Review on Young's Inequality

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CERTIFICATE

This is to certify that the thesis entitled **“REVIEW ON YOUNG’S INEQUALITY”** submitted by **Rajesh Moharana** (**Roll No: 412MA2072.**) in partial fulfilment of the requirements for the degree of Master of Science in Mathematics at the National Institute of Technology Rourkela is an authentic work carried out by him during **2nd year Project** under my supervision and guidance.

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Abstract

Young's inequality is a nice inequality which we are using in various concept of Mathematics. Some of its applications are envisaged for the development of proofs of other theorems and results. The main object of this project is to review and discuss such type of concepts and show its different kinds of proofs and applications. Here we develop the similar kinds of the inequality in different types of spaces ,i.e., finite dimensional as well as infinite dimensional spaces. In the beginning we start with the statement of Young's inequality which is already discussed by the Mathematician for the euclidian-space and Lebesgue space. We have extended this ideas to the abstract Banach spaces and studied its application by changing various condition and assumptions. In this sequel we have proved reverse Young's inequality and Fenchel- Young's inequality. Also we have investigated the affect of product and convolution of two functions on the Young's inequalities.

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Chapter 1

Introduction

In 1912, English Mathematician William Henry Young published the highly intuitive inequality, which is later named as Young's inequality. The most famous classical inequalities "Cauchy's inequality, Holder's inequality and Minkowski's inequality" can be deduced easily and quickly from the special case of Young's inequality. The detailed study on this concept is made by Elmer Tolsted in [7]. Tolsted derived Cauchy, Holder and Minkowski inequalities in a straight forward way from the Young's inequality by graphical method. A. Witkowski gave two proofs for Young's inequality as well as another one concerning its reverse [8]. While some others obtain the Young's inequality as a special case of quite complicated theorem. G. H. Hardy, J. E. Little wood and G. Polya included Young's inequality in their classic book "Inequalities"-[4], but there was no analytic proof until Diaz and Metacafe supplied one in 1970. An overview of available proofs and a complete proof of Young's inequality can be found in [2]. The aim of our work is to discuss the different kinds of Young's Inequality in differnt spaces with different conditions and also extend the entire discussion on the reverse inequalities formed by Young's inequality and their different results on L^p-space. We first briefly introduce the idea about the reverse inequality and develop the theory of some classical inequalities based on Young's inequality. This work involves generalization of some reverse inequalities given in [6] and [9].

In Chapter 2 and chapter 3, the mathematical theory that are needed to devolope the concept of Young's inequality and some special cases of this inequality is presented. Specially, some of its applications in different spaces with detail proof are given. Finally, the detail work of this paper and theorems are presented in Chapter 4 and some parts in the last of Chapter 3.

1.1 Preliminaries

The following preliminary concepts are used during this paper as follows,

Convolution:

Generally convolution operation is considered as the area of overlap between the function $f(x)$ and the function $g(x)$. A convolution is defined as the integral over all spaces of one function at x times another function at $u-x$. The integration is taken over the variable x. So, the convolution is a function of a new variable u . Mathematically,

$$C(u) = (f * g)(x) = \int_{-\infty}^{\infty} f(x) \cdot g(u-x)dx \tag{1.1.1}$$

Norm:

Generally norm is a function that generalizes the length of a vector in the plane or in spaces. Symbolically it is denoted as $\|\cdot\|$. A norm on a linear space X is a function $\|\cdot\| : X \to R$ with the following properties:

i. $\|x\| \geq 0, \forall x \in X$ [non-negative]

ii. $\|x\| = 0 \Rightarrow x = 0$ [strictly positive]

iii. $\|\lambda x\| = |\lambda|\|x\|\ , \forall\ x \in X\ and\ \lambda \in \mathbf{R}\,(or\,\mathbf{C})$ [homogeneous]

iv. $\|x+y\| \leq \|x\| + \|y\|, \forall x, y \in X$ [triangle inequality]

Lebesgue Space (L^p - space) :

If $[X, S, \mu]$ is a measurable space and $p > 0$ then L^p-space, can be written as $L^p(X, \mu)$ or $L^p(\mu)$ to be the class of measurable functions such that

$$\left\{f : \int |f|^p d\mu < \infty\right\} \quad \text{where} \quad f : X \to \mathbb{R}$$

Holder's inequality:

Holder's inequality, named after Otto Holder. Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ and let $f \in L^p(\mu), g \in L^q(\mu)$. Then it states that $fg \in L^1(\mu)$ and

$$\|fg\| \leq \|f\|_p \cdot \|g\|_q \tag{1.1.2}$$

Here norm denotes its usual meaning in Lebesgue-space and the numbers p and q are said to be Holder's conjugate of each other. Holder's inequality can be written for discrete type by using counting measure. In a sequence space Holder's inequality for counting measure is defined as,

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{\infty} |y_i|^q\right)^{\frac{1}{q}} \quad \forall\ (x_i)_{i\in\mathbf{N}},\ (y_i)_{i\in\mathbf{N}} \in \mathbf{R}^{\mathbf{N}} \text{or } \mathbf{C}^{\mathbf{N}} \tag{1.1.3}$$

Minkowski's inequality:

Let $p \geq 1$ and $f, g \in L^p(\mu)$; then $f + g \in L^p(\mu)$ and we have the inequality

