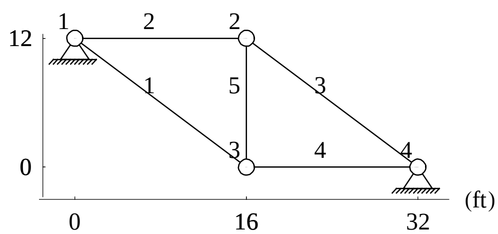
Example 4.3: Plane truss with temperature change (p. 233)

All members have the same cross-sectional area $A = 1/2\,\text{in}^2$ and are of the same material $E = 29,\,000\,\text{ksi}$ and $\alpha = 6.5\times10^{-6} \big/ {}^\circ\text{F}$. The first element undergoes a temperature rise of 100°F. The dimensions are shown in the figure.



For numerical calculations the k − in units are used.

Global equations at start of the element assembly process

$$\begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}\begin{pmatrix} u_1\\ v_1\\ u_2\\ v_2\\ u_3\\ v_3\\ u_4\\ v_4 \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}$$

Equations for element 1

$E = 29000 \qquad A = \frac{1}{2}$

$\alpha = 6.5\times10^{-6} \qquad \Delta T = 100 \qquad \epsilon_0 = 0.00065$

Element node	Global node number	x	y
1	1	0	144.
2	3	192.	0

$x_1 = 0 \qquad y_1 = 144. \qquad x_2 = 192. \qquad y_2 = 0$

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 240.$$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 0.8$ $\qquad m_s = \dfrac{y_2 - y_1}{L} = -0.6$

Substituting into the truss element equations we get

$$\begin{pmatrix} 38.6667 & -29. & -38.6667 & 29. \\ -29. & 21.75 & 29. & -21.75 \\ -38.6667 & 29. & 38.6667 & -29. \\ 29. & -21.75 & -29. & 21.75 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} -7.54 \\ 5.655 \\ 7.54 \\ -5.655 \end{pmatrix}$$

The element contributes to {1, 2, 5, 6} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 38.6667 & -29. & 0 & 0 & -38.6667 & 29. & 0 & 0 \\ -29. & 21.75 & 0 & 0 & 29. & -21.75 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -38.6667 & 29. & 0 & 0 & 38.6667 & -29. & 0 & 0 \\ 29. & -21.75 & 0 & 0 & -29. & 21.75 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} -7.54 \\ 5.655 \\ 0 \\ 0 \\ 7.54 \\ -5.655 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 2

$E = 29000$ $\qquad A = \dfrac{1}{2}$

Element node	Global node number	x	y
1	1	0	144.
2	2	192.	144.

$x_1 = 0$ $\qquad y_1 = 144.$ $\qquad x_2 = 192.$ $\qquad y_2 = 144.$

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 192.$$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 1.$ $\qquad m_s = \dfrac{y_2 - y_1}{L} = 0.$

Substituting into the truss element equations we get

$$\begin{pmatrix} 75.5208 & 0. & -75.5208 & 0. \\ 0. & 0. & 0. & 0. \\ -75.5208 & 0. & 75.5208 & 0. \\ 0. & 0. & 0. & 0. \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {1, 2, 3, 4} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 114.188 & -29. & -75.5208 & 0 & -38.6667 & 29. & 0 & 0 \\ -29. & 21.75 & 0 & 0 & 29. & -21.75 & 0 & 0 \\ -75.5208 & 0 & 75.5208 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -38.6667 & 29. & 0 & 0 & 38.6667 & -29. & 0 & 0 \\ 29. & -21.75 & 0 & 0 & -29. & 21.75 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} -7.54 \\ 5.655 \\ 0 \\ 0 \\ 7.54 \\ -5.655 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 3

$E = 29000 \qquad A = \frac{1}{2}$

Element node	Global node number	x	y
1	2	192.	144.
2	4	384.	0

$x_1 = 192. \qquad y_1 = 144. \qquad x_2 = 384. \qquad y_2 = 0$

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 240.$$

Direction cosines: $\ell_s = \frac{x_2 - x_1}{L} = 0.8 \qquad m_s = \frac{y_2 - y_1}{L} = -0.6$

