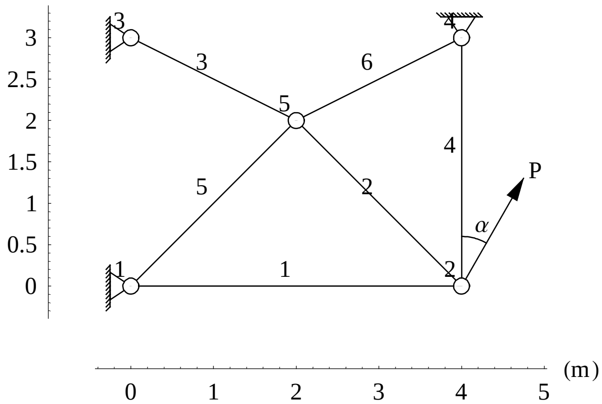
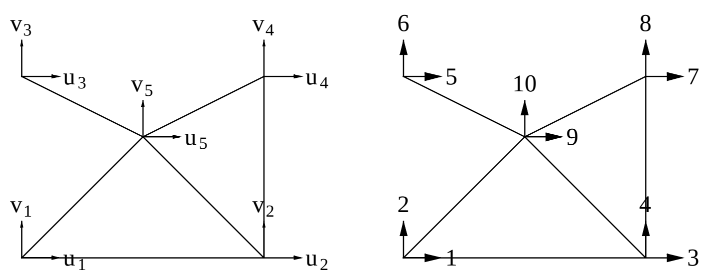
CHAPTER FOUR

Trusses, Beams, and Frames

Example 4.1: Six-bar truss (p. 226)

All members have the same cross-sectional area and are of the same material, $E = 200\,\text{GPa}$ and $A = 0.001\,m^2$. The load $P = 20\,\text{kN}$ and acts at an angle $\theta = 30\,°$. The dimensions in meters are shown in the figure.





For numerical calculations the $N - \text{mm}$ units are convenient. The displacements will be in mm and the stresses in MPa. The complete computations are as follows.

Specified nodal loads

Node	dof	Value
2	u_2	10000.
	v_2	17320.5

Global equations at start of the element assembly process

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} u_1\\ v_1\\ u_2\\ v_2\\ u_3\\ v_3\\ u_4\\ v_4\\ u_5\\ v_5 \end{pmatrix} = \begin{pmatrix} 0\\ 0\\ 10000.\\ 17320.5\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0 \end{pmatrix}$$

Equations for element 1

$E = 200000 \qquad A = 1000.$

Element node	Global node number	x	y
1	1	0	0
2	2	4000.	0

$x_1 = 0 \qquad y_1 = 0 \qquad x_2 = 4000. \qquad y_2 = 0$

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 4000.$$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 1. \qquad m_s = \dfrac{y_2 - y_1}{L} = 0$

Substituting into the truss element equations we get

$$\begin{pmatrix} 50000. & 0. & -50000. & 0.\\ 0. & 0. & 0. & 0.\\ -50000. & 0. & 50000. & 0.\\ 0. & 0. & 0. & 0. \end{pmatrix} \begin{pmatrix} u_1\\ v_1\\ u_2\\ v_2 \end{pmatrix} = \begin{pmatrix} 0.\\ 0.\\ 0.\\ 0. \end{pmatrix}$$

The element contributes to {1, 2, 3, 4} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 1\\ 2\\ 3\\ 4 \end{pmatrix}$

and to a global matrix: $\begin{pmatrix} [1,1] & [1,2] & [1,3] & [1,4]\\ [2,1] & [2,2] & [2,3] & [2,4]\\ [3,1] & [3,2] & [3,3] & [3,4]\\ [4,1] & [4,2] & [4,3] & [4,4] \end{pmatrix}$

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 50000. & 0 & -50000. & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -50000. & 0 & 50000. & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 10000. \\ 17320.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 2

$E = 200000$ $\qquad$ $A = 1000.$

Element node	Global node number	x	y
1	2	4000.	0
2	5	2000.	2000.

$x_1 = 4000.$ $\qquad$ $y_1 = 0$ $\qquad$ $x_2 = 2000.$ $\qquad$ $y_2 = 2000.$

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 2828.43$$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = -0.707107$ $\qquad$ $m_s = \dfrac{y_2 - y_1}{L} = 0.707107$

Substituting into the truss element equations we get

$$\begin{pmatrix} 35355.3 & -35355.3 & -35355.3 & 35355.3 \\ -35355.3 & 35355.3 & 35355.3 & -35355.3 \\ -35355.3 & 35355.3 & 35355.3 & -35355.3 \\ 35355.3 & -35355.3 & -35355.3 & 35355.3 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ u_5 \\ v_5 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {3, 4, 9, 10} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 3 \\ 4 \\ 9 \\ 10 \end{pmatrix}$

and to a global matrix: $\begin{pmatrix} [3, 3] & [3, 4] & [3, 9] & [3, 10] \\ [4, 3] & [4, 4] & [4, 9] & [4, 10] \\ [9, 3] & [9, 4] & [9, 9] & [9, 10] \\ [10, 3] & [10, 4] & [10, 9] & [10, 10] \end{pmatrix}$

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 50000. & 0 & -50000. & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -50000. & 0 & 85355.3 & -35355.3 & 0 & 0 & 0 & 0 & -35355.3 & 35355.3 \\ 0 & 0 & -35355.3 & 35355.3 & 0 & 0 & 0 & 0 & 35355.3 & -35355.3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -35355.3 & 35355.3 & 0 & 0 & 0 & 0 & 35355.3 & -35355.3 \\ 0 & 0 & 35355.3 & -35355.3 & 0 & 0 & 0 & 0 & -35355.3 & 35355.3 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 10000. \\ 17320.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 3

$E = 200000 \qquad A = 1000.$

Element node	Global node number	x	y
1	5	2000.	2000.
2	3	0	3000.

$x_1 = 2000. \qquad y_1 = 2000. \qquad x_2 = 0 \qquad y_2 = 3000.$

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 2236.07$$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = -0.894427 \qquad m_s = \dfrac{y_2 - y_1}{L} = 0.447214$

Substituting into the truss element equations we get

$$\begin{pmatrix} 71554.2 & -35777.1 & -71554.2 & 35777.1 \\ -35777.1 & 17888.5 & 35777.1 & -17888.5 \\ -71554.2 & 35777.1 & 71554.2 & -35777.1 \\ 35777.1 & -17888.5 & -35777.1 & 17888.5 \end{pmatrix} \begin{pmatrix} u_5 \\ v_5 \\ u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {9, 10, 5, 6} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 9 \\ 10 \\ 5 \\ 6 \end{pmatrix}$

and to a global matrix: $\begin{pmatrix} [9, 9] & [9, 10] & [9, 5] & [9, 6] \\ [10, 9] & [10, 10] & [10, 5] & [10, 6] \\ [5, 9] & [5, 10] & [5, 5] & [5, 6] \\ [6, 9] & [6, 10] & [6, 5] & [6, 6] \end{pmatrix}$

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 50000. & 0 & -50000. & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -50000. & 0 & 85355.3 & -35355.3 & 0 & 0 & 0 & 0 & -35355.3 & 35355.3 \\ 0 & 0 & -35355.3 & 35355.3 & 0 & 0 & 0 & 0 & 35355.3 & -35355.3 \\ 0 & 0 & 0 & 0 & 71554.2 & -35777.1 & 0 & 0 & -71554.2 & 35777.1 \\ 0 & 0 & 0 & 0 & -35777.1 & 17888.5 & 0 & 0 & 35777.1 & -17888.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -35355.3 & 35355.3 & -71554.2 & 35777.1 & 0 & 0 & 106910. & -71132.4 \\ 0 & 0 & 35355.3 & -35355.3 & 35777.1 & -17888.5 & 0 & 0 & -71132.4 & 53243.9 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 10000. \\ 17320.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 4

$E = 200000 \qquad A = 1000.$

Element node	Global node number	x	y
1	2	4000.	0
2	4	4000.	3000.

$x_1 = 4000. \qquad y_1 = 0 \qquad x_2 = 4000. \qquad y_2 = 3000.$

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 3000.$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 0. \qquad m_s = \dfrac{y_2 - y_1}{L} = 1.$

Substituting into the truss element equations we get

$$\begin{pmatrix} 0. & 0. & 0. & 0. \\ 0. & 66666.7 & 0. & -66666.7 \\ 0. & 0. & 0. & 0. \\ 0. & -66666.7 & 0. & 66666.7 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {3, 4, 7, 8} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 3 \\ 4 \\ 7 \\ 8 \end{pmatrix}$

and to a global matrix: $\begin{pmatrix} [3, 3] & [3, 4] & [3, 7] & [3, 8] \\ [4, 3] & [4, 4] & [4, 7] & [4, 8] \\ [7, 3] & [7, 4] & [7, 7] & [7, 8] \\ [8, 3] & [8, 4] & [8, 7] & [8, 8] \end{pmatrix}$

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 50000. & 0 & -50000. & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -50000. & 0 & 85355.3 & -35355.3 & 0 & 0 & 0 & 0 & -35355.3 & 35355.3 \\ 0 & 0 & -35355.3 & 102022. & 0 & 0 & 0 & -66666.7 & 35355.3 & -35355.3 \\ 0 & 0 & 0 & 0 & 71554.2 & -35777.1 & 0 & 0 & -71554.2 & 35777.1 \\ 0 & 0 & 0 & 0 & -35777.1 & 17888.5 & 0 & 0 & 35777.1 & -17888.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -66666.7 & 0 & 0 & 0 & 66666.7 & 0 & 0 \\ 0 & 0 & -35355.3 & 35355.3 & -71554.2 & 35777.1 & 0 & 0 & 106910. & -71132.4 \\ 0 & 0 & 35355.3 & -35355.3 & 35777.1 & -17888.5 & 0 & 0 & -71132.4 & 53243.9 \end{pmatrix}$$

$$\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 10000. \\ 17320.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 5

$E = 200000$ $\quad$ $A = 1000.$

Element node	Global node number	x	y
1	1	0	0
2	5	2000.	2000.

$x_1 = 0$ $\quad$ $y_1 = 0$ $\quad$ $x_2 = 2000.$ $\quad$ $y_2 = 2000.$

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 2828.43$$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 0.707107$ $\qquad$ $m_s = \dfrac{y_2 - y_1}{L} = 0.707107$

Substituting into the truss element equations we get

$$\begin{pmatrix} 35355.3 & 35355.3 & -35355.3 & -35355.3 \\ 35355.3 & 35355.3 & -35355.3 & -35355.3 \\ -35355.3 & -35355.3 & 35355.3 & 35355.3 \\ -35355.3 & -35355.3 & 35355.3 & 35355.3 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_5 \\ v_5 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {1, 2, 9, 10} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 1 \\ 2 \\ 9 \\ 10 \end{pmatrix}$

and to a global matrix: $\begin{pmatrix} [1, 1] & [1, 2] & [1, 9] & [1, 10] \\ [2, 1] & [2, 2] & [2, 9] & [2, 10] \\ [9, 1] & [9, 2] & [9, 9] & [9, 10] \\ [10, 1] & [10, 2] & [10, 9] & [10, 10] \end{pmatrix}$

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 85355.3 & 35355.3 & -50000. & 0 & 0 & 0 & 0 & 0 & -35355.3 & -35355.3 \\ 35355.3 & 35355.3 & 0 & 0 & 0 & 0 & 0 & 0 & -35355.3 & -35355.3 \\ -50000. & 0 & 85355.3 & -35355.3 & 0 & 0 & 0 & 0 & -35355.3 & 35355.3 \\ 0 & 0 & -35355.3 & 102022. & 0 & 0 & 0 & -66666.7 & 35355.3 & -35355.3 \\ 0 & 0 & 0 & 0 & 71554.2 & -35777.1 & 0 & 0 & -71554.2 & 35777.1 \\ 0 & 0 & 0 & 0 & -35777.1 & 17888.5 & 0 & 0 & 35777.1 & -17888.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -66666.7 & 0 & 0 & 0 & 66666.7 & 0 & 0 \\ -35355.3 & -35355.3 & -35355.3 & 35355.3 & -71554.2 & 35777.1 & 0 & 0 & 142265. & -35777.1 \\ -35355.3 & -35355.3 & 35355.3 & -35355.3 & 35777.1 & -17888.5 & 0 & 0 & -35777.1 & 88599.2 \end{pmatrix}$$

$$\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 10000. \\ 17320.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 6

$E = 200000 \qquad A = 1000.$

Element node	Global node number	x	y
1	5	2000.	2000.
2	4	4000.	3000.

$x_1 = 2000. \qquad y_1 = 2000. \qquad x_2 = 4000. \qquad y_2 = 3000.$

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 2236.07$$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 0.894427 \qquad m_s = \dfrac{y_2 - y_1}{L} = 0.447214$

Substituting into the truss element equations we get

$$\begin{pmatrix} 71554.2 & 35777.1 & -71554.2 & -35777.1 \\ 35777.1 & 17888.5 & -35777.1 & -17888.5 \\ -71554.2 & -35777.1 & 71554.2 & 35777.1 \\ -35777.1 & -17888.5 & 35777.1 & 17888.5 \end{pmatrix} \begin{pmatrix} u_5 \\ v_5 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {9, 10, 7, 8} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 9 \\ 10 \\ 7 \\ 8 \end{pmatrix}$

and to a global matrix: $\begin{pmatrix} [9, 9] & [9, 10] & [9, 7] & [9, 8] \\ [10, 9] & [10, 10] & [10, 7] & [10, 8] \\ [7, 9] & [7, 10] & [7, 7] & [7, 8] \\ [8, 9] & [8, 10] & [8, 7] & [8, 8] \end{pmatrix}$

Adding element equations into appropriate locations we have

$$\left(\begin{matrix} 85355.3 & 35355.3 & -50000. & 0 & 0 & 0 & 0 & 0 & -35355.3 & -35355 \\ 35355.3 & 35355.3 & 0 & 0 & 0 & 0 & 0 & 0 & -35355.3 & -35355 \\ -50000. & 0 & 85355.3 & -35355.3 & 0 & 0 & 0 & 0 & -35355.3 & 35355 \\ 0 & 0 & -35355.3 & 102022. & 0 & 0 & 0 & -66666.7 & 35355.3 & -35355 \\ 0 & 0 & 0 & 0 & 71554.2 & -35777.1 & 0 & 0 & -71554.2 & 35777 \\ 0 & 0 & 0 & 0 & -35777.1 & 17888.5 & 0 & 0 & 35777.1 & -17888 \\ 0 & 0 & 0 & 0 & 0 & 0 & 71554.2 & 35777.1 & -71554.2 & -35777 \\ 0 & 0 & 0 & -66666.7 & 0 & 0 & 35777.1 & 84555.2 & -35777.1 & -17888 \\ -35355.3 & -35355.3 & -35355.3 & 35355.3 & -71554.2 & 35777.1 & -71554.2 & -35777.1 & 213819. & 0 \\ -35355.3 & -35355.3 & 35355.3 & -35355.3 & 35777.1 & -17888.5 & -35777.1 & -17888.5 & 0 & 106488 \end{matrix}\right.$$

Essential boundary conditions

Node	dof	Value
1	u_1	0
	v_1	0
3	u_3	0
	v_3	0
4	u_4	0
	v_4	0

Remove {1, 2, 5, 6, 7, 8} rows and columns.

After adjusting for essential boundary conditions we have

$$\begin{pmatrix} 85355.3 & -35355.3 & -35355.3 & 35355.3 \\ -35355.3 & 102022. & 35355.3 & -35355.3 \\ -35355.3 & 35355.3 & 213819. & 0 \\ 35355.3 & -35355.3 & 0 & 106488. \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ u_5 \\ v_5 \end{pmatrix} = \begin{pmatrix} 10000. \\ 17320.5 \\ 0 \\ 0 \end{pmatrix}$$

Solving the final system of global equations we get

$$\{u_2 = 0.213105,\ v_2 = 0.249979,\ u_5 = -0.00609705,\ v_5 = 0.0122424\}$$

Complete table of nodal values

	u	v
1	0	0
2	0.213105	0.249979
3	0	0
4	0	0
5	−0.00609705	0.0122424

Computation of reactions

Equation numbers of dof with specified values: {1, 2, 5, 6, 7, 8}

Extracting equations {1, 2, 5, 6, 7, 8} from the global system we have

$$\begin{pmatrix} 85355.3 & 35355.3 & -50000. & 0 & 0 & 0 & 0 & 0 & -35355.3 & -35355.3 \\ 35355.3 & 35355.3 & 0 & 0 & 0 & 0 & 0 & 0 & -35355.3 & -35355.3 \\ 0 & 0 & 0 & 0 & 71554.2 & -35777.1 & 0 & 0 & -71554.2 & 35777.1 \\ 0 & 0 & 0 & 0 & -35777.1 & 17888.5 & 0 & 0 & 35777.1 & -17888.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 71554.2 & 35777.1 & -71554.2 & -35777.1 \\ 0 & 0 & 0 & -66666.7 & 0 & 0 & 35777.1 & 84555.2 & -35777.1 & -17888.5 \end{pmatrix}$$

$$\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \end{pmatrix} = \begin{pmatrix} R_1 + 0. \\ R_2 + 0. \\ R_3 + 0. \\ R_4 + 0. \\ R_5 + 0. \\ R_6 + 0. \end{pmatrix}$$

Substituting the nodal values and re−arranging

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{pmatrix} =$$

$$\begin{pmatrix} 85355.3 & 35355.3 & -50000. & 0 & 0 & 0 & 0 & 0 & -35355.3 & -35355.3 \\ 35355.3 & 35355.3 & 0 & 0 & 0 & 0 & 0 & 0 & -35355.3 & -35355.3 \\ 0 & 0 & 0 & 0 & 71554.2 & -35777.1 & 0 & 0 & -71554.2 & 35777.1 \\ 0 & 0 & 0 & 0 & -35777.1 & 17888.5 & 0 & 0 & 35777.1 & -17888.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 71554.2 & 35777.1 & -71554.2 & -35777.1 \\ 0 & 0 & 0 & -66666.7 & 0 & 0 & 35777.1 & 84555.2 & -35777.1 & -17888.5 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0.213105 \\ 0.249979 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.00609705 \\ 0.0122424 \end{pmatrix}$$

Carrying out computations, the reactions are as follows.

