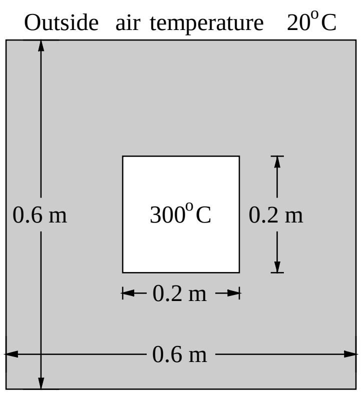
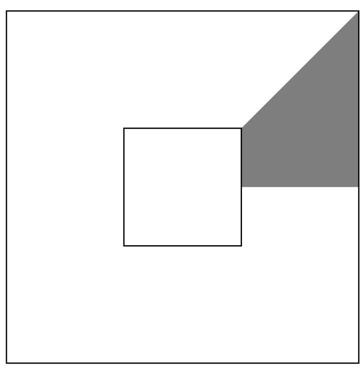
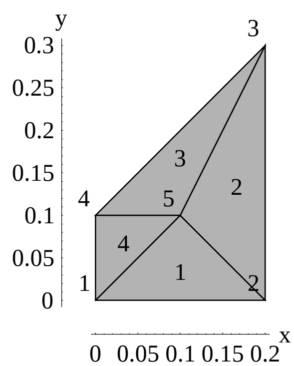
Square duct heat flow example: Examples 1.5 p. 28, 1.8 p. 44, and 1.11 p. 52

Cross-section of a 20 cm by 20 cm duct made of concrete walls 20 cm thick is shown in Figure. The inside surface of the duct is maintained at a temperature of 300 °C due to hot gases flowing from a furnace. On the outside the duct is exposed to air with an ambient temperature of 20 °C. The heat conduction coefficient of concrete is 1.4 $W/m.$°C. The average convection heat transfer coefficient on the outside of the duct is 27 $W/m.$°C.







Global equations at start of the element assembly process

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 1

$k_x = 1.4$ $\qquad k_y = 1.4$ $\qquad Q = 0$

Nodal coordinates

Element node	Global node number	x	y
1	1	0.	0.
2	2	0.2	0.
3	5	0.1	0.1

$x_1 = 0.$ $x_2 = 0.2$ $x_3 = 0.1$
$y_1 = 0.$ $y_2 = 0.$ $y_3 = 0.1$

Using these values we get

$b_1 = -0.1$ $b_2 = 0.1$ $b_3 = 0.$

$c_1 = -0.1$ $c_2 = -0.1$ $c_3 = 0.2$

$f_1 = 0.02$ $f_2 = 0.$ $f_3 = 0.$

Element area, $A = 0.01$

Substituting these values we get

$$\boldsymbol{k}_k = \begin{pmatrix} 0.7 & 0. & -0.7 \\ 0. & 0.7 & -0.7 \\ -0.7 & -0.7 & 1.4 \end{pmatrix} \qquad \boldsymbol{r}_Q = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 0.7 & 0. & -0.7 \\ 0. & 0.7 & -0.7 \\ -0.7 & -0.7 & 1.4 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to {1, 2, 5} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$

and to a global matrix: $\begin{pmatrix} [1, 1] & [1, 2] & [1, 5] \\ [2, 1] & [2, 2] & [2, 5] \\ [5, 1] & [5, 2] & [5, 5] \end{pmatrix}$

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 0.7 & 0 & 0 & 0 & -0.7 \\ 0 & 0.7 & 0 & 0 & -0.7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -0.7 & -0.7 & 0 & 0 & 1.4 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 2

$k_x = 1.4$ $k_y = 1.4$ $Q = 0$

Nodal coordinates

Element node	Global node number	x	y
1	2	0.2	0.
2	3	0.2	0.3
3	5	0.1	0.1

$x_1 = 0.2$ $x_2 = 0.2$ $x_3 = 0.1$
$y_1 = 0.$ $y_2 = 0.3$ $y_3 = 0.1$

Using these values we get

$b_1 = 0.2$ $\quad b_2 = 0.1$ $\quad b_3 = -0.3$

$c_1 = -0.1$ $\quad c_2 = 0.1$ $\quad c_3 = 0.$

$f_1 = -0.01$ $\quad f_2 = -0.02$ $\quad f_3 = 0.06$

Element area, $A = 0.015$

Substituting these values we get

$$\boldsymbol{k}_k = \begin{pmatrix} 1.16667 & 0.233333 & -1.4 \\ 0.233333 & 0.466667 & -0.7 \\ -1.4 & -0.7 & 2.1 \end{pmatrix} \qquad \boldsymbol{r}_Q = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Convection on side 1 (nodes {2, 3}) with $h = 27$ and $T_\infty = 20$

