Unitary ESPRIT-based DOA Estimation using Sparse Linear Dual Size Spatial Invariance Array

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Abstract — **We consider the direction-of-arrival estimation problem with a particular class of sparse linear arrays, composed of multiple widely separated subarrays. A new Unitary ESPRIT- based direction-finding approach is proposed for such class of sensor arrays. The unambiguous estimates of direction cosines based on a half-wavelength baseline (invariance) and ambiguous ones based on a larger baseline are combined to obtain a better estimation accuracy. Simulation results show that proposed technique outperforms the original Unitary ESPRIT which is used together with a ULA.**

I. INTRODUCTION

Sensor arrays are used for Direction-Of-Arrival (DOA) estimation in a wide range of applications such as radar, wireless communications, seismic analysis and medical imaging, sonar. Recently DOA estimation in large arrays composed of multiple subarrays has attracted attention of specialists. Application of the subarrays on a sparse grid extends the array aperture without a corresponding increase in hardware and software costs [1]. In its turn, a larger array aperture would produce more robust resolution of closely spaced sources and more accurate DOA estimates.

ESPRIT (estimation of signal parameters via rotational invariance techniques) -based algorithm for sparse rectangular dual-size spatial invariance array has been proposed in [1]. DOA estimation algorithm [1] is based on a combination of two estimates, one coarse non-ambiguous and one fine ambiguous estimate. The coarse estimate is used to disambiguate the fine estimate to obtain a fine non-ambiguous estimate.

Another particular solution to the problem of DOA estimation in subarray-based sensor arrays has been recently proposed in [2]. However, the RARE technique [2] does not enable the straightforward multidimensional extensions (where both azimuth and elevation angles are estimated).

Several authors have considered the related problem (multiple invariance ESPRIT [3] and parallel factor analysis approach [4]). However, these approaches require highly nonlinear optimization over a multidimensional parameter space. Existing optimization techniques all suffer from global convergence problem.

Unitary ESPRIT [5] retains the simplicity and high-resolution capability of the original ESPRIT, but attains a superior performance for correlated signals at a reduced computational cost. To prevent aliasing, a half-wavelength uniform liner array (ULA) was used with the Unitary ESPRIT.

In this paper we present the new Unitary ESPRIT-based approach to DOA estimation in subarray-based sparse array that can be viewed as an extension of the algorithm of [6]. The presented approach saves the advantages of the Unitary ESPRIT and enables straightforward multidimensional extension.

II. SIGNAL MODEL

Consider an array of M omnidirectional sensors that consist of Q identically oriented linear subarrays. Each subarray is a ULA, with a distance $\Delta_{ie} \le \lambda/2$ between adjacent elements, where λ is the wavelength of the impinging wavefront. The q th subarray ($q=1,\ldots,Q$) contains M_s sensors. An example of such array is shown in Fig. 1, where intersubarray displacement Δ_{is} is the distance between the phase centers of the subarrays. The spacing between subarrays is substantially larger than the signal wavelength.

Let this array receive V narrowband far-field signals impinging from the sources with the unknown DOAs $\{\theta_1,\theta_2,\ldots,\theta_V\}$. The $M_s\times 1$ steering vector of the subarray is given by

$$\mathbf{a}_{sub}(\mu_v)=[1,e^{j\omega_v},\ldots,e^{j(M_s-1)\omega_v}]^T, \quad (1)$$

where $\omega_v=(2\pi/\lambda)\Delta_{ie}\mu_v$, $\mu_v=\sin\theta_v$ is the direction cosine relative to the array axis, $(\cdot)^T$ stands for transpose. The $M_sQ\times 1$ steering vector of array can be presented as

$$\mathbf{a}(\mu_v)=\mathbf{a}_{sub}(\mu_v)\otimes\mathbf{a}_{sd}(\mu_v), \quad (2)$$

where $\mathbf{a}_{sd}(\mu_v)$ is the vector describing the phase shifts caused by subarray displacements

$$\mathbf{a}_{sd}(\mu_v)=[1,e^{j(2\pi/\lambda)\Delta_{is}\mu_v},\ldots,e^{j(2\pi/\lambda)(Q-1)\Delta_{is}\mu_v}]^T, \quad (3)$$

and $\otimes$ denotes the Kronecker product. The similar definition of $\mathbf{a}(\mu_v)$ is presented in [1].

