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REGULAR ARTICLE

# On circular correlation for data on the torus

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**Abstract** The circular correlation topic for data on the torus is studied. Firstly, the order for two points on the circumference is considered and an order function is defined. Then, an alternative moment coefficient to measure T-linear association between two circular variables based on the order function is proposed. After the concordant on the torus is explained, an alternative rank correlation coefficient on the torus is also proposed. A number of properties for the two coefficients are investigated and their comparisons with the existing alternatives are made. Two examples of real data analysis are presented to illustrate our results.

**Keywords** Circular correlation · Circular data · Kendall's $\tau$ · Circular order · T-linear

## 1 Introduction

Circular data collected in the form of angles apply broadly in a wide range of scientific research areas. A number of well-known examples include the wave directions in oceanography, the wind directions in meteorology and the animal movement directions in biology. Other examples are found in areas such as bioinformatics, evolution biology, molecular sciences and astronomy. Furthermore, circular data also arise from periodic data, for example, event times might be wrapped to a weekly period to give a circular

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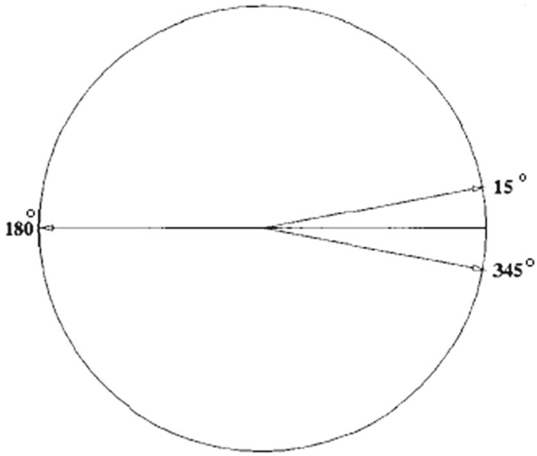
**Fig. 1** Mean direction for circular data




view of the pattern of event times. So such data can be seen in social sciences, medicine and ecology as well.

Circular data have an essential difference from linear data. It is obvious that the mean direction for 15° and 345° is 0°, but this is not the case for the arithmetic mean of the linear values 15 and 345; see Fig. 1.

The distance, order, summary statistics, distribution theory and correlation coefficients for circular data are all considered from a different point of view, see e.g. Jupp and Mardia (1980), Wehrly and Johnson (1980), Fisher and Lee (1982, 1983), Fisher and Hall (1989, 1990) and Shieh et al. (1994). The foundations and systematic studies can be found in e.g. Fisher (1993), Mardia and Jupp (2000) and Jammalamadaka and SenGupta (2001). An important and recent overall review of statistical methods for analyzing circular data was presented by Lee (2010) and Pewsey et al. (2013). Relevant studies include Abe and Pewsey (2011), Shieh et al. (2011), Kim and Sengupta (2013), Kato and Eguchi (2016), Liu et al. (2016) and SenGupta and Kim (2016).

The correlation relationship between two circular variables has little in common with its counterpart in the linear case. Bivariate circular data are expressed on a torus which is the product of two circles and therefore the four points $(0, 0)$, $(0, 2\pi)$, $(2\pi, 0)$, $(2\pi, 2\pi)$ are overlapping. Obviously, the torus has a different topological structure from the rectangle $[0, 2\pi) \times [0, 2\pi)$. In Fig. 2, when we plot the data on the rectangle, the correlation changes significantly after we move $P$ to $P^{'}$, and $Q$ to $Q^{'}$. When they are plotted on the torus, the changed points are very close to the original ones.

For linear data, there are two main approaches to describe the correlation relationship. One approach is moment correlation, and the most popular is Pearson's product moment correlation coefficient. The other one is rank correlation, including Spearman's $\rho$ and Kendall's $\tau$. It is then natural to consider these two approaches to study the correlation for circular data.

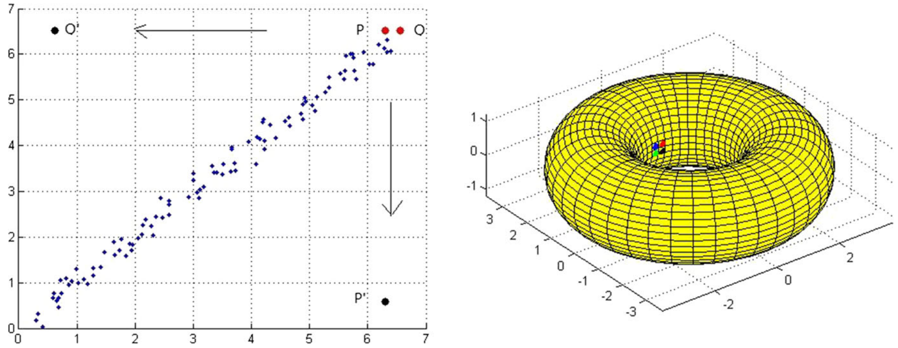



**Fig. 2** The different circumstances on the rectangle $[0, 2\pi) \times [0, 2\pi)$ and the torus

### 1.1 Moment correlation for circular data

There are several bivariate circular correlation coefficients which have previously been defined in the literature. These are the correlation coefficients of product moment type introduced by Downs (1974), Mardia (1975), Thompson (1975), Johnson and Wehrly (1977) and Mardia and Puri (1978). Jupp and Mardia (1980) gave a comprehensive review about these correlation coefficients and proposed a general correlation coefficient based on canonical correlations, followed by several concise forms defined in the literature.

Let $\Theta$ and $\Phi$ be two random circular variables with a joint distribution on the surface of a torus. A natural way of defining complete dependence of $\Theta$ and $\Phi$, corresponding to a linear relationship between two real random variables, is

$$\Theta = \Phi + \alpha_0 \quad \text{mod } (2\pi), \quad \text{positive association} \tag{1.1}$$

$$\Theta = -\Phi + \alpha_0 \quad \text{mod } (2\pi), \quad \text{negative association} \tag{1.2}$$

for some arbitrary fixed direction $\alpha_0$. Such dependence is referred to as toroidal-linear, or T-linear, corresponding to the strict linear relationship for linear data in Fisher and Lee (1983). Two examples are presented in Fig. 3.

Rivest (1982) proposed a signed measure of T-linear association and Jammalamadaka and Sarma (1988) defined a similar coefficient. Fisher and Lee (1983) discussed a very useful coefficient to measure association between angular variables. Let $(\Theta_1, \Phi_1)$ and $(\Theta_2, \Phi_2)$ be independently distributed as $(\Theta, \Phi)$, which are two angular random variables with a joint distribution on the surface of a torus. The correlation coefficient in Fisher and Lee (1983) has the following form

$$\rho_T = \frac{\mathrm{E}\{\sin(\Theta_1 - \Theta_2)\sin(\Phi_1 - \Phi_2)\}}{\sqrt{\mathrm{E}\{\sin^2(\Theta_1 - \Theta_2)\}\,\mathrm{E}\{\sin^2(\Phi_1 - \Phi_2)\}}}, \tag{1.3}$$

where E is the expectation. The correlation coefficient defined by Jammalamadaka and Sarma (1988) is as follows

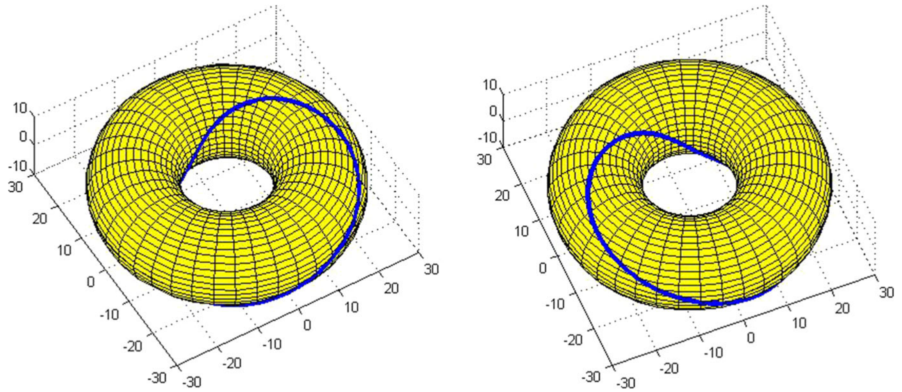


**Fig. 3** $\Theta = \Phi + 0.1\pi \mod (2\pi)$ and $\Theta = -\Phi + 0.1\pi \mod (2\pi)$

$$\rho_c = \frac{\mathrm{E}\{\sin(\Theta - \mu)\sin(\Phi - \nu)\}}{\sqrt{\mathrm{Var}\{\sin(\Theta - \mu)\}\,\mathrm{Var}\{\sin(\Phi - \nu)\}}}, \tag{1.4}$$

where Var is the variance, $\mu$ and $\nu$ are the circular means for $\Theta$ and $\Phi$, respectively.

