Example 4.14: Three dimensional frame (p. 290)

Analyze one story three dimensional frame shown in Figure. The height of the columns is 12 ft and the length of the beams is 10 ft. Each beam is subjected to a uniformly distributed load of 2 kip/ft in the downward direction. I-shape sections are used for both columns and beams with the arrangement as shown in the figure. The columns are connected to the foundation through simple connections that do not resist moments. The material is steel with $E = 29000\,\text{kip}\big/\text{in}^2$ and $G = 11200\,\text{kip}\big/\text{in}^2$. The section properties are as follows.

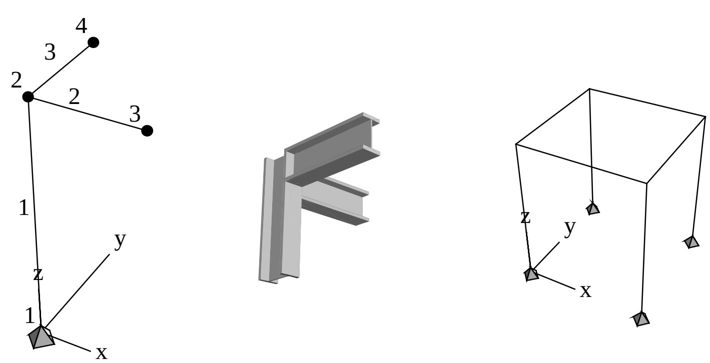
Beams: $A = 3.2\,\text{in}^2;\quad J = 43\,\text{in}^4;\quad I_{\max} = I_r = 450\,\text{in}^4;\quad I_{\min} = I_s = 32\,\text{in}^2$

Columns: $A = 4\,\text{in}^2;\quad J = 60\,\text{in}^4;\quad I_{\max} = I_r = 650\,\text{in}^4;\quad I_{\min} = I_s = 54\,\text{in}^2$

Taking advantage of symmetry we model a quarter of the frame using three elements. Because of symmetry, the boundary conditions at nodes 3 and 4 are as follows.

Node 3: $u = 0;\quad \theta_y = 0;\quad \theta_z = 0$

Node 4: $v = 0;\quad \theta_x = 0;\quad \theta_z = 0$



The distributed load is applied to the elements in their local coordinates. Therefore to assign proper direction and sign to the distributed loads we must carefully establish the local coordinates for the elements as follows.

Element 1: Nodes 1, 2, and 4

$\Longrightarrow$ $t-$axis along global z; $s-$axis along global x; $r-$axis along global y

Element 2: Nodes 2, 3, and 4

$\Longrightarrow$ $t-$axis along global x; $s-$axis along global $(-z)$; $r-$axis along global y

Distributed load: $q_r = 0$; $q_s = 2/12\ \text{kip/in}$

Element 3: Nodes 2, 4, and 3

$\Longrightarrow$ $t-$axis along global y; $s-$axis along global z; $r-$axis along global x

Distributed load: $q_r = 0$; $q_s = -2/12\ \text{kip/in}$

Global equations at start of the element assembly process

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix} u_1 \\ v_1 \\ w_1 \\ \theta x_1 \\ \theta y_1 \\ \theta z_1 \\ u_2 \\ v_2 \\ w_2 \\ \theta x_2 \\ \theta y_2 \\ \theta z_2 \\ u_3 \\ v_3 \\ w_3 \\ \theta x_3 \\ \theta y_3 \\ \theta z_3 \\ u_4 \\ v_4 \\ w_4 \\ \theta x_4 \\ \theta y_4 \\ \theta z_4 \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 1

$q_s = 0;$ $q_r = 0;$ $E = 29000.;$ $G = 11200.$

$A = 4;$ $J = 60;$ $I_r = 650;$ $I_s = 54$

Element nodal coordinates: $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 144. \\ 0 & 60. & 144. \end{pmatrix}$

Element length, $L = 144.$

Direction cosines: $\boldsymbol{H} = \begin{pmatrix} 0 & 0 & 1. \\ 1. & 0. & 0. \\ 0. & 1. & 0. \end{pmatrix}$