With the reference point at the center of the array, the subarray response vectors are conjugate centrosymmetric

$$\bar{\mathbf{a}}_{sub}(\mu_v)=[e^{-j((M_s-1)/2)\omega_v},\ldots,e^{j((M_s-1)/2)\omega_v}]^T. \quad (4)$$

Furthermore, $\mathbf{a}_{sd}(\mu_v)$ will have the following form

$$\bar{\mathbf{a}}_{sd}(\mu_v)=e^{-j((Q-1)/2)(2\pi/\lambda)\Delta_{is}\mu_v}\mathbf{a}_{sd}(\mu_v). \quad (5)$$

The data vector received by array can be expressed as

Fig. 1 Sparse linear array

$$\mathbf{x}(t)=\sum_{v=1}^{V}\overline{\mathbf{a}}(\mu_v)s_v(t)+\mathbf{n}(t), \tag{6}$$

where $\overline{\mathbf{a}}(\mu_v)=\overline{\mathbf{a}}_{sub}(\mu_v)\otimes\overline{\mathbf{a}}_{sd}(\mu_v)$. The complex samples of the V signals, $s_v(t)$, $v=1,\ldots,V$, are modeled as zero-mean Gaussian random variables, $\mathbf{n}(t)$ is the $M\times 1$ additive noise vector. The noise is assumed to be spatially and temporally white. In the case of $\Delta_{is}=M_s\Delta_{ie}$ $\overline{\mathbf{a}}(\mu_v)=\overline{\mathbf{a}}_{ULA}(\mu_v)$, where $\overline{\mathbf{a}}_{ULA}(\mu_v)$ is the steering vector for ULA of M sensors.

The batch DOA estimation problem considered below is that given the measurements $\mathbf{x}(t)$, $t=1,\ldots,N$, the unknown DOAs $\{\theta_1,\theta_2,\ldots,\theta_V\}$ should be estimated.

The array covariance matrix (CM) is given by

$$\mathbf{R}=E[\mathbf{x}(t)\mathbf{x}^{\mathrm{H}}(t)]=\overline{\mathbf{A}}\mathbf{S}\overline{\mathbf{A}}^{\mathrm{H}}+\sigma^2\mathbf{I}, \tag{7}$$

where $\overline{\mathbf{A}}=[\overline{\mathbf{a}}(\mu_1),...,\overline{\mathbf{a}}(\mu_V)]$ is the $M_sQ\times V$ matrix of source direction vectors, $\mathbf{S}=E[\mathbf{s}(t)\mathbf{s}^{\mathrm{H}}(t)]$ is the $V\times V$ signal covariance matrix, $\mathbf{s}(t)=[s_1(t),\ldots,s_V(t)]$ is the $V\times 1$ signal vector, σ^2 is the noise variance, $\mathbf{I}$ is the identity matrix, $E[\cdot]$ and $(\cdot)^H$ represent the expectation operator and the hermitian transpose, respectively.

We introduce the real-valued CM as in [5]

$$\mathbf{R}_u=\tfrac{1}{2}\mathbf{U}^H(\mathbf{R}+\mathbf{\Pi}\mathbf{R}^*\mathbf{\Pi})\mathbf{U}, \tag{8}$$

where $\mathbf{\Pi}$ is the exchange matrix with ones on its antidiagonal and zeros elsewhere, $(\cdot)^*$ stands for complex conjugate, and $\mathbf{U}$ is unitary, column conjugate symmetric matrix. For example,

$$\mathbf{U}_{M_sQ}=(1/\sqrt{2})\begin{bmatrix}\mathbf{I}_K & \mathbf{0} & j\mathbf{I}_K\\ \mathbf{0}^T & \sqrt{2} & \mathbf{0}^T\\ \mathbf{\Pi}_K & \mathbf{0} & -j\mathbf{\Pi}_K\end{bmatrix} \tag{9}$$

can be chosen for array with odd number of sensors ($M_sQ=2K+1$). Here $\mathbf{0}$ is the matrix whose elements are zeros.

