \\
0 & 0 & 0 & 0 & 51.949 & -27.3503 & 70.6763 & -103.968 & 49.623 & - \\
0 & 0 & 0 & 0 & -90.6349 & 14.6032 & -103.968 & 258.889 & -144.683 & \\
0 & 0 & 0 & 0 & 62.5992 & -42.2222 & 49.623 & -144.683 & 117.575 & - \\
0 & 0 & 0 & 0 & -32.2222 & -36.5079 & -11.2698 & 12.6984 & -12.8889 & 1 \\
22.0278 & -14.6825 & 21.4683 & -30.9127 & 72.8175 & -21.2698 & 26.6468 & -48.0159 & 25.3135 & \\
-46.6746 & 3.1746 & -28.4127 & 42.0635 & -33.6508 & 0 & 0 & 0 & 0 & \\
-16.5079 & -5.07937 & -19.3651 & 16.3492 & -156.349 & 65.3968 & -56.0317 & 101.111 & -55.3175 & -
\end{pmatrix}
$$

Essential boundary conditions

Node	dof	Value
9	T_9	110
10	T_{10}	110
11	T_{11}	110

Delete equations {9, 10, 11}.

$$
\begin{pmatrix}
57.3668 & -33.3821 & 23.1394 & -36.1508 & 30.4563 & 0 & 0 & 0 & 0 & \\
-33.3821 & 95.9594 & -20.5249 & 3.65079 & -28.0159 & 0 & 0 & 0 & 0 & \\
23.1394 & -20.5249 & 58.6435 & -68.1196 & 33.5836 & 0 & 0 & 0 & 0 & \\
-36.1508 & 3.65079 & -68.1196 & 156.313 & -82.6434 & 0 & 0 & 0 & 0 & \\
30.4563 & -28.0159 & 33.5836 & -82.6434 & 257.969 & -85.4456 & 51.949 & -90.6349 & 62.5992 & - \\
0 & 0 & 0 & 0 & -85.4456 & 133.896 & -27.3503 & 14.6032 & -42.2222 & - \\
0 & 0 & 0 & 0 & 51.949 & -27.3503 & 70.6763 & -103.968 & 49.623 & - \\
0 & 0 & 0 & 0 & -90.6349 & 14.6032 & -103.968 & 258.889 & -144.683 & \\
-46.6746 & 3.1746 & -28.4127 & 42.0635 & -33.6508 & 0 & 0 & 0 & 0 & \\
-16.5079 & -5.07937 & -19.3651 & 16.3492 & -156.349 & 65.3968 & -56.0317 & 101.111 & -55.3175 & -
\end{pmatrix}
$$

Extract columns {9, 10, 11}.

Multiply each column by its respective known value {110, 110, 110}.

Move all resulting vectors to the rhs.

After adjusting for essential boundary conditions we have

$$\left(\begin{array}{cccccccccc}
57.3668 & -33.3821 & 23.1394 & -36.1508 & 30.4563 & 0 & 0 & 0 & -46.6746 & - \\
-33.3821 & 95.9594 & -20.5249 & 3.65079 & -28.0159 & 0 & 0 & 0 & 3.1746 & \\
23.1394 & -20.5249 & 58.6435 & -68.1196 & 33.5836 & 0 & 0 & 0 & -28.4127 & - \\
-36.1508 & 3.65079 & -68.1196 & 156.313 & -82.6434 & 0 & 0 & 0 & 42.0635 & \\
30.4563 & -28.0159 & 33.5836 & -82.6434 & 257.969 & -85.4456 & 51.949 & -90.6349 & -33.6508 & -1 \\
0 & 0 & 0 & 0 & -85.4456 & 133.896 & -27.3503 & 14.6032 & 0 & \\
0 & 0 & 0 & 0 & 51.949 & -27.3503 & 70.6763 & -103.968 & 0 & - \\
0 & 0 & 0 & 0 & -90.6349 & 14.6032 & -103.968 & 258.889 & 0 & 1 \\
-46.6746 & 3.1746 & -28.4127 & 42.0635 & -33.6508 & 0 & 0 & 0 & 125.968 & - \\
-16.5079 & -5.07937 & -19.3651 & 16.3492 & -156.349 & 65.3968 & -56.0317 & 101.111 & -23.1746 & 3
\end{array}\right.$$

