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Non-linear systems under impulsive parametric input

M. Di Paola*, A. Pirrotta

Dipartimento di Ingegneria Strutturale e Geotecnica, Università degli Studi di Palermo, Viale delle Scienze, 1, 90128, Palermo, Italy

Abstract

In this paper the problem of the response of non-linear systems excited by an impulsive parametric input is treated. For such systems the response exhibits a jump depending on the amplitude of the impulse as well as on the value of the state variables immediately before the impulse occurrence. Recently, the jump prediction has been obtained in a series form. Here the incremental rule for any scalar real valued function is obtained in an analytical form involving the jump of the state variables. It is also shown that the formulation for the jump evaluation is also able to give a new step-by-step integration technique.

Keywords: Non-linear systems; Parametric input; Incremental rule

1. Introduction

Vibrations induced by a variation in the system parameters are called parametrically excited. Classical examples are a pendulum with a suspension point oscillating in the gravity field direction; or vibrating elastic structures surrounded by fluids. The main feature of parametrically excited systems is the Liapunov instability under proper coefficients (see e.g. [1,2]).

If the variations in the system parameters are Dirac's deltas, then such systems are called systems under impulsive parametric input. In correspondence with the impulse the state variables exhibit a finite jump depending on the intensity of the impulse and on the response itself. For the particular class of the so-called quasi-linear systems (or bilinear or simply linear systems), the solution in terms of the jump can exactly be evaluated [3,4].

If the amplitude of the instantaneous impulse depends on a general non-linear function of the response, the jump cannot be evaluated using the theory of distributions (see e.g. [5]). This happens because the response of the impulsive term is neither a Riemann nor a Fourier Stieltjess integral and the classical differential calculus cannot predict the jump.

Very recently, the exact jump prediction by means of a numerical series has been introduced, for any kind of non-linear system [6]. The only limitation is that the non-linear function of the response, that multiplies the Dirac's delta, belongs to the class C^{∞}.

Here the problem is reconsidered and an explicit form of the jump for any scalar real valued function

* Corresponding author. Tel.: 00 91 656 8419; fax: 00 91 656 8407; e-mail: dina@ipacuc.bitnet.

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of the response belonging to the class C^∞ is presented. Moreover, a step-by-step solution technique is also presented for non-linear systems excited by both parametric and non-parametric smooth or impulsive excitation. As, in fact, when the discretization procedure is performed, one can think that a parametric additive impulse is present whose intensity depends on the time step selected. As a result the exact response of dynamic non-linear systems in a numerical series is obtained independently of the size of the time step.

2. Parametric impulsive input

In this next section some preliminaries of the main results obtained by using the formulation in [6] are reported for clarity's sake.

Let the dynamical system be given in the form

$$\dot{z}(t) = f(z,t) + g(z,t)\gamma\delta(t-\bar{t}); \quad z(t_0) = z_0, \tag{1}$$

where $z(t)$ is a state variable, $f(z,t)$ and $g(z,t)$ are deterministic non-linear functions of z and t, belonging to C^∞ (class of infinite times differentiable function on z and t), $\delta(\cdot)$ is the Dirac's delta, $(\bar{t} > t_0)$, γ is the amplitude of external impulse and z_0 is the initial condition. The system (1) is called parametrically excited (or also loaded by multiplicative impulse) because the strength of the impulse depends on the response $z(t)$ itself.

If $g(z,t)$ is in dependent of $z(t)$, that is $g(z,t) = g(t)$ then Eq. (1) is called a non-linear system excited by an external impulsive load. If $f(z,t)$ and $g(z,t)$ are linear functions ($f(z,t) = \alpha(t)z$, $g(z,t) = z$) then the system is called linear (or quasi-linear or bilinear).

In order to evaluate the response $z(t)$, the time interval $[t_0, \infty)$ can be subdivided into three parts: the first one is $t_0 \leqslant t < \bar{t}^-$, the second one is $(\bar{t}^-, \bar{t}^+)$, and the third one is $t > \bar{t}^+$, where $\bar{t}^-$ and $\bar{t}^+$ indicate the time instants immediately before and after the impulse, respectively.

In the first and the third interval, the differential equation (1) is given in the form

$$\dot{z} = f(z,t), \quad \forall t\colon t_0 \leqslant t < \bar{t}^-, \tag{2a}$$

$$\dot{z} = f(z,t), \quad \forall t\colon t > \bar{t}^+. \tag{2b}$$

The appropriate initial conditions are $z(t_0) = z_0$ for Eq. (2a) and $z(\bar{t}^+)$ for Eq. (2b). In order to find the initial condition for Eq. (2b), it is necessary to solve the differential equation in the second interval $(\bar{t}^-, \bar{t}^+)$. Because of the impulse in $\bar{t}$, the response $z(\bar{t}^+)$ is different from $z(\bar{t}^-)$, that is the response exhibits a jump $J(\bar{t})$ given as