As we see, $\rho_T$ and $\rho_c$ have similar forms, however $\rho_c$ is very close to Pearson's product moment correlation coefficient. The two correlation values can take either 1 or $-1$, which corresponds to a strict positive and a strict negative T-linear association, respectively.

As expected, they maintain many of the properties for the classical product moment correlation coefficient $\rho$. However, the sine of an angle contains considerably less information than the angle itself. Moreover, it does not meet the regulation of circular data that the sine function is not monotone within an interval whose length is $\pi$. This may bring unreasonable results for some data analysis in terms of theory and practice in unusual situations.

Let us now look at sine in $\rho_T$, which is one of the most important and useful coefficients in directional statistics. One estimator of $\rho_T$ involving sine is

$$\hat{\rho}_T = \frac{\sum_{1\leqslant i<j\leqslant n}\sin(\theta_j - \theta_i)\sin(\phi_j - \phi_i)}{\sqrt{\{\sum_{1\leqslant i<j\leqslant n}\sin^2(\theta_j - \theta_i)\}\{\sum_{1\leqslant i<j\leqslant n}\sin^2(\phi_j - \phi_i)\}}}. \tag{1.5}$$

For convenience of explanation, we only choose three points $(\theta_1, \phi_1)$, $(\theta_2, \phi_2)$ and $(\theta_3, \phi_3)$, plotted on two concentric circles in Fig. 4. On the left, $\theta_1 = 0$, $\theta_2 = \frac{\pi}{8}$, $\theta_3 = \pi$ and $\phi_i = \theta_i + \frac{\pi}{2} \mod (2\pi)$ for $i = 1, 2, 3$. Obviously, there is a strict T-linear relationship between the two variables and the value of $\hat{\rho}_T$ is equal to 1. When $\theta_2$ is changed to be $\theta_2^* = \frac{7\pi}{8}$, and the other five values remain unchanged, as we can see, the strict T-linear relationship is broken, but the value calculated by Eq. (1.5) for $\hat{\rho}_T$ remains the same. This example indicates that the sine function being used in the coefficient does not perform as we hope.

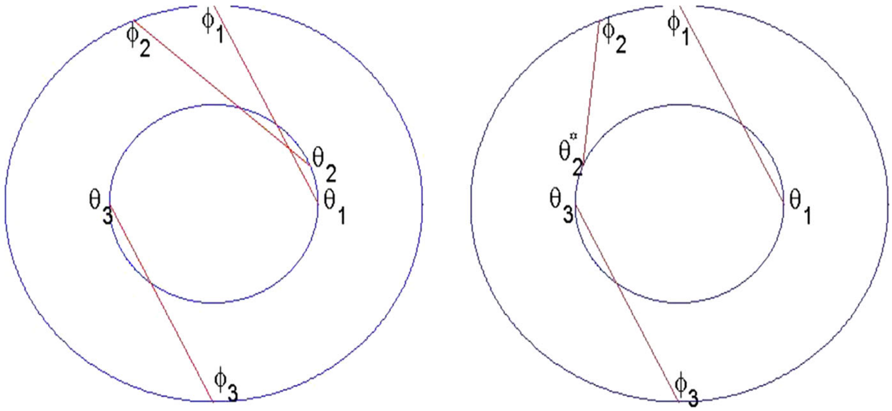



**Fig. 4** A simple example to show the defect of sine

## 1.2 Rank correlation for circular data

Fisher and Lee (1982) considered an important and useful nonparametric measure for circular-circular association. A concept 'T-concordant' was defined by a relationship $\phi = g(\theta)$ in their paper. As $\theta$ moves continuously through a complete revolution in a particular sense of direction, either clockwise or anticlockwise, $\phi$ also moves continuously in a particular sense of direction through a complete revolution. If the senses are the same, then the relationship $\phi = g(\theta)$ is said to be T-concordant. Otherwise, if one sense of direction is clockwise and the other one is anticlockwise, then $\phi = g(\theta)$ is T-discordant.

We let $P_i = (\theta_i, \phi_i)$ $(i = 1, \ldots, n)$ denote $n$ points on the torus, and then define the points to be T-concordant if there exists a T-concordant relationship $\phi = g(\theta)$ such that $\phi_i = g(\theta_i)$ $(i = 1, \ldots, n)$ and define the points to be discordant in a similar manner. Finally, we let $P = (\theta, \phi)$ be randomly distributed on the torus and measure the association between $\theta$ and $\phi$ by

$$\Delta = \mathrm{pr}(P_1, P_2, P_3 \text{ are T-concordant}) - \mathrm{pr}(P_1, P_2, P_3 \text{ are T-discordant}), \quad (1.6)$$

where $P_1$, $P_2$ and $P_3$ are independent random points distributed as $P$.

This measurement has a similar form to Kendall's $\tau$ in the linear case. A U-statistic for estimating $\Delta$ is

$$\hat{\Delta}_n = \binom{n}{3}^{-1} \sum_{1 \leqslant i < j < k \leqslant n} \delta_{p_i, p_j, p_k}, \quad (1.7)$$

where

$$\delta_{p_1, p_2, p_3} = \mathrm{sgn}(\theta_1 - \theta_2)\,\mathrm{sgn}(\theta_2 - \theta_3)\,\mathrm{sgn}(\theta_3 - \theta_1) \times \mathrm{sgn}(\phi_1 - \phi_2)\,\mathrm{sgn}(\phi_2 - \phi_3)\,\mathrm{sgn}(\phi_3 - \phi_1).$$

An analogue of Spearman's $\rho$ for assessing T-concordance was also discussed by Fisher and Lee (1982). The linear ranks for circular data were considered and the trigonometric functions were used to meet the periodicity. Shieh et al. (1994) proposed a weighted degenerate U-statistics, which combined together Kendall's $\tau$ and Spearman's $\rho$ in the linear case. After being multiplied by a constant related to sample size, it coincides with the statistic $\hat{\Delta}_n$ in Fisher and Lee (1982).

In this paper, we first define the order for two circular data, and then consider a new product moment type correlation coefficient which involves the angle itself only. It has the virtue of fast calculation and we hope it overcomes the non-monotonicity shortcoming for those situations of angular data analysis. Meanwhile, a toroidal analogue of Kendall's $\tau$ is defined based on the order for circular data. It is easy to understand and can maintain many of the properties of Kendall's $\tau$.

This paper proceeds in Sect. 2 by defining the order for the circular data. Section 3 presents our new correlation coefficient and its estimator. Section 4 presents the toroidal analogue of Kendall's $\tau$, discussing about the estimator and the properties. Some examples and the detailed presentation of the real data analysis are in Sect. 5. Section 6 concludes the paper.

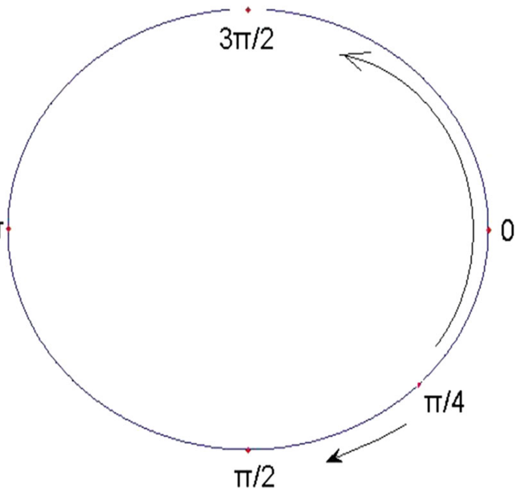
## 2 The circular order

The order of linear data is considered by their numerical values. Any sequence of linear data can be sorted as they are expressed in a straight line. For circular data, a similar approach for sorting sequences needs to be proposed. However, the most significant difference between the circular data and the linear data is that the circular data are expressed on the unit circle, and they have cyclic properties. Intuitively, the order for circular data is always described by clockwise direction and anticlockwise direction, but the order is not unique when we only consider two data points on the circumference. Accordingly, we set the order by rotating them along the shorter arc by default. For example, the order from $\frac{\pi}{4}$ to $\frac{\pi}{2}$ is clockwise and the order from $\frac{\pi}{4}$ to $\frac{3\pi}{2}$ is anticlockwise (Fig. 5). We compare the difference of two data points with $\pi$ to determine the order for two circular data ranged within $2\pi$. A specific definition is given as follows.