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p \tag{1.1.4}$$

which is known as Minkowski's inequality. It is a triangle inequality in $L^p(\mu)$. It establishes that the $L^p(\mu)$-spaces are normed vector spaces. Like Holder's inequality, the Minkowski's inequality can be specialized to sequences and vectors by using the counting measure:

$$\left(\sum_{i=1}^{\infty} |x_i + y_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty} |y_i|^p\right)^{\frac{1}{p}} \quad \forall\ (x_i)_{i\in\mathbf{N}},\ (y_i)_{i\in\mathbf{N}} \in \mathbf{R}^{\mathbf{N}} \text{or } \mathbf{C}^{\mathbf{N}} \tag{1.1.5}$$

Legendre duality:

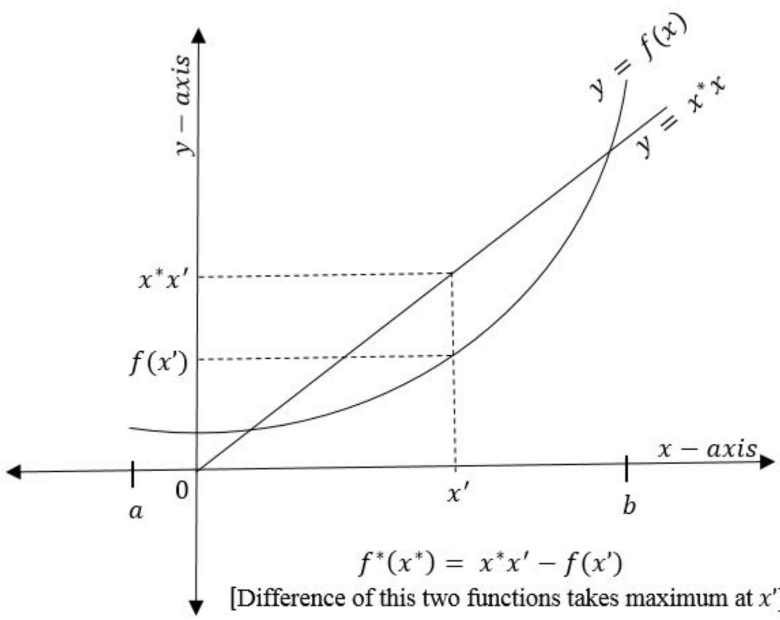
The legendre transformation is an involutive-transformation on the real valued convex functions of one real variable. Let $I \subset \mathbf{R}$ be a nondegenerate interval and $f : I \to \mathbf{R}$ is a convex function; then its Legendre transformation is the function, $f^* : I^* \to \mathbf{R}$ defined by,

$$f^*(x^*) = \sup_{x \in I} \left(x^* x - f(x) \right), \quad x^* \in I^*.$$

f^* is called the convex conjugate function of f. Domain of the function is,

$$I^* = \left\{ x^* : \sup_{x \in I} (x^* x - f(x)) < \infty \right\}$$

where I^* is an interval, f^* is convex on it.



$f^*(x^*) = x^* x' - f(x')$
[Difference of this two functions takes maximum at x']

The Legendre-transformation is an application of the duality relationship between points and lines. Also we can see that it fulfills the condition $f^{**} = f$ on I^* and it may differ atmost on their boundaries.

Example:-

Let $f(x) = cx^2$ defined on the whole $\mathbf{R}$, where $c > 0$ is a fixed constant. For x^* fixed, the function can be written as

$$\begin{aligned} f^*(x^*) &= x^*x - f(x) \\ &= x^*x - cx^2 \end{aligned}$$

$f^*(x^*)$ is a function of x has the first derivative $x^* - 2cx$ and the second drivative $-2c$, which is less than zero. So, it attains its maximum.

$$\begin{aligned} x^* - 2cx &= 0 \\ \Rightarrow x &= \frac{x^*}{2c}, \end{aligned}$$

which is always a maximum at $x = \frac{x^*}{2c}$.

Thus, $I^* = \mathbf{R}$ and

$$\begin{aligned} f^*(x^*) &= x^*x - cx^2 \\ &= \frac{(x^*)^2}{2c} - \frac{c(x^*)^2}{4c^2} \quad [\text{at } x = \tfrac{x^*}{2c}] \\ &= \frac{(x^*)^2}{4c} \\ &= c^*(x^*)^2 \quad [\text{where } c^* = \tfrac{1}{4c}]. \end{aligned}$$

According to the above calculation, clearly

$$\begin{aligned} f^{**}(x) &= \frac{1}{4c^*}x^2 = cx^2 \\ \Rightarrow f^{**}(x) &= f(x) \\ \therefore f^{**} &= f. \end{aligned}$$

Fenchel conjugate:

Let f be a function defined on a bannach space X, i.e. $f : X \to (-\infty, \infty]$. Then Fenchel conjugate of f is the function $f^* : X^* \to [-\infty, \infty]$ defined as,

$$f^*(x^*) = \sup_{x \in X} \{\langle x^*, x\rangle - f(x)\}\,; \quad \forall x \in X, x^* \in X^*$$

where, f^* is a convex function, defined on the dual space X^* of X. Also, we can derive fenchel conjugation of f^*, called the biconjugate of f and denoted by f^{**}. This is a function on X^{**}.