Solving the final system of global equations we get

$$\{T_1 = 156.441,\ T_2 = 150.756,\ T_3 = 149.196,\ T_4 = 144.225,\ T_5 = 133.843,\ T_6 = 124.002,\ T_7 = 121.746,\ T_8 = 119.148,\ T_{12} = 144.675,\ T_{13} = 129.132\}$$

Complete table of nodal values

	T
1	156.441
2	150.756
3	149.196
4	144.225
5	133.843
6	124.002
7	121.746
8	119.148
9	110
10	110
11	110
12	144.675
13	129.132

Solution for element 1

Element nodal values

Element node	Global node number	T
1	11	110
2	13	129.132
3	5	133.843
4	4	144.225
5	3	149.196
6	2	150.756
7	1	156.441
8	12	144.675

$\boldsymbol{d}^T = (\,110 \quad 129.132 \quad 133.843 \quad 144.225 \quad 149.196 \quad 150.756 \quad 156.441 \quad 144.675\,)$

Nodal values = (110   129.132   133.843   144.225   149.196   150.756   156.441   144.675)

Interpolation functions and their derivatives

$$\boldsymbol{N}^T = \Big\{-\frac{1}{4}(s-1)(t-1)(s+t+1),\ \frac{1}{2}\left(s^2-1\right)(t-1),\ \frac{1}{4}(t-1)\left(-s^2+t\,s+t+1\right),\ -\frac{1}{2}(s+1)\left(t^2-1\right),$$
$$\frac{1}{4}(s+1)(t+1)(s+t-1),\ -\frac{1}{2}\left(s^2-1\right)(t+1),\ \frac{1}{4}(s-1)(s-t+1)(t+1),\ \frac{1}{2}(s-1)\left(t^2-1\right)\Big\}$$

$$\partial\boldsymbol{N}^T/\partial s = \Big\{-\frac{1}{4}(t-1)(2\,s+t),\ s\,(t-1),\ -\frac{1}{4}(2\,s-t)(t-1),$$
$$\frac{1}{2}\left(1-t^2\right),\ \frac{1}{4}(t+1)(2\,s+t),\ -s\,(t+1),\ \frac{1}{4}(2\,s-t)(t+1),\ \frac{1}{2}\left(t^2-1\right)\Big\}$$

$$\partial\boldsymbol{N}^T/\partial t = \Big\{-\frac{1}{4}(s-1)(s+2\,t),\ \frac{1}{2}\left(s^2-1\right),\ -\frac{1}{4}(s+1)(s-2\,t),$$
$$-(s+1)\,t,\ \frac{1}{4}(s+1)(s+2\,t),\ \frac{1}{2}\left(1-s^2\right),\ \frac{1}{4}(s-1)(s-2\,t),\ (s-1)\,t\Big\}$$

Nodal coordinates

Element node	Global node number	x	y
1	11	0	0
2	13	0.015	0.0075
3	5	0.03	0.015
4	4	0.03	0.0225
5	3	0.03	0.03
6	2	0.015	0.03
7	1	0	0.03
8	12	0	0.015

Mapping to the master element

$$x(s,t) = 0.0075\left(1-s^2\right)(1-t) + 0.0075\left(1-s^2\right)(t+1) +$$
$$0.015\,(s+1)\left(1-t^2\right) + 0.03\left(\frac{1}{4}(s+1)(1-t) - \frac{1}{4}\left(1-s^2\right)(1-t) - \frac{1}{4}(s+1)\left(1-t^2\right)\right) +$$
$$0.03\left(\frac{1}{4}(s+1)(t+1) - \frac{1}{4}\left(1-s^2\right)(t+1) - \frac{1}{4}(s+1)\left(1-t^2\right)\right)$$

$$y(s,t) = 0.00375\left(1-s^2\right)(1-t) + 0.015\left(1-s^2\right)(t+1) + 0.0075\,(1-s)\left(1-t^2\right) +$$
$$0.01125\,(s+1)\left(1-t^2\right) + 0.03\left(\frac{1}{4}(1-s)(t+1) - \frac{1}{4}\left(1-s^2\right)(t+1) - \frac{1}{4}(1-s)\left(1-t^2\right)\right) +$$
$$0.015\left(\frac{1}{4}(s+1)(1-t) - \frac{1}{4}\left(1-s^2\right)(1-t) - \frac{1}{4}(s+1)\left(1-t^2\right)\right) +$$
$$0.03\left(\frac{1}{4}(s+1)(t+1) - \frac{1}{4}\left(1-s^2\right)(t+1) - \frac{1}{4}(s+1)\left(1-t^2\right)\right)$$

