APPM 5600 — HOMEWORK # 5

1. (15 points) Prove the following for $x \in \mathbb{C}^n$
   (a) (5 points) $\|x\|_\infty \leq \|x\|_1 \leq n\|x\|_\infty$
   (b) (5 points) $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty$
   (c) (5 points) $\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2$

2. (45 points)
   (a) (10 points) Let $\mathsf{A} \in \mathbb{R}^{n\times m}$ non-zero matrix with rank r. Write down the singular value decomposition of A. List the properties of the matrices you use in your decomposition.
   (b) (10 points) Show the $\mathbb{R}^m$ has an orthonormal basis $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m$, $\mathbb{R}^n$ has an orthonormal basis $\boldsymbol{u}_1, \dots, \boldsymbol{u}_n$ and there exists $\sigma_1 \geq \cdots \sigma_r \geq 0$ such that

$$\mathsf{A}\boldsymbol{v}_i = \begin{cases} \sigma_i \boldsymbol{u}_i & i = 1, \dots, r \\ 0 & i = r+1, \dots, m \end{cases}$$

$$\mathsf{A}^T\boldsymbol{u}_i = \begin{cases} \sigma_i \boldsymbol{v}_i & i = 1, \dots, r \\ 0 & i = r+1, \dots, n \end{cases}$$

   (c) (10 points) Agrue that

$$\begin{aligned} Range(\mathsf{A}) &= span\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_r\} \\ Null(\mathsf{A}) &= span\{\boldsymbol{v}_{r+1}, \dots, \boldsymbol{v}_m\} \\ Range(\mathsf{A}^T) &= span\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_r\} \\ Null(\mathsf{A}^T) &= span\{\boldsymbol{u}_{r+1}, \dots, \boldsymbol{u}_n\} \end{aligned}$$

   (d) (15 points) Now show that $Range(\mathsf{A}^T)$ is orthogonal to $Null(\mathsf{A})$.

3. (40 points)
   (a) (20 points) Let $\mathsf{A} \in \mathbb{R}^{n\times n}$ be nonsingular and $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$. Prove the following matrix identity (Sherman-Morrison).

$$(\mathsf{A} + \boldsymbol{u}\boldsymbol{v}^T)^{-1} = \mathsf{A}^{-1} - \frac{\mathsf{A}^{-1}\boldsymbol{u}\boldsymbol{v}^T\mathsf{A}^{-1}}{1 + \boldsymbol{v}^T\mathsf{A}^{-1}\boldsymbol{u}}$$

   (b) (10 points) Suppose that the LU factorization of A is available, e.g. because you computed it. Explain how the Sherman-Morrison identity can be used to solve the system $(\mathsf{A} + \boldsymbol{u}\boldsymbol{v}^T)\boldsymbol{x} = \boldsymbol{b}$.
   (c) (10 points) What is the operation count of the solution approach you made in part (b)? **NOTE:** The power of the Sherman-Morrison identity lies in this part. In some applications, one has to solve the linear system with many different $\boldsymbol{u}$ and $\boldsymbol{v}$. This identity saves from having to compute the dense inverse again.