Estimator for the Standard Deviation of Wind Direction Based on Moments of the Cartesian Components

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ABSTRACT

Mean and variance of horizontal wind direction are defined in a minimal variance sense. Starting from a theoretical model, termed anisotropic Gaussian model, of the probability density functions of the Cartesian wind components, the standard deviation of wind direction is calculated. This angular standard deviation is expressed as a function of the means and variances of the Cartesian wind components alone. These latter moments can be obtained by single-pass methods from measured data. Additionally, a parameterization of the angular standard deviation as a function of persistence of wind direction is given. A comparison with other estimation methods is made and some applications to experimental data are given.

1. Introduction

The definition of mean and standard deviation of horizontal wind direction is a long-standing problem in meteorology and climatology, whereby the principal difficulties come from the fact that the wind direction is a circular variable. This fact makes the development of single-pass estimators, which require only the performance of additions during data acquisition and not the storage of the individual data, a much more difficult task than in the case of linear variables. A large amount of mathematical literature is concerned with the statistical description of such circular variables (see Mardia 1972; Batschelet 1981; Upton and Fingleton 1989).

In this paper, we follow the pragmatic rules given in Fisher (1987) of how to construct estimates of the angular standard deviation. First, what purpose or interpretation our angular standard deviation shall serve is specified, and, according to that, an appropriate definition of angular standard deviation is chosen and methods to estimate it are given. We will develop a method to handle short time measurements (a few seconds apart), which are averaged over a certain sampling time of a few minutes to 1 h. The standard deviation of wind direction on such time scales is of great importance, for example, in the determination of the atmospheric stability class (Houghton 1985, p. 502) or in models of dry deposition (Hicks et al. 1987).

The angular distribution function on such short time scales is supposed to be unimodal and therefore allows a meaningful definition of standard deviation (Fisher 1987). If measurements over a longer time period, days or even years, are considered the distribution of angle will often have two or more local maxima. In such cases the definition of a single mean and standard deviation becomes meaningless. The first thing to do with such data is to inspect a graph of the angular distribution and to form appropriate classes of angles.

Section 2 gives some definitions of the important statistical quantities, especially our definition of the angular standard deviation. In section 3, we introduce a special model of the fluctuations of wind components which allows us to calculate the angular standard deviation. Section 4 describes how the application of the scheme can be done in practice: two expressions for the angular standard deviation are given, one in terms of the moments of Cartesian wind components, the other in terms of the persistence of the wind direction. Section 5 compares our method with other single-pass estimators. Section 6 gives some results of applications to measured data.

2. Definitions and basic properties of circular variables

Before describing the special properties of circular variables, some notations of statistics for linear random variables are reviewed. Given a random variable X, which may take any real value x and whose probability distribution is described by a probability density function (pdf) $p_X(x)$ having the following properties,

$$p_X(x) \geq 0 \text{ for all } x, \quad \int_{-\infty}^{\infty} p_X(x)dx = 1. \quad (1)$$

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Then the mean μ_X and the variance $\sigma_X{}^2$ of the random variable X can be defined in the usual way [see, for example, Taupin (1988)] as

$$\mu_X = \int_{-\infty}^{\infty} x p_X(x) dx,$$

$$\sigma_X{}^2 = \int_{-\infty}^{\infty} (x - \mu_X)^2 p_X(x) dx, \qquad (2)$$

where μ_X is the first moment and $\sigma_X{}^2$ the second centered moment of the linear random variable.

In a typical experimental situation, the pdf $p_X(x)$ is not known, and it would be a quite complicated procedure to extract an approximation of it from the measured data. However, an unbiased estimate of the mean and the variance based on a given sequence $X_1, \ldots, X_N$ of measured data may be obtained by

$$\bar{x} = \frac{1}{N} \sum_{i=1}^{N} X_i, \quad s_X{}^2 = \frac{1}{N-1} \sum_{i=1}^{N} (X_i - \bar{x})^2, \quad (3)$$

the so-called sample mean and variance. These definitions cannot be applied as simply as that to random variables such as the wind direction. The fundamental difficulties in calculating the mean and any other higher centered moment of the wind direction come from the fact that the angle is a circular (or periodic) variable, i.e., any physically meaningful function of the variable is periodic. If θ is a circular variable with period T (in the case of an angle $T = 2\pi$) then its pdf is a periodic function of the variable θ, and

$$p_\Theta(\theta) = p_\Theta(\theta + T) \text{ for all } \theta. \qquad (4)$$

This means that it is sufficient to map the circular random variable Θ, which may again take any real value θ, into an arbitrary positioned interval of length T by a suitable addition of integer multiples of the period T, for example, by virtue of the function

$$\left.\begin{array}{ll} \hat{\theta} = \theta_c + (\theta - \theta_c) \bmod T, & \theta \geqslant \theta_c \\ \hat{\theta} = \theta_c + T + (\theta - \theta_c) \bmod T, & \theta < \theta_c \end{array}\right\}, \qquad (5)$$

where θ_c is a cut point or branch point. This cut point itself is also only determined up to integer multiples of the period T.

The pdf is normalized in a slightly different way than in the case of linear variables, namely by

$$\int_0^T p_\Theta(\theta) d\theta = 1. \qquad (6)$$

In this normalization integral the absolute position of the integration interval is not important because the pdf $p_\Theta(\theta)$ is periodic; the interval has to be exactly one period T long.

If one tries to write the mean of an angle (a circular variable with period $T = 2\pi$) similar to the definition (2), which is used in the case of a linear variable, it is clear from the resulting expression

$$\mu_\theta = \int_{\theta_c}^{\theta_c + 2\pi} \theta p_\Theta(\theta) d\theta \qquad (7)$$

that the value of the so-defined mean μ_θ strongly depends on the position of the cut point θ_c. This remark is also true for the sample mean given analogously to the linear case (3) by

$$\bar{\theta} = \frac{1}{N} \sum_{i=1}^{N} \hat{\Theta}_i. \qquad (8)$$

As an example, suppose two angles, 30° and 350°, are measured; then the simple arithmetic mean of these values is $\bar{\theta} = 190°$, but an angle of 350° is equivalent to one of $-10°$. With that value, the mean of 30° and $-10°$ becomes $\bar{\theta} = 10°$. Here, it is quite obvious which value of the mean is more reasonable. In the case of more than two data points, however, this is not as obvious as is outlined in Essenwanger (1985).

An additional condition must be chosen that fixes the cut point in a certain way. One possibility of fixing the cut point is to impose the condition that the variance of θ, or equivalently its square root, the standard deviation, becomes minimal where the angular variance $\sigma_\theta{}^2$ is defined as

$$\sigma_\theta{}^2 = \int_{\theta_c}^{\theta_c + 2\pi} [\theta - \mu_\theta(\theta_c)]^2 p_\Theta(\theta) d\theta, \qquad (9)$$

a method described for discrete measured signals in Essenwanger (1985). One necessary condition for the angular variance $\sigma_\theta{}^2(\theta_c)$ to be a minimum is that it is stationary with respect to variations of the cut point; that is, $\partial\sigma_\theta{}^2(\theta_c)/\partial\theta_c = 0$ must hold, which leads to the following equation for the cut point,

$$[\pi + \theta_c - \mu_\theta(\theta_c)] p_\Theta(\theta_c) = 0. \qquad (10)$$

Unless the pdf $p_\Theta(\theta)$ becomes zero, this stationarity condition is equivalent to

$$\theta_c = \int_{\theta_c}^{\theta_c + 2\pi} \theta p_\Theta(\theta) d\theta - \pi, \qquad (11)$$

which is an implicit equation for the cut point. Having determined all values of θ_c that satisfy the stationarity condition (10), the one with the smallest variance must be chosen. However, there may exist several values of the cut point leading to the same minimal variance and consequently to different values of the mean. In contrast to the mean of a linear variable, which is uniquely defined by (2), the mean of a circular variable can thus be nonunique. As an example, we consider an angle, that takes in half the cases the value $\theta = 0$ and in the other half $\theta = \pi$; the pdf is, thus, bimodal. The minimal variance principle leads to either $\mu_\theta = \pi/2$ or $\mu_\theta = 3\pi/2$. Of course it is quite problematic to define a mean of such a multimodal distribution.