$$\boldsymbol{J} = \left(\begin{array}{c} -0.015\,s\,(1-t) - 0.015\,s\,(t+1) + 0.015\,(1-t^2) + 0.03\left(\tfrac{1}{2}\,s\,(1-t) + \tfrac{1-t}{4} + \cdot \right. \\ -0.0075\,s\,(1-t) - 0.03\,s\,(t+1) + 0.00375\,(1-t^2) + 0.03\left(\tfrac{1}{4}(-t-1) + \tfrac{1}{2}\,s\,(t+1) + \tfrac{1}{4}(1-t^2)\right) + 0.\right.$$

Solution at $\{s, t\} = \{0., 0.\} \Longrightarrow \{x, y\} = \{0.015, 0.01875\}$

Interpolation functions & their derivatives

$\boldsymbol{N}^T = \{-0.25, 0.5, -0.25, 0.5, -0.25, 0.5, -0.25, 0.5\}$
$\partial\boldsymbol{N}^T/\partial s = \{0., 0., 0., 0.5, 0., 0., 0., -0.5\}$
$\partial\boldsymbol{N}^T/\partial t = \{0., -0.5, 0., 0., 0., 0.5, 0., 0.\}$

Jacobian matrix, $\boldsymbol{J} = \begin{pmatrix} 0.015 & 0. \\ 0.00375 & 0.01125 \end{pmatrix}$; detJ = 0.00016875

$\partial\boldsymbol{N}^T/\partial x = \{0., 11.1111, 0., 33.3333, 0., -11.1111, 0., -33.3333\}$
$\partial\boldsymbol{N}^T/\partial y = \{0., -44.4444, 0., 0., 0., 44.4444, 0., 0.\}$

$T = 147.024$; $\partial T/\partial x = -255.296$; $\partial T/\partial y = 961.069$

Solution at $\{s, t\} = \{-1., -1.\} \Longrightarrow \{x, y\} = \{0., 0.\}$

Interpolation functions & their derivatives

$\boldsymbol{N}^T = \{1., 0., 0., 0., 0., 0., 0., 0.\}$
$\partial\boldsymbol{N}^T/\partial s = \{-1.5, 2., -0.5, 0., 0., 0., 0., 0.\}$
$\partial\boldsymbol{N}^T/\partial t = \{-1.5, 0., 0., 0., 0., 0., -0.5, 2.\}$

Jacobian matrix, $\boldsymbol{J} = \begin{pmatrix} 0.015 & 0. \\ 0.0075 & 0.015 \end{pmatrix}$; detJ = 0.000225

$\partial\boldsymbol{N}^T/\partial x = \{-50., 133.333, -33.3333, 0., 0., 0., 16.6667, -66.6667\}$
$\partial\boldsymbol{N}^T/\partial y = \{-100., 0., 0., 0., 0., 0., -33.3333, 133.333\}$

$T = 110.;\quad \partial T/\partial x = 218.472;\quad \partial T/\partial y = 3075.37$

Solution at {s, t} = {−1., 1.} ⟹ {x, y} = {0., 0.03}

Interpolation functions & their derivatives

$\boldsymbol{N}^{T} = \{0., 0., 0., 0., 0., 0., 1., 0.\}$
$\partial\boldsymbol{N}^{T}/\partial s = \{0., 0., 0., 0., -0.5, 2., -1.5, 0.\}$
$\partial\boldsymbol{N}^{T}/\partial t = \{0.5, 0., 0., 0., 0., 0., 1.5, -2.\}$

Jacobian matrix, $\boldsymbol{J} = \begin{pmatrix} 0.015 & 0. \\ 0. & 0.015 \end{pmatrix}$; $\quad \det J = 0.000225$

$\partial\boldsymbol{N}^{T}/\partial x = \{0., 0., 0., 0., -33.3333, 133.333, -100., 0.\}$
$\partial\boldsymbol{N}^{T}/\partial y = \{33.3333, 0., 0., 0., 0., 0., 100., -133.333\}$

$T = 156.441;\quad \partial T/\partial x = -516.458;\quad \partial T/\partial y = 20.6606$

