NOTES AND CORRESPONDENCE

On the Algorithms Used to Compute the Standard Deviation of Wind Direction

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ABSTRACT

The standard deviation of wind direction is a very important quantity in meteorology because in addition to being used to determine the dry deposition rate and the atmospheric stability class, it is also employed in the determination of the rate of horizontal diffusion, which in turn determines transport and dispersion of air pollutants. However, the computation of this quantity is rendered difficult by the fact that the horizontal wind direction is a circular variable having a discontinuity at 2π radians, beyond which the wind direction starts again from zero, thus preventing angular subtraction from being a straightforward procedure. In view of such a limitation, this work is meant to provide new mathematical expressions that simplify both the computational and analytical work involved in handling the standard deviation of wind direction. This is achieved by deriving a number of Fourier series and Taylor expansions that can represent the minimum angular distance and its powers. Using these expressions, the relation between two algorithms commonly used to determine the standard deviation of wind direction is analyzed. Furthermore, given that these trigonometric expansions effectively reduce the mathematical complexity involved when dealing with circular statistics, their potential application to solve other problems is discussed.

1. Introduction

Most variables, such as length, age, birth rate, and so on, are continuous and thus can be directly mapped onto the real number line. As a consequence, such variables are called linear variables, and for them the statistical treatment is well established. However, there exist other variables, such as the horizontal wind direction, that cannot be mapped onto a line because they are periodic, restarting from zero at a certain point. Such periodic variables need to be mapped on a circle, and as a result they are referred to as circular variables.

If one considers the statistical treatment of circular variables as is found in standard textbooks (Batschelet 1981; Fisher 1995; Mardia 1972), it becomes immediately apparent that the reasons for the differences between circular and linear variables are both qualitative and quantitative. Their dissimilarity can be traced back to the arbitrariness of the zero position, the fact that the notion of high and low values does not apply, and the periodicity of the circular variables. In particular, it is the last point that makes it so difficult to treat circular variables mathematically since their analysis has to be confined within an interval of 2π radians. This makes subtraction or addition a nontrivial matter. Thus, a simple quantity like the minimum angular distance $\Delta(\theta, \phi)$ that, as its name implies, gives the least angular distance between two angles (θ and ϕ), including the associated sign depending on which angle is larger, acquires the following complicated expression:

$$|\Delta(\theta, \phi)| = \min(|\theta - \phi| \bmod 2\pi,\ 2\pi - |\theta - \phi| \bmod 2\pi), \quad \text{with} \quad \Delta(\theta, \phi) \geq 0 \quad \text{if} \quad \phi \leq \theta < \phi + \pi,$$
$$\Delta(\theta, \phi) < 0 \quad \text{if} \quad \phi - \pi < \theta < \phi, \quad \text{and} \quad \Delta(\theta, \phi) = (\theta - \phi) \bmod 2\pi \quad \text{if} \quad |\Delta(\theta, \phi)| = \pi, \tag{1}$$

where angles are measured in radians and a mod$b$ gives the remainder obtained after dividing a by b. This quantity is very important because it is needed to compute the expectation values of fundamental statistics like

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the angular mean and standard deviation (Batschelet 1981; Essenwanger 1986; Mardia 1972; Yamartino 1984).

The study of circular statistics, of which wind direction forms a part, still lends itself to potential development in its mathematical treatment. The need for mathematical tools can be inferred from the fact that for each statistic employed for linear variables there might be more than one corresponding circular statistic. Furthermore, the relationship among the various circular statistics used as a measure for a given quantity is frequently not clear. The measures of dispersions proposed by Mardia (1972) and Batschelet (1981) are a case in point. Both statistics were developed using a linearization process as a measure of angular dispersion, but they differ by a multiplicative factor. It was only recently that these two quantities were related, after Farrugia and Micallef (2004) proposed a new geometric measure to describe circular variables.