Substituting into the truss element equations we get

$$\begin{pmatrix} 38.6667 & -29. & -38.6667 & 29. \\ -29. & 21.75 & 29. & -21.75 \\ -38.6667 & 29. & 38.6667 & -29. \\ 29. & -21.75 & -29. & 21.75 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {3, 4, 7, 8} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 114.188 & -29. & -75.5208 & 0 & -38.6667 & 29. & 0 & 0 \\ -29. & 21.75 & 0 & 0 & 29. & -21.75 & 0 & 0 \\ -75.5208 & 0 & 114.188 & -29. & 0 & 0 & -38.6667 & 29. \\ 0 & 0 & -29. & 21.75 & 0 & 0 & 29. & -21.75 \\ -38.6667 & 29. & 0 & 0 & 38.6667 & -29. & 0 & 0 \\ 29. & -21.75 & 0 & 0 & -29. & 21.75 & 0 & 0 \\ 0 & 0 & -38.6667 & 29. & 0 & 0 & 38.6667 & -29. \\ 0 & 0 & 29. & -21.75 & 0 & 0 & -29. & 21.75 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} -7.54 \\ 5.655 \\ 0 \\ 0 \\ 7.54 \\ -5.655 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 4

$E = 29000 \qquad A = \frac{1}{2}$

Element node	Global node number	x	y
1	3	192.	0
2	4	384.	0

$x_1 = 192. \qquad y_1 = 0 \qquad x_2 = 384. \qquad y_2 = 0$

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 192.$

Direction cosines: $\ell_s = \frac{x_2 - x_1}{L} = 1. \qquad m_s = \frac{y_2 - y_1}{L} = 0$

Substituting into the truss element equations we get

$$\begin{pmatrix} 75.5208 & 0. & -75.5208 & 0. \\ 0. & 0. & 0. & 0. \\ -75.5208 & 0. & 75.5208 & 0. \\ 0. & 0. & 0. & 0. \end{pmatrix} \begin{pmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {5, 6, 7, 8} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 114.188 & -29. & -75.5208 & 0 & -38.6667 & 29. & 0 & 0 \\ -29. & 21.75 & 0 & 0 & 29. & -21.75 & 0 & 0 \\ -75.5208 & 0 & 114.188 & -29. & 0 & 0 & -38.6667 & 29. \\ 0 & 0 & -29. & 21.75 & 0 & 0 & 29. & -21.75 \\ -38.6667 & 29. & 0 & 0 & 114.188 & -29. & -75.5208 & 0 \\ 29. & -21.75 & 0 & 0 & -29. & 21.75 & 0 & 0 \\ 0 & 0 & -38.6667 & 29. & -75.5208 & 0 & 114.188 & -29. \\ 0 & 0 & 29. & -21.75 & 0 & 0 & -29. & 21.75 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} -7.54 \\ 5.655 \\ 0 \\ 0 \\ 7.54 \\ -5.655 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 5

$E = 29000 \qquad A = \frac{1}{2}$

Element node	Global node number	x	y
1	2	192.	144.
2	3	192.	0

$x_1 = 192. \qquad y_1 = 144. \qquad x_2 = 192. \qquad y_2 = 0$

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 144.$

Direction cosines: $\ell_s = \frac{x_2 - x_1}{L} = 0. \qquad m_s = \frac{y_2 - y_1}{L} = -1.$

Substituting into the truss element equations we get

$$\begin{pmatrix} 0. & 0. & 0. & 0. \\ 0. & 100.694 & 0. & -100.694 \\ 0. & 0. & 0. & 0. \\ 0. & -100.694 & 0. & 100.694 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

The element contributes to {3, 4, 5, 6} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 114.188 & -29. & -75.5208 & 0 & -38.6667 & 29. & 0 & 0 \\ -29. & 21.75 & 0 & 0 & 29. & -21.75 & 0 & 0 \\ -75.5208 & 0 & 114.188 & -29. & 0 & 0 & -38.6667 & 29. \\ 0 & 0 & -29. & 122.444 & 0 & -100.694 & 29. & -21.75 \\ -38.6667 & 29. & 0 & 0 & 114.188 & -29. & -75.5208 & 0 \\ 29. & -21.75 & 0 & -100.694 & -29. & 122.444 & 0 & 0 \\ 0 & 0 & -38.6667 & 29. & -75.5208 & 0 & 114.188 & -29. \\ 0 & 0 & 29. & -21.75 & 0 & 0 & -29. & 21.75 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} -7.54 \\ 5.655 \\ 0 \\ 0 \\ 7.54 \\ -5.655 \\ 0 \\ 0 \end{pmatrix}$$

Essential boundary conditions

Node	dof	Value
1	u_1	0
	v_1	0
4	u_4	0
	v_4	0

Remove {1, 2, 7, 8} rows and columns.