Element equations in local coordinates

$$\left(\begin{array}{cccccccccc}
75.7539 & 0. & 0. & 0. & 5454.28 & 0. & -75.7539 & 0. & 0. & \\
0. & 6.2934 & 0. & -453.125 & 0. & 0. & 0. & -6.2934 & 0. & -4 \\
0. & 0. & 805.556 & 0. & 0. & 0. & 0. & 0. & -805.556 & \\
0. & -453.125 & 0. & 43500. & 0. & 0. & 0. & 453.125 & 0. & 217 \\
5454.28 & 0. & 0. & 0. & 523611. & 0. & -5454.28 & 0. & 0. & \\
0. & 0. & 0. & 0. & 0. & 4666.67 & 0. & 0. & 0. & \\
-75.7539 & 0. & 0. & 0. & -5454.28 & 0. & 75.7539 & 0. & 0. & \\
0. & -6.2934 & 0. & 453.125 & 0. & 0. & 0. & 6.2934 & 0. & 4 \\
0. & 0. & -805.556 & 0. & 0. & 0. & 0. & 0. & 805.556 & \\
0. & -453.125 & 0. & 21750. & 0. & 0. & 0. & 453.125 & 0. & 435 \\
5454.28 & 0. & 0. & 0. & 261806. & 0. & -5454.28 & 0. & 0. & \\
0. & 0. & 0. & 0. & 0. & -4666.67 & 0. & 0. & 0. &
\end{array}\right.$$

$$\text{Global to local transformation, } \boldsymbol{T} = \begin{pmatrix}
0 & 0 & 1. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1. & 0. & 0. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0. & 1. & 0. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1. & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1. & 0. & 0. & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0. & 1. & 0. & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1. & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1. & 0. & 0. & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0. & 1. & 0. & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1. & 0. & 0. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0. & 1. & 0.
\end{pmatrix}$$

Element equations in global coordinates

$$\left(\begin{array}{cccccccccc}
75.7539 & 0 & 0 & 0 & 5454.28 & 0 & -75.7539 & 0 & 0 & \\
0 & 6.2934 & 0 & -453.125 & 0 & 0 & 0 & -6.2934 & 0 & -4 \\
0 & 0 & 805.556 & 0 & 0 & 0 & 0 & 0 & -805.556 & \\
0 & -453.125 & 0 & 43500. & 0 & 0 & 0 & 453.125 & 0 & 217 \\
5454.28 & 0 & 0 & 0 & 523611. & 0 & -5454.28 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 4666.67 & 0 & 0 & 0 & \\
-75.7539 & 0 & 0 & 0 & -5454.28 & 0 & 75.7539 & 0 & 0 & \\
0 & -6.2934 & 0 & 453.125 & 0 & 0 & 0 & 6.2934 & 0 & 4 \\
0 & 0 & -805.556 & 0 & 0 & 0 & 0 & 0 & 805.556 & \\
0 & -453.125 & 0 & 21750. & 0 & 0 & 0 & 453.125 & 0 & 435 \\
5454.28 & 0 & 0 & 0 & 261806. & 0 & -5454.28 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & -4666.67 & 0 & 0 & 0 &
\end{array}\right.$$

The element contributes to {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\left(\begin{array}{cccccccccc}
75.7539 & 0 & 0 & 0 & 5454.28 & 0 & -75.7539 & 0 & 0 & \\
0 & 6.2934 & 0 & -453.125 & 0 & 0 & 0 & -6.2934 & 0 & -4 \\
0 & 0 & 805.556 & 0 & 0 & 0 & 0 & 0 & -805.556 & \\
0 & -453.125 & 0 & 43500. & 0 & 0 & 0 & 453.125 & 0 & 217 \\
5454.28 & 0 & 0 & 0 & 523611. & 0 & -5454.28 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 4666.67 & 0 & 0 & 0 & \\
-75.7539 & 0 & 0 & 0 & -5454.28 & 0 & 75.7539 & 0 & 0 & \\
0 & -6.2934 & 0 & 453.125 & 0 & 0 & 0 & 6.2934 & 0 & 4 \\
0 & 0 & -805.556 & 0 & 0 & 0 & 0 & 0 & 805.556 & \\
0 & -453.125 & 0 & 21750. & 0 & 0 & 0 & 453.125 & 0 & 435 \\
5454.28 & 0 & 0 & 0 & 261806. & 0 & -5454.28 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & -4666.67 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 &
\end{array}\right.$$

Equations for element 2

$q_s = 0.166667;$ $q_r = 0;$ $E = 29000.;$ $G = 11200.$

$A = 3.2;$ $J = 43;$ $I_r = 450;$ $I_s = 32$

Element nodal coordinates: $\begin{pmatrix} 0 & 0 & 144. \\ 60. & 0 & 144. \\ 0 & 60. & 144. \end{pmatrix}$

Element length, $L = 60.$

Direction cosines: $\boldsymbol{H} = \begin{pmatrix} 1. & 0 & 0. \\ 0. & 0. & -1. \\ 0. & 1. & 0. \end{pmatrix}$

