Taylor & Francis
Taylor & Francis Group

Communications in Statistics - Theory and Methods

ISSN: 0361-0926 (Print) 1532-415X (Online) Journal homepage: https://www.tandfonline.com/loi/lsta20

An Autoregressive Model for Time Series of Circular Data

Rinaldo Artes & Clelia M. C. Toloi

Communications in Statistics—Theory and Methods, 39: 186–194, 2010

ISSN: 0361-0926 print/1532-415X online
DOI: 10.1080/03610920802650338

An Autoregressive Model for Time Series of Circular Data

RINALDO ARTES[1] AND CLELIA M. C. TOLOI[2]

[1]Insper Institute of Education and Research, São Paulo, Brazil
[2]IME-USP, São Paulo, Brazil

This article focuses on estimating an autoregressive regression model for circular time series data. Simulation studies have shown the difficulties involved in obtaining good estimates from low concentration data or from small samples. It presents an application using real data.

Keywords Autoregressive models; Circular data; Circular mean; von Mises distribution.

Mathematics Subject Classification 62M10.

1. Introduction

Despite the fact that circular data studies are an old theme, there is still a lot of room for development in terms of proper analytical resources to analyze this type of variable, especially when compared to the techniques available for analyzing linear data. In this article, we developed a technique to analyze circular data with an autoregressive dependence structure, subject to the influence of the fixed covariates effects. In fact, such models are useful to analyze circular time series.

The study of regression models for circular data was initiated by Gould (1969), with the *Barber Pole* model; however, this model presented serious limitations, especially due to the problems related to the ability to identify some parameters. Subsequently, other models were presented, such as those of Fisher and Lee (1992), who developed regression models based on the assumption that a dependent variable followed a von Mises distribution (see Mardia and Jupp, 2000, for example). In this article, we developed models for modeling both the circular mean as well as the concentration parameter. Other models can be found in Jammalamadaka and SenGupta (2001).

Breckling (1989) developed a model to study a time series of wind direction data. Fisher and Lee (1994) presented some extensions of ARMA models for

Received September 16, 2008; Accepted November 26, 2008
Address correspondence to Rinaldo Artes, Rua Quata, 300, 04546-042, São Paulo, SP, Brazil; E-mail: rinaldoa@isper.org.br

directional data, specifically, they proposed autoregressive circular models (CAR) and the linked ARMA model (LARMA). Holzmann et al. (2006) proposed a hidden Markov approach to circular and linear-circular time series modeling.

Another attempt to model circular data with a dependence structure can be found in Artes et al. (2000). In that article, models for circular data with a panel structure can be found.

In our study, we propose a regression model for a time series with an autoregressive structure, which accommodates the effects of covariates.

In Sec. 2, we present the CAR model, which is the basis for the model to be developed in Sec. 3. In Sec. 4, we study, by simulations, the properties of the proposed model. In Sec. 5, we apply the model to a time series of real wind direction data; and finally, in Sec. 6, we present our final considerations.

2. CAR Model

In this model, $\{Y_t\}$ is a circular time series and $\Lambda_t^0 = \{y_{t-1}, \ldots, y_1\}$ is the set of information about the available series in time t. Consider that $Y_t \mid \Lambda_t^0$ follows a von Mises distribution with a concentration parameter κ and a circular mean μ_t. An autoregressive circular model of order p, CAR(p), is defined by

$$g(\mu_t - \mu) = \lambda_1 g(y_{t-1} - \mu) + \cdots + \lambda_p g(y_{t-p} - \mu),$$

where g is a real, invertible, and twice-differentiable, link function defined by a semi-open interval of length 2π ($S_{2\pi}$).

3. Autoregressive Models

In this section, we present two models for circular time series data with fixed covariates and autoregressive structures. In Sec. 3.1, we formulate a general model capable of assuming several different formats. In Sec. 3.3, we extend the CAR model (ECAR).