Label	dof	Reaction
R_1	u_1	−10872.5
R_2	v_1	−217.271
R_3	u_3	874.267
R_4	v_3	−437.133
R_5	u_4	−1.72786
R_6	v_4	−16666.1

Sum of Reactions

dof: u	−10000.
dof: v	−17320.5

Solution for element 1

Nodal coordinates

Element node	Global node number	x	y
1	1	0	0
2	2	4000.	0

$x_1 = 0 \qquad y_1 = 0 \qquad x_2 = 4000. \qquad y_2 = 0$

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 4000.$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 1. \qquad m_s = \dfrac{y_2 - y_1}{L} = 0$

Global to local transformation matrix, $\boldsymbol{T} = \begin{pmatrix} 1. & 0 & 0 & 0 \\ 0 & 0 & 1. & 0 \end{pmatrix}$

Element nodal displacements in global coordinates, $\boldsymbol{d} = \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0.213105 \\ 0.249979 \end{pmatrix}$

Element nodal displacements in local coordinates, $\boldsymbol{d}_\ell = \boldsymbol{T}\,\boldsymbol{d} = \begin{pmatrix} 0. \\ 0.213105 \end{pmatrix}$

Axial displacements at element ends, $d_1 = 0. \qquad d_2 = 0.213105$

$E = 200000 \qquad A = 1000.$

Axial strain, $\epsilon = (d_2 - d_1)/L = 0.0000532763$

Axial stress, $\sigma = E\epsilon = 10.6553 \qquad$ Axial force $= \sigma A = 10655.3$

Solution for element 2

Nodal coordinates

Element node	Global node number	x	y
1	2	4000.	0
2	5	2000.	2000.

$x_1 = 4000. \qquad y_1 = 0 \qquad x_2 = 2000. \qquad y_2 = 2000.$

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 2828.43$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = -0.707107 \qquad m_s = \dfrac{y_2 - y_1}{L} = 0.707107$

Global to local transformation matrix, $\boldsymbol{T} = \begin{pmatrix} -0.707107 & 0.707107 & 0 & 0 \\ 0 & 0 & -0.707107 & 0.707107 \end{pmatrix}$

Element nodal displacements in global coordinates, $\boldsymbol{d} = \begin{pmatrix} u_2 \\ v_2 \\ u_5 \\ v_5 \end{pmatrix} = \begin{pmatrix} 0.213105 \\ 0.249979 \\ -0.00609705 \\ 0.0122424 \end{pmatrix}$

Element nodal displacements in local coordinates, $\boldsymbol{d}_\ell = \boldsymbol{T}\ \boldsymbol{d} = \begin{pmatrix} 0.0260733 \\ 0.0129679 \end{pmatrix}$

Axial displacements at element ends, $d_1 = 0.0260733$ $\quad d_2 = 0.0129679$

$E = 200000$ $\quad A = 1000.$

Axial strain, $\epsilon = (d_2 - d_1)/L = -4.63345 \times 10^{-6}$

Axial stress, $\sigma = E\epsilon = -0.926689$ $\quad$ Axial force $= \sigma A = -926.689$

Solution for element 3

Nodal coordinates

Element node	Global node number	x	y
1	5	2000.	2000.
2	3	0	3000.

$x_1 = 2000.$ $\quad y_1 = 2000.$ $\quad x_2 = 0$ $\quad y_2 = 3000.$

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 2236.07$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = -0.894427$ $\quad m_s = \dfrac{y_2 - y_1}{L} = 0.447214$

Global to local transformation matrix, $\boldsymbol{T} = \begin{pmatrix} -0.894427 & 0.447214 & 0 & 0 \\ 0 & 0 & -0.894427 & 0.447214 \end{pmatrix}$

Element nodal displacements in global coordinates, $\boldsymbol{d} = \begin{pmatrix} u_5 \\ v_5 \\ u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} -0.00609705 \\ 0.0122424 \\ 0 \\ 0 \end{pmatrix}$

Element nodal displacements in local coordinates, $\boldsymbol{d}_\ell = \boldsymbol{T}\ \boldsymbol{d} = \begin{pmatrix} 0.0109283 \\ 0. \end{pmatrix}$

Axial displacements at element ends, $d_1 = 0.0109283$ $\quad d_2 = 0.$

$E = 200000$ $\quad A = 1000.$

Axial strain, $\epsilon = (d_2 - d_1)/L = -4.8873 \times 10^{-6}$

Axial stress, $\sigma = E\epsilon = -0.97746$ Axial force $= \sigma A = -977.46$

Solution for element 4

Nodal coordinates

Element node	Global node number	x	y
1	2	4000.	0
2	4	4000.	3000.

$x_1 = 4000.$ $y_1 = 0$ $x_2 = 4000.$ $y_2 = 3000.$

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 3000.$

Direction cosines: $\ell_s = \frac{x_2 - x_1}{L} = 0.$ $m_s = \frac{y_2 - y_1}{L} = 1.$

Global to local transformation matrix, $\boldsymbol{T} = \begin{pmatrix} 0. & 1. & 0 & 0 \\ 0 & 0 & 0. & 1. \end{pmatrix}$

Element nodal displacements in global coordinates, $\boldsymbol{d} = \begin{pmatrix} u_2 \\ v_2 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0.213105 \\ 0.249979 \\ 0 \\ 0 \end{pmatrix}$

Element nodal displacements in local coordinates, $\boldsymbol{d}_\ell = \boldsymbol{T}\ \boldsymbol{d} = \begin{pmatrix} 0.249979 \\ 0. \end{pmatrix}$

Axial displacements at element ends, $d_1 = 0.249979$ $d_2 = 0.$

$E = 200000$ $A = 1000.$

Axial strain, $\epsilon = (d_2 - d_1)/L = -0.0000833262$

Axial stress, $\sigma = E\epsilon = -16.6652$ Axial force $= \sigma A = -16665.2$

Solution for element 5

Nodal coordinates

Element node	Global node number	x	y
1	1	0	0
2	5	2000.	2000.

$x_1 = 0$ $y_1 = 0$ $x_2 = 2000.$ $y_2 = 2000.$

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 2828.43$

Direction cosines: $\ell_s = \frac{x_2 - x_1}{L} = 0.707107$ $m_s = \frac{y_2 - y_1}{L} = 0.707107$

Global to local transformation matrix, $\boldsymbol{T} = \begin{pmatrix} 0.707107 & 0.707107 & 0 & 0 \\ 0 & 0 & 0.707107 & 0.707107 \end{pmatrix}$

Element nodal displacements in global coordinates, $\boldsymbol{d} = \begin{pmatrix} u_1 \\ v_1 \\ u_5 \\ v_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -0.00609705 \\ 0.0122424 \end{pmatrix}$

Element nodal displacements in local coordinates, $\boldsymbol{d}_\ell = \boldsymbol{T}\ \boldsymbol{d} = \begin{pmatrix} 0. \\ 0.00434542 \end{pmatrix}$

Axial displacements at element ends, $d_1 = 0.$ $\quad d_2 = 0.00434542$

$E = 200000 \quad A = 1000.$

Axial strain, $\epsilon = (d_2 - d_1)/L = 1.53634 \times 10^{-6}$

Axial stress, $\sigma = E\epsilon = 0.307267$ $\quad$ Axial force $= \sigma A = 307.267$

Solution for element 6

Nodal coordinates

Element node	Global node number	x	y
1	5	2000.	2000.
2	4	4000.	3000.

$x_1 = 2000. \quad y_1 = 2000. \quad x_2 = 4000. \quad y_2 = 3000.$

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 2236.07$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 0.894427$ $\quad m_s = \dfrac{y_2 - y_1}{L} = 0.447214$

Global to local transformation matrix, $\boldsymbol{T} = \begin{pmatrix} 0.894427 & 0.447214 & 0 & 0 \\ 0 & 0 & 0.894427 & 0.447214 \end{pmatrix}$

Element nodal displacements in global coordinates, $\boldsymbol{d} = \begin{pmatrix} u_5 \\ v_5 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} -0.00609705 \\ 0.0122424 \\ 0 \\ 0 \end{pmatrix}$

Element nodal displacements in local coordinates, $\boldsymbol{d}_\ell = \boldsymbol{T}\ \boldsymbol{d} = \begin{pmatrix} 0.0000215983 \\ 0. \end{pmatrix}$

Axial displacements at element ends, $d_1 = 0.0000215983$ $\quad d_2 = 0.$

$E = 200000 \quad A = 1000.$

Axial strain, $\epsilon = (d_2 - d_1)/L = -9.65904 \times 10^{-9}$

Axial stress, $\sigma = E\epsilon = -0.00193181$ Axial force $= \sigma A = -1.93181$

Solution summary

Nodal solution

	x−coord	y−coord	u	v
1	0	0	0	0
2	4000.	0	0.213105	0.249979
3	0	3000.	0	0
4	4000.	3000.	0	0
5	2000.	2000.	−0.00609705	0.0122424

Element solution

	Stress	Axial force
1	10.6553	10655.3
2	−0.926689	−926.689
3	−0.97746	−977.46
4	−16.6652	−16665.2
5	0.307267	307.267
6	−0.00193181	−1.93181

Support reactions

Node	dof	Reaction
1	u_1	−10872.5
1	v_1	−217.271
3	u_3	874.267
3	v_3	−437.133
4	u_4	−1.72786
4	v_4	−16666.1

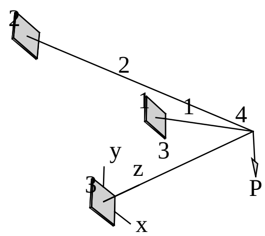
Sum of applied loads → (10000. 17320.5)

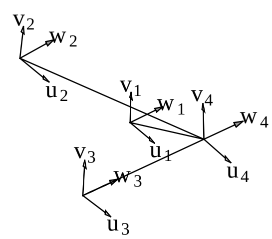
Sum of support reactions → (−10000. −17320.5)

Example 4.2: Three-bar truss (p. 230)

The cross-sectional areas of elements 1 and 2 is 200 mm^2 and that of element 3 is 600 mm^2. All elements are made of the same material with $E = 200$ GPa. The applied load is $P = 20$ kN. The nodal coordinates are as follows.

Node	$x(m)$	$y(m)$	$z(m)$
1	0.96	1.92	0
2	−1.44	1.44	0
3	0	0	0
4	0	0	2





The complete computations are as follows. The numerical values are in the N – mm units. The computed displacements are in mm and the stresses in MPa.

Specified nodal loads

Node	dof	Value
	u_4	0
4	v_4	−20000
	w_4	0

Global equations at start of the element assembly process

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} u_1\\ v_1\\ w_1\\ u_2\\ v_2\\ w_2\\ u_3\\ v_3\\ w_3\\ u_4\\ v_4\\ w_4 \end{pmatrix} = \begin{pmatrix} 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ -20000\\ 0 \end{pmatrix}$$

Equations for element 1

$E = 210000 \qquad A = 200$

Element node	Global node number	x	y	z
1	1	960.	1920.	0
2	4	0	0	2000.

Direction cosines, $\ell_s = -0.327205 \qquad m_s = -0.65441 \qquad n_s = 0.681677$

Substituting into the truss element equations we get

$$\begin{pmatrix} 1532.63 & 3065.27 & -3192.99 & -1532.63 & -3065.27 & 3192.99\\ 3065.27 & 6130.53 & -6385.97 & -3065.27 & -6130.53 & 6385.97\\ -3192.99 & -6385.97 & 6652.06 & 3192.99 & 6385.97 & -6652.06\\ -1532.63 & -3065.27 & 3192.99 & 1532.63 & 3065.27 & -3192.99\\ -3065.27 & -6130.53 & 6385.97 & 3065.27 & 6130.53 & -6385.97\\ 3192.99 & 6385.97 & -6652.06 & -3192.99 & -6385.97 & 6652.06 \end{pmatrix} \begin{pmatrix} u_1\\ v_1\\ w_1\\ u_4\\ v_4\\ w_4 \end{pmatrix} = \begin{pmatrix} 0.\\ 0.\\ 0.\\ 0.\\ 0.\\ 0. \end{pmatrix}$$

The element contributes to {1, 2, 3, 10, 11, 12} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 1532.63 & 3065.27 & -3192.99 & 0 & 0 & 0 & 0 & 0 & 0 & -1532.63 & -3065.27 & 3192.99 \\ 3065.27 & 6130.53 & -6385.97 & 0 & 0 & 0 & 0 & 0 & 0 & -3065.27 & -6130.53 & 6385.97 \\ -3192.99 & -6385.97 & 6652.06 & 0 & 0 & 0 & 0 & 0 & 0 & 3192.99 & 6385.97 & -6652.06 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1532.63 & -3065.27 & 3192.99 & 0 & 0 & 0 & 0 & 0 & 0 & 1532.63 & 3065.27 & -3192.99 \\ -3065.27 & -6130.53 & 6385.97 & 0 & 0 & 0 & 0 & 0 & 0 & 3065.27 & 6130.53 & -6385.97 \\ 3192.99 & 6385.97 & -6652.06 & 0 & 0 & 0 & 0 & 0 & 0 & -3192.99 & -6385.97 & 6652.06 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ u_3 \\ v_3 \\ w_3 \\ u_4 \\ v_4 \\ w_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -20000. \\ 0 \end{pmatrix}$$

Equations for element 2

$E = 210000$ $\quad A = 200$

Element node	Global node number	x	y	z
1	2	−1440.	1440.	0
2	4	0	0	2000.

Direction cosines, $\ell_s = 0.504497$ $\quad m_s = -0.504497$ $\quad n_s = 0.70069$

Substituting into the truss element equations we get

$$\begin{pmatrix} 3745.09 & -3745.09 & 5201.51 & -3745.09 & 3745.09 & -5201.51 \\ -3745.09 & 3745.09 & -5201.51 & 3745.09 & -3745.09 & 5201.51 \\ 5201.51 & -5201.51 & 7224.32 & -5201.51 & 5201.51 & -7224.32 \\ -3745.09 & 3745.09 & -5201.51 & 3745.09 & -3745.09 & 5201.51 \\ 3745.09 & -3745.09 & 5201.51 & -3745.09 & 3745.09 & -5201.51 \\ -5201.51 & 5201.51 & -7224.32 & 5201.51 & -5201.51 & 7224.32 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ w_2 \\ u_4 \\ v_4 \\ w_4 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {4, 5, 6, 10, 11, 12} global degrees of freedom.

Adding element equations into appropriate locations we have

$$
\left(\begin{array}{cccccccccccc}
1532.63 & 3065.27 & -3192.99 & 0 & 0 & 0 & 0 & 0 & 0 & -1532.63 & -3065.27 & 319 \\
3065.27 & 6130.53 & -6385.97 & 0 & 0 & 0 & 0 & 0 & 0 & -3065.27 & -6130.53 & 638 \\
-3192.99 & -6385.97 & 6652.06 & 0 & 0 & 0 & 0 & 0 & 0 & 3192.99 & 6385.97 & -665 \\
0 & 0 & 0 & 3745.09 & -3745.09 & 5201.51 & 0 & 0 & 0 & -3745.09 & 3745.09 & -520 \\
0 & 0 & 0 & -3745.09 & 3745.09 & -5201.51 & 0 & 0 & 0 & 3745.09 & -3745.09 & 520 \\
0 & 0 & 0 & 5201.51 & -5201.51 & 7224.32 & 0 & 0 & 0 & -5201.51 & 5201.51 & -722 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
-1532.63 & -3065.27 & 3192.99 & -3745.09 & 3745.09 & -5201.51 & 0 & 0 & 0 & 5277.72 & -679.818 & 200 \\
-3065.27 & -6130.53 & 6385.97 & 3745.09 & -3745.09 & 5201.51 & 0 & 0 & 0 & -679.818 & 9875.62 & -1158 \\
3192.99 & 6385.97 & -6652.06 & -5201.51 & 5201.51 & -7224.32 & 0 & 0 & 0 & 2008.52 & -11587.5 & 1387
\end{array}\right.
$$

Equations for element 3

$E = 210000$ $\qquad$ $A = 600$

Element node	Global node number	x	y	z
1	3	0	0	0
2	4	0	0	2000.

