APPM 5600 — HOMEWORK # 5

1. (15 points) Consider the problem of solving $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ where A is a nonsingular matrix. Determine an upper bound for the condition number of solving for $\boldsymbol{x}$ when there is a pertubation in $\boldsymbol{b}$.

2. (15 points) Consider solving $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ where $\mathsf{A} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1+10^{-10} & 1-10^{-10} \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The exact solution is $\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the inverse of A is $\begin{bmatrix} 1-10^{10} & 10^{10} \\ 1+10^{10} & -10^{10} \end{bmatrix}$. In this problem we will investigate a perturbation in $\boldsymbol{b}$ of $\begin{bmatrix} \Delta b_1 \\ \Delta b_2 \end{bmatrix}$ and the numerical effects of the condition number.

   (a) Find an exact formula for the change in the solution between the exact problem and the perturbed problem Δx.

   (b) What is the condition number of A?

   (c) Let Δb_1 and Δb_2 be of magnitude 10^{-5}. What is the relative error in the solution? How does the number of accurate digits match with the condition number of A?

3. (30 points) Let $\mathsf{A} \in \mathbb{R}^{n\times n}$ be a tridiagonal matrix where the diagonal entries are given by a_j for $j = 1, \ldots, n$, the lower diagonal entries are b_j for $j = 2, \ldots, n$ and the upper diagonal entries are c_j for $j = 1, \ldots, n-1$.

   (a) (10 points) For $n = 3$, derive the LU factorization of the matrix A.

   (b) (10 points) What is the extension of the LU factorization for general n?

   (c) (10 points) Theorem 8.2 in Atkinson states that Gaussian Elimination applied to a tridiagonal matrix satisfying certain diagonal-dominance conditions does not require pivoting. What is the operation count (give an exact formula) when applying Gaussian Elimination to a tridiagonal system without pivoting? **You must explain how you derive the operation count.**

4. (15 points) Consider the linear system

$$\begin{aligned} 6x + 2y + 2z &= -2 \\ 2x + 2/3y + 1/3z &= 1 \\ x + 2y - z &= 0 \end{aligned}$$

   (a) (3 points) Verify that $(x, y, z) = (2.6, -3.8, -5)$ is the exact solution.

   (b) (5 points) Using 4 digit floating point arithmetic with rounding, solve the system via Gaussian elimination without pivoting.

   (c) (5 points) Repart part (a) with partial pivoting.

   (d) (2 points) Which method is more accurate? i.e. stable.

(Remember to do the rounding to 4 significant digits as the machine would.)

5. (25 points) Consider the system $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ where

$$\mathsf{A} = \begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & -1 & 4 & -1 & 0 & -1 \\ -1 & 0 & -1 & 4 & -1 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix} \quad \text{and } \boldsymbol{b} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}.$$

Use the ones vector as $\boldsymbol{x}_0$. (i.e. $\boldsymbol{x}_0 = [1\ 1\ 1\ 1\ 1\ 1]^T$)

(a) (5 points) Use Gauss-Jacobi iteration to approximate the solution to this problem with $\epsilon = 1e - 7$.

(b) (5 points) Use Gauss-Siedel iteration to approximate the solution to this problem with $\epsilon = 1e - 7$.

(c) (5 points) Use SOR iteration with $\omega = 1.6735$ to approximate the solution to this problem with $\epsilon = 1e - 7$.

(d) (2 points) Which method converges faster? Do you expect this to always be true?

(e) (5 points) Set $c = \rho(\mathsf{B})$ (spectral radius). Use the following to error estimate to derive error bounds for the last computed approximations with all methods.

$$\|\boldsymbol{x}_{k+1} - \boldsymbol{x}\| \leq \frac{c}{1-c}\|\boldsymbol{x}_{k+1} - \boldsymbol{x}_k\|$$

(f) (3 points) What happens if you change the parameter ω for SOR?