The development of an algorithm that can be used to determine the standard deviation of wind direction has encountered the same difficulties, as has already been pointed out by Fisher (1987). In fact, a number of equations have been proposed as possible measures of angular dispersion (Essenwanger 1986; Irwin 1980; Nelson 1984; Skibin 1984; Turner 1986; Yamartino 1984). However, there seems to be no generally accepted equation and in addition little is known about the connection between these algorithms. Some information can be obtained from the work of Weber (1997) in which he analyzed quantitatively, using field data, the algorithms proposed by Yamartino (1984) and Essenwanger (1986), these being two of the most commonly adopted equations to calculate the standard deviation of wind direction. His results showed that the two algorithms agree over almost all of the range of angular dispersion, with the algorithm of Yamartino (1984) underpredicting that of Essenwanger (1986) at regimes of high dispersion. However, a direct mathematical analysis to establish a relation between the two equations has not been carried out so far.

The standard deviation of wind direction s_θ is a very important quantity in meteorology because of its link with the lateral turbulent intensity i_v, given by the equation $\tan(s_\theta) = i_v$ (Hanna 1983). The lateral turbulence intensity is in turn linked to the standard deviation of the lateral concentration distribution in the plume s_{Lat} through the equation $s_{\mathrm{Lat}} = i_v x f(x)$, where x is the downwind distance from the source and $f(x)$ is some function of x (Pasquill 1976). Thus, the lateral spread of a plume is linked to the angular standard deviation through the equation $s_{\mathrm{Lat}} = \tan(s_\theta) x f(x)$, which for small s_θ reduces to $s_{\mathrm{Lat}} = s_\theta x f(x)$. Even though there are various proposed forms for $f(x)$ (Irwin 1983), the use of schemes involving the horizontal standard deviation of wind direction provides the best way of estimating the cross-wind dispersion, especially for low wind speeds (Cirillo and Poli 1992; Essa et al. 2005; Sharan et al. 1995; Yadav and Sharan 1996). A practical application of this scheme is given by the work of Yamartino and Wiegand (1986), who use the standard deviation of wind direction in their Canyon Plume box model to determine the level of mixing and turbulence within the canyon. A similar approach was adopted by Hertel and Berkowicz (1989) and Berkowicz et al. (1997).

The wind standard deviation can also be used to determine the aerodynamic resistance and the quasi-laminar boundary layer resistance, which are quantities needed to determine the deposition velocity of particles (Hicks et al. 1987). Thus, there is a constant need to investigate and improve on the mathematical techniques available for use with the wind standard deviation.

Some simplifying methods that can be employed in the determination of the minimum angular distance have already been devised. These are also useful in the computation of the standard deviation of wind direction. To determine the absolute value of $\Delta(\theta, \phi)$, it is possible to use (Batschelet 1981)

$$|\Delta(\theta, \phi)| = \cos^{-1}[\cos(\theta - \phi)]; \tag{2}$$

this expression would leave only the associated sign for Eq. (1) to be set. A different approach was used by Fisher (1987) to determine the standard deviation from a known distribution that has a mean equal to zero. The procedure involves finding the expectation value of $f(x)$, where $f(x) = x^2$ for $-\pi < x \leq \pi$ and $f(x + 2\pi) = f(x)$, when expressed in terms of a Fourier expansion. In addition to these exact analytical tools, in the case of sample data many approximate techniques have been suggested to estimate the standard deviation of wind direction from data parameters that do not require excessive data storage (Ackermann 1983; Castans and Barquero 1994, 1998; Farrugia and Micallef 2006; Fisher 1983; Ibarra 1995; Leung and Liu 1996, 1998; Mori 1986; Turner 1986; Verrall and Williams 1982, 1983; Weber 1991, 1992, 1997; Yamartino 1984).

Considering the importance of the minimum angular distance in computing the wind standard deviation, in the following work an extensive array of alternative ways of expressing $\Delta(\theta, \phi)$ that can be useful in different instances is developed. This will be carried out using both Fourier series and Taylor expansions. In the process, in view of the fact that these expansions might need to be truncated, wherever possible the maximum error introduced when a finite number of terms are considered has been calculated. Following this, a mathematical analysis is carried out to establish the relation between the algorithms suggested by Yamartino (1984) and Essenwanger (1986).

