ECEN 5244 - Stochastic/Environmental Signal Processing

Homework #3

Issued 10/12/23
Due 11/14/23

3.1 In this problem we consider the impact of random error in the determination of the two standards (or, stimuli) used for calibration. Extend the analysis of Lecture 13 on instrument calibration to derive an expression for the measurement error standard deviation, Δx, that accounts for random errors of standard deviation σ_x in the otherwise presumably known calibration stimuli x_1 and x_2. (In other words, how do errors in the calibration source values affect the measurement error Δx ?)

(i) Find the error standard deviation in the case that the errors in x_1 and x_2 are uncorrelated ($\rho = 0$).

(ii) Find the error standard deviation in the case that the errors in x_1 and x_2 are completely correlated $(\rho = 1)$.

(iii) Find the error standard deviation in the case that the errors in x_1 and x_2 are completely anticorrelated $(\rho = -1)$.

3.2 Prove that the matrix $\overline{\overline{R}}_{xy}\overline{\overline{R}}_{yy}^{-1}\overline{\overline{R}}_{yx}$ is positive semidefinite regardless of the length of the vectors $\overline{x}$ and $\overline{y}$. This is the matrix used to determine the error covariance for the LMMSE estimator.

3.3 Consider the associated .mat or .txt file (HW3_3.xxx, both available on Canvas) containing a time segment of actual observed multichannel data matrix in the following form $\{\overline{y}_1, \overline{y}_2, \ldots, \overline{y}_k, \ldots\}$, where each of the the $\overline{y}_k$ are column vectors associated with one of 6 spectral channels.

(i) Compute and render the covariance and correlation matrices for the measured data. Are the data correlated across channels?

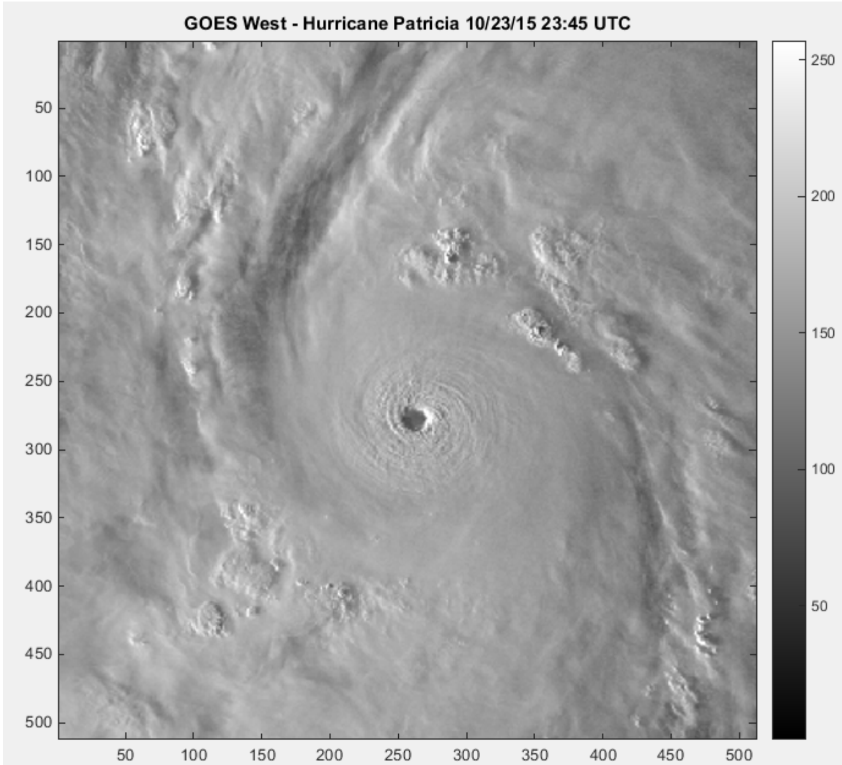
(ii) Find the principal spectral eigenvectors and eigenvalues of this data. Plot the three most dominant eigenmodes.

(iii) If the measurement noise is white and has a standard deviation of 6 (in the units of the data as provided) how many observable degrees of freedom are there in the data set?

(iv) Plot these most energetic principal components for these degrees of freedom as a function of time.

(v) Render scatter plots of the first three dominant principal components. Are all of these components mutually (a) uncorrelated, and (b) stastically independent? Discuss.