Direction cosines, $\ell_s = 0$ $\qquad m_s = 0$ $\qquad n_s = 1.$

Substituting into the truss element equations we get

$$
\begin{pmatrix}
0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 63000. & 0. & 0. & -63000. \\
0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & -63000. & 0. & 0. & 63000.
\end{pmatrix}
\begin{pmatrix} u_3 \\ v_3 \\ w_3 \\ u_4 \\ v_4 \\ w_4 \end{pmatrix}
=
\begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}
$$

The element contributes to {7, 8, 9, 10, 11, 12} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\left(\begin{array}{ccccccccccc}
1532.63 & 3065.27 & -3192.99 & 0 & 0 & 0 & 0 & 0 & 0 & -1532.63 & -3065.27 \\
3065.27 & 6130.53 & -6385.97 & 0 & 0 & 0 & 0 & 0 & 0 & -3065.27 & -6130.53 \\
-3192.99 & -6385.97 & 6652.06 & 0 & 0 & 0 & 0 & 0 & 0 & 3192.99 & 6385.97 \\
0 & 0 & 0 & 3745.09 & -3745.09 & 5201.51 & 0 & 0 & 0 & -3745.09 & 3745.09 \\
0 & 0 & 0 & -3745.09 & 3745.09 & -5201.51 & 0 & 0 & 0 & 3745.09 & -3745.09 \\
0 & 0 & 0 & 5201.51 & -5201.51 & 7224.32 & 0 & 0 & 0 & -5201.51 & 5201.51 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 63000. & 0 & 0 \\
-1532.63 & -3065.27 & 3192.99 & -3745.09 & 3745.09 & -5201.51 & 0 & 0 & 0 & 5277.72 & -679.818 \\
-3065.27 & -6130.53 & 6385.97 & 3745.09 & -3745.09 & 5201.51 & 0 & 0 & 0 & -679.818 & 9875.62 \\
3192.99 & 6385.97 & -6652.06 & -5201.51 & 5201.51 & -7224.32 & 0 & 0 & -63000. & 2008.52 & -11587.5
\end{array}\right.$$

Essential boundary conditions

Node	dof	Value
1	u_1	0
	v_1	0
	w_1	0
2	u_2	0
	v_2	0
	w_2	0
3	u_3	0
	v_3	0
	w_3	0

Remove {1, 2, 3, 4, 5, 6, 7, 8, 9} rows and columns.

After adjusting for essential boundary conditions we have

$$\begin{pmatrix} 5277.72 & -679.818 & 2008.52 \\ -679.818 & 9875.62 & -11587.5 \\ 2008.52 & -11587.5 & 76876.4 \end{pmatrix} \begin{pmatrix} u_4 \\ v_4 \\ w_4 \end{pmatrix} = \begin{pmatrix} 0 \\ -20000. \\ 0 \end{pmatrix}$$

Solving the final system of global equations we get

$$\{u_4 = -0.178143,\ v_4 = -2.46857,\ w_4 = -0.367431\}$$

Complete table of nodal values

	u	v	w
1	0	0	0
2	0	0	0
3	0	0	0
4	−0.178143	−2.46857	−0.367431

Computation of reactions

Equation numbers of dof with specified values: {1, 2, 3, 4, 5, 6, 7, 8, 9}

Extracting equations {1, 2, 3, 4, 5, 6, 7, 8, 9} from the global system we have

$$\begin{pmatrix} 1532.63 & 3065.27 & -3192.99 & 0 & 0 & 0 & 0 & 0 & 0 & -1532.63 & -3065.27 & 3 \\ 3065.27 & 6130.53 & -6385.97 & 0 & 0 & 0 & 0 & 0 & 0 & -3065.27 & -6130.53 & 6 \\ -3192.99 & -6385.97 & 6652.06 & 0 & 0 & 0 & 0 & 0 & 0 & 3192.99 & 6385.97 & -6 \\ 0 & 0 & 0 & 3745.09 & -3745.09 & 5201.51 & 0 & 0 & 0 & -3745.09 & 3745.09 & -5 \\ 0 & 0 & 0 & -3745.09 & 3745.09 & -5201.51 & 0 & 0 & 0 & 3745.09 & -3745.09 & 5 \\ 0 & 0 & 0 & 5201.51 & -5201.51 & 7224.32 & 0 & 0 & 0 & -5201.51 & 5201.51 & -7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 63000. & 0 & 0 & -63 \end{pmatrix}$$

Substituting the nodal values and re−arranging

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \\ R_9 \end{pmatrix} = \begin{pmatrix} 1532.63 & 3065.27 & -3192.99 & 0 & 0 & 0 & 0 & 0 & 0 & -1532.63 & -3065. \\ 3065.27 & 6130.53 & -6385.97 & 0 & 0 & 0 & 0 & 0 & 0 & -3065.27 & -6130. \\ -3192.99 & -6385.97 & 6652.06 & 0 & 0 & 0 & 0 & 0 & 0 & 3192.99 & 6385. \\ 0 & 0 & 0 & 3745.09 & -3745.09 & 5201.51 & 0 & 0 & 0 & -3745.09 & 3745. \\ 0 & 0 & 0 & -3745.09 & 3745.09 & -5201.51 & 0 & 0 & 0 & 3745.09 & -3745. \\ 0 & 0 & 0 & 5201.51 & -5201.51 & 7224.32 & 0 & 0 & 0 & -5201.51 & 5201. \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 63000. & 0 & 0 \end{pmatrix}$$

Carrying out computations, the reactions are as follows.

Label	dof	Reaction
R_1	u_1	6666.67
R_2	v_1	13333.3
R_3	w_1	−13888.9
R_4	u_2	−6666.67
R_5	v_2	6666.67
R_6	w_2	−9259.26
R_7	u_3	0
R_8	v_3	0
R_9	w_3	23148.1

Sum of Reactions

dof: u	0
dof: v	20000.
dof: w	0

Solution for element 1

Nodal coordinates

Element node	Global node number	x	y	z
1	1	960.	1920.	0
2	4	0	0	2000.

Direction cosines, $\ell_s = -0.327205$ $m_s = -0.65441$ $n_s = 0.681677$

Global to local transformation matrix, $\boldsymbol{T}$ =

$$\begin{pmatrix} -0.327205 & -0.65441 & 0.681677 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.327205 & -0.65441 & 0.681677 \end{pmatrix}$$

$$\text{Element nodal displacements in global coordinates, } \boldsymbol{d} = \begin{pmatrix} u_1 \\ v_1 \\ w_1 \\ u_4 \\ v_4 \\ w_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -0.178143 \\ -2.46857 \\ -0.367431 \end{pmatrix}$$

$$\text{Element nodal displacements in local coordinates, } \boldsymbol{d}_\ell = \boldsymbol{T}\,\boldsymbol{d} = \begin{pmatrix} 0. \\ 1.42328 \end{pmatrix}$$

Axial displacements at element ends, $d_1 = 0.$ $d_2 = 1.42328$

$E = 210000$ $A = 200$

Axial strain, $\epsilon = (d_2 - d_1)/L = 0.000485109$

Axial stress, $\sigma = E\epsilon = 101.873$ Axial force $= \sigma A = 20374.6$

Solution for element 2

Nodal coordinates

Element node	Global node number	x	y	z
1	2	−1440.	1440.	0
2	4	0	0	2000.

Direction cosines, $\ell_s = 0.504497$ $m_s = -0.504497$ $n_s = 0.70069$

Global to local transformation matrix, $\boldsymbol{T}$ =

$$\begin{pmatrix} 0.504497 & -0.504497 & 0.70069 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.504497 & -0.504497 & 0.70069 \end{pmatrix}$$

Element nodal displacements in global coordinates, $\boldsymbol{d} = \begin{pmatrix} u_2 \\ v_2 \\ w_2 \\ u_4 \\ v_4 \\ w_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -0.178143 \\ -2.46857 \\ -0.367431 \end{pmatrix}$

Element nodal displacements in local coordinates, $\boldsymbol{d}_\ell = \boldsymbol{T}\ \boldsymbol{d} = \begin{pmatrix} 0. \\ 0.89806 \end{pmatrix}$

Axial displacements at element ends, $d_1 = 0.$ $d_2 = 0.89806$

$E = 210000$ $A = 200$

Axial strain, $\epsilon = (d_2 - d_1)/L = 0.000314631$

Axial stress, $\sigma = E\epsilon = 66.0725$ Axial force $= \sigma A = 13214.5$

Solution for element 3

Nodal coordinates

Element node	Global node number	x	y	z
1	3	0	0	0
2	4	0	0	2000.

Direction cosines, $\ell_s = 0$ $m_s = 0$ $n_s = 1.$

Global to local transformation matrix, $\boldsymbol{T} = \begin{pmatrix} 0 & 0 & 1. & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1. \end{pmatrix}$

Element nodal displacements in global coordinates, $\boldsymbol{d} = \begin{pmatrix} u_3 \\ v_3 \\ w_3 \\ u_4 \\ v_4 \\ w_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -0.178143 \\ -2.46857 \\ -0.367431 \end{pmatrix}$

Element nodal displacements in local coordinates, $\boldsymbol{d}_\ell = \boldsymbol{T}\ \boldsymbol{d} = \begin{pmatrix} 0. \\ -0.367431 \end{pmatrix}$

Axial displacements at element ends, $d_1 = 0.$ $d_2 = -0.367431$

$E = 210000$ $A = 600$

Axial strain, $\epsilon = (d_2 - d_1)/L = -0.000183715$

Axial stress, $\sigma = E\epsilon = -38.5802$ Axial force $= \sigma A = -23148.1$

Solution summary

Nodal solution

	x−coord	y−coord	z−coord	u	v	w
1	960.	1920.	0	0	0	0
2	−1440.	1440.	0	0	0	0
3	0	0	0	0	0	0
4	0	0	2000.	−0.178143	−2.46857	−0.367431

Element solution

	Stress	Axial force
1	101.873	20374.6
2	66.0725	13214.5
3	−38.5802	−23148.1

Support reactions

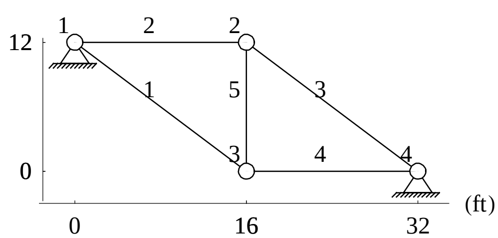
Node	dof	Reaction
1	u_1	6666.67
1	v_1	13333.3
1	w_1	−13888.9
2	u_2	−6666.67
2	v_2	6666.67
2	w_2	−9259.26
3	u_3	0.
3	v_3	0.
3	w_3	23148.1

Sum of applied loads → (0 −20000. 0)

Sum of support reactions → (0 20000. 0)

Example 4.3: Plane truss with temperature change (p. 233)

All members have the same cross-sectional area $A = 1/2\,\text{in}^2$ and are of the same material $E = 29,000$ ksi and $\alpha = 6.5\times10^{-6}/$°F. The first element undergoes a temperature rise of 100°F. The dimensions are shown in the figure.



For numerical calculations the k − in units are used.

Global equations at start of the element assembly process

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 1

$E = 29000 \qquad A = \frac{1}{2}$

$\alpha = 6.5 \times 10^{-6} \qquad \Delta T = 100 \qquad \epsilon_0 = 0.00065$

Element node	Global node number	x	y
1	1	0	144.
2	3	192.	0

$x_1 = 0 \qquad y_1 = 144. \qquad x_2 = 192. \qquad y_2 = 0$

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 240.$

Direction cosines: $\ell_s = \frac{x_2 - x_1}{L} = 0.8 \qquad m_s = \frac{y_2 - y_1}{L} = -0.6$

Substituting into the truss element equations we get

$$\begin{pmatrix} 38.6667 & -29. & -38.6667 & 29. \\ -29. & 21.75 & 29. & -21.75 \\ -38.6667 & 29. & 38.6667 & -29. \\ 29. & -21.75 & -29. & 21.75 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} -7.54 \\ 5.655 \\ 7.54 \\ -5.655 \end{pmatrix}$$

The element contributes to {1, 2, 5, 6} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 38.6667 & -29. & 0 & 0 & -38.6667 & 29. & 0 & 0 \\ -29. & 21.75 & 0 & 0 & 29. & -21.75 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -38.6667 & 29. & 0 & 0 & 38.6667 & -29. & 0 & 0 \\ 29. & -21.75 & 0 & 0 & -29. & 21.75 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} -7.54 \\ 5.655 \\ 0 \\ 0 \\ 7.54 \\ -5.655 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 2

$E = 29000 \qquad A = \frac{1}{2}$

Element node	Global node number	x	y
1	1	0	144.
2	2	192.	144.

$x_1 = 0 \qquad y_1 = 144. \qquad x_2 = 192. \qquad y_2 = 144.$

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 192.$

Direction cosines: $\ell_s = \frac{x_2 - x_1}{L} = 1. \qquad m_s = \frac{y_2 - y_1}{L} = 0.$

Substituting into the truss element equations we get

$$\begin{pmatrix} 75.5208 & 0. & -75.5208 & 0. \\ 0. & 0. & 0. & 0. \\ -75.5208 & 0. & 75.5208 & 0. \\ 0. & 0. & 0. & 0. \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {1, 2, 3, 4} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 114.188 & -29. & -75.5208 & 0 & -38.6667 & 29. & 0 & 0 \\ -29. & 21.75 & 0 & 0 & 29. & -21.75 & 0 & 0 \\ -75.5208 & 0 & 75.5208 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -38.6667 & 29. & 0 & 0 & 38.6667 & -29. & 0 & 0 \\ 29. & -21.75 & 0 & 0 & -29. & 21.75 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} -7.54 \\ 5.655 \\ 0 \\ 0 \\ 7.54 \\ -5.655 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 3

$E = 29000 \qquad A = \frac{1}{2}$

Element node	Global node number	x	y
1	2	192.	144.
2	4	384.	0

$x_1 = 192. \qquad y_1 = 144. \qquad x_2 = 384. \qquad y_2 = 0$

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 240.$

Direction cosines: $\ell_s = \frac{x_2 - x_1}{L} = 0.8 \qquad m_s = \frac{y_2 - y_1}{L} = -0.6$

Substituting into the truss element equations we get

$$\begin{pmatrix} 38.6667 & -29. & -38.6667 & 29. \\ -29. & 21.75 & 29. & -21.75 \\ -38.6667 & 29. & 38.6667 & -29. \\ 29. & -21.75 & -29. & 21.75 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {3, 4, 7, 8} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 114.188 & -29. & -75.5208 & 0 & -38.6667 & 29. & 0 & 0 \\ -29. & 21.75 & 0 & 0 & 29. & -21.75 & 0 & 0 \\ -75.5208 & 0 & 114.188 & -29. & 0 & 0 & -38.6667 & 29. \\ 0 & 0 & -29. & 21.75 & 0 & 0 & 29. & -21.75 \\ -38.6667 & 29. & 0 & 0 & 38.6667 & -29. & 0 & 0 \\ 29. & -21.75 & 0 & 0 & -29. & 21.75 & 0 & 0 \\ 0 & 0 & -38.6667 & 29. & 0 & 0 & 38.6667 & -29. \\ 0 & 0 & 29. & -21.75 & 0 & 0 & -29. & 21.75 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} -7.54 \\ 5.655 \\ 0 \\ 0 \\ 7.54 \\ -5.655 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 4

$E = 29000 \qquad A = \frac{1}{2}$

Element node	Global node number	x	y
1	3	192.	0
2	4	384.	0

$x_1 = 192. \qquad y_1 = 0 \qquad x_2 = 384. \qquad y_2 = 0$

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 192.$

Direction cosines: $\ell_s = \frac{x_2 - x_1}{L} = 1. \qquad m_s = \frac{y_2 - y_1}{L} = 0$

Substituting into the truss element equations we get

$$\begin{pmatrix} 75.5208 & 0. & -75.5208 & 0. \\ 0. & 0. & 0. & 0. \\ -75.5208 & 0. & 75.5208 & 0. \\ 0. & 0. & 0. & 0. \end{pmatrix} \begin{pmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {5, 6, 7, 8} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 114.188 & -29. & -75.5208 & 0 & -38.6667 & 29. & 0 & 0 \\ -29. & 21.75 & 0 & 0 & 29. & -21.75 & 0 & 0 \\ -75.5208 & 0 & 114.188 & -29. & 0 & 0 & -38.6667 & 29. \\ 0 & 0 & -29. & 21.75 & 0 & 0 & 29. & -21.75 \\ -38.6667 & 29. & 0 & 0 & 114.188 & -29. & -75.5208 & 0 \\ 29. & -21.75 & 0 & 0 & -29. & 21.75 & 0 & 0 \\ 0 & 0 & -38.6667 & 29. & -75.5208 & 0 & 114.188 & -29. \\ 0 & 0 & 29. & -21.75 & 0 & 0 & -29. & 21.75 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} -7.54 \\ 5.655 \\ 0 \\ 0 \\ 7.54 \\ -5.655 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 5

$E = 29000 \qquad A = \frac{1}{2}$

Element node	Global node number	x	y
1	2	192.	144.
2	3	192.	0

$x_1 = 192. \qquad y_1 = 144. \qquad x_2 = 192. \qquad y_2 = 0$

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 144.$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 0. \qquad m_s = \dfrac{y_2 - y_1}{L} = -1.$

Substituting into the truss element equations we get

$$\begin{pmatrix} 0. & 0. & 0. & 0. \\ 0. & 100.694 & 0. & -100.694 \\ 0. & 0. & 0. & 0. \\ 0. & -100.694 & 0. & 100.694 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {3, 4, 5, 6} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 114.188 & -29. & -75.5208 & 0 & -38.6667 & 29. & 0 & 0 \\ -29. & 21.75 & 0 & 0 & 29. & -21.75 & 0 & 0 \\ -75.5208 & 0 & 114.188 & -29. & 0 & 0 & -38.6667 & 29. \\ 0 & 0 & -29. & 122.444 & 0 & -100.694 & 29. & -21.75 \\ -38.6667 & 29. & 0 & 0 & 114.188 & -29. & -75.5208 & 0 \\ 29. & -21.75 & 0 & -100.694 & -29. & 122.444 & 0 & 0 \\ 0 & 0 & -38.6667 & 29. & -75.5208 & 0 & 114.188 & -29. \\ 0 & 0 & 29. & -21.75 & 0 & 0 & -29. & 21.75 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} -7.54 \\ 5.655 \\ 0 \\ 0 \\ 7.54 \\ -5.655 \\ 0 \\ 0 \end{pmatrix}$$

Essential boundary conditions

Node	dof	Value
1	u_1	0
	v_1	0
4	u_4	0
	v_4	0

Remove {1, 2, 7, 8} rows and columns.