$$J(\bar{t}) = z(\bar{t}^+) - z(\bar{t}^-). \tag{3}$$

It should be stressed that, only for a linear function of $g(z,t)$, the jump can be evaluated by means of elementary considerations, otherwise, in order to evaluate the jump, let $[\bar{t}-\varepsilon;\ \bar{t}+\varepsilon]$ be a small interval centered in $\bar{t}$, in which Eq. (1) may be expressed in the integral form as follows:

$$z(\bar{t}+\varepsilon) - z(\bar{t}-\varepsilon) = \int_{t-\varepsilon}^{\bar{t}+\varepsilon} f(z,t)\,\mathrm{d}t + \gamma \int_{t-\varepsilon}^{t+\varepsilon} g(z,t)\,\mathrm{d}H(t-\bar{t}), \tag{4}$$

where $H(\cdot)$ is the unit step function ($H(t-\bar{t}) = 1$ if $t \geqslant \bar{t}$, $H(t-\bar{t}) = 0$ if $t < \bar{t}$). The first integral in the right-hand side of Eq. (4) is a Riemann integral, the second integral is not a Riemann or Fourier–Stieltjes integral, in fact the limit of the Riemann sum does not achieve unique solution, but it depends on the intermediate point selected.

Implicitly, Pandit and Deo [7] selected the point $\bar{t}^+$, but this is an arbitrary choice, so that they only obtain an approximation of the jump, as shown in Appendix A. The key to find the jump for Eq. (1) is in writing the Taylor expansion of an increment of a real valued function $\Phi(z,t)$, belonging to the class C^∞, as follows:

$$\Delta\Phi(z,t) = \mathrm{d}\Phi(z,t) + \frac{1}{2!}\mathrm{d}^2\Phi(z,t) + \frac{1}{3!}\mathrm{d}^3\Phi(z,t) + \cdots = \sum_{j=1}^{\infty} \frac{\mathrm{d}^j\Phi(z,t)}{j!}. \tag{5}$$

By neglecting infinitesimals of higher order than $\mathrm{d}t$ the first few differentials appearing in Eq. (5) can be written in the form (with argument omitted)

$$\mathrm{d}\Phi = \frac{\partial \Phi}{\partial z}\,\mathrm{d}z + \frac{\partial \Phi}{\partial t}\,\mathrm{d}t,$$

$$\mathrm{d}^2\Phi = \frac{\partial(\mathrm{d}\Phi)}{\partial z}\,\mathrm{d}z = \frac{\partial^2 \Phi}{\partial z^2}(\mathrm{d}z)^2 + \frac{\partial \Phi}{\partial z}\,\mathrm{d}^2 z,$$

$$\mathrm{d}^3\Phi = \frac{\partial(\mathrm{d}^2\Phi)}{\partial z}\,\mathrm{d}z = \frac{\partial^3 \Phi}{\partial z^3}(\mathrm{d}z)^3 + 3\frac{\partial^2 \Phi}{\partial z^2}\,\mathrm{d}^2 z\,\mathrm{d}z + \frac{\partial \Phi}{\partial z}\,\mathrm{d}^3 z. \tag{6}$$

$$\vdots$$

The numerical series (5) proves that the increment and differential of the function $\Phi(z,t)$ do not coincide and in the following the rule given in Eq. (5) will be called the *incremental rule*. If $\Phi(z,t)$ is not a singular function, then $\Delta\Phi(z,t) = \mathrm{d}\Phi(z,t)$ because higher order differentials are infinitesimal of higher order than $\mathrm{d}t$. If $z(t)$ experiences a finite jump, as in the case of Eq. (1) in $\bar{t}$, higher order differentials are of same order of the first one and then they cannot be neglected. In fact, selecting $\Phi(z,t) = z$ and using Eq. (5) leads to

$$\Delta z = \mathrm{d}z + \frac{1}{2!}\mathrm{d}^2 z + \frac{1}{3!}\mathrm{d}^3 z + \cdots = \sum_{j=1}^{\infty} \frac{\mathrm{d}^j z}{j!}. \tag{7}$$

If z is the solution of Eq. (1) there are two possibilities, $t \neq \bar{t}$ or $t = \bar{t}$; in the first case

$$\mathrm{d}z = f(z,t)\mathrm{d}t, \tag{8}$$

then $\Delta z \equiv \mathrm{d}z$ because higher order differentials of $z(t)$ are infinitesimals of higher order than $\mathrm{d}t$. While in $\bar{t}$ one can write

$$\mathrm{d}z = f(z,t)\mathrm{d}t + \gamma g(z,t)\mathrm{d}H(t-\bar{t}), \tag{9}$$

then using Eq. (9) it follows that

$$\mathrm{d}^2 z = \gamma^2 \frac{\partial g(z,t)}{\partial z} g(z,t)(\mathrm{d}H(t-\bar{t}))^2,$$

$$\vdots \tag{10}$$

$$\mathrm{d}^j z = \gamma^j \frac{\partial g^{(j-1)}(z,t)}{\partial z} g(z,t)(\mathrm{d}H(t-\bar{t}))^j.$$