Element equations in local coordinates

$$\left(\begin{array}{cccccccccc}
1546.67 & 0. & 0. & 0. & 0. & 0. & -1546.67 & 0. & 0. & 0. \\
0. & 51.5556 & 0. & 0. & 0. & 1546.67 & 0. & -51.5556 & 0. & 0. \\
0. & 0. & 725. & 0. & -21750. & 0. & 0. & 0. & -725. & 0. \\
0. & 0. & 0. & 8026.67 & 0. & 0. & 0. & 0. & 0. & -8026.67 \\
0. & 0. & -21750. & 0. & 870000. & 0. & 0. & 0. & 21750. & 0. \\
0. & 1546.67 & 0. & 0. & 0. & 61866.7 & 0. & -1546.67 & 0. & 0. \\
-1546.67 & 0. & 0. & 0. & 0. & 0. & 1546.67 & 0. & 0. & 0. \\
0. & -51.5556 & 0. & 0. & 0. & -1546.67 & 0. & 51.5556 & 0. & 0. \\
0. & 0. & -725. & 0. & 21750. & 0. & 0. & 0. & 725. & 0. \\
0. & 0. & 0. & -8026.67 & 0. & 0. & 0. & 0. & 0. & 8026.67 \\
0. & 0. & -21750. & 0. & 435000. & 0. & 0. & 0. & 21750. & 0. \\
0. & 1546.67 & 0. & 0. & 0. & 30933.3 & 0. & -1546.67 & 0. & 0.
\end{array}\right.$$

Global to local transformation, $\boldsymbol{T} = \begin{pmatrix}
1. & 0 & 0. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0. & 0. & -1. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0. & 1. & 0. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1. & 0 & 0. & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0. & 0. & -1. & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0. & 1. & 0. & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1. & 0 & 0. & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0. & 0. & -1. & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0. & 1. & 0. & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1. & 0 & 0. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0. & 0. & -1. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0. & 1. & 0.
\end{pmatrix}$

Element equations in global coordinates

$$
\left(\begin{array}{cccccccccc}
1546.67 & 0 & 0 & 0 & 0 & 0 & -1546.67 & 0 & 0 & 0 \\
0 & 51.5556 & 0 & 0 & 0 & 1546.67 & 0 & -51.5556 & 0 & 0 \\
0 & 0 & 725. & 0 & -21750. & 0 & 0 & 0 & -725. & 0 \\
0 & 0 & 0 & 8026.67 & 0 & 0 & 0 & 0 & 0 & -8026.67 \\
0 & 0 & -21750. & 0 & 870000. & 0 & 0 & 0 & 21750. & 0 \\
0 & 1546.67 & 0 & 0 & 0 & 61866.7 & 0 & -1546.67 & 0 & 0 \\
-1546.67 & 0 & 0 & 0 & 0 & 0 & 1546.67 & 0 & 0 & 0 \\
0 & -51.5556 & 0 & 0 & 0 & -1546.67 & 0 & 51.5556 & 0 & 0 \\
0 & 0 & -725. & 0 & 21750. & 0 & 0 & 0 & 725. & 0 \\
0 & 0 & 0 & -8026.67 & 0 & 0 & 0 & 0 & 0 & 8026.67 \\
0 & 0 & -21750. & 0 & 435000. & 0 & 0 & 0 & 21750. & 0 \\
0 & 1546.67 & 0 & 0 & 0 & 30933.3 & 0 & -1546.67 & 0 & 0
\end{array}\right.
$$

The element contributes to {7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\left(\begin{array}{ccccccccc}
75.7539 & 0 & 0 & 0 & 5454.28 & 0 & -75.7539 & 0 & 0 \\
0 & 6.2934 & 0 & -453.125 & 0 & 0 & 0 & -6.2934 & 0 \\
0 & 0 & 805.556 & 0 & 0 & 0 & 0 & 0 & -805.556 \\
0 & -453.125 & 0 & 43500. & 0 & 0 & 0 & 453.125 & 0 \\
5454.28 & 0 & 0 & 0 & 523611. & 0 & -5454.28 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4666.67 & 0 & 0 & 0 \\
-75.7539 & 0 & 0 & 0 & -5454.28 & 0 & 1622.42 & 0 & 0 \\
0 & -6.2934 & 0 & 453.125 & 0 & 0 & 0 & 57.849 & 0 \\
0 & 0 & -805.556 & 0 & 0 & 0 & 0 & 0 & 1530.56 \\
0 & -453.125 & 0 & 21750. & 0 & 0 & 0 & 453.125 & 0 \\
5454.28 & 0 & 0 & 0 & 261806. & 0 & -5454.28 & 0 & -21750. \\
0 & 0 & 0 & 0 & 0 & -4666.67 & 0 & 1546.67 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1546.67 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -51.5556 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -725. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -21750. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1546.67 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right.$$