2. Derivation of expressions for the minimum angular distance

In this section, a number of alternative expressions for the minimum angular distance, its absolute value, and its powers in terms of trigonometric series are derived. Starting from $\Delta(\theta, \phi)$ and considering the graphical plot of this function against $\theta - \phi$ as shown in Fig. 1, it can be noted that the equation for the minimum angular distance is equivalent to the following periodic function:

$$f(\theta - \phi) = \theta - \phi \quad \text{for} \quad -\pi \le \theta - \phi \le \pi \quad \text{and}$$
$$f(\theta - \phi + 2\pi) = f(\theta - \phi), \tag{3}$$

which has a Fourier series given by

$$f(\theta - \phi) = 2\sum_{i=1}^{\infty} \frac{(-1)^{i+1}\sin[i(\theta - \phi)]}{i}. \tag{4}$$

As might have been expected, the expansion is in terms of sine functions because $\Delta(\theta, \phi)$ is an odd function.

Equation (4) is the most basic Fourier expansion that can be obtained for $\Delta(\theta, \phi)$. However, terms of the form $[\Delta(\theta, \phi)]^n$, where n is an integer, are frequently required, especially when determining expectation values. It is easy to see that the procedure to find the Fourier expansion for such a term is similar to the one just presented. The graphical representation of $[\Delta(\theta, \phi)]^n$ plotted against $\theta - \phi$ will be the same as that of the following function:

$$f(\theta - \phi) = (\theta - \phi)^n \quad \text{for} \quad -\pi \le \theta - \phi \le \pi \quad \text{and}$$
$$f(\theta - \phi + 2\pi) = f(\theta - \phi), \tag{5}$$

from which the Fourier series follows immediately.

The case in which $n = 2$ is of particular importance because $[\Delta(\theta, \phi)]^2$ is used to derive the standard deviation (Fisher 1987). The Fourier series in this case is as follows:

$$[\Delta(\theta, \phi)]^2 = \frac{\pi^2}{3} + 4\sum_{i=1}^{\infty} \frac{(-1)^i}{i^2}\cos[i(\theta - \phi)]. \tag{6}$$

The expansion is in terms of cosine functions, the reason being that $[\Delta(\theta, \phi)]^2$ is an even function. Considering that the series in Eq. (6) is absolutely convergent, it is possible to obtain an estimate of the error incurred in truncating the expansion using the first $k + 1$ terms. Taking the error to be

$$\text{Error} = \left|[\Delta(\theta, \phi)]^2 - \left\{\frac{\pi^2}{3} + 4\sum_{i=1}^{k} \frac{(-1)^i}{i^2}\cos[i(\theta - \phi)]\right\}\right| \tag{7}$$

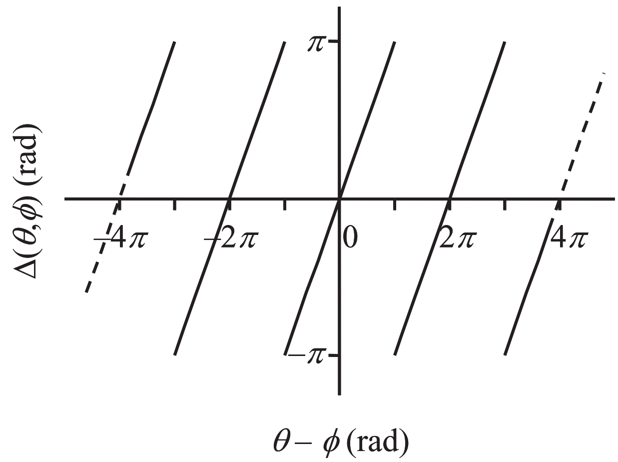

FIG. 1. Graph of $\Delta(\theta, \phi)$ against $\theta - \phi$. This is equivalent to the function $f(\theta - \phi) = \theta - \phi$ for $-\pi < \theta - \phi \le \pi$ and $f(\theta - \phi + 2\pi) = f(\theta - \phi)$.

and going through the algebra yields

$$\text{Error} \le 4\left|\frac{\pi^2}{6} - \sum_{i=1}^{k} \frac{1}{i^2}\right|. \tag{8}$$

Thus, the series converges as $1/i^2$. The same procedure cannot be used in the case of Eq. (4) because the series is not absolutely convergent.