3.4 Consider the the associated .mat file (HW3_4.mat, available on Canvas) containing a 2-dimensional visible satellite image of Hurricane Patricia, as observed by the NOAA GOES West satellite:



The axes units are pixel numbers. The data elements in these files are as follows:

d	Raw 512x512 image
dc	Blurred 512x512 image
dpn	Raw 512x512 image with additive noise (SNR=3dB)
$dcpn$	Blurred 512x512 image with additive noise (SNR=3dB)
Rxx_est	Estimated process radial autocovariance
$sigman$	Additive noise STD
$pattern$	Sensor blurring pattern (Gaussian)

The radial autocovariance is a model estimate of $R_{xx}(r)$, where $r = |\overline{x}_1 - \overline{x}_2|$ is the separation distance between pixels pairs at locations $\overline{x}_1$ and $\overline{x}_2$. This covariance excludes the average (DC) component of the signal so that $R_{xx}(r \to \infty) = 0$.

(i) From the data provided determine the optimal 2-dimensional non-casual Wiener filter to optimally smooth the noisy image contained in the matrix *dpn*. Apply this filter to the noisy image and calculate the resulting signal to noise ratio using the raw (uncorrupted) image. How does this resulting SNR compare to the noise variance and to the process variance?

(ii) Vary the presumed noise level from that determined by the SNR of 3 dB and see if you can find a more optimal filter.

(iii) Repeat parts (i) and (ii) to find and study the 2-D non-causal WF to optimally deconvolve the blurred and noisy image in the matrix *dcpn*. The sensor blurring function is a Gaussian pattern contained in the array variable *pattern*.

Sample Matlab code for displaying and manipulating these images is provided below:

```
%To display an image...
image(d);
colormap(repmat(linspace(0,1,256)',1,3));
colorbar
title('GOES West - Hurricane Patricia 10/23/15 23:45 UTC');

%To compute the process spectrum...
Rxx=zeros(512,512);
ixm=257; iym=257;
for iix=1:512
    for iiy=1:512
        Rxx(iix,iiy)=Rxx_est(min([round(sqrt((iix-ixm)^2+(iiy-iym)^2)),255])+1);
    end
end
Rxx=Rxx*10*round(Rxx(257,257)/10)/Rxx(257,257);
Sxx=fftshift(abs(fft2(Rxx)));
Sxx=Rxx_est(1)*Sxx./sum(sum(Sxx));

%To display a spectrum...
imagesc(10*log10(Sxx));
colormap('jet(256)');
title('GOES Hurricane Imagery Spectrum (256-level)');
colorbar
```

Be careful in using "fftshift" and "ifftshift", in properly treating the DC part of the image, and in scaling your variables.

3.5 Consider a white noise process driving a first order Butterworth filter of time constant τ, then sampled with period T and digitally averaged using a sliding window over N successive samples (see slides in Lecture 12 on low pass filtering and averaging).

$$\underline{v}_i = \frac{1}{N} \sum_{n=i}^{i+N-1} v\left(nT\right)$$

(a) First, find an expression for the correlation coefficient between any two samples $v\left(jT\right)$ and $v\left(kT\right)$.

Now, find the correlation coefficient between digitally averaged samples for the following two cases:

(b) Averaged samples $\underline{v}_i$ from a set shifted by a time T. That is, determine the correlation coefficient between $\underline{v}_i$ and $\underline{v}_{i+1}$.

(c) Averaged samples $\underline{v}_i$ from a set shifted by a time NT. That is, determine the correlation coefficient between $\underline{v}_i$ and $\underline{v}_{i+N}$.

3.6 (after Van Trees, Problem 2.4.5) A set of N measurements of a Gaussian random variable a, corrupted by additive Gaussian measurement noise n_i, are made:

$$y_i = a + n_i \qquad\qquad i = 1 \ldots N$$

The random variable a has a Gaussian probability density with standard deviation σ_a, and the measurement noise values are independent zero-mean random variables with standard deviation σ_n.

(i) Find the minimum mean square estimator $\widehat{a}_{SE}$.

(ii) Find the maximum a posteriori estimator $\widehat{a}_{MAP}$.

(iii) Find the mean square error using the above estimators.

(iv) Assuming that the variable a is nonrandom (i.e., no a priori information is available on a,which is equivalent to $\sigma_a \to \infty$), find the maximum likelihood estimator $\widehat{a}_{ML}$.