In general the solution of (11) and the determination of the zeros of the pdf are rather difficult. However, the calculations can be easily carried out if the pdf is symmetric about some value θ_s, that is, if it satisfies

$$p_\Theta(\theta_s + \theta) = p_\Theta(\theta_s - \theta) \text{ for all } \theta. \qquad (12)$$

In this case the best choice for the cut point is either one of the two values $\theta_c = \theta_s$, $\theta_c = \theta_s - \pi$ or a zero of the pdf $p_\Theta(\theta)$. Consider two examples of such symmetric distributions of an angle θ.

a. Example 1: Uniform distribution

In this case the pdf is constant $p_\Theta(\theta) = 1/(2\pi)$. As the distribution is rotational invariant, the mean is not defined; for all values of the cut point θ_c, the variance becomes identical and minimal. The mean results are $\mu_\theta = \theta_c + \pi$. The standard deviation then becomes $\sigma_\theta = \pi/3^{1/2} \triangleq 103.9°$, a value denoted as a maximal observable standard deviation in Yamartino (1984). However, there exist distributions with larger standard deviations as is shown in the next example.

b. Example 2: Symmetric distribution with maximal variance

Let the probability of finding a certain angle be defined in the following way,

$$\text{prob}(\theta) = \begin{cases} \frac{1}{4} + \delta, & \theta = \pm\epsilon \\ \frac{1}{4} - \delta, & \theta = \pi \pm \epsilon, \\ 0, & \text{else,} \end{cases} \qquad (13)$$

in which there exist only nonzero contributions to the pdf from four discrete values of the angle. The cut point that minimizes the variance is then $\theta_c = -\pi$, and the standard deviation becomes $\sigma_\theta = (\pi^2/2 - \pi\epsilon - 2\pi^2\delta + 4\pi\delta\epsilon + \epsilon^2)^{1/2}$. For very small δ and ϵ, this expression approaches $\sigma_{\max} = \pi/2^{1/2} \triangleq 127.3°$. If the two parameters δ and ϵ are exactly zero, then the value of the cut point jumps to $\theta_c = -\pi/2$ (or $\theta_c = \pi/2$), and the standard deviation becomes $\sigma_\theta = \pi/2$. Although there is a mean defined with the minimum variance principle, it is another question whether this mean is a useful quantity to describe the distribution law (13), which mimics a bimodal distribution.

This method of minimal variance can now be used to get estimates of the mean and the variance from a sequence $\Theta_1, \ldots, \Theta_N$ of measured data by a self-consistent determination of the cut point θ_c and the mean $\bar{\theta}$ by

$$\bar{\theta} = \frac{1}{N}\sum_{i=1}^{N} \hat{\Theta}_i \quad \text{and}$$

$$s_\theta{}^2 = \frac{1}{N-1}\sum_{i=1}^{N} (\hat{\Theta}_i - \bar{\theta})^2 = \text{minimal}, \qquad (14)$$

where the mapping (5) of the angle to the appropriate interval of length 2π is used. This method is outlined in Essenwanger (1985); its weakness lies in the fact that it is not a single-pass estimator. Because of the required selfconsistency one has to store all the data and to do several summation loops over them until the minimum of the variance is found. An efficient algorithm, using a not selfconsistently determined mean value, is described in Nelson (1984).

To avoid these multipass methods, one can consider a suitable periodic function of the circular variable with period T. Then an integral like the mean (7) becomes independent of the position of the integration interval. That is exactly what is done in so-called circular statistics (Mardia 1972, 1975; Batschelet 1981; Upton and Fingleton 1989) by taking the sine and cosine of the circular variable.

However, it is not obvious how to extract an estimate of the above defined standard deviation of the circular variable from the moments of the trigonometric functions. That can only be done by first considering a specific circular distribution, then computing the trigonometric moments and the circular variance, and finally fitting the latter one to the trigonometric moments. This is, for example, done by taking a wrapped normal distribution in Mardia (1972, p. 74) (see also Fisher 1983). This paper will not take trigonometric functions of the angle but Cartesian components of the wind vector as the periodic functions of the angle.

3. Anisotropic Gaussian model

The main idea followed in this paper is to get an estimator of the standard deviation of wind direction in the following way: assume that the joint pdf of the Cartesian components u and v of the wind vector is known. Then a simple variable substitution leads to the joint pdf of the polar components r and θ of the wind vector. A subsequent integration over the radial variable r leads then to the pdf of the angle θ, which is nothing else than a projection of the joint pdf of the polar components of the wind vector onto the θ axis. As outlined earlier, the mean and the standard deviation of the angle can then be calculated by virtue of (7), (9), and (10), that is, by minimizing the variance. As a result of this way of proceeding we clearly have a direct relation of the angular variance to the moments of the Cartesian wind components.

In order to carry out the outlined scheme, the joint pdf of the Cartesian components should first be known. Unfortunately, this function is in general unknown, and it would be a very cumbersome procedure to extract it from experimental data. Therefore, this paper constructs a theoretical joint pdf, which allows the calculation of the angular standard deviation as outlined earlier.

The construction of the theoretical joint pdf is mainly based on the following two assumptions:

- A1: the longitudinal (alongwind) and the lateral (crosswind) component of the wind vector are both random variables that are described by a Gaussian distribution function, and
- A2: the longitudinal and the lateral component are statistically independent of each other.

Here it is silently assumed that there exists a nonvanishing mean wind vector that allows the definition of a longitudinal and a lateral direction.

We term this model anisotropic Gaussian model because no restrictions are imposed on the standard deviations of the longitudinal and the lateral fluctuations. It is a generalization of an earlier model of McWilliams et al. (1979). In their model it is additionally assumed that the variances of the longitudinal and lateral components are equal, and the distribution function is thus isotropic in the fluctuation space. In McWilliams et al. (1979), the model was used to calculate the distribution function of the wind speed and the wind direction. A comparison coincides well with experimental data. In a second paper (McWilliams and Sprevak 1980) an attempt was made to compute moments of the angle, although without correctly considering the range of integration that led to the strange result that the mean of the angle depended on the variance of the longitudinal component. The range of integration should be adjusted to the requirement of minimal variance (10), which then would lead to the result that the mean angle is just the angle of the mean wind vector that defines the longitudinal direction.

