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Logarithmic Convexity and Inequalities for the Gamma Function*

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We propose a method, based on logarithmic convexity, for producing sharp bounds for the ratio $\Gamma(x+\beta)/\Gamma(x)$. As an application, we present an inequality that sharpens and generalizes inequalities due to Gautschi, Chu, Boyd, Lazarević-Lupaş, and Kershaw. © 1996 Academic Press, Inc.

1. SOME CONVEX AND CONCAVE FUNCTIONS

It is well known that the second derivative of the function $x \mapsto \log\Gamma(x)$ can be expressed in terms of the series (see [2] or [10])

$$\frac{d^2}{dx^2}\log\Gamma(x) = \frac{1}{x^2} + \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \cdots \qquad (x \neq 0, -1, -2, \ldots), \quad (1)$$

so the gamma function is a log-convex one. The following theorem gives a supply of convex and concave functions related to $\log\Gamma$.

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THEOREM 1. *Let B_{2k} be Bernoulli numbers and let L and R be generic notations for the sums*

$$L(x) = L_{2N}(x) = -\sum_{k=1}^{2N} \frac{B_{2k}}{2k(2k-1)x^{2k-1}}$$
$$(N = 1, 2, \ldots),\ L_0(x) = 0,$$

$$R(x) = R_{2N+1}(x) = -\sum_{k=1}^{2N+1} \frac{B_{2k}}{2k(2k-1)x^{2k-1}} \qquad (N = 0, 1, 2, \ldots).$$

(i) *The functions*

$$F_1(x) = \log \Gamma(x)$$
$$F_2(x) = \log \Gamma(x) - x \log x$$
$$F_3(x) = \log \Gamma(x) - \left(x - \frac{1}{2}\right)\log x$$
$$F(x) = \log \Gamma(x) - \left(x - \frac{1}{2}\right)\log x + L(x)$$

are convex on $x > 0$.

(ii) *The functions*

$$G(x) = \log \Gamma(x) - \left(x - \frac{1}{2}\right)\log x + R(x)$$

are concave on $x > 0$.

Proof. We will apply the Euler−MacLaurin summation formula to the series in (1). For a finite number of terms, m say, we have

$$\sum_{k=0}^{m} f(x+k) = \int_x^{x+m} f(t)\,dt + \frac{1}{2}\big(f(x+m) + f(x)\big)$$
$$+ \sum_{k=1}^{N} \frac{B_{2k}}{(2k)!}\big(f^{(2k-1)}(x+m) - f^{(2k-1)}(x)\big)$$
$$+ \frac{B_{2N+2}}{(2N+2)!} \sum_{k=0}^{m-1} f^{(2N+2)}(x+k+\theta), \tag{2}$$

for some $\theta \in (0, 1)$, where B_j are Bernoulli numbers defined by

$$\frac{t}{e^t - 1} = \sum_{j=0}^{\infty} B_j \frac{t^j}{j!}.$$

First five Bernoulli numbers with even indices are

$$B_2 = \frac{1}{6}, \qquad B_4 = -\frac{1}{30}, \qquad B_6 = \frac{1}{42}, \qquad B_8 = -\frac{1}{30}, \qquad B_{10} = \frac{5}{66}.$$

Letting $f(x) = x^{-2}$ in (2), we obtain (see also [7, 5.8] for this particular case)

$$\sum_{k=0}^{m} \frac{1}{(x+k)^2} = S(N) + T(m,N) + E(m,N) \tag{3}$$

where

$$S(N) = \frac{1}{x} + \frac{1}{2x^2} + \sum_{k=1}^{N} \frac{B_{2k}}{x^{2k+1}}$$

$$T(m,N) = \frac{1}{2(x+m)^2} - \frac{1}{x+m} - \sum_{k=1}^{N} \frac{B_{2k}}{(x+m)^{2k+1}}$$

$$E(m,N) = (2N+3) B_{2N+2} \sum_{k=0}^{m-1} \frac{1}{(x+k+\theta)^{2N+4}} \qquad (0 < \theta < 1)$$

and the sign of $E(m,N)$ is clearly equal to the sign of B_{2N+2}, which is $(-1)^N$ (see [5, Sect. 449]). This implies that for every $m \geq 1$, $N \geq 1$, and $x > 0$ we have

$$S(2N) + T(m,2N) \leq \sum_{k=0}^{m} \frac{1}{(x+k)^2} \leq S(2N+1) + T(m,2N+1). \tag{4}$$