Solution at {s, t} = {1., −1.} ⟹ {x, y} = {0.03, 0.015}

Interpolation functions & their derivatives

$\boldsymbol{N}^{T} = \{0., 0., 1., 0., 0., 0., 0., 0.\}$
$\partial\boldsymbol{N}^{T}/\partial s = \{0.5, -2., 1.5, 0., 0., 0., 0., 0.\}$
$\partial\boldsymbol{N}^{T}/\partial t = \{0., 0., -1.5, 2., -0.5, 0., 0., 0.\}$

Jacobian matrix, $\boldsymbol{J} = \begin{pmatrix} 0.015 & 0. \\ 0.0075 & 0.0075 \end{pmatrix}$; $\quad \det J = 0.0001125$

$\partial\boldsymbol{N}^{T}/\partial x = \{33.3333, -133.333, 200., -133.333, 33.3333, 0., 0., 0.\}$
$\partial\boldsymbol{N}^{T}/\partial y = \{0., 0., -200., 266.667, -66.6667, 0., 0., 0.\}$

$T = 133.843;\quad \partial T/\partial x = -1039.01;\quad \partial T/\partial y = 1744.8$

Solution at {s, t} = {1., 1.} ⟹ {x, y} = {0.03, 0.03}

Interpolation functions & their derivatives

$\boldsymbol{N}^{T} = \{0., 0., 0., 0., 1., 0., 0., 0.\}$
$\partial\boldsymbol{N}^{T}/\partial s = \{0., 0., 0., 0., 1.5, -2., 0.5, 0.\}$
$\partial\boldsymbol{N}^{T}/\partial t = \{0., 0., 0.5, -2., 1.5, 0., 0., 0.\}$

Jacobian matrix, $\boldsymbol{J} = \begin{pmatrix} 0.015 & 0. \\ 0. & 0.0075 \end{pmatrix}$; $\quad \det J = 0.0001125$

$\partial\boldsymbol{N}^{T}/\partial x = \{0., 0., 0., 0., 100., -133.333, 33.3333, 0.\}$
$\partial\boldsymbol{N}^{T}/\partial y = \{0., 0., 66.6667, -266.667, 200., 0., 0., 0.\}$

$T = 149.196;\quad \partial T/\partial x = 33.5225;\quad \partial T/\partial y = 302.296$

Solution for element 2

Element nodal values

Element node	Global node number	T
1	11	110
2	10	110
3	9	110
4	8	119.148
5	7	121.746
6	6	124.002
7	5	133.843
8	13	129.132

$\boldsymbol{d}^{T} = (\,110 \quad 110 \quad 110 \quad 119.148 \quad 121.746 \quad 124.002 \quad 133.843 \quad 129.132\,)$

Nodal values = (110   110   110   119.148   121.746   124.002   133.843   129.132)

Interpolation functions and their derivatives

$$\boldsymbol{N}^{T} = \left\{-\frac{1}{4}(s-1)(t-1)(s+t+1),\ \frac{1}{2}\left(s^2-1\right)(t-1),\ \frac{1}{4}(t-1)\left(-s^2+ts+t+1\right),\ -\frac{1}{2}(s+1)\left(t^2-1\right),\right.$$
$$\left.\frac{1}{4}(s+1)(t+1)(s+t-1),\ -\frac{1}{2}\left(s^2-1\right)(t+1),\ \frac{1}{4}(s-1)(s-t+1)(t+1),\ \frac{1}{2}(s-1)\left(t^2-1\right)\right\}$$

$$\partial\boldsymbol{N}^{T}/\partial s = \left\{-\frac{1}{4}(t-1)(2s+t),\ s(t-1),\ -\frac{1}{4}(2s-t)(t-1),\right.$$
$$\left.\frac{1}{2}\left(1-t^2\right),\ \frac{1}{4}(t+1)(2s+t),\ -s(t+1),\ \frac{1}{4}(2s-t)(t+1),\ \frac{1}{2}\left(t^2-1\right)\right\}$$

$$\partial\boldsymbol{N}^{T}/\partial t = \left\{-\frac{1}{4}(s-1)(s+2t),\ \frac{1}{2}\left(s^2-1\right),\ -\frac{1}{4}(s+1)(s-2t),\right.$$
$$\left.-(s+1)t,\ \frac{1}{4}(s+1)(s+2t),\ \frac{1}{2}\left(1-s^2\right),\ \frac{1}{4}(s-1)(s-2t),\ (s-1)t\right\}$$

Nodal coordinates

Element node	Global node number	x	y
1	11	0	0
2	10	0.03	0
3	9	0.06	0
4	8	0.06	0.0075
5	7	0.06	0.015
6	6	0.045	0.015
7	5	0.03	0.015
8	13	0.015	0.0075