3.1. *General Model*

Now, assume that Y_t is influenced both by the past data of the series as well as by the behavior of a set of fixed covariates, $\{\mathbf{x}_t\}$, where, $\mathbf{x}_t$ is a p-dimensional vector. The set of available information at instant t is given by $\Lambda_t = \{y_{t-1}, \ldots, y_1, \mathbf{x}_t, \ldots, \mathbf{x}_1\}$. Consider that $Y_t \mid \Lambda_t$ follows a von Mises distribution with a concentration parameter κ and a circular mean μ_t such that

$$g(\mu_t - \mu) = \mathbf{x}_t^\top \boldsymbol{\beta} + \sum_{i=1}^{m} \lambda_i f_i(\Lambda_t), \tag{1}$$

where $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)^\top$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)^\top$ are parametric vectors; g is a twice-differentiable and invertible link function and f_i, $i = 1, 2, \ldots, m$, are twice-differentiable and invertible functions independent from $\boldsymbol{\beta}$. This model defines a type of autoregressive process with covariates. A particular case can be obtained if we consider that $f_i(\Lambda_t) = g(y_{t-i})$, in which case we will have:

$$g(\mu_t - \mu) = \mathbf{x}_t^\top \boldsymbol{\beta} + \sum_{i=1}^{m} \lambda_i g(y_{t-i}). \tag{2}$$

Model (2) is a simplification of CAR models with covariates.

In order to make theoretical development easier, consider $\eta_t = \mathbf{x}_t^\top \boldsymbol{\beta} + \sum_{i=1}^{m} \lambda_i f_i(\Lambda_t)$, h denoting the inverse function of g and $\boldsymbol{\theta} = (\mu, \boldsymbol{\beta}^\top, \boldsymbol{\lambda}^\top)^\top$ a parametric vector. We have considered that the first m components of the series were given (not random).

3.2. *Estimation*

According to model (1), the probability density function of $Y_t \mid \Lambda_t$ is given by

$$p_{Y_t \mid \Lambda_t}(y_t) = \frac{1}{2\pi I_0(\kappa)} \exp\{\kappa \cos(y_t - \mu_t)\},$$

where I_r is the modified Bessel function of order r (details can be found in Abramowitz and Stegun, 1970, for example). After n consecutive observations of time series data, the loglikelihood function of $\boldsymbol{\theta}$ is given by

$$L(\boldsymbol{\theta}) = -(n - m)\log\{2\pi I_0(\kappa)\} + \kappa \sum_{t=m+1}^{n} \cos(y_t - \mu_t),$$

and so the score function is

$$\frac{\partial L}{\partial \boldsymbol{\theta}}(\boldsymbol{\theta}) = \kappa \begin{pmatrix} \mathbf{1}^\top \mathbf{u} \\ \mathbf{A} \end{pmatrix}, \tag{3}$$

such that $\mathbf{1}$ is the unit vector, $u_t = \sin(y_t - \mu_t)$, $\mathbf{u} = (u_{m+1}, \ldots, u_n)^\top$,

$$\mathbf{A} = \begin{pmatrix} \mathbf{X}^\top \\ \mathbf{F}^\top \end{pmatrix} \mathbf{H}\mathbf{u},$$

$\dot{h}_j = \frac{\partial h}{\partial x}(x)|_{x=\eta_j}$, $\mathbf{H} = \operatorname{diag}\{\dot{h}_{m+1}, \ldots, \dot{h}_n\}$, $\mathbf{X} = (\mathbf{x}_{m+1}, \ldots, \mathbf{x}_n)^\top$ and

$$\mathbf{F} = \begin{Bmatrix} f_1(\Lambda_{m+1}) & f_2(\Lambda_{m+1}) & \ldots & f_m(\Lambda_{m+1}) \\ f_1(\Lambda_{m+2}) & f_2(\Lambda_{m+2}) & \ldots & f_m(\Lambda_{m+2}) \\ \vdots & \vdots & \ddots & \vdots \\ f_1(\Lambda_n) & f_2(\Lambda_n) & \ldots & f_m(\Lambda_n). \end{Bmatrix}$$

Since $\sin(y_t - \mu_t) = \sin\left[(y_t - h(\mathbf{x}_t^\top \beta + \sum_{i=1}^{m} \lambda_i f_i(\Lambda_{t-i})) - \mu\right]$, by setting the score function to zero, the μ estimator corresponds to the circular mean of $\left\{y_t - h(\mathbf{x}_t^\top \beta + \sum_{i=1}^{m} \lambda_i g(y_{t-i}))\right\}$ for $t = m+1, \ldots, n$. Define this mean by circ.mean$\left\{y_t - h(\mathbf{x}_t^\top \beta + \sum_{i=1}^{m} \lambda_i f_i(\Lambda_{t-i})), t = m+1, \ldots, n\right\}$.

