The ESPRIT algorithm. Variants and precision.

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***Abstract* – since wireless communications began to take an abrupt rise, with the development of techniques which were introduced for the growth of data transfer, antenna systems were considered to be the ideal choice to obtain the wanted results. Nevertheless these techniques required the development of certain algorithms also, to facilitate the recognition of the signals at the reception side (direction of arrival (DOA), phase, frequency, etc.)**

The object of this article is the ESPRIT algorithm, and its variants, with the focus on the precision of the identification of signals at the reception end. The performance evaluation is based on the simulation of the algorithms and the estimation of the parameters. The antenna system, or antenna array, used for these simulations is a linear array with a variable number of elements.

Keywords: ESPRIT, frequency estimation, direction of arrival estimation

I. INTRODUCTION

The high accuracy of signal estimation at the reception end was always the middle point of telecommunications, radar systems and also commercial and military applications.

Frequency estimation, DOA of a signal, using a simple, fixed antenna, may present a series of limitations which can negatively affect the precision of results. To eliminate this aspect it is preferable to use antenna arrays, which permit a much higher performance of the estimation process.

Besides these antenna arrays, to further increase the precision a various number of algorithms were developed.

Among these algorithms we encounter the MUSIC (MUltiple SIgnal Classification) algorithm, which was a widely spread algorithm, its applicability being extensively studied for years. Although the advantages of this algorithm are notable, it requires vast computational and memory resources.

To eliminate the previously mentioned issues of the MUSIC algorithm, the development of a new algorithm was inevitable: the ESPRIT (Estimation of Signal Parameters via Rotational Invariance Techniques) algorithm, which, thanks to its robustness, compensates the small imperfections of the antenna array; this characteristic is not available with other algorithms such as MUSIC.

II. THE ESPRIT ALGORITHM

The ESPRIT algorithm was developed as an improved/alternative for the DOA estimation method. In comparison with other algorithms, other methods, ESPRIT presents a series of advantages, amongst which one of the most important is the fact that the calibration of the antenna array is not necessary (this is obligatory for the MUSIC algorithm).

<u>Description</u>

The ESPRIT algorithm is based on an analysis of a subspace used to localize a source or estimate parameters of the signal (frequency, phase, etc). The basic idea of this algorithm is to "split" the antenna array (AA) in to subarrays (SA) separated by an equivalent displacement.

Below we will explain the ESPRIT algorithm in the context of a uniform linear AA with N elements. The number of elements of the SA will be noted with N_s. D_s is the displacement parameter and should be considered as the distance between the two SAs. The d_s parameter is expressed in units of distance between elements of the AA; in the following examples we consider the first element of the first SA as the first element of the original AA and first element of the second SA is the (d_s+1)th element of the original AA.

AA:

• • • • • • • • • •
1 2 3 4 5 6 7 8 9 10

Examples of splits could be one of the following variants, and not only:

<u>*Ex1*</u>. $d_s=1$

- SA a:

• • • • • • • • •
1 2 3 4 5 6 7 8 9

(the selection matrix is $J_{sa} = [I_{9\times9}\ 0_{9\times1}]$)

- SA b:

• • • • • • • • •
2 3 4 5 6 7 8 9 10

(the selection matrix is $J_{sb} = [0_{9\times1}\ I_{9\times9}]$)

<u>*Ex2*</u>. $d_s=3$

- SA a:

• • • • • • •
1 2 3 4 5 6 7

(the selection matrix is $J_{sa} = [I_{7\times7}\ 0_{7\times3}]$)

- SA b:

• • • • • • •

4 5 6 7 8 9 10

(the selection matrix is $J_{sb} = [0_{7\times3}\ I_{7\times7}]$)

In the examples above we have noted with I the identity matrix and with 0 the zero matrix. It can be observed that the selection matrix for the two SA can be written in a generic form as:

$$J_{sa} = [I_{Ns\times Ns}\ 0_{Ns\times ds}],$$
$$J_{sa} = [0_{Ns\times ds}\ I_{Ns\times Ns}]$$