Fubini's Theorem:

This theorem induced by Guido Fubini (in 1907). It is also known as Tonelli's theorem.

Definition:- If $f(x, y)$ is continuous on $R = [a, b] \times [c, d]$ i.e. on the rectangular region $R : a \leq x \leq b,\ c \leq y \leq d$ then

$$\int_R \int f(x, y)\, d(x, y) = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

In general,

$$\int_X \left(\int_Y f(x, y)\, dy \right) dx = \int_Y \left(\int_X f(x, y)\, dx \right) dy = \int_{X \times Y} f(x, y)\, d(x, y)$$

Here, it doesn't matter which variable we integrate with respect to first, we will get the same answer if we consider any order of integration.

Example:- Evaluate $\int_R \int 6xy^2\ dA,\ \ R = [2, 4] \times [1, 2]$.

1st case:-(Integrate with respect to y first then x)

$$\begin{aligned}
\int_R\int 6xy^2\,dA &= \int_2^4\int_1^2 6xy^2\,dy\,dx \\
&= \int_2^4 [2xy^3]_1^2\,dx \\
&= \int_2^4 14x\,dx \\
&= [7x^2]_2^4 \\
&= 84
\end{aligned}$$

2nd case:-(Integrate with respect to x first then y)

$$\begin{aligned}
\int_R\int 6xy^2\,dA &= \int_1^2\int_2^4 6xy^2\,dx\,dy \\
&= \int_1^2 [3x^2y^2]_2^4\,dy \\
&= \int_1^2 36y^2\,dy \\
&= [12y^3]_1^2 \\
&= 84
\end{aligned}$$

So, we can do the integration in any order.

Pseudo-inverse function

Let $f : [a,b] \to [c,d]$ be a monotone function defined between the two closed subintervals of the real line. The pseudo-inverse function to f is the function $f^{-1} : [c,d] \to [a,b]$ defined as

$$f^{-1}(y) = \begin{cases} \sup\{x \in [a,b] \mid f(x) < y\} & \text{for } f \text{ non-decreasing} \\ \sup\{x \in [a,b] \mid f(x) > y\} & \text{for } f \text{ non-increasing} \end{cases} \tag{1.1.6}$$

Chapter 2

Over view on Young's inequality

In this chapter, we discuss the necessary idea about Young's inequality and its reverse generaalization.

In mathematics, Young's inequality is of two types: One about the product of two numbers and other one about the convolution of two functions.

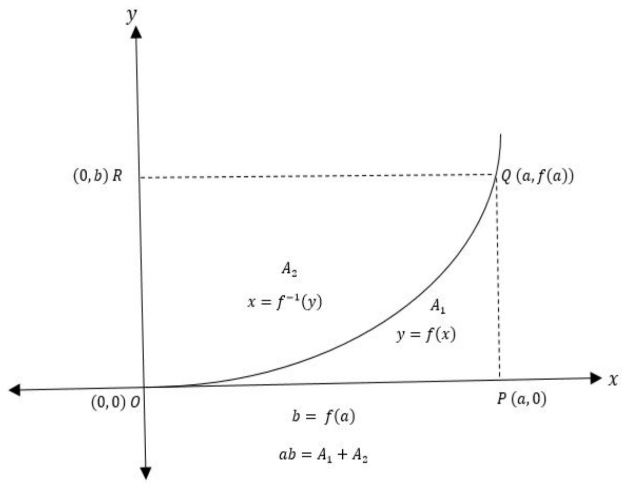
2.1 Young's Inequality for product of two numbers:

Young's inequality states that every strictly increasing continuous function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $\lim_{x \to \infty} f(x) = \infty$ verifies an inequality of the following form

$$ab \leq \int_0^a f(x)\, dx + \int_0^b f^{-1}(x)\, dx \tag{2.1.1}$$

Wherever a and b are non negative real numbers.The equality occurs if and only if $f(a) = b$.

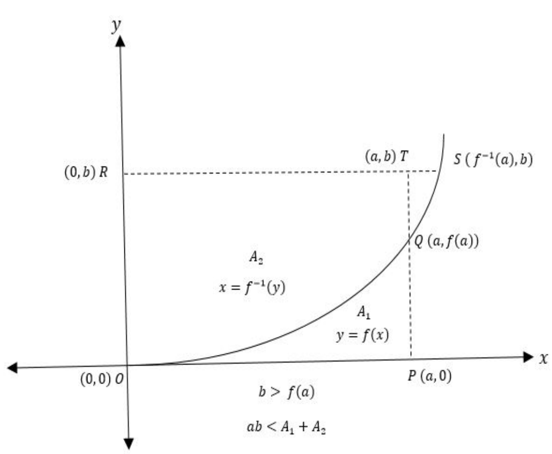
Proof. Young's inequality can be prove in many ways, but we can see its proof easily by the graphical method (in $\mathbf{R^2}$).