Moreover, $g^{(j)}(z,t)$ can be evaluated by the recursive relationship

$$g^{(j)}(z,t) = \frac{\partial g^{(j-1)}(z,t)}{\partial z} g(z,t), \quad g^{(1)}(z,t) = g(z,t) \tag{11}$$

and being in $t = \bar{t}$, $(\mathrm{d}H(t-\bar{t}))^j = 1$ and $(\mathrm{d}H(t-\bar{t}))^j = 0$ elsewhere, then the increment Δz, that is the jump in correspondence with the Dirac's occurrence, is given as

$$J(\bar{t}) = \Delta z = \sum_{j=1}^{\infty} \gamma^j \frac{g^{(j)}(z(\bar{t}^-),\bar{t})}{j!}. \tag{12}$$

It will be noted that in Eq. (12), the jump depends only on the value of the state variable immediately before the impulse occurrence, that is already known by solving the differential equation (2a), as well as on the intensity of the impulse. Once the jump is evaluated the value of the state variable after the impulse can be evaluated and then the initial condition for Eq. (2b) is known. The jump evaluated by Eq. (12) could be a divergent quantity and this means that the system experiences an instantaneous instability.

3. Incremental rule

In this section, on the basis of the main results of the previous section, the incremental rule given in Eq. (5) will be expressed in an explicit form. The incremental rule of a non-linear scalar real valued function belonging to the class C^∞ is given in Eq. (5), where the various differentials are defined in Eq. (6). However, it can be rewritten in a more expressive form, as in fact by inserting Eq. (6) into Eq. (5) the incremental rule proves to be written in the form

$$\Delta\Phi(z,t) = \frac{\partial \Phi(z,t)}{\partial t}\mathrm{d}t + \frac{\partial \Phi(z,t)}{\partial z}\left(\sum_{j=1}^{\infty}\frac{\mathrm{d}^j z}{j!}\right) + \frac{1}{2!}\frac{\partial^2 \Phi(z,t)}{\partial z^2}\left(\sum_{j=1}^{\infty}\frac{\mathrm{d}^j z}{j!}\right)^2 + \cdots + \frac{1}{k!}\frac{\partial^k \Phi(z,t)}{\partial z^k}\left(\sum_{j=1}^{\infty}\frac{\mathrm{d}^j z}{j!}\right)^k. \tag{13}$$

By substituting Eq. (7) into Eq. (13) it leads to

$$\Delta\Phi(z,t)=\frac{\partial\Phi(z,t)}{\partial t}\,\mathrm{d}t+\sum_{k=1}^{\infty}\frac{1}{k!}\frac{\partial^k\Phi(z,t)}{\partial z^k}(\Delta z)^k. \quad (14)$$

Eq. (14) is the explicit form of the incremental rule. If $t\neq\bar{t}$, then $\Delta z\equiv \mathrm{d}z$, is an infinitesimal of first order, it follows that by neglecting infinitesimals of higher order, $\Delta\Phi(z,t)\equiv \mathrm{d}\Phi(z,t)$, and Eq. (14) reduces to the classical rule of differentiation of composite functions, that is

$$\Delta\Phi(z,t)=\mathrm{d}\Phi(z,t)=\frac{\partial\Phi(z,t)}{\partial t}\,\mathrm{d}t+\frac{\partial\Phi(z,t)}{\partial z}\,\mathrm{d}z,\quad \forall t\neq\bar{t}. \quad (15)$$

In contrast if the increment Δz is evaluated in $\bar{t}$, then Δz is a finite quantity, the term $(\partial\Phi(z,t)/\partial t)\mathrm{d}t$ can be neglected because it is an infinitesimal quantity with respect to all higher order terms that are all of finite quantity. The jump of the function $\Phi(z(\bar{t}^-),\bar{t})$, in correspondence with the impulse is given as

$$\Delta\Phi(z,\bar{t})=\Phi(z(\bar{t}^+),\bar{t})-\Phi(z(\bar{t}^-),\bar{t})=\sum_{k=1}^{\infty}\frac{1}{k!}\frac{\partial^k\Phi(z(\bar{t}^-),\bar{t})}{\partial z^k}(\Delta z)^k,\quad t=\bar{t}. \quad (16)$$

As an example if $\Phi(z,t)=z$, then the jump of z is exactly given as in Eq. (12). If $\Phi(z,t)=z^2$, Eq. (16) leads to

$$\Delta(z^2)=(z(\bar{t}^+))^2-(z(\bar{t}^-))^2=2z(\bar{t}^-)\times\left(\sum_{j=1}^{\infty}\gamma^j\frac{g^{(j)}(z(\bar{t}^-),\bar{t})}{j!}\right)+\left(\sum_{j=1}^{\infty}\gamma^j\frac{g^{(j)}(z(\bar{t}^-),\bar{t})}{j!}\right)^2. \quad (17)$$

Eq. (16) is very useful for manipulating non-linear differential equations driven by parametric impulses.