Equations for element 3

$q_s = -0.166667;\quad q_r = 0;\quad E = 29000.;\quad G = 11200.$

$A = 3.2;\quad J = 43;\quad I_r = 450;\quad I_s = 32$

Element nodal coordinates: $\begin{pmatrix} 0 & 0 & 144. \\ 0 & 60. & 144. \\ 60. & 0 & 144. \end{pmatrix}$

Element length, L = 60.

Direction cosines: $\boldsymbol{H} = \begin{pmatrix} 0 & 1. & 0. \\ 0. & 0. & 1. \\ 1. & 0. & 0. \end{pmatrix}$

Element equations in local coordinates

$$\left(\begin{array}{cccccccccc}
51.5556 & 0. & 0. & 0. & 0. & -1546.67 & -51.5556 & 0. & 0. & 0. \\
0. & 1546.67 & 0. & 0. & 0. & 0. & 0. & -1546.67 & 0. & 0. \\
0. & 0. & 725. & 21750. & 0. & 0. & 0. & 0. & -725. & 21750. \\
0. & 0. & 21750. & 870000. & 0. & 0. & 0. & 0. & -21750. & 435000. \\
0. & 0. & 0. & 0. & 8026.67 & 0. & 0. & 0. & 0. & 0. \\
-1546.67 & 0. & 0. & 0. & 0. & 61866.7 & 1546.67 & 0. & 0. & 0. \\
-51.5556 & 0. & 0. & 0. & 0. & 1546.67 & 51.5556 & 0. & 0. & 0. \\
0. & -1546.67 & 0. & 0. & 0. & 0. & 0. & 1546.67 & 0. & 0. \\
0. & 0. & -725. & -21750. & 0. & 0. & 0. & 0. & 725. & -21750. \\
0. & 0. & 21750. & 435000. & 0. & 0. & 0. & 0. & -21750. & 870000. \\
0. & 0. & 0. & 0. & -8026.67 & 0. & 0. & 0. & 0. & 0. \\
-1546.67 & 0. & 0. & 0. & 0. & 30933.3 & 1546.67 & 0. & 0. & 0.
\end{array}\right.$$

$$\text{Global to local transformation, } \boldsymbol{T} = \begin{pmatrix}
0 & 1. & 0. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0. & 0. & 1. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1. & 0. & 0. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1. & 0. & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0. & 0. & 1. & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1. & 0. & 0. & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1. & 0. & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0. & 0. & 1. & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1. & 0. & 0. & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1. & 0. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0. & 0. & 1. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1. & 0. & 0.
\end{pmatrix}$$

Element equations in global coordinates

$$\left(\begin{array}{cccccccccc}
51.5556 & 0 & 0 & 0 & 0 & -1546.67 & -51.5556 & 0 & 0 & 0 \\
0 & 1546.67 & 0 & 0 & 0 & 0 & 0 & -1546.67 & 0 & 0 \\
0 & 0 & 725. & 21750. & 0 & 0 & 0 & 0 & -725. & 21750. \\
0 & 0 & 21750. & 870000. & 0 & 0 & 0 & 0 & -21750. & 435000. \\
0 & 0 & 0 & 0 & 8026.67 & 0 & 0 & 0 & 0 & 0 \\
-1546.67 & 0 & 0 & 0 & 0 & 61866.7 & 1546.67 & 0 & 0 & 0 \\
-51.5556 & 0 & 0 & 0 & 0 & 1546.67 & 51.5556 & 0 & 0 & 0 \\
0 & -1546.67 & 0 & 0 & 0 & 0 & 0 & 1546.67 & 0 & 0 \\
0 & 0 & -725. & -21750. & 0 & 0 & 0 & 0 & 725. & -21750. \\
0 & 0 & 21750. & 435000. & 0 & 0 & 0 & 0 & -21750. & 870000. \\
0 & 0 & 0 & 0 & -8026.67 & 0 & 0 & 0 & 0 & 0 \\
-1546.67 & 0 & 0 & 0 & 0 & 30933.3 & 1546.67 & 0 & 0 & 0
\end{array}\right.$$

The element contributes to {7, 8, 9, 10, 11, 12, 19, 20, 21, 22, 23, 24} global degrees of freedom.