If we consider V the array manifold matrix of the AA and with V_i the array manifold matrix for the SA of the i[th] order (in our case i = a, b) we can express: $V_a = J_{sa}V$ si $V_b = J_{sb}V$

We introduce the Φ matrix of the form:

$$\Phi = \begin{bmatrix} e^{jd_s\psi_1} & 0 & \cdots & 0 \\ 0 & e^{jd_s\psi_2} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & e^{jd_s\psi_D} \end{bmatrix}$$

Were ψ_i is the number of the D signals in the ψ space. Thus we can write:

$$V_b = V_a\ \Phi.$$

Keep in mind that for a uniform linear AA we have the following expression:

$$\psi = \frac{2\pi}{\lambda} d \cos\theta$$

If we note T a DxD matrix (not unique), then we can write:

$$U_s = V\ T$$

This relationship says that the signal eigenvectors are linear combinations of the array manifold vectors of the D sources.

$$U_{sa} = J_{sa}U_s = J_{sa}VT = V_aT$$
$$U_{sb} = J_{sb}U_s = J_{sb}VT = V_bT$$

The relation $U_{sa} = V_aT$ implies $V_a = U_{sa}T^{-1}$, and also the relation $U_{sb} = V_bT = V_a\Phi T$ implies $U_{sb} = U_{sa}T^{-1}\Phi T$.

We define $\Psi=T^{-1}\Phi T$, therefore we can express de signal subspace of the first SA as a function of the signal subspace of the second SA:

$$U_{sa} = \Psi\ U_{sb}$$

We take into account the fact that $N_s>D$, therefore in practice instead of the theoretical values of U_{sa} and U_{sb} we will have to use their estimates:

$$U'_{sa} = J_{sa}\ U'_s$$
$$U'_{sb} = J_{sb}\ U'_s$$

Therefore $U'_{sa}\Psi' = U'_{sb}$

III. VARIANTS OF THE ESPRIT ALGORITHM

Various types of ESPRIT were developed: LS-ESPRIT (Least Square ESPRIT), TLS-ESPRIT (Total LS-ESPRIT), URV-ESPRIT, M-ESPRIT (Modified ESPRIT), HO-ESPRIT (Higher Order ESPRIT), V-ESPRIT (Virtual ESPRIT), etc.

We will discuss the first three.

A. LS-ESPRIT

In this method the difference between $U'_{sa}\Psi'$ and U'_{sb} is minimized.

$$\Psi' = \arg\min_\Psi \| U'_{sb} - U'_{sa}\Psi\|^2_F =$$
$$= \arg\min_\Psi \mathrm{tr}\{[U'_{sb} - U'_{sa}\Psi]^H[U'_{sb} - U'_{sa}\Psi]\} \quad (1.1)$$

therefore $\Psi' = [U'^H_{sa}U'_{sa}]^{-1}U'^H_{sa}U'_{sb}$ (1.2)

The steps of the LS-ESPRIT are summarized as follows:

- Perform the eigendecomposition on C_x to obtain U'_s
- Find U'_{sa} and U'_{sb}
- Find the LS estimate for $\Psi'_{LS}=[U'^H_{sa}U'_{sa}]^{-1}U'^H_{sa}U'_{sb}$
- Find the eigenvalues of Ψ'_{LS}
- Find the estimates in ψ-space by using $\psi'_i=(\arg\lambda'_i)/d_s$, i=1,2,…D

B. TLS-ESPRIT

Because both U'_{sa} and U'_{sb} are estimates containing errors, Golub and Van Loan suggest that a total least squares (TLS) approach is more appropriate. If the TLS approach is used,

$$\Psi'_{TLS}=-V_{12}V^{-1}_{22} \quad (2.1)$$

where V_{12} and V_{22} are matrices defined by the eigendecomposition of the 2D × 2D matrix.

In this TLS variant, to obtain the value of ψ'_i we will use Ψ'_{TLS} in the steps 3 and 4 from the previous, LS, variant.

<u>Running repeated simulations</u> with the two algorithms, willing to estimate the frequency (normalized) of the signal at arrival, the results were noticeably satisfactory (Figure 1).