4. Step-by-step solution procedure

On the basis of the results of the previous section the step-by-step solution procedure, for both smooth and impulsive inputs, will be revised.

Let the equation of motion be given in form (1), in order to perform the step-by-step integration procedure; let the time axis $[0,T]$, be subdivided into small intervals, and let $t_0\equiv 0, t_1,\ldots,t_r\equiv T$, be the subdivision times. Moreover, let $\Delta t_k=t_{k+1}-t_k$ and $z(t_k)$ be the solution at the time instant t_k. For every time $t_k\neq\bar{t}$ it is necessary to estimate the solution at the time instant t_{k+1} by solving the differential equation

$$\mathrm{d}z=f(z,t)\mathrm{d}t,\quad z(t_k)=z_k,\ t_k\neq\bar{t}. \quad (18)$$

Although in Eq. (18) no impulsive parametric input is present in its discretized form $\Delta z=f(z,t)\Delta t$, it can be figured out as a differential equation driven by an impulse whose strength is Δt. Thus the value of the response at the time instant t_{k+1} can be evaluated by Eq. (12) and it is given by

$$z_{k+1}-z_k=\sum_{j=1}^{\infty}\frac{f^{(j)}(z_k,t_k)}{j!}(\Delta t_k)^j, \quad (19)$$

where $f^{(j)}(z,t)$ is given by the recursive form

$$f^{(j)}(z,t)=\frac{\partial f^{(j-1)}(z,t)}{\partial z}f(z,t)+\frac{\partial f^{(j-1)}(z,t)}{\partial t},$$
$$f^{(1)}(z,t)=f(z,t), \quad (20)$$

where the last term in Eq. (20) accounts for the time evolution of the function $f(z,t)$ in the interval Δt_k that is now a finite quantity.

From Eq. (19) it is worth noting that by truncating the series up to the first term it leads to the central difference scheme. By truncating the series up to the second term it leads to a comparable result of the trapezoidal rule. Moreover, from Eq. (19) it is apparent that smaller the interval the lesser the terms which influence the response at the time z_{k+1} in a significative manner. In any case Eq. (19) allows to quantify the error connected by truncating the series up to a fixed order, depending on the time step selected.

In the instant location $\bar{t}$ the response z exhibits a jump that can be evaluated by Eq. (12), then the step-by-step solution is given in the form

$$z_{k+1}=z_k+\sum_{j=1}^{\infty}\frac{f^{(j)}(z_k,t_k)}{j!}(\Delta t_k)^j+\left[\sum_{j=1}^{\infty}\frac{\gamma^j}{j!}g^{(j)}(z(\bar{t}^-),\bar{t})\right]H(t-\bar{t}). \quad (21)$$

It will be emphasized that if all terms of Eq. (21) are retained, the solution obtained by Eq. (21) is exact for any time step selected.

5. Multi-degree-of-freedom system

In this section the extension of the previous concepts to multi-degree-of-freedom non-linear systems is provided.

A dynamical system, enforced by parametric impulsive input, including structural ones, can be cast in the form

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}, t) + \mathbf{g}(\mathbf{z}, t)\gamma\delta(t - \bar{t}), \quad \mathbf{z}(t_0) = \mathbf{z}_0, \tag{22}$$

where $\mathbf{z}$ is an n-vector of state variables, $\mathbf{f}(\mathbf{z}, t)$ and $\mathbf{g}(\mathbf{z}, t)$ are non-linear differentiable n-vector functions of $\mathbf{z}$ and t.

In the time intervals $[0; \bar{t}^-)$ the solution of Eq. (22) can be reached by solving the differential equations

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}, t), \quad \mathbf{z}(t_0) = \mathbf{z}_0, \quad \forall t\colon t_0 \leqslant t < \bar{t}^-, \tag{23}$$

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}, t), \quad \mathbf{z}(\bar{t}^+), \quad \forall t\colon t > \bar{t}^+,$$

while the jump can be evaluated in the form

$$\mathbf{z}(\bar{t}^+) - \mathbf{z}(\bar{t}^-) = \sum \frac{\gamma^{(j)}}{j!}\,\mathbf{g}^{(j)}(\mathbf{z}(\bar{t}^-), \bar{t}), \tag{24}$$

where $\mathbf{g}^{(j)}$ can be evaluated in recursive form as follows:

$$\mathbf{g}^{(j)}(\mathbf{z}, t) = (\nabla\mathbf{g}^{(j-1)}(\mathbf{z}, t))\mathbf{g}^{(1)}(\mathbf{z}, t), \tag{25}$$