In the anisotropic Gaussian model, the longitudinal component x and the lateral component y are, thus, according to the assumption A1, described by the following pdf's

$$p_X(x) = \frac{1}{(2\pi)^{1/2}\sigma_x} \exp\left[-\frac{(x-\mu_x)^2}{2\sigma_x^2}\right] \qquad (15)$$

$$p_Y(y) = \frac{1}{(2\pi)^{1/2}\sigma_y} \exp\left(-\frac{y^2}{2\sigma_y^2}\right). \qquad (16)$$

The mean values μ_x, μ_y and the standard deviations σ_x, σ_y of these two wind components are given by

$$\mu_x, \quad \mu_y = 0, \quad \sigma_x, \quad \sigma_y. \qquad (17)$$

Because of the assumption A2, the statistical independence of the x and y component, the joint probability density can be written as a product of the two pdf's of the components

$$p_{XY}(x, y) = p_X(x)p_Y(y). \qquad (18)$$

After a transformation to polar coordinates,

$$x = r\cos\theta, \quad y = r\sin\theta, \qquad (19)$$

the joint probability density becomes

$$p_{R\Theta}(r, \theta) = \frac{1}{2\pi\sigma_x\sigma_y} r \times \exp\left[-\frac{(r\cos\theta - \mu_x)^2}{2\sigma_x^2} - \frac{r^2\sin^2\theta}{2\sigma_y^2}\right]. \qquad (20)$$

The pdf of the angle θ can be written as a projection of this joint pdf onto the θ axis

$$p_\Theta(\theta) = \int_0^\infty p_{R\Theta}(r, \theta)dr. \qquad (21)$$

This integration can be carried out in closed form (Prudnikov et al. 1986) yielding the following expression

$$p_\Theta(\theta) = \frac{1}{2\pi}\exp(-\gamma^2/2)\frac{\delta}{\delta^2\sin^2\theta + \cos^2\theta} \times [1 - \pi^{1/2}\zeta\exp(\zeta^2)\,\mathrm{erfc}(\zeta)], \qquad (22)$$

where the following abbreviations are used,

$$\gamma = \mu_x/\sigma_x, \quad \delta = \sigma_x/\sigma_y \qquad (23)$$

$$\zeta = \frac{-\gamma\cos\theta}{(2\cos^2\theta + 2\delta^2\sin^2\theta)^{1/2}}, \qquad (24)$$

and where $\mathrm{erfc}(z)$ denotes the complementary error function

$$\mathrm{erfc}(z) = 1 - \mathrm{erf}(z) = \frac{2}{\pi^{1/2}}\int_z^\infty \exp(-t^2)dt. \qquad (25)$$

This pdf is a special case of the offset normal distribution given in Mardia [1972, Eq. (3.4.17)], namely, with his parameters ν and ρ set to zero. The angular pdf depends only on the two dimensionless parameters γ and δ, where γ is the longitudinal signal-to-noise ratio (or the reciprocal of the longitudinal turbulence intensity), and δ is a measure of the anisotropy of the fluctuations.

First, we check whether the pdf (22) is a unimodal function on the interval $(-\pi, \pi)$. Because (22) is a symmetric function of the angle θ about $\theta = 0$, it must be stationary at $\theta = 0$. For fixed γ, the pdf (22) becomes unimodal with a peak at $\theta = 0$ in the limit $\delta \to \infty$, and it becomes bimodal with two peaks at $\delta = \pm\pi/2$ for $\delta \to 0$. From these considerations and from inspection of graphs of the pdf, one concludes that the transition from unimodal to bimodal behavior takes place when the point $\theta = 0$ changes from a local maximum to a local minimum. Whether it is a minimum or a maximum can be decided by calculating the sign of the second-order derivative of the pdf (22) at $\theta = 0$. This second-order derivative is negative, and the point $\theta = 0$ therefore results to be a maximum if the parameter δ is larger than a critical value δ_c. This critical value is given by

$$\delta_c = \left(\frac{1 + 2A\gamma}{1 + 3A\gamma + \gamma^2/2 + A\gamma^3}\right)^{1/2}, \quad (26)$$

with

$$A = (\pi/8)^{1/2} \exp(\gamma^2/2) \operatorname{erfc}(-2^{-1/2}\gamma). \quad (27)$$

The line of the critical values δ_c is shown as a dashed line in Fig. 1 as a function of the parameter γ. For parameter values δ above the dashed line, the pdf (22) is unimodal. In Fig. 2, the angular pdf (22) is plotted for a fixed value of the parameter $\gamma = 1$ and different values of δ. For δ larger than δ_c ($\delta_c = 0.7277$ for $\gamma = 1$) the pdf is in fact unimodal, for δ smaller than δ_c it is bimodal.

Because the pdf (22) is symmetric with respect to $\theta = 0$, the cut point that leads to minimal variance is either $\theta_c = -\pi$ or $\theta_c = 0$ as discussed in section 2, and the resulting mean is then either $\mu_\theta = 0$ or $\mu_\theta = \pi$. If δ is larger than δ_c, then the pdf has its maximum at $\theta = 0$, and therefore the mean is zero. It is directly seen from (17) that the mean wind vector is $(\mu_x, 0)$ and the longitudinal direction has the angle $\mu_\theta = 0$.

The angular variance (9) can now be calculated by virtue of

$$\sigma_\theta{}^2 = \int_{-\pi}^{\pi} \theta^2 p_\Theta(\theta) d\theta, \quad (28)$$

which, however, cannot be integrated in closed form for arbitrary values of the parameters γ and δ.

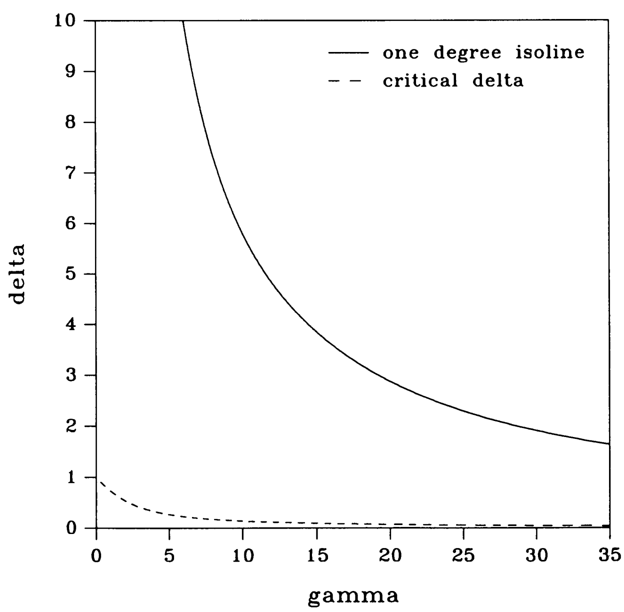

FIG. 1. Critical anisotropy parameter δ_c (dashed line) above which the angular probability density function is unimodal. The solid line is the 1° isoline of the angular standard deviation.

