APPM 5600 — HOMEWORK # 6

1. (15 points) In class we showed that update to the basis for the residual space is given by

$$\boldsymbol{p}_{k+1} = \boldsymbol{r}_{k+1} - \frac{< \boldsymbol{p}_k, \boldsymbol{r}_{k+1} >_{\mathsf{A}}}{\|\boldsymbol{p}_k\|_{\mathsf{A}}^2} \boldsymbol{p}_k. \tag{1}$$

   In this exercise, we will remove the need to apply A to create this update (less computing $\boldsymbol{r}_{k+1}$).

   (a) (6 points) Using the fact that $\boldsymbol{r}_{k+1} = \boldsymbol{r}_k - \alpha_k \mathsf{A} \boldsymbol{p}_k$ and $\boldsymbol{r}_{k+1}^T \boldsymbol{r}_k = 0$, show that $< \boldsymbol{p}_k, \boldsymbol{r}_{k+1} >_{\mathsf{A}} = -\frac{\|\boldsymbol{r}_{k+1}\|_2^2}{\alpha_k}$.

   (b) (6 points) Rewrite $\|\boldsymbol{p}_k\|_{\mathsf{A}}^2$ in terms of $\boldsymbol{r}_k$ and α_k.

   (c) (3 points) Plug these expressions into (1) to get a technique for evaluating the next basis vector for the residual space without any applications of the matrix A.

2. (50 points) Consider a sparse 500×500 matrix A constructed as follows.

   - Put a 1 in each diagonal entry.
   - In each off-diagonal entry put a random number from the uniform distribution on $[-1, 1]$ but make sure to maintain symmetry. Then replace each off-diagonal entry with $|a_{ij}| > \tau$ by 0, where τ is a parameter. For $\tau = 0.01, 0.05, 0.1$, and 0.2. (You should have 4 matrices A_j. One for each τ.)

   Take the right hand side to be a random vector $\boldsymbol{b}$ and set the tolerance to 10^{-10}.

   (a) (10 points) Write a Steepest Descent and Conjugate Gradient solver.

   (b) (10 points) Apply Steepest Descent to solve each of the linear systems and plot the residual for each iteration $\|\boldsymbol{r}_n\|$ versus the iteration n on a semilogy scale. Put all of these plots on the same graph. linear system.

   (c) (10 points) Apply Conjugate Gradient to solve each of the linear systems and plot the residual for each iteration $\|\boldsymbol{r}_n\|$ versus the iteration n on a semilogy scale. Put all of these plots on the same graph.

   (d) (10 points) What do you observe about the convergence of these methods? If the methods do not converge for any choices of τ explain what happening.

   (e) (10 points) How does the residual compare with the error bounds provided in class? (This is strictly comparing the numbers that you get.)

3. (15 points) Suppose Conjugate Gradient is applied to a symmetric positive definite matrix A with the result $\|\boldsymbol{e}_0\|_{\mathsf{A}} = 1$, and $\|\boldsymbol{e}_{10}\|_{\mathsf{A}} = 2 \times 2^{-10}$, where $\|\boldsymbol{e}_k\|_{\mathsf{A}} = \|\boldsymbol{x}_k - \boldsymbol{x}^*\|_{\mathsf{A}}$ and $\boldsymbol{x}^*$ is the true solution. Based solely on this data,

   (a) (7 points) What bound can you give on $\kappa(\mathsf{A})$?

   (b) (8 points) What bound can you give on $\|\boldsymbol{e}_{20}\|_{\mathsf{A}}$?

   Related note: Theorem 38.2 from Trefethen and Bau gives a slightly more interesting bound.

Thm 38.2 Let the conjugate gradient iteration be applied to a symmetric positive definite matrix problem $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$. If the iteration has not already converged (i.e. $\boldsymbol{r}_{n-1} \neq \mathbf{0}$), then $\boldsymbol{x}_n$ is the unique point in the Krylov space that minimizes $\|\boldsymbol{e}_n\|_{\mathsf{A}}$. The convergence is monotonic,

$$\|\boldsymbol{e}_n\|_{\mathsf{A}} \leq \|\boldsymbol{e}_{n-1}\|_{\mathsf{A}}$$

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4. (20 points) Consider the task of solving the following system of nonlinear equations.

$$f_1(x, y) = 3x^2 + 4y^2 - 1 = 0 \text{ and } f_2(x, y) = y^3 - 8x^3 - 1 = 0$$

for the solution $\boldsymbol{\alpha}$ near $(x, y) = (-0.5, 0.25)$.

(a) (10 points) Apply the fixed point iteration with $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{x} - \begin{bmatrix} 0.016 & -0.17 \\ 0.52 & -0.26 \end{bmatrix} \begin{bmatrix} 3x^2 + 4y^2 - 1 \\ y^3 - 8x^3 - 1 \end{bmatrix}$. You can use $(-0.5, 0.25)$ as the initial condition. How many steps are needed to get an approximation to 7 digits of accuracy?

(b) (10 points) Why is this a good choice for $\boldsymbol{g}(\boldsymbol{x})$?