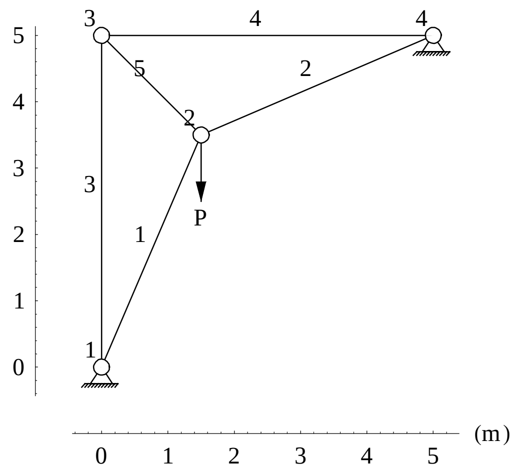
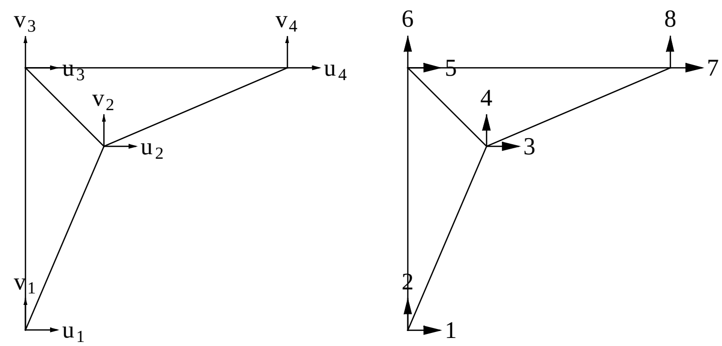
CHAPTER ONE

Finite Element Method
The Big Picture

Five-bar truss example: Examples 1.4 p. 25, 1.7 p. 42, and 1.10 p. 50

The area of cross-section for elements 1 and 2 is 40 cm^2, for elements 3 and 4 is 30 cm^2 and for element 5 is 20 cm^2. The first four elements are made of a material with $E = 200$ GPa and the last one with $E = 70$ GPa. The applied load $P = 150$ kN.





Specified nodal loads

Node	dof	Value
2	u_2	0
	v_2	-150000

Global equations at start of the element assembly process

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -150000 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 1

$E = 200000 \qquad A = 4000$

Element node	Global node number	x	y
1	1	0	0
2	2	1500.	3500.

$x_1 = 0 \qquad y_1 = 0 \qquad x_2 = 1500. \qquad y_2 = 3500.$

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 3807.89$$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 0.393919 \qquad m_s = \dfrac{y_2 - y_1}{L} = 0.919145$

Substituting into the truss element equations we get

$$\begin{pmatrix} 32600.2 & 76067.2 & -32600.2 & -76067.2 \\ 76067.2 & 177490. & -76067.2 & -177490. \\ -32600.2 & -76067.2 & 32600.2 & 76067.2 \\ -76067.2 & -177490. & 76067.2 & 177490. \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {1, 2, 3, 4} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$

and to a global matrix: $\begin{pmatrix} [1,1] & [1,2] & [1,3] & [1,4] \\ [2,1] & [2,2] & [2,3] & [2,4] \\ [3,1] & [3,2] & [3,3] & [3,4] \\ [4,1] & [4,2] & [4,3] & [4,4] \end{pmatrix}$

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 32600.2 & 76067.2 & -32600.2 & -76067.2 & 0 & 0 & 0 & 0 \\ 76067.2 & 177490. & -76067.2 & -177490. & 0 & 0 & 0 & 0 \\ -32600.2 & -76067.2 & 32600.2 & 76067.2 & 0 & 0 & 0 & 0 \\ -76067.2 & -177490. & 76067.2 & 177490. & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -150000. \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 2

$E = 200000 \qquad A = 4000$

Element node	Global node number	x	y
1	2	1500.	3500.
2	4	5000	5000

$x_1 = 1500. \qquad y_1 = 3500. \qquad x_2 = 5000 \qquad y_2 = 5000$

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 3807.89$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 0.919145 \qquad m_s = \dfrac{y_2 - y_1}{L} = 0.393919$

Substituting into the truss element equations we get

$$\begin{pmatrix} 177490. & 76067.2 & -177490. & -76067.2 \\ 76067.2 & 32600.2 & -76067.2 & -32600.2 \\ -177490. & -76067.2 & 177490. & 76067.2 \\ -76067.2 & -32600.2 & 76067.2 & 32600.2 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {3, 4, 7, 8} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 3 \\ 4 \\ 7 \\ 8 \end{pmatrix}$

and to a global matrix: $\begin{pmatrix} [3,3] & [3,4] & [3,7] & [3,8] \\ [4,3] & [4,4] & [4,7] & [4,8] \\ [7,3] & [7,4] & [7,7] & [7,8] \\ [8,3] & [8,4] & [8,7] & [8,8] \end{pmatrix}$

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 32600.2 & 76067.2 & -32600.2 & -76067.2 & 0 & 0 & 0 & 0 \\ 76067.2 & 177490. & -76067.2 & -177490. & 0 & 0 & 0 & 0 \\ -32600.2 & -76067.2 & 210090. & 152134. & 0 & 0 & -177490. & -76067.2 \\ -76067.2 & -177490. & 152134. & 210090. & 0 & 0 & -76067.2 & -32600.2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -177490. & -76067.2 & 0 & 0 & 177490. & 76067.2 \\ 0 & 0 & -76067.2 & -32600.2 & 0 & 0 & 76067.2 & 32600.2 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -150000. \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 3

$E = 200000 \qquad A = 3000$

Element node	Global node number	x	y
1	1	0	0
2	3	0	5000

$x_1 = 0 \qquad y_1 = 0 \qquad x_2 = 0 \qquad y_2 = 5000$

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 5000$$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 0 \qquad m_s = \dfrac{y_2 - y_1}{L} = 1$

Substituting into the truss element equations we get

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 120000 & 0 & -120000 \\ 0 & 0 & 0 & 0 \\ 0 & -120000 & 0 & 120000 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to {1, 2, 5, 6} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 1 \\ 2 \\ 5 \\ 6 \end{pmatrix}$

and to a global matrix: $\begin{pmatrix} [1,1] & [1,2] & [1,5] & [1,6] \\ [2,1] & [2,2] & [2,5] & [2,6] \\ [5,1] & [5,2] & [5,5] & [5,6] \\ [6,1] & [6,2] & [6,5] & [6,6] \end{pmatrix}$

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 32600.2 & 76067.2 & -32600.2 & -76067.2 & 0 & 0 & 0 & 0 \\ 76067.2 & 297490. & -76067.2 & -177490. & 0 & -120000 & 0 & 0 \\ -32600.2 & -76067.2 & 210090. & 152134. & 0 & 0 & -177490. & -76067.2 \\ -76067.2 & -177490. & 152134. & 210090. & 0 & 0 & -76067.2 & -32600.2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -120000 & 0 & 0 & 0 & 120000 & 0 & 0 \\ 0 & 0 & -177490. & -76067.2 & 0 & 0 & 177490. & 76067.2 \\ 0 & 0 & -76067.2 & -32600.2 & 0 & 0 & 76067.2 & 32600.2 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -150000. \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 4

$E = 200000 \qquad A = 3000$

Element node	Global node number	x	y
1	3	0	5000
2	4	5000	5000

$x_1 = 0 \qquad y_1 = 5000 \qquad x_2 = 5000 \qquad y_2 = 5000$

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 5000$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 1 \qquad m_s = \dfrac{y_2 - y_1}{L} = 0$

Substituting into the truss element equations we get

$$\begin{pmatrix} 120000 & 0 & -120000 & 0 \\ 0 & 0 & 0 & 0 \\ -120000 & 0 & 120000 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to {5, 6, 7, 8} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 5 \\ 6 \\ 7 \\ 8 \end{pmatrix}$

and to a global matrix: $\begin{pmatrix} [5,5] & [5,6] & [5,7] & [5,8] \\ [6,5] & [6,6] & [6,7] & [6,8] \\ [7,5] & [7,6] & [7,7] & [7,8] \\ [8,5] & [8,6] & [8,7] & [8,8] \end{pmatrix}$

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 32600.2 & 76067.2 & -32600.2 & -76067.2 & 0 & 0 & 0 & 0 \\ 76067.2 & 297490. & -76067.2 & -177490. & 0 & -120000 & 0 & 0 \\ -32600.2 & -76067.2 & 210090. & 152134. & 0 & 0 & -177490. & -76067.2 \\ -76067.2 & -177490. & 152134. & 210090. & 0 & 0 & -76067.2 & -32600.2 \\ 0 & 0 & 0 & 0 & 120000 & 0 & -120000 & 0 \\ 0 & -120000 & 0 & 0 & 0 & 120000 & 0 & 0 \\ 0 & 0 & -177490. & -76067.2 & -120000 & 0 & 297490. & 76067.2 \\ 0 & 0 & -76067.2 & -32600.2 & 0 & 0 & 76067.2 & 32600.2 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -150000. \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 5

$E = 70000 \qquad A = 2000$

Element node	Global node number	x	y
1	2	1500.	3500.
2	3	0	5000

$x_1 = 1500. \qquad y_1 = 3500. \qquad x_2 = 0 \qquad y_2 = 5000$

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 2121.32$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = -0.707107 \qquad m_s = \dfrac{y_2 - y_1}{L} = 0.707107$

Substituting into the truss element equations we get

$$\begin{pmatrix} 32998.3 & -32998.3 & -32998.3 & 32998.3 \\ -32998.3 & 32998.3 & 32998.3 & -32998.3 \\ -32998.3 & 32998.3 & 32998.3 & -32998.3 \\ 32998.3 & -32998.3 & -32998.3 & 32998.3 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {3, 4, 5, 6} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$

and to a global matrix: $\begin{pmatrix} [3,3] & [3,4] & [3,5] & [3,6] \\ [4,3] & [4,4] & [4,5] & [4,6] \\ [5,3] & [5,4] & [5,5] & [5,6] \\ [6,3] & [6,4] & [6,5] & [6,6] \end{pmatrix}$

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 32600.2 & 76067.2 & -32600.2 & -76067.2 & 0 & 0 & 0 & 0 \\ 76067.2 & 297490. & -76067.2 & -177490. & 0 & -120000 & 0 & 0 \\ -32600.2 & -76067.2 & 243089. & 119136. & -32998.3 & 32998.3 & -177490. & -76067.2 \\ -76067.2 & -177490. & 119136. & 243089. & 32998.3 & -32998.3 & -76067.2 & -32600.2 \\ 0 & 0 & -32998.3 & 32998.3 & 152998. & -32998.3 & -120000 & 0 \\ 0 & -120000 & 32998.3 & -32998.3 & -32998.3 & 152998. & 0 & 0 \\ 0 & 0 & -177490. & -76067.2 & -120000 & 0 & 297490. & 76067.2 \\ 0 & 0 & -76067.2 & -32600.2 & 0 & 0 & 76067.2 & 32600.2 \end{pmatrix}$$

$$\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -150000. \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Essential boundary conditions

Node	dof	Value
1	u_1	0
	v_1	0
4	u_4	0
	v_4	0

Remove {1, 2, 7, 8} rows and columns.

After adjusting for essential boundary conditions we have

$$\begin{pmatrix} 243089. & 119136. & -32998.3 & 32998.3 \\ 119136. & 243089. & 32998.3 & -32998.3 \\ -32998.3 & 32998.3 & 152998. & -32998.3 \\ 32998.3 & -32998.3 & -32998.3 & 152998. \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -150000. \\ 0 \\ 0 \end{pmatrix}$$

Solving the final system of global equations we get

$$\{u_2 = 0.538954,\ v_2 = -0.953061,\ u_3 = 0.264704,\ v_3 = -0.264704\}$$

Complete table of nodal values

	u	v
1	0	0
2	0.538954	−0.953061
3	0.264704	−0.264704
4	0	0

Computation of reactions

Equation numbers of dof with specified values: {1, 2, 7, 8}

Extracting equations {1, 2, 7, 8} from the global system we have

$$\begin{pmatrix} 32600.2 & 76067.2 & -32600.2 & -76067.2 & 0 & 0 & 0 & 0 \\ 76067.2 & 297490. & -76067.2 & -177490. & 0 & -120000 & 0 & 0 \\ 0 & 0 & -177490. & -76067.2 & -120000 & 0 & 297490. & 76067.2 \\ 0 & 0 & -76067.2 & -32600.2 & 0 & 0 & 76067.2 & 32600.2 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} R_1 + 0. \\ R_2 + 0. \\ R_3 + 0. \\ R_4 + 0. \end{pmatrix}$$

Substituting the nodal values and re−arranging

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{pmatrix} = \begin{pmatrix} 32600.2 & 76067.2 & -32600.2 & -76067.2 & 0 & 0 & 0 & 0 \\ 76067.2 & 297490. & -76067.2 & -177490. & 0 & -120000 & 0 & 0 \\ 0 & 0 & -177490. & -76067.2 & -120000 & 0 & 297490. & 76067.2 \\ 0 & 0 & -76067.2 & -32600.2 & 0 & 0 & 76067.2 & 32600.2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0.538954 \\ -0.953061 \\ 0.264704 \\ -0.264704 \\ 0 \\ 0 \end{pmatrix}$$

Carrying out computations, the reactions are as follows.