After adjusting for essential boundary conditions we have

$$\begin{pmatrix} 114.188 & -29. & 0 & 0 \\ -29. & 122.444 & 0 & -100.694 \\ 0 & 0 & 114.188 & -29. \\ 0 & -100.694 & -29. & 122.444 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 7.54 \\ -5.655 \end{pmatrix}$$

Solving the final system of global equations we get

$$\{u_2 = -0.0308148,\ v_2 = -0.121333,\ u_3 = 0.0308148,\ v_3 = -0.138667\}$$

Complete table of nodal values

	u	v
1	0	0
2	−0.0308148	−0.121333
3	0.0308148	−0.138667
4	0	0

Computation of reactions

Equation numbers of dof with specified values: {1, 2, 7, 8}

Extracting equations {1, 2, 7, 8} from the global system we have

$$\begin{pmatrix} 114.188 & -29. & -75.5208 & 0 & -38.6667 & 29. & 0 & 0 \\ -29. & 21.75 & 0 & 0 & 29. & -21.75 & 0 & 0 \\ 0 & 0 & -38.6667 & 29. & -75.5208 & 0 & 114.188 & -29. \\ 0 & 0 & 29. & -21.75 & 0 & 0 & -29. & 21.75 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} R_1 - 7.54 \\ R_2 + 5.655 \\ R_3 + 0. \\ R_4 + 0. \end{pmatrix}$$

Substituting the nodal values and re−arranging

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{pmatrix} =$$

$$\begin{pmatrix} 114.188 & -29. & -75.5208 & 0 & -38.6667 & 29. & 0 & 0 \\ -29. & 21.75 & 0 & 0 & 29. & -21.75 & 0 & 0 \\ 0 & 0 & -38.6667 & 29. & -75.5208 & 0 & 114.188 & -29. \\ 0 & 0 & 29. & -21.75 & 0 & 0 & -29. & 21.75 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -0.0308148 \\ -0.121333 \\ 0.0308148 \\ -0.138667 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} -7.54 \\ 5.655 \\ 0. \\ 0. \end{pmatrix}$$

Carrying out computations, the reactions are as follows.

Label	dof	Reaction
R_1	u_1	4.65432
R_2	v_1	−1.74537
R_3	u_4	−4.65432
R_4	v_4	1.74537

Sum of Reactions

dof: u	0
dof: v	0

Solution for element 1

Nodal coordinates

Element node	Global node number	x	y
1	1	0	144.
2	3	192.	0

$x_1 = 0$ $\quad y_1 = 144.$ $\quad x_2 = 192.$ $\quad y_2 = 0$

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 240.$$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 0.8$ $\qquad m_s = \dfrac{y_2 - y_1}{L} = -0.6$

Global to local transformation matrix, $\boldsymbol{T} = \begin{pmatrix} 0.8 & -0.6 & 0 & 0 \\ 0 & 0 & 0.8 & -0.6 \end{pmatrix}$

Element nodal displacements in global coordinates, $\boldsymbol{d} = \begin{pmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0.0308148 \\ -0.138667 \end{pmatrix}$

Element nodal displacements in local coordinates, $\boldsymbol{d}_\ell = \boldsymbol{T}\ \boldsymbol{d} = \begin{pmatrix} 0. \\ 0.107852 \end{pmatrix}$

Axial displacements at element ends, $d_1 = 0.$ $\quad d_2 = 0.107852$

$E = 29000 \quad A = \frac{1}{2}$

$\alpha = 6.5 \times 10^{-6} \quad \Delta T = 100 \quad \epsilon_0 = 0.00065$

Axial strain, $\epsilon = (d_2 - d_1)/L - \epsilon_0 = 0.000449383$

Axial stress, $\sigma = E\epsilon = -5.8179$ $\quad$ Axial force $= \sigma A = -2.90895$

Solution for element 2

Nodal coordinates

Element node	Global node number	x	y
1	1	0	144.
2	2	192.	144.

$x_1 = 0 \quad y_1 = 144. \quad x_2 = 192. \quad y_2 = 144.$

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 192.$

Direction cosines: $\ell_s = \frac{x_2 - x_1}{L} = 1. \quad m_s = \frac{y_2 - y_1}{L} = 0.$

Global to local transformation matrix, $\boldsymbol{T} = \begin{pmatrix} 1. & 0. & 0 & 0 \\ 0 & 0 & 1. & 0. \end{pmatrix}$

Element nodal displacements in global coordinates, $\boldsymbol{d} = \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -0.0308148 \\ -0.121333 \end{pmatrix}$

Element nodal displacements in local coordinates, $\boldsymbol{d}_\ell = \boldsymbol{T}\ \boldsymbol{d} = \begin{pmatrix} 0. \\ -0.0308148 \end{pmatrix}$

Axial displacements at element ends, $d_1 = 0.$ $\quad d_2 = -0.0308148$

$E = 29000 \quad A = \frac{1}{2}$

Axial strain, $\epsilon = (d_2 - d_1)/L = -0.000160494$

Axial stress, $\sigma = E\epsilon = -4.65432$ Axial force $= \sigma A = -2.32716$

Solution for element 3

Nodal coordinates

Element node	Global node number	x	y
1	2	192.	144.
2	4	384.	0

$x_1 = 192.$ $y_1 = 144.$ $x_2 = 384.$ $y_2 = 0$

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 240.$$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 0.8$ $m_s = \dfrac{y_2 - y_1}{L} = -0.6$

Global to local transformation matrix, $\boldsymbol{T} = \begin{pmatrix} 0.8 & -0.6 & 0 & 0 \\ 0 & 0 & 0.8 & -0.6 \end{pmatrix}$

Element nodal displacements in global coordinates, $\boldsymbol{d} = \begin{pmatrix} u_2 \\ v_2 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} -0.0308148 \\ -0.121333 \\ 0 \\ 0 \end{pmatrix}$

Element nodal displacements in local coordinates, $\boldsymbol{d}_\ell = \boldsymbol{T}\,\boldsymbol{d} = \begin{pmatrix} 0.0481481 \\ 0. \end{pmatrix}$

Axial displacements at element ends, $d_1 = 0.0481481$ $d_2 = 0.$

$E = 29000$ $A = \dfrac{1}{2}$

Axial strain, $\epsilon = (d_2 - d_1)/L = -0.000200617$

Axial stress, $\sigma = E\epsilon = -5.8179$ Axial force $= \sigma A = -2.90895$

Solution for element 4

Nodal coordinates

Element node	Global node number	x	y
1	3	192.	0
2	4	384.	0

$x_1 = 192.$ $y_1 = 0$ $x_2 = 384.$ $y_2 = 0$

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 192.$$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 1.$ $m_s = \dfrac{y_2 - y_1}{L} = 0$

Global to local transformation matrix, $\boldsymbol{T} = \begin{pmatrix} 1. & 0 & 0 & 0 \\ 0 & 0 & 1. & 0 \end{pmatrix}$

Element nodal displacements in global coordinates, $\boldsymbol{d} = \begin{pmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0.0308148 \\ -0.138667 \\ 0 \\ 0 \end{pmatrix}$

Element nodal displacements in local coordinates, $\boldsymbol{d}_\ell = \boldsymbol{T}\ \boldsymbol{d} = \begin{pmatrix} 0.0308148 \\ 0. \end{pmatrix}$

Axial displacements at element ends, $d_1 = 0.0308148$ $\quad d_2 = 0.$

$E = 29000 \qquad A = \frac{1}{2}$

Axial strain, $\epsilon = (d_2 - d_1)/L = -0.000160494$

Axial stress, $\sigma = E\epsilon = -4.65432$ $\quad$ Axial force $= \sigma A = -2.32716$

Solution for element 5

Nodal coordinates

Element node	Global node number	x	y
1	2	192.	144.
2	3	192.	0

$x_1 = 192. \qquad y_1 = 144. \qquad x_2 = 192. \qquad y_2 = 0$

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 144.$

Direction cosines: $\ell_s = \frac{x_2 - x_1}{L} = 0. \qquad m_s = \frac{y_2 - y_1}{L} = -1.$

Global to local transformation matrix, $\boldsymbol{T} = \begin{pmatrix} 0. & -1. & 0 & 0 \\ 0 & 0 & 0. & -1. \end{pmatrix}$

Element nodal displacements in global coordinates, $\boldsymbol{d} = \begin{pmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} -0.0308148 \\ -0.121333 \\ 0.0308148 \\ -0.138667 \end{pmatrix}$

Element nodal displacements in local coordinates, $\boldsymbol{d}_\ell = \boldsymbol{T}\ \boldsymbol{d} = \begin{pmatrix} 0.121333 \\ 0.138667 \end{pmatrix}$

Axial displacements at element ends, $d_1 = 0.121333$ $\quad d_2 = 0.138667$

$E = 29000 \qquad A = \frac{1}{2}$

Axial strain, $\epsilon = (d_2 - d_1)/L = 0.00012037$

Axial stress, $\sigma = E\epsilon = 3.49074$ Axial force $= \sigma A = 1.74537$

Solution summary

Nodal solution

	x−coord	y−coord	u	v
1	0	144.	0	0
2	192.	144.	−0.0308148	−0.121333
3	192.	0	0.0308148	−0.138667
4	384.	0	0	0

Element solution

	Stress	Axial force
1	−5.8179	−2.90895
2	−4.65432	−2.32716
3	−5.8179	−2.90895
4	−4.65432	−2.32716
5	3.49074	1.74537

Support reactions

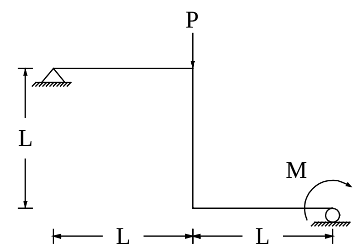
Node	dof	Reaction
1	u_1	4.65432
1	v_1	−1.74537
4	u_4	−4.65432
4	v_4	1.74537

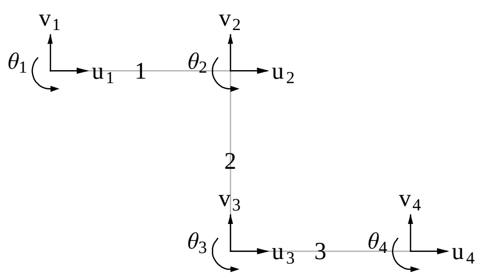
Sum of applied loads → (0   0)

Sum of support reactions → (0   0)

Example 4.10 Three element frame (p. 270)

$$M = 20\,\text{kN} - m;\; P = 10\,\text{kN};\; L = 1\,m;\; E = 210\,\text{GPa};\; A = 4 \text{x} 10^{-2}\,m^2;\; I = 4 \text{x} 10^{-4}\,m^4$$





Use kN $-$ m units for numerical computations. The computed displacements will be in m and stresses in kN$/m^2$.

Specified nodal loads

Node	dof	Value
2	v_2	-10
4	θ_4	-20

Global equations at start of the element assembly process

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} u_1\\ v_1\\ \theta_1\\ u_2\\ v_2\\ \theta_2\\ u_3\\ v_3\\ \theta_3\\ u_4\\ v_4\\ \theta_4 \end{pmatrix} = \begin{pmatrix} 0\\ 0\\ 0\\ 0\\ -10\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ -20 \end{pmatrix}$$

Equations for element 1

$E = 2.1\times 10^8$; $I = 0.0004$; $A = 0.04$; $q = \{0., 0.\}$

Nodal coordinates

Element node	Global node number	x	y
1	1	0	0
2	2	1	0

Length = 1; Direction cosines: $\ell_s = 1$ $m_s = 0$

Element equations in local coordinates

$$10^{6.} \begin{pmatrix} 8.4 & 0 & 0 & -8.4 & 0 & 0\\ 0 & 1.008 & 0.504 & 0 & -1.008 & 0.504\\ 0 & 0.504 & 0.336 & 0 & -0.504 & 0.168\\ -8.4 & 0 & 0 & 8.4 & 0 & 0\\ 0 & -1.008 & -0.504 & 0 & 1.008 & -0.504\\ 0 & 0.504 & 0.168 & 0 & -0.504 & 0.336 \end{pmatrix} \begin{pmatrix} d_1\\ d_2\\ d_3\\ d_4\\ d_5\\ d_6 \end{pmatrix} = \begin{pmatrix} 0.\\ 0.\\ 0.\\ 0.\\ 0.\\ 0. \end{pmatrix}$$

Global to local transformation, $\boldsymbol{T} = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$

Element equations in global coordinates

$$10^{6.}\begin{pmatrix} 8.4 & 0 & 0 & -8.4 & 0 & 0 \\ 0 & 1.008 & 0.504 & 0 & -1.008 & 0.504 \\ 0 & 0.504 & 0.336 & 0 & -0.504 & 0.168 \\ -8.4 & 0 & 0 & 8.4 & 0 & 0 \\ 0 & -1.008 & -0.504 & 0 & 1.008 & -0.504 \\ 0 & 0.504 & 0.168 & 0 & -0.504 & 0.336 \end{pmatrix}\begin{pmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {1, 2, 3, 4, 5, 6} global degrees of freedom.

Adding element equations into appropriate locations we have

$$10^{6}\begin{pmatrix} 8.4 & 0 & 0 & -8.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.008 & 0.504 & 0 & -1.008 & 0.504 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.504 & 0.336 & 0 & -0.504 & 0.168 & 0 & 0 & 0 & 0 & 0 & 0 \\ -8.4 & 0 & 0 & 8.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.008 & -0.504 & 0 & 1.008 & -0.504 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.504 & 0.168 & 0 & -0.504 & 0.336 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \\ u_3 \\ v_3 \\ \theta_3 \\ u_4 \\ v_4 \\ \theta_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -10. \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -20 \end{pmatrix}$$

Equations for element 2

$E = 2.1\times 10^{8}$; $I = 0.0004$; $A = 0.04$; $q = \{0., 0.\}$

Nodal coordinates

Element node	Global node number	x	y
1	2	1	0
2	3	1	−1

Length = 1; Direction cosines: $\ell_s = 0$ $m_s = -1$

Element equations in local coordinates

$$10^{6.}\begin{pmatrix} 8.4 & 0 & 0 & -8.4 & 0 & 0 \\ 0 & 1.008 & 0.504 & 0 & -1.008 & 0.504 \\ 0 & 0.504 & 0.336 & 0 & -0.504 & 0.168 \\ -8.4 & 0 & 0 & 8.4 & 0 & 0 \\ 0 & -1.008 & -0.504 & 0 & 1.008 & -0.504 \\ 0 & 0.504 & 0.168 & 0 & -0.504 & 0.336 \end{pmatrix}\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

Global to local transformation, $\boldsymbol{T} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Element equations in global coordinates

$$10^{6.}\begin{pmatrix} 1.008 & 0 & 0.504 & -1.008 & 0 & 0.504 \\ 0 & 8.4 & 0 & 0 & -8.4 & 0 \\ 0.504 & 0 & 0.336 & -0.504 & 0 & 0.168 \\ -1.008 & 0 & -0.504 & 1.008 & 0 & -0.504 \\ 0 & -8.4 & 0 & 0 & 8.4 & 0 \\ 0.504 & 0 & 0.168 & -0.504 & 0 & 0.336 \end{pmatrix}\begin{pmatrix} u_2 \\ v_2 \\ \theta_2 \\ u_3 \\ v_3 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {4, 5, 6, 7, 8, 9} global degrees of freedom.