Mapping to the master element

$$x(s,t) = 0.015\left(1-s^2\right)(1-t) + 0.0225\left(1-s^2\right)(t+1) + 0.0075\,(1-s)\left(1-t^2\right) +$$
$$0.03\,(s+1)\left(1-t^2\right) + 0.03\left(\frac{1}{4}(1-s)(t+1) - \frac{1}{4}\left(1-s^2\right)(t+1) - \frac{1}{4}(1-s)\left(1-t^2\right)\right) +$$
$$0.06\left(\frac{1}{4}(s+1)(1-t) - \frac{1}{4}\left(1-s^2\right)(1-t) - \frac{1}{4}(s+1)\left(1-t^2\right)\right) +$$
$$0.06\left(\frac{1}{4}(s+1)(t+1) - \frac{1}{4}\left(1-s^2\right)(t+1) - \frac{1}{4}(s+1)\left(1-t^2\right)\right)$$

$$y(s,t) = 0.0075\left(1-s^2\right)(t+1) + 0.00375\,(1-s)\left(1-t^2\right) +$$
$$0.00375\,(s+1)\left(1-t^2\right) + 0.015\left(\frac{1}{4}(1-s)(t+1) - \frac{1}{4}\left(1-s^2\right)(t+1) - \frac{1}{4}(1-s)\left(1-t^2\right)\right) +$$
$$0.015\left(\frac{1}{4}(s+1)(t+1) - \frac{1}{4}\left(1-s^2\right)(t+1) - \frac{1}{4}(s+1)\left(1-t^2\right)\right)$$

$$\boldsymbol{J} = \begin{pmatrix} -0.03\,s\,(1-t) - 0.045\,s\,(t+1) + 0.0225\,(1-t^2) + 0.03\left(\tfrac{1}{4}(-t-1) + \tfrac{1}{2}\,s\,(t+1) + \tfrac{1}{4}(1-t^2)\right) + 0.06 \\ -0.015\,s\,(t+1) + 0.015\left(\tfrac{1}{4}(-t-1) + \tfrac{1}{2}\,s\,(t+1) + \tfrac{1}{4}(1-t^2)\right) + 0. \end{pmatrix}$$

Solution at $\{s, t\} = \{0., 0.\} \Longrightarrow \{x, y\} = \{0.0375, 0.0075\}$

Interpolation functions & their derivatives

$\boldsymbol{N}^T = \{-0.25, 0.5, -0.25, 0.5, -0.25, 0.5, -0.25, 0.5\}$
$\partial\boldsymbol{N}^T/\partial s = \{0., 0., 0., 0.5, 0., 0., 0., -0.5\}$
$\partial\boldsymbol{N}^T/\partial t = \{0., -0.5, 0., 0., 0., 0.5, 0., 0.\}$

Jacobian matrix, $\boldsymbol{J} = \begin{pmatrix} 0.0225 & 0.0075 \\ 0. & 0.0075 \end{pmatrix}$; detJ = 0.00016875

$\partial\boldsymbol{N}^T/\partial x = \{0., 0., 0., 22.2222, 0., 0., 0., -22.2222\}$
$\partial\boldsymbol{N}^T/\partial y = \{0., -66.6667, 0., -22.2222, 0., 66.6667, 0., 22.2222\}$

$T = 122.244$; $\partial T/\partial x = -221.864$; $\partial T/\partial y = 1155.33$

Solution at $\{s, t\} = \{-1., -1.\} \Longrightarrow \{x, y\} = \{0., 0.\}$

Interpolation functions & their derivatives

$\boldsymbol{N}^T = \{1., 0., 0., 0., 0., 0., 0., 0.\}$
$\partial\boldsymbol{N}^T/\partial s = \{-1.5, 2., -0.5, 0., 0., 0., 0., 0.\}$
$\partial\boldsymbol{N}^T/\partial t = \{-1.5, 0., 0., 0., 0., 0., -0.5, 2.\}$

Jacobian matrix, $\boldsymbol{J} = \begin{pmatrix} 0.03 & 0.015 \\ 0. & 0.0075 \end{pmatrix}$; detJ = 0.000225

$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{x} = \{-50., 66.6667, -16.6667, 0., 0., 0., 0., 0.\}$
$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{y} = \{-100., -133.333, 33.3333, 0., 0., 0., -66.6667, 266.667\}$