Label	dof	Reaction
R_1	u_1	54926.7
R_2	v_1	159927.
R_3	u_4	−54926.7
R_4	v_4	−9926.67

Sum of Reactions

dof: u	0
dof: v	150000.

Solution for element 1

Nodal coordinates

Element node	Global node number	x	y
1	1	0	0
2	2	1500.	3500.

$x_1 = 0 \quad y_1 = 0 \quad x_2 = 1500. \quad y_2 = 3500.$

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 3807.89$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 0.393919 \qquad m_s = \dfrac{y_2 - y_1}{L} = 0.919145$

Global to local transformation matrix, $\boldsymbol{T} = \begin{pmatrix} 0.393919 & 0.919145 & 0 & 0 \\ 0 & 0 & 0.393919 & 0.919145 \end{pmatrix}$

Element nodal displacements in global coordinates, $\boldsymbol{d} = \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0.538954 \\ -0.953061 \end{pmatrix}$

Element nodal displacements in local coordinates, $\boldsymbol{d}_\ell = \boldsymbol{T}\,\boldsymbol{d} = \begin{pmatrix} 0. \\ -0.663697 \end{pmatrix}$

Axial displacements at element ends, $d_1 = 0. \qquad d_2 = -0.663697$

$E = 200000 \qquad A = 4000$

Axial strain, $\epsilon = (d_2 - d_1)/L = -0.000174295$

Axial stress, $\sigma = E\epsilon = -34.8591 \qquad$ Axial force $= \sigma A = -139436.$

Solution for element 2

Nodal coordinates

Element node	Global node number	x	y
1	2	1500.	3500.
2	4	5000	5000

$x_1 = 1500. \quad y_1 = 3500. \quad x_2 = 5000 \quad y_2 = 5000$

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 3807.89$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 0.919145 \qquad m_s = \dfrac{y_2 - y_1}{L} = 0.393919$

Global to local transformation matrix, $\boldsymbol{T} = \begin{pmatrix} 0.919145 & 0.393919 & 0 & 0 \\ 0 & 0 & 0.919145 & 0.393919 \end{pmatrix}$

Element nodal displacements in global coordinates, $\boldsymbol{d} = \begin{pmatrix} u_2 \\ v_2 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0.538954 \\ -0.953061 \\ 0 \\ 0 \end{pmatrix}$

Element nodal displacements in local coordinates, $\boldsymbol{d}_\ell = \boldsymbol{T}\,\boldsymbol{d} = \begin{pmatrix} 0.119947 \\ 0. \end{pmatrix}$

Axial displacements at element ends, $d_1 = 0.119947 \qquad d_2 = 0.$

$E = 200000 \qquad A = 4000$

Axial strain, $\epsilon = (d_2 - d_1)/L = -0.0000314997$

Axial stress, $\sigma = E\epsilon = -6.29994 \qquad$ Axial force $= \sigma A = -25199.8$

Solution for element 3

Nodal coordinates

Element node	Global node number	x	y
1	1	0	0
2	3	0	5000

$x_1 = 0$ $\quad y_1 = 0$ $\quad x_2 = 0$ $\quad y_2 = 5000$

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 5000$

Direction cosines: $\ell_s = \frac{x_2 - x_1}{L} = 0$ $\qquad m_s = \frac{y_2 - y_1}{L} = 1$

Global to local transformation matrix, $\boldsymbol{T} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Element nodal displacements in global coordinates, $\boldsymbol{d} = \begin{pmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0.264704 \\ -0.264704 \end{pmatrix}$

Element nodal displacements in local coordinates, $\boldsymbol{d}_\ell = \boldsymbol{T}\,\boldsymbol{d} = \begin{pmatrix} 0. \\ -0.264704 \end{pmatrix}$

Axial displacements at element ends, $d_1 = 0.$ $\qquad d_2 = -0.264704$

$E = 200000$ $\qquad A = 3000$

Axial strain, $\epsilon = (d_2 - d_1)/L = -0.0000529407$

Axial stress, $\sigma = E\epsilon = -10.5881$ $\qquad$ Axial force $= \sigma A = -31764.4$

Solution for element 4

Nodal coordinates

Element node	Global node number	x	y
1	3	0	5000
2	4	5000	5000

$x_1 = 0$ $\quad y_1 = 5000$ $\quad x_2 = 5000$ $\quad y_2 = 5000$

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 5000$

Direction cosines: $\ell_s = \frac{x_2 - x_1}{L} = 1$ $\qquad m_s = \frac{y_2 - y_1}{L} = 0$

Global to local transformation matrix, $\boldsymbol{T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Element nodal displacements in global coordinates, $\boldsymbol{d} = \begin{pmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0.264704 \\ -0.264704 \\ 0 \\ 0 \end{pmatrix}$

Element nodal displacements in local coordinates, $\boldsymbol{d}_\ell = \boldsymbol{T}\,\boldsymbol{d} = \begin{pmatrix} 0.264704 \\ 0. \end{pmatrix}$

Axial displacements at element ends, $d_1 = 0.264704$ $\qquad d_2 = 0.$

$E = 200000$ $\qquad A = 3000$

Axial strain, $\epsilon = (d_2 - d_1)/L = -0.0000529407$

Axial stress, $\sigma = E\epsilon = -10.5881$ Axial force $= \sigma A = -31764.4$

Solution for element 5

Nodal coordinates

Element node	Global node number	x	y
1	2	1500.	3500.
2	3	0	5000

$x_1 = 1500.$ $y_1 = 3500.$ $x_2 = 0$ $y_2 = 5000$

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 2121.32$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = -0.707107$ $m_s = \dfrac{y_2 - y_1}{L} = 0.707107$

Global to local transformation matrix, $\boldsymbol{T} = \begin{pmatrix} -0.707107 & 0.707107 & 0 & 0 \\ 0 & 0 & -0.707107 & 0.707107 \end{pmatrix}$

Element nodal displacements in global coordinates, $\boldsymbol{d} = \begin{pmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0.538954 \\ -0.953061 \\ 0.264704 \\ -0.264704 \end{pmatrix}$

Element nodal displacements in local coordinates, $\boldsymbol{d}_\ell = \boldsymbol{T}\,\boldsymbol{d} = \begin{pmatrix} -1.05501 \\ -0.374347 \end{pmatrix}$

Axial displacements at element ends, $d_1 = -1.05501$ $d_2 = -0.374347$

$E = 70000$ $A = 2000$

Axial strain, $\epsilon = (d_2 - d_1)/L = 0.000320869$

Axial stress, $\sigma = E\epsilon = 22.4608$ Axial force $= \sigma A = 44921.7$

Solution summary

Nodal solution

	x−coord	y−coord	u	v
1	0	0	0	0
2	1500.	3500.	0.538954	−0.953061
3	0	5000	0.264704	−0.264704
4	5000	5000	0	0

Element solution

	Stress	Axial force
1	−34.8591	−139436.
2	−6.29994	−25199.8
3	−10.5881	−31764.4
4	−10.5881	−31764.4
5	22.4608	44921.7

Support reactions

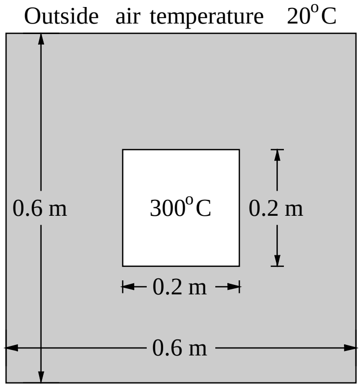
Node	dof	Reaction
1	u_1	54926.7
1	v_1	159927.
4	u_4	−54926.7
4	v_4	−9926.67

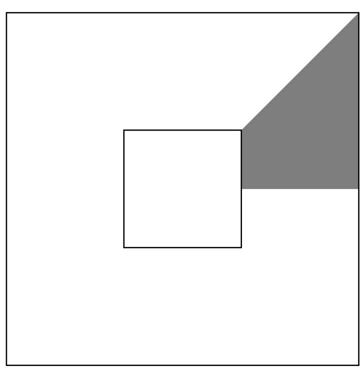
Sum of applied loads → (0   −150000.)

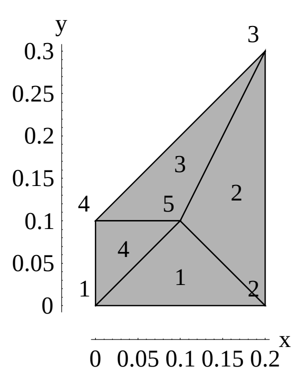
Sum of support reactions → (0   150000.)

Square duct heat flow example: Examples 1.5 p. 28, 1.8 p. 44, and 1.11 p. 52

Cross-section of a 20 cm by 20 cm duct made of concrete walls 20 cm thick is shown in Figure. The inside surface of the duct is maintained at a temperature of 300 °C due to hot gases flowing from a furnace. On the outside the duct is exposed to air with an ambient temperature of 20 °C. The heat conduction coefficient of concrete is 1.4 W/m.°C. The average convection heat transfer coefficient on the outside of the duct is 27 W/m.°C.







Global equations at start of the element assembly process

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 1

$k_x = 1.4 \qquad k_y = 1.4 \qquad Q = 0$

Nodal coordinates

Element node	Global node number	x	y
1	1	0.	0.
2	2	0.2	0.
3	5	0.1	0.1

$x_1 = 0. \qquad x_2 = 0.2 \qquad x_3 = 0.1$
$y_1 = 0. \qquad y_2 = 0. \qquad y_3 = 0.1$

Using these values we get

$b_1 = -0.1 \qquad b_2 = 0.1 \qquad b_3 = 0.$

$c_1 = -0.1 \qquad c_2 = -0.1 \qquad c_3 = 0.2$

$f_1 = 0.02 \qquad f_2 = 0. \qquad f_3 = 0.$

Element area, $A = 0.01$

Substituting these values we get

$$\boldsymbol{k}_k = \begin{pmatrix} 0.7 & 0. & -0.7 \\ 0. & 0.7 & -0.7 \\ -0.7 & -0.7 & 1.4 \end{pmatrix} \qquad \boldsymbol{r}_Q = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 0.7 & 0. & -0.7 \\ 0. & 0.7 & -0.7 \\ -0.7 & -0.7 & 1.4 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to {1, 2, 5} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$

and to a global matrix: $\begin{pmatrix} [1,1] & [1,2] & [1,5] \\ [2,1] & [2,2] & [2,5] \\ [5,1] & [5,2] & [5,5] \end{pmatrix}$

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 0.7 & 0 & 0 & 0 & -0.7 \\ 0 & 0.7 & 0 & 0 & -0.7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -0.7 & -0.7 & 0 & 0 & 1.4 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 2

$k_x = 1.4 \qquad k_y = 1.4 \qquad Q = 0$

Nodal coordinates

Element node	Global node number	x	y
1	2	0.2	0.
2	3	0.2	0.3
3	5	0.1	0.1

$x_1 = 0.2 \qquad x_2 = 0.2 \qquad x_3 = 0.1$
$y_1 = 0. \qquad y_2 = 0.3 \qquad y_3 = 0.1$

Using these values we get

$b_1 = 0.2 \qquad b_2 = 0.1 \qquad b_3 = -0.3$

$c_1 = -0.1 \qquad c_2 = 0.1 \qquad c_3 = 0.$

$f_1 = -0.01 \qquad f_2 = -0.02 \qquad f_3 = 0.06$

Element area, A = 0.015

Substituting these values we get

$$\boldsymbol{k}_k = \begin{pmatrix} 1.16667 & 0.233333 & -1.4 \\ 0.233333 & 0.466667 & -0.7 \\ -1.4 & -0.7 & 2.1 \end{pmatrix} \qquad \boldsymbol{r}_Q = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Convection on side 1 (nodes {2, 3}) with h = 27 and $T_\infty = 20$

End nodal coordinates: ({0.2, 0.} {0.2, 0.3}) giving side length, L = 0.3

Using these values we get

$$\boldsymbol{k}_h = \begin{pmatrix} 2.7 & 1.35 & 0 \\ 1.35 & 2.7 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \boldsymbol{r}_h = \begin{pmatrix} 81. \\ 81. \\ 0 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 3.86667 & 1.58333 & -1.4 \\ 1.58333 & 3.16667 & -0.7 \\ -1.4 & -0.7 & 2.1 \end{pmatrix} \begin{pmatrix} T_2 \\ T_3 \\ T_5 \end{pmatrix} = \begin{pmatrix} 81. \\ 81. \\ 0 \end{pmatrix}$$

The element contributes to {2, 3, 5} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$

and to a global matrix: $\begin{pmatrix} [2,2] & [2,3] & [2,5] \\ [3,2] & [3,3] & [3,5] \\ [5,2] & [5,3] & [5,5] \end{pmatrix}$

