1 A) i) $\mathbb{E}[x^3] = \int_{\mathbb{R}} \frac{x^3}{\sqrt{2\pi\sigma_x^2}} \exp\left(-\frac{1}{2\sigma^2} x^2\right) dx$

$= \lim_{R\to\infty} \int_{-R}^{R} \frac{x^3 \exp\left(-\frac{1}{2}\left(\frac{x}{\sigma_x}\right)^2\right)}{\sqrt{2\pi\sigma_x^2}} dx$. Since the integrand

is an odd function this integral is $0 \ \forall R$,

Therefore $\mathbb{E}[x^3] = \lim_{R\to\infty} \int_{-R}^{R} \frac{x^3}{\sqrt{2\pi\sigma_x^2}} \exp\left(-\frac{1}{2}\left(\frac{x}{\sigma_x}\right)^2\right) dx = 0$

ii) $\mathbb{E}[x^4] = \frac{1}{\sqrt{2\pi\sigma_x^2}} \int_{\mathbb{R}} x^4 \exp\left[\frac{-1}{2\sigma_x^2} x^2\right] dx = \frac{2}{\sqrt{2\pi\sigma_x^2}} \int_0^\infty x^4 \exp\left(\frac{-1}{2\sigma_x^2} x^2\right) dx.$

Let $z = \frac{x^2}{2\sigma_x^2}$, then $dz = \frac{x}{\sigma_x^2} dx$ and the integral

becomes $\frac{2}{\sqrt{2\pi\sigma_x^2}} \int_0^\infty x^4 \left(\frac{\sigma_x^2}{x}\right) \exp(-z) dz$

$= 2\sqrt{\frac{\sigma_x^2}{2\pi}} \int_0^\infty x^3 \exp(-z) dy \quad \to x = \sqrt{2\sigma_x^2 z}$

$= 2\sqrt{\frac{\sigma_x^2}{2\pi}} \int_0^\infty (2\sigma_x^2)^{3/2} z^{3/2} \exp(-z) dz$

$= \frac{4}{\sqrt{\pi}} \sigma_x^4 \int_0^\infty z^{3/2} \exp(-z) dz = \frac{4\sigma_x^4}{\sqrt{\pi}} \Gamma(5/2)$ so

$\boxed{\mathbb{E}(x^4) = 3\sigma_x^4}$ as desired, as $\Gamma(5/2) = \frac{3}{4}\sqrt{\pi}$

iii) $\mathbb{E}[x^6] = \frac{1}{\sqrt{2\pi\sigma_x^2}} \int_{\mathbb{R}} x^6 \exp\left(-\frac{x^2}{2\sigma_x^2}\right) dx = \sqrt{\frac{2}{\pi\sigma_x^2}} \int_0^\infty x^6 \exp\left(-\frac{x^2}{2\sigma_x^2}\right) dx$

Again let $z = \frac{x^2}{2\sigma_x^2}$, then $dz = \frac{x}{\sigma_x^2} dx$ and the integral

becomes $\sqrt{\frac{2\sigma_x^2}{\pi}} \int_0^\infty (2\sigma_x^2)^{5/2} z^{5/2} \exp(-z) dz$

$= \frac{8\sigma_x^6}{\sqrt{\pi}} \Gamma(7/2) = 15\sigma_x^6$ since $\Gamma(7/2) = \frac{15\sqrt{\pi}}{8}$

So $\boxed{\mathbb{E}[x^6] = 15\sigma_x^6}$

iv) $\mathbb{E}[x^2y^4]$: We apply Isserlis' Theorem:

$\mathbb{E}[x^2y^4] = \sum \prod \mathbb{E}[X_iX_j]$. This sum has $5!! = 15$ terms. Only one of these terms will be of the form $\mathbb{E}[x^2]\mathbb{E}[y^2]^2$, and then for each other term, we have the form $\mathbb{E}[xy]^2\mathbb{E}[y^2]$. So, we can write that

$\mathbb{E}[x^2y^4] = \mathbb{E}[x^2]\mathbb{E}[y^2]^2 + 14\mathbb{E}[xy]^2\mathbb{E}[y^2]$

$= \sigma_x^2\sigma_y^4 + 14\sigma_y^2[\rho\sigma_x\sigma_y]^2 = \sigma_x^2\sigma_y^4(1+14\rho^2)$

Thus $\boxed{\mathbb{E}[x^2y^4] = \sigma_x^2\sigma_y^4(1+14\rho^2)}$