Adding element equations into appropriate locations we have

$$10^{6}\begin{pmatrix} 8.4 & 0 & 0 & -8.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.008 & 0.504 & 0 & -1.008 & 0.504 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.504 & 0.336 & 0 & -0.504 & 0.168 & 0 & 0 & 0 & 0 & 0 & 0 \\ -8.4 & 0 & 0 & 9.408 & 0 & 0.504 & -1.008 & 0 & 0.504 & 0 & 0 & 0 \\ 0 & -1.008 & -0.504 & 0 & 9.408 & -0.504 & 0 & -8.4 & 0 & 0 & 0 & 0 \\ 0 & 0.504 & 0.168 & 0.504 & -0.504 & 0.672 & -0.504 & 0 & 0.168 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.008 & 0 & -0.504 & 1.008 & 0 & -0.504 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -8.4 & 0 & 0 & 8.4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.504 & 0 & 0.168 & -0.504 & 0 & 0.336 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \\ u_3 \\ v_3 \\ \theta_3 \\ u_4 \\ v_4 \\ \theta_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -10. \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -20 \end{pmatrix}$$

Equations for element 3

$E = 2.1\times 10^{8}$; $I = 0.0004$; $A = 0.04$; $q = \{0., 0.\}$

Nodal coordinates

Element node	Global node number	x	y
1	3	1	−1
2	4	2	−1

Length = 1; Direction cosines: $\ell_s = 1$ $m_s = 0$

Element equations in local coordinates

$$10^{6.}\begin{pmatrix} 8.4 & 0 & 0 & -8.4 & 0 & 0 \\ 0 & 1.008 & 0.504 & 0 & -1.008 & 0.504 \\ 0 & 0.504 & 0.336 & 0 & -0.504 & 0.168 \\ -8.4 & 0 & 0 & 8.4 & 0 & 0 \\ 0 & -1.008 & -0.504 & 0 & 1.008 & -0.504 \\ 0 & 0.504 & 0.168 & 0 & -0.504 & 0.336 \end{pmatrix}\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

Global to local transformation, $\boldsymbol{T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Element equations in global coordinates

$$10^{6.}\begin{pmatrix} 8.4 & 0 & 0 & -8.4 & 0 & 0 \\ 0 & 1.008 & 0.504 & 0 & -1.008 & 0.504 \\ 0 & 0.504 & 0.336 & 0 & -0.504 & 0.168 \\ -8.4 & 0 & 0 & 8.4 & 0 & 0 \\ 0 & -1.008 & -0.504 & 0 & 1.008 & -0.504 \\ 0 & 0.504 & 0.168 & 0 & -0.504 & 0.336 \end{pmatrix}\begin{pmatrix} u_3 \\ v_3 \\ \theta_3 \\ u_4 \\ v_4 \\ \theta_4 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {7, 8, 9, 10, 11, 12} global degrees of freedom.

Adding element equations into appropriate locations we have

$$
10^6\begin{pmatrix}
8.4 & 0 & 0 & -8.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1.008 & 0.504 & 0 & -1.008 & 0.504 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.504 & 0.336 & 0 & -0.504 & 0.168 & 0 & 0 & 0 & 0 & 0 & 0 \\
-8.4 & 0 & 0 & 9.408 & 0 & 0.504 & -1.008 & 0 & 0.504 & 0 & 0 & 0 \\
0 & -1.008 & -0.504 & 0 & 9.408 & -0.504 & 0 & -8.4 & 0 & 0 & 0 & 0 \\
0 & 0.504 & 0.168 & 0.504 & -0.504 & 0.672 & -0.504 & 0 & 0.168 & 0 & 0 & 0 \\
0 & 0 & 0 & -1.008 & 0 & -0.504 & 9.408 & 0 & -0.504 & -8.4 & 0 & 0 \\
0 & 0 & 0 & 0 & -8.4 & 0 & 0 & 9.408 & 0.504 & 0 & -1.008 & 0.504 \\
0 & 0 & 0 & 0.504 & 0 & 0.168 & -0.504 & 0.504 & 0.672 & 0 & -0.504 & 0.168 \\
0 & 0 & 0 & 0 & 0 & 0 & -8.4 & 0 & 0 & 8.4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.008 & -0.504 & 0 & 1.008 & -0.504 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.504 & 0.168 & 0 & -0.504 & 0.336
\end{pmatrix}
$$

$$
\begin{pmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \\ u_3 \\ v_3 \\ \theta_3 \\ u_4 \\ v_4 \\ \theta_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -10. \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -20. \end{pmatrix}
$$

Essential boundary conditions

Node	dof	Value
1	u_1	0
	v_1	0
4	v_4	0

Remove {1, 2, 11} rows and columns.

After adjusting for essential boundary conditions we have

$$10^6\begin{pmatrix} 0.336 & 0 & -0.504 & 0.168 & 0 & 0 & 0 & 0 & 0 \\ 0 & 9.408 & 0 & 0.504 & -1.008 & 0 & 0.504 & 0 & 0 \\ -0.504 & 0 & 9.408 & -0.504 & 0 & -8.4 & 0 & 0 & 0 \\ 0.168 & 0.504 & -0.504 & 0.672 & -0.504 & 0 & 0.168 & 0 & 0 \\ 0 & -1.008 & 0 & -0.504 & 9.408 & 0 & -0.504 & -8.4 & 0 \\ 0 & 0 & -8.4 & 0 & 0 & 9.408 & 0.504 & 0 & 0.504 \\ 0 & 0.504 & 0 & 0.168 & -0.504 & 0.504 & 0.672 & 0 & 0.168 \\ 0 & 0 & 0 & 0 & -8.4 & 0 & 0 & 8.4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.504 & 0.168 & 0 & 0.336 \end{pmatrix}\begin{pmatrix} \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \\ u_3 \\ v_3 \\ \theta_3 \\ u_4 \\ \theta_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -10. \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -20. \end{pmatrix}$$

Solving the final system of global equations we get

$$\{\theta_1 = 0.0000784722,\ u_2 = 0,\ v_2 = 0.0000685516,\ \theta_2 = 0.0000487103,\ u_3 = 0.0000189484,$$
$$v_3 = 0.0000703373,\ \theta_3 = -0.0000108135,\ u_4 = 0.0000189484,\ \theta_4 = -0.000159623\}$$

Complete table of nodal values

	u	v	θ
1	0	0	0.0000784722
2	0	0.0000685516	0.0000487103
3	0.0000189484	0.0000703373	−0.0000108135
4	0.0000189484	0	−0.000159623

Computation of reactions

Equation numbers of dof with specified values: {1, 2, 11}

Extracting equations {1, 2, 11} from the global system we have

$$10^6\begin{pmatrix} 8.4 & 0 & 0 & -8.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.008 & 0.504 & 0 & -1.008 & 0.504 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.008 & -0.504 & 0 & 1.008 & -0.504 \end{pmatrix}\begin{pmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \\ u_3 \\ v_3 \\ \theta_3 \\ u_4 \\ v_4 \\ \theta_4 \end{pmatrix} = \begin{pmatrix} R_1 + 0. \\ R_2 + 0. \\ R_3 + 0. \end{pmatrix}$$

Substituting the nodal values and re−arranging

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} = 10^6$$

$$\begin{pmatrix} 8.4 & 0 & 0 & -8.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.008 & 0.504 & 0 & -1.008 & 0.504 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.008 & -0.504 & 0 & 1.008 & -0.504 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0.0000784722 \\ 0 \\ 0.0000685516 \\ 0.0000487103 \\ 0.0000189484 \\ 0.0000703373 \\ -0.0000108135 \\ 0.0000189484 \\ 0 \\ -0.000159623 \end{pmatrix}$$

Carrying out computations, the reactions are as follows.

Label	dof	Reaction
R_1	u_1	0
R_2	v_1	−5.
R_3	v_4	15.

Sum of Reactions

dof: u	0
dof: v	10.
dof: θ	0

Solution for element 1

$E = 2.1 \times 10^8$; $I = 0.0004$; $A = 0.04$; $q = \{0., 0.\}$

Length = 1; Direction cosines: $\ell_s = 1$ $m_s = 0$

Nodal values in global coordinates, $\boldsymbol{d}^T = (0 \quad 0 \quad 0.0000784722 \quad 0 \quad 0.0000685516 \quad 0.0000487103)$

Global to local transformation, $\boldsymbol{T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Nodal values in local coordinates, $\boldsymbol{d}_\ell^{\mathrm{T}} = \boldsymbol{T}\boldsymbol{d} = (\,0 \quad 0 \quad 0.0000784722 \quad 0 \quad 0.0000685516 \quad 0.0000487103\,)$

Axial displacement interpolation functions, $\boldsymbol{N}_{\mathrm{u}}^{\mathrm{T}} = \{1 - \mathrm{s},\ \mathrm{s}\}$

Axial displacement, $\mathrm{u(s)} = \boldsymbol{N}_{\mathrm{u}}^{\mathrm{T}} \begin{pmatrix} d_1 \\ d_4 \end{pmatrix} = 0$

Axial force, EA du(s)/ds = 0

Beam bending interpolation functions, $\boldsymbol{N}_{\mathrm{v}}^{\mathrm{T}} = \{2\,s^3 - 3\,s^2 + 1,\ s^3 - 2\,s^2 + s,\ 3\,s^2 - 2\,s^3,\ s^3 - s^2\}$

Transverse displacement, $\mathrm{v(s)} = \boldsymbol{N}_{\mathrm{v}}^{\mathrm{T}} \begin{pmatrix} d_2 \\ d_3 \\ d_5 \\ d_6 \end{pmatrix} = 0.0000784722\,s - 9.92063 \times 10^{-6}\,s^3$

Fixed−end displacement solution, $= 0.\,(1 - s)^2\,s^2$

Total transverse displacement, $\mathrm{v(s)} = 0.0000784722\,s - 9.92063 \times 10^{-6}\,s^3$

Bending moment, $M = EI\,d^2v(s)/ds^2 = -5.\,s$

Shear force, $V(s) = dM/ds = -5.$

Solution for element 2

$E = 2.1 \times 10^8$; $I = 0.0004$; $A = 0.04$; $q = \{0.,\ 0.\}$

Length = 1; Direction cosines: $\ell_s = 0$ $m_s = -1$

Nodal values in global coordinates, $\boldsymbol{d}^{\mathrm{T}} =$
$(\,0 \quad 0.0000685516 \quad 0.0000487103 \quad 0.0000189484 \quad 0.0000703373 \quad -0.0000108135\,)$

Global to local transformation, $\boldsymbol{T} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Nodal values in local coordinates, $\boldsymbol{d}_\ell^{\mathrm{T}} = \boldsymbol{T}\boldsymbol{d}$ =
$(-0.0000685516 \quad 0 \quad 0.0000487103 \quad -0.0000703373 \quad 0.0000189484 \quad -0.0000108135)$

Axial displacement interpolation functions, $\boldsymbol{N}_u^{\mathrm{T}} = \{1 - s,\ s\}$

Axial displacement, $u(s) = \boldsymbol{N}_u^{\mathrm{T}} \begin{pmatrix} d_1 \\ d_4 \end{pmatrix} = -1.78571 \times 10^{-6}\, s - 0.0000685516$

Axial force, EA du(s)/ds = −15.

Beam bending interpolation functions, $\boldsymbol{N}_v^{\mathrm{T}} = \left\{2 s^3 - 3 s^2 + 1,\ s^3 - 2 s^2 + s,\ 3 s^2 - 2 s^3,\ s^3 - s^2\right\}$

Transverse displacement, $v(s) = \boldsymbol{N}_v^{\mathrm{T}} \begin{pmatrix} d_2 \\ d_3 \\ d_5 \\ d_6 \end{pmatrix} = 0.0000487103\, s - 0.0000297619\, s^2$

Fixed−end displacement solution, $= 0.\,(1 - s)^2 s^2$

Total transverse displacement, $v(s) = 0.0000487103\, s - 0.0000297619\, s^2$

Bending moment, $M = EI\ d^2 v(s)/ds^2 = -5.$

Shear force, $V(s) = dM/ds = 0$

Solution for element 3

$E = 2.1 \times 10^8$; $I = 0.0004$; $A = 0.04$; $q = \{0.,\ 0.\}$

Length = 1; Direction cosines: $\ell_s = 1$ $\quad m_s = 0$

Nodal values in global coordinates, $\boldsymbol{d}^{\mathrm{T}}$ =
$(0.0000189484 \quad 0.0000703373 \quad -0.0000108135 \quad 0.0000189484 \quad 0 \quad -0.000159623)$

Global to local transformation, $\boldsymbol{T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Nodal values in local coordinates, $\boldsymbol{d}_\ell^{\mathrm{T}} = \boldsymbol{T}\boldsymbol{d}$ =
$(0.0000189484 \quad 0.0000703373 \quad -0.0000108135 \quad 0.0000189484 \quad 0 \quad -0.000159623)$

Axial displacement interpolation functions, $\boldsymbol{N}_u^{\mathrm{T}} = \{1 - s,\ s\}$

Axial displacement, $u(s) = \boldsymbol{N}_u^{\mathrm{T}} \begin{pmatrix} d_1 \\ d_4 \end{pmatrix} = 0.0000189484$

Axial force, EA du(s)/ds = 0

Beam bending interpolation functions, $\boldsymbol{N}_v^T = \{2\,s^3 - 3\,s^2 + 1,\ s^3 - 2\,s^2 + s,\ 3\,s^2 - 2\,s^3,\ s^3 - s^2\}$

Transverse displacement, $v(s) = \boldsymbol{N}_v^T \begin{pmatrix} d_2 \\ d_3 \\ d_5 \\ d_6 \end{pmatrix} =$

$-0.0000297619\,s^3 - 0.0000297619\,s^2 - 0.0000108135\,s + 0.0000703373$

Fixed−end displacement solution, $= 0.\,(1 - s)^2\,s^2$

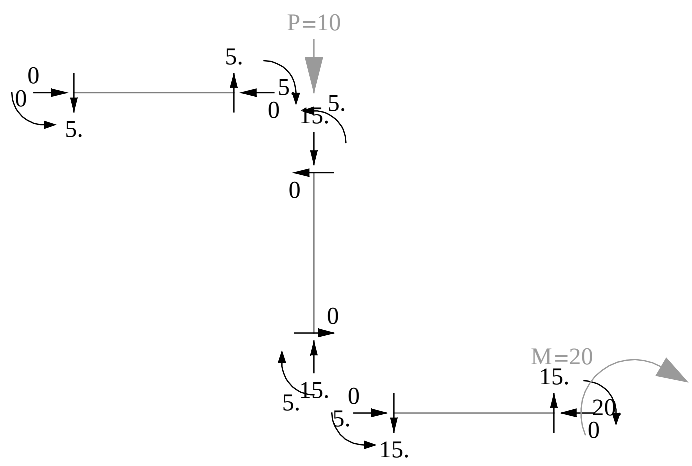
Total transverse displacement, $v(s) = -0.0000297619\,s^3 - 0.0000297619\,s^2 - 0.0000108135\,s + 0.0000703373$

Bending moment, $M = EI\,d^2v(s)/ds^2 = -15.\,s - 5.$

Shear force, $V(s) = dM/ds = -15.$

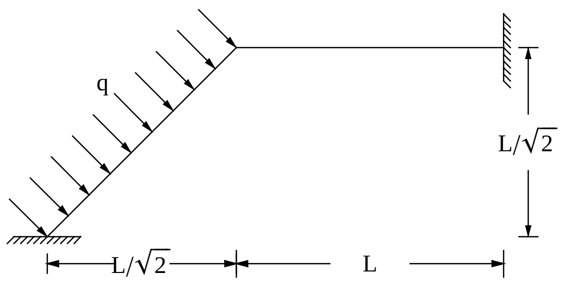
Forces at element ends

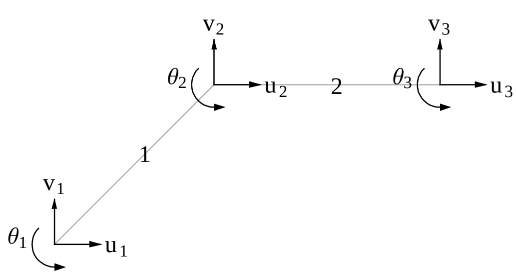
	x	y	Axial force	Bending moment	Shear force
1	0	0	0	0	−5.
	1	0	0	−5.	−5.
2	1	0	−15.	−5.	0
	1	−1	−15.	−5.	0
3	1	−1	0	−5.	−15.
	2	−1	0	−20.	−15.