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 0.7 & 0 & 0 & 0 & -0.7 \\ 0 & 4.56667 & 1.58333 & 0 & -2.1 \\ 0 & 1.58333 & 3.16667 & 0 & -0.7 \\ 0 & 0 & 0 & 0 & 0 \\ -0.7 & -2.1 & -0.7 & 0 & 3.5 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 81. \\ 81. \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 3

$k_x = 1.4 \qquad k_y = 1.4 \qquad Q = 0$

Nodal coordinates

Element node	Global node number	x	y
1	3	0.2	0.3
2	4	0.	0.1
3	5	0.1	0.1

$x_1 = 0.2$ $x_2 = 0.$ $x_3 = 0.1$
$y_1 = 0.3$ $y_2 = 0.1$ $y_3 = 0.1$

Using these values we get

$b_1 = 0.$ $b_2 = -0.2$ $b_3 = 0.2$

$c_1 = 0.1$ $c_2 = 0.1$ $c_3 = -0.2$

$f_1 = -0.01$ $f_2 = 0.01$ $f_3 = 0.02$

Element area, $A = 0.01$

Substituting these values we get

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 0.35 & 0.35 & -0.7 \\ 0.35 & 1.75 & -2.1 \\ -0.7 & -2.1 & 2.8 \end{pmatrix} \qquad \boldsymbol{r}_{\mathrm{Q}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 0.35 & 0.35 & -0.7 \\ 0.35 & 1.75 & -2.1 \\ -0.7 & -2.1 & 2.8 \end{pmatrix} \begin{pmatrix} T_3 \\ T_4 \\ T_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to {3, 4, 5} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$

and to a global matrix: $\begin{pmatrix} [3, 3] & [3, 4] & [3, 5] \\ [4, 3] & [4, 4] & [4, 5] \\ [5, 3] & [5, 4] & [5, 5] \end{pmatrix}$

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 0.7 & 0 & 0 & 0 & -0.7 \\ 0 & 4.56667 & 1.58333 & 0 & -2.1 \\ 0 & 1.58333 & 3.51667 & 0.35 & -1.4 \\ 0 & 0 & 0.35 & 1.75 & -2.1 \\ -0.7 & -2.1 & -1.4 & -2.1 & 6.3 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 81. \\ 81. \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 4

$k_x = 1.4$ $k_y = 1.4$ $Q = 0$

Nodal coordinates

Element node	Global node number	x	y
1	1	0.	0.
2	5	0.1	0.1
3	4	0.	0.1

$x_1 = 0.$ $x_2 = 0.1$ $x_3 = 0.$
$y_1 = 0.$ $y_2 = 0.1$ $y_3 = 0.1$

Using these values we get

$b_1 = 0.$ $\quad b_2 = 0.1$ $\quad b_3 = -0.1$

$c_1 = -0.1$ $\quad c_2 = 0.$ $\quad c_3 = 0.1$

$f_1 = 0.01$ $\quad f_2 = 0.$ $\quad f_3 = 0.$

Element area, $A = 0.005$

Substituting these values we get

$$\boldsymbol{k}_\mathrm{k} = \begin{pmatrix} 0.7 & 0. & -0.7 \\ 0. & 0.7 & -0.7 \\ -0.7 & -0.7 & 1.4 \end{pmatrix} \qquad \boldsymbol{r}_\mathrm{Q} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 0.7 & 0. & -0.7 \\ 0. & 0.7 & -0.7 \\ -0.7 & -0.7 & 1.4 \end{pmatrix} \begin{pmatrix} T_1 \\ T_5 \\ T_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to {1, 5, 4} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$

and to a global matrix: $\begin{pmatrix} [1,1] & [1,5] & [1,4] \\ [5,1] & [5,5] & [5,4] \\ [4,1] & [4,5] & [4,4] \end{pmatrix}$

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 1.4 & 0 & 0 & -0.7 & -0.7 \\ 0 & 4.56667 & 1.58333 & 0 & -2.1 \\ 0 & 1.58333 & 3.51667 & 0.35 & -1.4 \\ -0.7 & 0 & 0.35 & 3.15 & -2.8 \\ -0.7 & -2.1 & -1.4 & -2.8 & 7. \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 81. \\ 81. \\ 0 \\ 0 \end{pmatrix}$$

Essential boundary conditions

Node	dof	Value
1	T_1	300
4	T_4	300

Delete equations {1, 4}.

$$\begin{pmatrix} 0 & 4.56667 & 1.58333 & 0 & -2.1 \\ 0 & 1.58333 & 3.51667 & 0.35 & -1.4 \\ -0.7 & -2.1 & -1.4 & -2.8 & 7. \end{pmatrix} \begin{pmatrix} 300 \\ T_2 \\ T_3 \\ 300 \\ T_5 \end{pmatrix} = \begin{pmatrix} 81. \\ 81. \\ 0 \end{pmatrix}$$

Extract columns {1, 4}.

Multiply each column by its respective known value {300, 300}.

Move all resulting vectors to the rhs.

After adjusting for essential boundary conditions we have

$$\begin{pmatrix} 4.56667 & 1.58333 & -2.1 \\ 1.58333 & 3.51667 & -1.4 \\ -2.1 & -1.4 & 7. \end{pmatrix} \begin{pmatrix} T_2 \\ T_3 \\ T_5 \end{pmatrix} = \begin{pmatrix} 81. \\ -24. \\ 1050. \end{pmatrix}$$

Solving the final system of global equations we get

$\{T_2 = 93.5466,\ T_3 = 23.8437,\ T_5 = 182.833\}$

Complete table of nodal values

	T
1	300
2	93.5466
3	23.8437
4	300
5	182.833

Computation of reactions

Equation numbers of dof with specified values: {1, 4}

Extracting equations {1, 4} from the global system we have

$$\begin{pmatrix} 1.4 & 0 & 0 & -0.7 & -0.7 \\ -0.7 & 0 & 0.35 & 3.15 & -2.8 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$$

Substituting the nodal values and re−arranging

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \begin{pmatrix} 1.4 & 0 & 0 & -0.7 & -0.7 \\ -0.7 & 0 & 0.35 & 3.15 & -2.8 \end{pmatrix} \begin{pmatrix} 300 \\ 93.5466 \\ 23.8437 \\ 300 \\ 182.833 \end{pmatrix}$$

Carrying out computations, the reactions are as follows.

Label	dof	Reaction
R_1	T_1	82.0171
R_2	T_4	231.414

Sum of Reactions

313.431

Solution for element 1

Nodal coordinates

Element node	Global node number	x	y
1	1	0.	0.
2	2	0.2	0.
3	5	0.1	0.1

$x_1 = 0.$ $\quad x_2 = 0.2$ $\quad x_3 = 0.1$

$y_1 = 0.$ $\quad y_2 = 0.$ $\quad y_3 = 0.1$

Using these values we get

$b_1 = -0.1$ $b_2 = 0.1$ $b_3 = 0.$

$c_1 = -0.1$ $c_2 = -0.1$ $c_3 = 0.2$

$f_1 = 0.02$ $f_2 = 0.$ $f_3 = 0.$

Element area, $A = 0.01$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^T = \{-5.\,x - 5.\,y + 1.,\ 5.\,x - 5.\,y,\ 10.\,y\}$

From global solution the temperatures at the element nodes are
(from nodes {1, 2, 5}), $\boldsymbol{d}^T = \{300,\ 93.5466,\ 182.833\}$

Thus the temperature distribution over the element, $T(x,y) = \boldsymbol{N}^T\boldsymbol{d} = -1032.27\,x - 139.406\,y + 300.$

Differentiating with respect to x and y, $\partial T/\partial x = -1032.27$ and $\partial T/\partial y = -139.406$

Solution for element 2

Nodal coordinates

Element node	Global node number	x	y
1	2	0.2	0.
2	3	0.2	0.3
3	5	0.1	0.1

$x_1 = 0.2$ $x_2 = 0.2$ $x_3 = 0.1$
$y_1 = 0.$ $y_2 = 0.3$ $y_3 = 0.1$

Using these values we get

$b_1 = 0.2$ $b_2 = 0.1$ $b_3 = -0.3$

$c_1 = -0.1$ $c_2 = 0.1$ $c_3 = 0.$

$f_1 = -0.01$ $f_2 = -0.02$ $f_3 = 0.06$

Element area, $A = 0.015$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^T = \{6.66667\,x - 3.33333\,y - 0.333333,\ 3.33333\,x + 3.33333\,y - 0.666667,\ 2. - 10.\,x\}$

From global solution the temperatures at the element nodes are
(from nodes {2, 3, 5}), $\boldsymbol{d}^T = \{93.5466,\ 23.8437,\ 182.833\}$

Thus the temperature distribution over the element, $T(x,y) = \boldsymbol{N}^T\boldsymbol{d} = -1125.2\,x - 232.343\,y + 318.587$

Differentiating with respect to x and y, $\partial T/\partial x = -1125.2$ and $\partial T/\partial y = -232.343$

Solution for element 3

Nodal coordinates

Element node	Global node number	x	y
1	3	0.2	0.3
2	4	0.	0.1
3	5	0.1	0.1

$x_1 = 0.2$ $x_2 = 0.$ $x_3 = 0.1$
$y_1 = 0.3$ $y_2 = 0.1$ $y_3 = 0.1$

Using these values we get

$b_1 = 0.$ $\quad b_2 = -0.2$ $\quad b_3 = 0.2$

$c_1 = 0.1$ $\quad c_2 = 0.1$ $\quad c_3 = -0.2$

$f_1 = -0.01$ $\quad f_2 = 0.01$ $\quad f_3 = 0.02$

Element area, A = 0.01

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^T = \{5.\,y - 0.5,\ -10.\,x + 5.\,y + 0.5,\ 10.\,x - 10.\,y + 1.\}$

From global solution the temperatures at the element nodes are
(from nodes {3, 4, 5}), $\boldsymbol{d}^T = \{23.8437,\ 300,\ 182.833\}$

Thus the temperature distribution over the element, $T(x,y) = \boldsymbol{N}^T\boldsymbol{d} = -1171.67\,x - 209.109\,y + 320.911$

Differentiating with respect to x and y, $\partial T/\partial x = -1171.67$ and $\partial T/\partial y = -209.109$

Solution for element 4

Nodal coordinates

Element node	Global node number	x	y
1	1	0.	0.
2	5	0.1	0.1
3	4	0.	0.1

$x_1 = 0.$ $\quad x_2 = 0.1$ $\quad x_3 = 0.$
$y_1 = 0.$ $\quad y_2 = 0.1$ $\quad y_3 = 0.1$

Using these values we get

$b_1 = 0.$ $\quad b_2 = 0.1$ $\quad b_3 = -0.1$

$c_1 = -0.1$ $\quad c_2 = 0.$ $\quad c_3 = 0.1$

$f_1 = 0.01$ $\quad f_2 = 0.$ $\quad f_3 = 0.$

Element area, A = 0.005

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^T = \{1. - 10.\,y,\ 10.\,x,\ 10.\,y - 10.\,x\}$

From global solution the temperatures at the element nodes are
(from nodes {1, 5, 4}), $\boldsymbol{d}^T = \{300,\ 182.833,\ 300\}$

Thus the temperature distribution over the element, $T(x,y) = \boldsymbol{N}^T\boldsymbol{d} = 300. - 1171.67\,x$

Differentiating with respect to x and y, $\partial T/\partial x = -1171.67$ and $\partial T/\partial y = 0$

Solution summary

Nodal temperatures

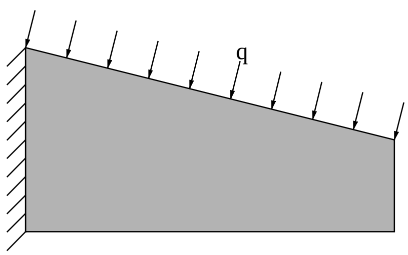
Node	Temperature
1	300
2	93.5466
3	23.8437
4	300
5	182.833

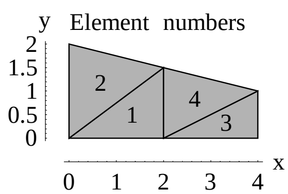
Element solution

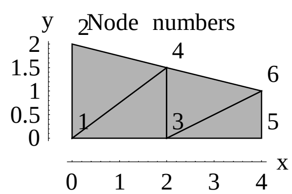
	T(x,y)	∂T/∂x	∂T/∂y
1	−1032.27 x − 139.406 y + 300.	−1032.27	−139.406
2	−1125.2 x − 232.343 y + 318.587	−1125.2	−232.343
3	−1171.67 x − 209.109 y + 320.911	−1171.67	−209.109
4	300. − 1171.67 x	−1171.67	0

Stress analysis of a bracket: Examples 1.6 p. 32, 1.9 p. 46, and 1.12 p. 55

Top surface of a thin cantilever bracket is subjected to normal pressure $q = 20\,\text{lb}/\text{in}^2$ as shown in Figure. The bracket is 4 in long and is 2 in wide at the base and 1 in wide at the free end. The thickness of the bracket perpendicular to the plane of paper is $1/4$ in. The material properties are $E = 10^4\,\text{lb}/\text{in}^2$ and $\nu = 0.2$.