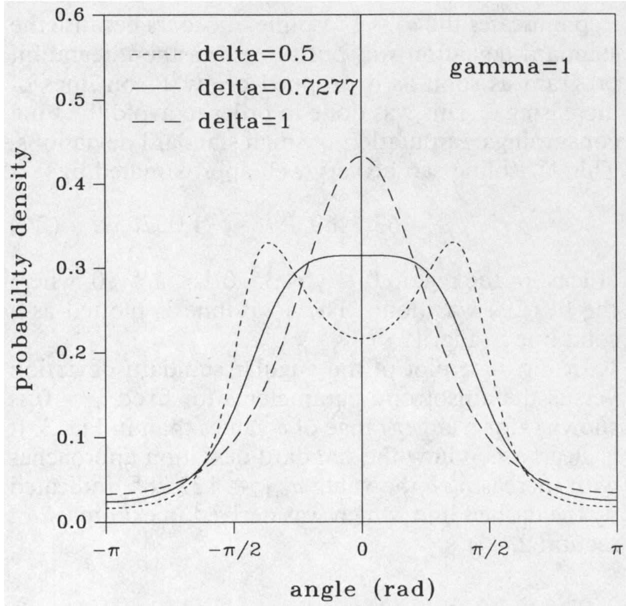

FIG. 2. Angular probability density function of the anisotropic Gaussian model for a fixed value of $\gamma = 1$ and different values of δ. The solid line represents the critical δ_c ($\delta_c = 0.7277$ for $\gamma = 1$), the dotted curve gives the probability density for $\delta = 0.5$, and the dashed curve gives the probability density for $\delta = 1$.

Only in a few special cases can the angular standard deviation σ_θ be explicitly determined: (i) if the anisotropy parameter δ goes to zero, then $\sigma_\theta \to \pi/2$, and (ii) if $\delta = 1$ and $\gamma \to 0$, then the angular distribution becomes isotropic and $\sigma_\theta \to \pi/3^{1/2}$.

For arbitrary values of the two parameters, γ and δ, the integral (28) is evaluated numerically. The result is shown in Fig. 3, where the angular standard deviation σ_θ is plotted versus the parameters γ and δ. The curved

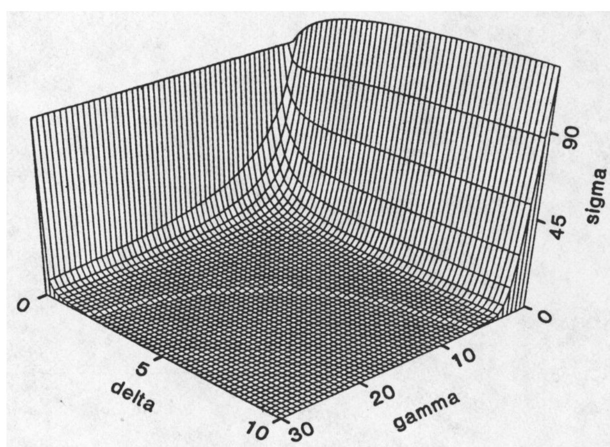

FIG. 3. Angular standard deviation of the anisotropic Gaussian model as a function of the longitudinal signal-to-noise ratio $\gamma = \mu_x/\sigma_x$ and of the anisotropy parameter $\delta = \sigma_x/\sigma_y$. The step indicates the 1° isoline of the angular standard deviation.

step indicates the $\sigma_\theta = 1°$ isoline, it occurs because the standard deviation was put to zero in the integration program as soon as it dropped below 1° on lines of increasing δ. This was done in order to avoid the time consuming computation of small standard deviations. This 1° isoline can be very well approximated by

$$\delta_{1°} = a\gamma^b, \quad a = 63.7469, \quad b = -1.037659, \quad (29)$$

at least in the region $0 \leqslant \gamma \leqslant 35$, $0.1 \leqslant \delta \leqslant 10$, where the fit (29) was done. The 1° isoline is plotted as a solid line in Fig. 1.

In Fig. 4, a plot of the angular standard deviation versus the anisotropy parameter δ for fixed $\gamma = 0$ is shown—for a larger range of δ values than in Fig. 3. It is clearly seen how the standard deviation approaches with increasing δ the value $\sigma_{\max} = 127.28°$, indicated by the dashed line, which was derived in example 2 of section 2.

4. Practical computation of the angular standard deviation

Until now, it was assumed that the mean and the standard deviation of the longitudinal and the lateral components were given. In fact, these quantities were first computed from the corresponding values of the Cartesian components u and v of the wind vector. Suppose that after N consecutive measurements, the unbiased estimates (3) of the means ($\bar{u}$, $\bar{v}$) and variances (s_u, s_v) of the Cartesian components are stored. From these the direction of the mean wind vector is calculated as

$$\bar{\theta} = \arctan(\bar{v}/\bar{u}) \quad (30)$$

and taken as an estimate of the mean angle.

The transformation to the coordinate system defined by the longitudinal and lateral components x and y is a simple rotation. Therefore, the mean and the covariance matrix transform in a very simple manner (e.g., Flury and Riedwyl 1988) by

$$\bar{x} = \bar{u}\cos\bar{\theta} + \bar{v}\sin\bar{\theta}, \quad \bar{y} = 0 \quad (31)$$

$$s_x{}^2 = (s_u{}^2\cos^2\bar{\theta} - s_v{}^2\sin^2\bar{\theta})(\cos^2\bar{\theta} - \sin^2\bar{\theta})^{-1} \quad (32)$$

$$s_y{}^2 = (s_v{}^2\cos^2\bar{\theta} - s_u{}^2\sin^2\bar{\theta})(\cos^2\bar{\theta} - \sin^2\bar{\theta})^{-1}. \quad (33)$$

The assumption that the x and y component are statistically independent implies $s_{xy} = 0$. Furthermore, from $s_{xy} = 0$ it follows that the relation

$$(s_v{}^2 - s_u{}^2)\cos\bar{\theta}\sin\bar{\theta} = s_{uv}(\sin^2\bar{\theta} - \cos^2\bar{\theta}) \quad (34)$$

is valid for the elements of the covariance matrix of the Cartesian u and v components.

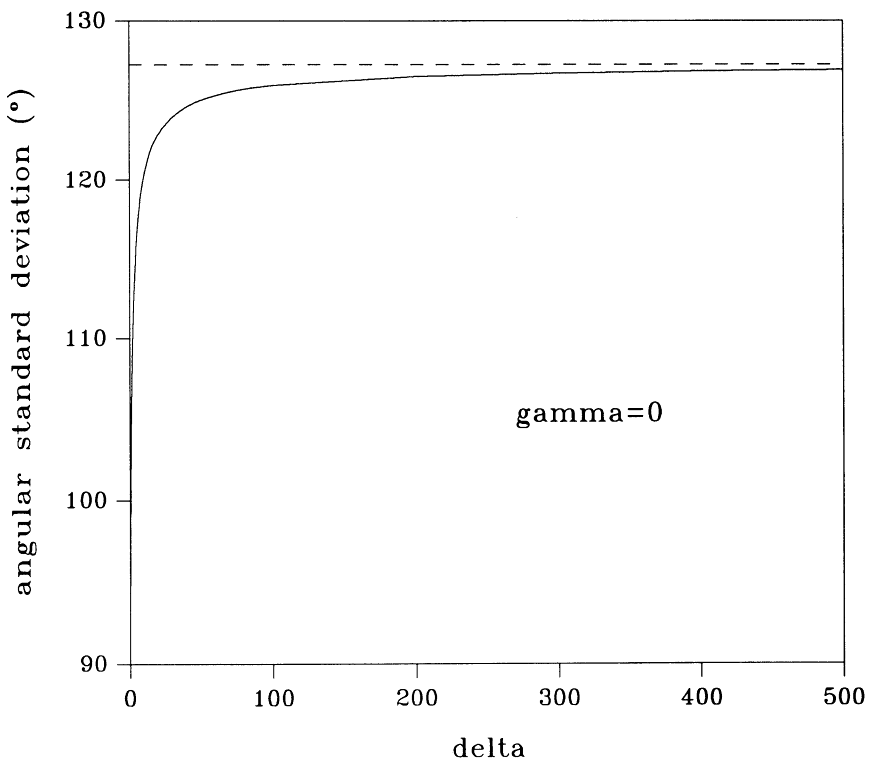

FIG. 4. Angular standard deviation for a fixed value of $\gamma = 0$ as a function of the anisotropy parameter δ. The dashed line marks the value of $\sigma_{\max} = 127.28°$.