**Definition 1** (*Clockwise and anticlockwise*) Let $\alpha$ and $\beta$ be two circular data points and set $0 \le \alpha, \beta < 2\pi$, so $-2\pi < \alpha - \beta < 2\pi$. When $-\pi < \alpha - \beta \le 0$ or $\alpha - \beta > \pi$, we say that the order from $\alpha$ to $\beta$ is clockwise. When $\alpha - \beta \le -\pi$ or $0 < \alpha - \beta \le \pi$, we say that the order from $\alpha$ to $\beta$ is anticlockwise.

Obviously, the order defined above is free of the choice of origin but it is worth noting that it has no transitivity. That means we can sort two circular data points, but cannot give a rank to a series containing more than two observations. For example, let us consider the three data points in the previous paragraph. The order from $\frac{\pi}{4}$ to $\pi$ is clockwise, the order from $\pi$ to $\frac{3\pi}{2}$ is clockwise, but the order from $\frac{\pi}{4}$ to $\frac{3\pi}{2}$ is anticlockwise. In addition, when the arc distance for $\alpha$ and $\beta$ is $\pi$, the order from $\alpha$ to $\beta$ is neither clockwise nor anticlockwise.

Fig. 5 The order on the circumference



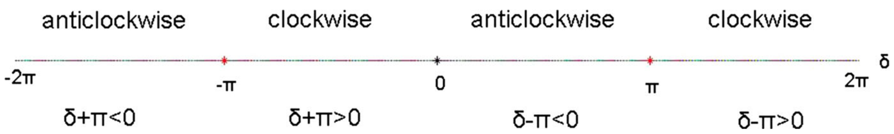




**Fig. 6** The order considered by $\delta$

As we can see, the interval $(-2\pi, 2\pi)$ is divided into four subintervals in the definition, now we treat $\alpha - \beta$ as a whole and let $\delta = \alpha - \beta$. When $\delta$ lies in intervals $(-2\pi, -\pi] \cup (0, \pi]$ the order from $\alpha$ to $\beta$ is anticlockwise, and when delta lies in the other two intervals the order is clockwise (Fig. 6). In order to show this more clearly, we choose two points $-\pi$ and $\pi$ as centers and then consider the span of $\delta$ in two intervals $(-2\pi, 0]$ and $(0, 2\pi)$. In the first interval, the sign of $\delta + \pi$ is considered, with positive corresponding to clockwise and negative corresponding to anticlockwise. In the second interval, the sign of $\delta - \pi$ is considered, with positive corresponding to clockwise and negative corresponding to anticlockwise. Obviously, a function needs to be defined for combining the two circumstances together. Considering the cyclic behavior of circular data, we use the modulus operator who return to a remainder to propose an order function.

**Definition 2** (*The order function*) Let $\alpha$ and $\beta$ be two circular data and set $0 \le \alpha, \beta < 2\pi$. An order function h is defined as

$$h(\alpha, \beta) = \left[(\alpha - \beta + 2\pi) \mod 2\pi\right] - \pi, \tag{2.1}$$

or changing the function argument into $\delta$,

$$h(\delta) = [(\delta + 2\pi) \mod 2\pi] - \pi = \begin{cases} \delta + \pi & -2\pi < \delta < 0, \\ \delta - \pi & 0 \le \delta < 2\pi. \end{cases} \tag{2.2}$$

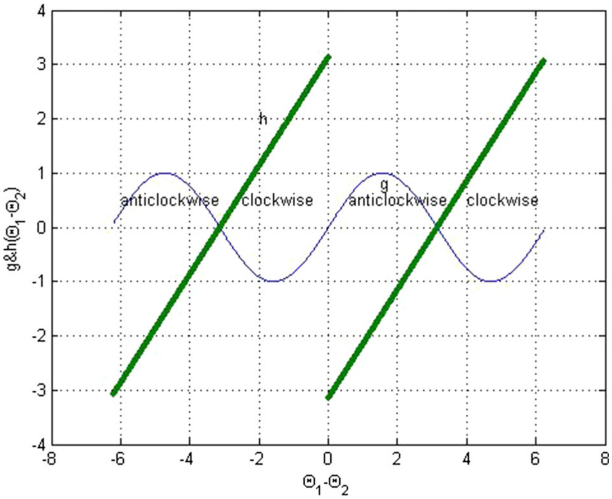



**Fig. 7** Two functions $g(\Theta_1, \Theta_2) = \sin(\Theta_1 - \Theta_2)$ and $h(\Theta_1, \Theta_2) = \left[(\Theta_1 - \Theta_2 + 2\pi) \mod 2\pi\right] - \pi$

We name $h$ order function because it can be used to identify the order of two circular observations (Fig. 7). When the order from $\alpha$ to $\beta$ is clockwise, $h(\alpha, \beta) > 0$. Otherwise, when the order from $\alpha$ to $\beta$ is anticlockwise, $h(\alpha, \beta) < 0$. However, special attention needs to be paid to the value zero. Now to summarize, the order from $\alpha$ to $\beta$ corresponding to the sign of $h$ is as follows

$$\begin{array}{ll} h(\alpha, \beta) > 0 & \text{clockwise,} \\ h(\alpha, \beta) < 0 & \text{anticlockwise.} \end{array} \tag{2.3}$$

Let us go back to see the data mentioned above. The order from $\frac{\pi}{4}$ to $\frac{\pi}{2}$ is clockwise and $h(\frac{\pi}{4}, \frac{\pi}{2}) = -\frac{\pi}{4} + \pi = \frac{3\pi}{4} > 0$. The order from $\frac{\pi}{4}$ to $\pi$ is clockwise and $h(\frac{\pi}{4}, \pi) = -\frac{3\pi}{4} + \pi = \frac{\pi}{4} > 0$. The order from $\frac{\pi}{4}$ to $\frac{3\pi}{2}$ is anticlockwise and $h(\frac{\pi}{4}, \frac{3\pi}{2}) = -\frac{5\pi}{4} + \pi = -\frac{\pi}{4} < 0$. In summary, the order clockwise or anticlockwise from $\alpha$ to $\beta$ strictly corresponds to the sign of $h(\alpha, \beta)$.

## 3 A new product moment circular-circular correlation coefficient

Pearson's product moment correlation coefficient is widely used in the sciences as a measure of the degree of linear dependence between two variables. Based on the order function, a new product moment circular-circular correlation coefficient corresponding to the T-linear relationship for circular data is proposed in this section. The definition and properties are given in Sect. 3.1, and the estimation problems are discussed in Sect. 3.2.

### 3.1 A new product moment circular-circular correlation coefficient

A new circular correlation coefficient using the order function $h$ is given in the following definition.

**Definition 3** (*A new circular-circular correlation coefficient*) Suppose that $(\Theta_1, \Phi_1)$ and $(\Theta_2, \Phi_2)$ are independently distributed as $(\Theta, \Phi)$. Then, a circular correlation coefficient is defined by

$$\rho_o = \frac{\mathrm{E}\left\{h(\Theta_1, \Theta_2)h(\Phi_1, \Phi_2)\right\}}{\sqrt{\mathrm{E}\left\{\left[h(\Theta_1, \Theta_2)\right]^2\right\}\mathrm{E}\left\{\left[h(\Phi_1, \Phi_2)\right]^2\right\}}}, \tag{3.1}$$

where E is the expectation and h is the same as defined in (2.1).

Details of the calculation can be seen in Appendix 1.

Obviously, if $h(\Theta_1, \Theta_2)h(\Phi_1, \Phi_2) > 0$ then $\rho_o > 0$. Here $\rho_o > 0$ indicates the two variables $\Theta$ and $\Phi$ move on the circumference in the same direction and $\rho_o < 0$ indicates the opposite.

**Proposition 1** *The measure $\rho_o$ also has the following properties.*

1. $-1 \leq \rho_o \leq 1$.
2. $\rho_o = 1$ *if and only if $\Theta$ and $\Phi$ are related by (1.1), and $\rho_o = -1$ if and only if (1.2) holds.*
3. $\rho_o$ *is invariant under choice of origin for $\Theta$ and $\Phi$.*
4. *If $\Theta$ and $\Phi$ are independent, then* $\rho_o = 0$.
5. *If the distributions of $\Theta$ and $\Phi$ are each unimodal and highly concentrated,* $\rho_o(\Theta, \Phi) \approx \rho(\Theta, \Phi)$.

