A Simple Proof of the Hölder and the Minkowski Inequality

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The proofs as well as the extensions, inverses and applications of the well-known Hölder and Minkowski inequalities can be found in many books about real functions, analysis, functional analysis or L_p-spaces (cf. [**Mi**]). The aim of this note is to give another proof of these classical inequalities. The following lemma will be a main step in our simple proof of these inequalities. This lemma was motivated by considerations in [**KPS**], [**M**] and [**MP**].

Lemma. *For* $1 \le p < \infty$ *and any* $a, b > 0$, *we have*

$$\text{(i)} \qquad \inf_{t>0}\left[\frac{1}{p}t^{1/p-1}a + \left(1-\frac{1}{p}\right)t^{1/p}b\right] = a^{1/p}b^{1-1/p}.$$

$$\text{(ii)} \qquad \inf_{0<t<1}\left[t^{1-p}a^p + (1-t)^{1-p}b^p\right] = (a+b)^p.$$

First proof. In these proofs we will use calculus.

(i) Let, for $t > 0$, the function f be defined by

$$f(t) = \frac{1}{p}t^{1/p-1}a + \left(1-\frac{1}{p}\right)t^{1/p}b.$$

Then the derivative f' satisfies

$$f'(t) = \frac{1}{p}\left(\frac{1}{p}-1\right)t^{1/p-2}a + \left(1-\frac{1}{p}\right)\frac{1}{p}t^{1/p-1}b = \frac{1}{p}\left(\frac{1}{p}-1\right)t^{1/p-2}(a-tb),$$

and so f' is negative for $t < t_0 = a/b$, zero for $t = t_0$ and positive for $t > t_0$. Hence, f has its minimum at the point $t_0 = a/b$ and this minimum is equal to

$$f(t_0) = f\left(\frac{a}{b}\right) = \frac{1}{p}\left(\frac{a}{b}\right)^{1/p-1}a + \left(1-\frac{1}{p}\right)\left(\frac{a}{b}\right)^{1/p}b = a^{1/p}b^{1-1/p}.$$

(ii) Let, for $0 < t < 1$, the function g be defined by

$$g(t) = t^{1-p}a^p + (1-t)^{1-p}b^p.$$

Then the derivative g' satisfies the equation

$$g'(t) = (1-p)t^{-p}a^p - (1-p)(1-t)^{-p}b^p = 0$$

only when $t = t_1 = a/(a+b)$. Since

$$g''(t) = (1-p)(-p)t_1^{-p-1}a^p - (1-p)(-p)(1-t_1)^{-p-1}b^p > 0,$$

it follows that g has its local minimum at $t_1 = a/(a+b)$, which is equal to

$$g(t_1) = g\left(\frac{a}{a+b}\right) = \left(\frac{a}{a+b}\right)^{1-p} a^p + \left(1 - \frac{a}{a+b}\right)^{1-p} b^p$$
$$= \left(\frac{a}{a+b}\right)^{1-p} a^p + \left(\frac{b}{a+b}\right)^{1-p} b^p = (a+b)^p.$$

This local minimum of the function g is equal to its global minimum because g is continuous on $(0,1)$ and $\lim_{t\to 0^+} g(t) = \lim_{t \to 1^-} g(t) = +\infty$.

Second proof. In these proofs we will use convexity of some functions.

(i) The function $\varphi(u) = \exp(u)$ is convex on R. Thus

$$\begin{aligned}
a^{1/p}b^{1-1/p} &= \left[t^{1/p-1}a\right]^{1/p}\left[t^{1/p}b\right]^{1-1/p} \\
&= \exp\left[\frac{1}{p}\ln(t^{1/p-1}a) + \left(1 - \frac{1}{p}\right)\ln(t^{1/p}b)\right] \\
&\leq \frac{1}{p}\exp\left[\ln(t^{1/p-1}a)\right] + \left(1 - \frac{1}{p}\right)\exp\left[\ln(t^{1/p}b)\right] \\
&= \frac{1}{p}t^{1/p-1}a + \left(1 - \frac{1}{p}\right)t^{1/p}b
\end{aligned}$$

for every $t > 0$. For $t = a/b$ we have equality.

(ii) The function $\psi(u) = u^p$ for $p > 1$ is convex on $[0, \infty)$. Therefore,

$$\begin{aligned}
(a+b)^p &= \left[t\frac{a}{t} + (1-t)\frac{b}{1-t}\right]^p \\
&\leq t\left(\frac{a}{t}\right)^p + (1-t)\left(\frac{b}{1-t}\right)^p = t^{1-p}a^p + (1-t)^{1-p}b^p
\end{aligned}$$

for every $0 < t < 1$. For $t = a/(a+b)$ we have equality.

Remark 1. If $0 < p < 1$ and we change in the equalities (i) and (ii) the infimum into supremum, then our Lemma is still true.