After adjusting for essential boundary conditions we have

$$\begin{pmatrix} 114.188 & -29. & 0 & 0 \\ -29. & 122.444 & 0 & -100.694 \\ 0 & 0 & 114.188 & -29. \\ 0 & -100.694 & -29. & 122.444 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 7.54 \\ -5.655 \end{pmatrix}$$

Solving the final system of global equations we get

$$\{u_2 = -0.0308148,\ v_2 = -0.121333,\ u_3 = 0.0308148,\ v_3 = -0.138667\}$$

Complete table of nodal values

	u	v
1	0	0
2	−0.0308148	−0.121333
3	0.0308148	−0.138667
4	0	0

Computation of reactions

Equation numbers of dof with specified values: {1, 2, 7, 8}

Extracting equations {1, 2, 7, 8} from the global system we have

$$\begin{pmatrix} 114.188 & -29. & -75.5208 & 0 & -38.6667 & 29. & 0 & 0 \\ -29. & 21.75 & 0 & 0 & 29. & -21.75 & 0 & 0 \\ 0 & 0 & -38.6667 & 29. & -75.5208 & 0 & 114.188 & -29. \\ 0 & 0 & 29. & -21.75 & 0 & 0 & -29. & 21.75 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} R_1 - 7.54 \\ R_2 + 5.655 \\ R_3 + 0. \\ R_4 + 0. \end{pmatrix}$$

Substituting the nodal values and re−arranging

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{pmatrix} =$$

$$\begin{pmatrix} 114.188 & -29. & -75.5208 & 0 & -38.6667 & 29. & 0 & 0 \\ -29. & 21.75 & 0 & 0 & 29. & -21.75 & 0 & 0 \\ 0 & 0 & -38.6667 & 29. & -75.5208 & 0 & 114.188 & -29. \\ 0 & 0 & 29. & -21.75 & 0 & 0 & -29. & 21.75 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -0.0308148 \\ -0.121333 \\ 0.0308148 \\ -0.138667 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} -7.54 \\ 5.655 \\ 0. \\ 0. \end{pmatrix}$$

Carrying out computations, the reactions are as follows.

Label	dof	Reaction
R_1	u_1	4.65432
R_2	v_1	−1.74537
R_3	u_4	−4.65432
R_4	v_4	1.74537

Sum of Reactions

dof: u	0
dof: v	0

Solution for element 1

Nodal coordinates

Element node	Global node number	x	y
1	1	0	144.
2	3	192.	0

$x_1 = 0$ $\quad y_1 = 144.$ $\quad x_2 = 192.$ $\quad y_2 = 0$

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 240.$$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 0.8$ $\qquad m_s = \dfrac{y_2 - y_1}{L} = -0.6$

Global to local transformation matrix, $\boldsymbol{T} = \begin{pmatrix} 0.8 & -0.6 & 0 & 0 \\ 0 & 0 & 0.8 & -0.6 \end{pmatrix}$

Element nodal displacements in global coordinates, $\boldsymbol{d} = \begin{pmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0.0308148 \\ -0.138667 \end{pmatrix}$

Element nodal displacements in local coordinates, $\boldsymbol{d}_\ell = \boldsymbol{T}\ \boldsymbol{d} = \begin{pmatrix} 0. \\ 0.107852 \end{pmatrix}$

Axial displacements at element ends, $d_1 = 0.$ $\quad d_2 = 0.107852$

$E = 29000 \quad A = \frac{1}{2}$

$\alpha = 6.5 \times 10^{-6} \quad \Delta T = 100 \quad \epsilon_0 = 0.00065$

Axial strain, $\epsilon = (d_2 - d_1)/L - \epsilon_0 = 0.000449383$

Axial stress, $\sigma = E\epsilon = -5.8179$ $\quad$ Axial force $= \sigma A = -2.90895$

Solution for element 2

Nodal coordinates

Element node	Global node number	x	y
1	1	0	144.
2	2	192.	144.

$x_1 = 0 \quad y_1 = 144. \quad x_2 = 192. \quad y_2 = 144.$

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 192.$

Direction cosines: $\ell_s = \frac{x_2 - x_1}{L} = 1.$ $\quad m_s = \frac{y_2 - y_1}{L} = 0.$

Global to local transformation matrix, $\boldsymbol{T} = \begin{pmatrix} 1. & 0. & 0 & 0 \\ 0 & 0 & 1. & 0. \end{pmatrix}$

Element nodal displacements in global coordinates, $\boldsymbol{d} = \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -0.0308148 \\ -0.121333 \end{pmatrix}$

Element nodal displacements in local coordinates, $\boldsymbol{d}_\ell = \boldsymbol{T}\ \boldsymbol{d} = \begin{pmatrix} 0. \\ -0.0308148 \end{pmatrix}$

Axial displacements at element ends, $d_1 = 0.$ $\quad d_2 = -0.0308148$

$E = 29000 \quad A = \frac{1}{2}$

Axial strain, $\epsilon = (d_2 - d_1)/L = -0.000160494$

Axial stress, $\sigma = E\epsilon = -4.65432$ Axial force $= \sigma A = -2.32716$