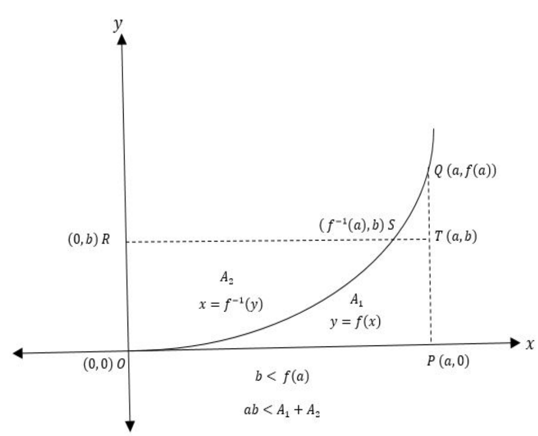


From the above graph we can directly conclude the equality of Young's inequality. Let $y = f(x)$ be a strictly increasing continuous function for $x \geq 0$ with $f(0) = 0$ and $f(a) = b$, where a and b are any positive real numbers. Also assuming its inverse function $x = f^{-1}(y)$ Inverse function f^{-1} also strictly increasing continuous function with $f^{-1}(0) = 0$ and $f^{-1}(b) = a$. Consider area of $A_1 = \int_0^a f(x)\,dx$ and area of $A_2 = \int_0^b f^{-1}(x)\,dx$.
Area of the rectangular $OPQR = A_1 \cdot A_2$

$$ab = \int_0^a f(x)\,dx + \int_0^b f^{-1}(x)\,dx$$

Now, suppose that $b \neq f(a)$.





In this two graph area of the rectangle $OPTR$ formed by a and b is smaller than the area formed by the functions f and f^{-1}. Some extra additional area QST are present, clearly shown in the figure. Hence,

$$ab < \int_0^a f(x)dx + \int_0^b f^{-1}(x)dx$$

Combining these two inequalities we will get the desired result, which is commonly known as Young's inequality. □

Corollary 2.1.1. *A useful consequence of this definition is Young's inequality of the form*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{2.1.2}$$

where p and q both are two positive real numbers in $[1, \infty)$ provided by $\frac{1}{p} + \frac{1}{q} = 1$. With equality if and only if $a^p = b^q$, a fact derived from W. H. Young by taking $f(x) = x^\alpha$. Also we can say it is an application of young's inequality because by using this inequality we can derive Cauchy' inequality and holder's inequality easily.

For the proof of this above inequality , other applications and extensions of Young's inequality refer [5], [7], [8].

2.2 Young's Inequality for convolution of two functions:

Let f be in $L^p(\mathbb{R}^n)$ and g be in $L^q(\mathbb{R}^n)$ and $1 \leq p, q, r \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$ then

$$\|f * g\|_r \leq \|f\|_p \cdot \|g\|_q \tag{2.2.1}$$

Here the star denotes convolution. $f * g$ denotes the convolution product of two functions. L^p is lebesgue space and $\|f\|_p = \left(\int |f(x)|^p\, dx\right)^{\frac{1}{p}}$ denotes the usual $L^p - norm$.

Proof. Let $1 \leq p', q' \leq \infty$ be two real numbers such that $\frac{1}{p'} + \frac{1}{q'} + \frac{1}{r} = 1$

$$\text{where} \quad \frac{1}{p'} = \frac{1}{q} - \frac{1}{r} \qquad \text{and} \qquad \frac{1}{q'} = \frac{1}{p} - \frac{1}{r}.$$
$$\Rightarrow q = p'(1 - \frac{q}{r}) \qquad \text{and} \qquad p = q'(1 - \frac{p}{r}).$$

$$\begin{aligned} \text{Now, } |f * g(x)| &= \int |f(x-y)| \cdot |g(y)| dy \\ &= \int |f(x-y)|^{\frac{p}{r}} \cdot |f(x-y)|^{1-\frac{p}{r}} \cdot |g(y)|^{\frac{q}{r}} \cdot |g(y)|^{1-\frac{q}{r}} dy \\ &= \int (|f(x-y)|^p |g(y)|^q)^{\frac{1}{r}} \cdot |f(x-y)|^{1-\frac{p}{r}} \cdot |g(y)|^{1-\frac{q}{r}} dy \\ &\leq \left(\int |f(x-y)|^p |g(y)|^q \right)^{\frac{1}{r}} \cdot \left(\int |f(x-y)|^{(1-\frac{p}{r})q'} dy \right)^{\frac{1}{q'}} \cdot \left(\int |g(y)|^{(1-\frac{q}{r})p'} dy \right)^{\frac{1}{p'}} \end{aligned}$$

[by Holder's inequality]

Thus we have, $\quad |f * g(x)| = \left(\int |f(x-y)|^p \cdot |g(y)|^q dy \right)^{\frac{1}{r}} \cdot \|f\|_p^{1-\frac{p}{r}} \cdot \|g\|_q^{1-\frac{q}{r}}$

Take the rth-power of the above equation in both the side we get,

$$\Rightarrow \quad |f * g(x)|^r \leq \left(\int |f(x-y)|^p \cdot |g(y)|^q dy \right) \cdot \|f\|_p^{r-p} \cdot \|g\|_q^{r-q}$$