Example 4.11 Frame with distributed load (p. 274)

$$q = 1\,k/\text{ft};\ \ L = 15\,\text{ft};\ E = 30\times 10^3\,k\big/\text{in}^2;\ A = 100\,\text{in}^2;\ I = 1000\,\text{in}^4$$





Use k-in units

Global equations at start of the element assembly process

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \\ u_3 \\ v_3 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 1

E = 30000; I = 1000; A = 100; q = {0., −0.0833333}

Nodal coordinates

Element node	Global node number	x	y
1	1	0.	0.
2	2	127.279	127.279

Length = 180.; Direction cosines: $\ell_s = 0.707107$ $m_s = 0.707107$

Element equations in local coordinates

$$\begin{pmatrix} 16666.7 & 0 & 0 & -16666.7 & 0 & 0 \\ 0 & 61.7284 & 5555.56 & 0 & -61.7284 & 5555.56 \\ 0 & 5555.56 & 666667. & 0 & -5555.56 & 333333. \\ -16666.7 & 0 & 0 & 16666.7 & 0 & 0 \\ 0 & -61.7284 & -5555.56 & 0 & 61.7284 & -5555.56 \\ 0 & 5555.56 & 333333. & 0 & -5555.56 & 666667. \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{pmatrix} = \begin{pmatrix} 0. \\ -7.5 \\ -225. \\ 0. \\ -7.5 \\ 225. \end{pmatrix}$$

$$\text{Global to local transformation, } \boldsymbol{T} = \begin{pmatrix} 0.707107 & 0.707107 & 0 & 0 & 0 & 0 \\ -0.707107 & 0.707107 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.707107 & 0.707107 & 0 \\ 0 & 0 & 0 & -0.707107 & 0.707107 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Element equations in global coordinates

$$\begin{pmatrix} 8364.2 & 8302.47 & -3928.37 & -8364.2 & -8302.47 & -3928.37 \\ 8302.47 & 8364.2 & 3928.37 & -8302.47 & -8364.2 & 3928.37 \\ -3928.37 & 3928.37 & 666667. & 3928.37 & -3928.37 & 333333. \\ -8364.2 & -8302.47 & 3928.37 & 8364.2 & 8302.47 & 3928.37 \\ -8302.47 & -8364.2 & -3928.37 & 8302.47 & 8364.2 & -3928.37 \\ -3928.37 & 3928.37 & 333333. & 3928.37 & -3928.37 & 666667. \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 5.3033 \\ -5.3033 \\ -225. \\ 5.3033 \\ -5.3033 \\ 225. \end{pmatrix}$$

The element contributes to {1, 2, 3, 4, 5, 6} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 8364.2 & 8302.47 & -3928.37 & -8364.2 & -8302.47 & -3928.37 & 0 & 0 & 0 \\ 8302.47 & 8364.2 & 3928.37 & -8302.47 & -8364.2 & 3928.37 & 0 & 0 & 0 \\ -3928.37 & 3928.37 & 666667. & 3928.37 & -3928.37 & 333333. & 0 & 0 & 0 \\ -8364.2 & -8302.47 & 3928.37 & 8364.2 & 8302.47 & 3928.37 & 0 & 0 & 0 \\ -8302.47 & -8364.2 & -3928.37 & 8302.47 & 8364.2 & -3928.37 & 0 & 0 & 0 \\ -3928.37 & 3928.37 & 333333. & 3928.37 & -3928.37 & 666667. & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \\ u_3 \\ v_3 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} 5.3033 \\ -5.3033 \\ -225. \\ 5.3033 \\ -5.3033 \\ 225. \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 2

$E = 30000;$ $I = 1000;$ $A = 100;$ $q = \{0, 0\}$

Nodal coordinates

Element node	Global node number	x	y
1	2	127.279	127.279
2	3	307.279	127.279

Length = 180.; Direction cosines: $\ell_s = 1.$ $m_s = 0.$

Element equations in local coordinates

$$\begin{pmatrix} 16666.7 & 0 & 0 & -16666.7 & 0 & 0 \\ 0 & 61.7284 & 5555.56 & 0 & -61.7284 & 5555.56 \\ 0 & 5555.56 & 666667. & 0 & -5555.56 & 333333. \\ -16666.7 & 0 & 0 & 16666.7 & 0 & 0 \\ 0 & -61.7284 & -5555.56 & 0 & 61.7284 & -5555.56 \\ 0 & 5555.56 & 333333. & 0 & -5555.56 & 666667. \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Global to local transformation, $\boldsymbol{T} = \begin{pmatrix} 1. & 0. & 0 & 0 & 0 & 0 \\ 0. & 1. & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1. & 0. & 0 \\ 0 & 0 & 0 & 0. & 1. & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Element equations in global coordinates

$$\begin{pmatrix} 16666.7 & 0 & 0 & -16666.7 & 0 & 0 \\ 0 & 61.7284 & 5555.56 & 0 & -61.7284 & 5555.56 \\ 0 & 5555.56 & 666667. & 0 & -5555.56 & 333333. \\ -16666.7 & 0 & 0 & 16666.7 & 0 & 0 \\ 0 & -61.7284 & -5555.56 & 0 & 61.7284 & -5555.56 \\ 0 & 5555.56 & 333333. & 0 & -5555.56 & 666667. \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ \theta_2 \\ u_3 \\ v_3 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to {4, 5, 6, 7, 8, 9} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\left(\begin{matrix} 8364.2 & 8302.47 & -3928.37 & -8364.2 & -8302.47 & -3928.37 & 0 & 0 & \\ 8302.47 & 8364.2 & 3928.37 & -8302.47 & -8364.2 & 3928.37 & 0 & 0 & \\ -3928.37 & 3928.37 & 666667. & 3928.37 & -3928.37 & 333333. & 0 & 0 & \\ -8364.2 & -8302.47 & 3928.37 & 25030.9 & 8302.47 & 3928.37 & -16666.7 & 0 & \\ -8302.47 & -8364.2 & -3928.37 & 8302.47 & 8425.93 & 1627.18 & 0 & -61.7284 & 5 \\ -3928.37 & 3928.37 & 333333. & 3928.37 & 1627.18 & 1.33333\times 10^6 & 0 & -5555.56 & 33 \\ 0 & 0 & 0 & -16666.7 & 0 & 0 & 16666.7 & 0 & \\ 0 & 0 & 0 & 0 & -61.7284 & -5555.56 & 0 & 61.7284 & -5 \\ 0 & 0 & 0 & 0 & 5555.56 & 333333. & 0 & -5555.56 & 66 \end{matrix}\right.$$

Essential boundary conditions

Node	dof	Value
1	u_1	0
	v_1	0
	θ_1	0
3	u_3	0
	v_3	0
	θ_3	0

Remove {1, 2, 3, 7, 8, 9} rows and columns.

After adjusting for essential boundary conditions we have

$$\begin{pmatrix} 25030.9 & 8302.47 & 3928.37 \\ 8302.47 & 8425.93 & 1627.18 \\ 3928.37 & 1627.18 & 1.33333\times 10^6 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 5.3033 \\ -5.3033 \\ 225. \end{pmatrix}$$

Solving the final system of global equations we get

$\{u_2 = 0.000601607,\ v_2 = -0.00125474,\ \theta_2 = 0.000168509\}$

Complete table of nodal values

	u	v	θ
1	0	0	0
2	0.000601607	−0.00125474	0.000168509
3	0	0	0

Computation of reactions

Equation numbers of dof with specified values: {1, 2, 3, 7, 8, 9}

Extracting equations {1, 2, 3, 7, 8, 9} from the global system we have

$$\begin{pmatrix} 8364.2 & 8302.47 & -3928.37 & -8364.2 & -8302.47 & -3928.37 & 0 & 0 & 0 \\ 8302.47 & 8364.2 & 3928.37 & -8302.47 & -8364.2 & 3928.37 & 0 & 0 & 0 \\ -3928.37 & 3928.37 & 666667. & 3928.37 & -3928.37 & 333333. & 0 & 0 & 0 \\ 0 & 0 & 0 & -16666.7 & 0 & 0 & 16666.7 & 0 & 0 \\ 0 & 0 & 0 & 0 & -61.7284 & -5555.56 & 0 & 61.7284 & -5555.56 \\ 0 & 0 & 0 & 0 & 5555.56 & 333333. & 0 & -5555.56 & 666667. \end{pmatrix}$$

$$\begin{pmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \\ u_3 \\ v_3 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} R_1 + 5.3033 \\ R_2 - 5.3033 \\ R_3 - 225. \\ R_4 \\ R_5 \\ R_6 \end{pmatrix}$$

Substituting the nodal values and re−arranging

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{pmatrix} = \begin{pmatrix} 8364.2 & 8302.47 & -3928.37 & -8364.2 & -8302.47 & -3928.37 & 0 & 0 & 0 \\ 8302.47 & 8364.2 & 3928.37 & -8302.47 & -8364.2 & 3928.37 & 0 & 0 & 0 \\ -3928.37 & 3928.37 & 666667. & 3928.37 & -3928.37 & 333333. & 0 & 0 & 0 \\ 0 & 0 & 0 & -16666.7 & 0 & 0 & 16666.7 & 0 & 0 \\ 0 & 0 & 0 & 0 & -61.7284 & -5555.56 & 0 & 61.7284 & -5555 \\ 0 & 0 & 0 & 0 & 5555.56 & 333333. & 0 & -5555.56 & 66666 \end{pmatrix}$$

Carrying out computations, the reactions are as follows.

Label	dof	Reaction
R_1	u_1	−0.579812
R_2	v_1	11.4653
R_3	θ_1	288.462
R_4	u_3	−10.0268
R_5	v_3	−0.858707
R_6	θ_3	49.1988

Sum of Reactions

dof: u	−10.6066
dof: v	10.6066
dof: θ	337.661

Solution for element 1

$E = 30000;$ $I = 1000;$ $A = 100;$ $q = \{0., -0.0833333\}$

$\text{Length} = 180.;$ Direction cosines: $\ell_s = 0.707107$ $m_s = 0.707107$

Nodal values in global coordinates, $\boldsymbol{d}^T = (0 \quad 0 \quad 0 \quad 0.000601607 \quad -0.00125474 \quad 0.000168509)$

Global to local transformation, $\boldsymbol{T} = \begin{pmatrix} 0.707107 & 0.707107 & 0 & 0 & 0 & 0 \\ -0.707107 & 0.707107 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.707107 & 0.707107 & 0 \\ 0 & 0 & 0 & -0.707107 & 0.707107 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Nodal values in local coordinates, $\boldsymbol{d}_\ell^T = \boldsymbol{T}\boldsymbol{d} = (0 \quad 0 \quad 0 \quad -0.000461833 \quad -0.00131263 \quad 0.000168509)$

Axial displacement interpolation functions, $\boldsymbol{N}_u^T = \{1. - 0.00555556\,s,\ 0.00555556\,s\}$

Axial displacement, $u(s) = \boldsymbol{N}_u^T \begin{pmatrix} d_1 \\ d_4 \end{pmatrix} = -2.56574 \times 10^{-6}\, s$

Axial force, $EA\, du(s)/ds = -7.69721$

Beam bending interpolation functions, $\boldsymbol{N}_v^T =$

$$\{3.42936 \times 10^{-7} s^3 - 0.0000925926 s^2 + 1,\ 0.0000308642 s^3 - 0.0111111 s^2 + s,$$
$$0.0000925926 s^2 - 3.42936 \times 10^{-7} s^3,\ 0.0000308642 s^3 - 0.00555556 s^2\}$$

Transverse displacement, $v(s) = \boldsymbol{N}_v^T \begin{pmatrix} d_2 \\ d_3 \\ d_5 \\ d_6 \end{pmatrix} = 5.65104 \times 10^{-9} s^3 - 1.0577 \times 10^{-6} s^2$

Fixed−end displacement solution, $= -1.15741\times 10^{-10}\,(180.-s)^2\,s^2$

Total transverse displacement, $v(s) = -1.15741\times 10^{-10}\,s^4 + 4.73177\times 10^{-8}\,s^3 - 4.8077\times 10^{-6}\,s^2$

Bending moment, $M = EI\ d^2v(s)/ds^2 = -0.0416667\,s^2 + 8.51719\,s - 288.462$

Shear force, $V(s) = dM/ds = 8.51719 - 0.0833333\,s$

Solution for element 2

$E = 30000;$ $I = 1000;$ $A = 100;$ $q = \{0, 0\}$

Length = 180.; Direction cosines: $\ell_s = 1.$ $m_s = 0.$

Nodal values in global coordinates, $\boldsymbol{d}^T = (\ 0.000601607\quad -0.00125474\quad 0.000168509\quad 0\quad 0\quad 0\ )$

$$\text{Global to local transformation, } \boldsymbol{T} = \begin{pmatrix} 1. & 0. & 0 & 0 & 0 & 0 \\ 0. & 1. & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1. & 0. & 0 \\ 0 & 0 & 0 & 0. & 1. & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Nodal values in local coordinates, $\boldsymbol{d}_\ell^T = \boldsymbol{T}\boldsymbol{d} = (\ 0.000601607\quad -0.00125474\quad 0.000168509\quad 0\quad 0\quad 0\ )$

Axial displacement interpolation functions, $\boldsymbol{N}_u^T = \{1. - 0.00555556\,s,\ 0.00555556\,s\}$

$$\text{Axial displacement, } u(s) = \boldsymbol{N}_u^T \begin{pmatrix} d_1 \\ d_4 \end{pmatrix} = 0.000601607 - 3.34226\times 10^{-6}\,s$$

Axial force, $EA\ du(s)/ds = -10.0268$

Beam bending interpolation functions, $\boldsymbol{N}_v^T =$

$$\{3.42936\times 10^{-7}\,s^3 - 0.0000925926\,s^2 + 1,\ 0.0000308642\,s^3 - 0.0111111\,s^2 + s,$$
$$0.0000925926\,s^2 - 3.42936\times 10^{-7}\,s^3,\ 0.0000308642\,s^3 - 0.00555556\,s^2\}$$

$$\text{Transverse displacement, } v(s) = \boldsymbol{N}_v^T \begin{pmatrix} d_2 \\ d_3 \\ d_5 \\ d_6 \end{pmatrix} =$$

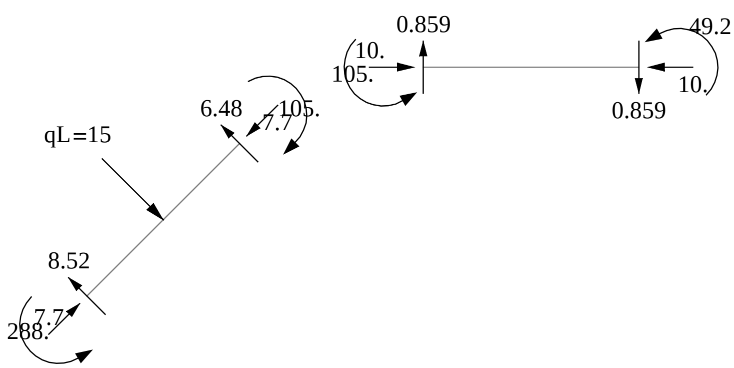
$$4.77059\times 10^{-9}\,s^3 - 1.75614\times 10^{-6}\,s^2 + 0.000168509\,s - 0.00125474$$

Bending moment, $M = EI\ d^2v(s)/ds^2 = 0.858707\,s - 105.368$

Shear force, $V(s) = dM/ds = 0.858707$

Forces at element ends

	x	y	Axial force	Bending moment	Shear force
1	0	0	−7.69721	−288.462	8.51719
	127.279	127.279	−7.69721	−105.368	−6.48281
2	127.279	127.279	−10.0268	−105.368	0.858707
	307.279	127.279	−10.0268	49.1988	0.858707



Example 4.14: Three dimensional frame (p. 290)

Analyze one story three dimensional frame shown in Figure. The height of the columns is 12 ft and the length of the beams is 10 ft. Each beam is subjected to a uniformly distributed load of 2 kip/ft in the downward direction. I-shape sections are used for both columns and beams with the arrangement as shown in the figure. The columns are connected to the foundation through simple connections that do not resist moments. The material is steel with $E = 29000\,\text{kip}\big/\text{in}^2$ and $G = 11200\,\text{kip}\big/\text{in}^2$. The section properties are as follows.

