APPM 5600 — HOMEWORK # 4

1. (10 pts) Let x_0, x_1 be two successive points from a secant method applied to solving $f(x) = 0$ with $f_0 = f(x_0), f_1 = f(x_1)$. Show that regardless of which point x_0 or x_1 is regarded as the most recent point, the new point derived from the secant step will be the same.

2. (10 points) Determine whether the following sets of vectors are dependent or linearly independent:

   (a) (1,2,-1,3), (3,-1,1,1), (1,9,-5,11)

   (b) (1,1,0), (0,1,1), (1,0,1)

3. (10 points) Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k$ be linearly independent vectors in $\mathbb{R}^n$ and let A be a nonsingular $n \times n$ matrix. Define $\boldsymbol{y}_i = \mathsf{A}\boldsymbol{x}_i$ for $i = 1, 2, \ldots, k$. Show that $\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_k$ are linearly independent.

4. (10 points) Given the orthogonal vectors

   $$\boldsymbol{u}_1 = (1, 2, -1) \quad \boldsymbol{u}_2 = (1, 1, 3)$$

   produce a third vector $\boldsymbol{u}_3$ such that $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$. Normalize the vectors to create an orthonormal basis.

5. (20 points) Prove that similar matrices have the same eigenvalues and that there is a one-to-one correspondence of the eigenvectors.

6. (25 points) A matrix $\mathsf{A} \in \mathbb{R}^{n\times n}$ is <u>positive definite</u> if and only if $< \mathsf{A}\boldsymbol{x}, \boldsymbol{x} >> 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$; $x \neq 0$.
   Prove that if A is positive definite, then A is non-singular.

7. (15 points) Let M be any real $n \times n$ non-singular matrix and let $\mathsf{A} = \mathsf{M}^T\mathsf{M}$. Prove that A is positive definite.
   Fun fact: any positive definite matrix can be written in the form $\mathsf{A} = \mathsf{R}^*\mathsf{R}$ for some upper-triangular matrix R. This important factorization is Cholesky Decomposition.