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Bivariate models for dependence of angular observations and a related Markov process

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Summary

Some new distributions having specified marginal distributions are proposed for dependent random variables taking values on the torus. A related family of Markov processes taking values on the circle is constructed, and statistical inference for these processes is studied.

Some key words: Bivariate distribution; Dependence; Directional data; Markov process.

A limited number of bivariate circular distributions have been proposed in the literature. Thompson (1975) briefly mentioned the wrapped bivariate normal distribution. Johnson & Wehrly (1977) discussed statistical inference for this distribution. Mardia (1975a) obtained a family of generalized von Mises–Fisher distributions as particular cases of maximum entropy densities. These distributions do provide models for circular dependence, but in general the marginal distributions are not of types commonly employed for directional data.

We have studied the problem of creating families of bivariate circular distributions with specified marginal distributions, a requirement consonant with standard techniques for generating bivariate distributions. The following theorem provides a method for obtaining such families. In particular, bivariate distributions with von Mises marginal distributions can be constructed. This method is similar to one used by Johnson & Wehrly (1978) for angular-linear distributions. The straightforward proof is omitted.

THEOREM. *Let $f_1(\theta)$ and $f_2(\eta)$ be specified densities on the circle and $F_1(\theta)$ and $F_2(\eta)$ be their distribution functions defined with respect to fixed, arbitrary, origins. Also, let $g(.)$ be a density on the circle. Then*

$$f(\theta,\eta) = 2\pi g[2\pi\{F_1(\theta)-F_2(\eta)\}]f_1(\theta)f_2(\eta), \tag{1}$$

$$f(\theta,\eta) = 2\pi g[2\pi\{F_1(\theta)+F_2(\eta)\}]f_1(\theta)f_2(\eta), \tag{2}$$

where $0 \leqslant \theta$, $\eta < 2\pi$, are densities on the torus having the specified marginal densities $f_1(\theta)$ and $f_2(\eta)$.

The distributions (1) or (2) lead naturally to a family of distributions for a Markov process on the circle. Let $\theta_0, \theta_1, \ldots$ be random variables taking values on the circle such that

$$p(\theta_0) = f(\theta_0), \quad p(\theta_n \mid \theta_0, \ldots, \theta_{n-1}) = p(\theta_n \mid \theta_{n-1}) = 2\pi g[2\pi\{F(\theta_n) - F(\theta_{n-1})\}]f(\theta_n), \tag{3}$$

where $f(.)$ and $g(.)$ are densities on the circle and

$$F(\theta) = \int_0^\theta f(\xi)\,d\xi.$$

Then $p(\theta_0)$ is the initial distribution, and $p(\theta_n \mid \theta_{n-1})$ is the stationary transition density. This family will enable us to do statistical inference for a stationary Markov process on the circle and should prove useful to applied researchers working with time series of angular observations.

Let $\{\theta_n\}$, $n \geqslant 0$, be distributed according to (3), $g(\xi) = \{2\pi I_0(\kappa)\}^{-1} \exp\{\kappa \cos(\xi - \mu)\}$, and $f(\theta)$ be a completely specified distribution.

(i) If $\mu \in [0, 2\pi)$ is a known parameter, then the uniformly most powerful test for $\kappa = 0$ versus $\kappa > 0$ rejects when $\Sigma \cos[2\pi\{F(\theta_i) - F(\theta_{i-1})\} - \mu] > k$.

(ii) If μ is not known, the likelihood ratio test rejects the hypothesis that $\kappa = 0$ for large values of

$$R^2 = \left(\sum_{i=1}^{n} \cos[2\pi\{F(\theta_i) - F(\theta_{i-1})\}]\right)^2 + \left(\sum_{i=1}^{n} \sin[2\pi\{F(\theta_i) - F(\theta_{i-1})\}]\right)^2.$$

(iii) Using the exponential family theory, one can also construct uniformly most powerful unbiased tests of $\kappa = \kappa_0$ versus $\kappa \neq \kappa_0$ for $\kappa_0 > 0$. Confidence intervals for κ follow directly by inverting the class of tests.

(iv) The concept of contiguity and its applications to optimal statistical procedures in Markov processes (Johnson & Roussas, 1970) provide the large sample result that in distribution

$$n^{-\frac{1}{2}} \sum_{i=1}^{n} \{\cos[2\pi\{F(\theta_i) - F(\theta_{i-1})\} - \mu] - A(\kappa)\} \to N\{0, \Gamma(\kappa)\}, \tag{4}$$

where $\Gamma(\kappa) = 1 - A(\kappa)/\kappa - A(\kappa)$, $A(\kappa) = I_1(\kappa)/I_0(\kappa)$ and $I_p(\kappa)$ is the modified Bessel function of the first kind and order p.

We apply these results to a model with uniform marginal distributions and set μ equal to zero to obtain a test of serial dependence for the sequence $\{\theta_n\}$, $n \geqslant 0$, which rejects independence for large values of $\Sigma \cos(\theta_i - \theta_{i-1})$. This test statistic corresponds exactly to one proposed by Watson & Beran (1967), who used the permutation distribution for testing in small samples and a normal approximation for large samples. However, the normal approximation using contiguity results is computationally simpler, and expressions for asymptotic efficiency follow directly. Also, note that the likelihood ratio test is similar in form to the nonparametric test for serial correlation presented by Mardia (1975b).

As an example, Johnson & Wehrly (1977, Table 2) gave data on the wind direction at noon for 21 consecutive days at a weather station in Milwaukee. Examination suggested that the stationary distribution could be assumed to be a circular uniform distribution. To test for serial dependence in the sequence of wind directions, we set $\mu = 0$ and use the test in (i) where F is uniform. The test statistic is $C\sqrt{(2/n)} = \Sigma \cos(\theta_i - \theta_{i-1})\sqrt{(2/n)}$. According to (4), its asymptotic null distribution is standard normal. For this data set, $C\sqrt{(2/n)} = 0{\cdot}934$. Thus, there is no strong indication of serial correlation in the sequence of wind direction.

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