$\nabla\mathbf{g}^{(j-1)}(\mathbf{z}, t)$ being now the gradient operator, that is

$$\nabla\mathbf{g}^{(j-1)} = \begin{bmatrix} \dfrac{\partial g_1^{(j-1)}}{\partial z_1} & \dfrac{\partial g_1^{(j-1)}}{\partial z_2} & \cdots & \dfrac{\partial g_1^{(j-1)}}{\partial z_n} \\ \dfrac{\partial g_2^{(j-1)}}{\partial z_1} & \dfrac{\partial g_2^{(j-1)}}{\partial z_2} & \cdots & \dfrac{\partial g_2^{(j-1)}}{\partial z_n} \\ \vdots & \vdots & \cdots & \vdots \\ \dfrac{\partial g_n^{(j-1)}}{\partial z_1} & \dfrac{\partial g_n^{(j-1)}}{\partial z_2} & \cdots & \dfrac{\partial g_n^{(j-1)}}{\partial z_n} \end{bmatrix}, \tag{26}$$

where $g_s^{(j-1)}(\mathbf{z}, t)$ is the sth component of the vector $\mathbf{g}^{(j-1)}(\mathbf{z}, t)$.

The incremental rule for any scalar real valued function of $\mathbf{z}$ and t, $\Phi(\mathbf{z}, t)$, is given in the form

$$\Delta\Phi(\mathbf{z}, t) = \frac{\partial\Phi(\mathbf{z}, t)}{\partial t} + \sum_{j=1}^{n} \frac{\partial\Phi(\mathbf{z}, t)}{\partial z_j}\Delta z_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2\Phi(\mathbf{z}, t)}{\partial z_j \partial z_k}\Delta z_j \Delta z_k + \cdots \tag{27}$$

Δz_j and Δz_k being the increments of the state variable z_j and z_k, respectively.

The step-by-step solution procedure can be performed by extending the concepts expressed in Section 4. That is for all intervals $[t_{k+1}, t_k]$ in which no Dirac's delta is present, one can write

$$\mathbf{z}_{k+1} - \mathbf{z}_k = \sum_{j=1}^{\infty} \frac{\mathbf{f}^{(j)}(\mathbf{z}_k, t_k)}{j!}(\Delta t_k)^j, \tag{28}$$

where

$$\mathbf{f}^{(j)}(\mathbf{z}, t) = (\nabla\mathbf{f}^{(j-1)}(\mathbf{z}, t))\,\mathbf{f}^{(1)}(\mathbf{z}, t) + \frac{\partial}{\partial t}\mathbf{f}^{(j-1)}(\mathbf{z}, t),$$

$$\mathbf{f}^{(1)}(\mathbf{z}, t) = \mathbf{f}(\mathbf{z}, t). \tag{29}$$

6. Examples

In this section three examples are proposed, a quasi-linear system, a Bernoulli equation and the Duffing oscillator.

6.1. Quasi-linear system

Let the equation of motion be given in the form

$$\dot{z} = \alpha z + \gamma z\delta(t - \bar{t}), \quad z(0) = z_0, \quad \bar{t} > 0. \tag{30}$$

This is a quasi-linear also called bilinear or, simply, linear system.

For every $t < \bar{t}$ the equation of motion is $\dot{z} = \alpha z$ and then

$$z(t) = z_0 \exp(\alpha t), \quad z(\bar{t}^-) = z_0 \exp(\alpha\bar{t}). \tag{31}$$

The jump can now be evaluated by means of Eq. (12):

$$\Delta z = z(\bar{t}^+) - z(\bar{t}^-) = z_0 \exp(\alpha\bar{t}) \sum_{j=1}^{\infty} \frac{\gamma^j}{j!} = z_0 (\exp(\gamma) - 1)\exp(\alpha\bar{t}). \tag{32}$$

Then the response immediately after the impulse occurrence is written as

$$z(\bar{t}^+) = z_0 \exp(\alpha\bar{t}) \exp(\gamma). \tag{33}$$

Let $\Phi(z) = z^2$, in order to evaluate $z^2(\bar{t}^+) - z^2(\bar{t}^-)$, since $z(\bar{t}^+)$ and $z(\bar{t}^-)$ are known in closed form solution, then it is possible to evaluate immediately $z^2(\bar{t}^+)$ and $z^2(\bar{t}^-)$ by using Eqs. (31) and (33), that is

$$\begin{aligned}\Delta\Phi(z) &= z^2(\bar{t}^+) - z^2(\bar{t}^-) \\ &= z_0^2 \exp(2\alpha\bar{t}) \exp(2\gamma) - z_0^2 \exp(2\alpha\bar{t}) \\ &= z_0^2 \exp(2\alpha\bar{t})(\exp(2\gamma) - 1).\end{aligned} \tag{34}$$

In order to test the validity of incremental rule (14), use is made of the latter one for evaluating the jumps of z^2, that is

$$\Delta\Phi(z) = \Delta(z^2) = 2z(\bar{t}^-)\Delta z + (\Delta z)^2. \tag{35}$$

Then by substituting Eq. (32) into Eq. (35) we have

$$\Delta\Phi(z) = z_0^2 \exp(2\alpha\bar{t})(\exp(2\gamma) - 1) \tag{36}$$

that exactly coincides with the jump z^2 of evaluate in Eq. (34).