At this point we have to make a remark about the application limitations of our anisotropic Gaussian model. Because of the assumption of statistical independence of the longitudinal and lateral wind component, (32) and (33) can be used to transform the variances of the Cartesian components to the corresponding variances of the longitudinal and lateral components. However, this assumption is not strictly fulfilled with real data; the crossvariance term s_{xy} will in general be small, but not precisely zero. It can therefore occur that the transformations (32) and (33) give a negative value for one of the two variances, s_x^2 and s_y^2, which are of course meaningless. In other words, the eigendirections of the covariance matrix are not parallel to the longitudinal and lateral direction of wind in such cases.

It is easy to calculate from (32) and (33) the regions in $(\bar{\theta}, s_u/s_v)$ space where one of the estimates according to (32) and (33) of the variances becomes negative. The following result is obtained after having introduced the abbreviation $A = s_u/s_v$:

$$A < 1\colon 0.5 < \cos^2\bar{\theta} < 1/(A^2+1) \Rightarrow s_x^2 < 0.$$

$$A^2/(A^2+1) < \cos^2\bar{\theta} < 0.5 \Rightarrow s_y^2 < 0.$$

$$A > 1\colon 1/(A^2+1) < \cos^2\bar{\theta} < 0.5 \Rightarrow s_x^2 < 0.$$

$$0.5 < \cos^2\bar{\theta} < A^2/(A^2+1) \Rightarrow s_y^2 < 0.$$

In Fig. 5, the region leading to a negative estimate of s_x^2 is shown as a light-shaded area and the region with

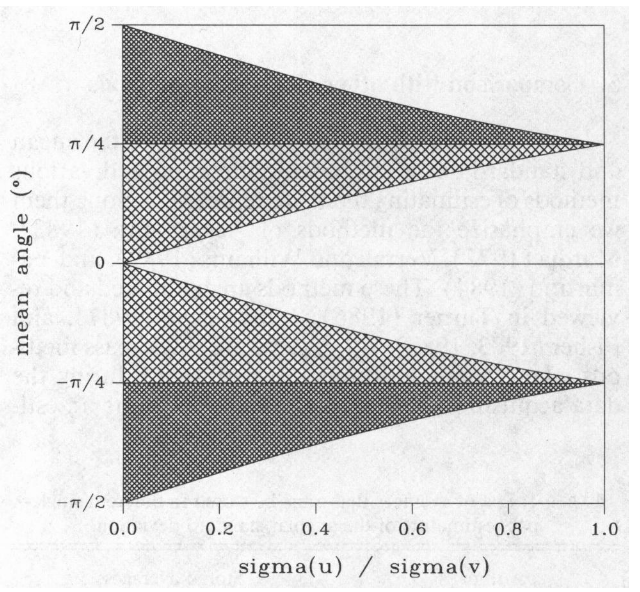

FIG. 5. Regions in $(s_u/s_v, \bar{\theta})$ space, where the transformation (32) or (33) gives negative values of the variances s_x^2 (light-shaded regions) or s_y^2 (dense-shaded regions). The picture is periodic with period π in the $\bar{\theta}$ coordinate. For values of s_u/s_v larger than 1, the reciprocal value s_v/s_u has to be taken as an abscissa value and the role of the light- and dense-shaded regions should be interchanged.

a negative estimate of s_y^2 as a dense-shaded area. The picture can be periodically continued with period π along the $\bar{\theta}$ axis. If s_u/s_v is larger than 1, then its reciprocal value s_v/s_u can be taken as an abscissa value and the role of the light- and dense-shaded regions must be interchanged.

If the estimate of one of the two variances s_x^2 and s_y^2 gets negative, then the anisotropic Gaussian model cannot be applied. Fortunately, the parameter σ_u/σ_v is in most cases close to 1, and there is no danger of negative values of the variances.

This remark does not only concern our model but is also of importance in Lagrangian models of atmospheric dispersion, where it is also assumed that the fluctuations of longitudinal and lateral wind components are uncorrelated (e.g., Zannetti 1984).

In some meteorological databases, the north n and east e component are given instead of the Cartesian components u and v. These latter ones indicate the direction from where the wind is blowing. The transformation to the Cartesian components of the wind vector can be done by

$$u = -e, \quad v = -n \tag{35}$$

$$s_u = s_e, \quad s_v = s_n. \tag{36}$$

So far, the mean and variance of the longitudinal and lateral components have been calculated. From these estimates, two sample quantities $\gamma = \bar{x}/s_x$ and $\delta = s_x/s_y$ can be determined. Because the angular standard deviation cannot be given in closed form as function of γ and δ, an approximative method has to be used. If the stored data are analyzed off-line, the easiest way may be to take tabulated values of the angular standard deviation and interpolate for the desired parameter values. For an on-line application, the storage of the tabulated data may be problematic. We have therefore tried to find an appropriate fit function, which needs only a few fit parameters to be stored and allows a very quick evaluation.

The search of a suitable fit function was restricted to parameter values in the range $0 \leqslant \gamma \leqslant 30$ and $0.1 \leqslant \delta \leqslant 10$. Taking an array of 150×100 values of the parameters γ and δ, it was found that the following function fits the angular standard deviation σ_θ, for $\sigma_\theta \geqslant 1°$, within an accuracy of 3°:

$$\sigma_\theta = \frac{A + B\gamma + C\gamma^2}{1 + D\gamma + E\gamma^2}$$

$$A = \frac{90 + 172.24\delta + 32.427\delta^2}{1 + 1.6067\delta + 0.25135\delta^2}$$

$$B = \frac{3366.68 - 1934.33\delta - 316.342\delta^2}{75.0227\delta + 1.69087\delta^2}$$

$$C = -0.0113449 + 0.57699\delta$$

$$D = \frac{188.451 - 49.908\delta - 1.42925\delta^2}{377.822\delta + 43.8346\delta^2}$$

$$\begin{aligned} E = {} & 0.64449 - 0.293313\delta + 0.329844\delta^2 \\ & - 0.172855\delta^3 + 0.0440614\delta^4 - 5.8839 \times 10^{-3}\delta^5 \\ & + 3.96817 \times 10^{-4}\delta^6 - 1.06722 \times 10^{-5}\delta^7. \end{aligned} \quad (37)$$

In a computer program that calculates the angular standard deviation according to the above outlined procedure, the following steps should be executed:

(i) compute an estimate of the mean angle according to (30) and then transform the Cartesian means and variances to the corresponding longitudinal and lateral quantities (31), (32), and (33);

(ii) check whether the variances ${s_x}^2$ and ${s_y}^2$ are positive;

(iii) check whether γ and δ are in the appropriate range: $0 \leqslant \gamma \leqslant 30$ and $0.1 \leqslant \delta \leqslant 10$;

(iv) compute the 1° isoline value (29): if $\delta > \delta_{1^\circ}$ then put $\sigma_\theta = 0$; and

(v) use the fitted function (37) to get the angular standard deviation.