Another series of interest is the Fourier expansion of $|\Delta(\theta, \phi)|$, which can once again be inferred from a graphical representation of the function (Fig. 2) as follows:

$$|\Delta(\theta, \phi)| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{i=0}^{\infty} \frac{\cos[(2i + 1)(\theta - \phi)]}{(2i + 1)^2}. \tag{9}$$

Its importance stems from the fact that it converges as $1/(2i + 1)^2$, which is faster than the convergence of Eq. (6). This can be verified from the maximum error incurred in truncating the series using the first $k + 1$ terms:

$$\text{Error} \le \frac{4}{\pi}\left|\frac{\pi^2}{8} - \sum_{i=0}^{k} \frac{1}{(2i + 1)^2}\right|. \tag{10}$$

The absolute value of the minimum angular difference can have another representation, this time in terms of powers of $\cos(\theta - \phi)$. Noting that from Eq. (2) $|\Delta(\theta, \phi)|$ can be algorithmically implemented as $\cos^{-1}[\cos(\theta - \phi)]$ and using the Taylor expansion of the arccosine function, it is easy to show that

$$|\Delta(\theta, \phi)| = \frac{1}{2}\pi - \sum_{i=0}^{\infty} \frac{(2i)!\cos^{2i+1}(\theta - \phi)}{2^{2i}(i!)^2(2i + 1)}, \tag{11}$$

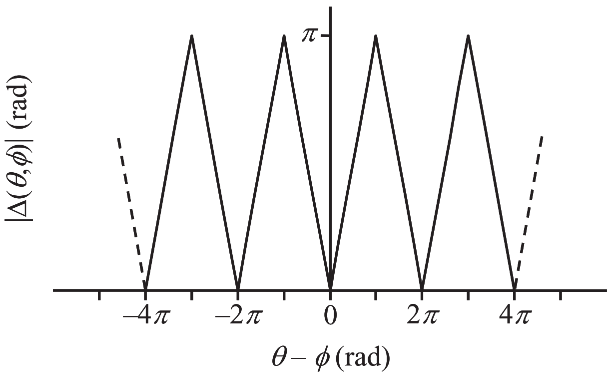

FIG. 2. Graph of $|\Delta(\theta, \phi)|$ against $\theta - \phi$. This is equivalent to the function $f(\theta - \phi) = |\theta - \phi|$ for $-\pi < \theta - \phi \leq \pi$ and $f(\theta - \phi + 2\pi) = f(\theta - \phi)$.

where the maximum error in truncating the expansion after $k + 2$ terms is given by

$$\text{Error} \leq \left| \frac{1}{2}\pi - \sum_{i=0}^{k} \frac{[(2i)!]^2}{2^{2i}(i!)^2(2i+1)!} \right|. \qquad (12)$$

The advantage of this representation becomes apparent when trying to determine the standard deviation from distributions that are expressed in terms of powers of $\cos(\theta - \phi)$, which can be the case if the distribution is symmetric about some value. For such situations, it should be simpler to work with Eq. (11).

From Fig. 1 it is easy to verify that $\Delta(\theta, \phi)$ is also equivalent to the following function:

$$\Delta(\theta, \phi) = 2\tan^{-1}\{\tan[0.5(\theta - \phi)]\}. \qquad (13)$$

This expression gives an easy single-step method of determining $\Delta(\theta, \phi)$—something that up till now was not available. Equation (13) can be used in numerical work, making computation simpler and more efficient and reducing the possibility of introducing errors during the programming stage. It also provides a way of expressing $\Delta(\theta, \phi)$ in terms of powers of $\tan[0.5(\theta - \phi)]$ using the Taylor expansion of the arctangent:

$$\Delta(\theta, \phi) = 2\sum_{i=1}^{\infty} \frac{(-1)^{i+1}\tan^{2i+1}[0.5(\theta - \phi)]}{2i+1}. \qquad (14)$$

