A Visual Explanation of Jensen's Inequality

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"This theorem is so fundamental that we propose to give a number of proofs, of varying degrees of simplicity and generality." So say Hardy, Littlewood, and Pólya ([**1**], p. 17) of the theorem of the arithmetic and geometric means. True to their word, they proceed to give eleven (!) different proofs of the fact that for non-negative x_i,

$$\sqrt[n]{x_1 \cdot x_2 \cdots x_n} \leq \left(\frac{x_1 + x_2 + \cdots + x_n}{n} \right), \tag{1}$$

with equality *iff* $x_1 = x_2 = \cdots = x_n$. For elegant applications (suitable for the classroom) of this result to elementary geometry, see [**2**].

One of the simplest proofs of (1) consists in recognizing it to be merely a special case of Jensen's inequality [**3**]. This widely used result (e.g., probability theory [**4**]) states that if the graph of a real continuous function $f(x)$ is *concave down* then

$$\frac{\sum f(x_i)}{n} \leq f\left(\frac{\sum x_i}{n} \right), \tag{2}$$

with equality *iff* the x's are all equal. If the graph is concave up, the inequality is reversed. To obtain (1) we need only put $f(x) = \ln x$ and note that its graph is concave down. Very neat, but where did (2) come from? This note describes a particularly simple way of *seeing* its truth, which we hope may be of value in the classroom. Indeed, we believe it could even be used successfully in high schools.

We have given no formal definition of a graph being "concave down," and when presenting the following argument to young students we shall suppose that none *will* be given; what matters is that they know what one looks like. With more mature students we may define the graph of f to be "concave down" if the region $\{(x, y)\colon\ y \leq f(x)\}$ below the graph is convex. This is not one of the standard definitions, but it is a visually compelling inference from any other reasonable definition.

Consider a set of n point particles in the plane, of equal mass and with position vectors $\mathbf{r}_i$. The center of mass therefore has position vector

$$\mathbf{c} = \frac{1}{n} \sum \mathbf{r}_i,$$

from which it follows easily that

$$\sum (\mathbf{r}_i - \mathbf{c}) = \mathbf{0}.$$

In other words (see FIGURE 1), *the vectors from c to the particles cancel.*

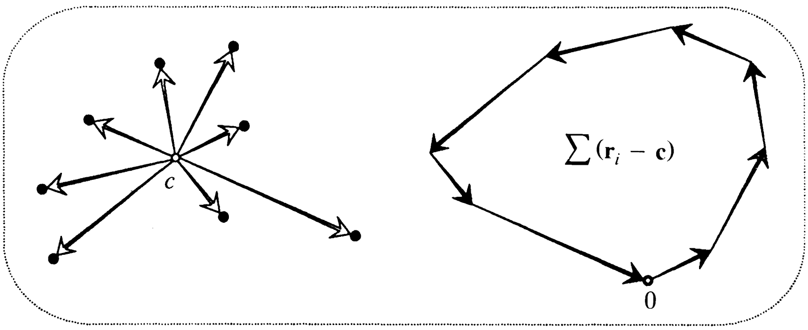

Figure 1

Imagining pegs sticking out of the plane at the locations of the particles, stretch a rubber band so as to enclose all the pegs. When released, the rubber band will contract into the dashed polygon H of FIGURE 2. This is the "convex hull" of the set of particles. The key point is this: *c must lie in the shaded interior of H*. For if p is outside this set, we see that the vectors from p to the particles cannot possibly cancel, as they must do for c. More formally, we take it as visually evident that through any exterior point p we may draw a line L such that H and its shaded interior lie entirely on one side of L. [Alternatively, this property may be taken as a (non-standard) *definition* of a convexity for a closed planar set.] The impossibility of the vectors cancelling now follows from their lying entirely on this side of L, for they all must have positive components in the direction of the normal vector $\mathbf{n}$. Except when the particles are collinear (in which case H collapses to a line-segment), the same reasoning forbids c from lying *on H*.