It can be easily seen that the jump for any function of z can be evaluated following the concept outlined above.

6.2. Bernoulli equation

Let the equation of motion be given in the form

$$\dot{z} = az + \gamma z^3 \delta(t - \bar{t}), \quad z(0) = z_0, \quad \bar{t} > 0. \tag{37}$$

Following the procedure already described in Section 2, the value immediately before the Dirac's delta is given by $(z_0 e^{a\bar{t}})$, then the jump, evaluated by means of Eq. (12) is given as

$$\Delta z = (z_0 e^{a\bar{t}}) \sum_{j=1}^{\infty} \frac{\gamma^j \Gamma(2j-1)}{j!} (z_0 e^{a\bar{t}})^{2j}, \tag{38}$$

where $\Gamma(2j-1) = (2j-1)(2j-3) \cdots 5 \cdot 3 \cdot 1$.

Since Eq. (37) is a Bernoulli equation, it can be exactly solved, to aim at this, the following non-linear transformations can be made:

$$y = z^{-2}, \quad \dot{y} = -2z^{-3}\dot{z}, \quad y(0) = z_0^{-2}. \tag{39}$$

By substituting Eq. (39) into Eq. (37) it follows that

$$\dot{y} = -2ay - 2\gamma\delta(t - \bar{t}), \quad y(0) = z_0^{-2}. \tag{40}$$

Eq. (40) is a linear differential equation enforced by an external input, hence the solution of this differential equation can be obtained by using the classical differential calculus:

$$y(t) = y(0)e^{-2at}, \quad \forall t < \bar{t}. \tag{41}$$

Then in $\bar{t}$ the response $y(t)$ exhibits a jump given as

$$\Delta y = y(\bar{t}^+) - y(\bar{t}^-) = (-2\gamma), \tag{42}$$

that is, the value immediately after the impulse is given as

$$y(\bar{t}^+) = y_0 e^{-2a\bar{t}} - 2\gamma. \tag{43}$$

This is the initial condition for $t > \bar{t}$, then the response after the impulse is given in the form

$$y(t) = y_0 e^{-2a\underline{t}} - 2\gamma e^{-2a(t-\bar{t})}, \quad \forall t > \bar{t}. \tag{44}$$

Now from Eq. (39) the variable is given as $z = y^{(-1/2)}$ and by using the incremental rule given in Eq. (14), it is possible to evaluate the jump of z that proves to be written exactly as in Eq. (38).

6.3. Duffing oscillator

As a third example the resonant Duffing oscillator is examined. The equation of motion is

$$\begin{aligned}&\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2(x + \beta x^3) = \omega_0^2 r_0 \sin(\omega_0 t), \\ &\quad x(0) = \dot{x}(0) = 0.\end{aligned} \tag{45}$$

Although this equation is driven by an external smooth forcing function, in its discretized version, it can be considered as a non-linear system enforced by impulses and it can be solved by using the step-by-step procedure described in Section 4.

This equation can be recast in the canonical form

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}, t), \quad \mathbf{z}(t_0) = \mathbf{0}, \tag{46}$$

where $\mathbf{z}$ is the state variable vector $\mathbf{z}^T = [x \;\; \dot{x}] = [z_1 \;\; z_2]$, while $\mathbf{f}(\mathbf{z}, t)$ is given as

$$\mathbf{f}(\mathbf{z}, t) = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\xi\omega_0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -\omega_0^2 \beta z_1^3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \omega_0^2 r_0 \sin(\omega_0 t). \tag{47}$$

