SIAM J. SCI. STAT. COMPUT.
Vol. 5, No. 2, June 1984

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A METHOD FOR CONSTRUCTING LOCAL MONOTONE PIECEWISE CUBIC INTERPOLANTS*

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Abstract. A method is described for producing monotone piecewise cubic interpolants to monotone data which is completely local and which is extremely simple to implement.

1980 AMS mathematics subject classifications. Primary 65D05, secondary 65D07

Key words. monotone interpolation, cubic Hermite interpolation

In [4] Fritsch and Carlson gave necessary and sufficient conditions for a piecewise cubic interpolant to monotone data to be monotonic. The algorithm which they proposed for computing such an interpolant suffers from three defects: (1) it requires two passes over the data; (2) the result is dependent upon the order in which the data are processed; and (3) it is potentially nonlocal (i.e., a correction introduced in the first interval might ripple through the entire interpolant).

The purpose of this note is to acquaint the mathematical community with a technique proposed by Butland [2] for obtaining monotone piecewise cubic interpolants which avoids all of these problems. Before we describe Butland's method, we recall some notation from [4]. Let $x_1 < x_2 < \cdots < x_n$ $(n > 2)$ and let $f_i = f(x_i)$ $i = 1, \cdots, n$ be the values of a monotone function at these points. Let $p(x)$ be a piecewise cubic function such that $p(x_i) = f_i$ and $p'(x_i) = d_i$, $i = 1, \cdots, n$. Let $\Delta_i = (f_{i+1} - f_i)/(x_{i+1} - x_i)$, $i = 1, \cdots, n-1$.

A piecewise cubic interpolation scheme is a method for selecting the values of the derivatives d_i. Butland's idea is to construct a function G such that

$$d_i = G(\Delta_{i-1}, \Delta_i), \qquad i = 2, \cdots, n-1, \tag{1}$$

and $p(x)$ is monotonic. Since d_i depends only on neighboring slopes, a method based on formula (1) is necessarily one-pass and local. If G is a symmetric function of its arguments, the result will also be independent of the direction of processing, as desired.

Sufficient conditions for an acceptable G-function are given in [2], where Butland observes that the harmonic mean

$$G_H(S_1, S_2) = \begin{cases} \dfrac{2S_1S_2}{S_1 + S_2} & \text{if } S_1S_2 > 0, \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

satisfies all of these conditions. Unfortunately, formula (2) tends to produce interpolants that are "too flat", because the values $(\alpha_i, \beta_i) = (d_i/\Delta_i, d_{i+1}/\Delta_i)$ are restricted to the small square $[0, 2]\times[0, 2]$, contained in the monotonicity region. We have experimented with various G-functions that fill out the square[1] $[0, 3]\times[0, 3]$ more completely, thus producing more "reasonable" curves. One such family of functions

* Received by the editors February 25, 1983.

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[1] This is the largest square contained in the monotonicity region.

is described in [3], where the recommended formula is

$$G_3(S_1, S_2) = \begin{cases} 0 & \text{if } S_1S_2 \leqq 0, \\ \text{sign}\,(S_1)\dfrac{3|S_1|\,|S_2|}{|S_1|+2|S_2|} & \text{if } |S_2| \leqq |S_1|, \\ G(S_2, S_1) & \text{otherwise.} \end{cases} \tag{3}$$

One negative aspect of (1) is that it gives the same value for d_i regardless of the relative spacing of the surrounding x-values. One way to remedy this is to replace (1) with

$$d_i = G(\Delta_{i-1}, \Delta_i, h_{i-1}, h_i), \qquad i = 1, \cdots, n-1, \tag{4}$$

where $h_j = x_{j+1} - x_j$, $j = 1, \cdots, n-1$. The conditions for an acceptable G-function now are that, for all S_1, S_2, and all positive h_1 and h_2:

A. $$G(S_2, S_1, h_2, h_1) = G(S_1, S_2, h_1, h_2).$$

[This makes the formula independent of the order of the data points.]

B. $$\min\,(S_1, S_2) \leqq G(S_1, S_2, h_1, h_2) \leqq \max\,(S_1, S_2).$$

[Thus the slope of the curve lies between the slopes of the two adjacent data segments. While not necessary for monotonicity, this condition seems intuitively reasonable.]