The Fisher information matrix of $\boldsymbol{\theta}$ is given by

$$\mathbf{J}(\boldsymbol{\theta}) = \kappa \mathsf{A}_1(\kappa) \sum_{t=m+1}^{n} E \begin{pmatrix} 1 & \mathbf{x}_t^\top \dot{h}_t & \mathbf{f}_t^\top \dot{h}_t \\ \mathbf{x}_t \dot{h}_t & \mathbf{x}_t \dot{h}_t^2 \mathbf{x}_t^\top & \mathbf{x}_t \dot{h}_t^2 \mathbf{f}_t^\top \\ \mathbf{f}_t \dot{h}_t & \mathbf{f}_t \dot{h}_t^2 \mathbf{x}_t^\top & \mathbf{f}_t \dot{h}_t^2 \mathbf{f}_t^\top \end{pmatrix}, \tag{4}$$

with $\mathsf{A}_1(\kappa) = I_0(\kappa)/I_1(\kappa)$ and $\mathbf{f}_t$ is a m-dimensional vector with $f_i(\Lambda_t)$ components.

Define

$$\mathbf{B}_t = \begin{pmatrix} \mathbf{x}_t \dot{h}_t^2 \mathbf{x}_t^\top & \mathbf{x}_t \dot{h}_t^2 \mathbf{f}_t^\top \\ \mathbf{f}_t \dot{h}_t^2 \mathbf{x}_t^\top & \mathbf{f}_t \dot{h}_t^2 \mathbf{f}_t^\top \end{pmatrix}.$$

Therefore,

$$\mathbf{B} = \sum_{t=m+1}^{n} \mathbf{B}_t = \begin{pmatrix} \mathbf{X}^\top \mathbf{H}^2 \mathbf{X} & \mathbf{X}^\top \mathbf{H}^2 \mathbf{F} \\ \mathbf{F}^\top \mathbf{H}^2 \mathbf{X} & \mathbf{F}^\top \mathbf{H}^2 \mathbf{F} \end{pmatrix}.$$

The step $(k+1)$ of the Newton scoring algorithm is given by

$$\boldsymbol{\theta}^{(k+1)} = \boldsymbol{\theta}^{(k)} + \mathbf{J}^{-1}(\boldsymbol{\theta}) \frac{\partial L}{\partial \boldsymbol{\theta}}(\boldsymbol{\theta}^{(k)}). \tag{5}$$

The Fisher information matrix can be used to obtain asymptotic standard errors for $\hat{\boldsymbol{\theta}}$. The asymptotic covariance matrix is given by $\mathrm{Cov}_a = \mathbf{J}^{(-1)}(\boldsymbol{\theta})$. This matrix can be consistently estimated by substituting the parameters with their maximum likelihood estimates. Such a result makes it possible to use Wald tests to verify the hypothesis about the parameters of the model.

3.3. *Extended CAR Model*

Expression (2) is called the Extended CAR Model (ECAR). From a practical point of view, it is difficult to use all the results presented in Sec. 3.2. The main difficulty lies in determining $\mathbf{J}(\boldsymbol{\theta})$, especially in obtaining the expected value for $\dot{h}(t)$ and $\mathbf{f}_t$. Therefore, we suggest a few alterations in the algorithm given in (5).

1. Determine initial values for $\boldsymbol{\beta}$ and $\boldsymbol{\lambda}$, and call them $\boldsymbol{\beta}^{(0)}$ and $\boldsymbol{\lambda}^{(0)}$, respectively.
2. Obtain a first estimate of μ, given by $\mu^{(0)} = \text{circ.mean}\{y_t - h(\mathbf{x}_t^\top \boldsymbol{\beta}^{(0)} + \sum_{i=1}^{m} \boldsymbol{\lambda}_i^{(0)} g(y_{t-i})), t = m+1, \ldots, n\}$.
3. Estimate κ through

$$\kappa^{(0)} = \sum_{t=m+1}^{n} \frac{\cos(y_t - \mu_t^{(0)})}{n-m},$$

   where $\mu_t^{(0)}$ is the value of μ_t evaluated in $\mu^{(0)}$, $\boldsymbol{\beta}^{(0)}$, and $\boldsymbol{\lambda}^{(0)}$.
4. Update the estimates of $\boldsymbol{\beta}$ and $\boldsymbol{\lambda}$ through

$$\begin{pmatrix} \boldsymbol{\beta}^{(1)} \\ \boldsymbol{\lambda}^{(1)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\beta}^{(0)} \\ \boldsymbol{\lambda}^{(0)} \end{pmatrix} - \kappa^{(0)} \mathsf{A}_1(\kappa^{(0)}) (\mathbf{B}^{(0)})^{-1} \mathbf{A}^{(0)}, \tag{6}$$

where $\mathbf{B}^{(0)}$ and $\mathbf{A}^{(0)}$ are the values of $\mathbf{B}$ and $\mathbf{A}$, respectively, evaluated in $\mu^{(0)}$, $\boldsymbol{\beta}^{(0)}$, and $\boldsymbol{\lambda}^{(0)}$.