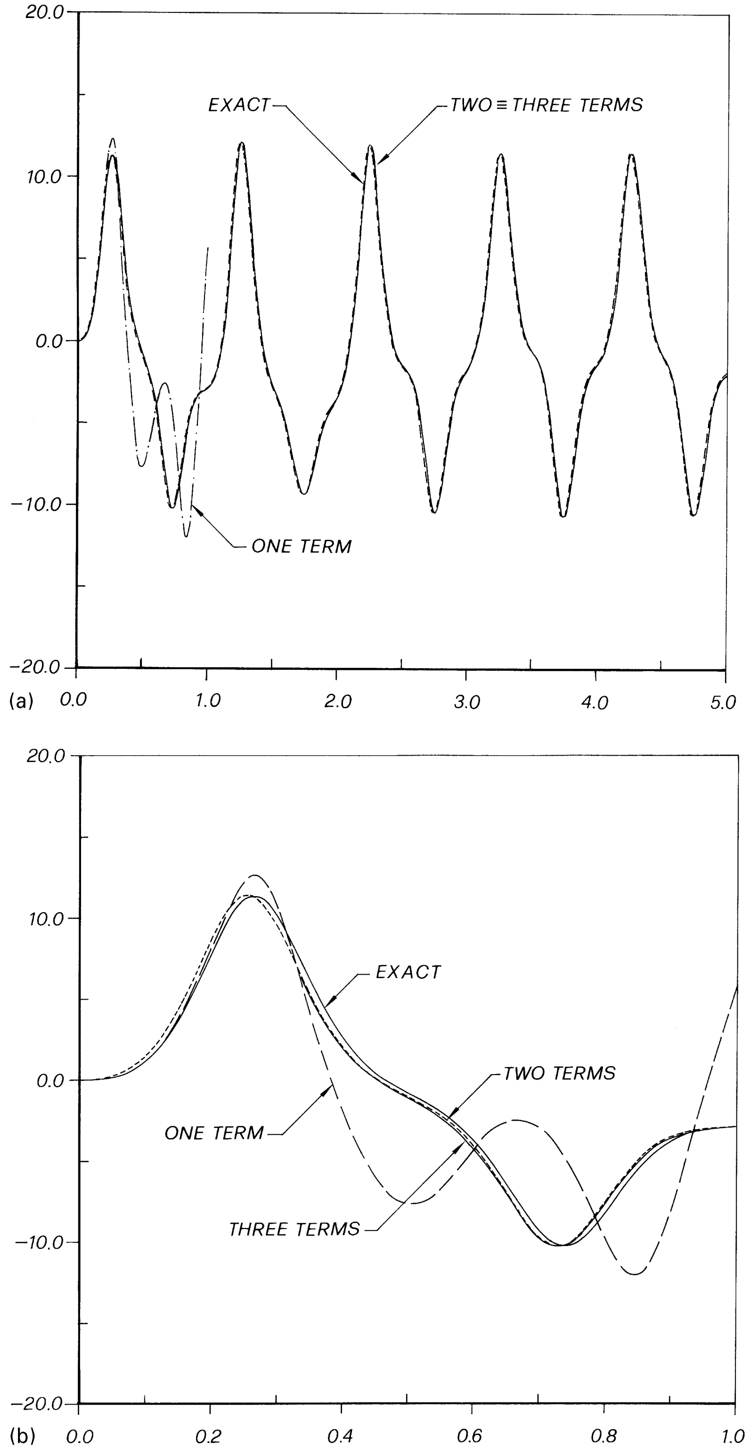

Fig. 1. (a) Displacement time hystory of a resonant Duffing oscillator: comparison among the exact solution with the proposed results including one, two and three terms; (b) a magnification of the previous figure.

Eq. (46) is a non-linear differential equation enforced by external excitation. Using a step-by-step solution given in Eq. (28) the response in the discrete points: $t_0 = 0, t_1 = \Delta t, t_2 = 2\Delta t, \ldots$ is obtained in the form

$$\mathbf{z}_{k+1} = \mathbf{z}_k + \sum_{j=1}^{\infty} \frac{\mathbf{f}^{(j)}(\mathbf{z}_k, t_k)}{j!} (\Delta t_k)^j, \tag{48}$$

where the first two functions $\mathbf{f}^{(j)}$ are reported in Appendix B.

The analysis has been performed by using the following parameters $\zeta = 0.05$, $\beta = 0.05$, $r_0 = 30$ and $\omega_0 = 2\pi$. In Fig. 1a and Fig. 1b the exact solution is reported. Moreover, the results obtained by a large time step ($\Delta t = 0.01$) are also reported by including one, two and three terms in the summation (48). From this figure one can see that at a parity of the time step, increasing the number of terms in the summation the step-by-step integration method tends towards the exact solution.

7. Conclusions

Non-linear systems excited by a parametric impulse exhibit a jump at the Dirac's delta occurrences. The jump can be exactly evaluated by means of a numerical series involving the strength of the impulse and the response immediately before the impulse. Using a step-by-step procedure for evaluating the response of a non-linear system even in those time intervals in which no impulses are present, because of discretization, one can think that an impulse is present. Then a new version of the step-by-step procedure to integrate a non-linear differential system driven by both smooth and impulsive noise has been presented. Test performed on simple cases reveals that the proposed step-by-step procedure leads to very accurate results using a very large time step depending on the number of terms included in the series. Moreover, prediction of the total error in the time lag of integration can be easily predicted depending on the time step, on the number of terms included in the series and on the total length of the time integration.

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Appendix A

In this appendix the main results of Pandit and Deo [7], are reported and compared with those obtained from elementary considerations and with those obtained here. For simplicitly the scalar case and a single impulse is treated.

Let the differential equation be given in the form

$$\dot{z} = az + \gamma az\delta(t - \bar{t}), \quad z(0) = z_0, \; 0 < \bar{t}. \tag{A.1}$$

From [7] it can be written as

$$z(t) = z_0 \exp(at), \quad \forall t < \bar{t}. \tag{A.2}$$

The value immediately after the impulse is evaluated in the form

$$z(\bar{t}^+) = z_0 \exp(a\bar{t}) + \gamma az(\bar{t}^+), \tag{A.3}$$

then the result obtained in [7] is

$$z(\bar{t}^+) = \frac{z_0}{1 - \gamma a} \exp(a\bar{t}) \tag{A.4}$$

and the jump is given as

$$j = z_0 \exp(a\bar{t}) \left[\frac{\gamma a}{1 - \gamma a} \right]. \tag{A.5}$$

For the case of Eq. (A.1), since the system is linear, the jump can be exactly evaluated on the basis of elementary considerations. In order to do this, let the Dirac's delta be defined as a window function in $\bar{t} \div \bar{t} + \varepsilon$ whose amplitude is $1/\varepsilon$.