C. $$G(S_1, S_2, h_1, h_2) = 0 \text{ if } S_1S_2 \leqq 0.$$

[For $S_1S_2 = 0$, this is a necessary condition for monotonicity. For $S_1S_2 < 0$, this implies that the curve stays within the limits of the data.]

D. $$|G(S_1, S_2, h_1, h_2)| \leqq \min\,(3|S_1|, 3|S_2|).$$

[This insures that $(\alpha_i, \beta_i) = (d_i/\Delta_i, d_{i+1}/\Delta_i)$ lies inside the square $[0, 3]\times[0, 3]$.]

When coupled with appropriate boundary conditions, (4) with any acceptable G-function gives a monotone piecewise cubic interpolant to the given data which has all the properties we desire.

In the discussion of [1], Brodlie proposed a formula which can be written in the form of (4) with

$$G(S_1, S_2, h_1, h_2) = \begin{cases} \dfrac{S_1S_2}{\alpha S_2 + (1-\alpha)S_1} & \text{if } S_1S_2 > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

where

$$\alpha = \frac{1}{3}\left(1 + \frac{h_2}{h_1+h_2}\right) = \frac{h_1+2h_2}{3(h_1+h_2)}.$$

We observe that this G-function satisfies conditions A–D and it reduces to (2) when $h_1 = h_2$.

Much experimentation indicates that (4) and (5), when coupled with the boundary conditions in either [2] or [4], generally produce interpolants that are at least as "visually pleasing" as (1) and (3). Furthermore, it can be shown that for uniformly spaced data (5) gives an $O(h^2)$ approximation to $f'(x_i)$, whereas (3) is only $O(h)$.

In Figs. 1–4, we exhibit the curves produced by the four methods discussed here when applied to the data set Akima 3 of [4]. We conclude that the technique described

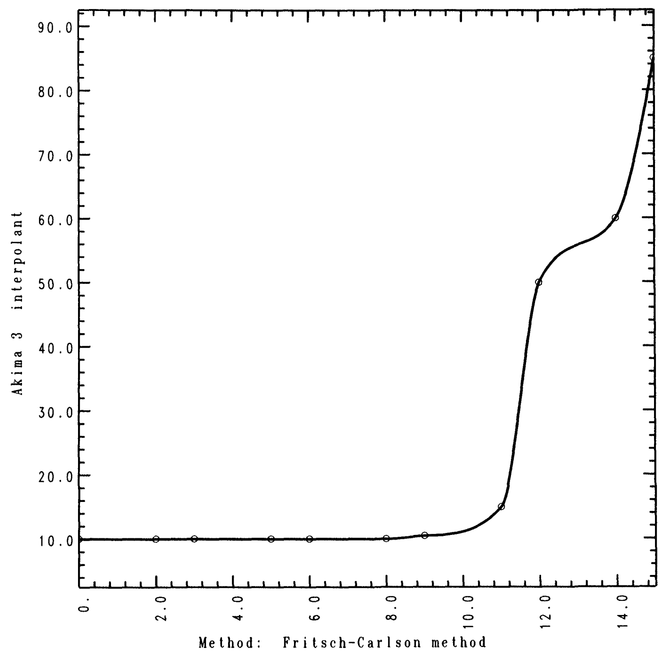

FIG. 1. *Result from the algorithm proposed in* [4] *when applied to the* Akima 3 *data.* (*This and all following examples used boundary derivatives computed by the standard noncentral three-point formula.*)