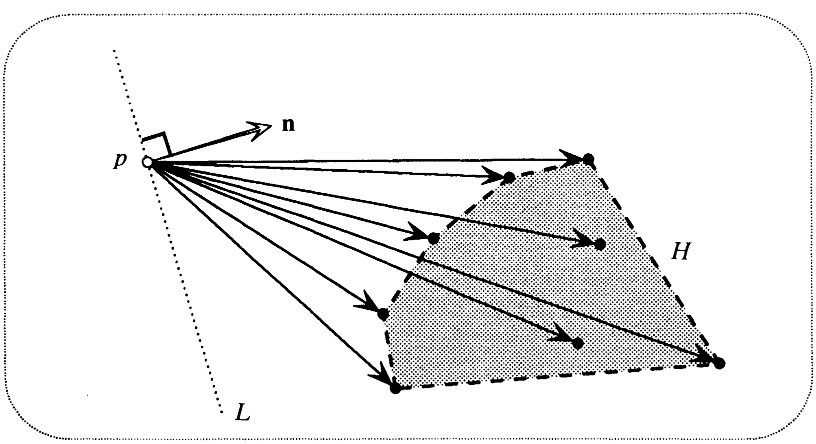

Figure 2

Next, suppose that the particles are distributed along a convex curve K. See FIGURE 3. The shaded interior of H now lies entirely on the concave side of K, and consequently so too must c. Furthermore, we see that c can only lie on K in the degenerate case that all the particles coalesce. Finally, take K to be the graph of a function $f(x)$. If this graph is *concave down* [*up*], then c lies *below* [*above*] K. Thus, with the particles located at $(x_i, f[x_i])$, we conclude that if the graph is

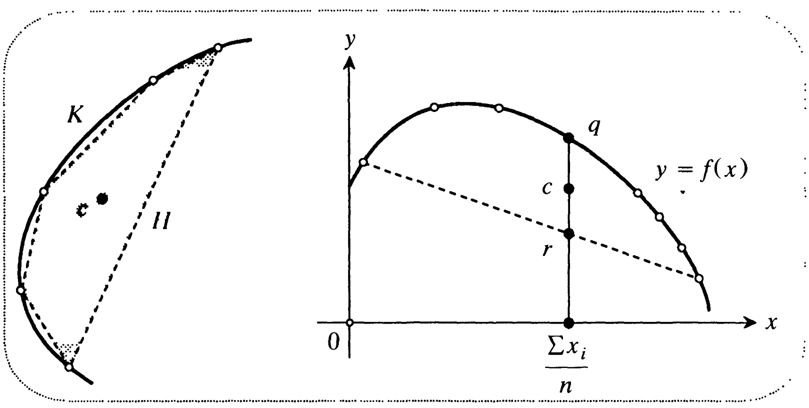

Figure 3

concave down,

$$\frac{\sum f(x_i)}{n} = \textit{height of } c \leq \textit{height of } q = f\left(\frac{\sum x_i}{n}\right),$$

with equality *iff* the x's are all equal. If the graph is concave up, the inequality is simply reversed.

As a bonus, observe that c must also lie on or above the dashed chord connecting the two end particles. Thus if $y = g(x)$ is the equation of this chord, we obtain

$$g\left(\frac{\sum x_i}{n}\right) = \textit{height of } r \leq \textit{height of } c = \frac{\sum f(x_i)}{n}.$$

I do not know if this result has a name.

We note that the above ideas can be generalized in at least two directions:

(1) The positive masses m_i of the particles need not be equal for the argument to work. Thus, once again taking the graph to be concave down,

$$\frac{\sum m_i f(x_i)}{M} \leq f\left(\frac{\sum m_i x_i}{M}\right),$$

where M denotes the total mass. This is essentially the form that is used in probability theory, for we are free to interpret (m_i/M) as a probability distribution for x_i, yielding

$$\mathscr{E}[f(x)] \leq f(\mathscr{E}[x]),$$

where $\mathscr{E}$ stands for the expected value. Also, by allowing the number of particles to increase without limit, we may pass from a discrete probability distribution to a continuous one.

(2) The argument is equally applicable to a set of particles in three-dimensional space. Thus, taking these particles (of equal mass, say) to be distributed over a surface $z = f(x, y)$ that is concave down, we deduce that

$$\frac{\sum f(x_i, y_i)}{n} \leq f\left(\frac{\sum x_i}{n}, \frac{\sum y_i}{n}\right).$$

Of course this too may be generalized to unequal masses and be given a probabilistic interpretation.

I do not wish to claim that the above is more original than it really is. In particular, the argument associated with FIGURE 2 is very old; I merely rediscovered it. The first important application of this idea that I know of occurred in 1874 when F. Lucas used it (see [**5**]) to demonstrate a complex analogue of Rolle's theorem: *the critical points of a polynomial in the complex plane must all lie within the convex hull of its zeros*. This follows from FIGURE 2 by observing [Gauss, 1816] that if $P(z)$ is the factorized polynomial, the conjugate of the logarithmic derivative $[P'(z)/P(z)]$ is a weighted sum of vectors from z to the zeros.

Also, consideration of centers of mass is certainly not new in the context of Jensen's inequality, and thus it is hard to believe that so simple a line of thought can have escaped notice. Nevertheless, it would appear that in the literature (e.g., [**1**], p. 71) the location of the center of mass is merely used as an *interpretation* of (2), rather than as the source of an explanation.

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The Fine Memorial Mathematics Hall, which will be erected at Princeton University at a cost of $400,000 in memory of the late Henry B. Fine, for many years a professor of mathematics and dean of science, will be started in the near future.

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