### 3.2 Estimation of the new correlation coefficient

Given a random sample $P_i = (\theta_i, \phi_i)$ $(i = 1, \ldots, n)$ from a distribution on the torus, a natural estimator of $\rho_o$ is as follows.

**Proposition 2** (Estimator for the new coefficient)

$$\hat{\rho}_o = \frac{\sum_{1\leqslant i<j\leqslant n} h(\theta_i, \theta_j)h(\phi_i, \phi_j)}{\sqrt{\{\sum_{1\leqslant i<j\leqslant n}[h(\theta_i, \theta_j)]^2\}\{\sum_{1\leqslant i<j\leqslant n}[h(\phi_i, \phi_j)]^2\}}}. \tag{3.2}$$

*The estimator loses no information as it is about the angles themselves. Furthermore, the calculation is convenient and quick.*

**Theorem 1** (Asymptotic properties of $\hat{\rho}_o$) *$\hat{\rho}_o$ is a consistent estimator for $\rho_o$ and it shares Properties 1 to 3 of $\rho_o$. $\sqrt{n(n-1)}(\hat{\rho}_o - \rho_o)$ converges in distribution to the normal law with mean zero and variance $2(\gamma - \rho_o^2)$, where $\gamma > \rho_o^2$ is a constant.*

**Table 1** Comparison of $\hat{\rho}_o$ with sine-based $\hat{\rho}_T$

| Variables | $\Theta$ | $\Phi$ | $\Theta$ | $\Phi$ |
|---|---|---|---|---|
| Observations | 0 | $\pi/2$ | 0 | $\pi/2$ |
| | $\pi/8$ | $5\pi/8$ | $7\pi/8$ | $5\pi/8$ |
| | $\pi$ | $3\pi/2$ | $\pi$ | $3\pi/2$ |
| $\hat{\rho}_T$ | 1 | | 1 | |
| $\hat{\rho}_o$ | 1 | | 0.28 | |

The specific form of $\gamma$ and the proof of Theorem 1 are given in Appendix 2.

As for a time series $\{\theta_t, t \in T\}$, we define the moment autoregressive coefficient of lag $v$ as follows,

$$\hat{\rho}_o^v = \frac{\sum\sum_{1\le t<s\le n-v} h(\theta_t, \theta_s)h(\theta_{t+v}, \theta_{s+v})}{\sqrt{\sum\sum_{1\le t<s\le n-v}[h(\theta_t, \theta_s)]^2 \sum\sum_{1\le t<s\le n-v}[h(\theta_{t+v}, \theta_{s+v})]^2}}. \tag{3.3}$$

### 3.3 Comparison of $\rho_o$ with $\rho_T$

The estimator for $\rho_T$ is used widely in circular statistics for its simple form and convenient calculation. Compared with it, the new coefficient can not only have these advantages but also overcome its defects. We define a function $g(\Theta_1, \Theta_2) = \sin(\Theta_1 - \Theta_2)$ (Fig. 7). As we see, $\sin(\Theta_1 - \Theta_2) = \sin(\pi - (\Theta_1 - \Theta_2))$, when $\Theta_1^* = \pi - \Theta_1 + 2\Theta_2$, $g(\Theta_1^*, \Theta_2) = g(\Theta_1, \Theta_2)$. In other words, the function $g$ is not monotone within the range of $\pi$ and there will be some unreasonable results in both theory and practice. Let us reconsider the example in Sect. 1.1 and the correlation results are listed in Table 1.

As we can see, though the strict T-linear relationship is broken, the value for sine-based $\hat{\rho}_T$ remains unchanged, but the new coefficient $\hat{\rho}_o$ changes to be 0.28. So our new coefficient can cope with data in such situations to avoid the defects of sine.

## 4 A toroidal analogue of Kendall's $\tau$

Generally speaking, the random variables are said to be associated when they are not independent. There are a number of concepts of association in linear statistics. Kendall's $\tau$ concentrates on concordance which aims to capture the fact that the probability of having 'large' (or 'small') values of both variables X and Y is high, while the probability of having 'large' values of X together with 'small' values of Y, or vice versa, is low. On the torus, we may consider this association problem according to the clockwise and anticlockwise order for the circular data. In this section, we first discuss the concordant on the torus and then give the definition for Kendall's $\tau$ on the torus to consider the properties and the estimation.

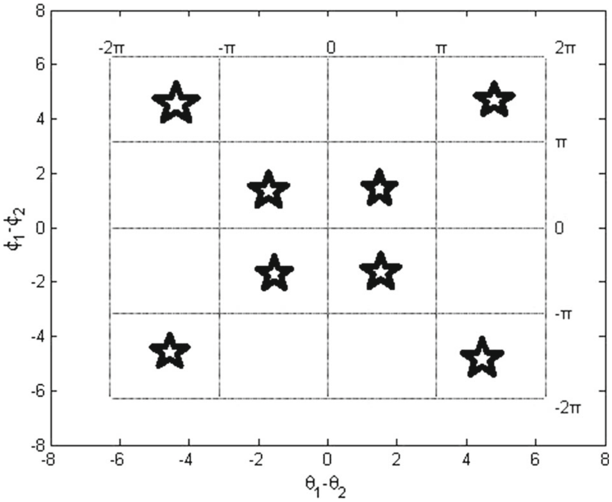



**Fig. 8** $(\theta_1, \phi_1)$ and $(\theta_2, \phi_2)$ are concordant when $(\theta_1 - \theta_2, \phi_1 - \phi_2)$ lies in the areas with a star, and discordant when $(\theta_1 - \theta_2, \phi_1 - \phi_2)$ lies in the in other areas

### 4.1 Concordant on the torus

Let $P_1 = (\theta_1, \phi_1)$ and $P_2 = (\theta_2, \phi_2)$ be two points on the torus. We define the two points to be *concordant* if the order from $\theta_1$ to $\theta_2$ and the order from $\phi_1$ to $\phi_2$ are both clockwise or anticlockwise. We define the two points to be *discordant* if the order from $\theta_1$ to $\theta_2$ is clockwise while the order from $\phi_1$ to $\phi_2$ is anticlockwise, and vice versa.

We may here recall the order function $h(\theta_1, \theta_2)$ defined in Definition 2. Its sign exactly corresponds to the order from $\theta_1$ to $\theta_2$, so the concordant on the torus can be defined by $h$ in an immediate way. If $h(\theta_1, \theta_2)$ and $h(\phi_1, \phi_2)$ have the same signs, we then define the two points to be *concordant*. If $h(\theta_1, \theta_2)$ and $h(\phi_1, \phi_2)$ have different signs we then define the two points to be *discordant*. Or in a simple form,

$$\begin{array}{ll} h(\theta_1, \theta_2)h(\phi_1, \phi_2) > 0 & \text{concordant}, \\ h(\theta_1, \theta_2)h(\phi_1, \phi_2) < 0 & \text{discordant}. \end{array} \tag{4.1}$$

See Fig. 8.

### 4.2 A toroidal analogue of Kendall's $\tau$

Now we transplant Kendall's $\tau$ in the linear case to the circular data.

**Definition 4** (*Toroidal analogue of Kendall's* $\tau$) Suppose that $(\Theta_1, \Phi_1)$ and $(\Theta_2, \Phi_2)$ are independently distributed as $(\Theta, \Phi)$. We define a toroidal analogue of Kendall's $\tau$, $\tau_o$ to be the difference between the probabilities of concordant and discordant of $(\Theta_1, \Phi_1)$ and $(\Theta_2, \Phi_2)$, i.e.,

$$\tau_o = \tau_{\Theta,\Phi} = \mathrm{pr}\big[h(\Theta_1, \Theta_2)h(\Phi_1, \Phi_2) > 0\big] - \mathrm{pr}\big[h(\Theta_1, \Theta_2)h(\Phi_1, \Phi_2) < 0\big]. \tag{4.2}$$

More details can be seen in Appendix 3. We can see that the range of $\tau_o$ is $[-1, 1]$ and a positive value indicates that the probability of both $\Theta$ and $\Phi$ rotating clockwise (or anticlockwise) is high, while the probability of $\Theta$ rotating clockwise together with $\Phi$ rotating anticlockwise, or vice versa, is low.

As we know, copulas are useful tools for describing multivariate data with complex dependence and in the linear case Kendall's $\tau$ has a concise calculating formula in the copula frame. It is worth mentioning that Kendall's $\tau$ on the torus defined above has an identical form.