Global equations at start of the element assembly process

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} u_1\\ v_1\\ u_2\\ v_2\\ u_3\\ v_3\\ u_4\\ v_4\\ u_5\\ v_5\\ u_6\\ v_6 \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}$$

Equations for element 1

h = 0.25; E = 10000; $\nu = 0.2$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 10416.7 & 2083.33 & 0 \\ 2083.33 & 10416.7 & 0 \\ 0 & 0 & 4166.67 \end{pmatrix}$

Nodal coordinates

Element node	Global node number	x	y
1	1	0.	0.
2	3	2.	0.
3	4	2.	1.5

$x_1 = 0.$ $x_2 = 2.$ $x_3 = 2.$
$y_1 = 0.$ $y_2 = 0.$ $y_3 = 1.5$

Using these values we get

$b_1 = -1.5$ $b_2 = 1.5$ $b_3 = 0.$

$c_1 = 0.$ $c_2 = -2.$ $c_3 = 2.$

$f_1 = 3.$ $f_2 = 0.$ $f_3 = 0.$

Element area, A = 1.5

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -0.5 & 0 & 0.5 & 0 & 0. & 0 \\ 0 & 0. & 0 & -0.666667 & 0 & 0.666667 \\ 0. & -0.5 & -0.666667 & 0.5 & 0.666667 & 0. \end{pmatrix}$$

Thus the element stiffness matrix is

$$\boldsymbol{k} = \mathrm{h}\mathrm{A}\boldsymbol{BCB}^{\mathrm{T}} = \begin{pmatrix} 976.563 & 0 & -976.563 & 260.417 & 0 & -260.417 \\ 0 & 390.625 & 520.833 & -390.625 & -520.833 & 0 \\ -976.563 & 520.833 & 1671.01 & -781.25 & -694.444 & 260.417 \\ 260.417 & -390.625 & -781.25 & 2126.74 & 520.833 & -1736.11 \\ 0 & -520.833 & -694.444 & 520.833 & 694.444 & 0 \\ -260.417 & 0 & 260.417 & -1736.11 & 0 & 1736.11 \end{pmatrix}$$

Complete equations for element 1

$$\begin{pmatrix} 976.563 & 0 & -976.563 & 260.417 & 0 & -260.417 \\ 0 & 390.625 & 520.833 & -390.625 & -520.833 & 0 \\ -976.563 & 520.833 & 1671.01 & -781.25 & -694.444 & 260.417 \\ 260.417 & -390.625 & -781.25 & 2126.74 & 520.833 & -1736.11 \\ 0 & -520.833 & -694.444 & 520.833 & 694.444 & 0 \\ -260.417 & 0 & 260.417 & -1736.11 & 0 & 1736.11 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {1, 2, 5, 6, 7, 8} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 1 \\ 2 \\ 5 \\ 6 \\ 7 \\ 8 \end{pmatrix}$

$$\text{and to a global matrix: } \begin{pmatrix} [1,1] & [1,2] & [1,5] & [1,6] & [1,7] & [1,8] \\ [2,1] & [2,2] & [2,5] & [2,6] & [2,7] & [2,8] \\ [5,1] & [5,2] & [5,5] & [5,6] & [5,7] & [5,8] \\ [6,1] & [6,2] & [6,5] & [6,6] & [6,7] & [6,8] \\ [7,1] & [7,2] & [7,5] & [7,6] & [7,7] & [7,8] \\ [8,1] & [8,2] & [8,5] & [8,6] & [8,7] & [8,8] \end{pmatrix}$$

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 976.563 & 0 & 0 & 0 & -976.563 & 260.417 & 0 & -260.417 & 0 & 0 & 0 & 0 \\ 0 & 390.625 & 0 & 0 & 520.833 & -390.625 & -520.833 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -976.563 & 520.833 & 0 & 0 & 1671.01 & -781.25 & -694.444 & 260.417 & 0 & 0 & 0 & 0 \\ 260.417 & -390.625 & 0 & 0 & -781.25 & 2126.74 & 520.833 & -1736.11 & 0 & 0 & 0 & 0 \\ 0 & -520.833 & 0 & 0 & -694.444 & 520.833 & 694.444 & 0 & 0 & 0 & 0 & 0 \\ -260.417 & 0 & 0 & 0 & 260.417 & -1736.11 & 0 & 1736.11 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 2

$h = 0.25$; $E = 10000$; $\nu = 0.2$

$$\text{Plane stress constitutive matrix, } \boldsymbol{C} = \begin{pmatrix} 10416.7 & 2083.33 & 0 \\ 2083.33 & 10416.7 & 0 \\ 0 & 0 & 4166.67 \end{pmatrix}$$

Nodal coordinates

Element node	Global node number	x	y
1	4	2.	1.5
2	2	0.	2.
3	1	0.	0.

$x_1 = 2.$ $x_2 = 0.$ $x_3 = 0.$
$y_1 = 1.5$ $y_2 = 2.$ $y_3 = 0.$

Using these values we get

$b_1 = 2.$ $b_2 = -1.5$ $b_3 = -0.5$

$c_1 = 0.$ $c_2 = 2.$ $c_3 = -2.$

$f_1 = 0.$ $f_2 = 0.$ $f_3 = 4.$

Element area, $A = 2.$

$$\boldsymbol{B}^T = \begin{pmatrix} 0.5 & 0 & -0.375 & 0 & -0.125 & 0 \\ 0 & 0. & 0 & 0.5 & 0 & -0.5 \\ 0. & 0.5 & 0.5 & -0.375 & -0.5 & -0.125 \end{pmatrix}$$

Thus the element stiffness matrix is

$$\boldsymbol{k} = \mathrm{hA}\boldsymbol{BCB}^{\mathrm{T}} = \begin{pmatrix} 1302.08 & 0 & -976.563 & 260.417 & -325.521 & -260.417 \\ 0 & 520.833 & 520.833 & -390.625 & -520.833 & -130.208 \\ -976.563 & 520.833 & 1253.26 & -585.938 & -276.693 & 65.1042 \\ 260.417 & -390.625 & -585.938 & 1595.05 & 325.521 & -1204.43 \\ -325.521 & -520.833 & -276.693 & 325.521 & 602.214 & 195.313 \\ -260.417 & -130.208 & 65.1042 & -1204.43 & 195.313 & 1334.64 \end{pmatrix}$$

Load vector due to distributed load on side 1 (nodes {4, 2})

Specified load components: $q_n = -20$; $q_t = 0$

End nodal coordinates: ({2., 1.5} {0., 2.}) giving side length, L = 2.06155

Components of unit normal to the side: $n_x = 0.242536$; $n_y = 0.970143$

Using these values we get
$\boldsymbol{r}_q^{\mathrm{T}} = (\,-1.25 \quad -5. \quad -1.25 \quad -5. \quad 0 \quad 0\,)$

Complete equations for element 2

$$\begin{pmatrix} 1302.08 & 0 & -976.563 & 260.417 & -325.521 & -260.417 \\ 0 & 520.833 & 520.833 & -390.625 & -520.833 & -130.208 \\ -976.563 & 520.833 & 1253.26 & -585.938 & -276.693 & 65.1042 \\ 260.417 & -390.625 & -585.938 & 1595.05 & 325.521 & -1204.43 \\ -325.521 & -520.833 & -276.693 & 325.521 & 602.214 & 195.313 \\ -260.417 & -130.208 & 65.1042 & -1204.43 & 195.313 & 1334.64 \end{pmatrix} \begin{pmatrix} u_4 \\ v_4 \\ u_2 \\ v_2 \\ u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} -1.25 \\ -5. \\ -1.25 \\ -5. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {7, 8, 3, 4, 1, 2} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 7 \\ 8 \\ 3 \\ 4 \\ 1 \\ 2 \end{pmatrix}$

and to a global matrix: $\begin{pmatrix} [7,7] & [7,8] & [7,3] & [7,4] & [7,1] & [7,2] \\ [8,7] & [8,8] & [8,3] & [8,4] & [8,1] & [8,2] \\ [3,7] & [3,8] & [3,3] & [3,4] & [3,1] & [3,2] \\ [4,7] & [4,8] & [4,3] & [4,4] & [4,1] & [4,2] \\ [1,7] & [1,8] & [1,3] & [1,4] & [1,1] & [1,2] \\ [2,7] & [2,8] & [2,3] & [2,4] & [2,1] & [2,2] \end{pmatrix}$

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 1578.78 & 195.313 & -276.693 & 325.521 & -976.563 & 260.417 & -325.521 & -781.25 & 0 & 0 & 0 & 0 \\ 195.313 & 1725.26 & 65.1042 & -1204.43 & 520.833 & -390.625 & -781.25 & -130.208 & 0 & 0 & 0 & 0 \\ -276.693 & 65.1042 & 1253.26 & -585.938 & 0 & 0 & -976.563 & 520.833 & 0 & 0 & 0 & 0 \\ 325.521 & -1204.43 & -585.938 & 1595.05 & 0 & 0 & 260.417 & -390.625 & 0 & 0 & 0 & 0 \\ -976.563 & 520.833 & 0 & 0 & 1671.01 & -781.25 & -694.444 & 260.417 & 0 & 0 & 0 & 0 \\ 260.417 & -390.625 & 0 & 0 & -781.25 & 2126.74 & 520.833 & -1736.11 & 0 & 0 & 0 & 0 \\ -325.521 & -781.25 & -976.563 & 260.417 & -694.444 & 520.833 & 1996.53 & 0 & 0 & 0 & 0 & 0 \\ -781.25 & -130.208 & 520.833 & -390.625 & 260.417 & -1736.11 & 0 & 2256.94 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1.25 \\ -5. \\ 0 \\ 0 \\ -1.25 \\ -5. \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 3

$h = 0.25;$ $E = 10000;$ $\nu = 0.2$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 10416.7 & 2083.33 & 0 \\ 2083.33 & 10416.7 & 0 \\ 0 & 0 & 4166.67 \end{pmatrix}$

Nodal coordinates

Element node	Global node number	x	y
1	3	2.	0.
2	5	4.	0.
3	6	4.	1.

$x_1 = 2.$ $x_2 = 4.$ $x_3 = 4.$
$y_1 = 0.$ $y_2 = 0.$ $y_3 = 1.$

Using these values we get

$b_1 = -1.$ $b_2 = 1.$ $b_3 = 0.$

$c_1 = 0.$ $c_2 = -2.$ $c_3 = 2.$

$f_1 = 4.$ $f_2 = -2.$ $f_3 = 0.$

Element area, A = 1.

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -0.5 & 0 & 0.5 & 0 & 0. & 0 \\ 0 & 0. & 0 & -1. & 0 & 1. \\ 0. & -0.5 & -1. & 0.5 & 1. & 0. \end{pmatrix}$$

Thus the element stiffness matrix is

$$\boldsymbol{k} = \mathrm{hA}\boldsymbol{BCB}^{\mathrm{T}} = \begin{pmatrix} 651.042 & 0 & -651.042 & 260.417 & 0 & -260.417 \\ 0 & 260.417 & 520.833 & -260.417 & -520.833 & 0 \\ -651.042 & 520.833 & 1692.71 & -781.25 & -1041.67 & 260.417 \\ 260.417 & -260.417 & -781.25 & 2864.58 & 520.833 & -2604.17 \\ 0 & -520.833 & -1041.67 & 520.833 & 1041.67 & 0 \\ -260.417 & 0 & 260.417 & -2604.17 & 0 & 2604.17 \end{pmatrix}$$

Complete equations for element 3

$$\begin{pmatrix} 651.042 & 0 & -651.042 & 260.417 & 0 & -260.417 \\ 0 & 260.417 & 520.833 & -260.417 & -520.833 & 0 \\ -651.042 & 520.833 & 1692.71 & -781.25 & -1041.67 & 260.417 \\ 260.417 & -260.417 & -781.25 & 2864.58 & 520.833 & -2604.17 \\ 0 & -520.833 & -1041.67 & 520.833 & 1041.67 & 0 \\ -260.417 & 0 & 260.417 & -2604.17 & 0 & 2604.17 \end{pmatrix} \begin{pmatrix} u_3 \\ v_3 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {5, 6, 9, 10, 11, 12} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 5 \\ 6 \\ 9 \\ 10 \\ 11 \\ 12 \end{pmatrix}$

and to a global matrix: $\begin{pmatrix} [5,5] & [5,6] & [5,9] & [5,10] & [5,11] & [5,12] \\ [6,5] & [6,6] & [6,9] & [6,10] & [6,11] & [6,12] \\ [9,5] & [9,6] & [9,9] & [9,10] & [9,11] & [9,12] \\ [10,5] & [10,6] & [10,9] & [10,10] & [10,11] & [10,12] \\ [11,5] & [11,6] & [11,9] & [11,10] & [11,11] & [11,12] \\ [12,5] & [12,6] & [12,9] & [12,10] & [12,11] & [12,12] \end{pmatrix}$

Adding element equations into appropriate locations we have

$$\left(\begin{matrix} 1578.78 & 195.313 & -276.693 & 325.521 & -976.563 & 260.417 & -325.521 & -781.25 & 0 & 0 \\ 195.313 & 1725.26 & 65.1042 & -1204.43 & 520.833 & -390.625 & -781.25 & -130.208 & 0 & 0 \\ -276.693 & 65.1042 & 1253.26 & -585.938 & 0 & 0 & -976.563 & 520.833 & 0 & 0 \\ 325.521 & -1204.43 & -585.938 & 1595.05 & 0 & 0 & 260.417 & -390.625 & 0 & 0 \\ -976.563 & 520.833 & 0 & 0 & 2322.05 & -781.25 & -694.444 & 260.417 & -651.042 & 260. \\ 260.417 & -390.625 & 0 & 0 & -781.25 & 2387.15 & 520.833 & -1736.11 & 520.833 & -260. \\ -325.521 & -781.25 & -976.563 & 260.417 & -694.444 & 520.833 & 1996.53 & 0 & 0 & 0 \\ -781.25 & -130.208 & 520.833 & -390.625 & 260.417 & -1736.11 & 0 & 2256.94 & 0 & 0 \\ 0 & 0 & 0 & 0 & -651.042 & 520.833 & 0 & 0 & 1692.71 & -781. \\ 0 & 0 & 0 & 0 & 260.417 & -260.417 & 0 & 0 & -781.25 & 2864. \\ 0 & 0 & 0 & 0 & 0 & -520.833 & 0 & 0 & -1041.67 & 520. \\ 0 & 0 & 0 & 0 & -260.417 & 0 & 0 & 0 & 260.417 & -2604. \end{matrix}\right.$$

Equations for element 4

$h = 0.25;$ $E = 10000;$ $\nu = 0.2$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 10416.7 & 2083.33 & 0 \\ 2083.33 & 10416.7 & 0 \\ 0 & 0 & 4166.67 \end{pmatrix}$

Nodal coordinates

Element node	Global node number	x	y
1	6	4.	1.
2	4	2.	1.5
3	3	2.	0.

