Given this, we have that, letting $\bar{y} = \frac{Z}{N}$,

$$y = \sqrt{\langle a\rangle^2\left(\frac{\sigma_n^2 - N\sigma_a^2}{\sigma_n^2 + N\sigma_a^2}\right) + \frac{2N\sigma_a^2\langle a\rangle\bar{y}}{N\sigma_a^2+\sigma_n^2} + \frac{N^2\sigma_a^4(\bar{y}-\langle a\rangle)^2}{(\sigma_n^2+N\sigma_a^2)^2}}$$

Now $\hat{a} = \int_{-\infty}^{\infty} \frac{a}{\sqrt{2\pi s^2}} \exp\left(\frac{-1}{2s^2}(a-y)^2\right) da$, so

$\hat{a}_{SE} = y$

ii) Now we have from the notes that given a linear process with Gaussian noise, $\hat{a}_{SE} = \hat{a}_{MAP}$, thus $\hat{a}_{MAP} = y$ as found in part i.

iii) The mean-square error is $\mathbb{E}\left[(\hat{a}_{SE} - a)^2\right]$

$$= \frac{1}{\sqrt{2\pi s^2}} \int_{-\infty}^{\infty} (a-y)^2 \exp\left(\frac{-1}{2s^2}(a-y)^2\right) da = s^2$$

$$= \frac{\sigma_n^2\sigma_a^2}{\sigma_n^2 + N\sigma_a^2}$$

iv) $\left(\lim_{\sigma_a^2\to\infty} y\right)^2 = \lim_{\sigma_a^2\to\infty} y^2$ by continuity, so we have that

$$\lim_{\sigma_a^2\to\infty} y^2 = \lim_{\sigma_a^2\to\infty}\left(\langle a\rangle^2\left(\frac{\sigma_n^2 - N\sigma_a^2}{\sigma_n^2+N\sigma_a^2}\right) + \frac{2N\sigma_a^2\langle a\rangle\bar{y}}{N\sigma_a^2+\sigma_n^2} + \frac{N^2\sigma_a^4(\bar{y}-\langle a\rangle)^2}{(\sigma_n^2+N\sigma_a^2)^2}\right)$$

$$= -\langle a\rangle^2 + 2\langle a\rangle\bar{y} + (\bar{y}-\langle a\rangle)^2$$

$$= -\langle a\rangle^2 + 2\langle a\rangle\bar{y} + \bar{y}^2 - 2\langle a\rangle\bar{y} + \langle a\rangle^2 = \bar{y}^2,$$ thus

$\lim_{\sigma_a^2\to\infty} y = \lim_{\sigma_a^2\to\infty} \hat{a}_{SE} = \bar{y} = \frac{1}{N}\sum_{i=1}^{N} y_i$ which is the sample mean.