ECEN 5244 - Stochastic/Environmental Signal Processing

Homework #2

Issued 9/21/23
Due 10/12/23

2.1 (a) The sample variance $\underline{\sigma}_N^2$ of a zero-mean Gaussian random variable y_i with actual variance σ^2 is:

$$\underline{\sigma}_N^2 = \frac{1}{N-1}\sum_{n=1}^{N} y_i^2$$

This sample variance approaches the true variance σ^2 in the limit as $N \to \infty$.

Note that the sample variance summation is essentially equal to that of the χ^2 metric for a perfect N-point single-parameter (i.e., $M = 1$) fit to the $\{y_i\}$ for which the measurement errors are each independent zero mean and identially distributed random variables of STD σ:

$$\chi^2 = \sum_{n=1}^{N}\left(\frac{y_i - 0}{\sigma}\right)^2$$

Given the above, use the fact that the variance of the χ^2 metric for a "good" fit is $2\nu = 2(N-M)$ to show that the standard deviation of the error in the sample variance estimate is:

$$\sigma_{\underline{\sigma}_N^2} = \sqrt{\frac{2}{N-1}}\sigma^2$$

(b) Using Fisher's expression for the error variance of a sample correlation coefficient, show that if the errors in the correlation coefficient estimates of a sample autocorrelation function are themselves uncorrelated, then the following lemma holds:

$$E\left[\left|\overline{U}_k\, \overline{\overline{w}}\, \delta\overline{\rho}_x\right|^2\right] = \sum_{n=-N+4}^{N-4} w_n^2 \frac{\left(1-\rho_{x_n}^2\right)^2}{N-3-|n|}$$

where the quantities in the above expression are defined in the slides for Lecture #8. Here, N is the length of the sample sequence $\{x_i\}$ (i.e., $i = 0 \ldots N-1$), and assume that the last three autcorrelation lag samples are zeroed due to excessive noise. Consider using appropriate approximations and use of Fisher's standard deviation for the error in the sample correlation coefficient estimate.

2.2 Consider the associated .mat or .txt file (HW2_2.xxx, both available on Canvas) containing 2^{10} length vectors of frequency (ω) and complex transfer function data ($H = H_R + jH_I$) in the following form: $\{\omega, H_R, H_I\}$.

(A) Using transfer function system identification techniques, estimate the poles and zeros of the underlying transfer function for ths data. How many poles and zeros are required to model this data?

(B) Now, add random complex noise with successively higher standard deviation values to the measured TF data (e.g.):

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H = H + sigma*(randn(size(H))+j*randn(size(H)))/sqrt(2);
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then repeat your estimation process. At what value of noise STD does your estimate of the poles and zeros break down?

(C) Can you suggest any means of improving the robustness of your pole/zero estimates in the presence of measurement noise?

(D) Which goodness of fit metric (χ^2 or Pearson's sample correlation coefficient $\underline{\rho}_N$) is a more useful measure of the validity of this transfer function model?

2.3 Consider the associated .mat or .txt file (HW2_3.xxx, both available on Canvas) containing 2^{16} length vectors of time (in seconds) and sampled signal data.

(A) Using a suitable data window, implement a Welch periodogram spectral estimate of this data.

(B) Using a suitable data window, implement a Blackman-Tukey spectral estimate of this data.

(C) Using a suitable all-pole model, implement an autoregressive spectral estimate of this data.

(D) Summarize your results from the parts (A-C) into a table quantitatively describing the essential signals and/or spectral behavior found and the estimated accuracy of your descriptors. How many distinct signals or noise processes are there in the sample?

2.4 An infrared Fourier transform spectrometer based on a Mach-Zender interferometer is required to have a spectral resolution of $\Delta f = 1$ GHz at an IR wavelength of $\lambda = 3\ \mu$m in order to resolve atmospheric spectral lines. Further, it must provide radiances that are accurate to within 1%.

From the above, determine the allowable tolerance on the mechnical components used to generate the optical delay in the MZ interferometer, and the length over which such tolerance must be maintained. (Hint: Recall that the FT spectrometer measures the autocorrelation function of an optical signal by a controlled optical delay between two inteferometer paths, then use your knowledge of the effects of sampling grid error on measured spectra to determine the required precision in the optical delay line components.)

2.5 (A) Determine the autocorrelation functions for a white noise signal passed through the following filters:

(A) A first-order Butterworth low pass filter of time constant τ.

(B) An ideal bandpass filter of bandwidth B and center frequency f_o, where $f_o > B/2$.

2.6 A band-limited zero-mean random voltage signal of power 1 mW into a 50 Ω resistance is sampled at it's Nyquist rate using a 12-bit A/D converter.

(A) Determine the voltage range of the A/D converter assuming that the discretization noise variance in the digitized signal is no greater than 1% of the signal variance.

(B) In part (A) the spectrum of the discretized noise is impicitly presumed to be flat (i.e., the discretization noise is presumed to be white). Can you think if a situation in which the discretization noise spectrum would not be white? Describe in enough detail to justify your assertion.