**Theorem 2** *Let $(\Theta_1, \Phi_1)$, $(\Theta_2, \Phi_2)$ be independent vectors of continuous random variables with joint distributions $H_1$ and $H_2$, respectively, with common marginal cumulative distribution functions F (of $\Theta_1$ and $\Theta_2$) and G (of $\Phi_1$ and $\Phi_2$). Let $C_1$ and $C_2$ denote the copulas of $(\Theta_1, \Phi_1)$ and $(\Theta_2, \Phi_2)$, respectively, so that $H_1(\theta, \phi) = C_1\big(F(\theta), G(\phi)\big)$ and $H_2(\theta, \phi) = C_2\big(F(\theta), G(\phi)\big)$. Using the probability transforms $u = F(\theta)$ and $v = G(\phi)$, we have,*

$$\tau_o = 4\int\int_{I^2} C_2(u, v)dC_1(u, v) - 1. \tag{4.3}$$

*If the data are from the same joint distribution, the theorem is reduced to the following form.*

**Theorem 3** *Let $\Theta$ and $\Phi$ be continuous random variables whose copula is C. Then the population version of Kendall's $\tau$ for $\Theta$ and $\Phi$ (which we will denote by $\tau_C$) is given by*

$$\tau_C = 4\,\mathrm{E}\left(C(U, V)\right) - 1, \tag{4.4}$$

*where* E *is the expectation, $U = F(\Theta)$ and $V = G(\Phi)$.*

The proofs for the two theorems above are given in Appendix 4.

### 4.3 Estimation of the toroidal analogue of Kendall's $\tau$

**Proposition 3** (Estimator for $\tau_o$) *Given a random sample $P_i = (\theta_i, \phi_i)$ $(i = 1, \ldots, n)$ from a distribution on the torus, a natural estimator of $\tau_o$ is as follows:*

$$\hat{\tau}_o = \binom{n}{2}^{-1} \sum_{1 \leqslant i < j \leqslant n} \mathrm{sgn}\left(h(\theta_1, \theta_2)\right) \mathrm{sgn}\left(h(\phi_1, \phi_2)\right). \tag{4.5}$$

*A detailed form is in Appendix 5 and the asymptotic property of $\hat{\tau}_o$ is given in the following theorem.*

**Theorem 4** (Asymptotic property of $\hat{\tau}_o$) *$\hat{\tau}_o$ is a consistent estimator for $\tau_o$ and $\sqrt{n(n-1)}(\hat{\tau}_o - \tau_o)$converges in distribution to the normal law with mean 0 and variance $2(1 - \tau_o^2)$.*

*The proof of Theorem 4 is given in Appendix 6.*

**Table 2** Comparison of $\hat{\tau}_o$ with $\hat{\Delta}_n$

| Variables | $\Theta$ | $\Phi$ | $\Theta$ | $\Phi$ |
|---|---|---|---|---|
| Observations | 0 | $\pi/2$ | 0 | $\pi/2$ |
| | $\pi/8$ | $5\pi/8$ | $9\pi/8$ | $5\pi/8$ |
| | $11\pi/8$ | $15\pi/8$ | $11\pi/8$ | $15\pi/8$ |
| $\hat{\Delta}_n$ | 1 | | 1 | |
| $\hat{\tau}_o$ | 1 | | $-0.33$ | |

**Proposition 4** *Similar to the linear case, $\hat{\tau}_o$ has the following properties.*

1. $-1 \leq \hat{\tau}_o \leq 1$.
2. $\hat{\tau}_o = 1$ *if and only if* $P_1, \ldots, P_n$ *are all concordant, and* $\hat{\tau}_o = -1$ *if and only if* $P_1, \ldots, P_n$ *are all discordant.*
3. $\hat{\tau}_o$ *is invariant under choice of origin for* $\Theta$ *and* $\Phi$.

*As for a time series* $\{\theta_t, t \in T\}$, *the rank autocorrelation coefficient of lag* $v$ *is defined as*

$$\hat{\tau}_o^v = \binom{n}{2}^{-1} \sum_{1 \leqslant t < s \leqslant n-v} \operatorname{sgn}\left(h(\theta_t, \theta_s)\right) \operatorname{sgn}\left(h(\phi_{t+v}, \phi_{s+v})\right). \tag{4.6}$$

### 4.4 Comparison of $\tau_o$ with $\Delta$

The estimator $\hat{\Delta}_n$ in Eq. (1.7) is called 'Circular Kendall's Tau' by Shieh et al. (1994). All three points are used to calculate $\hat{\Delta}_n$ while two points are used to calculate $\hat{\tau}_o$. Usually the calculated values for the two estimators are different. An example is presented in Table 2. When the second value for $\theta$ changes from $\pi/8$ to $9\pi/8$ and the other five values remain the same, the value for $\hat{\Delta}_n$ remains unchanged but the new coefficient $\hat{\tau}_o$ changes to be $-0.33$.

## 5 Two real data examples

*Example 1* (Downs 1974) The peak times for two successive measurements of blood pressure, converted into angles, of 10 medical students were recorded (Fig. 9). The estimated value $\hat{\rho}_o$ for these data is 0.8204, and $\hat{\tau}_o$ is 0.8667.

*Example 2* (*Wind direction*) The data are from a survey of the real wind direction data compiled by the Arctic RIMS project.[1] There are 39926 observation stations, on which wind direction data from January 1980 to March 2006 have been recorded daily. We calculate $\hat{\rho}_o$ and $\hat{\tau}_o$ in three cases.

[1] http://rims.unh.edu (A Regional, Integrated Hydrological Monitoring System for the Pan-Arctic Land Mass).

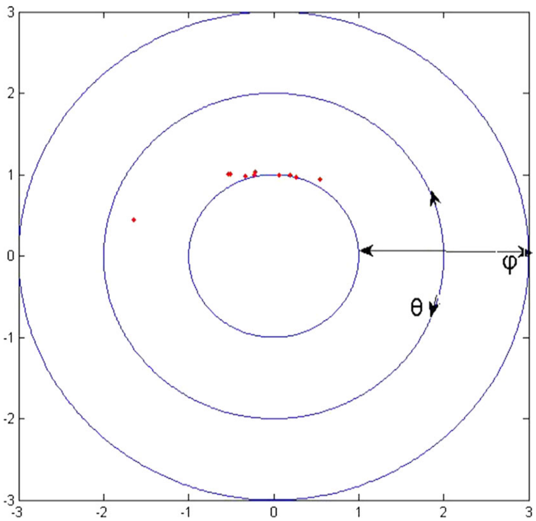



**Fig. 9** Peak times of two successive measurements $\theta$ and $\phi$, of blood pressure (Downs 1974). Data are projected from the surface of a torus onto a plane normal to the axis of the torus

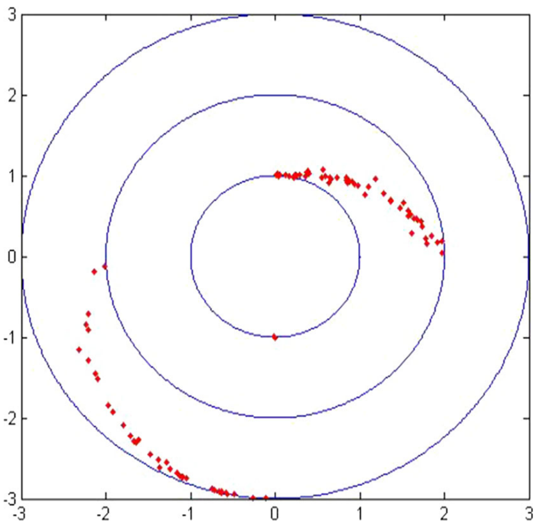


**Fig. 10** The wind direction data of observation stations 1 and 2 are plotted. Projection with data on concealed half of torus appearing as open circles

First, we calculate the correlation coefficients between different stations. A plot of the wind direction data for stations 1 and 2 are shown in Fig. 10. For convenience, we choose station 1 whose location (latitude, longitude) is (56.5398, 201.656) to be the comparison object and do the calculations using the wind direction data of 9587 days. The correlation coefficients between station 1 and five stations nearby are calculated in Table 3.