$T = 110.;$ $\partial T/\partial \mathrm{x} = -9.09495\times 10^{-13};$ $\partial T/\partial \mathrm{y} = 3512.32$

Solution at $\{\mathrm{s}, \mathrm{t}\} = \{-1., 1.\} \Longrightarrow \{\mathrm{x}, \mathrm{y}\} = \{0.03, 0.015\}$

Interpolation functions & their derivatives

$\boldsymbol{N}^{\mathrm{T}} = \{0., 0., 0., 0., 0., 0., 1., 0.\}$
$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{s} = \{0., 0., 0., 0., -0.5, 2., -1.5, 0.\}$
$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{t} = \{0.5, 0., 0., 0., 0., 0., 1.5, -2.\}$

Jacobian matrix, $\boldsymbol{J} = \begin{pmatrix} 0.015 & 0.015 \\ 0. & 0.0075 \end{pmatrix};$ $\det J = 0.0001125$

$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{x} = \{0., 0., 0., 0., -33.3333, 133.333, -100., 0.\}$
$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{y} = \{66.6667, 0., 0., 0., 66.6667, -266.667, 400., -266.667\}$

$T = 133.843;$ $\partial T/\partial \mathrm{x} = -908.945;$ $\partial T/\partial \mathrm{y} = 1484.68$

Solution at $\{\mathrm{s}, \mathrm{t}\} = \{1., -1.\} \Longrightarrow \{\mathrm{x}, \mathrm{y}\} = \{0.06, 0.\}$

Interpolation functions & their derivatives

$\boldsymbol{N}^{\mathrm{T}} = \{0., 0., 1., 0., 0., 0., 0., 0.\}$
$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{s} = \{0.5, -2., 1.5, 0., 0., 0., 0., 0.\}$
$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{t} = \{0., 0., -1.5, 2., -0.5, 0., 0., 0.\}$

Jacobian matrix, $\boldsymbol{J} = \begin{pmatrix} 0.03 & 0. \\ 0. & 0.0075 \end{pmatrix};$ $\det J = 0.000225$

$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{x} = \{16.6667, -66.6667, 50., 0., 0., 0., 0., 0.\}$
$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{y} = \{0., 0., -200., 266.667, -66.6667, 0., 0., 0.\}$

$T = 110.;$ $\partial T/\partial \mathrm{x} = 9.09495\times 10^{-13};$ $\partial T/\partial \mathrm{y} = 1656.41$

Solution at $\{\mathrm{s}, \mathrm{t}\} = \{1., 1.\} \Longrightarrow \{\mathrm{x}, \mathrm{y}\} = \{0.06, 0.015\}$

Interpolation functions & their derivatives

$\boldsymbol{N}^{\mathrm{T}} = \{0., 0., 0., 0., 1., 0., 0., 0.\}$
$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{s} = \{0., 0., 0., 0., 1.5, -2., 0.5, 0.\}$
$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{t} = \{0., 0., 0.5, -2., 1.5, 0., 0., 0.\}$

Jacobian matrix, $\boldsymbol{J} = \begin{pmatrix} 0.015 & 0. \\ 0. & 0.0075 \end{pmatrix};$ $\det J = 0.0001125$

$$\partial \boldsymbol{N}^{\mathrm{T}}/\partial x = \{0., 0., 0., 0., 100., -133.333, 33.3333, 0.\}$$
$$\partial \boldsymbol{N}^{\mathrm{T}}/\partial y = \{0., 0., 66.6667, -266.667, 200., 0., 0., 0.\}$$

$T = 121.746;$ $\quad \partial T/\partial x = 102.484;$ $\quad \partial T/\partial y = -90.2303$

Solution summary

Nodal solution

	x	y	T
1	0	0.03	156.441
2	0.015	0.03	150.756
3	0.03	0.03	149.196
4	0.03	0.0225	144.225
5	0.03	0.015	133.843
6	0.045	0.015	124.002
7	0.06	0.015	121.746
8	0.06	0.0075	119.148
9	0.06	0	110
10	0.03	0	110
11	0	0	110
12	0	0.015	144.675
13	0.015	0.0075	129.132

Solution at selected points on the elements

	x	y	T	$\partial T/\partial x$	$\partial T/\partial y$
1	0.015	0.01875	147.024	−255.296	961.069
2	0.0375	0.0075	122.244	−221.864	1155.33