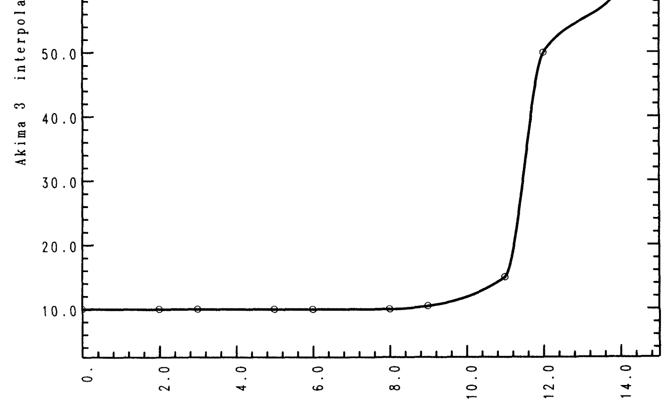

FIG. 2. *Result of applying formulas* (1) *and* (2) *to the* Akima 3 *data.*

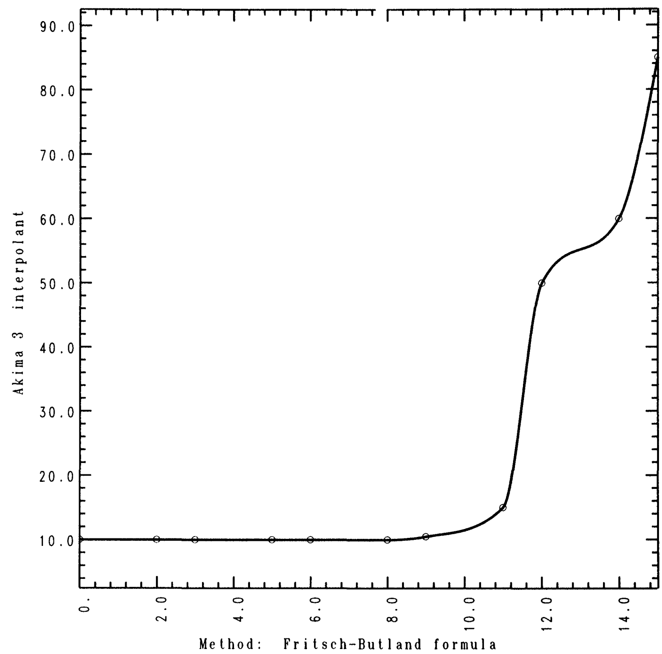

FIG. 3. *Result of applying formulas* (1) *and* (3) *to the* Akima 3 *data.*

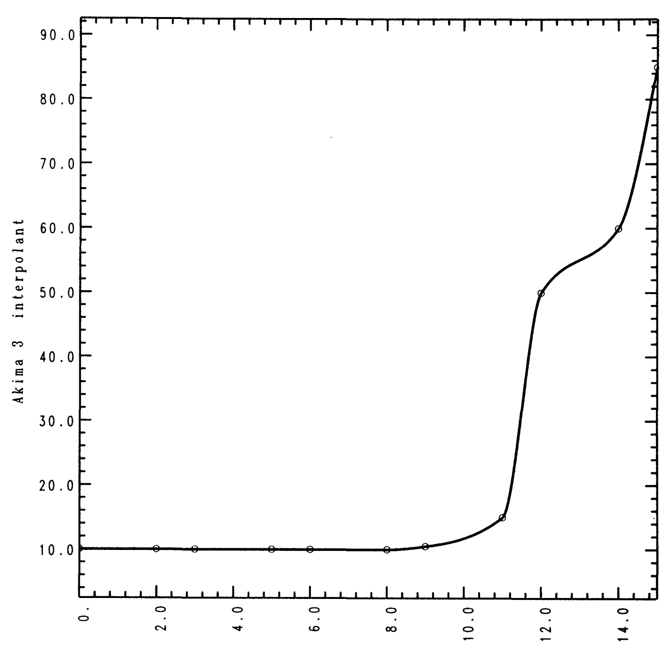

FIG. 4. *Result of applying formulas* (4) *and* (5) *to the* Akima 3 *data.*

here leads to a method for computing monotone piecewise cubic interpolants which is simple, symmetrical, and completely local. We remark that the method also produces reasonable results when applied to piecewise monotone data. Software implementing this algorithm may be obtained by writing to the first author.

Acknowledgment. The first author wishes to thank R. E. Carlson and J. M. Hyman for valuable discussions during the course of this work.

REFERENCES

[1] K. W. BRODLIE, *A review of methods for curve and function drawing*, in Mathematical Methods in Computer Graphics and Design, K. W. Brodlie, ed., Academic Press, London, 1980, pp. 1–37.

[2] J. BUTLAND, *A method of interpolating reasonable-shaped curves through any data*, Proc. of Computer Graphics 80, Online Publications Ltd., Northwood Hills, Middlesex, UK, 1980, pp. 409–422.

[3] F. N. FRITSCH AND J. BUTLAND, *An improved monotone piecewise cubic interpolation algorithm*, Rep. UCRL-85104, Lawrence Livermore Laboratory, Livermore, CA, October 1980.

[4] F. N. FRITSCH AND R. E. CARLSON, *Monotone piecewise cubic interpolation*, SIAM J. Numer. Anal., 17 (1980), pp. 238–246.