Another way to parameterize the angular standard deviation is to use the means of the persistence with one of the two parameters γ or δ. The persistence is defined as the ratio of the vector mean v_v and the scalar mean v_s of wind speed

$$P = v_v/v_s, \quad \text{where}$$

$$v_v = (\bar{u}^2 + \bar{v}^2)^{1/2} \quad \text{and} \quad v_s = \overline{(u^2 + v^2)^{1/2}}. \quad (38)$$

In Yamartino (1984), the following expression for the persistence is given

$$P^2 = \overline{\sin\theta}^2 + \overline{\cos\theta}^2, \quad (39)$$

which holds under the stronger precondition that the fluctuations in wind speed and the fluctuations in wind direction are statistically independent. This does not hold under the weaker precondition that the fluctuations in wind speed are uncorrelated with the fluctuations in wind direction. For our anisotropic Gaussian model, the cross variance matrix element

$$\sigma_{r\theta} = \int_0^\infty dr \int_{-\pi}^{\pi} p_{R\Theta}(r, \theta) r\theta d\theta \quad (40)$$

becomes zero because the joint pdf $p_{R\Theta}(r, \theta)$ (20) is an even function of the angle θ. Therefore, the wind speed and wind direction are uncorrelated. But they are not statistically independent as the joint pdf (20) cannot, at least for all values of γ and δ, be written as a product of two univariate pdf's $[p_{R\Theta}(r, \theta) \neq p_R(r)p_\Theta(\theta)]$. Only in the special case $\gamma = 0$ and $\delta = 1$ is such a factorization possible. Thus, the two sides of (39) are, in general, not equal.

Instead of using the two parameters γ and δ to determine the value of σ_θ, P and δ can also be chosen as independent parameters. Again, a suitable function is fitted to $\sigma_\theta(P, \delta)$. The best choice found was

$$\sigma_\theta = A(1 - P)^B$$

$$A = \frac{-0.262964 + 50.21289\delta + 124.9263\delta^2}{-6.414684 \times 10^{-3} + 0.6601226\delta + \delta^2}$$

$$B = \frac{0.6787763 + 0.8351036\delta + 0.311525\delta^2}{1.189552 + 1.230715\delta + \delta^2}, \quad (41)$$

which reproduces the angular standard deviation within an accuracy of 6°. Whereas the persistence is quite often computed and stored in meteorological measurements, the anisotropy parameter δ is hardly available. However, it is shown by Mori (1986) that for a large dataset the experimental data gave an anisotropy parameter close to 1, that is, $s_u \approx s_v$ (Mori's Fig. 2). In cases where only the persistence P is given, one can take the fit (41) with $\delta = 1$

$$\sigma_\theta = 105.75(1 - P)^{0.5337} \quad (42)$$

as a rough estimate of the angular standard deviation.

5. Comparison with other single-pass methods

In the literature, different definitions of the mean and standard deviation of wind direction and various methods of estimating them are reported. Among them we emphasize the methods of Ackermann (1983), Mardia (1972), Verrall and Williams (1982), and Yamartino (1984). These methods are compared and reviewed in Turner (1986), Mori (1986, 1987), and Fisher (1983, 1987). All of these are single-pass methods where some additions are performed during the data acquisition, and after the sampling time an esti-

TABLE 1. List of averages that must be stored in different single-pass estimators of the angular standard deviation.

Author	Stored averages				
Ackermann (1983)	$\bar{u}$	$\bar{v}$	$\overline{u^2}$	$\overline{v^2}$	$\overline{uv}$
Mardia (1972)	$\overline{\cos\theta}$	$\overline{\sin\theta}$			
Verrall and Williams (1982)	$\overline{\cos\theta}$	$\overline{\sin\theta}$	$\overline{\cos^2\theta}$	$\overline{\sin^2\theta}$	
Yamartino (1984)	$\overline{\cos\theta}$	$\overline{\sin\theta}$			
Present paper	$\bar{u}$	$\bar{v}$	$\overline{u^2}$	$\overline{v^2}$	

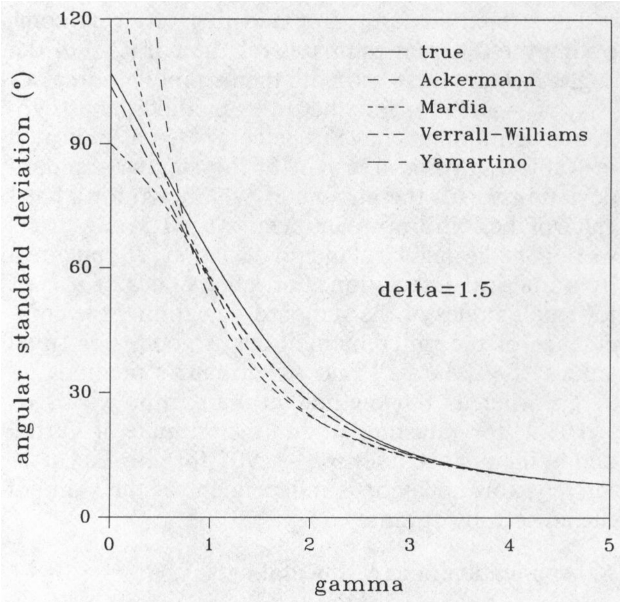

FIG. 6. Comparison of different single-pass estimators of angular standard deviation, assuming that the Cartesian coordinates of the wind vector are distributed according to the anisotropic Gaussian model. The solid line is the integral (28) denoted as true value. The estimates of Ackermann (1983, dashed line) (43), Mardia (1972, dotted line) (44), Verall and Williams (1982, dot–dashed) (47), and Yamartino (1984, long dashes) (49) are shown.

mate of the mean and standard deviation is calculated from these sums. Table 1 gives a summary of the quantities that must be computed and stored for the different methods. The averaging is always done over the sampling time. Suppose N data are measured during the sampling time, then the mean $\bar{x}$ and the variance s_x^2 of a measured quantity are obtained by the unbiased estimators (3).

From these averages, Ackermann (1983) estimates a linear angular standard deviation of the angle by

$$\sigma_A = (\bar{u}^2 + \bar{v}^2)^{-1} (\bar{u}^2 s_u^2 + \bar{v}^2 s_u^2 - 2\bar{u}\bar{v} s_{uv})^{1/2}, \quad (43)$$

which is based on the propagation law of errors and holds only for small values of σ_A. If transformations (31), (32), and (33) are performed to longitudinal and lateral coordinates, the estimate (43) becomes simply $\sigma_A = s_y/\bar{x}$.

In Mardia (1972, p. 74) the following formula is proposed to get an estimate of "the standard deviation on the line" of the angle

$$\sigma_M = [-\log(\overline{\cos\theta}^2 + \overline{\sin\theta}^2)]^{1/2}, \quad (44)$$

which is derived for a wrapped Gaussian distribution of the angle.