3. The mathematical relation between the algorithms of Essenwanger and Yamartino

As of now there is no generally accepted equation to compute the standard deviation of wind direction, and researchers can choose between different algorithms for this purpose. Even though some qualitative work has been carried out to relate two of these equations (Weber 1997), without mathematical confirmation there can still be some concern as to the validity of the results obtained when comparing findings of work done with different algorithms. For this reason, in this section an analysis of the mathematical relation between two commonly adopted algorithms that are used to determine the standard deviation of wind direction is carried out. The two algorithms are those proposed by Yamartino (1984) and Essenwanger (1986). This will be accomplished using the expansions derived above.

For a sample of wind angles θ_i having size n, the unbiased standard deviation of wind direction s_Y—or, to be more precise, the variance of the wind direction, which is nothing more than the square of the standard deviation s_Y^2—as suggested by Yamartino (1984) is the following:

$$s_Y^2 = \frac{n}{n-1}\left\{\frac{1}{n}\sum_{i=1}^{n}[\Delta(\theta_i, \theta_a)]^2 - \left[\frac{1}{n}\sum_{i=1}^{n}\Delta(\theta_i, \theta_a)\right]^2\right\}, \qquad (15)$$

where θ_a is the mean angle of the sample that can be computed by mapping each angle θ_i to $[\cos(\theta_i), \sin(\theta_i)]$, a unit vector pointing in that direction, and then calculating the angle that the mean vector makes with the x direction. In algorithmic terms, this can be computed using the following:

$$\theta_a = \arctan\left(\frac{\overline{S}}{\overline{C}}\right), \qquad (16)$$

where $\overline{C}$ and $\overline{S}$ are the means of the cosines and the sines, respectively, given by

$$\overline{C} = \frac{1}{n}\sum_{i=1}^{n}\cos(\theta_i) \quad \text{and} \qquad (17)$$

$$\overline{S} = \frac{1}{n}\sum_{i=1}^{n}\sin(\theta_i). \qquad (18)$$

In the computation, the arctangent needs to be calculated taking into account in which quadrant $\overline{C}$ and $\overline{S}$ are located.

On the other hand, Essenwanger (1986) proposed the use of the following expression for the sample variance:

$$s_E^2 = \frac{1}{n-1}\sum_{i=1}^{n}[\Delta(\theta_i, \theta_M)]^2, \qquad (19)$$

where the value of θ_M is varied until Eq. (19) attains a minimum value, in analogy with the same property of the linear variables. Considering that in linear statistics the arithmetic mean $\overline{X}$ for a discrete random variable X of size n minimizes the population variance—that is,

$$\sigma_L^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - Y)^2 \tag{20}$$

attains its minimum for $Y = \overline{X}$—Essenwanger (1986) also proposes to use θ_M as the measure of the mean.

The first step needed to relate Eqs. (19) and (15) is to minimize Eq. (19). Replacing $[\Delta(\theta_i, \theta_M)]^2$ in Eq. (19) by the equivalent expression given by Eq. (6) gives the following:

$$s_E^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left\{\frac{\pi^2}{3} + 4\sum_{j=1}^{\infty}\frac{(-1)^j}{j^2}\cos[j(\theta_i - \theta_M)]\right\}. \tag{21}$$

Differentiating it with respect to θ_M and then setting the derivative equal to zero to determine the turning points yields

$$\frac{ds_E^2}{d\theta_M} = \frac{4}{n-1}\sum_{i=1}^{n}\sum_{j=1}^{\infty}\frac{(-1)^j}{j}\sin[j(\theta_i - \theta_M)] = 0. \tag{22}$$

To find a solution to this equation, a first-order approximation, valid in principle for small angles, is considered:

$$\frac{ds_E^2}{d\theta_M} \approx \frac{4}{n-1}\sum_{i=1}^{n}\sin(\theta_i - \theta_M) = 0. \tag{23}$$