$x_1 = 4.$ $x_2 = 2.$ $x_3 = 2.$
$y_1 = 1.$ $y_2 = 1.5$ $y_3 = 0.$

Using these values we get

$b_1 = 1.5$ $b_2 = -1.$ $b_3 = -0.5$

$c_1 = 0.$ $c_2 = 2.$ $c_3 = -2.$

$f_1 = -3.$ $f_2 = 2.$ $f_3 = 4.$

Element area, $A = 1.5$

$$\boldsymbol{B}^T = \begin{pmatrix} 0.5 & 0 & -0.333333 & 0 & -0.166667 & 0 \\ 0 & 0. & 0 & 0.666667 & 0 & -0.666667 \\ 0. & 0.5 & 0.666667 & -0.333333 & -0.666667 & -0.166667 \end{pmatrix}$$

Thus the element stiffness matrix is

$$\boldsymbol{k} = hA\boldsymbol{BCB}^T = \begin{pmatrix} 976.563 & 0 & -651.042 & 260.417 & -325.521 & -260.417 \\ 0 & 390.625 & 520.833 & -260.417 & -520.833 & -130.208 \\ -651.042 & 520.833 & 1128.47 & -520.833 & -477.431 & 0 \\ 260.417 & -260.417 & -520.833 & 1909.72 & 260.417 & -1649.31 \\ -325.521 & -520.833 & -477.431 & 260.417 & 802.951 & 260.417 \\ -260.417 & -130.208 & 0 & -1649.31 & 260.417 & 1779.51 \end{pmatrix}$$

Load vector due to distributed load on side 1 (nodes {6, 4})

Specified load components: $q_n = -20;$ $q_t = 0$

End nodal coordinates: ({4., 1.} {2., 1.5}) giving side length, $L = 2.06155$

Components of unit normal to the side: $n_x = 0.242536;$ $n_y = 0.970143$

Using these values we get
$\boldsymbol{r}_q^T = (-1.25 \quad -5. \quad -1.25 \quad -5. \quad 0 \quad 0)$

Complete equations for element 4

$$\begin{pmatrix} 976.563 & 0 & -651.042 & 260.417 & -325.521 & -260.417 \\ 0 & 390.625 & 520.833 & -260.417 & -520.833 & -130.208 \\ -651.042 & 520.833 & 1128.47 & -520.833 & -477.431 & 0 \\ 260.417 & -260.417 & -520.833 & 1909.72 & 260.417 & -1649.31 \\ -325.521 & -520.833 & -477.431 & 260.417 & 802.951 & 260.417 \\ -260.417 & -130.208 & 0 & -1649.31 & 260.417 & 1779.51 \end{pmatrix} \begin{pmatrix} u_6 \\ v_6 \\ u_4 \\ v_4 \\ u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} -1.25 \\ -5. \\ -1.25 \\ -5. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {11, 12, 7, 8, 5, 6} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 11 \\ 12 \\ 7 \\ 8 \\ 5 \\ 6 \end{pmatrix}$

and to a global matrix: $\begin{pmatrix} [11, 11] & [11, 12] & [11, 7] & [11, 8] & [11, 5] & [11, 6] \\ [12, 11] & [12, 12] & [12, 7] & [12, 8] & [12, 5] & [12, 6] \\ [7, 11] & [7, 12] & [7, 7] & [7, 8] & [7, 5] & [7, 6] \\ [8, 11] & [8, 12] & [8, 7] & [8, 8] & [8, 5] & [8, 6] \\ [5, 11] & [5, 12] & [5, 7] & [5, 8] & [5, 5] & [5, 6] \\ [6, 11] & [6, 12] & [6, 7] & [6, 8] & [6, 5] & [6, 6] \end{pmatrix}$

Adding element equations into appropriate locations we have

$$\left(\begin{array}{cccccccccc} 1578.78 & 195.313 & -276.693 & 325.521 & -976.563 & 260.417 & -325.521 & -781.25 & 0 & \\ 195.313 & 1725.26 & 65.1042 & -1204.43 & 520.833 & -390.625 & -781.25 & -130.208 & 0 & \\ -276.693 & 65.1042 & 1253.26 & -585.938 & 0 & 0 & -976.563 & 520.833 & 0 & \\ 325.521 & -1204.43 & -585.938 & 1595.05 & 0 & 0 & 260.417 & -390.625 & 0 & \\ -976.563 & 520.833 & 0 & 0 & 3125. & -520.833 & -1171.88 & 520.833 & -651.042 & 2 \\ 260.417 & -390.625 & 0 & 0 & -520.833 & 4166.67 & 520.833 & -3385.42 & 520.833 & -2 \\ -325.521 & -781.25 & -976.563 & 260.417 & -1171.88 & 520.833 & 3125. & -520.833 & 0 & \\ -781.25 & -130.208 & 520.833 & -390.625 & 520.833 & -3385.42 & -520.833 & 4166.67 & 0 & \\ 0 & 0 & 0 & 0 & -651.042 & 520.833 & 0 & 0 & 1692.71 & -7 \\ 0 & 0 & 0 & 0 & 260.417 & -260.417 & 0 & 0 & -781.25 & 28 \\ 0 & 0 & 0 & 0 & -325.521 & -781.25 & -651.042 & 260.417 & -1041.67 & 5 \\ 0 & 0 & 0 & 0 & -781.25 & -130.208 & 520.833 & -260.417 & 260.417 & -26 \end{array}\right.$$

Essential boundary conditions

Node	dof	Value
1	u_1	0
	v_1	0
2	u_2	0
	v_2	0

Remove {1, 2, 3, 4} rows and columns.

After adjusting for essential boundary conditions we have

$$\begin{pmatrix} 3125. & -520.833 & -1171.88 & 520.833 & -651.042 & 260.417 & -325.521 & -781.25 \\ -520.833 & 4166.67 & 520.833 & -3385.42 & 520.833 & -260.417 & -781.25 & -130.208 \\ -1171.88 & 520.833 & 3125. & -520.833 & 0 & 0 & -651.042 & 520.833 \\ 520.833 & -3385.42 & -520.833 & 4166.67 & 0 & 0 & 260.417 & -260.417 \\ -651.042 & 520.833 & 0 & 0 & 1692.71 & -781.25 & -1041.67 & 260.417 \\ 260.417 & -260.417 & 0 & 0 & -781.25 & 2864.58 & 520.833 & -2604.17 \\ -325.521 & -781.25 & -651.042 & 260.417 & -1041.67 & 520.833 & 2018.23 & 0 \\ -781.25 & -130.208 & 520.833 & -260.417 & 260.417 & -2604.17 & 0 & 2994.79 \end{pmatrix}$$

$$\begin{pmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2.5 \\ -10. \\ 0 \\ 0 \\ -1.25 \\ -5. \end{pmatrix}$$

Solving the final system of global equations we get

$$\{u_3 = -0.0103553,\ v_3 = -0.0255297,\ u_4 = 0.00472765,\ v_4 = -0.0247357,\ u_5 = -0.0131394,\ v_5 = -0.0554931,\ u_6 = 0.0000838902,\ v_6 = -0.0555664\}$$

Complete table of nodal values

	u	v
1	0	0
2	0	0
3	−0.0103553	−0.0255297
4	0.00472765	−0.0247357
5	−0.0131394	−0.0554931
6	0.0000838902	−0.0555664

Computation of reactions

Equation numbers of dof with specified values: {1, 2, 3, 4}

Extracting equations {1, 2, 3, 4} from the global system we have

$$\begin{pmatrix} 1578.78 & 195.313 & -276.693 & 325.521 & -976.563 & 260.417 & -325.521 & -781.25 & 0 & 0 & 0 & 0 \\ 195.313 & 1725.26 & 65.1042 & -1204.43 & 520.833 & -390.625 & -781.25 & -130.208 & 0 & 0 & 0 & 0 \\ -276.693 & 65.1042 & 1253.26 & -585.938 & 0 & 0 & -976.563 & 520.833 & 0 & 0 & 0 & 0 \\ 325.521 & -1204.43 & -585.938 & 1595.05 & 0 & 0 & 260.417 & -390.625 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \end{pmatrix} =$$

$$\begin{pmatrix} R_1 + 0. \\ R_2 + 0. \\ R_3 - 1.25 \\ R_4 - 5. \end{pmatrix}$$

Substituting the nodal values and re−arranging

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{pmatrix} = \begin{pmatrix} 1578.78 & 195.313 & -276.693 & 325.521 & -976.563 & 260.417 & -325.521 & -781.25 & 0 & 0 & 0 & 0 \\ 195.313 & 1725.26 & 65.1042 & -1204.43 & 520.833 & -390.625 & -781.25 & -130.208 & 0 & 0 & 0 & 0 \\ -276.693 & 65.1042 & 1253.26 & -585.938 & 0 & 0 & -976.563 & 520.833 & 0 & 0 & 0 & 0 \\ 325.521 & -1204.43 & -585.938 & 1595.05 & 0 & 0 & 260.417 & -390.625 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -0.0103553 \\ -0.0255297 \\ 0.00472765 \\ -0.0247357 \\ -0.0131394 \\ -0.0554931 \\ 0.0000838902 \\ -0.0555664 \end{pmatrix} - \begin{pmatrix} 0. \\ 0. \\ -1.25 \\ -5. \end{pmatrix}$$

Carrying out computations, the reactions are as follows.

Label	dof	Reaction
R_1	u_1	21.25
R_2	v_1	4.10648
R_3	u_2	−16.25
R_4	v_2	15.8935

Sum of Reactions

dof: u	5.
dof: v	20.