Now for $x > 0$ and $N \geq 1$ being fixed, let $m \to \infty$ in (4). Then $T(m,2N)$ and $T(m,2N+1)$ converge to 0 and we obtain

$$\frac{1}{x} + \frac{1}{2x^2} + \sum_{k=1}^{2N} \frac{B_{2k}}{x^{2k+1}} < \sum_{k=0}^{\infty} \frac{1}{(x+k)^2} < \frac{1}{x} + \frac{1}{2x^2} + \sum_{k=1}^{2N+1} \frac{B_{2k}}{x^{2k+1}}. \tag{5}$$

From (5) we see that, for positive x, $F''(x) > 0$ and $G''(x) < 0$.

As we already noticed, the expansion (1) implies that F_1 is a convex function, so it remains to show convexity of F_2 and F_3. Convexity of F_3

follows from

$$
\begin{aligned}
F_3''(x) &= \sum_{k=0}^{\infty} \frac{1}{(x+k)^2} - \frac{1}{x} - \frac{1}{2x^2} \\
&= \sum_{k=0}^{\infty} \left(\frac{1}{(x+k)^2} - \frac{1}{x+k} + \frac{1}{x+k+1} \right. \\
&\qquad \left. - \frac{1}{2(x+k)^2} + \frac{1}{2(x+k+1)^2} \right) \\
&= \sum_{k=0}^{\infty} \frac{1}{2(x+k)^2(x+k+1)^2} > 0. \qquad (6)
\end{aligned}
$$

Since $F_2''(x) > F_3''(x)$ for $x > 0$, the function F_2 is also convex, which ends the proof. ∎

The functions F and G introduced in Theorem 1 are closely related to the well known asymptotic expansion

$$
\log \Gamma(x) \sim \left(x - \frac{1}{2} \right) \log x - x + \frac{1}{2} \log 2\pi + \sum_{k=1}^{+\infty} \frac{B_{2k}}{2k(2k-1)x^{2k-1}} \qquad (x \to +\infty).
$$

In fact, it follows from Theorem 1 that the function

$$
x \mapsto \log \Gamma(x) - \left(x - \frac{1}{2} \right) \log x + x - \frac{1}{2} \log 2\pi - \sum_{k=1}^{n} \frac{B_{2k}}{2k(2k-1)x^{2k-1}}
$$

is convex on $(0, +\infty)$ if n is even and it is concave for odd n.

2. BOUNDS FOR $\Gamma(x+\beta)/\Gamma(x)$

In this section we will use the equalities

$$
x = \beta(x - 1 + \beta) + (1 - \beta)(x + \beta) \qquad (7)
$$

$$
x + \beta = (1 - \beta)x + \beta(x + 1). \qquad (8)
$$

Let us also introduce the following notation: For a function S ($= L$ or R) let

$$
U(x, \beta, S) = S(x) - \beta S(x - 1 + \beta) - (1 - \beta) S(x + \beta) \qquad (9)
$$

$$
\begin{aligned}
V(x, \beta, S) &= -U(x + \beta, 1 - \beta, S) \\
&= (1 - \beta) S(x) + \beta S(x + 1) - S(x + \beta). \qquad (10)
\end{aligned}
$$

Further, let $A(x,\beta)$ and $B(x,\beta)$ (do not confuse the latter notation with the beta function) be defined by

$$A(x,\beta) = \frac{\left((x-1+\beta)^{\beta}(x+\beta)^{1-\beta}\right)^{x-1/2+\beta}}{x^{x-1/2}} \qquad (x > 1-\beta), \quad (11)$$

$$B(x,\beta) = \frac{x}{A(x+\beta, 1-\beta)}$$

$$= \frac{(x+\beta)^{x+\beta-1/2}x^{\beta}}{x^{(1-\beta)(x-1/2)}(x+1)^{\beta(x+1/2)}} \qquad (x > 0). \qquad (12)$$

The ratio $\Gamma(x+\beta)/\Gamma(x)$ will be denoted by $Q(x,\beta)$.