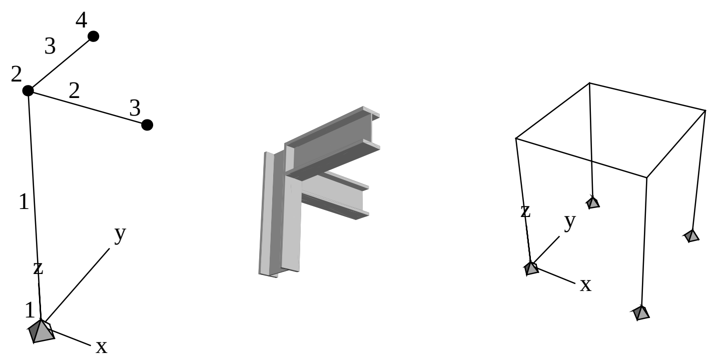
Beams: $A = 3.2\,\text{in}^2$; $J = 43\,\text{in}^4$; $I_{\text{max}} = I_r = 450\,\text{in}^4$; $I_{\text{min}} = I_s = 32\,\text{in}^2$

Columns: $A = 4\,\text{in}^2$; $J = 60\,\text{in}^4$; $I_{\text{max}} = I_r = 650\,\text{in}^4$; $I_{\text{min}} = I_s = 54\,\text{in}^2$

Taking advantage of symmetry we model a quarter of the frame using three elements. Because of symmetry, the boundary conditions at nodes 3 and 4 are as follows.

Node 3: $u = 0$; $\theta_y = 0$; $\theta_z = 0$

Node 4: $v = 0$; $\theta_x = 0$; $\theta_z = 0$



The distributed load is applied to the elements in their local coordinates. Therefore to assign proper direction and sign to the distributed loads we must carefully establish the local coordinates for the elements as follows.

Element 1: Nodes 1, 2, and 4

$\Longrightarrow$ t – axis along global z; s – axis along global x; r – axis along global y

Element 2: Nodes 2, 3, and 4

$\Longrightarrow$ t – axis along global x; s – axis along global $(-z)$; r – axis along global y

Distributed load: $q_r = 0$; $q_s = 2/12$ kip / in

Element 3: Nodes 2, 4, and 3

$\Longrightarrow$ t – axis along global y; s – axis along global z; r – axis along global x

Distributed load: $q_r = 0$; $q_s = -2/12$ kip / in

Global equations at start of the element assembly process

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ w_1 \\ \theta x_1 \\ \theta y_1 \\ \theta z_1 \\ u_2 \\ v_2 \\ w_2 \\ \theta x_2 \\ \theta y_2 \\ \theta z_2 \\ u_3 \\ v_3 \\ w_3 \\ \theta x_3 \\ \theta y_3 \\ \theta z_3 \\ u_4 \\ v_4 \\ w_4 \\ \theta x_4 \\ \theta y_4 \\ \theta z_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 1

$q_s = 0;$ $q_r = 0;$ $E = 29000.;$ $G = 11200.$

$A = 4;$ $J = 60;$ $I_r = 650;$ $I_s = 54$

Element nodal coordinates: $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 144. \\ 0 & 60. & 144. \end{pmatrix}$

Element length, $L = 144.$

Direction cosines: $\boldsymbol{H} = \begin{pmatrix} 0 & 0 & 1. \\ 1. & 0. & 0. \\ 0. & 1. & 0. \end{pmatrix}$

Element equations in local coordinates

$$
\left(\begin{array}{cccccccccc}
75.7539 & 0. & 0. & 0. & 5454.28 & 0. & -75.7539 & 0. & 0. & \\
0. & 6.2934 & 0. & -453.125 & 0. & 0. & 0. & -6.2934 & 0. & -4 \\
0. & 0. & 805.556 & 0. & 0. & 0. & 0. & 0. & -805.556 & \\
0. & -453.125 & 0. & 43500. & 0. & 0. & 0. & 453.125 & 0. & 217 \\
5454.28 & 0. & 0. & 0. & 523611. & 0. & -5454.28 & 0. & 0. & \\
0. & 0. & 0. & 0. & 0. & 4666.67 & 0. & 0. & 0. & \\
-75.7539 & 0. & 0. & 0. & -5454.28 & 0. & 75.7539 & 0. & 0. & \\
0. & -6.2934 & 0. & 453.125 & 0. & 0. & 0. & 6.2934 & 0. & 4 \\
0. & 0. & -805.556 & 0. & 0. & 0. & 0. & 0. & 805.556 & \\
0. & -453.125 & 0. & 21750. & 0. & 0. & 0. & 453.125 & 0. & 435 \\
5454.28 & 0. & 0. & 0. & 261806. & 0. & -5454.28 & 0. & 0. & \\
0. & 0. & 0. & 0. & 0. & -4666.67 & 0. & 0. & 0. &
\end{array}\right.
$$

$$
\text{Global to local transformation, } \boldsymbol{T} = \begin{pmatrix}
0 & 0 & 1. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1. & 0. & 0. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0. & 1. & 0. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1. & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1. & 0. & 0. & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0. & 1. & 0. & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1. & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1. & 0. & 0. & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0. & 1. & 0. & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1. & 0. & 0. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0. & 1. & 0.
\end{pmatrix}
$$

Element equations in global coordinates

$$
\left(\begin{array}{cccccccccc}
75.7539 & 0 & 0 & 0 & 5454.28 & 0 & -75.7539 & 0 & 0 & \\
0 & 6.2934 & 0 & -453.125 & 0 & 0 & 0 & -6.2934 & 0 & -4 \\
0 & 0 & 805.556 & 0 & 0 & 0 & 0 & 0 & -805.556 & \\
0 & -453.125 & 0 & 43500. & 0 & 0 & 0 & 453.125 & 0 & 217 \\
5454.28 & 0 & 0 & 0 & 523611. & 0 & -5454.28 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 4666.67 & 0 & 0 & 0 & \\
-75.7539 & 0 & 0 & 0 & -5454.28 & 0 & 75.7539 & 0 & 0 & \\
0 & -6.2934 & 0 & 453.125 & 0 & 0 & 0 & 6.2934 & 0 & 4 \\
0 & 0 & -805.556 & 0 & 0 & 0 & 0 & 0 & 805.556 & \\
0 & -453.125 & 0 & 21750. & 0 & 0 & 0 & 453.125 & 0 & 435 \\
5454.28 & 0 & 0 & 0 & 261806. & 0 & -5454.28 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & -4666.67 & 0 & 0 & 0 &
\end{array}\right.
$$

The element contributes to {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix}
75.7539 & 0 & 0 & 0 & 5454.28 & 0 & -75.7539 & 0 & 0 & \\
0 & 6.2934 & 0 & -453.125 & 0 & 0 & 0 & -6.2934 & 0 & -4 \\
0 & 0 & 805.556 & 0 & 0 & 0 & 0 & 0 & -805.556 & \\
0 & -453.125 & 0 & 43500. & 0 & 0 & 0 & 453.125 & 0 & 217 \\
5454.28 & 0 & 0 & 0 & 523611. & 0 & -5454.28 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 4666.67 & 0 & 0 & 0 & \\
-75.7539 & 0 & 0 & 0 & -5454.28 & 0 & 75.7539 & 0 & 0 & \\
0 & -6.2934 & 0 & 453.125 & 0 & 0 & 0 & 6.2934 & 0 & 4 \\
0 & 0 & -805.556 & 0 & 0 & 0 & 0 & 0 & 805.556 & \\
0 & -453.125 & 0 & 21750. & 0 & 0 & 0 & 453.125 & 0 & 435 \\
5454.28 & 0 & 0 & 0 & 261806. & 0 & -5454.28 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & -4666.67 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 &
\end{pmatrix}$$

Equations for element 2

$q_s = 0.166667$; $q_r = 0$; $E = 29000.$; $G = 11200.$

$A = 3.2$; $J = 43$; $I_r = 450$; $I_s = 32$

Element nodal coordinates: $\begin{pmatrix} 0 & 0 & 144. \\ 60. & 0 & 144. \\ 0 & 60. & 144. \end{pmatrix}$

Element length, $L = 60.$

Direction cosines: $\boldsymbol{H} = \begin{pmatrix} 1. & 0 & 0. \\ 0. & 0. & -1. \\ 0. & 1. & 0. \end{pmatrix}$

Element equations in local coordinates

$$\left(\begin{array}{cccccccccc} 1546.67 & 0. & 0. & 0. & 0. & 0. & -1546.67 & 0. & 0. & 0. \\ 0. & 51.5556 & 0. & 0. & 0. & 1546.67 & 0. & -51.5556 & 0. & 0. \\ 0. & 0. & 725. & 0. & -21750. & 0. & 0. & 0. & -725. & 0. \\ 0. & 0. & 0. & 8026.67 & 0. & 0. & 0. & 0. & 0. & -8026.67 \\ 0. & 0. & -21750. & 0. & 870000. & 0. & 0. & 0. & 21750. & 0. \\ 0. & 1546.67 & 0. & 0. & 0. & 61866.7 & 0. & -1546.67 & 0. & 0. \\ -1546.67 & 0. & 0. & 0. & 0. & 0. & 1546.67 & 0. & 0. & 0. \\ 0. & -51.5556 & 0. & 0. & 0. & -1546.67 & 0. & 51.5556 & 0. & 0. \\ 0. & 0. & -725. & 0. & 21750. & 0. & 0. & 0. & 725. & 0. \\ 0. & 0. & 0. & -8026.67 & 0. & 0. & 0. & 0. & 0. & 8026.67 \\ 0. & 0. & -21750. & 0. & 435000. & 0. & 0. & 0. & 21750. & 0. \\ 0. & 1546.67 & 0. & 0. & 0. & 30933.3 & 0. & -1546.67 & 0. & 0. \end{array} \right.$$

Global to local transformation, $\boldsymbol{T} = \begin{pmatrix} 1. & 0 & 0. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0. & 0. & -1. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0. & 1. & 0. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1. & 0 & 0. & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0. & 0. & -1. & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0. & 1. & 0. & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1. & 0 & 0. & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0. & 0. & -1. & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0. & 1. & 0. & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1. & 0 & 0. \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0. & 0. & -1. \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0. & 1. & 0. \end{pmatrix}$

Element equations in global coordinates

$$
\left(\begin{array}{cccccccccc}
1546.67 & 0 & 0 & 0 & 0 & 0 & -1546.67 & 0 & 0 & 0 \\
0 & 51.5556 & 0 & 0 & 0 & 1546.67 & 0 & -51.5556 & 0 & 0 \\
0 & 0 & 725. & 0 & -21750. & 0 & 0 & 0 & -725. & 0 \\
0 & 0 & 0 & 8026.67 & 0 & 0 & 0 & 0 & 0 & -8026.67 \\
0 & 0 & -21750. & 0 & 870000. & 0 & 0 & 0 & 21750. & 0 \\
0 & 1546.67 & 0 & 0 & 0 & 61866.7 & 0 & -1546.67 & 0 & 0 \\
-1546.67 & 0 & 0 & 0 & 0 & 0 & 1546.67 & 0 & 0 & 0 \\
0 & -51.5556 & 0 & 0 & 0 & -1546.67 & 0 & 51.5556 & 0 & 0 \\
0 & 0 & -725. & 0 & 21750. & 0 & 0 & 0 & 725. & 0 \\
0 & 0 & 0 & -8026.67 & 0 & 0 & 0 & 0 & 0 & 8026.67 \\
0 & 0 & -21750. & 0 & 435000. & 0 & 0 & 0 & 21750. & 0 \\
0 & 1546.67 & 0 & 0 & 0 & 30933.3 & 0 & -1546.67 & 0 & 0
\end{array}\right.
$$

The element contributes to {7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\left(\begin{array}{ccccccccc}
75.7539 & 0 & 0 & 0 & 5454.28 & 0 & -75.7539 & 0 & 0 \\
0 & 6.2934 & 0 & -453.125 & 0 & 0 & 0 & -6.2934 & 0 \\
0 & 0 & 805.556 & 0 & 0 & 0 & 0 & 0 & -805.556 \\
0 & -453.125 & 0 & 43500. & 0 & 0 & 0 & 453.125 & 0 \\
5454.28 & 0 & 0 & 0 & 523611. & 0 & -5454.28 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4666.67 & 0 & 0 & 0 \\
-75.7539 & 0 & 0 & 0 & -5454.28 & 0 & 1622.42 & 0 & 0 \\
0 & -6.2934 & 0 & 453.125 & 0 & 0 & 0 & 57.849 & 0 \\
0 & 0 & -805.556 & 0 & 0 & 0 & 0 & 0 & 1530.56 \\
0 & -453.125 & 0 & 21750. & 0 & 0 & 0 & 453.125 & 0 \\
5454.28 & 0 & 0 & 0 & 261806. & 0 & -5454.28 & 0 & -21750. \\
0 & 0 & 0 & 0 & 0 & -4666.67 & 0 & 1546.67 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1546.67 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -51.5556 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -725. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -21750. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1546.67 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right.$$

Equations for element 3

$q_s = -0.166667;$ $\quad q_r = 0;$ $\quad E = 29000.;$ $\quad G = 11200.$

$A = 3.2;$ $\quad J = 43;$ $\quad I_r = 450;$ $\quad I_s = 32$

Element nodal coordinates: $\begin{pmatrix} 0 & 0 & 144. \\ 0 & 60. & 144. \\ 60. & 0 & 144. \end{pmatrix}$

Element length, $L = 60.$

Direction cosines: $\boldsymbol{H} = \begin{pmatrix} 0 & 1. & 0. \\ 0. & 0. & 1. \\ 1. & 0. & 0. \end{pmatrix}$

Element equations in local coordinates

$$\left(\begin{array}{cccccccccc}
51.5556 & 0. & 0. & 0. & 0. & -1546.67 & -51.5556 & 0. & 0. & 0. \\
0. & 1546.67 & 0. & 0. & 0. & 0. & 0. & -1546.67 & 0. & 0. \\
0. & 0. & 725. & 21750. & 0. & 0. & 0. & 0. & -725. & 21750. \\
0. & 0. & 21750. & 870000. & 0. & 0. & 0. & 0. & -21750. & 435000. \\
0. & 0. & 0. & 0. & 8026.67 & 0. & 0. & 0. & 0. & 0. \\
-1546.67 & 0. & 0. & 0. & 0. & 61866.7 & 1546.67 & 0. & 0. & 0. \\
-51.5556 & 0. & 0. & 0. & 0. & 1546.67 & 51.5556 & 0. & 0. & 0. \\
0. & -1546.67 & 0. & 0. & 0. & 0. & 0. & 1546.67 & 0. & 0. \\
0. & 0. & -725. & -21750. & 0. & 0. & 0. & 0. & 725. & -21750. \\
0. & 0. & 21750. & 435000. & 0. & 0. & 0. & 0. & -21750. & 870000. \\
0. & 0. & 0. & 0. & -8026.67 & 0. & 0. & 0. & 0. & 0. \\
-1546.67 & 0. & 0. & 0. & 0. & 30933.3 & 1546.67 & 0. & 0. & 0.
\end{array}\right.$$

$$\text{Global to local transformation, } \boldsymbol{T} = \begin{pmatrix}
0 & 1. & 0. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0. & 0. & 1. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1. & 0. & 0. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1. & 0. & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0. & 0. & 1. & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1. & 0. & 0. & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1. & 0. & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0. & 0. & 1. & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1. & 0. & 0. & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1. & 0. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0. & 0. & 1. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1. & 0. & 0.
\end{pmatrix}$$

Element equations in global coordinates

$$\left(\begin{array}{cccccccccc}
51.5556 & 0 & 0 & 0 & 0 & -1546.67 & -51.5556 & 0 & 0 & 0 \\
0 & 1546.67 & 0 & 0 & 0 & 0 & 0 & -1546.67 & 0 & 0 \\
0 & 0 & 725. & 21750. & 0 & 0 & 0 & 0 & -725. & 21750. \\
0 & 0 & 21750. & 870000. & 0 & 0 & 0 & 0 & -21750. & 435000. \\
0 & 0 & 0 & 0 & 8026.67 & 0 & 0 & 0 & 0 & 0 \\
-1546.67 & 0 & 0 & 0 & 0 & 61866.7 & 1546.67 & 0 & 0 & 0 \\
-51.5556 & 0 & 0 & 0 & 0 & 1546.67 & 51.5556 & 0 & 0 & 0 \\
0 & -1546.67 & 0 & 0 & 0 & 0 & 0 & 1546.67 & 0 & 0 \\
0 & 0 & -725. & -21750. & 0 & 0 & 0 & 0 & 725. & -21750. \\
0 & 0 & 21750. & 435000. & 0 & 0 & 0 & 0 & -21750. & 870000. \\
0 & 0 & 0 & 0 & -8026.67 & 0 & 0 & 0 & 0 & 0 \\
-1546.67 & 0 & 0 & 0 & 0 & 30933.3 & 1546.67 & 0 & 0 & 0
\end{array}\right.$$

The element contributes to {7, 8, 9, 10, 11, 12, 19, 20, 21, 22, 23, 24} global degrees of freedom.