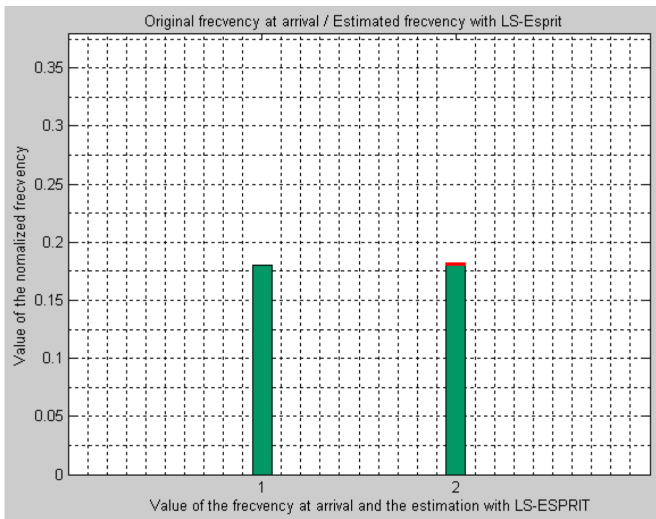

Figure 1. LS-ESPRIT result of simulation. The first value is the normalized frequency value of the real signal and the second value represents the value of estimated normalized frequency

The results are very acceptable: with an accuracy of 0.63%, meaning the estimated value is equal to the real value + / - 0.63%. In the image above (Fig. 1), the upper part of the estimated frequency is exactly that + / - 0.63% mentioned before. The calculus was performed on a series of 30 simulations per value of real frequency.

The results obtained with TLS-ESPRIT are very close to those obtained with the LS-ESPRIT, yet with a more favorable percentage: an accuracy of 0.51%, meaning the estimated value is equal to the real value + / - 0.51%. The calculus was performed on a series of 30 simulations per value of real frequency.

C. URV-ESPRIT

G.W. Stewart introduced this new approach, using URV decompositions (decomposition of the matrix that produces the signal subspace and noise). This allows real-time applications, of course, if the URV decomposition can be successfully substituted. (This method was applied to accelerate the MUSIC algorithm).

URV decomposition is reflected in the value of X^H in the form:

$$X^H = URV^H = U\begin{pmatrix} R & F \\ 0 & G \end{pmatrix} V^H \qquad (3.1)$$

Where the columns U and V are orthonormal, R and G are upper triangular of orders *d* and *m-d*, and F and G are small in norm. This decomposition reveals that X falls within $\sqrt{\|F\|^2 + \|G\|^2}$ of the matrix X' of rank *d* obtained by setting the values of F and G in 3.1 to zero.

The column space X' is the space spanned by the first *d* columns of V, and those columns are therefore a natural candidate for the basis required by the ESPRIT algorithm.

The U matrix is not required and is not stored or updated.

Similarly, URV decomposition of the matrix (V_1 V_2) can be used to determine the matrices W_1 and W_2 of the ESPRIT algorithm.

The URV decomposition can be updated in $O(m^2)$ time (and in O(m) time on a linear array of m processors). The updating procedure consists of two parts: an incorporation step and a deflation step. The incorporation is analogous to the standard update of QR decomposition; however, special care is taken that only the first column of F and G increase in norm.

This corresponds to the fact that adding a row to a matrix can increase its rank by at most one. After the update, a condition estimator is used to test R for rank degeneracy, and a deflation step reduces the norm to the last column of R.

If degeneracy is detected, a refinement step is performed to bring the decomposition closer to diagonal form.

All transformations are accomplished by plane rotations.

To determine the rank, ||G|| is the analogous of $\sqrt{\delta_{d+1}^2 + \cdots + \delta_m^2}$ for SVD. In consequence, we try to pick a value for *d* so that

$$\|G\| \le \psi_{d^\sigma} \sqrt{n(m-d)}$$

for the rectangular variant, and

$$\|G\| \le \psi_{d^\sigma} \sqrt{\frac{m-d}{1-\mu^2}}$$

for the exponential variant.