In this section we will derive general inequalities for $Q(x,\beta)$, using LC with functions F and G as defined in Theorem 1.

THEOREM 2. *For $\beta \in [0,1]$ and for $x > 1-\beta$ we have*

$$A(x,\beta)\exp(U(x,\beta,L)) \le Q(x,\beta) \le B(x,\beta)\exp(V(x,\beta,L)) \quad (13)$$

$$B(x,\beta)\exp(V(x,\beta,R)) \le Q(x,\beta) \le A(x,\beta)\exp(U(x,\beta,R)) \quad (14)$$

with equalities if and only if $\beta = 0$ and $\beta = 1$.

As $x \to +\infty$, the absolute and relative error in all four inequalities tends to zero.

Proof. If φ is a convex function on $[s,t]$, where $s < t$, then by Jensen's inequality we have

$$\varphi(\lambda s + (1-\lambda)t) \le \lambda\varphi(s) + (1-\lambda)\varphi(t). \qquad (15)$$

If, in this inequality, we put $\varphi = F$, $s = x-1+\beta$, $t = x+\beta$, $\lambda = \beta$, then by (7) we get

$$\begin{aligned}
&\log\Gamma(x) - \left(x - \frac{1}{2}\right)\log x + L(x) \\
&\le \beta\Bigg(\log\Gamma(x-1+\beta) - \left(x - \frac{3}{2} + \beta\right) \\
&\qquad \times \log(x-1+\beta) + L(x-1+\beta)\Bigg) \\
&\quad + (1-\beta)\Bigg(\log\Gamma(x+\beta) - \left(x+\beta-\frac{1}{2}\right) \\
&\qquad \times \log(x+\beta) + L(x+\beta)\Bigg), \qquad (16)
\end{aligned}$$

or, after some manipulations,

$$\frac{\Gamma^{\beta}(x-1+\beta)\Gamma^{1-\beta}(x+\beta)}{\Gamma(x)} \geq \frac{(x-1+\beta)^{(x-3/2+\beta)\beta}(x+\beta)^{(1-\beta)(x+\beta-1/2)}}{x^{x-1/2}} \exp(U(x,\beta,L)). \tag{17}$$

Since $\Gamma(x-1+\beta) = \Gamma(x+\beta)/(x-1+\beta)$, we obtain

$$\frac{\Gamma(x+\beta)}{\Gamma(x)} \geq \frac{\left((x-1+\beta)^{\beta}(x+\beta)^{1-\beta}\right)^{x-1/2+\beta}}{x^{x-1/2}} \exp(U(x,\beta,L)), \tag{18}$$

which is the left inequality in (13).

Similarly, the right inequality in (13) is obtained with F and (8). Applying Jensen's inequality to (concave function) G and using (7), one gets the right hand side of (14), and the left hand side of (14) is obtained with G and (8). Since neither of the functions used for Jensen's inequality is a straight line, the equality in the obtained inequalities is possible if and only if $\beta = 0$ or $\beta = 1$.