Using basic trigonometry, it can be shown that two values of θ_M that would satisfy Eq. (23) are θ_a and $\pi + \theta_a$—something that can be verified by direct substitution. To find which one needs to be adopted, the first two terms of Eq. (21) are considered:

$$s_E^2 \approx \frac{n}{n-1}\frac{\pi^2}{3} - \frac{1}{n-1}\sum_{i=1}^{n}4\cos(\theta_i - \theta_M). \tag{24}$$

Using a first-order approximation requires that the magnitude of the $\Delta(\theta_i, \theta_M)$s has to be small, which means that the angles θ_i are all clustered around a single direction. Considering this procedure and the vectorial method used to obtain θ_a, it can be concluded that θ_a is somewhere in the middle of the cluster of the angles and hence $|\Delta(\theta_i, \theta_M)| < 0.5\pi$. Then, using $\theta_M = \theta_a$, $\cos(\theta_i - \theta_a) > 0$ for all values of i and Eq. (24) would take a value smaller than $[n/(n-1)](\pi^2/3)$. On the other hand, taking $\theta_M = \theta_a + \pi$, then $\cos(\theta_i - \theta_a + \pi) = -\cos(\theta_i - \theta_a) < 0$ for all values of i and Eq. (24) would take a value larger than $[n/(n-1)](\pi^2/3)$. Because θ_M should minimize s_E, we can deduce that, to a first-order approximation,

$$\theta_M \approx \theta_a. \tag{25}$$

This confirms the quantitative relationship in Weber (1997), where it was shown that, apart from some outliers that are located mainly where the values involved were large, these two angles show good agreement. Thus, the theoretical relation derived above seems to hold relatively well in a wide range of situations.

Once we have discriminated between the two possibilities that θ_M can attain, it is also possible to equate θ_M to what Yamartino (1984) defined to be the arithmetic mean angle $\overline{\theta}$, defined as

$$\sum_{i=1}^{n}\Delta(\theta_i, \overline{\theta}) = 0; \tag{26}$$

this can be directly verified by differentiating Eq. (19) with respect to θ_M and setting the result equal to zero. Thus, it follows that the difference between θ_M and θ_a is (Yamartino 1984)

$$\theta_a - \theta_M = \tan^{-1}\left\{\frac{\sum_{i=1}^{n}\sin[\Delta(\theta_i, \theta_M)]}{\sum_{i=1}^{n}\cos[\Delta(\theta_i, \theta_M)]}\right\}. \tag{27}$$

As a result, as mentioned by Yamartino (1984), the difference between the two angles depends on the level of skewness of the distribution. In practice, the skewness of the distribution for a sample of wind angles is very small. In fact, the isotropic Gaussian model, which gives a combined model for the wind speed and direction, predicts a symmetric distribution for the wind direction (Castans and Barquero 1994; Ibarra 1995). Also, in their work Verrall and Williams (1982) assumed the probability density for the angle to be a normal distribution; hence, the expected difference between θ_M and θ_a is small. This explains why they agree over a wide range of values.

Once the relation between θ_M and θ_a has been established, it follows that, to a first-order approximation, the second term in Yamartino's algorithm [Eq. (15)] is zero so that

$$s_Y \approx s_E. \tag{28}$$

This confirms the quantitative results obtained by Weber (1997) that the two algorithms have a similar behavior over the whole range of angular dispersion, with the algorithm of Essenwanger (1986) overestimating relative to that of Yamartino (1984) when the angular dispersion is high. Thus, once again, even though the relation was obtained by assuming small angles, it seems to hold well for most practical situations.

In the limiting case in which the angular distribution is perfectly symmetric around θ_a and n is sufficiently large, then

$$\sum_{i=1}^{n} \sin[j(\theta_i - \theta_a)] \to 0 \tag{29}$$

for any integer j. Hence, it follows from Eq. (22) that in this limit $\theta_a = \theta_M$ and, as a consequence, the algorithms given by Essenwanger (1986) and Yamartino (1984) would be exactly identical so that

$$s_Y = s_E. \tag{30}$$

This once again indicates that the difference in the values given by the two algorithms can be directly linked to the skewness of the distribution from which the sample has been taken.