The eigendecomposition of $\mathbf{R}_u$ can be defined in a standard way

$$\mathbf{R}_u=\mathbf{E}_{su}\Lambda_{su}\mathbf{E}_{su}^{\mathrm{H}}+\sigma^2\mathbf{E}_{nu}\mathbf{E}_{nu}^{\mathrm{H}}, \tag{10}$$

where $\mathbf{E}_{su}$ is $M_sQ\times V$ and contains the eigenvectors of $\mathbf{R}_u$ associated with the V largest eigenvalues, Λ_{su} is a diagonal matrix whose elements are the V largest eigenvalues (signal subspace eigenvalues), and the remaining (M_sQ-V) eigenvectors (all of whose eigenvalues are equal to σ^2) form the columns of $\mathbf{E}_{nu}$. The real-valued sample CM can be obtained as $\hat{\mathbf{R}}_u=\tfrac{1}{2}\mathbf{U}^H(\hat{\mathbf{R}}+\mathbf{\Pi}\hat{\mathbf{R}}^*\mathbf{\Pi})\mathbf{U}$ (where $\hat{\mathbf{R}}$ is the estimate of the CM (7), $\hat{\mathbf{R}}=\frac{1}{N}\sum_{t=1}^{N}\mathbf{x}(t)\mathbf{x}^{\mathrm{H}}(t)$) or $\hat{\mathbf{R}}_u=\mathrm{Re}(\mathbf{U}^H\hat{\mathbf{R}}\mathbf{U})$. The signal and noise subspaces can be estimated from the eigendecomposition of the sample CM $\hat{\mathbf{R}}_u$. We will denote the matrices made from eigenvectors of $\hat{\mathbf{R}}_u$ associated with the V largest eigenvalues and (M_sQ-V) smallest eigenvalues by $\hat{\mathbf{E}}_{su}$ and $\hat{\mathbf{E}}_{nu}$, respectively.

III. DOA ESTIMATION

It is known that original ESPRIT exploits the translational invariance in an array composed of two identical subarrays separated by a known displacement vector. The chosen array geometry often admits more than one displacement invariance. For instance, the array structure shown in Fig.1 combines two spatial invariances - Δ_{ie} and Δ_{is}.

The half-wavelength invariance (Δ_{ie}) yields unambiguous but high-variance DOA estimates to disambiguate low variance but cyclically ambiguous estimates obtained using the larger invariance (Δ_{is}). It should be noted, that when using array configuration with dual size spatial invariance (Fig.1) all array elements contribute toward the derivation of the coarse estimates as well as the fine estimates [1,6].

A. Derivation of the coarse reference estimates

The first M_s-1 and last M_s-1 elements of each subarray are used to form two identical M_s-1 element subarrays (at this point, M_s element subarray can be considered as the independent array). The response vector of subarray of M_s elements satisfies the so - called invariance relation

$$\exp(j\omega)\mathbf{J}_{c1}\overline{\mathbf{a}}_{sub}(\mu)=\mathbf{J}_{c2}\overline{\mathbf{a}}_{sub}(\mu), \tag{11}$$

where $\mathbf{J}_{c1}$ and $\mathbf{J}_{c2}$ are the selection matrices of size $(M_s-1)\times M_s$, such that $\mathbf{J}_{c1}=[\mathbf{I}_{M_s-1\times M_s-1}\ \mathbf{0}_{M_s-1\times 1}]$ and $\mathbf{J}_{c2}=[\mathbf{0}_{M_s-1\times 1}\ \mathbf{I}_{M_s-1\times M_s-1}]$. The matrices $\mathbf{J}_{c1}$ and $\mathbf{J}_{c2}$ select the first and last M_s-1 components of steering vector $\overline{\mathbf{a}}_{sub}(\mu)$, respectively. However, it is possible to extend the equation (11) for all Q subarrays. As a result, the generalized invariance relation can be written as

$$\exp(j\omega)\,\mathrm{bdiag}(\mathbf{J}_{c1})\overline{\mathbf{a}}(\mu)=\mathrm{bdiag}(\mathbf{J}_{c2})\overline{\mathbf{a}}(\mu), \tag{12}$$

where $\mathrm{bdiag}(\cdot)$ denotes the block-diagonal matrix operator. For example, $\mathrm{bdiag}(\mathbf{J}_{c1})$ is the $(M_s-1)Q\times M_sQ$ matrix with Q diagonal blocks equal to the matrix $\mathbf{J}_{c1}$.