Adding element equations into appropriate locations we have

$$\left(\begin{array}{ccccccccc}
75.7539 & 0 & 0 & 0 & 5454.28 & 0 & -75.7539 & 0 & 0 \\
0 & 6.2934 & 0 & -453.125 & 0 & 0 & 0 & -6.2934 & 0 \\
0 & 0 & 805.556 & 0 & 0 & 0 & 0 & 0 & -805.556 \\
0 & -453.125 & 0 & 43500. & 0 & 0 & 0 & 453.125 & 0 \\
5454.28 & 0 & 0 & 0 & 523611. & 0 & -5454.28 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4666.67 & 0 & 0 & 0 \\
-75.7539 & 0 & 0 & 0 & -5454.28 & 0 & 1673.98 & 0 & 0 \\
0 & -6.2934 & 0 & 453.125 & 0 & 0 & 0 & 1604.52 & 0 \\
0 & 0 & -805.556 & 0 & 0 & 0 & 0 & 0 & 2255.56 \\
0 & -453.125 & 0 & 21750. & 0 & 0 & 0 & 453.125 & 21750. \\
5454.28 & 0 & 0 & 0 & 261806. & 0 & -5454.28 & 0 & -21750. \\
0 & 0 & 0 & 0 & 0 & -4666.67 & -1546.67 & 1546.67 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1546.67 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -51.5556 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -725. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -21750. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1546.67 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -51.5556 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1546.67 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -725. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 21750. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1546.67 & 0 & 0
\end{array}\right.$$

Essential boundary conditions

Node	dof	Value
1	u_1	0
1	v_1	0
1	w_1	0
3	u_3	0
3	θy_3	0
3	θz_3	0
4	v_4	0
4	θx_4	0
4	θz_4	0

Remove {1, 2, 3, 13, 17, 18, 20, 22, 24} rows and columns.

After adjusting for essential boundary conditions we have

$$\left(\begin{array}{ccccccccc}
43500. & 0 & 0 & 0 & 453.125 & 0 & 21750. & 0 & \\
0 & 523611. & 0 & -5454.28 & 0 & 0 & 0 & 261806. & \\
0 & 0 & 4666.67 & 0 & 0 & 0 & 0 & 0 & \\
0 & -5454.28 & 0 & 1673.98 & 0 & 0 & 0 & -5454.28 & \\
453.125 & 0 & 0 & 0 & 1604.52 & 0 & 453.125 & 0 & \\
0 & 0 & 0 & 0 & 0 & 2255.56 & 21750. & -21750. & \\
21750. & 0 & 0 & 0 & 453.125 & 21750. & 921527. & 0 & \\
0 & 261806. & 0 & -5454.28 & 0 & -21750. & 0 & 1.40164\times 10^{6} & \\
0 & 0 & -4666.67 & -1546.67 & 1546.67 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -51.5556 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & -725. & 0 & 21750. & \\
0 & 0 & 0 & 0 & 0 & 0 & -8026.67 & 0 & \\
0 & 0 & 0 & -51.5556 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & -725. & -21750. & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -8026.67 &
\end{array}\right.$$

Solving the final system of global equations we get

$$\{\theta x_1 = 0.000398634,\ \theta y_1 = -0.00015917,\ \theta z_1 = 0,\ u_2 = 0.000575402,\ v_2 = 0.000117025,$$
$$w_2 = -0.0248276,\ \theta x_2 = -0.000799706,\ \theta y_2 = 0.000330328,\ \theta z_2 = 0,\ v_3 = 0.000117025,$$
$$w_3 = -0.041634,\ \theta x_3 = -0.000799706,\ u_4 = 0.000575402,\ w_4 = -0.0557153,\ \theta y_4 = 0.000330328\}$$

Complete table of nodal values

	u	v	w	θx	θy	θz
1	0	0	0	0.000398634	−0.00015917	0
2	0.000575402	0.000117025	−0.0248276	−0.000799706	0.000330328	0
3	0	0.000117025	−0.041634	−0.000799706	0	0
4	0.000575402	0	−0.0557153	0	0.000330328	0

Computation of reactions

Equation numbers of dof with specified values: {1, 2, 3, 13, 17, 18, 20, 22, 24}

Extracting equations {1, 2, 3, 13, 17, 18, 20, 22, 24} from the global system we have

$$\left(\begin{array}{ccccccccccc}
75.7539 & 0 & 0 & 0 & 5454.28 & 0 & -75.7539 & 0 & 0 & 0 & 5 \\
0 & 6.2934 & 0 & -453.125 & 0 & 0 & 0 & -6.2934 & 0 & -453.125 & \\
0 & 0 & 805.556 & 0 & 0 & 0 & 0 & 0 & -805.556 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & -1546.67 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -21750. & 0 & 435 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1546.67 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1546.67 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 21750. & 435000. & \\
0 & 0 & 0 & 0 & 0 & 0 & -1546.67 & 0 & 0 & 0 &
\end{array}\right.$$