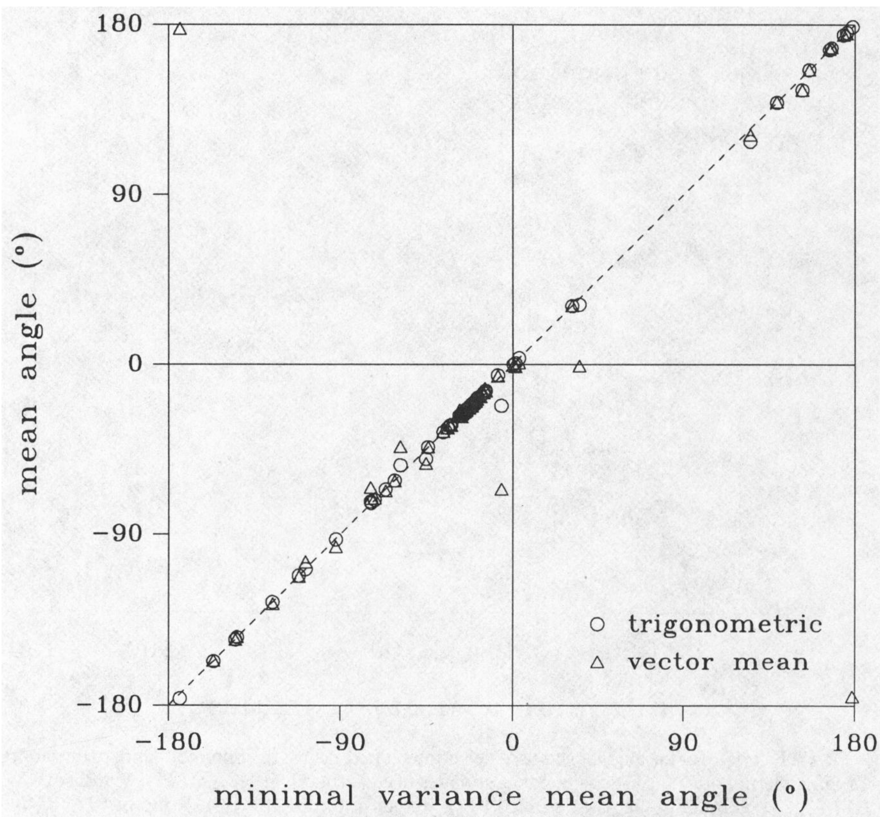

FIG. 7. Trigonometric mean of the angle (○) and vector mean of the angle (△) plotted versus the minimal variance mean; computed from our experimental data.

The method of Verrall and Williams (1982) is based on geometrical considerations and requires the computation of two vectors

$$\mathbf{e}_1 = (\overline{\sin\theta} - s_{\sin\theta}, \overline{\cos\theta} + s_{\cos\theta}) \qquad (45)$$

$$\mathbf{e}_2 = (\overline{\sin\theta} + s_{\sin\theta}, \overline{\cos\theta} - s_{\cos\theta}), \qquad (46)$$

from which a standard deviation of wind direction is computed by

$$\sigma_{VW} = \frac{1}{2}\arccos\left(\frac{\mathbf{e}_1 \cdot \mathbf{e}_2}{|\mathbf{e}_1| \cdot |\mathbf{e}_2|}\right). \qquad (47)$$

Yamartino (1984) calculates first the quantity

$$\epsilon = [1 - (\overline{\sin\theta}^2 + \overline{\cos\theta}^2)]^{1/2} \qquad (48)$$

and proposes as a best fit of the angular standard deviation, as defined by the minimum variance principle, the function

$$\sigma_Y = (1 + 0.1547\epsilon^3)\arcsin(\epsilon). \qquad (49)$$

These methods are compared first of all by computing the different estimates for the pdf (22) of the angle. Ackermann's estimate then simply becomes $\sigma_A = s_y/\bar{x} = 1/(\gamma\delta)$. For the other methods, the trigonometric moments have to be computed as indicated in Table 1. In Fig. 6, a plot of the angular standard deviation versus the parameter γ is shown for a fixed value of the anisotropy parameter δ ($\delta = 1.5$). By "true" we denote the angular standard deviation (28) obtained by numerical integration. For large values of γ (i.e., for small values of the standard deviation), the coincidence of the different methods is good. For small values of γ, Mardia's and Ackermann's methods diverge; whereas the method of Yamartino yields $\sigma_Y = 103.9°$ for vanishing γ, and the estimate of Verrall and Williams becomes $\sigma_{VW} = 90°$ for $\gamma = 0$. Qualitatively, this behavior is independent of the value of the anisotropy parameter δ.

6. Application to experimental data

The estimation methods of angular standard deviation described in the last section are applied to data

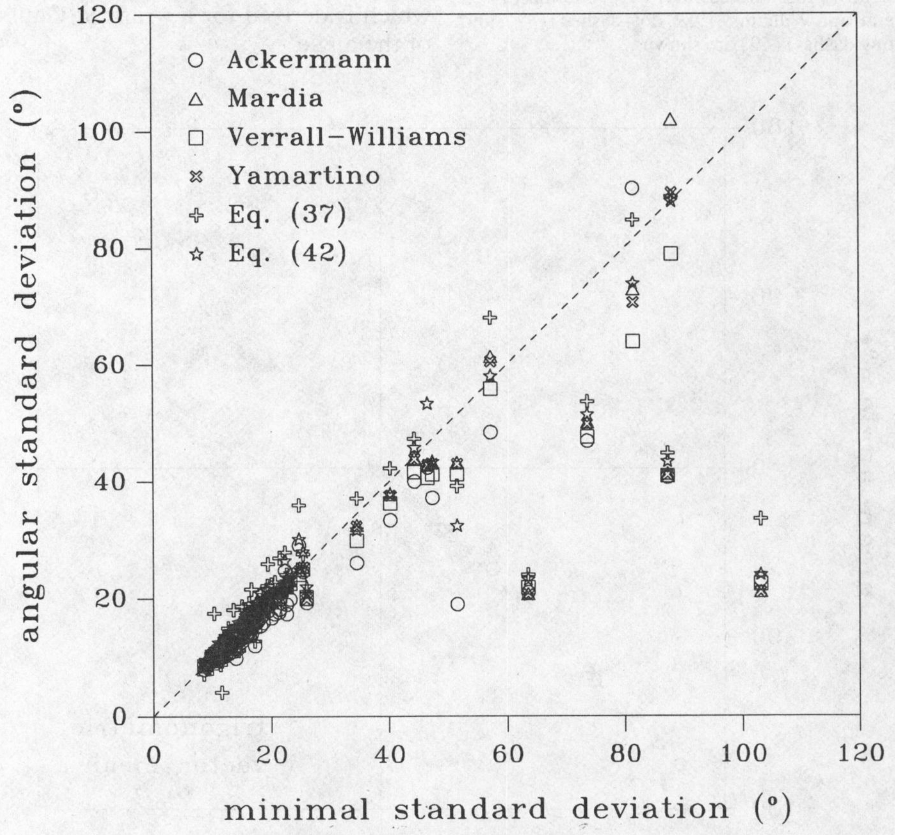

FIG. 8. Estimates of the angular standard deviation plotted against the minimal standard deviation for our experimental data. Shown are the estimates of Ackermann (1983, ○) (43), Mardia (1972, △) (44), Verall and Williams (1982, □) (47), Yamartino (1984, ×) (49), the fit (37) of the present paper (+), and the persistence fit (42) (☆).