Remark 2. The second proof of (i) gives also a different proof of the arithmetic-geometric mean inequality

$$a^{1/p}b^{1-1/p} \leq \frac{1}{p}a + \left(1 - \frac{1}{p}\right)b$$

(put $t = 1$) as well as a different proof of the Young inequality

$$ab \leq \frac{1}{p}a^p + \left(1 - \frac{1}{p}\right)b^{1/(1-1/p)}.$$

The classical *Hölder inequality* states: *Let* $1 \leq p < \infty$ *and* $1/p + 1/q = 1$. *If* $x \in L_p(\mu)$ *and* $y \in L_q(\mu)$, *then* $xy \in L_1(\mu)$ *and*

(HI) $$\|xy\|_1 \leq \|x\|_p \|y\|_q.$$

Equivalently, if $x, y \in L_1(\mu)$, *then* $|x|^{1/p}|y|^{1-1/p} \in L_1(\mu)$ *and*

(HI$_1$) $$\left\| \, |x|^{1/p}|y|^{1-1/p} \right\|_1 \leq \|x\|_1^{1/p}\|y\|_1^{1-1/p}.$$

Proof: According to our Lemma the inequality

$$a^{1/p}b^{1-1/p} \leq \frac{1}{p}t^{1/p-1}a + \left(1 - \frac{1}{p}\right)t^{1/p}b$$

holds for all $t > 0$ and it follows that

$$\begin{aligned}
\| \, |x|^{1/p}|y|^{1-1/p} \|_1 &= \int_\Omega |x(s)|^{1/p} |y(s)|^{1-1/p} \, d\mu(s) \\
&\leq \int_\Omega \left[\frac{1}{p} t^{1/p-1}|x(s)| + \left(1 - \frac{1}{p}\right) t^{1/p}|y(s)| \right] d\mu(s) \\
&= \frac{1}{p}t^{1/p-1} \int_\Omega |x(s)| \, d\mu(s) + \left(1 - \frac{1}{p}\right) t^{1/p} \int_\Omega |y(s)| \, d\mu(s) \\
&= \frac{1}{p}t^{1/p-1}\|x\|_1 + \left(1 - \frac{1}{p}\right)t^{1/p}\|y\|_1.
\end{aligned}$$

Taking the infimum over all $t > 0$ and using our Lemma again we obtain

$$\| \, |x|^{1/p}|y|^{1-1/p} \|_1 \leq \|x\|_1^{1/p}\|y\|_1^{1-1/p},$$

which proves inequality (HI_1).

Remark 3. Our proof of (HI_1) still works for a general Banach function space $X(\mu)$ instead of the $L_1(\mu)$-space, i.e., if $x, y \in X(\mu)$, then $|x|^{1/p}|y|^{1-1/p} \in X(\mu)$ and

$$(\mathrm{HI}_X) \qquad \| \, |x|^{1/p}|y|^{1-1/p} \|_X \leq \|x\|_X^{1/p}\|y\|_X^{1-1/p}.$$

Equivalently (cf. **[MP]**), if $|x|^p \in X(\mu)$ and $|y|^q \in X(\mu)$, $1/p + 1/q = 1$, *then* $xy \in X(\mu)$ *and*

$$(\mathrm{HI}) \qquad \|xy\|_X \leq \| \, |x|^p \|_X^{1/p} \| \, |y|^q \|_X^{1/q}.$$

The classical *Minkowski inequality* states: *Let* $1 \leq p < \infty$. *If* $x, y \in L_p(\mu)$, *then* $x + y \in L_p(\mu)$ *and*

$$(\mathrm{MI}) \qquad \|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

Proof: By using the second part of our Lemma, i.e. the inequality

$$(a + b)^p \leq t^{1-p}a^p + (1 - t)^{1-p}b^p$$

we find that for all t, $0 < t < 1$,

$$\begin{aligned}
\|x + y\|_p^p &= \int_\Omega |x(s) + y(s)|^p \, d\mu(s) \leq \int_\Omega [|x(s)| + |y(s)|]^p \, d\mu(s) \\
&\leq \int_\Omega \left[t^{1-p}|x(s)|^p + (1 - t)^{1-p}|y(s)|^p \right] d\mu(s) \\
&= t^{1-p} \int_\Omega |x(s)|^p \, d\mu(s) + (1 - t)^{1-p} \int_\Omega |y(s)|^p \, d\mu(s) \\
&= t^{1-p}\|x\|_p^p + (1 - t)^{1-p}\|y\|_p^p.
\end{aligned}$$

Taking the infimum over $0 < t < 1$ and using our Lemma again we obtain

$$\|x + y\|_p^p \le \left(\|x\|_p + \|y\|_p\right)^p,$$

which is inequality (MI).

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[M] Maligranda, L., *Calderón-Lozanovskii spaces and interpolation of operators*, Semesterbericht Funktionalanalysis, Tübingen 8 (1985), 83–92.

[MP] Maligranda, L., Persson, L. E., *Generalized duality of some Banach function spaces*, Indagationes Math. 51 (1989), 323–338.

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Never Too Late

There is a slip in Williamson's excellent article [2]. Although nearly 50 years have gone by it is never too late to restore the article to perfection.

The 8 by 8 determinant displayed on page 433 has the value 44, instead of 56 as stated.

From the material in the article that immediately precedes one can deduce that 56 is attainably by appropriately bordering the incidence matrix of the seven point projective plane:

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{vmatrix}$$

Some followup thoughts arise at once but this is not the place to explore them.

Years later Ehlich and Zeller [1] showed that 56 is the largest possible value for the determinant of an 8 by 8 matrix consisting entirely of 0's and 1's.

1. H. Ehlich and K. Zeller, Binäre Matrizen, Zeit Angew. Math. Mech. 42(1962), pages 20–21 of the Sonderheft.

2. J. Williamson, Determinants whose elements are 0 and 1, Amer. Math. Monthly 53(1946), 427–434.

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