However, a decision must be made, when executing the incorporation, to establish if G rose in norm due to a rise in rank. Here, we will use the same criteria, but with a different Ψ_u factor, replacing Ψ_d.

With the increase of Ψ_u, the signal subspace changes more rarely. Therefore Ψ_u can be looked upon as a factor which controls the subspace precision of the estimated signal.

Simulations performed to determine, this time, the DOA (angle of arrival), had as results values very close to actual values.

Ex1. For 16 antenna elements, with 4 static signal sources, the results are shown in Figure 2.

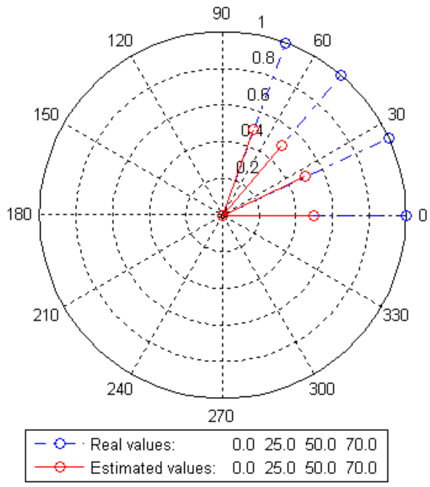

Figure 2. URV-ESPRIT: result of simulation. Actual/real and estimated angles of arrival, 16 antenna elements, 4 sources, static

It is obvious that values are identical.

Ex2. For 16 antenna elements, with 4 sources (three static and one moving, but without intersection of sources), the results looks like this:

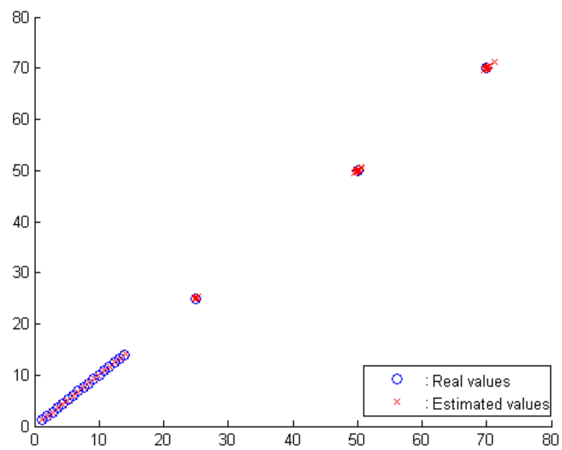

Figure 3. URV-ESPRIT: result of simulation. 16 antenna elements, 4 sources, static+dynamic

For a better understanding, the following table of values should be considered: it contains the values represented.

TABLE I. URV-ESPRIT: RESULT OF SIMULATION. 16 ANTENNA ELEMENTS, 4 SOURCES, STATIC+DYNAMIC

Step	Source 1 Real Value	Source 1 Estim. Value	Source 2 Real Value	Source 2 Estim. Value	Source 3 Real Value	Source 3 Estim. Value	Source 4 Real Value	Source 4 Estim. Value
1	1.2000	1.2232	25.000	24.875	50.000	49.613	70.000	70.108
2	2.0000	2.0520	25.000	24.939	50.000	50.595	70.000	71.175
3	2.8000	2.7926	25.000	24.985	50.000	50.518	70.000	69.745
4	3.6000	3.5865	25.000	25.314	50.000	49.852	70.000	70.146
5	4.4000	4.4221	25.000	25.151	50.000	49.579	70.000	69.599
6	5.2000	5.1394	25.000	24.946	50.000	49.937	70.000	69.641
7	6.0000	6.0371	25.000	24.885	50.000	50.001	70.000	70.005
8	6.8000	6.6861	25.000	25.242	50.000	49.424	70.000	70.582
9	7.6000	7.7083	25.000	24.962	50.000	49.867	70.000	69.838
10	8.4000	8.4248	25.000	25.017	50.000	49.812	70.000	70.275
11	9.2000	9.2382	25.000	25.219	50.000	50.020	70.000	70.240
12	10.000	9.9538	25.000	24.824	50.000	50.056	70.000	69.861
13	10.800	10.843	25.000	25.041	50.000	49.988	70.000	70.212
14	11.600	11.665	25.000	25.191	50.000	49.844	70.000	69.968
15	12.400	12.436	25.000	24.959	50.000	50.242	70.000	70.110
16	13.200	13.305	25.000	24.975	50.000	49.859	70.000	69.810
17	14.000	14.039	25.000	24.994	50.000	50.095	70.000	69.928