Let us now prove that absolute errors in inequalities converge to zero as $x \to +\infty$. This is a consequence of the fact that second derivatives of the convex function F and of the concave function G converge to zero as their arguments converge to $+\infty$. Indeed, for any convex function u, define $r = r(x, y, \lambda, u)$ by

$$\lambda u(x) + (1-\lambda)u(y) = u(\lambda x + (1-\lambda)y) + r(x,y,\lambda,u).$$

From the Taylor formula with the integral form of the remainder, it follows

$$u(x) = u(a) + u'(a)(x-a) + \int_a^x (x-t)u''(t)\,dt \tag{19}$$

$$u(y) = u(a) + u'(a)(y-a) + \int_a^y (y-t)u''(t)\,dt. \tag{20}$$

Letting $a = \lambda x + (1-\lambda)y$ ($\lambda \in [0,1]$), multiplying (19) by λ and (20) by $1-\lambda$ and adding, we obtain

$$0 \leq r(x,y,\lambda,u) = \lambda\int_x^{\lambda x+(1-\lambda)y}(t-x)u''(t)\,dt + (1-\lambda)\int_{\lambda x+(1-\lambda)y}^{y}(y-t)u''(t)\,dt. \tag{21}$$

If we let $x \to +\infty$, $y = x + 1$, and if we assume that $u''(t) \to 0$ as $t \to +\infty$, we get that $r \to 0$. Now replace u by F or $-G$ to get the desired conclusion.

Since $Q(x, \beta) \to +\infty$ as $x \to +\infty$ and $0 < \beta < 1$, relative errors also converge to zero. ∎

As one can see from the above proof, the restriction $x > 1 - \beta$ applies only to the left inequality in (13) and to the right inequality in (14), as $A(x,\beta)$ is defined for $x > 1 - \beta$. The remaining two inequalities hold for $x > 0$.

Theorem 2 gives a variety of inequalities that can be produced by taking various number of terms in L and in R. For instance, the left inequality in (14), with $R = R_1 = -1/12x$ reads

$$\frac{(x+\beta)^{x+\beta-1/2}x^{\beta}}{x^{(1-\beta)(x-1/2)}(x+1)^{\beta(x+1/2)}}\exp\left(-\frac{\beta(1-\beta)}{12x(x+1)(x+\beta)}\right) < Q(x,\beta), \tag{22}$$

for $x > 0$, $\beta \in (0, 1)$.

Using the same technique with functions F_1 and F_2 (which cannot be considered as particular forms of F and G), one can also obtain bounds for Q. For example, the celebrated Walter Gautschi's inequality [6]

$$Q(x,\beta) \ge (x+\beta-1)^{\beta} \tag{23}$$

follows easily upon applying Jensen's inequality with $F_1 = \log\Gamma$ and using (7).

It turns out that a great deal of known bounds for the ratio Q can be derived from Theorem 2, or from logarithmic convexity in general. Some inequalities for Q that involve the digamma function, as in [1], can also be produced in this way. These topics will be considered in detail in our forthcoming papers.

In the next section we give a new sharp bound for Q, based on Theorem 2, and we show that this generalizes and sharpens several known inequalities.

3. A NEW INEQUALITY

Gautschi's inequality (23), published in 1959, has been an object of various improvements. We focus on one particular branch among many of them.

J. T. Chu in the article [4] gives the result

$$\frac{\Gamma(n/2)}{\Gamma(n/2-1/2)} > \sqrt{\frac{n-1}{2}}\sqrt{\frac{2n-3}{2n-2}} \tag{24}$$

$(n = 2, 3, \dots)$, which, after letting $x = (n-1)/2$, becomes

$$\frac{\Gamma(x+1/2)}{\Gamma(x)} > \sqrt{x - \frac{1}{4}}\,. \tag{25}$$

Chu indicates that, for $x = 1, 2, \dots,$ there is a sharper lower bound:

$$\frac{\Gamma(x+1/2)}{\Gamma(x)} > \sqrt{x - \frac{1}{4} + \frac{1}{(4x+2)^2}}\,. \tag{26}$$

In 1967, Boyd [3] gives an inequality in the same spirit. The lower bound in our notation reads

$$\frac{\Gamma(x+1/2)}{\Gamma(x)} > \sqrt{x - \frac{1}{4} + \frac{1}{32x+16}}\,, \tag{27}$$

for $x = m + 1/2$, $m = 1, 2, \dots$.