Solution for element 1

$h = 0.25;$ $E = 10000;$ $\nu = 0.2$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 10416.7 & 2083.33 & 0 \\ 2083.33 & 10416.7 & 0 \\ 0 & 0 & 4166.67 \end{pmatrix}$

Element nodes: First node (node # 1): {0., 0.}
Second node (node # 3): {2., 0.} Third node (node # 4): {2., 1.5}

$x_1 = 0.$ $x_2 = 2.$ $x_3 = 2.$
$y_1 = 0.$ $y_2 = 0.$ $y_3 = 1.5$

Using these values we get

$b_1 = -1.5$ $b_2 = 1.5$ $b_3 = 0.$

$c_1 = 0.$ $c_2 = -2.$ $c_3 = 2.$

$f_1 = 3.$ $f_2 = 0.$ $f_3 = 0.$

Element area, A = 1.5

$$\boldsymbol{B}^T = \begin{pmatrix} -0.5 & 0 & 0.5 & 0 & 0. & 0 \\ 0 & 0. & 0 & -0.666667 & 0 & 0.666667 \\ 0. & -0.5 & -0.666667 & 0.5 & 0.666667 & 0. \end{pmatrix}$$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\{1. - 0.5\,x,\ 0.5\,x - 0.666667\,y,\ 0.666667\,y\}$

$$\boldsymbol{N}^T = \begin{pmatrix} 1. - 0.5\,x & 0 & 0.5\,x - 0.666667\,y & 0 & 0.666667\,y & 0 \\ 0 & 1. - 0.5\,x & 0 & 0.5\,x - 0.666667\,y & 0 & 0.666667\,y \end{pmatrix}$$

From global solution the displacements at the element nodes are
(displacements at nodes {1, 3, 4}):
$\boldsymbol{d}^T = \{0, 0, -0.0103553, -0.0255297, 0.00472765, -0.0247357\}$

The displacement distribution over the element is

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \boldsymbol{N}^T\boldsymbol{d} = \begin{pmatrix} 0.0100553\,y - 0.00517764\,x \\ 0.000529362\,y - 0.0127648\,x \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^T\boldsymbol{d} = (\ -0.00517764 \quad 0.000529362 \quad -0.00270956\ )$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\ -52.8309 \quad -5.27256 \quad -11.2898\ )$

Computing out−of−plane strain and stress components
using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^T = (\ -0.00517764 \quad 0.000529362 \quad 0.00116207 \quad -0.00270956 \quad 0 \quad 0\ )$

$\boldsymbol{\sigma}^T = (\ -52.8309 \quad -5.27256 \quad 0 \quad -11.2898 \quad 0 \quad 0\ )$

Substituting these stress components into appropriate formulas

Principal stresses = $(\ 0 \quad -2.72856 \quad -55.3749\ )$

Effective stress (von Mises) = 54.0623

Solution for element 2

$h = 0.25;$ $E = 10000;$ $\nu = 0.2$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 10416.7 & 2083.33 & 0 \\ 2083.33 & 10416.7 & 0 \\ 0 & 0 & 4166.67 \end{pmatrix}$

Element nodes: First node (node # 4): {2., 1.5}
Second node (node # 2): {0., 2.} Third node (node # 1): {0., 0.}

$x_1 = 2.$ $x_2 = 0.$ $x_3 = 0.$
$y_1 = 1.5$ $y_2 = 2.$ $y_3 = 0.$

Using these values we get

$b_1 = 2.$ $b_2 = -1.5$ $b_3 = -0.5$

$c_1 = 0.$ $c_2 = 2.$ $c_3 = -2.$

$f_1 = 0.$ $f_2 = 0.$ $f_3 = 4.$

Element area, A = 2.

$$\boldsymbol{B}^{T} = \begin{pmatrix} 0.5 & 0 & -0.375 & 0 & -0.125 & 0 \\ 0 & 0. & 0 & 0.5 & 0 & -0.5 \\ 0. & 0.5 & 0.5 & -0.375 & -0.5 & -0.125 \end{pmatrix}$$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\{0.5\,x,\ 0.5\,y - 0.375\,x,\ -0.125\,x - 0.5\,y + 1.\}$

$$\boldsymbol{N}^{T} = \begin{pmatrix} 0.5\,x & 0 & 0.5\,y - 0.375\,x & 0 & -0.125\,x - 0.5\,y + 1. & 0 \\ 0 & 0.5\,x & 0 & 0.5\,y - 0.375\,x & 0 & -0.125\,x - 0.5\,y + 1. \end{pmatrix}$$

From global solution the displacements at the element nodes are
(displacements at nodes {4, 2, 1}):
$\boldsymbol{d}^{T} = \{0.00472765, -0.0247357, 0, 0, 0, 0\}$

The displacement distribution over the element is

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \boldsymbol{N}^{T}\boldsymbol{d} = \begin{pmatrix} 0.00236383\,x \\ -0.0123678\,x \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{T}\boldsymbol{d} = (\, 0.00236383 \quad 0 \quad -0.0123678 \,)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\, 24.6232 \quad 4.92464 \quad -51.5326 \,)$

Computing out−of−plane strain and stress components
using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{T} = (\, 0.00236383 \quad 0 \quad -0.000590956 \quad -0.0123678 \quad 0 \quad 0 \,)$

$\boldsymbol{\sigma}^{T} = (\, 24.6232 \quad 4.92464 \quad 0 \quad -51.5326 \quad 0 \quad 0 \,)$

Substituting these stress components into appropriate formulas

Principal stresses = (67.2393   0   −37.6915)

Effective stress (von Mises) = 92.0659

Solution for element 3

$h = 0.25;$ $E = 10000;$ $\nu = 0.2$

Plane stress constitutive matrix, $\boldsymbol{C} = \begin{pmatrix} 10416.7 & 2083.33 & 0 \\ 2083.33 & 10416.7 & 0 \\ 0 & 0 & 4166.67 \end{pmatrix}$

Element nodes: First node (node # 3): {2., 0.}
Second node (node # 5): {4., 0.} Third node (node # 6): {4., 1.}

$x_1 = 2.$ $x_2 = 4.$ $x_3 = 4.$
$y_1 = 0.$ $y_2 = 0.$ $y_3 = 1.$

Using these values we get

$b_1 = -1.$ $b_2 = 1.$ $b_3 = 0.$

$c_1 = 0.$ $c_2 = -2.$ $c_3 = 2.$

$f_1 = 4.$ $f_2 = -2.$ $f_3 = 0.$

Element area, A = 1.

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} -0.5 & 0 & 0.5 & 0 & 0. & 0 \\ 0 & 0. & 0 & -1. & 0 & 1. \\ 0. & -0.5 & -1. & 0.5 & 1. & 0. \end{pmatrix}$$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, {2. − 0.5 x, 0.5 x − 1. y − 1., 1. y}

$$\boldsymbol{N}^{\mathrm{T}} = \begin{pmatrix} 2.-0.5\,x & 0 & 0.5\,x-1.\,y-1. & 0 & 1.\,y & 0 \\ 0 & 2.-0.5\,x & 0 & 0.5\,x-1.\,y-1. & 0 & 1.\,y \end{pmatrix}$$

From global solution the displacements at the element nodes are
(displacements at nodes {3, 5, 6}):
$\boldsymbol{d}^{\mathrm{T}} = \{-0.0103553, -0.0255297, -0.0131394, -0.0554931, 0.0000838902, -0.0555664\}$

The displacement distribution over the element is

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{d} = \begin{pmatrix} -0.00139207\,x + 0.0132233\,y - 0.00757114 \\ -0.0149817\,x - 0.0000732667\,y + 0.00443371 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{d} = (\,-0.00139207 \quad -0.0000732667 \quad -0.0017584\,)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\,-14.6533 \quad -3.66334 \quad -7.32667\,)$

Computing out−of−plane strain and stress components
using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{\mathrm{T}} = (\,-0.00139207 \quad -0.0000732667 \quad 0.000366334 \quad -0.0017584 \quad 0 \quad 0\,)$

$\boldsymbol{\sigma}^{\mathrm{T}} = (\,-14.6533 \quad -3.66334 \quad 0 \quad -7.32667 \quad 0 \quad 0\,)$

Substituting these stress components into appropriate formulas

Principal stresses = $(\,0 \quad 0 \quad -18.3167\,)$

Effective stress (von Mises) = 18.3167

Solution for element 4

$h = 0.25;$ $E = 10000;$ $\nu = 0.2$

$$\text{Plane stress constitutive matrix, } \boldsymbol{C} = \begin{pmatrix} 10416.7 & 2083.33 & 0 \\ 2083.33 & 10416.7 & 0 \\ 0 & 0 & 4166.67 \end{pmatrix}$$

Element nodes: First node (node # 6): {4., 1.}
Second node (node # 4): {2., 1.5} Third node (node # 3): {2., 0.}

$x_1 = 4.$ $\quad x_2 = 2.$ $\quad x_3 = 2.$
$y_1 = 1.$ $\quad y_2 = 1.5$ $\quad y_3 = 0.$

Using these values we get

$b_1 = 1.5 \quad b_2 = -1. \quad b_3 = -0.5$

$c_1 = 0. \quad c_2 = 2. \quad c_3 = -2.$

$f_1 = -3. \quad f_2 = 2. \quad f_3 = 4.$

Element area, A = 1.5

$$\boldsymbol{B}^{\mathrm{T}} = \begin{pmatrix} 0.5 & 0 & -0.333333 & 0 & -0.166667 & 0 \\ 0 & 0. & 0 & 0.666667 & 0 & -0.666667 \\ 0. & 0.5 & 0.666667 & -0.333333 & -0.666667 & -0.166667 \end{pmatrix}$$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\{0.5\,x - 1.,\ -0.333333\,x + 0.666667\,y + 0.666667,\ -0.166667\,x - 0.666667\,y + 1.33333\}$

$$\boldsymbol{N}^{\mathrm{T}} = \begin{pmatrix} 0.5\,x - 1. & 0 & -0.333333\,x + 0.666667\,y + 0.666667 & 0 & -0.1 \\ 0 & 0.5\,x - 1. & 0 & -0.333333\,x + 0.666667\,y + 0.666667 \end{pmatrix}$$

From global solution the displacements at the element nodes are
(displacements at nodes {6, 4, 3}):
$\boldsymbol{d}^{\mathrm{T}} = \{0.0000838902,\ -0.0555664,\ 0.00472765,\ -0.0247357,\ -0.0103553,\ -0.0255297\}$

The displacement distribution over the element is

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{d} = \begin{pmatrix} 0.000191941\,x + 0.0100553\,y - 0.0107392 \\ -0.015283\,x + 0.000529362\,y + 0.00503634 \end{pmatrix}$$

In−plane strain components, $\boldsymbol{\epsilon} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{d} = (\,0.000191941 \quad 0.000529362 \quad -0.00522773\,)$

In−plane stress components, $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} = (\,3.10223 \quad 5.91407 \quad -21.7822\,)$

Computing out−of−plane strain and stress components
using appropriate formulas, the complete strain and stress vectors are as follows.

$\boldsymbol{\epsilon}^{\mathrm{T}} = (\,0.000191941 \quad 0.000529362 \quad -0.000180326 \quad -0.00522773 \quad 0 \quad 0\,)$

$\boldsymbol{\sigma}^{\mathrm{T}} = (\,3.10223 \quad 5.91407 \quad 0 \quad -21.7822 \quad 0 \quad 0\,)$

Substituting these stress components into appropriate formulas

Principal stresses = $(\,26.3357 \quad 0 \quad -17.3194\,)$

Effective stress (von Mises) = 38.0742

Solution summary

Nodal solution

	x	y	u	v
1	0.	0.	0	0
2	0.	2.	0	0
3	2.	0.	−0.0103553	−0.0255297
4	2.	1.5	0.00472765	−0.0247357
5	4.	0.	−0.0131394	−0.0554931
6	4.	1.	0.0000838902	−0.0555664

Solution at element centers

	Coord	Disp	Stresses	Principal stresses	Effective Stress
1	1.33333 0.5	−0.00187588 −0.0167551	−52.8309 −5.27256 0 −11.2898 0 0	0 −2.72856 −55.3749	54.0623
2	0.666667 1.16667	0.00157588 −0.00824522	24.6232 4.92464 0 −51.5326 0 0	67.2393 0 −37.6915	92.0659
3	3.33333 0.333333	−0.0078036 −0.0455297	−14.6533 −3.66334 0 −7.32667 0 0	0 0 −18.3167	18.3167
4	2.66667 0.833333	−0.00184791 −0.0352772	3.10223 5.91407 0 −21.7822 0 0	26.3357 0 −17.3194	38.0742

Support reactions

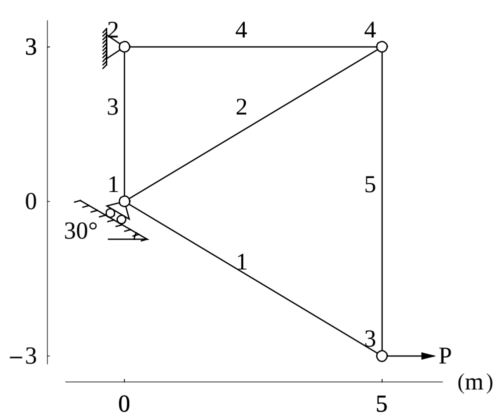
Node	dof	Reaction
1	1	21.25
1	2	4.10648
2	1	−16.25
2	2	15.8935

Sum of applied loads → (−5.   −20.)

Sum of support reactions → (5.   20.)

Example 1.17: Five bar truss with inclined support (p. 76)

Consider a five bar pin-jointed structure shown in Figure. All members have the same cross-sectional area and are of the same material, $E = 70$ GPa and $A = 10^{-3}\ m^2$. The load $P = 20$ kN.



For numerical calculations use the N – mm units are convenient. The displacements will be in mm and the stresses in MPa. The complete computations are as follows.

Specified nodal loads

Node	dof	Value
3	u_3	20000.

Global equations at start of the element assembly process

$$\begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}\begin{pmatrix} u_1\\ v_1\\ u_2\\ v_2\\ u_3\\ v_3\\ u_4\\ v_4 \end{pmatrix} = \begin{pmatrix} 0\\ 0\\ 0\\ 0\\ 20000.\\ 0\\ 0\\ 0 \end{pmatrix}$$

Equations for element 1

$E = 70000 \qquad A = 1000$

Element node	Global node number	x	y
1	1	0	0
2	3	5000.	−3000.