5. Return to step 2 until the algorithm converges.

4. Simulation Studies

Simulation studies have been conducted to verify the performance of parameter estimators of the ECAR model, expression (2), where $m=1$, a single covariate and $h = 2\text{atan}(\cdot)$, under controlled situations. The von Mises distribution was simulated according to the algorithm described by Fisher (1993). In this study, we simulated series of three sizes ($n = 100$, 200, and 500), four values for the autoregressive parameter ($\lambda = 0.01, 0.1, 0.5$, and 0.9), two values for the concentration parameter ($\kappa = 2$, showing low concentration, and 5, showing high concentration), and two values for the covariate parameter ($\beta = 0.5$ and 2). The covariate was generated according to a uniform distribution in the interval ranging from -1 to 1. For each combination of n, κ, λ, and β, we have simulated 1,000 time series. The results found for $\beta = 0.5$ and $\beta = 2$ were similar; therefore, we decided to present only the results found for $\beta = 2$.

The simulation was done using a macro developed in R; and a maximum of 200 interactions were allowed. We considered that there was convergence in cases in which the percentage variation in successive parameter estimates was less than 0.1%. The algorithm proved to be highly dependent on the initial value proposed for the parameter. When we started the interaction process with the parameter values, in the best case ($n = 500$, $\kappa = 5$, $\lambda = 0.01$), there was convergence in 93% of the cases, and in its worst case ($n = 100$, $\kappa = 2$, $\lambda = 0.9$), we achieved convergence in only 49% of the cases. Generally speaking, the best convergences were found when the autoregressive parameter was closer to zero and the time series size and concentration parameter were large.

Table 1 summarizes the simulation results. In columns $\hat{\beta}$ and $\hat{\lambda}$, respectively, it is possible to see the mean values for the estimates of β and λ; columns MSE($\hat{\beta}$) and MSE($\hat{\lambda}$) show the mean square errors estimated for these estimators and columns $V(\hat{\beta})$ and $V(\hat{\lambda})$ correspond to the absolute bias (in percentage) estimated for such estimators.

In general, situations with larger values of n and with $\kappa = 5$ presented better performance. These effects had already been expected due to the asymptotic properties of the maximum likelihood estimators and to the fact that in a von Mises distribution sample, usually, the higher the concentration of data, the better the performance of statistical methods. In this case, such a distribution comes closer to a normal distribution (see Mardia and Jupp, 2000, for example). An especially bad performance was observed in situations where $\lambda = 0.90$.

Overall, the magnitude of errors was high in all the situations studied, which suggests the need for a large series to obtain reliable estimates.

5. Application

We analyzed a series of wind orientation data (from 0–180°) obtained from the daily data for wind direction compiled by the ArticRIMS Project (Regional Integrated Hydrological Monitoring System for the Pan-Artic Land Mass) site.[1] Gross data

[1]rims.unh.edu accessed in January 2006.