Integrate both the side with respect to x,

$$\begin{aligned} \int |f * g(x)|^r dx &\leq \|f\|_p^{r-p} \cdot \|g\|_q^{r-q} \cdot \int \left(\int |f(x-y)|^p \cdot |g(y)|^q dy \right) dx \\ &= \|f\|_p^{r-p} \cdot \|g\|_q^{r-q} \cdot \int \left(|g(y)|^q \int |f(x-y)|^p dx \right) dy \quad \text{[by Fubini's theorem]} \\ &= \|f\|_p^{r-p} \cdot \|g\|_q^{r-q} \cdot \int |g(y)|^q \cdot \left(\|f\|_p^p \right) dy \\ &= \|f\|_p^{r-p} \cdot \|g\|_q^{r-q} \cdot \|f\|_p^p \cdot \int |g(y)|^q dy \\ &= \|f\|_p^{r-p} \cdot \|f\|_p^p \cdot \|g\|_q^{r-q} \cdot \|g\|_q^q \\ &= \|f\|_p^r \cdot \|g\|_q^r \\ \Rightarrow \left(\int |f * g(x)|^r dx \right)^{\frac{1}{r}} &\leq \|f\|_p \cdot \|g\|_q \\ \Rightarrow \|f * g(x)\|_r &\leq \|f\|_p \cdot \|g\|_q \end{aligned}$$

Hence proved. $\square$

2.3 Reverse Young's inequality:

if f(x)is a strictly increasing continuous function defined as $f : [0, \infty) \to [0, \infty)$ with f(0)=0 and $\lim_{x \to \infty} f(x) = \infty$ then we can rewrite Young's inequality as:

$$\min\left\{1, \frac{b}{f(a)}\right\} \int_0^a f(t)dt + \min\left\{1, \frac{a}{f(b)}\right\} \int_0^b f^{-1}(t)dt \leq ab \tag{2.3.1}$$

The inequality holds equality if and only if b=f(a).

Proof. Proof of this inequality is shown by A. Witkowski in his paper [8]. $\square$

Chapter 3

Extension of Young's Inequality by using some different concepts:

In this chapter, we will describe the different kinds of young's inequality and some of its relation between other mathematical concept and also its extension in Banach space.

3.1 Similarity between Young's inequality and legendre duality

Youngs inequality is an illustration of the Legendre duality. Take, $F(a) = \int_0^a f(x)dx$ and $G(b) = \int_0^b f^{-1}(x)dx$. The functions $F(a)$ and $G(b)$ are both continuous and convex on $[0, \infty)$. Youngs inequality can be restated as

$$ab \leq F(a) + G(b), \quad \forall b \in [0, \infty) \tag{3.1.1}$$

Equality holds if and only if $f(a) = b$. For the equality case, the inequality (3.1.1) leads to the following connections between the functions F and G.

$$F(a) = \sup\{ab - G(b) : b \geq 0\}, \quad G(b) = \sup\{ab - F(a) : a \geq 0\} \tag{3.1.2}$$

Which is nothing but a similar concept of the Legendre duality.

3.2 Modified Young's inequality by using the concept of pseudo-inverse function

Youngs Inequality can be modified by using the concept of pseudo-inverse function and lebesgue locally integrable function. let $f : [0, \infty) \to [0, \infty)$ be a non decreasing function such that $f(0) = 0$ and $\lim_{x \to \infty} f(x) = \infty$. since f is not necessarily injective we will attach to a pesuido-inverse function f by the following formula,

$$f_{sup}^{-1} : [0, \infty) \to [0, \infty) \quad \text{by the rule} \quad f_{sup}^{-1}(y) = \inf\{x \geq 0 : f(x) \geq y\}.$$

Clearly, f_{sup}^{-1} is non decreasing and $f_{sup}^{-1}(f(x)) \geq x, \ \forall x$.

$$f_{sup}^{-1}(y) = \sup\{x : y \in [f(x-), f(x+)\}$$

Here $f(x-)$ and $f(x+)$ represent the lateral limits at x. When f is continuous,

$$f_{sup}^{-1}(y) = \max\{x \geq 0 : y = f(x)\}.$$

If $0 \leq a \leq b$ the epigraph and the hypo-graph of $f|_{[a,b]}$ is,

$$epif|_{[a,b]} = \{(x, y) \in [a, b] \times [f(a), f(b)] : y \geq f(x)\}.$$

$$hypf|_{[a,b]} = \{(x, y) \in [a, b] \times [f(a), f(b)] : y \leq f(x)\}.$$

Graph of $f|_{[a,b]}$ is

$$graphf|_{[a,b]} = \{(x, y) \in [a, b] \times [f(a), f(b)] : y = f(x)\}$$

Let us consider a measure ρ on $[o, \infty) \times [0, \infty)$ with respect to the Lebesgue-measure $dx\, dy$ i.e.

$$\rho(A) = \int_A K(x, y)\, dx\, dy$$

where $K : [0,\infty) \times [0,\infty) \to [0,\infty)$ is a lebesgue locally integrable function and A is any compact subset of $[0,\infty) \times [0,\infty)$.

$$\begin{aligned}\rho\left(hypf|_{[a,b]}\right) &= \int_a^b \left(\int_{f(a)}^{f(b)} K(x,y)\,dy\right) dx \\ \rho\left(epif|_{[a,b]}\right) &= \int_{f(a)}^{f(b)} \left(\int_a^{f_{sup}^{-1}(y)} K(x,y)\,dx\right) dy\end{aligned}$$

Clearly,

$$\begin{aligned}\rho\left(hypf|_{[a,b]}\right) + \rho\left(epif|_{[a,b]}\right) &= \rho\left([a,b] \times [f(a), f(b)]\right) \\ &= \int_a^b \int_{f(a)}^{f(b)} K(x,y)\,dy\,dx\end{aligned}$$