4. Conclusions

In this work a number of methods for computing the minimum difference between two angles, its absolute value, and its powers are derived. This is carried out by expanding these quantities in terms of Fourier series and Taylor expansions. Furthermore, a simple single-step expression for calculating the minimum angular distance together with the relevant sign is given.

These expressions are then used to establish a mathematical relation between two different algorithms that are frequently used to determine the standard deviation of wind direction [i.e., those given by Essenwanger (1986) and Yamartino (1984)]. It is shown that the algorithms agree at least to a first approximation and that any difference is due to the skewness of the angular distribution. In the case in which the distribution of the angles is exactly symmetrical about the mean angle and the sample number is large, it has been shown that the two algorithms would give the same result. Such an outcome is significant because it gives backing to the qualitative findings by Weber (1997) and establishes that results obtained with one algorithm are consistent with those of the other.

The expansions presented here are meant to address the analytical and computational problems encountered when dealing with the calculation of the minimum angular distance and hence with the wind standard deviation. Their application to the problem of establishing an analytical relation between the algorithms given by Essenwanger (1986) and Yamartino (1984) is just one of their possible applications. In fact, they can also be employed to find exact analytical expressions from distributions, especially if these are expressed in terms of trigonometric functions as has already been carried out by Fisher (1987). A simple result can be inferred for distributions symmetric about the population mean μ. Considering that $[\Delta(\theta, \mu)]^n$ would be expressed as a cosine series if n is even and as a sine series if n is odd, it follows that the expectation values of $[\Delta(\theta, \mu)]^{2n+1}$ for symmetric distributions are always zero because in this case there will be an odd function integrated over a symmetric interval. Apart from this, if the computation of exact analytical results becomes too complicated, approximations may be adopted, and the number of terms needed for the required accuracy can be calculated beforehand using the expressions for the maximum errors derived in this work.

The Fourier series and Taylor expansions presented in this work can also find potential use in remote data acquisition systems employed in the gathering of wind statistics. For such systems, the storage capacity might be a limiting factor, and in such cases it would be inconvenient to store all of the data, which is in fact a necessary step needed to compute the multipass algorithms used to calculate the wind standard deviation. On the other hand, the inclusion of computation capabilities to reduce the datasets might not be practical. Thus, the expansions presented in this work can be used to derive direct approximations to these multipass algorithms by truncating the associated series so as to reduce the number of terms that need to be stored for each dataset. The number of terms can be chosen as a tradeoff between the impositions posed by the limited data capacity and the need for accuracy.

Given the extensive possible applications of the expansions and expressions derived here, it is envisaged that this work would in the future become an integral part of standard textbooks on the subject.

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APPENDIX

List of Symbols Used

$\overline{C}$	The average of the cosines of the angles of a sample
$f(x)$	A function defined as appropriate
i_v	The lateral turbulence intensity

n	A general integer or the size of a sample/population
$\overline{S}$	The average of the sines of the angles of a sample
$s_E\,(s_E^2)$	The angular standard deviation (variance) proposed by Essenwanger (1986)
s_{Lat}	The standard deviation representing the lateral dispersion of a plume
s_θ	General representation of the wind angle standard deviation
$s_Y\,(s_Y^2)$	The angular standard deviation (variance) proposed by Yamartino (1984)
x	A parameter defined as appropriate
$\overline{X}$	The population mean of the linear discrete random variable X
X_i	The values of the linear discrete random variable X
Y	A general number
$\Delta(\theta, \phi)$	The minimum angular distance between two angles θ and ϕ
μ	The mean angle of the population
$\overline{\theta}$	Arithmetic mean angle
θ_a	The mean angle of the sample computed using trigonometric moments
θ_i	The ith angle of a sample
θ_M	The mean angle computed from the minimization of the algorithm given by Essenwanger (1986) to compute the standard deviation of wind direction
θ and ϕ	General angles
σ_L^2	The population variance of the linear discrete random variable X

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