$x_1 = 0 \qquad y_1 = 0 \qquad x_2 = 5000. \qquad y_2 = -3000.$

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 5830.95$$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 0.857493 \qquad m_s = \dfrac{y_2 - y_1}{L} = -0.514496$

Substituting into the truss element equations we get

$$\begin{pmatrix} 8827.13 & -5296.28 & -8827.13 & 5296.28\\ -5296.28 & 3177.77 & 5296.28 & -3177.77\\ -8827.13 & 5296.28 & 8827.13 & -5296.28\\ 5296.28 & -3177.77 & -5296.28 & 3177.77 \end{pmatrix}\begin{pmatrix} u_1\\ v_1\\ u_3\\ v_3 \end{pmatrix} = \begin{pmatrix} 0.\\ 0.\\ 0.\\ 0. \end{pmatrix}$$

The element contributes to {1, 2, 5, 6} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 8827.13 & -5296.28 & 0 & 0 & -8827.13 & 5296.28 & 0 & 0\\ -5296.28 & 3177.77 & 0 & 0 & 5296.28 & -3177.77 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ -8827.13 & 5296.28 & 0 & 0 & 8827.13 & -5296.28 & 0 & 0\\ 5296.28 & -3177.77 & 0 & 0 & -5296.28 & 3177.77 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} u_1\\ v_1\\ u_2\\ v_2\\ u_3\\ v_3\\ u_4\\ v_4 \end{pmatrix} = \begin{pmatrix} 0\\ 0\\ 0\\ 0\\ 20000.\\ 0\\ 0\\ 0 \end{pmatrix}$$

Equations for element 2

$E = 70000 \qquad A = 1000$

Element node	Global node number	x	y
1	1	0	0
2	4	5000.	3000.

$x_1 = 0 \qquad y_1 = 0 \qquad x_2 = 5000. \qquad y_2 = 3000.$

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 5830.95$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 0.857493 \qquad m_s = \dfrac{y_2 - y_1}{L} = 0.514496$

Substituting into the truss element equations we get

$$\begin{pmatrix} 8827.13 & 5296.28 & -8827.13 & -5296.28 \\ 5296.28 & 3177.77 & -5296.28 & -3177.77 \\ -8827.13 & -5296.28 & 8827.13 & 5296.28 \\ -5296.28 & -3177.77 & 5296.28 & 3177.77 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {1, 2, 7, 8} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 17654.3 & 0 & 0 & 0 & -8827.13 & 5296.28 & -8827.13 & -5296.28 \\ 0 & 6355.54 & 0 & 0 & 5296.28 & -3177.77 & -5296.28 & -3177.77 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -8827.13 & 5296.28 & 0 & 0 & 8827.13 & -5296.28 & 0 & 0 \\ 5296.28 & -3177.77 & 0 & 0 & -5296.28 & 3177.77 & 0 & 0 \\ -8827.13 & -5296.28 & 0 & 0 & 0 & 0 & 8827.13 & 5296.28 \\ -5296.28 & -3177.77 & 0 & 0 & 0 & 0 & 5296.28 & 3177.77 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 20000. \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 3

$E = 70000 \qquad A = 1000$

Element node	Global node number	x	y
1	1	0	0
2	2	0	3000.

$x_1 = 0 \qquad y_1 = 0 \qquad x_2 = 0 \qquad y_2 = 3000.$

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 3000.$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 0 \qquad m_s = \dfrac{y_2 - y_1}{L} = 1.$

Substituting into the truss element equations we get

$$\begin{pmatrix} 0. & 0. & 0. & 0. \\ 0. & 23333.3 & 0. & -23333.3 \\ 0. & 0. & 0. & 0. \\ 0. & -23333.3 & 0. & 23333.3 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {1, 2, 3, 4} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 17654.3 & 0 & 0 & 0 & -8827.13 & 5296.28 & -8827.13 & -5296.28 \\ 0 & 29688.9 & 0 & -23333.3 & 5296.28 & -3177.77 & -5296.28 & -3177.77 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -23333.3 & 0 & 23333.3 & 0 & 0 & 0 & 0 \\ -8827.13 & 5296.28 & 0 & 0 & 8827.13 & -5296.28 & 0 & 0 \\ 5296.28 & -3177.77 & 0 & 0 & -5296.28 & 3177.77 & 0 & 0 \\ -8827.13 & -5296.28 & 0 & 0 & 0 & 0 & 8827.13 & 5296.28 \\ -5296.28 & -3177.77 & 0 & 0 & 0 & 0 & 5296.28 & 3177.77 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 20000. \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 4

$E = 70000 \qquad A = 1000$

Element node	Global node number	x	y
1	2	0	3000.
2	4	5000.	3000.

$x_1 = 0 \qquad y_1 = 3000. \qquad x_2 = 5000. \qquad y_2 = 3000.$

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 5000.$$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 1. \qquad m_s = \dfrac{y_2 - y_1}{L} = 0.$

Substituting into the truss element equations we get

$$\begin{pmatrix} 14000. & 0. & -14000. & 0. \\ 0. & 0. & 0. & 0. \\ -14000. & 0. & 14000. & 0. \\ 0. & 0. & 0. & 0. \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {3, 4, 7, 8} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 17654.3 & 0 & 0 & 0 & -8827.13 & 5296.28 & -8827.13 & -5296.28 \\ 0 & 29688.9 & 0 & -23333.3 & 5296.28 & -3177.77 & -5296.28 & -3177.77 \\ 0 & 0 & 14000. & 0 & 0 & 0 & -14000. & 0 \\ 0 & -23333.3 & 0 & 23333.3 & 0 & 0 & 0 & 0 \\ -8827.13 & 5296.28 & 0 & 0 & 8827.13 & -5296.28 & 0 & 0 \\ 5296.28 & -3177.77 & 0 & 0 & -5296.28 & 3177.77 & 0 & 0 \\ -8827.13 & -5296.28 & -14000. & 0 & 0 & 0 & 22827.1 & 5296.28 \\ -5296.28 & -3177.77 & 0 & 0 & 0 & 0 & 5296.28 & 3177.77 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 20000. \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 5

$E = 70000 \qquad A = 1000$

Element node	Global node number	x	y
1	3	5000.	−3000.
2	4	5000.	3000.

$x_1 = 5000. \qquad y_1 = -3000. \qquad x_2 = 5000. \qquad y_2 = 3000.$

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 6000.$$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 0. \qquad m_s = \dfrac{y_2 - y_1}{L} = 1.$

Substituting into the truss element equations we get

$$\begin{pmatrix} 0. & 0. & 0. & 0. \\ 0. & 11666.7 & 0. & -11666.7 \\ 0. & 0. & 0. & 0. \\ 0. & -11666.7 & 0. & 11666.7 \end{pmatrix} \begin{pmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {5, 6, 7, 8} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 17654.3 & 0 & 0 & 0 & -8827.13 & 5296.28 & -8827.13 & -5296.28 \\ 0 & 29688.9 & 0 & -23333.3 & 5296.28 & -3177.77 & -5296.28 & -3177.77 \\ 0 & 0 & 14000. & 0 & 0 & 0 & -14000. & 0 \\ 0 & -23333.3 & 0 & 23333.3 & 0 & 0 & 0 & 0 \\ -8827.13 & 5296.28 & 0 & 0 & 8827.13 & -5296.28 & 0 & 0 \\ 5296.28 & -3177.77 & 0 & 0 & -5296.28 & 14844.4 & 0 & -11666.7 \\ -8827.13 & -5296.28 & -14000. & 0 & 0 & 0 & 22827.1 & 5296.28 \\ -5296.28 & -3177.77 & 0 & 0 & 0 & -11666.7 & 5296.28 & 14844.4 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 20000. \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Essential boundary conditions

Node	dof	Value
2	u_2	0
	v_2	0

Remove {3, 4} rows and columns.

After adjusting for essential boundary conditions we have

$$\begin{pmatrix} 17654.3 & 0 & -8827.13 & 5296.28 & -8827.13 & -5296.28 \\ 0 & 29688.9 & 5296.28 & -3177.77 & -5296.28 & -3177.77 \\ -8827.13 & 5296.28 & 8827.13 & -5296.28 & 0 & 0 \\ 5296.28 & -3177.77 & -5296.28 & 14844.4 & 0 & -11666.7 \\ -8827.13 & -5296.28 & 0 & 0 & 22827.1 & 5296.28 \\ -5296.28 & -3177.77 & 0 & -11666.7 & 5296.28 & 14844.4 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 20000. \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Multipoint constraint due to inclined support at node 1: $u_1 \sin(\pi/6) + v_1 \cos(\pi/6) = 0$. The augmented global equations with the Lagrange multiplier are as follows.

$$\begin{pmatrix} 17654.3 & 0 & -8827.13 & 5296.28 & -8827.13 & -5296.28 & 1/2 \\ 0 & 29688.9 & 5296.28 & -3177.77 & -5296.28 & -3177.77 & \frac{\sqrt{3}}{2} \\ -8827.13 & 5296.28 & 8827.13 & -5296.28 & 0 & 0 & 0 \\ 5296.28 & -3177.77 & -5296.28 & 14844.4 & 0 & -11666.7 & 0 \\ -8827.13 & -5296.28 & 0 & 0 & 22827.1 & 5296.28 & 0 \\ -5296.28 & -3177.77 & 0 & -11666.7 & 5296.28 & 14844.4 & 0 \\ 1/2 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 20000. \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving the final system of global equations we get

$$\{u_1 = 5.14286,\ v_1 = -2.96923,\ u_3 = 16.8629,\ v_3 = 12.788,\ u_4 = -1.42857,\ v_4 = 11.7594,\ \lambda = 80000.\}$$

Solution for element 1

Nodal coordinates

Element node	Global node number	x	y
1	1	0	0
2	3	5000.	−3000.

$x_1 = 0$ $y_1 = 0$ $x_2 = 5000.$ $y_2 = -3000.$

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 5830.95$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 0.857493$ $m_s = \dfrac{y_2 - y_1}{L} = -0.514496$

Global to local transformation matrix, $\boldsymbol{T} = \begin{pmatrix} 0.857493 & -0.514496 & 0 & 0 \\ 0 & 0 & 0.857493 & -0.514496 \end{pmatrix}$

Element nodal displacements in global coordinates, $\boldsymbol{d} = \begin{pmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 5.14286 \\ -2.96923 \\ 16.8629 \\ 12.788 \end{pmatrix}$

Element nodal displacements in local coordinates, $\boldsymbol{d}_\ell = \boldsymbol{T}\,\boldsymbol{d} = \begin{pmatrix} 5.93762 \\ 7.88048 \end{pmatrix}$

$E = 70000$ $A = 1000$

Axial strain, $\epsilon = (d_2 - d_1)/L = 0.000333197$

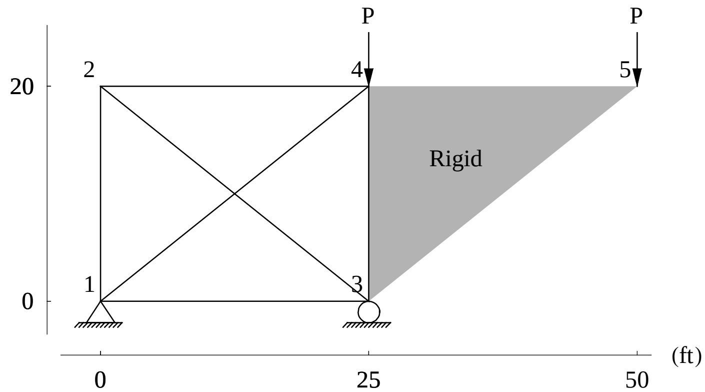
Axial stress, $\sigma = E\epsilon = 23.3238$ Axial force $= \sigma A = 23323.8$

In a similar manner we can compute the solutions over the remaining elements.

	Stress	Axial force
1	23.3238	23323.8
2	23.3238	23323.8
3	69.282	69282.
4	−20.	−20000.
5	−12.	−12000.

Example 1.18: Truss supporting a rigid plate (p. 80)

A plane truss is designed to support a rigid triangular plate as shown in Figure. All members have the same cross-sectional area $A = 1\,\text{in}^2$ and are of the same material, $E = 29,000$ ksi. The load $P = 20$ kip. The dimensions in ft are shown in the figure. Note there is no connection between the diagonal members where they cross each other.