Finally, Lazarević and Lupaş' result [9] from 1979 reads

$$\frac{\Gamma(x+\beta)}{\Gamma(x)} \ge \left(x - \frac{1-\beta}{2}\right)^{\beta}, \tag{28}$$

for $x > (1-\beta)/2$ and $\beta \in [0,1]$. This inequality was rediscovered by Kershaw [8] in 1983.

As we can see, (28) coincides with (25) where the latter holds, but it is much more general than (25). On the other hand, (27) is sharper than (28) for a particular choice of x and β. We now give an inequality which is a generalization of Boyd's result and a sharpening of the inequalities of Gautschi, Chu, and Lazarević and Lupaş.

THEOREM 3. *For any* $x \ge (1-\beta)/2$ *and* $\beta \in [0,1]$,

$$Q(x,\beta) \ge \left(x - \frac{1-\beta}{2} + \frac{1-\beta^2}{24x+12}\right)^{\beta}, \tag{29}$$

with equality if and only if $\beta = 0$ *or* $\beta = 1$.

Proof. If $\beta = 0$ or $\beta = 1$ it is easy to see that equality occurs in (29). Therefore, we have to show that (29) holds as a strict inequality for $\beta \in (0,1)$.

Let us denote the right hand side in (29) by $C_*(x, \beta)$ and let

$$\rho(x, \beta) = Q(x, \beta)/C_*(x, \beta).$$

Borrowing an idea from Kershaw [8], we will show that (29) holds by showing:

(i) $\lim_{x \to +\infty} \rho(x, \beta) = 1$, for every $\beta \in (0,1)$.

(ii) $\rho(x+1, \beta) < \rho(x, \beta)$, for every $\beta \in (0,1)$ and every $x \geq (1-\beta)/2$.

Indeed, from (ii) it follows that for every natural number n, $\rho(x+n, \beta) < \rho(x, \beta)$ and by letting $n \to +\infty$ and using (i) we get

$$\rho(x, \beta) > \lim_{x \to +\infty} \rho(x, \beta) = 1,$$

which gives the inequality (29).

Proof of (i). Letting $t = 1/x$ and using Taylor's expansion we find, after some elaboration,

$$\log C_*(x, \beta) = \beta\left(-\log t - \frac{1-\beta}{2}t + \frac{-2\beta^2 + 3\beta - 1}{12}t^2\right) + o(t^2) \qquad (t \to 0). \quad (30)$$

Let us denote the lower bound in inequality (22) by $C(x, \beta)$ and let $t = 1/x$ again. Then we have

$$\log C(x, \beta) = -\beta \log t + \left(\frac{1}{t} + \beta - \frac{1}{2}\right)\log(1+\beta t) - \beta\left(\frac{1}{t} + \frac{1}{2}\right)\log(1+t) - \frac{\beta(1-\beta)t^3}{12(1+t)(1+\beta t)}. \quad (31)$$

By the Taylor expansion in (31), one can see that the expansion (30) holds for $\log C$ too; therefore,

$$\log C_*(x, \beta) - \log C(x, \beta) = o(1/x^2) \qquad (x \to +\infty). \quad (32)$$

From the Theorem 2 we have that

$$\lim_{x \to +\infty} \frac{Q(x, \beta)}{C(x, \beta)} = 1$$

and by (32) we conclude that also

$$\lim_{x \to +\infty} \frac{Q(x, \beta)}{C_*(x, \beta)} = 1,$$

which ends the proof of (i).