Substituting the nodal values and re−arranging

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \\ R_9 \end{pmatrix} = \begin{pmatrix} 75.7539 & 0 & 0 & 0 & 5454.28 & 0 & -75.7539 & 0 & 0 & 0 \\ 0 & 6.2934 & 0 & -453.125 & 0 & 0 & 0 & -6.2934 & 0 & -453.1 \\ 0 & 0 & 805.556 & 0 & 0 & 0 & 0 & 0 & -805.556 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1546.67 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -21750. & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1546.67 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1546.67 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 21750. & 435000. \\ 0 & 0 & 0 & 0 & 0 & 0 & -1546.67 & 0 & 0 & 0 \end{pmatrix}$$

Carrying out computations, the reactions are as follows.

Label	dof	Reaction
R_1	u_1	0.889955
R_2	v_1	0.180999
R_3	w_1	20.
R_4	u_3	−0.889955
R_5	θy_3	−171.846
R_6	θz_3	0
R_7	v_4	−0.180999
R_8	θx_4	273.936
R_9	θz_4	0

Sum of Reactions

dof: u	0
dof: v	0
dof: w	20.
dof: θx	273.936
dof: θy	−171.846
dof: θz	0

Solution for element 1

Nodal values in global coordinates, $\boldsymbol{d}^{\mathrm{T}} = \{0, 0, 0, 0.000398634, -0.00015917, 0, 0.000575402, 0.000117025, -0.0248276, -0.000799706, 0.000330328, 0\}$

$$\text{Global to local transformation, } \boldsymbol{T} = \begin{pmatrix} 0 & 0 & 1. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1. & 0. & 0. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0. & 1. & 0. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1. & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1. & 0. & 0. & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0. & 1. & 0. & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1. & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1. & 0. & 0. & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0. & 1. & 0. & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1. \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1. & 0. & 0. \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0. & 1. & 0. \end{pmatrix}$$

Nodal values in local coordinates, $\boldsymbol{d}_{\ell}^{\mathrm{T}} = \boldsymbol{T}\boldsymbol{d} = \{0., 0., 0., 0., 0.000398634, -0.00015917, -0.0248276, 0.000575402, 0.000117025, 0., -0.000799706, 0.000330328\}$

Axial effects:

Interpolation functions, $\boldsymbol{N}_{\mathrm{u}}^{\mathrm{T}} = \{1. - 0.00694444\,\mathrm{t},\ 0.00694444\,\mathrm{t}\}$

Axial displacement, $\mathrm{u(t)} = \boldsymbol{N}_{\mathrm{u}}^{\mathrm{T}} \begin{pmatrix} \mathrm{d}_1 \\ \mathrm{d}_7 \end{pmatrix} = -0.000172414\,\mathrm{t}$

Axial force, EA du(t)/dt = −20.

Torsional effects:

Twist angle, $\psi(\mathrm{t}) = \boldsymbol{N}_{\mathrm{u}}^{\mathrm{T}} \begin{pmatrix} \mathrm{d}_4 \\ \mathrm{d}_{10} \end{pmatrix} = 0$

Twisting moment, GJ dψ(t)/dt = 0.

Bending about r−axis:

$$\boldsymbol{N}_v^T = \{6.69796\times10^{-7}\,t^3 - 0.000144676\,t^2 + 1,\ 0.0000482253\,t^3 - 0.0138889\,t^2 + t,\ 0.000144676\,t^2 - 6.69796\times10^{-7}\,t^3,\ 0.0000482253\,t^3 - 0.00694444\,t^2\}$$

$$v(t) = \boldsymbol{N}_v^T \begin{pmatrix} d_2 \\ d_6 \\ d_8 \\ d_{12} \end{pmatrix} = 7.86875\times10^{-9}\,t^3 - 0.00015917\,t$$

Bending moment, $M_r = E\,I_r\ d^2v(t)/dt^2 = 0.889955\,t$

Shear force, $V_s = dM_r/dt = 0.889955$

Bending about s−axis:

$$\boldsymbol{N}_w^T = \{6.69796\times10^{-7}\,t^3 - 0.000144676\,t^2 + 1,\ -0.0000482253\,t^3 + 0.0138889\,t^2 - t,\ 0.000144676\,t^2 - 6.69796\times10^{-7}\,t^3,\ 0.00694444\,t^2 - 0.0000482253\,t^3\}$$

$$w(t) = \boldsymbol{N}_w^T \begin{pmatrix} d_3 \\ d_5 \\ d_9 \\ d_{11} \end{pmatrix} = -1.94202\times10^{-8}\,t^3 + 3.38615\times10^{-8}\,t^2 + 0.000398634\,t$$