Solution for element 3

Nodal coordinates

Element node	Global node number	x	y
1	2	192.	144.
2	4	384.	0

$x_1 = 192.$ $y_1 = 144.$ $x_2 = 384.$ $y_2 = 0$

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 240.$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 0.8$ $m_s = \dfrac{y_2 - y_1}{L} = -0.6$

Global to local transformation matrix, $\boldsymbol{T} = \begin{pmatrix} 0.8 & -0.6 & 0 & 0 \\ 0 & 0 & 0.8 & -0.6 \end{pmatrix}$

Element nodal displacements in global coordinates, $\boldsymbol{d} = \begin{pmatrix} u_2 \\ v_2 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} -0.0308148 \\ -0.121333 \\ 0 \\ 0 \end{pmatrix}$

Element nodal displacements in local coordinates, $\boldsymbol{d}_\ell = \boldsymbol{T}\,\boldsymbol{d} = \begin{pmatrix} 0.0481481 \\ 0. \end{pmatrix}$

Axial displacements at element ends, $d_1 = 0.0481481$ $d_2 = 0.$

$E = 29000$ $A = \dfrac{1}{2}$

Axial strain, $\epsilon = (d_2 - d_1)/L = -0.000200617$

Axial stress, $\sigma = E\epsilon = -5.8179$ Axial force $= \sigma A = -2.90895$

Solution for element 4

Nodal coordinates

Element node	Global node number	x	y
1	3	192.	0
2	4	384.	0

$x_1 = 192.$ $y_1 = 0$ $x_2 = 384.$ $y_2 = 0$

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 192.$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 1.$ $m_s = \dfrac{y_2 - y_1}{L} = 0$

Global to local transformation matrix, $\boldsymbol{T} = \begin{pmatrix} 1. & 0 & 0 & 0 \\ 0 & 0 & 1. & 0 \end{pmatrix}$

Element nodal displacements in global coordinates, $\boldsymbol{d} = \begin{pmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0.0308148 \\ -0.138667 \\ 0 \\ 0 \end{pmatrix}$

Element nodal displacements in local coordinates, $\boldsymbol{d}_\ell = \boldsymbol{T}\ \boldsymbol{d} = \begin{pmatrix} 0.0308148 \\ 0. \end{pmatrix}$

Axial displacements at element ends, $d_1 = 0.0308148$ $\quad d_2 = 0.$

$E = 29000 \qquad A = \frac{1}{2}$

Axial strain, $\epsilon = (d_2 - d_1)/L = -0.000160494$

Axial stress, $\sigma = E\epsilon = -4.65432$ $\quad$ Axial force $= \sigma A = -2.32716$

Solution for element 5

Nodal coordinates

Element node	Global node number	x	y
1	2	192.	144.
2	3	192.	0

$x_1 = 192. \qquad y_1 = 144. \qquad x_2 = 192. \qquad y_2 = 0$

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 144.$

Direction cosines: $\ell_s = \frac{x_2 - x_1}{L} = 0. \qquad m_s = \frac{y_2 - y_1}{L} = -1.$

Global to local transformation matrix, $\boldsymbol{T} = \begin{pmatrix} 0. & -1. & 0 & 0 \\ 0 & 0 & 0. & -1. \end{pmatrix}$

Element nodal displacements in global coordinates, $\boldsymbol{d} = \begin{pmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} -0.0308148 \\ -0.121333 \\ 0.0308148 \\ -0.138667 \end{pmatrix}$

Element nodal displacements in local coordinates, $\boldsymbol{d}_\ell = \boldsymbol{T}\ \boldsymbol{d} = \begin{pmatrix} 0.121333 \\ 0.138667 \end{pmatrix}$

Axial displacements at element ends, $d_1 = 0.121333$ $\quad d_2 = 0.138667$

$E = 29000 \qquad A = \frac{1}{2}$

Axial strain, $\epsilon = (d_2 - d_1)/L = 0.00012037$

Axial stress, $\sigma = E\epsilon = 3.49074$ Axial force $= \sigma A = 1.74537$

Solution summary

Nodal solution

	x−coord	y−coord	u	v
1	0	144.	0	0
2	192.	144.	−0.0308148	−0.121333
3	192.	0	0.0308148	−0.138667
4	384.	0	0	0

Element solution

	Stress	Axial force
1	−5.8179	−2.90895
2	−4.65432	−2.32716
3	−5.8179	−2.90895
4	−4.65432	−2.32716
5	3.49074	1.74537

Support reactions

Node	dof	Reaction
1	u_1	4.65432
1	v_1	−1.74537
4	u_4	−4.65432
4	v_4	1.74537

Sum of applied loads → (0 0)

Sum of support reactions → (0 0)