End nodal coordinates: ({0.2, 0.} {0.2, 0.3}) giving side length, $L = 0.3$

Using these values we get

$$\boldsymbol{k}_h = \begin{pmatrix} 2.7 & 1.35 & 0 \\ 1.35 & 2.7 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \boldsymbol{r}_h = \begin{pmatrix} 81. \\ 81. \\ 0 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 3.86667 & 1.58333 & -1.4 \\ 1.58333 & 3.16667 & -0.7 \\ -1.4 & -0.7 & 2.1 \end{pmatrix} \begin{pmatrix} T_2 \\ T_3 \\ T_5 \end{pmatrix} = \begin{pmatrix} 81. \\ 81. \\ 0 \end{pmatrix}$$

The element contributes to {2, 3, 5} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$

and to a global matrix: $\begin{pmatrix} [2,2] & [2,3] & [2,5] \\ [3,2] & [3,3] & [3,5] \\ [5,2] & [5,3] & [5,5] \end{pmatrix}$

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 0.7 & 0 & 0 & 0 & -0.7 \\ 0 & 4.56667 & 1.58333 & 0 & -2.1 \\ 0 & 1.58333 & 3.16667 & 0 & -0.7 \\ 0 & 0 & 0 & 0 & 0 \\ -0.7 & -2.1 & -0.7 & 0 & 3.5 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 81. \\ 81. \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 3

$k_x = 1.4$ $\quad k_y = 1.4$ $\quad Q = 0$

Nodal coordinates

Element node	Global node number	x	y
1	3	0.2	0.3
2	4	0.	0.1
3	5	0.1	0.1

$x_1 = 0.2$ $\quad x_2 = 0.$ $\quad x_3 = 0.1$

$y_1 = 0.3$ $\quad y_2 = 0.1$ $\quad y_3 = 0.1$

Using these values we get

$b_1 = 0.$ $\quad b_2 = -0.2$ $\quad b_3 = 0.2$

$c_1 = 0.1$ $\quad c_2 = 0.1$ $\quad c_3 = -0.2$

$f_1 = -0.01$ $\quad f_2 = 0.01$ $\quad f_3 = 0.02$

Element area, $A = 0.01$

Substituting these values we get

$$\boldsymbol{k}_k = \begin{pmatrix} 0.35 & 0.35 & -0.7 \\ 0.35 & 1.75 & -2.1 \\ -0.7 & -2.1 & 2.8 \end{pmatrix} \qquad \boldsymbol{r}_Q = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 0.35 & 0.35 & -0.7 \\ 0.35 & 1.75 & -2.1 \\ -0.7 & -2.1 & 2.8 \end{pmatrix} \begin{pmatrix} T_3 \\ T_4 \\ T_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to {3, 4, 5} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$

and to a global matrix: $\begin{pmatrix} [3,3] & [3,4] & [3,5] \\ [4,3] & [4,4] & [4,5] \\ [5,3] & [5,4] & [5,5] \end{pmatrix}$

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 0.7 & 0 & 0 & 0 & -0.7 \\ 0 & 4.56667 & 1.58333 & 0 & -2.1 \\ 0 & 1.58333 & 3.51667 & 0.35 & -1.4 \\ 0 & 0 & 0.35 & 1.75 & -2.1 \\ -0.7 & -2.1 & -1.4 & -2.1 & 6.3 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 81. \\ 81. \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 4

$k_x = 1.4$ $\quad k_y = 1.4$ $\quad Q = 0$

Nodal coordinates

Element node	Global node number	x	y
1	1	0.	0.
2	5	0.1	0.1
3	4	0.	0.1

$x_1 = 0.$ $\quad x_2 = 0.1$ $\quad x_3 = 0.$
$y_1 = 0.$ $\quad y_2 = 0.1$ $\quad y_3 = 0.1$

Using these values we get

$b_1 = 0.$ $\quad b_2 = 0.1$ $\quad b_3 = -0.1$

$c_1 = -0.1$ $\quad c_2 = 0.$ $\quad c_3 = 0.1$

$f_1 = 0.01$ $\quad f_2 = 0.$ $\quad f_3 = 0.$

Element area, $A = 0.005$

Substituting these values we get

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 0.7 & 0. & -0.7 \\ 0. & 0.7 & -0.7 \\ -0.7 & -0.7 & 1.4 \end{pmatrix} \qquad \boldsymbol{r}_{\mathrm{Q}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Complete element equations

$$\begin{pmatrix} 0.7 & 0. & -0.7 \\ 0. & 0.7 & -0.7 \\ -0.7 & -0.7 & 1.4 \end{pmatrix} \begin{pmatrix} T_1 \\ T_5 \\ T_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The element contributes to {1, 5, 4} global degrees of freedom.