By using this concept one of the important lemma which we are using for derive Young's inequality can be written as,

Lemma 3.2.1. *Let $f : [0,\infty) \to [0,\infty)$ be a non decreasing function such that $f(0) = 0$ and $\lim_{x\to\infty} f(x) = \infty$. Then for every Lebesgue locally integrabble function $K : [0,\infty) \times [0,\infty) \to [0,\infty)$ and every pair of non negative numbers $a < b$ we have,*

$$\int_a^b \left(\int_{f(a)}^{f(x)} K(x,y)dy\right) dx + \int_{f(a)}^{f(b)} \left(\int_a^{f_{sup}^{-1}(y)} K(x,y)dx\right) dy = \int_a^b \left(\int_{f(a)}^{f(b)} K(x,y)dy\right) dx \tag{3.2.1}$$

Now from the above lemma we can conclude the statement of Young's inequality for the non-decreasing function is,

Consider all the assumptions of above lemma, where $a < b$. Let us consider a number $c \geq f(a)$, then the inequality obtained

$$\int_a^b \left(\int_{f(a)}^{c} K(x,y)dy\right) dx \leq \int_a^b \left(\int_{f(a)}^{f(x)} K(x,y)dy\right) dx + \int_{f(a)}^{c} \left(\int_a^{f_{sup}^{-1}(y)} K(x,y)dx\right) dy \tag{3.2.2}$$

If K is strictly positive almost every where, then the equality occurs if and only if $c \in [f(b-), f(b+)]$.

Detail prove of these inequalities are given in [2].

When we will discuss this to derive Young's inequality for continuous increasing function, it becomes

$$\int_a^b \left(\int_{f(a)}^c K(x,y)dy \right) dx \leq \int_a^b \left(\int_{f(a)}^{f(x)} K(x,y)dy \right) dx + \int_{f(a)}^c \left(\int_a^{f(y)} K(x,y)dx \right) dy \quad (3.2.3)$$

for every real number $c \geq f(a)$.

Where $f : [o, \infty) \to [0, \infty)$ is a continuous and increasing function and K is strictly positive almost everywhere. The equality occurs if and only if $c = f(b)$.

3.3 Fenchel Young's inequality

Again Young's inequality can be derive in the form of Fenchel Young's inequality by using Fenchel's conjugate. This is an extension of Young's inequality in bannach space

Theorem 3.3.1. *Let X be a banach space and $f : X \to \mathbb{R}$ be a convex function. X^* be a dual space of X. Suppose that $x^* \epsilon X^*$ and $x \epsilon X$. Then it satisfy the inequality*

$$f(x) + f^*(x^*) \geq \langle x^*, x \rangle \quad (3.3.1)$$

Equality holds if and only if $x^* \in \partial f(x)$. where $\partial f(x)$ is the subdifferential.

3.4 A new type of inequality deducing by changing the (parameter or) necessary condition of the convolution of Young's inequality

Let $\alpha, \beta \in [0, 1]$ be any two real numbers and $r, p_1, p_2 \in [1, \infty)$ such that $\frac{1}{p_1} + \frac{1}{P_2} = 1 - \frac{1}{r}$, i.e., $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{r} = 1$. Since convolution are commutative, hence $f * g(x) = \int |f(x-y) \cdot g(y)| dy$.

Thus,

$$\begin{aligned}
|f*g(x)| &= \int |f(x-y)\cdot g(y)|dy \\
&= \int |f(x-y)|\cdot|g(y)|dy \\
&= \int |f(x-y)|^{1-\alpha}\cdot|f(x-y)|^{\alpha}\cdot|g(y)|^{1-\beta}\cdot|g(y)|^{\beta}dy \\
&= \int \left(|f(x-y)|^{1-\alpha}|g(y)|^{1-\beta}\right)\cdot|f(x-y)|^{\alpha}\cdot|g(y)|^{\beta}dy \\
&\leq \left(\int |f(x-y)|^{(1-\alpha)r}|g(y)|^{(1-\beta)r}dy\right)^{\frac{1}{r}}\cdot\left(\int |f(x-y)|^{(\alpha p_1)}dy\right)^{\frac{1}{p_1}}\left(\int |g(y)|^{(\beta p_2)}dy\right)^{\frac{1}{p_2}} \\
&= \left(\int |f(x-y)|^{(1-\alpha)r}\cdot|g(y)|^{(1-\beta)r}dy\right)^{\frac{1}{r}}\cdot\|f\|^{\alpha}_{(\alpha p_1)}\cdot\|g\|^{\beta}_{(\beta p_2)} \\
\Rightarrow |f*g(x)|^r &\leq \left(\int |f(x-y)|^{(1-\alpha)r}\cdot|g(y)|^{(1-\beta)r}dy\right)\cdot\|f\|^{\alpha r}_{(\alpha p_1)}\cdot\|g\|^{\beta r}_{(\beta p_2)}.
\end{aligned}$$

Now integrate both the side with respect to x, we have,

$$\int |f*g(x)|^r dx \leq \|f\|^{(\alpha r)}_{(\alpha p_1)}\cdot\|g\|^{(\beta r)}_{(\beta p_2)}\cdot\int\left(\int |f(x-y)|^{(1-\alpha)r}\cdot|g(y)|^{(1-\beta)r}dy\right)dx$$