Table 1
Simulation results

n	κ	λ	$\hat{\beta}$	$V(\hat{\beta})$	MSE($\hat{\beta}$)	$\hat{\lambda}$	$V(\hat{\lambda})$	MSE($\hat{\lambda}$)	Convergence
100	2	0.01	1.85	7.6	0.48	0.014	39.3	0.0020	562
100	2	0.10	1.92	3.9	0.78	0.098	1.7	0.0054	630
100	2	0.25	2.06	2.8	1.36	0.251	0.4	0.0245	593
100	2	0.50	2.10	5.0	0.69	0.500	0.0	0.0609	604
100	2	0.75	2.05	2.6	0.48	0.730	2.7	0.1018	641
100	2	0.90	2.04	2.0	0.44	0.879	2.4	0.1311	656
500	2	0.01	1.95	2.7	0.02	0.010	1.8	0.0000	847
500	2	0.10	1.91	4.3	0.08	0.094	5.8	0.0008	783
500	2	0.25	2.00	0.2	0.22	0.237	5.1	0.0092	603
500	2	0.50	2.00	0.0	0.16	0.481	3.8	0.0182	603
500	2	0.75	1.97	1.6	0.12	0.725	3.3	0.0324	589
500	2	0.90	1.94	3.0	0.11	0.857	4.8	0.0380	522
1000	2	0.01	1.98	1.1	0.01	0.010	2.0	0.0000	911
1000	2	0.10	1.94	3.1	0.09	0.097	3.1	0.0006	789
1000	2	0.25	2.01	0.5	0.24	0.236	5.5	0.0074	581
1000	2	0.50	2.01	0.4	0.06	0.498	0.4	0.0087	538
1000	2	0.75	1.96	2.2	0.05	0.721	3.9	0.0150	487
1000	2	0.90	1.94	2.9	0.07	0.864	4.0	0.0246	434
100	5	0.01	1.87	6.4	0.14	0.012	17.2	0.0005	777
100	5	0.10	1.89	5.3	0.37	0.098	2.1	0.0029	707
100	5	0.25	2.03	1.3	0.63	0.243	2.7	0.0148	662
100	5	0.50	2.13	6.7	0.32	0.513	2.6	0.0364	704
100	5	0.75	2.07	3.4	0.21	0.747	0.4	0.0519	711
100	5	0.90	2.04	2.0	0.17	0.903	0.4	0.0563	703
500	5	0.01	1.96	2.2	0.01	0.010	2.4	0.0000	942
500	5	0.10	1.92	3.9	0.14	0.096	4.4	0.0009	820
500	5	0.25	2.03	1.4	0.26	0.243	2.7	0.0076	653
500	5	0.50	2.04	2.0	0.04	0.509	1.8	0.0063	671
500	5	0.75	1.99	0.5	0.03	0.740	1.3	0.0091	623
500	5	0.90	2.00	0.1	0.04	0.897	0.3	0.0150	584
1000	5	0.01	1.98	1.2	0.00	0.010	1.2	0.0000	925
1000	5	0.10	1.96	2.0	0.01	0.098	1.6	0.0003	780
1000	5	0.25	2.03	1.3	0.08	0.246	1.6	0.0046	649
1000	5	0.50	2.00	0.2	0.02	0.498	0.4	0.0034	652
1000	5	0.75	1.98	0.8	0.02	0.740	1.3	0.0057	513
1000	5	0.90	1.99	0.5	0.02	0.892	0.9	0.0082	463

$V(\cdot)$: absolute bias estimate (in percentege); MSE($\cdot$): mean square error estimate; Convergence: number of cases in which the algorithm converged.

refer to the daily mean direction of wind, in a place located at latitude 56.5398 and longitude 201.65601, for the period beginning on October 1, 1992 and ending on September 30, 2005.

Table 2
Estimated parameters of the wind orientation model

Parameter	Estimate	Standard-error	p-value
β_1	−0.02476	0.00458	<0.001
β_2	0.00385	0.00457	0.402
λ_1	0.00062	0.00017	<0.001
μ	−1.91°		
κ	5.58		

In order to adjust possible seasonality, the days of the year, d, were numbered from 1 (October 1) to 365 (September 30), and the following variables were created:

$$x_{1_t} = \sin\left(\frac{2d_t\pi}{365}\right) \quad x_{2t} = \cos\left(\frac{2d_t\pi}{365}\right),$$

where d_t is the day of the year (starting on October 1) that corresponds to t-th observation of the series. For leap years, we replaced 365 with 366.

Models of type (7) were adjusted with m ranging from 1–8. Thus,

$$g(\mu_t - \mu) = x_{1t}\beta_1 + x_{2t}\beta_2 + \sum_{i=1}^{m} \lambda_i g(y_{t-i}), \tag{7}$$

where $h = 2\text{atan}(\cdot)$.

As for the model with $m = 8$, the non significant autoregressive terms of the highest order were removed one by one. According to this criterion, the best model was the order 1 autoregressive model. Table 2 shows the estimates for these model parameters. The seasonal variation of wind orientation was detected by the model.

6. Final Considerations

The ECAR model developed by this article is especially useful to estimate intervention models in circular time series. There is still room for improvement in this method, especially in regard to the development of a more efficient estimation method.

We would also like to highlight that, for the purposes of this study, we considered the homogeneity of the concentration parameter along time. However, this hypothesis may no be valid in studies regarding the direction taken by animals, in which concentration tends to increase with time.