The real-valued manifold relation for the Unitary ESPRIT [5] translates into the following relation

$$\mathrm{tg}(\omega/2)\mathbf{K}_{1c}\mathbf{d}(\mu)=\mathbf{K}_{2c}\mathbf{d}(\mu), \tag{13}$$

where $\mathbf{K}_{1c} = \mathrm{Re}(\mathbf{U}^H_{(M_s-1)Q}\,\mathrm{bdiag}(\mathbf{J}_{c2})\mathbf{U}_{M_sQ})$ and $\mathbf{K}_{2c} = \mathrm{Im}(\mathbf{U}^H_{(M_s-1)Q}\,\mathrm{bdiag}(\mathbf{J}_{c2})\mathbf{U}_{M_sQ})$ are real-valued $(M_s-1)Q \times M_sQ$ matrices, $\mathbf{d}(\mu) = \mathbf{U}^H\bar{\mathbf{a}}(\mu)$ is the real-valued steering vector. For all V sources the real-valued relation (13) becomes the DOA matrix relation

$$\mathbf{K}_{1c}\mathbf{D}\mathbf{\Omega}_c = \mathbf{K}_{2c}\mathbf{D}, \tag{14}$$

where $\mathbf{\Omega}_c = \mathrm{diag}[\mathrm{tg}(\omega_v/2)]_{v=1}^{V}$ contains the desired information about DOAs, $\mathbf{D} = [\mathbf{d}(\mu_1),\ldots,\mathbf{d}(\mu_V)]$ is the $M_sQ \times V$ real-valued DOA matrix.

If there exists no noise or if there are infinite number of snapshots available $\mathbf{E}_{su}$ and $\mathbf{D}$ are related by unknown $V \times V$ nonsingular matrix $\mathbf{T}$ as $\mathbf{E}_{su} = \mathbf{DT}$ (in the more realistic cases this relation would becomes only approximate). So, we can write

$$\mathbf{K}_{1c}\mathbf{E}_{su}\mathbf{\Psi}_c = \mathbf{K}_{2c}\mathbf{E}_{su}, \tag{15}$$

where $\mathbf{\Psi}_c = \mathbf{T}_c^{-1}\mathbf{\Omega}\mathbf{T}_c$, $\mathbf{T}_c = \mathbf{F}_c\mathbf{T}$, $\mathbf{F}_c$ represents some unknown permutation matrix. The matrix equation (15) can be solved by least squares (LS) or total least squares (TLS) method [5]. When using the least squares method, the matrix of interest $\mathbf{\Psi}_c$ can be determined as $\mathbf{\Psi}_c = ((\mathbf{K}_{1c})^H\mathbf{K}_{1c})^{-1}(\mathbf{K}_{1c})^H\mathbf{K}_{2c}$.

The eigenvalues $\gamma_v^c, v=1,\ldots,V$ of $V \times V$ real-valued matrix $\mathbf{\Psi}_c$ are $\mathrm{tg}(\omega_v/2)$, $v=1,\ldots,V$. From these, a set of unambiguous estimates of the direction cosines may be determined

$$\mu_v^c = \frac{\omega_v}{2\pi\Delta_{ie}/\lambda} = \frac{2\,\mathrm{arctg}(\gamma_v^c)}{2\pi\Delta_{ie}/\lambda},\ v=1,\ldots,V. \tag{16}$$

Clearly, that simplification of this relationship yields $\mu_v^c = \mathrm{arctg}(\gamma_v^c)/(\pi\Delta_{ie}/\lambda)$.

B. Derivation of the fine estimates

In this case, the first $Q-1$ subarrays and last $Q-1$ subarrays would be used to form two identical subarrays of $M_s(Q-1)$ elements. The two subarray manifolds are related as

$$\exp(j\omega^f)\mathbf{J}_{f1}\bar{\mathbf{a}}(\mu) = \mathbf{J}_{f2}\bar{\mathbf{a}}(\mu), \tag{17}$$

where $\mathbf{J}_{f1} = [\mathbf{I}_{M_s(Q-1)\times M_s(Q-1)}\ \mathbf{0}_{M_s(Q-1)\times M_s}]$ and $\mathbf{J}_{f2} = [\mathbf{0}_{M_s(Q-1)\times M_s}\ \mathbf{I}_{M_s(Q-1)\times M_s(Q-1)}]$ are the corresponding $M_s(Q-1)\times M_sQ$ selection matrices, $\omega^{\mathrm{f}} = (2\pi/\lambda)\Delta_{\mathrm{is}}\mu$. Furthermore, the invariance relationship similar to the (13) can be presented as

$$\mathrm{tg}(\omega^{\mathrm{f}}/2)\mathbf{K}_{1\mathrm{f}}\mathbf{d}(\mu) = \mathbf{K}_{2\mathrm{f}}\mathbf{d}(\mu), \tag{18}$$

where matrices $\mathbf{K}_{1f} = \mathrm{Re}(\mathbf{U}^H_{M_s(Q-1)}\mathbf{J}_{f2}\mathbf{U}_{M_sQ})$ and $\mathbf{K}_{2f} = \mathrm{Im}(\mathbf{U}^H_{M_s(Q-1)}\mathbf{J}_{f2}\mathbf{U}_{M_sQ})$ are the real-valued $M_s(Q-1)\times M_sQ$ matrices.