The model consists of 5 nodes and thus the global system of equations before boundary conditions will be 10×10. The equations for the six truss elements are written as in the previous examples and assembled in the usual manner to give the following system of equations.

$$\text{Element 1, k} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 120.833 & 0 & -120.833 \\ 0 & 0 & 0 & 0 \\ 0 & -120.833 & 0 & 120.833 \end{pmatrix}$$

$$\text{Global K} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 120.833 & 0 & -120.833 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -120.833 & 0 & 120.833 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Element 2, k} = \begin{pmatrix} 96.6667 & 0 & -96.6667 & 0 \\ 0 & 0 & 0 & 0 \\ -96.6667 & 0 & 96.6667 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Global K} = \begin{pmatrix} 96.6667 & 0 & 0 & 0 & -96.6667 & 0 & 0 & 0 & 0 & 0 \\ 0 & 120.833 & 0 & -120.833 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -120.833 & 0 & 120.833 & 0 & 0 & 0 & 0 & 0 & 0 \\ -96.6667 & 0 & 0 & 0 & 96.6667 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Element 3, k} = \begin{pmatrix} 46.0268 & 36.8215 & -46.0268 & -36.8215 \\ 36.8215 & 29.4572 & -36.8215 & -29.4572 \\ -46.0268 & -36.8215 & 46.0268 & 36.8215 \\ -36.8215 & -29.4572 & 36.8215 & 29.4572 \end{pmatrix}$$

$$\text{Global K} = \begin{pmatrix} 142.693 & 36.8215 & 0 & 0 & -96.6667 & 0 & -46.0268 & -36.8215 & 0 & 0 \\ 36.8215 & 150.29 & 0 & -120.833 & 0 & 0 & -36.8215 & -29.4572 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -120.833 & 0 & 120.833 & 0 & 0 & 0 & 0 & 0 & 0 \\ -96.6667 & 0 & 0 & 0 & 96.6667 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -46.0268 & -36.8215 & 0 & 0 & 0 & 0 & 46.0268 & 36.8215 & 0 & 0 \\ -36.8215 & -29.4572 & 0 & 0 & 0 & 0 & 36.8215 & 29.4572 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Element 4, k} = \begin{pmatrix} 96.6667 & 0. & -96.6667 & 0. \\ 0. & 0. & 0. & 0. \\ -96.6667 & 0. & 96.6667 & 0. \\ 0. & 0. & 0. & 0. \end{pmatrix}$$

$$\text{Global K} = \begin{pmatrix} 142.693 & 36.8215 & 0 & 0 & -96.6667 & 0 & -46.0268 & -36.8215 & 0 & 0 \\ 36.8215 & 150.29 & 0 & -120.833 & 0 & 0 & -36.8215 & -29.4572 & 0 & 0 \\ 0 & 0 & 96.6667 & 0. & 0 & 0 & -96.6667 & 0. & 0 & 0 \\ 0 & -120.833 & 0. & 120.833 & 0 & 0 & 0. & 0. & 0 & 0 \\ -96.6667 & 0 & 0 & 0 & 96.6667 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -46.0268 & -36.8215 & -96.6667 & 0. & 0 & 0 & 142.693 & 36.8215 & 0 & 0 \\ -36.8215 & -29.4572 & 0. & 0. & 0 & 0 & 36.8215 & 29.4572 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Element 5, k} = \begin{pmatrix} 46.0268 & -36.8215 & -46.0268 & 36.8215 \\ -36.8215 & 29.4572 & 36.8215 & -29.4572 \\ -46.0268 & 36.8215 & 46.0268 & -36.8215 \\ 36.8215 & -29.4572 & -36.8215 & 29.4572 \end{pmatrix}$$

$$\text{Global K} = \begin{pmatrix} 142.693 & 36.8215 & 0 & 0 & -96.6667 & 0 & -46.0268 & -36.8215 & 0 & 0 \\ 36.8215 & 150.29 & 0 & -120.833 & 0 & 0 & -36.8215 & -29.4572 & 0 & 0 \\ 0 & 0 & 142.693 & -36.8215 & -46.0268 & 36.8215 & -96.6667 & 0. & 0 & 0 \\ 0 & -120.833 & -36.8215 & 150.29 & 36.8215 & -29.4572 & 0. & 0. & 0 & 0 \\ -96.6667 & 0 & -46.0268 & 36.8215 & 142.693 & -36.8215 & 0 & 0 & 0 & 0 \\ 0 & 0 & 36.8215 & -29.4572 & -36.8215 & 29.4572 & 0 & 0 & 0 & 0 \\ -46.0268 & -36.8215 & -96.6667 & 0. & 0 & 0 & 142.693 & 36.8215 & 0 & 0 \\ -36.8215 & -29.4572 & 0. & 0. & 0 & 0 & 36.8215 & 29.4572 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Element 6, k} = \begin{pmatrix} 0. & 0. & 0. & 0. \\ 0. & 120.833 & 0. & -120.833 \\ 0. & 0. & 0. & 0. \\ 0. & -120.833 & 0. & 120.833 \end{pmatrix}$$

$$\text{Global K} = \begin{pmatrix} 142.693 & 36.8215 & 0 & 0 & -96.6667 & 0 & -46.0268 & -36.8215 & 0 & 0 \\ 36.8215 & 150.29 & 0 & -120.833 & 0 & 0 & -36.8215 & -29.4572 & 0 & 0 \\ 0 & 0 & 142.693 & -36.8215 & -46.0268 & 36.8215 & -96.6667 & 0. & 0 & 0 \\ 0 & -120.833 & -36.8215 & 150.29 & 36.8215 & -29.4572 & 0. & 0. & 0 & 0 \\ -96.6667 & 0 & -46.0268 & 36.8215 & 142.693 & -36.8215 & 0. & 0. & 0 & 0 \\ 0 & 0 & 36.8215 & -29.4572 & -36.8215 & 150.29 & 0. & -120.833 & 0 & 0 \\ -46.0268 & -36.8215 & -96.6667 & 0. & 0. & 0. & 142.693 & 36.8215 & 0 & 0 \\ -36.8215 & -29.4572 & 0. & 0. & 0. & -120.833 & 36.8215 & 150.29 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 142.693 & 36.8215 & 0 & 0 & -96.6667 & 0 & -46.0268 & -36.8215 & 0 & 0 \\ 36.8215 & 150.29 & 0 & -120.833 & 0 & 0 & -36.8215 & -29.4572 & 0 & 0 \\ 0 & 0 & 142.693 & -36.8215 & -46.0268 & 36.8215 & -96.6667 & 0. & 0 & 0 \\ 0 & -120.833 & -36.8215 & 150.29 & 36.8215 & -29.4572 & 0. & 0. & 0 & 0 \\ -96.6667 & 0 & -46.0268 & 36.8215 & 142.693 & -36.8215 & 0. & 0. & 0 & 0 \\ 0 & 0 & 36.8215 & -29.4572 & -36.8215 & 150.29 & 0. & -120.833 & 0 & 0 \\ -46.0268 & -36.8215 & -96.6667 & 0. & 0. & 0. & 142.693 & 36.8215 & 0 & 0 \\ -36.8215 & -29.4572 & 0. & 0. & 0. & -120.833 & 36.8215 & 150.29 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -20 \\ 0 \\ -20 \end{pmatrix}$$

The essential boundary conditions at nodes 1 ($u_1 = v_1 = 0$) are incorporated by removing the corresponding rows and columns in the usual way. Node 3 also has zero vertical displacement. However since this node is connected to the rigid plate as well, the boundary condition $v_3 = 0$, will be imposed later as part of the multi-point constraints. Removing the first two rows and columns, the global system of equations is as follows.

$\boldsymbol{K}\,\boldsymbol{d} = \boldsymbol{R} \Longrightarrow$

$$\begin{pmatrix} 142.693 & -36.8215 & -46.0268 & 36.8215 & -96.6667 & 0. & 0 & 0 \\ -36.8215 & 150.29 & 36.8215 & -29.4572 & 0. & 0. & 0 & 0 \\ -46.0268 & 36.8215 & 142.693 & -36.8215 & 0. & 0. & 0 & 0 \\ 36.8215 & -29.4572 & -36.8215 & 150.29 & 0. & -120.833 & 0 & 0 \\ -96.6667 & 0. & 0. & 0. & 142.693 & 36.8215 & 0 & 0 \\ 0. & 0. & 0. & -120.833 & 36.8215 & 150.29 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ -20. \\ 0. \\ -20. \end{pmatrix}$$

The rigid plate is connected between nodes 3, 5 and 4. Treating u_3, v_3, and u_5 as independent degrees of freedom, the multi-point constraints are as follows.

$$\begin{pmatrix} v_5 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} \frac{x_3-x_5}{y_3-y_5} & 1 & \frac{x_5-x_3}{y_3-y_5} \\ \frac{y_4-y_5}{y_3-y_5} & 0 & \frac{y_3-y_4}{y_3-y_5} \\ \frac{x_3-x_4}{y_3-y_5} & 1 & \frac{x_4-x_3}{y_3-y_5} \end{pmatrix} \begin{pmatrix} u_3 \\ v_3 \\ u_5 \end{pmatrix} = \begin{pmatrix} \frac{5}{4} & 1 & -\frac{5}{4} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} u_3 \\ v_3 \\ u_5 \end{pmatrix}$$

Expanding and re-arranging we have

$$
\begin{aligned}
&-\frac{5\,u_3}{4}+\frac{5\,u_5}{4}-v_3+v_5=0\\
&u_4-u_5=0\\
&v_4-v_3=0
\end{aligned}
$$

To this list we must also add the roller support constraint that $v_3 = 0$. Thus the complete set of constraint equations, expanded to include all degrees of freedom present in the global equations, we have

$$
\boldsymbol{C}\,\boldsymbol{d}=\boldsymbol{q} \Longrightarrow \begin{pmatrix} 0 & 0 & -\frac{5}{4} & -1 & 0 & 0 & \frac{5}{4} & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

For using the penalty function approach we choose the penalty parameter μ equal to 10^5 times the largest number in the global $\boldsymbol{K}$ matrix.

$$
\mu = 150.29\times 10^5 = 1.5029\times 10^7
$$

Incorporating the constraints into the global equations with this value of μ, the final system of equations is as follows.

$$
(\boldsymbol{K}+\mu\,\boldsymbol{C}^T\,\boldsymbol{C})\,\boldsymbol{d}=\boldsymbol{R}+\mu\,\boldsymbol{C}^T\boldsymbol{q} \Longrightarrow
$$

$$
10^5\left(\begin{matrix}
0.00142693 & -0.000368215 & -0.000460268 & 0.000368215 & -0.000966667 & 0. & 0 \\
-0.000368215 & 0.0015029 & 0.000368215 & -0.000294572 & 0. & 0. & 0 \\
-0.000460268 & 0.000368215 & -234.827 & -187.863 & 0. & 0. & 234.829 \\
0.000368215 & -0.000294572 & -187.863 & -450.87 & 0. & 150.289 & 187.863 \\
-0.000966667 & 0. & 0. & 0. & -150.289 & 0.000368215 & 150.29 \\
0. & 0. & 0. & 150.289 & 0.000368215 & -150.289 & 0 \\
0 & 0 & 234.829 & 187.863 & 150.29 & 0 & -385.119 \\
0 & 0 & 187.863 & 150.29 & 0 & 0 & -187.863
\end{matrix}\right.
$$

Solving this system of linear equations we get

$$
\begin{aligned}
&\{u_2 = 0.172845,\ v_2 = 0.076446,\ u_3 = -0.139174,\ v_3 = 3.99227\times 10^{-6},\\
&\ \ u_4 = 0.292292,\ v_4 = 6.03917\times 10^{-6},\ u_5 = 0.29229,\ v_5 = -0.539324\}
\end{aligned}
$$

Substituting these values into the constraint equations we can see that the constraints are reasonably satisfied.

$$\boldsymbol{C\,d} = \begin{pmatrix} 1.33076\times10^{-6} \\ 1.66345\times10^{-6} \\ 2.0469\times10^{-6} \\ 3.99227\times10^{-6} \end{pmatrix} \approx \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Knowing the nodal values, the element solutions can be computed in the usual manner.

	Strain	Stress	Axial force
1	0.000318525	9.23722	9.23722
2	−0.000463913	−13.4535	−13.4535
3	0.000594098	17.2289	17.2289
4	0.000398156	11.5465	11.5465
5	−0.000509888	−14.7868	−14.7868
6	8.52877×10^{-9}	0.000247334	0.000247334

Using the Lagrange multipliers method, the solution is obtained as follows.

Augmented system of equations

$$\begin{pmatrix}
142.693 & -36.8215 & -46.0268 & 36.8215 & -96.6667 & 0. & 0 & 0 & 0 & 0 & 0 & 0 \\
-36.8215 & 150.29 & 36.8215 & -29.4572 & 0. & 0. & 0 & 0 & 0 & 0 & 0 & 0 \\
-46.0268 & 36.8215 & 142.693 & -36.8215 & 0. & 0. & 0 & 0 & -\frac{5}{4} & 0 & 0 & 0 \\
36.8215 & -29.4572 & -36.8215 & 150.29 & 0. & -120.833 & 0 & 0 & -1 & 0 & -1 & 1 \\
-96.6667 & 0. & 0. & 0. & 142.693 & 36.8215 & 0 & 0 & 0 & 1 & 0 & 0 \\
0. & 0. & 0. & -120.833 & 36.8215 & 150.29 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{4} & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -\frac{5}{4} & -1 & 0 & 0 & \frac{5}{4} & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix}
=
\begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ -20. \\ 0. \\ -20. \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solution

$\{d_1 = 0.172849,\ d_2 = 0.0764461,\ d_3 = -0.139174,\ d_4 = -4.68418\times10^{-18},\ d_5 = 0.292296,$
$d_6 = -1.62088\times10^{-17},\ d_7 = 0.292296,\ d_8 = -0.539337,\ \lambda_1 = -20.,\ \lambda_2 = -25.,\ \lambda_3 = -30.7628,\ \lambda_4 = -60.\}$