It is clear that the accuracy is slightly damaged in comparison with the previous example, when all sources were static. The dynamic source has a certain influence on the results. On the statistical precision of values for the source motion is + / - 0.83%. Regarding the static sources, the accuracy is + / - 0.41%.

Ex3. A case, closer to reality, is if all sources are dynamic, therefore all in motion. In this case the 16 antenna elements are not sufficient for acceptable results. For this example we considered 28 antenna elements, and 4 sources in motion, with possible intersections between them.

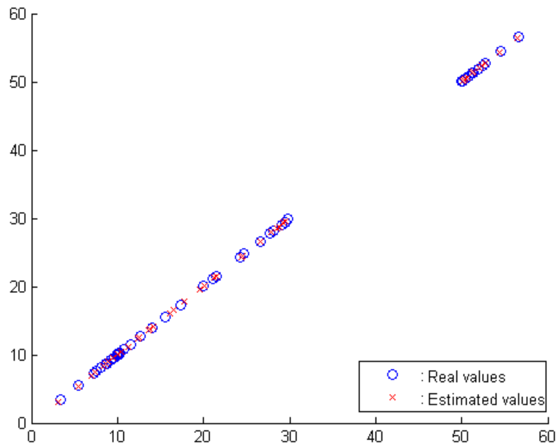

Figure 4. URV-ESPRIT: result of simulation. 16 antenna elements, 4 sources, dynamic

For a better understanding, the following table of values should be considered: it contains the values represented.

TABLE II. URV-ESPRIT: RESULT OF SIMULATION. 16 ANTENNA ELEMENTS, 4 SOURCES, DYNAMIC

Step	Source 1 Real Value	Source 1 Estim. Value	Source 2 Real Value	Source 2 Estim. Value	Source 3 Real Value	Source 3 Estim. Value	Source 4 Real Value	Source 4 Estim. Value
1	3.4029	3.0725	17.2856	17.7961	29.4550	28.7695	56.5971	56.4911
2	5.4695	5.4092	15.5771	16.0740	29.8296	29.3065	54.5305	54.3717
3	7.2443	7.4003	14.0168	14.1236	27.7232	27.9072	52.7557	52.7545
4	8.6326	8.6666	12.6568	13.8057	24.2178	24.3795	51.3674	51.4147
5	9.5604	10.5039	11.5427	12.5611	21.1143	21.2784	50.4396	50.4086
6	9.9781	10.0531	10.7119	11.2211	20.0069	20.1663	50.0219	50.0569
7	9.8636	10.0267	10.1921	19.5101	21.4645	21.4700	50.1364	50.2181
8	9.2229	9.5156	10.0009	16.5803	24.7383	24.5049	50.7771	50.6963
9	8.0902	8.3670	10.1444	12.3331	28.1466	28.5240	51.9098	52.0136
10	8.7148	8.9593	10.0308	13.6295	26.5451	26.6021	51.2852	51.2287
11	7.5771	7.0016	10.2763	10.4089	29.0451	29.4751	52.4229	52.3373

Accuracy is damaged this case, but still remains within normal limits. To note is that if you increase the number of antenna elements the discrepancies disappear.

In this case, the worst values occur with the second source. Between steps 7 and 10, we can observe a very poor estimation of the actual values. But if those steps are neglected, the error is about 4%, which is also the error on source number 4. The last two sources error values are under 1%.

IV. CONCLUSIONS

From all of the above we can see that for the different type of algorithms developed in the previous years, there are positive and negative sides. Nevertheless the opportunities offered by them are highly valuable and the applicability is vast.

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