Then, we calculate some first-lag autocorrelation coefficients for the data from the first 100 stations. The first seven coefficients i.e. the correlation coefficient between the first day and the second day, the correlation coefficient between the second day and the third day and so on up to the correlation coefficient between the seventh day

**Table 3** The correlation coefficients between station 1 and five stations nearby

| Station | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| Latitude | 56.6259 | 56.7107 | 56.6703 | 56.7584 | 56.8451 |
| Longitude | 201.2912 | 200.9245 | 202.1664 | 201.8014 | 201.4346 |
| $\hat{\rho}_o$ | 0.8492 | 0.6954 | 0.7989 | 0.9024 | 0.8554 |
| $\hat{\tau}_o$ | 0.8909 | 0.7907 | 0.8640 | 0.9196 | 0.8580 |

**Table 4** Seven first-order autocorrelation coefficients

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $\hat{\rho}_o$ | 0.4906 | 0.2940 | 0.2490 | −0.2073 | −0.4819 | −0.1570 | 0.5364 |
| $\hat{\rho}_T$ | 0.0174 | 0.0497 | 0.1687 | −0.2656 | −0.0629 | −0.2690 | 0.0891 |
| $\hat{\tau}_o$ | 0.2861 | 0.1929 | 0.2220 | −0.1881 | −0.1335 | −0.2014 | 0.2131 |
| $\hat{\Delta}_n$ | 0.5526 | 0.1602 | 0.0365 | −0.1602 | −0.2888 | −0.0642 | 0.4393 |

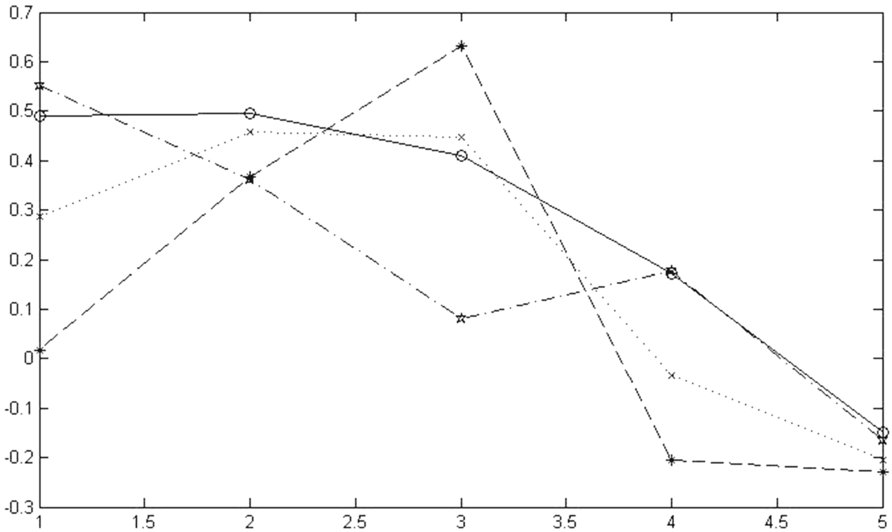



**Fig. 11** Autocorrelation coefficients for the data of the first day from order 1 to order 5. The *solid line with circles* is for $\hat{\rho}_o$, *long-dashed line with stars* is for $\hat{\rho}_T$, *short-dashed line with multiplication signs* is for $\hat{\tau}_o$ and the *line with pentagrams* is for $\hat{\Delta}_n$

and the eighth day, are listed in Table 4, with the values for $\hat{\rho}_T$ and $\hat{\Delta}_n$ calculated for comparison purposes.

We find that the values for $\hat{\rho}_o$ are more concentrated than $\hat{\rho}_T$ which may make a visible difference. However, the values for $\hat{\tau}_o$ are mostly stable and a stable first-order autocorrelation coefficient is reasonable for a stationary time series.

Finally, using the data of the first 100 stations, the autocorrelation coefficients for higher orders are also calculated. The coefficients from order 1 to order 5 for the wind direction of the first day i.e. the autocorrelation coefficient between the first day and the second day, the autocorrelation coefficient between the first day and the third day and so on up to the autocorrelation coefficient between the first day and the sixth day are plotted in Fig. 11.

As we can see, the values for $\hat{\rho}_o$ progressively decreases as the order increases and this is reasonable for a stationary time series. The performance for $\hat{\rho}_T$ is quite irregular.

## 6 Concluding remarks

In this paper, a moment correlation coefficient to measure T-linear association between circular measurements on the torus and a rank correlation based on the order function are studied. The order function introduced is the key to our discussion. In addition, the correlation coefficients' estimators and properties are studied. Compared with the alternative counterparts, our estimators can overcome those problems caused by non-monotonicity of sine function, and perform very well in the real data analysis.

Further research and extension can be considered for correlations in three dimensions for spherical data and for correlations involving both circular and linear data.

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## Appendices

### Appendix 1: Calculation of the new circular-circular correlation coefficient

The new correlation coefficient is defined as

$$\rho_o = \frac{\mathrm{E}\{h(\Theta_1, \Theta_2)h(\Phi_1, \Phi_2)\}}{\sqrt{\mathrm{E}\{[h(\Theta_1, \Theta_2)]^2\}E\{[h(\Phi_1, \Phi_2)]^2\}}}. \tag{6.1}$$

In order to calculate the correlation coefficient, we divide the region $(-2\pi, 2\pi) \times (-2\pi, 2\pi)$ into four subregions (Fig. 12), and then the modular function can be replaced. We denote the point $(\Theta_1 - \Theta_2, \Phi_1 - \Phi_2)$ as $P$, and then get

$$h(\Theta_1, \Theta_2)h(\Phi_1, \Phi_2) = \begin{cases} (\Theta_1 - \Theta_2 + \pi)(\Phi_1 - \Phi_2 + \pi) & P \in D_1, \\ (\Theta_1 - \Theta_2 + \pi)(\Phi_1 - \Phi_2 - \pi) & P \in D_2, \\ (\Theta_1 - \Theta_2 - \pi)(\Phi_1 - \Phi_2 + \pi) & P \in D_3, \\ (\Theta_1 - \Theta_2 - \pi)(\Phi_1 - \Phi_2 - \pi) & P \in D_4. \end{cases} \tag{6.2}$$

The expression of the correlation coefficient becomes the following form

$$\rho_o = \frac{\mathrm{E}[(\Theta_1 - \Theta_2)(\Phi_1 - \Phi_2)]}{(V_\Theta)^{1/2}(V_\Phi)^{1/2}}. \tag{6.3}$$

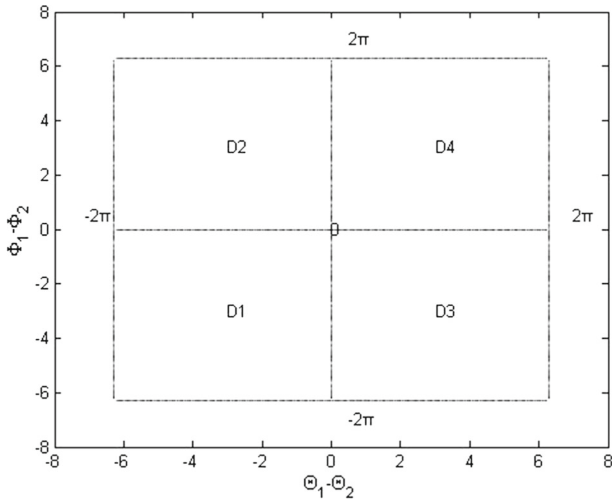


**Fig. 12** Four subregions which marked by $D_1$, $D_2$, $D_3$ and $D_4$

Suppose $\Theta$ and $\Phi$ are continuous variables and their density functions are $f$ and $g$, respectively. Then,

$$V_\Theta = 2\pi \left[ \int\int_{\theta_1 - \theta_2 < 0} f(\theta_1) f(\theta_2) d\theta_1 d\theta_2 - \int\int_{\theta_1 - \theta_2 \geq 0} f(\theta_1) f(\theta_2) d\theta_1 d\theta_2 \right] + \pi^2,$$

$$V_\Phi = 2\pi \left[ \int\int_{\phi_1 - \phi_2 < 0} g(\phi_1) g(\phi_2) d\theta_1 d\theta_2 - \int\int_{\phi_1 - \phi_2 \geq 0} g(\phi_1) g(\phi_2) d\theta_1 d\theta_2 \right] + \pi^2.$$