Now apply Fubini's theorem on the above,

$$\begin{aligned}
\|f*g\|_r^r &\leq \|f\|^{(\alpha r)}_{(\alpha p_1)}\cdot\|g\|^{(\beta r)}_{(\beta p_2)}\cdot\int |g(y)|^{(1-\beta)r}\left(\int |f(x-y)|^{(1-\alpha)r}dx\right)dy \\
&= \|f\|^{(\alpha r)}_{(\alpha p_1)}\cdot\|g\|^{(\beta r)}_{(\beta p_2)}\cdot\|f\|^{(1-\alpha)r}_{(1-\alpha)r}\cdot\int |g(y)|^{(1-\beta)r}dy \\
&= \|f\|^{(\alpha r)}_{(\alpha p_1)}\cdot\|g\|^{(\beta r)}_{(\beta p_2)}\cdot\|f\|^{(1-\alpha)r}_{(1-\alpha)r}\cdot\|g\|^{(1-\beta)r}_{(1-\beta)r} \\
\Rightarrow \|f*g\|_r &\leq \|f\|^{(\alpha)}_{(\alpha p_1)}\cdot\|g\|^{(\beta)}_{(\beta p_2)}\cdot\|f\|^{(1-\alpha)}_{(1-\alpha)r}\cdot\|g\|^{(1-\beta)}_{(1-\beta)r}.
\end{aligned}$$

Hence, desired result.

Chapter 4

Derivation of Young's inequality for convolutin by using the Riesz-Thorin convexity theorem

Theorem 4.0.1. *Let (X, μ) be a measure space.*

Let $1 \leq p \leq \infty$ and $C > 0$. Suppose K is a measurable function on $X \times X$ such that

$$\sup_{x \in X} \int_X |K(x,y)|\, d\mu(y) \leq C$$

$$\sup_{y \in X} \int_X |K(x,y)|\, d\mu(x) \leq C$$

Define a operator $T : L^p(X) \to L^p(X)$. If $f \in L^p(X)$, then the function Tf is defined by

$$Tf(x) = \int_X K(x,y) \cdot f(y)\, d\mu(y)$$

Tf is well defined almost every where in $L^p(X)$ and $\|Tf\|_p \leq C\|f\|_p$.

Proof. Suppose $1 < p < \infty$ and let q be the conjgate exponent of p. Hence, $\frac{1}{p} + \frac{1}{q} = 1$.

$$\begin{aligned}
|Tf(x)| &= \int_X |K(x,y)| \cdot |f(y)| \, d\mu(y) \\
&= \int_X |K(x,y)|^{\frac{1}{q}} \cdot |K(x,y|^{1-\frac{1}{q}} \cdot |f(y)| \, d\mu(y) \\
\Rightarrow |Tf(x)| &\leq \left(\int_X |K(x,y)| \, d\mu \right)^{\frac{1}{q}} \cdot \left(\int_X |K(x,y)|^{(1-\frac{1}{q})p} \cdot |f(y)|^p \, d\mu \right)^{\frac{1}{p}} \\
&\leq C^{\frac{1}{q}} \cdot \left(\int_X |K(x,y)| \cdot |f(y)|^p \, d\mu \right)^{\frac{1}{p}} \\
\Rightarrow |Tf(x)|^p &\leq C^{\frac{p}{q}} \cdot \left(\int_X |K(x,y)| \cdot |f(y)|^p \, d\mu \right)
\end{aligned}$$

Integrate both the side with respect to $d\mu(x)$ we get,

$$\begin{aligned}
\Rightarrow \int_X |Tf(x)|^p \, d\mu(x) &\leq C^{\frac{p}{q}} \cdot \int_X \left(\int_X |K(x,y)| \cdot |f(y)|^p \, d\mu(y) \right) d\mu(x) \\
\Rightarrow \|Tf\|_p^p &\leq C^{\frac{p}{q}} \cdot \int_X |f(y)|^p \left(\int_X |K(x,y)| \, d\mu(x) \right) d\mu(y) \\
&\leq C^{\frac{p}{q}} \cdot C \cdot \int_X |f(y)|^p \, d\mu(y) \\
&= C^{(1+\frac{p}{q})} \cdot \|f(y)\|_p^p \\
\Rightarrow \|Tf\|_p &\leq C^{(1+\frac{p}{q})\frac{1}{p}} \cdot \|f(y)\|_p \\
\Rightarrow \|Tf\|_p &\leq C^{\frac{1}{p}+\frac{1}{q}} \cdot \|f(y)\|_p \\
\Rightarrow \|Tf\|_p &\leq C \cdot \|f(y)\|_p
\end{aligned}$$

For the case $p = 1$ and $p = \infty$ this result is trivial. $\square$

Some corollary which are deducted from the above theorem by assuming the $X = R^n$ and $K(x,y) = f(x-y)$.