Proof of (ii). Let $D(x, \beta) = \log \rho(x, \beta) - \log \rho(x + 1, \beta)$. We have to show that

$$D(x, \beta) > 0 \qquad \text{for } x \geq (1 - \beta)/2. \tag{33}$$

It is clear from the previous considerations that $\lim_{x \to +\infty} D(x, \beta) = 0$. Therefore, to show (33) it suffices to show

$$\frac{\partial}{\partial x} D(x, \beta) < 0 \qquad \text{for } x \geq (1 - \beta)/2, \beta \in (0, 1). \tag{34}$$

Writing $D(x, \beta)$ in the form

$$\begin{aligned} D(x, \beta) &= \beta \log\left(1 + \frac{48x + 12\beta + 24}{24x^2 + 12\beta x - (1 - \beta)(5 - \beta)}\right) \\ &\quad + \beta \log\left(1 - \frac{2}{2x + 3}\right) - \log\left(1 + \frac{\beta}{x}\right), \end{aligned}$$

we find the derivative

$$\frac{\partial}{\partial x} D(x, \beta) = \frac{\beta(1 - \beta^2)}{x(2x + 3)(2x + 1)(x + \beta)} \cdot \frac{U(x, \beta)}{V(x, \beta) W(x, \beta)}, \tag{35}$$

where

$$\begin{aligned} U(x, \beta) &= 576x^3(\beta - 2) + 8x^2(11\beta^2 + 120\beta - 299) \\ &\quad - 4x(\beta^3 - 22\beta^2 - 115\beta + 370) - 3(19 - \beta)(5 - \beta), \\ V(x, \beta) &= 24x^2 + 12x(\beta + 4) - \beta^2 + 18\beta + 19, \\ W(x, \beta) &= 24x^2 + 12\beta x - (1 - \beta)(5 - \beta). \end{aligned}$$

Let $x_0 = (1 - \beta)/2$. The sign of the expression in (35) depends only on the sign of U, V, W. We will show that for every $\beta \in (0, 1)$ and $x \geq x_0$, the terms V and W are positive and the term U is negative.

Term V. It is not difficult to see that $V(x_0, \beta) > 0$ for every $\beta \in (0, 1)$. Further, for $x > x_0$ we have

$$\frac{\partial}{\partial x} V(x, \beta) = 48x + 12(\beta + 4) > 12(6 - \beta) > 0,$$

thus $V(x, \beta) > 0$ for every $\beta \in (0, 1)$ and $x \geq x_0$.

Term W. By $W(x_0, \beta) = 1 - \beta^2 > 0$ and

$$\frac{\partial}{\partial x} W(x, \beta) = 48x + 12\beta > 0,$$

we see that $W(x, \beta) > 0$ for $\beta \in (0, 1)$ and $x \geq x_0$.

Term U. Here we evaluate $U(x, \beta)$ at x_0 as

$$U(x_0, \beta) = -3\big(16\beta^4 - 170\beta^3 + 631\beta^2 - 994\beta + 589\big). \tag{36}$$

By localizing zeros of the polynomial in (36), we conclude that $U(x_0, \beta) < 0$ for $\beta \in (0, 1)$. Further,

$$\begin{aligned} \frac{\partial}{\partial x} U(x, \beta) & \\ &= -1728x^2(2 - \beta) + 16x\big(11\beta^2 + 120\beta - 299\big) \\ &\quad - 4\big(\beta^3 - 22\beta^2 - 115\beta + 370\big). \end{aligned} \tag{37}$$

Both zeros of the quadratic trinomial in (37) are negative for $\beta \in (0, 1)$; therefore the derivative of U with respect to $x > 0$ is negative, and from $U(x_0, \beta) < 0$ it follows that $U(x, \beta) < 0$ for $\beta \in (0, 1)$ and $x \geq x_0$.

This ends the proof of the theorem. ∎

Let us remark that the bound in (29) agrees with the bound (22) to the order of magnitude of $1/x^2$ as $x \to +\infty$. Therefore, according to Theorem 2, the error converges to zero as $x \to +\infty$. In the following table we provide some sample numerical values of relative errors in (29), for $x = 1$ and $x = 3$.

Relative Errors in (29), in %

β	0.1	0.3	0.5	0.7	0.9
$x = 1$	0.50	0.96	0.93	0.64	0.22
$x = 3$	0.016	0.036	0.041	0.031	0.012

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