A. Results

Properties:

$$E(\sin(y_t - \mu_t)) = E\{E(\sin(y_t - \mu_t)) \mid \Lambda_t\} = 0.$$

$$E(\cos(y_t - \mu_t)) = E\{E(\cos(y_t - \mu_t)) \mid \Lambda_t\} = \mathsf{A}_1(\kappa).$$

Partial derivatives of the loglikelihood function:

$$\frac{L}{\partial\kappa} = -(n-m)\frac{I_1(\kappa)}{I_0(\kappa)} = (n-m)\mathsf{A}_1(\kappa).$$

$$\frac{\partial L}{\partial\mu} = \kappa\sum_{t=m+1}^{n}\sin(y_t-\mu_t).$$

$$\frac{\partial L}{\partial\mu_t} = \kappa\sin(y_t-\mu_t).$$

$$\frac{\partial\mu_t}{\partial\beta_k} = x_{tk}\dot{h}_t.$$

$$\frac{\partial L}{\partial\beta_k} = \kappa\sum_{t=m+1}^{n}x_{tk}\dot{h}_t\sin(y_t-\mu_t).$$

$$\frac{\partial\mu_t}{\partial\lambda_r} = f_i(\Lambda_t)\dot{h}_t.$$

$$\frac{\partial L}{\partial\lambda_r} = \kappa\sum_{t=m+1}^{n}f_r(\Lambda_t)\dot{h}_t\sin(y_t-\mu_t).$$

$$E\left(\frac{\partial^2 L}{\partial\mu^2}\right) = -\kappa\sum_{i=m+1}^{n}E\{\cos(y_t-\mu_t)\} = (n-m)\kappa\mathsf{A}_1(\kappa).$$

$$E\left\{\frac{\partial^2 l}{\partial\mu\partial\beta_k}\right\} = -\kappa\sum_{t=m+1}^{n}x_{tk}E\big[E\{\dot{h}_t\cos(y_t-\mu_t)\mid\Lambda_t\}\big]$$

$$= -\kappa\mathsf{A}_1(\kappa)\sum_{t=m+1}^{n}x_{tk}E\{\dot{h}_t\}.$$

$$E\left\{\frac{\partial^2 L}{\partial\mu\partial\lambda_i}\right\} = -\kappa\mathsf{A}_1(\kappa)\sum_{t=m+1}^{n}x_{tk}E\{f_i(\Lambda_t)\dot{h}_t\}.$$

$$E\left\{\frac{\partial^2 L}{\partial\beta_k\partial\beta_s}\right\} = -\kappa\mathsf{A}_1(\kappa)\sum_{t=m+1}^{n}x_{tk}x_{ts}E\{\dot{h}_t^2\}.$$

$$E\left\{\frac{\partial^2 L}{\partial\beta_k\partial\lambda_i}\right\} = -\kappa\mathsf{A}_1(\kappa)\sum_{t=m+1}^{n}x_{tk}E\{f_i(\Lambda_t)\dot{h}_t^2\}.$$

$$E\left\{\frac{\partial^2 L}{\partial\lambda_i\partial\lambda_r}\right\} = -\kappa\mathsf{A}_1(\kappa)\sum_{t=m+1}^{n}E\big\{f_i(\Lambda_t)\dot{h}_t^2 f_r(\Lambda_t)\big\}.$$

Acknowledgment

The second author acknowledges the support of FAPESP (grants 03/10105-2 and 08/51097-6).

References

Abramowitz, M., Stegun, I. A. (1970). *Handbook of Mathematical Functions*. New York: Dover Publishers.

Artes, R., Paula, G. A., Ranvaud, R. (2000). Analysis of circular longitudinal data based on generalized estimating equations. *Austral. N.Z. J. Statist.* 3:347–358.

Breckling, J. (1989). *Analysis of Directional Time Series: Application to Wind Speed and Direction*. Lecture Notes in Statistics, Vol. 61. Berlin: Springer-Verlag.

Fisher, N. I. (1993). *Statistical Analysis of Circular Data*. Cambridge: Cambridge University Press.

Fisher, N. I., Lee, A. J. (1992). Regression models for an angular response. *Biometrics* 48:665–677.

Fisher, N. I., Lee, A. J. (1994). Time series analysis of circular data. *J. Roy. Statist. Soc.* 56:327–639.

Gould, A. L. (1969). A regression technique for angular variates. *Biometrics* 25:683–700.

Holzmann, H., Munk, A., Suster, M., Zucchini, W. (2006). Hidden Markov models for circular and linear-circular time series. *Environ. Ecol. Statist.* 13:325–347.

Jammalamadaka, S. R., SenGupta, A. (2001). *Topics in Circular Statistics*. Singapore: World Scientific.

Mardia, K. V., Jupp, P. F. (2000). *Directional Statistics*. Chechester: Wiley.