3.6 i) We have that $\hat{a} = \mathbb{E}[a|\vec{y}]$

$$= \int_{-\infty}^{\infty} a\, p(a|\vec{y})\, da = \int_{-\infty}^{\infty} a \frac{p(\vec{y}|a)\, p(a)}{p(\vec{y})} d\vec{y}$$

$a \sim N(\langle a \rangle, \sigma_a^2)$ and $\vec{y}|a \sim N(a, \sigma_n^2 I)$. Now since $\vec{y} = a\mathbb{1} + \vec{n}$, $\langle \vec{y} \rangle = \langle a \rangle \mathbb{1}$ and

$$\Sigma_{yy} = \langle (\vec{y} - \langle y \rangle)(\vec{y} - \langle y \rangle)^* \rangle = \langle ((a - \langle a \rangle)\mathbb{1} + \vec{n})((a - \langle a \rangle)\mathbb{1} + \vec{n})^* \rangle$$

$= \sigma_a^2 \mathbb{1}\mathbb{1}^* + \sigma_n^2 I$, which is invertible so long as $\sigma_n^2 \neq 0$. Thus $\vec{y} \sim N(\langle a \rangle \mathbb{1}, \sigma_a^2 \mathbb{1}\mathbb{1}^* + \sigma_n^2 I)$, and the determinant of the covariance matrix is given by $\sigma_n^{2N} + \sigma_n^{2N}\left(\frac{N\sigma_a^2}{\sigma_n^2}\right) = \sigma_n^{2N}\left(\frac{\sigma_n^2 + N\sigma_a^2}{\sigma_n^2}\right) = |\Sigma_{yy}|$.

Then
$$\hat{a} = \int_{-\infty}^{\infty} \frac{a\,(2\pi)^{N/2}\sigma_n^N \sqrt{\frac{\sigma_n^2 + N\sigma_a^2}{\sigma_n^2}}}{(2\pi)^{N/2}\sigma_n^N \sigma_a \sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{a - \langle a \rangle}{\sigma_a}\right)^2\right] \exp\left[\frac{-1}{2\sigma_n^2} \sum_{i=1}^{N} (y_i - a)^2\right]$$
$$\exp\left[\frac{1}{2}(\vec{y} - \langle a \rangle \mathbb{1})^* \Sigma_{yy}^{-1} (\vec{y} - \langle a \rangle \mathbb{1})^*\right] da$$

$$= \int_{-\infty}^{\infty} \frac{a}{\sqrt{2\pi}\sqrt{\frac{\sigma_n^2\sigma_a^2}{N\sigma_a^2 + \sigma_n^2}}} \exp\left[-\frac{1}{2}\left(\left(\frac{a - \langle a \rangle}{\sigma_a}\right)^2 + \frac{1}{\sigma_n^2}\sum_{i=1}^{N}(y_i - a)^2 - (\vec{y} - \langle a \rangle \mathbb{1})^* \Sigma_{yy}^{-1} (\vec{y} - \langle a \rangle \mathbb{1})^*\right)\right] da$$

We wish to write that $a|\hat{y} \sim N(\mu, s^2)$. Clearly then we need $s^2 = \frac{\sigma_n^2 \sigma_a^2}{N\sigma_a^2 + \sigma_n^2}$ and that, since this is true when $a = 0$,

$$\frac{\mu^2}{s^2} = \frac{\langle a \rangle^2}{\sigma_a^2} + \vec{y}^* \Sigma_{nn}^{-1} \vec{y} - (\vec{y} - \langle a \rangle \mathbb{1})^* \Sigma_{yy}^{-1} (\vec{y} - \langle a \rangle \mathbb{1})^*.$$

Here we can apply the Sherman-Morrison formula: Since $\Sigma_{yy} = \Sigma_{nn} + \sigma_a^2 \mathbb{1}\mathbb{1}^*$,

$$\Sigma_{yy}^{-1} = \Sigma_{nn}^{-1} - \frac{\sigma_a^2 \Sigma_{nn}^{-1} \mathbb{1}\mathbb{1}^* \Sigma_{nn}^{-1}}{1 + \sigma_a^2 \mathbb{1}^* \Sigma_{nn}^{-1} \mathbb{1}}$$

and so we find that