DOA matrix $\mathbf{D}$ satisfies

$$\mathbf{K}_{1f}\mathbf{D}\mathbf{\Omega}_f = \mathbf{K}_{2f}\mathbf{D}. \tag{19}$$

Now, it is easy to demonstrate that

$$\mathbf{K}_{1f}\mathbf{E}_{su}\mathbf{\Psi}_f = \mathbf{K}_{2f}\mathbf{E}_{su}, \tag{20}$$

where $\mathbf{\Psi}_c = \mathbf{T}_f^{-1}\mathbf{\Omega}_f\mathbf{T}_f$, $\mathbf{T}_f = \mathbf{F}_f\mathbf{T}_f$, $\mathbf{F}_f$ represents some unknown permutation matrix. Similar to (15) the matrix equation (20) can be solved by LS or TLS method. The eigenvalues $\gamma_v^f, v=1,\ldots,V$ of the $V \times V$ real-valued matrix $\mathbf{\Psi}_f$ are $\mathrm{tg}(\omega_v^f/2)$, $v=1,\ldots,V$. The set of low-variance (fine) but cyclically ambiguous direction cosine estimates μ_v^f may be derived in such a way

$$\mu_v^f = \frac{\omega_v^f}{2\pi\Delta_{is}/\lambda} = \frac{2\,\mathrm{arctg}(\gamma_v^f)}{2\pi\Delta_{is}/\lambda}. \tag{21}$$

C. Disambiguation of the fine estimates

The fine disambiguated estimates are

$$\mu_v^{fd} = \mu_v^f + n^*\frac{\lambda}{\Delta_{is}}, \tag{22}$$

where n^* is estimated as

$$n^* = \arg\min_n \left| \mu_v^c - \mu_v^f - \frac{\lambda}{\Delta_{is}}n \right|. \tag{23}$$

The bounds for n are determined from

$$\left\lceil \frac{\Delta_{is}}{\lambda}(-1-\mu_v^f) \right\rceil \le n \le \left\lfloor \frac{\Delta_{is}}{\lambda}(1-\mu_v^f) \right\rfloor, \tag{24}$$

where $\lceil z \rceil$ ($\lfloor z \rfloor$) denotes the smallest (largest) integer greater (less) than z.

The proposed approach is summarized below.

Step 1. Obtain the sample covariance matrix $\hat{\mathbf{R}}_u$ as an estimate of $\mathbf{R}_u$.

Step 2. Perform eigendecomposition of the $M_sQ \times M_sQ$ matrix $\hat{\mathbf{R}}_u$. Apply an appropriate detection technique to get an estimate $\hat{V}$ of the number of sources and obtain the $M_sQ \times \hat{V}$ matrix $\hat{\mathbf{E}}_{su}$.

Step 3. Find the coarse reference estimates of the direction cosines $\hat{\mu}_v^c$, $v=1,\ldots,\hat{V}$.

Step 4. Find the low-variance (fine) but cyclically ambiguous direction cosine estimates $\hat{\mu}_v^f$, $v=1,\ldots,\hat{V}$.

Step 5. Perform the disambiguation procedure (22-24) for each of $\hat{V}$ sources and obtain the fine disambiguated estimates $\hat{\mu}_v^{fd}$, $v=1,\ldots,\hat{V}$.

Step 6. Compute the DOA estimates $\hat{\theta}_v = \mathrm{asin}(\hat{\mu}_v^{fd})$.

It is known that in nonadaptive scenarios the computational load of eigenstructure-based estimators is usually dominated by formation of the respective covariance matrix and calculation eigendecomposition of this matrix.