Corollary 4.0.1. *let $f \in L^1$ and $g \in L^p$ with $1 \leq p \leq \infty$ then $f * g \in L^p$ and*

$$\|f * g\|_p \leq \|f\|_1 \cdot \|g\|_p \tag{4.0.1}$$

Corollary 4.0.2. *let $f \in L^q$ and $g \in L^p$ with $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ then $f * g \in L^\infty$ and*

$$\|f * g\|_\infty \leq \|f\|_q \cdot \|g\|_p \tag{4.0.2}$$

We can rewrite the inequalities of the above two corollaries as follows:

Let $g \in L^p$ be fixed and the operator T is equal to the convolution operator $f * g$. i.e. $Tf = f * g$.

Hence,

$$T : L^1 \to L^p \quad \text{and} \quad \|Tf\|_p \leq \|g\|_p \cdot \|f\|_1$$

$$T : L^q \to L^\infty \quad \text{and} \quad \|Tf\|_\infty \leq \|g\|_p \cdot \|f\|_q$$

provided $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 4.0.2. *(Riesz-Thorin convexity theorem:)*

Let $\mathcal{A}$ be a suitable linear Hausdorff space containing all $L^p(R^n)$ and $1 \leq p \leq \infty$. Let T be a operator defined as $T : \mathcal{A} \to \mathcal{A}$. Suppose $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ where q_0, q_1 are the conjugate exponent of p_0, p_1 respectively. $T : L^{p_0} \to L^{q_0}$ and $T : L^{p_1} \to L^{q_1}$. Hence, for $i = 0, 1$, $T : L^{p_i} \to L^{q_i}$ and $\|T|_{L^{p_i}}\| = M_i$, ,i.e., $\|T|_{L^{p_0}}\| = M_0$ and $\|T|_{L^{p_1}}\| = M_1$. Moreover for $t \in [0, 1]$ define,

$$p_t = \frac{1}{\frac{1-t}{p_0} + \frac{t}{p_1}}, \quad q_t = \frac{1}{\frac{1-t}{q_0} + \frac{t}{q_1}}$$

$$\begin{aligned}
\frac{1}{p_t} + \frac{1}{q_t} &= \frac{1-t}{p_0} + \frac{t}{p_1} + \frac{1-t}{q_0} + \frac{t}{q_1} \\
&= (1-t)\left(\frac{1}{p_0} + \frac{1}{q_0}\right) + t\left(\frac{1}{p_1} + \frac{1}{q_1}\right) \\
&= (1-t) + t \\
&= 1
\end{aligned}$$

We have $T : L^{p_t} \to L^{q_t}$, where $\|T|_{L^{p_t}}\| = M_t$ with $M_t \leq M_0^{1-t} M_1^t$.

As per the above, $T : L^{p_0} \to L^{q_0}$ and $\|Tf\|_{q_0} \leq M_0 \|f\|_{p_0}$

Simillarly, $T : L^{p_1} \to L^{q_1}$ and $\|Tf\|_{q_1} \leq M_1 \|f\|_{p_1}$.

If we write it in terms of p_t and q_t then,

$$T : L^{p_t} \to L^{q_t}, \quad \|Tf\|_{q_t} \leq M \|f\|_{p_t} \quad \text{and} \quad M \leq M_0^{1-t} M_1^t$$

Our aim is to prove Young's inequality for convolution by using the conditions of Riesz-Thorin Convexity's theorem.

That is to prove, Suppose $1 \leq p, q, r \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$. If $f \in L^q$ and $g \in L^p$ then $f * g \in L^r$ and

$$\|f * g\|_r \leq \|f\|_q \|g\|_p$$

Proof. Already we have the following two inequalities,

$$T : L^1 \to L^p \quad \text{and} \quad \|Tf\|_p \leq \|g\|_p \|f\|_1$$

$$T : L^{q'} \to L^\infty \quad \text{and} \quad \|Tf\|_\infty \leq \|g\|_p \|f\|_{q'}$$

with $\frac{1}{p} + \frac{1}{q'} = 1$ and $\|g\|_p$ is fixed.

Take,

$$q = \frac{1}{\frac{1-t}{1} + \frac{t}{q'}} \quad \text{and} \quad r = \frac{1}{\frac{1-t}{p} + \frac{t}{\infty}} = \frac{p}{1-t}$$

$$\begin{aligned}
\text{Hence,} \quad \frac{1}{p} + \frac{1}{q} &= \frac{1}{p} + \left(\frac{1-t}{1} + \frac{t}{q'} \right) \\
&= \frac{1}{p} + \frac{1-t}{1} + t \left(\frac{p-1}{p} \right) \\
&= \frac{1 + (1-t)p + t(p-1)}{p} \\
&= \frac{p+1-t}{p} \\
&= 1 + \frac{1-t}{p} \\
&= 1 + \frac{1}{r} \\
\Rightarrow \frac{1}{p} + \frac{1}{q} &= \frac{1}{r} + 1
\end{aligned}$$

By Riesz-Thorin Convexity theorem, we have

$$T : L^q \to L^r \quad \text{and} \quad \|Tf\|_r \le M\|f\|_q \quad \text{with} \quad M \le M_0^{1-t} M_1^t.$$

But already we assumed $\|g\|_p$ is the fixed for the operator T. So, $M_0 = M_1 = \|g\|_p$.

$$\begin{aligned} \text{Thus,} \quad M &\le \|g\|_p^{1-t} \cdot \|g\|_p^t \\ &= \|g\|_p \end{aligned}$$

Hence, $\|Tf\|_r \le \|g\|_p \cdot \|f\|_q$. $\square$

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