For every $t < \bar{t}$ the response is that given in Eq. (A.2). In $\bar{t} \leqslant t \leqslant \bar{t} + \varepsilon$ the differential equation is

$$\dot{z} = az + \frac{(\gamma a z)}{\varepsilon}, \quad z(\bar{t}^-) = z_0 \exp(a(\bar{t})). \tag{A.6}$$

Then the response is

$$z(t) = z(\bar{t}^-) \exp\left[\left(a + \frac{\gamma a}{\varepsilon}\right)(t - \bar{t})\right],$$
$$\forall t\colon \bar{t} \leqslant t \leqslant \bar{t} + \varepsilon \tag{A.7}$$

in $t = \bar{t} + \varepsilon$

$$z(\bar{t} + \varepsilon) = z(\bar{t}^-) \exp[a\varepsilon + \gamma a], \tag{A.8}$$

if $\varepsilon \to 0$ then

$$z(\bar{t}^+) = z(\bar{t}^-) \exp[a\gamma] = z_0 \exp(a\bar{t}) \exp(a\gamma) \tag{A.9}$$

by comparing Eqs. (A.9) and (A.4) it can be recognized that the two results are totally different from each other. This happens because the result in [7] is obtained as follows: for $t < \bar{t}$ the response is given in Eq. (A.2), in order to find the jump in [7] it may be written as

$$z(\bar{t}^+) = z_0 \exp(a(\bar{t})) + \gamma a z(\bar{t}^+),$$

$$z(\bar{t}^+) = z(\bar{t}^-) + \int_{\bar{t}-\varepsilon}^{\bar{t}} a\gamma z(t)\, \mathrm{d}H(t - \bar{t})$$
$$= z(\bar{t}) + a\gamma z(\bar{t}^+) \tag{A.10}$$

that is in [7] the intermediate selected point in the Riemann sum is the final point for each partition interval, but this is an arbitrary choice since $z(t)$ in $\bar{t}$ is a discontinuous function.

In contrast, by using Eq. (12) it is easy to see that the result in terms of jump exactly coincides with the result obtained in Eq. (A.9).

Appendix B

Here all components of $\mathbf{f}(\mathbf{z}, t)$ are reported, that were used for the example in Section 6.3 regarding the Duffing oscillator:

$$\begin{aligned}
f_1^{(1)}(\mathbf{z}, t) &= z_2, \\
f_2^{(1)}(\mathbf{z}, t) &= r_0\omega_0^2 \sin(\omega_0 t) - 2\zeta\omega_0 z_2 - \omega_0^2(z_1 + \beta z_1^3), \\
f_1^{(2)}(\mathbf{z}, t) &= r_0\omega_0^2 \sin(\omega_0 t) - 2\zeta\omega_0 z_2 - \omega_0^2(z_1 + \beta z_1^3), \\
f_2^{(2)}(\mathbf{z}, t) &= r_0\omega_0^3 \cos(\omega_0 t) - 2\zeta r_0\omega_0^3 \sin(\omega_0 t) \\
&\quad + \omega_0^2(2\zeta\beta\omega_0 z_1^3 - 3\beta z_1^2 z_2 + 2\zeta\omega_0 z_1 \\
&\quad + z_2(4\zeta^2 - 1)), \qquad \text{(B.1)} \\
f_1^{(3)}(\mathbf{z}, t) &= r_0\omega_0^3 \cos(\omega_0 t) - 2\zeta r_0\omega_0^3 \sin(\omega_0 t) \\
&\quad + \omega_0^2(2\zeta\beta\omega_0 z_1^3 - 3\beta z_1^2 z_2 + 2\zeta\omega_0 z_1 \\
&\quad + z_2(4\zeta^2 - 1)), \\
f_2^{(3)}(\mathbf{z}, t) &= \omega_0^2 - (3\beta^2\omega_0^2 z_1^5 + 4\beta\omega_0^2 z_1^3(1 - \zeta^2) \\
&\quad + 12\zeta\beta\omega_0 z_1^2 z_2 - z_1(6\beta z_2^2 + \omega_0^2(4\zeta^2 - 1)) \\
&\quad + 4\zeta\omega_0 z_2(1 - 2\zeta^2)) - r_0\omega_0^4(3\beta z_1^2 \\
&\quad - 4\zeta^2 + 1)(\sin(\omega_0 t)) \\
&\quad - 2r_0\zeta\omega_0^4 \cos(\omega_0 t) - r_0\omega_0^4 \sin(\omega_0 t).
\end{aligned}$$