Adding element equations into appropriate locations we have

$$
\left(\begin{array}{ccccccccc}
75.7539 & 0 & 0 & 0 & 5454.28 & 0 & -75.7539 & 0 & 0 \\
0 & 6.2934 & 0 & -453.125 & 0 & 0 & 0 & -6.2934 & 0 \\
0 & 0 & 805.556 & 0 & 0 & 0 & 0 & 0 & -805.556 \\
0 & -453.125 & 0 & 43500. & 0 & 0 & 0 & 453.125 & 0 \\
5454.28 & 0 & 0 & 0 & 523611. & 0 & -5454.28 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4666.67 & 0 & 0 & 0 \\
-75.7539 & 0 & 0 & 0 & -5454.28 & 0 & 1673.98 & 0 & 0 \\
0 & -6.2934 & 0 & 453.125 & 0 & 0 & 0 & 1604.52 & 0 \\
0 & 0 & -805.556 & 0 & 0 & 0 & 0 & 0 & 2255.56 \\
0 & -453.125 & 0 & 21750. & 0 & 0 & 0 & 453.125 & 21750. \\
5454.28 & 0 & 0 & 0 & 261806. & 0 & -5454.28 & 0 & -21750. \\
0 & 0 & 0 & 0 & 0 & -4666.67 & -1546.67 & 1546.67 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1546.67 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -51.5556 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -725. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -21750. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1546.67 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -51.5556 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1546.67 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -725. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 21750. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1546.67 & 0 & 0
\end{array}\right.
$$

Essential boundary conditions

Node	dof	Value
1	u_1	0
1	v_1	0
1	w_1	0
3	u_3	0
3	θy_3	0
3	θz_3	0
4	v_4	0
4	θx_4	0
4	θz_4	0

Remove {1, 2, 3, 13, 17, 18, 20, 22, 24} rows and columns.

After adjusting for essential boundary conditions we have

$$\left(\begin{array}{ccccccccc}
43500. & 0 & 0 & 0 & 453.125 & 0 & 21750. & 0 & \\
0 & 523611. & 0 & -5454.28 & 0 & 0 & 0 & 261806. & \\
0 & 0 & 4666.67 & 0 & 0 & 0 & 0 & 0 & \\
0 & -5454.28 & 0 & 1673.98 & 0 & 0 & 0 & -5454.28 & \\
453.125 & 0 & 0 & 0 & 1604.52 & 0 & 453.125 & 0 & \\
0 & 0 & 0 & 0 & 0 & 2255.56 & 21750. & -21750. & \\
21750. & 0 & 0 & 0 & 453.125 & 21750. & 921527. & 0 & \\
0 & 261806. & 0 & -5454.28 & 0 & -21750. & 0 & 1.40164\times 10^{6} & \\
0 & 0 & -4666.67 & -1546.67 & 1546.67 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -51.5556 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & -725. & 0 & 21750. & \\
0 & 0 & 0 & 0 & 0 & 0 & -8026.67 & 0 & \\
0 & 0 & 0 & -51.5556 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & -725. & -21750. & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -8026.67 &
\end{array}\right.$$

Solving the final system of global equations we get

$$\{\theta x_1 = 0.000398634,\ \theta y_1 = -0.00015917,\ \theta z_1 = 0,\ u_2 = 0.000575402,\ v_2 = 0.000117025,$$
$$w_2 = -0.0248276,\ \theta x_2 = -0.000799706,\ \theta y_2 = 0.000330328,\ \theta z_2 = 0,\ v_3 = 0.000117025,$$
$$w_3 = -0.041634,\ \theta x_3 = -0.000799706,\ u_4 = 0.000575402,\ w_4 = -0.0557153,\ \theta y_4 = 0.000330328\}$$

Complete table of nodal values

	u	v	w	θx	θy	θz
1	0	0	0	0.000398634	−0.00015917	0
2	0.000575402	0.000117025	−0.0248276	−0.000799706	0.000330328	0
3	0	0.000117025	−0.041634	−0.000799706	0	0
4	0.000575402	0	−0.0557153	0	0.000330328	0

Computation of reactions

Equation numbers of dof with specified values: {1, 2, 3, 13, 17, 18, 20, 22, 24}

Extracting equations {1, 2, 3, 13, 17, 18, 20, 22, 24} from the global system we have

$$
\left(\begin{array}{ccccccccccc}
75.7539 & 0 & 0 & 0 & 5454.28 & 0 & -75.7539 & 0 & 0 & 0 & 5 \\
0 & 6.2934 & 0 & -453.125 & 0 & 0 & 0 & -6.2934 & 0 & -453.125 & \\
0 & 0 & 805.556 & 0 & 0 & 0 & 0 & 0 & -805.556 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & -1546.67 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -21750. & 0 & 435 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1546.67 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1546.67 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 21750. & 435000. & \\
0 & 0 & 0 & 0 & 0 & 0 & -1546.67 & 0 & 0 & 0 &
\end{array}\right.
$$

Substituting the nodal values and re−arranging

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \\ R_9 \end{pmatrix} = \begin{pmatrix} 75.7539 & 0 & 0 & 0 & 5454.28 & 0 & -75.7539 & 0 & 0 & 0 \\ 0 & 6.2934 & 0 & -453.125 & 0 & 0 & 0 & -6.2934 & 0 & -453.1 \\ 0 & 0 & 805.556 & 0 & 0 & 0 & 0 & 0 & -805.556 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1546.67 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -21750. & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1546.67 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1546.67 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 21750. & 435000. \\ 0 & 0 & 0 & 0 & 0 & 0 & -1546.67 & 0 & 0 & 0 \end{pmatrix}$$

Carrying out computations, the reactions are as follows.

Label	dof	Reaction
R_1	u_1	0.889955
R_2	v_1	0.180999
R_3	w_1	20.
R_4	u_3	−0.889955
R_5	θy_3	−171.846
R_6	θz_3	0
R_7	v_4	−0.180999
R_8	θx_4	273.936
R_9	θz_4	0

Sum of Reactions

dof: u	0
dof: v	0
dof: w	20.
dof: θx	273.936
dof: θy	−171.846
dof: θz	0

Solution for element 1

Nodal values in global coordinates, $\boldsymbol{d}^{\mathrm{T}}$ = {0, 0, 0, 0.000398634, −0.00015917, 0, 0.000575402, 0.000117025, −0.0248276, −0.000799706, 0.000330328, 0}

$$\text{Global to local transformation, } \boldsymbol{T} = \begin{pmatrix} 0 & 0 & 1. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1. & 0. & 0. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0. & 1. & 0. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1. & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1. & 0. & 0. & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0. & 1. & 0. & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1. & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1. & 0. & 0. & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0. & 1. & 0. & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1. \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1. & 0. & 0. \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0. & 1. & 0. \end{pmatrix}$$

Nodal values in local coordinates, $\boldsymbol{d}_{\ell}^{\mathrm{T}} = \boldsymbol{T}\boldsymbol{d}$ = {0., 0., 0., 0., 0.000398634, −0.00015917, −0.0248276, 0.000575402, 0.000117025, 0., −0.000799706, 0.000330328}

Axial effects:

Interpolation functions, $\boldsymbol{N}_{\mathrm{u}}^{\mathrm{T}} = \{1. - 0.00694444\,\mathrm{t},\ 0.00694444\,\mathrm{t}\}$

Axial displacement, $\mathrm{u(t)} = \boldsymbol{N}_{\mathrm{u}}^{\mathrm{T}} \begin{pmatrix} d_1 \\ d_7 \end{pmatrix} = -0.000172414\,\mathrm{t}$

Axial force, EA du(t)/dt = −20.

Torsional effects:

Twist angle, $\psi(\mathrm{t}) = \boldsymbol{N}_{\mathrm{u}}^{\mathrm{T}} \begin{pmatrix} d_4 \\ d_{10} \end{pmatrix} = 0$

Twisting moment, GJ dψ(t)/dt = 0.

Bending about r−axis:

$$\boldsymbol{N}_v^T = \{6.69796\times 10^{-7}\, t^3 - 0.000144676\, t^2 + 1,\ 0.0000482253\, t^3 - 0.0138889\, t^2 + t,\ 0.000144676\, t^2 - 6.69796\times 10^{-7}\, t^3,\ 0.0000482253\, t^3 - 0.00694444\, t^2\}$$

$$v(t) = \boldsymbol{N}_v^T \begin{pmatrix} d_2 \\ d_6 \\ d_8 \\ d_{12} \end{pmatrix} = 7.86875\times 10^{-9}\, t^3 - 0.00015917\, t$$

Bending moment, $M_r = E\, I_r\ d^2v(t)/dt^2 = 0.889955\, t$

Shear force, $V_s = dM_r/dt = 0.889955$

Bending about s−axis:

$$\boldsymbol{N}_w^T = \{6.69796\times 10^{-7}\, t^3 - 0.000144676\, t^2 + 1,\ -0.0000482253\, t^3 + 0.0138889\, t^2 - t,\ 0.000144676\, t^2 - 6.69796\times 10^{-7}\, t^3,\ 0.00694444\, t^2 - 0.0000482253\, t^3\}$$

$$w(t) = \boldsymbol{N}_w^T \begin{pmatrix} d_3 \\ d_5 \\ d_9 \\ d_{11} \end{pmatrix} = -1.94202\times 10^{-8}\, t^3 + 3.38615\times 10^{-8}\, t^2 + 0.000398634\, t$$

Bending moment, $M_s = -EI_s\ d^2w(t)/dt^2 = -0.180999\, t$

Shear force, $V_r = -dM_s/dt = 0.180999$

Solution for element 2

Nodal values in global coordinates, $\boldsymbol{d}^T = \{0.000575402, 0.000117025, -0.0248276, -0.000799706, 0.000330328, 0, 0, 0.000117025, -0.041634, -0.000799706, 0, 0\}$

$$\text{Global to local transformation, } \boldsymbol{T} = \begin{pmatrix} 1. & 0 & 0. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0. & 0. & -1. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0. & 1. & 0. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1. & 0 & 0. & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0. & 0. & -1. & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0. & 1. & 0. & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1. & 0 & 0. & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0. & 0. & -1. & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0. & 1. & 0. & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1. & 0 & 0. \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0. & 0. & -1. \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0. & 1. & 0. \end{pmatrix}$$

Nodal values in local coordinates, $\boldsymbol{d}_\ell^T = \boldsymbol{T}\boldsymbol{d}$ = {0.000575402, 0.0248276, 0.000117025, −0.000799706, 0., 0.000330328, 0., 0.041634, 0.000117025, −0.000799706, 0., 0.}

Axial effects:

Interpolation functions, $\boldsymbol{N}_u^T = \{1. - 0.0166667\,t,\ 0.0166667\,t\}$

Axial displacement, $u(t) = \boldsymbol{N}_u^T \begin{pmatrix} d_1 \\ d_7 \end{pmatrix} = 0.000575402 - 9.59003\times10^{-6}\,t$

Axial force, EA $du(t)/dt = -0.889955$

Torsional effects:

Twist angle, $\psi(t) = \boldsymbol{N}_u^T \begin{pmatrix} d_4 \\ d_{10} \end{pmatrix} = -0.000799706$

Twisting moment, GJ $d\psi(t)/dt = 8.70253\times10^{-16}$

Bending about r−axis:

$$\boldsymbol{N}_v^T = \{9.25926\times10^{-6}\,t^3 - 0.000833333\,t^2 + 1,\ 0.000277778\,t^3 - 0.0333333\,t^2 + t,\ 0.000833333\,t^2 - 9.25926\times10^{-6}\,t^3,\ 0.000277778\,t^3 - 0.0166667\,t^2\}$$

$$v(t) = \boldsymbol{N}_v^T \begin{pmatrix} d_2 \\ d_6 \\ d_8 \\ d_{12} \end{pmatrix} = -6.3857\times10^{-8}\,t^3 + 2.99439\times10^{-6}\,t^2 + 0.000330328\,t + 0.0248276$$

Fixed−end displacement solution, $= 5.32141\times10^{-10}\,(60. - t)^2\,t^2$

Transverse displacement, $v(t) = 5.32141\times10^{-10}\,t^4 - 1.27714\times10^{-7}\,t^3 + 4.9101\times10^{-6}\,t^2 + 0.000330328\,t + 0.0248276$

Bending moment, $M_r = E\,I_r\,d^2v(t)/dt^2 = 1.305\times10^7\,(6.3857\times10^{-9}\,t^2 - 7.66284\times10^{-7}\,t + 9.8202\times10^{-6})$

Shear force, $V_s = dM_r/dt = 0.166667\,t - 10.$

Bending about s−axis:

$$\boldsymbol{N}_w^T = \{9.25926\times10^{-6}\,t^3 - 0.000833333\,t^2 + 1,\ -0.000277778\,t^3 + 0.0333333\,t^2 - t,\ 0.000833333\,t^2 - 9.25926\times10^{-6}\,t^3,\ 0.0166667\,t^2 - 0.000277778\,t^3\}$$

$$w(t) = \boldsymbol{N}_w^T \begin{pmatrix} d_3 \\ d_5 \\ d_9 \\ d_{11} \end{pmatrix} = 0.000117025$$

Bending moment, $M_s = -EI_s\,d^2w(t)/dt^2 = 0$

Shear force, $V_r = -dM_s/dt = 0$

Solution for element 3

Nodal values in global coordinates, $\boldsymbol{d}^T = \{0.000575402, 0.000117025, -0.0248276, -0.000799706, 0.000330328, 0, 0.000575402, 0, -0.0557153, 0, 0.000330328, 0\}$

$$\text{Global to local transformation, } \boldsymbol{T} = \begin{pmatrix} 0 & 1. & 0. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0. & 0. & 1. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1. & 0. & 0. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1. & 0. & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0. & 0. & 1. & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1. & 0. & 0. & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1. & 0. & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0. & 0. & 1. & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1. & 0. & 0. & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1. & 0. \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0. & 0. & 1. \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1. & 0. & 0. \end{pmatrix}$$

Nodal values in local coordinates, $\boldsymbol{d}_\ell^T = \boldsymbol{T}\boldsymbol{d} = \{0.000117025, -0.0248276, 0.000575402, 0.000330328, 0., -0.000799706, 0., -0.0557153, 0.000575402, 0.000330328, 0., 0.\}$

Axial effects:

Interpolation functions, $\boldsymbol{N}_u^T = \{1. - 0.0166667\,t, 0.0166667\,t\}$

Axial displacement, $u(t) = \boldsymbol{N}_u^T \begin{pmatrix} d_1 \\ d_7 \end{pmatrix} = 0.000117025 - 1.95042 \times 10^{-6}\,t$

Axial force, EA $du(t)/dt = -0.180999$

Torsional effects:

Twist angle, $\psi(t) = \boldsymbol{N}_u^T \begin{pmatrix} d_4 \\ d_{10} \end{pmatrix} = 0.000330328$

Twisting moment, GJ $d\psi(t)/dt = 0.$

Bending about r−axis:

$$\boldsymbol{N}_v^T = \{9.25926 \times 10^{-6}\,t^3 - 0.000833333\,t^2 + 1, 0.000277778\,t^3 - 0.0333333\,t^2 + t, 0.000833333\,t^2 - 9.25926 \times 10^{-6}\,t^3, 0.000277778\,t^3 - 0.0166667\,t^2\}$$

$$v(t) = \boldsymbol{N}_v^T \begin{pmatrix} d_2 \\ d_6 \\ d_8 \\ d_{12} \end{pmatrix} = 6.3857 \times 10^{-8}\, t^3 + 9.17092 \times 10^{-7}\, t^2 - 0.000799706\, t - 0.0248276$$

Fixed−end displacement solution, $= -5.32141 \times 10^{-10}\,(60. - t)^2\, t^2$

Transverse displacement, $v(t) =$
$-5.32141 \times 10^{-10}\, t^4 + 1.27714 \times 10^{-7}\, t^3 - 9.98617 \times 10^{-7}\, t^2 - 0.000799706\, t - 0.0248276$

Bending moment, $M_r = E\, I_r\, d^2 v(t)/d$
$t^2 = 1.305 \times 10^7 \left(-6.3857 \times 10^{-9}\, t^2 + 7.66284 \times 10^{-7}\, t - 1.99723 \times 10^{-6}\right)$

Shear force, $V_s = dM_r/dt = 10. - 0.166667\, t$

Bending about s−axis:

$\boldsymbol{N}_w^T = \{9.25926 \times 10^{-6}\, t^3 - 0.000833333\, t^2 + 1,\ -0.000277778\, t^3 + 0.0333333\, t^2 - t,$
$0.000833333\, t^2 - 9.25926 \times 10^{-6}\, t^3,\ 0.0166667\, t^2 - 0.000277778\, t^3\}$

$$w(t) = \boldsymbol{N}_w^T \begin{pmatrix} d_3 \\ d_5 \\ d_9 \\ d_{11} \end{pmatrix} = 0.000575402$$

Bending moment, $M_s = -EI_s\, d^2 w(t)/dt^2 = 0$

Shear force, $V_r = -dM_s/dt = 0$

Forces & Moments at element ends

	x	y	z	Axial force	V_s	V_r	M_r	M_s	M_t
1	0	0	0	−20.	0.889955	0.180999	0	0	0
	0	0	144.	−20.	0.889955	0.180999	128.154	−26.0639	0
2	0	0	144.	−0.889955	−10.	0	128.154	0	0
	60.	0	144.	−0.889955	0	0	−171.846	0	0
3	0	0	144.	−0.180999	10.	0	−26.0639	0	0
	0	60.	144.	−0.180999	0	0	273.936	0	0