Locations for element contributions to a global vector: $\begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$

and to a global matrix: $\begin{pmatrix} [1, 1] & [1, 5] & [1, 4] \\ [5, 1] & [5, 5] & [5, 4] \\ [4, 1] & [4, 5] & [4, 4] \end{pmatrix}$

Adding element equations into appropriate locations we have

$$\begin{pmatrix} 1.4 & 0 & 0 & -0.7 & -0.7 \\ 0 & 4.56667 & 1.58333 & 0 & -2.1 \\ 0 & 1.58333 & 3.51667 & 0.35 & -1.4 \\ -0.7 & 0 & 0.35 & 3.15 & -2.8 \\ -0.7 & -2.1 & -1.4 & -2.8 & 7. \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 81. \\ 81. \\ 0 \\ 0 \end{pmatrix}$$

Essential boundary conditions

Node	dof	Value
1	T_1	300
4	T_4	300

Delete equations {1, 4}.

$$\begin{pmatrix} 0 & 4.56667 & 1.58333 & 0 & -2.1 \\ 0 & 1.58333 & 3.51667 & 0.35 & -1.4 \\ -0.7 & -2.1 & -1.4 & -2.8 & 7. \end{pmatrix} \begin{pmatrix} 300 \\ T_2 \\ T_3 \\ 300 \\ T_5 \end{pmatrix} = \begin{pmatrix} 81. \\ 81. \\ 0 \end{pmatrix}$$

Extract columns {1, 4}.

Multiply each column by its respective known value {300, 300}.

Move all resulting vectors to the rhs.

After adjusting for essential boundary conditions we have

$$\begin{pmatrix} 4.56667 & 1.58333 & -2.1 \\ 1.58333 & 3.51667 & -1.4 \\ -2.1 & -1.4 & 7. \end{pmatrix} \begin{pmatrix} T_2 \\ T_3 \\ T_5 \end{pmatrix} = \begin{pmatrix} 81. \\ -24. \\ 1050. \end{pmatrix}$$

Solving the final system of global equations we get

$\{T_2 = 93.5466,\ T_3 = 23.8437,\ T_5 = 182.833\}$

Complete table of nodal values

	T
1	300
2	93.5466
3	23.8437
4	300
5	182.833

Computation of reactions

Equation numbers of dof with specified values: {1, 4}

Extracting equations {1, 4} from the global system we have

$$\begin{pmatrix} 1.4 & 0 & 0 & -0.7 & -0.7 \\ -0.7 & 0 & 0.35 & 3.15 & -2.8 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$$

Substituting the nodal values and re−arranging

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \begin{pmatrix} 1.4 & 0 & 0 & -0.7 & -0.7 \\ -0.7 & 0 & 0.35 & 3.15 & -2.8 \end{pmatrix} \begin{pmatrix} 300 \\ 93.5466 \\ 23.8437 \\ 300 \\ 182.833 \end{pmatrix}$$

Carrying out computations, the reactions are as follows.

Label	dof	Reaction
R_1	T_1	82.0171
R_2	T_4	231.414

Sum of Reactions

313.431

Solution for element 1

Nodal coordinates

Element node	Global node number	x	y
1	1	0.	0.
2	2	0.2	0.
3	5	0.1	0.1

$x_1 = 0.$ $\quad x_2 = 0.2$ $\quad x_3 = 0.1$
$y_1 = 0.$ $\quad y_2 = 0.$ $\quad y_3 = 0.1$

Using these values we get

$b_1 = -0.1$ $\quad b_2 = 0.1$ $\quad b_3 = 0.$

$c_1 = -0.1$ $\quad c_2 = -0.1$ $\quad c_3 = 0.2$

$f_1 = 0.02$ $\quad f_2 = 0.$ $\quad f_3 = 0.$

Element area, $A = 0.01$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^{\mathrm{T}} = \{-5.\,x - 5.\,y + 1.,\ 5.\,x - 5.\,y,\ 10.\,y\}$

From global solution the temperatures at the element nodes are
(from nodes {1, 2, 5}), $\boldsymbol{d}^{\mathrm{T}} = \{300,\ 93.5466,\ 182.833\}$

Thus the temperature distribution over the element, $T(x,y) = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{d} = -1032.27\,x - 139.406\,y + 300.$