Bending moment, $M_s = -EI_s\ d^2w(t)/dt^2 = -0.180999\,t$

Shear force, $V_r = -dM_s/dt = 0.180999$

Solution for element 2

Nodal values in global coordinates, $\boldsymbol{d}^T = \{0.000575402,\ 0.000117025,\ -0.0248276,\ -0.000799706,\ 0.000330328,\ 0,\ 0,\ 0.000117025,\ -0.041634,\ -0.000799706,\ 0,\ 0\}$

Global to local transformation,
$$\boldsymbol{T} = \begin{pmatrix}
1. & 0 & 0. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0. & 0. & -1. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0. & 1. & 0. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1. & 0 & 0. & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0. & 0. & -1. & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0. & 1. & 0. & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1. & 0 & 0. & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0. & 0. & -1. & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0. & 1. & 0. & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1. & 0 & 0. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0. & 0. & -1. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0. & 1. & 0.
\end{pmatrix}$$

Nodal values in local coordinates, $\boldsymbol{d}_\ell^{\mathrm{T}} = \boldsymbol{Td}$ = {0.000575402, 0.0248276, 0.000117025, −0.000799706, 0., 0.000330328, 0., 0.041634, 0.000117025, −0.000799706, 0., 0.}

Axial effects:

Interpolation functions, $\boldsymbol{N}_u^{\mathrm{T}} = \{1. - 0.0166667\, t,\ 0.0166667\, t\}$

Axial displacement, $u(t) = \boldsymbol{N}_u^{\mathrm{T}} \begin{pmatrix} d_1 \\ d_7 \end{pmatrix} = 0.000575402 - 9.59003 \times 10^{-6}\, t$

Axial force, EA du(t)/dt = −0.889955

Torsional effects:

Twist angle, $\psi(t) = \boldsymbol{N}_u^{\mathrm{T}} \begin{pmatrix} d_4 \\ d_{10} \end{pmatrix} = -0.000799706$

Twisting moment, GJ $d\psi(t)/dt = 8.70253 \times 10^{-16}$

Bending about r−axis:

$\boldsymbol{N}_v^{\mathrm{T}} = \{9.25926 \times 10^{-6}\, t^3 - 0.000833333\, t^2 + 1,\ 0.000277778\, t^3 - 0.0333333\, t^2 + t,\ 0.000833333\, t^2 - 9.25926 \times 10^{-6}\, t^3,\ 0.000277778\, t^3 - 0.0166667\, t^2\}$

$$v(t) = \boldsymbol{N}_v^{\mathrm{T}} \begin{pmatrix} d_2 \\ d_6 \\ d_8 \\ d_{12} \end{pmatrix} = -6.3857 \times 10^{-8}\, t^3 + 2.99439 \times 10^{-6}\, t^2 + 0.000330328\, t + 0.0248276$$

Fixed−end displacement solution, $= 5.32141 \times 10^{-10}\, (60. - t)^2\, t^2$

Transverse displacement, v(t) =
$5.32141 \times 10^{-10}\, t^4 - 1.27714 \times 10^{-7}\, t^3 + 4.9101 \times 10^{-6}\, t^2 + 0.000330328\, t + 0.0248276$

Bending moment, $M_r = E\, I_r\, d^2 v(t)/dt^2 = 1.305 \times 10^7 \left(6.3857 \times 10^{-9}\, t^2 - 7.66284 \times 10^{-7}\, t + 9.8202 \times 10^{-6}\right)$

Shear force, $V_s = dM_r/dt = 0.166667\, t - 10.$

Bending about s−axis:

$\boldsymbol{N}_w^{\mathrm{T}} = \{9.25926 \times 10^{-6}\, t^3 - 0.000833333\, t^2 + 1,\ -0.000277778\, t^3 + 0.0333333\, t^2 - t,\ 0.000833333\, t^2 - 9.25926 \times 10^{-6}\, t^3,\ 0.0166667\, t^2 - 0.000277778\, t^3\}$

$$w(t) = \boldsymbol{N}_w^{\mathrm{T}} \begin{pmatrix} d_3 \\ d_5 \\ d_9 \\ d_{11} \end{pmatrix} = 0.000117025$$