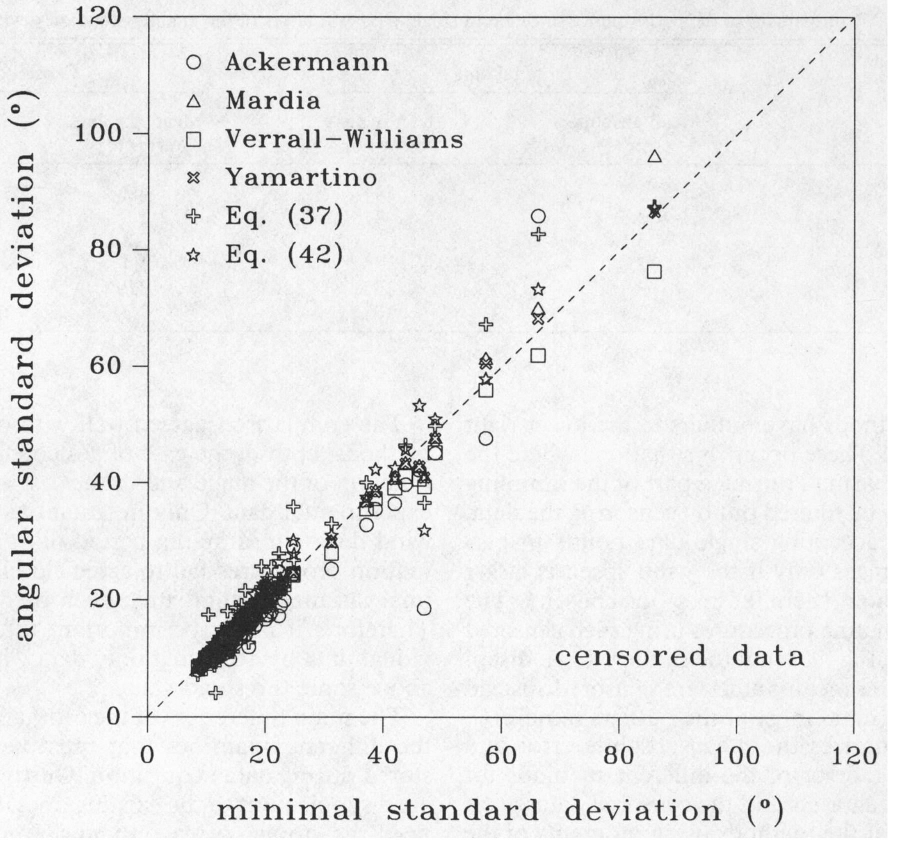

FIG. 9. Estimates of the angular standard deviation as in Fig. 8, but only taking into account data points with a wind speed larger than 0.5 m s^{-1}.

that were provided by Baraldi. The data consist of 1-s and 5-s measurements of the horizontal wind components measured with a Vaisala cup anemometer WAA 15 and a Vaisala wind vane WAV 15. The instrument were placed on a small mast on top of a high building in Basel, Switzerland. Data were taken during test runs of a newly developed data logger on 14 and 15 August 1990.

From these fast-sampled data, 5-min averages were taken in the case of the 1-s measurements, and 30-min averages were taken in the case of the 5-s data, in order to test the estimation procedures.

First, we checked how the different methods of determining the mean angle coincided. The minimal variance mean θ_{MV} was determined with a multipass method according to (14). The range of angle was divided into 360 classes, cut points were chosen in the middle of each class, and the cut point that yielded the smallest variance was determined. The trigonometric mean $\theta_T = \overline{\sin\theta}/\overline{\cos\theta}$ and the vector mean $\theta_V = \bar{v}/\bar{u}$ can be obtained, of course, by performing the corresponding sums in a single run. In Fig. 7, the trigonometric and vector mean of the angle are plotted against the minimal variance mean as obtained from our datasets. The circles mark the trigonometric mean and the triangles the vector mean of the angle; the dotted line is the bisection. The coincidence is reasonably good for both single-pass methods, except for two outliers (in the middle of the picture). These occurred in the 30-min averages where the wind speed changed very markedly during the sampling time. Only the directions with high wind speed contribute in such cases significantly to the vector mean. The values in the higher left and lower right corner are not outliers, but must be mapped by adding or subtracting 360°.

In Fig. 8, the results of the different estimation procedures for the angular standard deviation are plotted against the minimal standard deviation obtained via the minimal variance procedure (14). There are some remarks to be made about this scatterplot. First of all, most of the data points are centered around the diagonal, indicating that, in general, the different methods yield reasonable values of the angular standard deviation. A closer inspection of the data shows that none of the different estimators consistently over- or underestimates; one very large value (170°) obtained by Ackermann's method is not plotted. For small values of the angular standard deviation, the estimates are more precise. Even our crude estimate (42) using the persistence alone as parameter gives good results.

TABLE 2. Comparison of the performance of different single-pass estimators of the angular standard deviation.

	Original data		Censored data	
Method	Mean absolute error (°)	Mean relative error (%)	Mean absolute error (°)	Mean relative error (%)
Ackermann (1983)	5.5	14	4.2	13
Mardia (1972)	2.8	4	1.5	3
Verrall and Williams (1982)	3.1	5	1.8	4
Yamartino (1984)	2.7	5	1.4	3
Equation (37)	3.9	13	2.7	12
Equation (42)	3.4	10	2.2	8

However, all methods have outliers in the lower right half of the plane. These occur in situations where the wind speed is very small during a part of the sampling period. They can be filtered out by censoring the data. This is done by accepting single data points in performing the averages only if the wind speed is larger than some threshold (here 0.5 m s^{-1} is chosen). The result of the estimating procedures using such censored data is shown in Fig. 9. Now the outliers have disappeared. If only the mean values are censored instead of the individual data, most of the outliers remain.

Table 2 summarizes the mean absolute error and the mean relative error of the different methods for both the original datasets and the censored datasets. It is clearly seen that the methods using moments of the trigonometric functions sin and cos are in better agreement with the minimal variance method than the estimators based on moments of the Cartesian components. This can be easily understood by noting that the latter methods are, roughly speaking, weighting the angle with the wind speed, whereas the former two kinds of methods are considering the angle alone. Deciding which of these two ways of treating wind direction is more appropriate depends on the question one is posing. Yet the difference of the angular standard deviation obtained by the different methods is quite small, in general, below 5°. Surprisingly, our estimate (42) based on the persistence alone gave a better result than the fit (37). That may be an indication that the determination of the parameters γ and δ is less stable than the determination of persistence.

7. Conclusions

In this paper, we have defined the mean wind direction and the angular standard deviation in a minimal variance sense. Starting from a theoretical distribution function of the Cartesian wind components, we derived two fits of angular standard deviation, one in terms of the means and variances of the Cartesian components, the other in terms of the persistence. Surprisingly, the crude parameterization of the persistency alone showed quite good results.

The comparison agreed well with other estimation methods, both in the case of a theoretical distribution function of the angle and in the case of application to experimental data. Only in situations with very weak wind during a sampling period did the different estimation procedures fail to agree closely with the minimal variance method, unless censored data were used. Therefore, it might be important to censor the individual data by accepting only data with a wind speed above some threshold value.

The main differences of the estimation methods are the different quantities that must be calculated and stored during data acquisition. Our two new methods are a supplement to the existing ones. Since they don't need the storage of trigonometric moments, but use the mean and variance of the Cartesian wind components or the persistence, they may be useful in application to already existing databases because these latter quantities are more easily accessible than trigonometric moments. Persistence is a quantity that has been stored for a long time.

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