3.1 i) We now suppose that the calibration stimuli are $X_i+\delta_i$, where δ_i has standard deviation σ_X. If we assume additionally that there is noise in the measurements of V_{OK}, call the noise n_K, we find that

$$\hat{m}-m = \left[\frac{X_2-X_1}{X_2-X_1+\delta_2-\delta_1+m(n_2-n_1)}\right]m - m$$

$$= \frac{-m(\delta_2-\delta_1)-m^2(n_2-n_1)}{X_2-X_1+(\delta_2-\delta_1)+m(n_2-n_1)}$$

$$= -m\left[\frac{\delta_2-\delta_1}{\Delta x+\delta_2-\delta_1+m(n_2-n_1)} + \frac{m(n_2-n_1)}{\Delta x+\delta_2-\delta_1+m(n_2-n_1)}\right]$$

Now Taylor Expanding each term, this becomes, following the notes,

$$\hat{m}-m \simeq -\frac{m(\delta_2-\delta_1)}{X_2-X_1} - \frac{m^2(n_2-n_1)}{X_2-X_1}.$$ Now if all noises are uncorrelated, we can calculate that

$$\mathbb{E}[(\hat{m}-m)^2] = \mathbb{E}\left[m^2\left(\frac{\delta_2-\delta_1}{X_2-X_1}+\frac{m(n_2-n_1)}{X_2-X_1}\right)^2\right]$$

$$= m^2\,\mathbb{E}\left[\frac{(\delta_2-\delta_1)^2}{(X_2-X_1)^2} + \frac{2m(n_2-n_1)(\delta_2-\delta_1)}{(X_2-X_1)^2} + m^2\left(\frac{n_2-n_1}{X_2-X_1}\right)^2\right]$$

The n_K are uncorrelated with each other and with the δ_K, which are uncorrelated, so this becomes

$$m^2\left[\frac{2\sigma_X^2}{(X_2-X_1)^2} + \frac{2\Delta x^2 m^2}{m^2(X_2-X_1)^2}\right]$$ since $\mathbb{E}[n_K^2] = \frac{\Delta x^2}{m^2}$, and we find

$$\mathbb{E}[(\hat{m}-m)^2] = \frac{2m^2}{(X_2-X_1)^2}\left[\sigma_X^2+\Delta x^2\right]$$ and the effective RMS error is

thus $\sqrt{\sigma_X^2+\Delta x^2}$.