Bending moment, $M_s = -EI_s\, d^2 w(t)/dt^2 = 0$

Shear force, $V_r = -dM_s/dt = 0$

Solution for element 3

Nodal values in global coordinates, $\boldsymbol{d}^T = \{0.000575402, 0.000117025, -0.0248276, -0.000799706, 0.000330328, 0, 0.000575402, 0, -0.0557153, 0, 0.000330328, 0\}$

$$\text{Global to local transformation, } \boldsymbol{T} = \begin{pmatrix} 0 & 1. & 0. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0. & 0. & 1. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1. & 0. & 0. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1. & 0. & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0. & 0. & 1. & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1. & 0. & 0. & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1. & 0. & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0. & 0. & 1. & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1. & 0. & 0. & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1. & 0. \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0. & 0. & 1. \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1. & 0. & 0. \end{pmatrix}$$

Nodal values in local coordinates, $\boldsymbol{d}_\ell^T = \boldsymbol{T}\boldsymbol{d} = \{0.000117025, -0.0248276, 0.000575402, 0.000330328, 0., -0.000799706, 0., -0.0557153, 0.000575402, 0.000330328, 0., 0.\}$

Axial effects:

Interpolation functions, $\boldsymbol{N}_u^T = \{1. - 0.0166667\,t,\ 0.0166667\,t\}$

Axial displacement, $u(t) = \boldsymbol{N}_u^T \begin{pmatrix} d_1 \\ d_7 \end{pmatrix} = 0.000117025 - 1.95042 \times 10^{-6}\,t$

Axial force, $EA\ du(t)/dt = -0.180999$

Torsional effects:

Twist angle, $\psi(t) = \boldsymbol{N}_u^T \begin{pmatrix} d_4 \\ d_{10} \end{pmatrix} = 0.000330328$

Twisting moment, $GJ\ d\psi(t)/dt = 0.$

Bending about r–axis:

$$\boldsymbol{N}_v^T = \{9.25926 \times 10^{-6}\,t^3 - 0.000833333\,t^2 + 1,\ 0.000277778\,t^3 - 0.0333333\,t^2 + t,\ 0.000833333\,t^2 - 9.25926 \times 10^{-6}\,t^3,\ 0.000277778\,t^3 - 0.0166667\,t^2\}$$

$$v(t) = \boldsymbol{N}_v^T \begin{pmatrix} d_2 \\ d_6 \\ d_8 \\ d_{12} \end{pmatrix} = 6.3857 \times 10^{-8} t^3 + 9.17092 \times 10^{-7} t^2 - 0.000799706\, t - 0.0248276$$

Fixed−end displacement solution, $= -5.32141 \times 10^{-10} (60. - t)^2 t^2$

Transverse displacement, v(t) =
$-5.32141 \times 10^{-10} t^4 + 1.27714 \times 10^{-7} t^3 - 9.98617 \times 10^{-7} t^2 - 0.000799706\, t - 0.0248276$

Bending moment, $M_r = E\, I_r\, d^2 v(t)/d$
$t^2 = 1.305 \times 10^7 \left(-6.3857 \times 10^{-9} t^2 + 7.66284 \times 10^{-7} t - 1.99723 \times 10^{-6}\right)$

Shear force, $V_s = dM_r/dt = 10. - 0.166667\, t$

Bending about s−axis:

$\boldsymbol{N}_w^T = \{9.25926 \times 10^{-6} t^3 - 0.000833333\, t^2 + 1,\ -0.000277778\, t^3 + 0.0333333\, t^2 - t,$
$0.000833333\, t^2 - 9.25926 \times 10^{-6} t^3,\ 0.0166667\, t^2 - 0.000277778\, t^3\}$

$$w(t) = \boldsymbol{N}_w^T \begin{pmatrix} d_3 \\ d_5 \\ d_9 \\ d_{11} \end{pmatrix} = 0.000575402$$

Bending moment, $M_s = -EI_s\, d^2 w(t)/dt^2 = 0$

Shear force, $V_r = -dM_s/dt = 0$

Forces & Moments at element ends

	x	y	z	Axial force	V_s	V_r	M_r	M_s	M_t
1	0	0	0	−20.	0.889955	0.180999	0	0	0
	0	0	144.	−20.	0.889955	0.180999	128.154	−26.0639	0
2	0	0	144.	−0.889955	−10.	0	128.154	0	0
	60.	0	144.	−0.889955	0	0	−171.846	0	0
3	0	0	144.	−0.180999	10.	0	−26.0639	0	0
	0	60.	144.	−0.180999	0	0	273.936	0	0