## Appendix 2: Proof of Theorem 1

We have

$$\binom{n}{2}^{-1} \sum_{1 \leqslant i < j \leqslant n} h(\theta_i, \theta_j) h(\phi_i, \phi_j) \xrightarrow{P} E\{h(\Theta_1, \Theta_2) h(\Phi_1, \Phi_2)\},$$

$$\binom{n}{2}^{-1} \sum_{1 \leqslant i < j \leqslant n} [h(\theta_i, \theta_j)]^2 \xrightarrow{P} E\{[h(\Theta_1, \Theta_2)]^2\},$$

and

$$\binom{n}{2}^{-1} \sum_{1 \leqslant i < j \leqslant n} [h(\phi_i, \phi_j)]^2 \xrightarrow{P} E\{[h(\Phi_1, \Phi_2)]^2\},$$

then $\hat{\rho}_o \xrightarrow{P} \rho_o$, where $\hat{\rho}_o$ is a consistent estimator for $\rho_o$. By the central limit theorem, $\binom{n}{2}^{-1}\sum_{1\leqslant i<j\leqslant n} h(\theta_i, \theta_j)h(\phi_i, \phi_j)$ converges in distribution to the normal law with mean $E[h(\Theta_1, \Theta_2)h(\Phi_1, \Phi_2)]$ and variance $\binom{n}{2}^{-1}E\{[h(\Theta_1, \Theta_2)h(\Phi_1, \Phi_2)]^2\} - \binom{n}{2}^{-1}E\{[h(\Theta_1, \Theta_2)h(\Phi_1, \Phi_2)]\}^2$. By the Slutsky Lemma, we have

$$\sqrt{\binom{n}{2}}(\hat{\rho}_o - \rho_o) \xrightarrow{L} N(0, \frac{\mathrm{E}\{[h(\Theta_1, \Theta_2)h(\Phi_1, \Phi_2)]^2\}}{\mathrm{E}\{[h(\Theta_1, \Theta_2)]^2\,\mathrm{E}\{[h(\Phi_1, \Phi_2)]^2\}} - \rho_o^2).$$

We use $\gamma$ to denote $\frac{\mathrm{E}\{[h(\Theta_1,\Theta_2)h(\Phi_1,\Phi_2)]^2\}}{\mathrm{E}\{[h(\Theta_1,\Theta_2)]^2\,\mathrm{E}\{[h(\Phi_1,\Phi_2)]^2\}}$, and then establish that $\sqrt{n(n-1)}(\hat{\rho}_o - \rho_o)$ converges in distribution to the normal law with mean 0 and variance $2(\gamma - \rho_o^2)$.

## Appendix 3: Calculation of the toroidal analogue of Kendall's $\tau$

Kendall's $\tau$ on the torus defined by $h$ is as follows

$$\tau_o = \tau_{\Theta,\Phi} = P[h(\Theta_1, \Theta_2)h(\Phi_1, \Phi_2) > 0] - P[h(\Theta_1, \Theta_2)h(\Phi_1, \Phi_2) < 0]. \quad (6.4)$$

Furthermore it can be specify to be

$$\begin{aligned}\tau_o = &\, P[(\Theta_1 - \Theta_2 - \pi)(\Phi_1 - \Phi_2 - \pi) > 0; \Theta_1 - \Theta_2 > 0, \Phi_1 - \Phi_2 > 0] \\ &- P[(\Theta_1 - \Theta_2 - \pi)(\Phi_1 - \Phi_2 - \pi) < 0; \Theta_1 - \Theta_2 > 0, \Phi_1 - \Phi_2 > 0] \\ &+ P[(\Theta_1 - \Theta_2 + \pi)(\Phi_1 - \Phi_2 - \pi) > 0; \Theta_1 - \Theta_2 < 0, \Phi_1 - \Phi_2 > 0] \\ &- P[(\Theta_1 - \Theta_2 + \pi)(\Phi_1 - \Phi_2 - \pi) < 0; \Theta_1 - \Theta_2 < 0, \Phi_1 - \Phi_2 > 0] \\ &+ P[(\Theta_1 - \Theta_2 + \pi)(\Phi_1 - \Phi_2 + \pi) > 0; \Theta_1 - \Theta_2 < 0, \Phi_1 - \Phi_2 < 0] \\ &- P[(\Theta_1 - \Theta_2 + \pi)(\Phi_1 - \Phi_2 + \pi) < 0; \Theta_1 - \Theta_2 < 0, \Phi_1 - \Phi_2 < 0] \\ &+ P[(\Theta_1 - \Theta_2 - \pi)(\Phi_1 - \Phi_2 + \pi) > 0; \Theta_1 - \Theta_2 > 0, \Phi_1 - \Phi_2 < 0] \\ &- P[(\Theta_1 - \Theta_2 - \pi)(\Phi_1 - \Phi_2 + \pi) < 0; \Theta_1 - \Theta_2 > 0, \Phi_1 - \Phi_2 < 0].\end{aligned} \quad (6.5)$$

If $\Theta$ and $\Phi$ are continuous, Eq. (6.5) is specified to be

$$\begin{aligned}\tau_o = 2\{&P[(\Theta_1 - \Theta_2 - \pi)(\Phi_1 - \Phi_2 - \pi) > 0; \Theta_1 - \Theta_2 > 0, \Phi_1 - \Phi_2 > 0] \\ &+ P[(\Theta_1 - \Theta_2 + \pi)(\Phi_1 - \Phi_2 - \pi) > 0; \Theta_1 - \Theta_2 < 0, \Phi_1 - \Phi_2 > 0] \\ &+ P[(\Theta_1 - \Theta_2 + \pi)(\Phi_1 - \Phi_2 + \pi) > 0; \Theta_1 - \Theta_2 < 0, \Phi_1 - \Phi_2 < 0] \\ &+ P[(\Theta_1 - \Theta_2 - \pi)(\Phi_1 - \Phi_2 + \pi) > 0; \Theta_1 - \Theta_2 > 0, \Phi_1 - \Phi_2 < 0]\} - 1.\end{aligned} \quad (6.6)$$

## Appendix 4: Proof of Theorem 2

Let us begin with Eq. (6.6) and denote the integral region by $C^2 = (0, 2\pi) \times (0, 2\pi)$. Let $C_1$ and $C_2$ denote the copulas of $(\Theta_1, \Phi_1)$ and $(\Theta_2, \Phi_2)$, respectively, so that $H_1(\theta, \phi) = C_1(F(\theta), G(\phi))$, $H_2(\theta, \phi) = C_2(F(\theta), G(\phi))$. Then,

$$
\begin{aligned}
&P[(\Theta_1 - \Theta_2 - \pi)(\Phi_1 - \Phi_2 - \pi) > 0; \Theta_1 - \Theta_2 > 0, \Phi_1 - \Phi_2 > 0] \\
&\quad = \int\int_{C^2} P(\Theta_1 > \theta + \pi, \Phi_1 > \phi + \pi) dC_2(F(\theta), G(\phi)) \\
&\qquad + \int\int_{C^2} P(\theta < \Theta_1 < \theta + \pi, \phi < \Phi_1 < \phi + \pi) dC_2(F(\theta), G(\phi)) \\
&\quad = \int\int_{C^2} [1 - F(\theta + \pi) - G(\phi + \pi) + C_1(F(\theta + \pi), G(\phi + \pi)) \\
&\qquad + C_1(F(\theta + \pi), G(\phi + \pi)) - C_1(F(\theta), G(\phi + \pi)) \\
&\qquad - C_1(F(\theta + \pi), G(\phi)) + C_1(F(\theta), G(\phi))] dC_2(F(\theta), G(\phi)),
\end{aligned}
$$

$$
\begin{aligned}
&P[(\Theta_1 - \Theta_2 + \pi)(\Phi_1 - \Phi_2 + \pi) > 0; \Theta_1 - \Theta_2 < 0, \Phi_1 - \Phi_2 < 0] \\
&\quad = \int\int_{C^2} P(\theta - \pi < \Theta_1 < \theta, \phi - \pi < \Phi_1 < \phi) dC_2(F(\theta), G(\phi)) \\
&\qquad + \int\int_{C^2} P(\Theta_1 < \theta - \pi, \Phi_1 < \phi - \pi) dC_2(F(\theta), G(\phi)) \\
&\quad = \int\int_{C^2} [C_1(F(\theta), G(\phi)) - C_1(F(\theta - \pi), G(\phi)) \\
&\qquad - C_1(F(\theta), G(\phi - \pi)) + C_1(F(\Theta_2 - \pi), G(\Phi_2 - \pi)) \\
&\qquad + C_1(F(\Theta_2 - \pi), G(\Phi_2 - \pi))] dC_2(F(\theta), G(\phi)),
\end{aligned}
$$