Differentiating with respect to x and y, $\partial T/\partial x = -1032.27$ and $\partial T/\partial y = -139.406$

Solution for element 2

Nodal coordinates

Element node	Global node number	x	y
1	2	0.2	0.
2	3	0.2	0.3
3	5	0.1	0.1

$x_1 = 0.2 \quad x_2 = 0.2 \quad x_3 = 0.1$
$y_1 = 0. \quad y_2 = 0.3 \quad y_3 = 0.1$

Using these values we get

$b_1 = 0.2 \quad b_2 = 0.1 \quad b_3 = -0.3$

$c_1 = -0.1 \quad c_2 = 0.1 \quad c_3 = 0.$

$f_1 = -0.01 \quad f_2 = -0.02 \quad f_3 = 0.06$

Element area, $A = 0.015$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^{\mathrm{T}} = \{6.66667\,x - 3.33333\,y - 0.333333,\ 3.33333\,x + 3.33333\,y - 0.666667,\ 2. - 10.\,x\}$

From global solution the temperatures at the element nodes are
(from nodes {2, 3, 5}), $\boldsymbol{d}^{\mathrm{T}} = \{93.5466,\ 23.8437,\ 182.833\}$

Thus the temperature distribution over the element, $T(x,y) = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{d} = -1125.2\,x - 232.343\,y + 318.587$

Differentiating with respect to x and y, $\partial T/\partial x = -1125.2$ and $\partial T/\partial y = -232.343$

Solution for element 3

Nodal coordinates

Element node	Global node number	x	y
1	3	0.2	0.3
2	4	0.	0.1
3	5	0.1	0.1

$x_1 = 0.2 \quad x_2 = 0. \quad x_3 = 0.1$
$y_1 = 0.3 \quad y_2 = 0.1 \quad y_3 = 0.1$

Using these values we get

$b_1 = 0. \quad b_2 = -0.2 \quad b_3 = 0.2$

$c_1 = 0.1 \quad c_2 = 0.1 \quad c_3 = -0.2$

$f_1 = -0.01 \quad f_2 = 0.01 \quad f_3 = 0.02$

Element area, $A = 0.01$

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^{T} = \{5.\,y - 0.5,\ -10.\,x + 5.\,y + 0.5,\ 10.\,x - 10.\,y + 1.\}$

From global solution the temperatures at the element nodes are
(from nodes {3, 4, 5}), $\boldsymbol{d}^{T} = \{23.8437,\ 300,\ 182.833\}$

Thus the temperature distribution over the element, $T(x,y) = \boldsymbol{N}^{T}\boldsymbol{d} = -1171.67\,x - 209.109\,y + 320.911$

Differentiating with respect to x and y, $\partial T/\partial x = -1171.67$ and $\partial T/\partial y = -209.109$

Solution for element 4

Nodal coordinates

Element node	Global node number	x	y
1	1	0.	0.
2	5	0.1	0.1
3	4	0.	0.1

$x_1 = 0.$ $x_2 = 0.1$ $x_3 = 0.$
$y_1 = 0.$ $y_2 = 0.1$ $y_3 = 0.1$

Using these values we get

$b_1 = 0.$ $b_2 = 0.1$ $b_3 = -0.1$

$c_1 = -0.1$ $c_2 = 0.$ $c_3 = 0.1$

$f_1 = 0.01$ $f_2 = 0.$ $f_3 = 0.$

Element area, A = 0.005

Substituting these into the formulas for triangle interpolation functions we get

Interpolation functions, $\boldsymbol{N}^{T} = \{1. - 10.\,y,\ 10.\,x,\ 10.\,y - 10.\,x\}$

From global solution the temperatures at the element nodes are
(from nodes {1, 5, 4}), $\boldsymbol{d}^{T} = \{300,\ 182.833,\ 300\}$

Thus the temperature distribution over the element, $T(x,y) = \boldsymbol{N}^{T}\boldsymbol{d} = 300. - 1171.67\,x$

Differentiating with respect to x and y, $\partial T/\partial x = -1171.67$ and $\partial T/\partial y = 0$

Solution summary

Nodal temperatures

Node	Temperature
1	300
2	93.5466
3	23.8437
4	300
5	182.833

Element solution

	T(x,y)	$\partial T/\partial x$	$\partial T/\partial y$
1	$-1032.27\,x - 139.406\,y + 300.$	-1032.27	-139.406
2	$-1125.2\,x - 232.343\,y + 318.587$	-1125.2	-232.343
3	$-1171.67\,x - 209.109\,y + 320.911$	-1171.67	-209.109
4	$300. - 1171.67\,x$	-1171.67	0