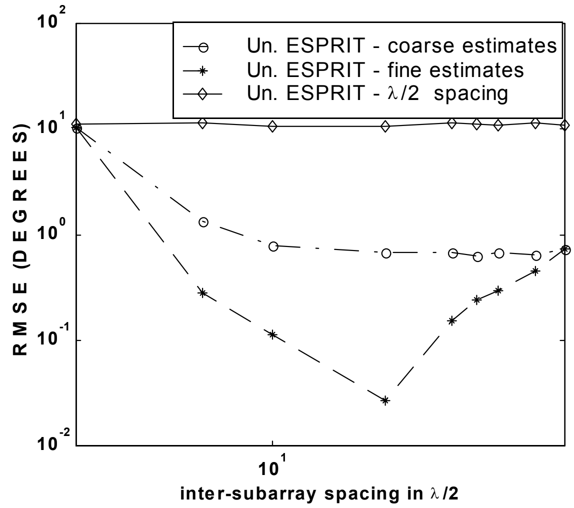

Fig.2 DOA estimation RMSEs versus intersubarray displacement.

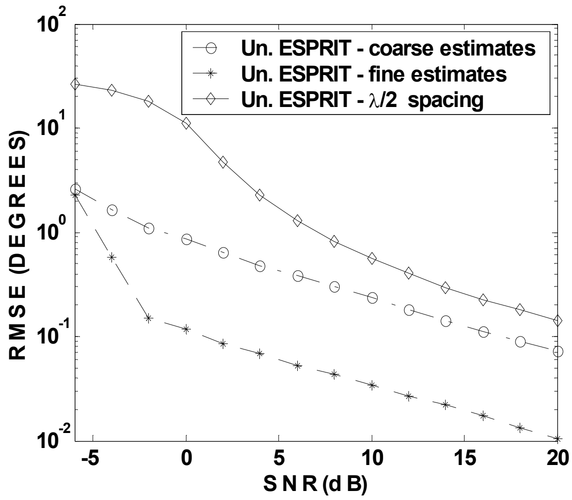

Fig.3 DOA estimation RMSEs versus SNR; intersubarray displacement is $10(\lambda/2)$

Although the batch DOA estimation problem is considered in the paper, real-time estimation can be realized using the eigendecomposition or covariance matrix updating methods.

IV. Simulations

In this section, we illustrate the performance of the proposed algorithm. Simulation results are presented in Figs.2,3. Throughout simulations, we assume two uncorrelated sources impinging on an array of $M=12$ sensors from DOAs $\theta_1=36^\circ$ and $\theta_2=38^\circ$. Array is composed of $Q=4$ subarrays. $N=100$ snapshots are taken in each example and all results are averaged over 1000 simulation runs. LS Unitary ESPRIT is used. In Figs.2 and 3, the RMSE curves are averaged over the sources. The sample RMSE was computed as

$$RMSE=\sqrt{\frac{1}{ZV}\sum_{z=1}^{Z}\sum_{v=1}^{V}[(\hat{\theta}_v(z)-\theta_v)^2]}\,, \qquad (25)$$

where $\hat{\theta}_v(z)$ denote the DOA estimate of the vth source obtained from a particular algorithm at the zth run, whereas θ_v is the corresponding true DOA. In addition, it was assumed in all case that the number of emitters had been correctly estimated (i.e. $\hat{V}=V$).

In order to realize the Unitary ESPRIT with $\lambda/2$ spacing ULA of 12 elements was used. So, the hardware and computational complexity are comparable (the number of used antenna elements is the same in both cases). The coarse estimates of direction cosines were defined from formula (16).

Fig. 2 shows DOA estimation RMSE's versus intersubarray displacement. RMSE of fine estimates decreases as intersubarray distance increases from $\lambda/2$ to $\Delta_{is}=20(\lambda/2)$. From the other side, after $\Delta_{is}=20(\lambda/2)$ RMSE of fine estimates increases and approximates the RMSE of coarse estimates. Here the signal-to-noise ratios of the sources were SNR=0 dB.

For case in Fig.3 value of intersubarray distance was taken $\Delta_{is}=10(\lambda/2)$. Fig. 3 shows DOA estimation RMSE versus SNR. It is seen that the estimation performance of the proposed approach with sparse array (composed of multiple identical translated subarrays) is better than the estimation performance of the Unitary ESPRIT with ULA.

V. Conclusions

We have considered the direction-finding problem in sparse array with dual – size spatial invariance. A new Unitary ESPRIT-based approach to DOA estimation has been proposed. Simulation results illustrate the performance of the proposed approach compare to the performance of the original Unitary ESPRIT. The presented approach can be easy extended to 2D DOA estimation (azimuth and elevation angles).

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