$$
\begin{aligned}
&P[(\Theta_1 - \Theta_2 + \pi)(\Phi_1 - \Phi_2 + \pi) > 0; \Theta_1 - \Theta_2 < 0, \Phi_1 - \Phi_2 < 0] \\
&\quad = \int\int_{C^2} P(\theta - \pi < \Theta_1 < \theta, \Phi_1 > \phi + \pi) dC_2(F(\theta), G(\phi)) \\
&\qquad + \int\int_{C^2} P(\Theta_1 < \theta - \pi, \phi < \Phi_1 < \phi + \pi) dC_2(F(\theta), G(\phi)) \\
&\quad = \int\int_{C^2} [F(\theta) - F(\theta - \pi) - C_1(F(\theta), G(\phi + \pi)) + C_1(F(\theta - \pi), G(\phi + \pi)) \\
&\qquad + C_1(F(\theta - \pi), G(\phi + \pi)) - C_1(F(\theta - \pi), G(\phi))] dC_2(F(\theta), G(\phi)),
\end{aligned}
$$

$$
\begin{aligned}
&P[(\Theta_1 - \Theta_2 - \pi)(\Phi_1 - \Phi_2 + \pi) > 0; \Theta_1 - \Theta_2 > 0, \Phi_1 - \Phi_2 < 0] \\
&\quad = \int\int_{C^2} P(\Theta_1 > \theta + \pi, \phi - \pi < \Phi_1 < \phi) dC_2(F(\theta), G(\phi)) \\
&\qquad + \int\int_{C^2} P(\theta < \Theta_1 < \theta + \pi, \Phi_1 < \phi - \pi) dC_2(F(\theta), G(\phi)) \\
&\quad = \int\int_{C^2} [G(\phi) - G(\phi - \pi) - C_1(\theta + \pi, \phi) + C_1(\theta + \pi, \phi - \pi) \\
&\qquad + C_1(\theta + \pi, \phi - \pi) - C_1(\theta, \phi - \pi)] dC_2(F(\theta), G(\phi)),
\end{aligned}
$$

The sum of the four equations above is

$$\int\int_{C^2}[1+F(\theta)-2F(\theta+\pi)+G(\phi)-2G(\phi+\pi)+2C_1(F(\theta),G(\phi))$$
$$-4C_1(F(\theta),G(\phi+\pi))-4C_1(F(\theta+\pi),G(\phi))+4C_1(F(\theta+\pi),G(\phi-\pi))$$
$$+4C_1(F(\theta+\pi),G(\phi+\pi))]dC_2(F(\theta),G(\phi)).$$

We use the probability transforms $u = F(\theta)$ and $v = G(\phi)$ in the calculation, then $4\int\int_{I^2} C_1(u,v)dC_2(u,v)-1$ is obtained. If $C_1 = C_2 = C$, Eq. (4.4) in Theorem 2 is established.

## Appendix 5: Estimator of $\tau_o$

Using Eq. (6.2), we define $\tau_{1,2}$ for two points $(\theta_1,\phi_1)$ and $(\theta_2,\phi_2)$ on the torus as follows

$$\begin{aligned}\tau_{1,2} = {} & \operatorname{sgn}(\theta_1-\theta_2-\pi)\operatorname{sgn}(\phi_1-\phi_2-\pi)I(\theta_1-\theta_2>0)I(\phi_1-\phi_2>0)\\ & +\operatorname{sgn}(\theta_1-\theta_2+\pi)\operatorname{sgn}(\phi_1-\phi_2-\pi)I(\theta_1-\theta_2<0)I(\phi_1-\phi_2>0)\\ & +\operatorname{sgn}(\theta_1-\theta_2+\pi)\operatorname{sgn}(\phi_1-\phi_2+\pi)I(\theta_1-\theta_2<0)I(\phi_1-\phi_2<0)\\ & +\operatorname{sgn}(\theta_1-\theta_2-\pi)\operatorname{sgn}(\phi_1-\phi_2+\pi)I(\theta_1-\theta_2>0)I(\phi_1-\phi_2<0).\end{aligned} \tag{6.7}$$

Given a random sample $P_i = (\theta_i,\phi_i)$ $(i = 1,\ldots,n)$ from a distribution on the torus, a natural estimator of $\tau_o$ is obtained as follows

$$\hat{\tau_o} = \binom{n}{2}^{-1}\sum_{1\leqslant i<j\leqslant n}\tau_{i,j}. \tag{6.8}$$

## Appendix 6: Proof of Theorem 4

Let us begin with Eq. (6.7)

$$\mathrm{E}(\tau_{1,2}) = \mathrm{pr}[h(\Theta_1,\Theta_2)h(\Phi_1,\Phi_2)>0]-\mathrm{pr}[h(\Theta_1,\Theta_2)h(\Phi_1,\Phi_2)<0] = \tau_o.$$

Then $\hat{\tau_o}$ is a consistent estimator for $\tau_o$. As noted previously $\tau_{1,2}^2 = 1$, we have

$$\mathrm{Var}(\tau_{1,2}) = \mathrm{E}[\tau_{1,2}^2]-[\mathrm{E}(\tau_{1,2})]^2 = 1-\tau_o^2,$$

and then $\sqrt{n(n-1)}(\hat{\tau}_o-\tau_o)$ converges in distribution to the normal law with mean zero and variance $2(1-\tau_o^2)$.

## References

Abe T, Pewsey A (2011) Sine-skewed circular distributions. Stat Pap 52(3):683–707

Downs TD (1974) Rotational angular correlations. In: Ferin M, Halberg F, Richart R, Vande Wiele R (eds) Biorhythms for human reproduction. Wiley, New York, pp 97–104

Fisher NI (1993) Statistical analysis of circular data. Cambridge University Press, Cambridge

Fisher NI, Lee AJ (1982) Nonparametric measures of angular–angular association. Biometrika 69(2):315–321

Fisher NI, Lee AJ (1983) A correlation coefficient for circular data. Biometrika 70(2):327–332

Fisher NI, Hall PG (1989) Bootstrap confidence regions for directional data. J Am Stat Assoc 84:996–1002

Fisher NI, Hall PG (1990) New statistical methods for directional data I. Bootstrap comparison of mean directions and the fold test in palaeomagnetism. Geophys J Int 101(2):305–313

Jammalamadaka SR, Sarma YR (1988) A correlation coefficient for angular variables. In: Matusita K (ed) Statistical theory and data analysis II. North Holland, Amsterdam, pp 349–364

Jammalamadaka SR, SenGupta A (2001) Topics in circular statistics. World Scientific, Singapore

Johnson RA, Wehrly TE (1977) Measures and models for angular correlation and angular–linear correlation. J R Stat Soc B 39:222–229

Jupp PE, Mardia KV (1980) A general correlation coefficient for directional data and related regression problems. Biometrika 67(1):163–173

Kato S, Eguchi S (2016) Robust estimation of location and concentration parameters for the von Mises–Fisher distribution. Stat Pap 57(1):205–234

Kim S, Sengupta A (2013) A three-parameter generalized von mises distribution. Stat Pap 54(3):685–693

Lee AJ (2010) Circular data. Wiley Interdiscip Rev 2:477–486

Liu S, Ma T, SenGupta A, Shimizu K, Wang M-Z (2016) Influence diagnostics in possibly asymmetric circular-linear multivariate regression models. Sankhyā. doi:10.1007/s13571-016-0116-8

Mardia KV (1975) Statistics of directional data (with discussion). J R Stat Soc B 37:349–393

Mardia KV, Puri ML (1978) A spherical correlation coefficient robust against scale. Biometrika 65(2):391–395

Mardia KV, Jupp PE (2000) Directional statistics. Wiley, Chichester

Pewsey A, Neuhäuser M, Ruxton GD (2013) Circular statistics in R. Oxford University Press, Oxford

Rivest LP (1982) Some statistical methods for bivariate circular data. J R Stat Soc B 44:81–90

SenGupta A, Kim S (2016) Statistical inference for homologous gene pairs between two circular genomes: a new circular-circular regression model. Stat Methods Appl 3:1–12

Shieh GS, Johnson RA, Frees EW (1994) Testing independence of bivariate circular data and weighted degenerate U-statistics. Stat Sin 4:729–747

Shieh GS, Zheng S, Johnson RA, Chang Y-F, Shimizu K, Wang C-C, Tang S-L (2011) Modeling and comparing the organization of circular genomes. Bioinformatics 27:912–918

Thompson J (1975) Discussion of paper by K. V. Mardia. J R Stat Soc B 37:379

Wehrly TE, Johnson RA (1980) Bivariate models for dependence of angular observations and a related Markov process. Biometrika 67(1):255–256