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Direct evaluation of jumps for nonlinear systems under external and multiplicative impulses

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Abstract

In this paper the problem of the response evaluation of nonlinear systems under multiplicative impulsive input is treated. Such systems exhibit a jump at each impulse occurrence, whose value cannot be predicted through the classical differential calculus. In this context here the correct jump evaluation of nonlinear systems is obtained in closed form for two general classes of nonlinear multiplicative functions. Analysis has been performed to show the different typical behaviors of the response, which in some cases could diverge or converge to zero instantaneously, depending on the amplitude of the Dirac's delta.

Keywords

Nonlinear systems, multiplicative impulsive input, Dirac's delta

1. Introduction

The determination of the system response to an impulsive force plays a fundamental role in structural dynamic. In fact it is well known that from the impulse response function one can get the system response to any forcing function, and in this regard the Duhamel integral provides the solution for linear systems.

A strong kick, that is an impulse, can be considered as a constant force applied to the system for a very short time interval ε and, from a mathematical point of view, it is modeled through a Dirac's delta function. The real challenge of researchers concerned with this topic is the jump prediction, since in correspondence of the impulse arrival, the state variables of the system exhibit a finite jump.

If the impulse is external, that is modulated by a function independent on the state variables, this evaluation is very simple since the jump coalesces with the amplitude of the impulse itself. However, for nonlinear systems driven by impulses modulated by a function of the response, say multiplicative (parametric) impulses, this evaluation is not straight.

Problems involving the response determination of nonlinear systems under parametric impulsive input are rather common in mechanical and structural engineering. Parametric impulses arise for instance when dealing with problems regarding impacts between objects or objects with a rigid barrier (Dimentberg and Iourtchenko, 2004, Xu et al., 2013, 2014) and more generally when dealing with the so-called non-smooth mechanical systems (Popp, 2000). For example the problem of an elastically restrained and damped bar subjected to a periodic impact load is addressed in Hsu (1997) and Hsu and Cheng (1973, 1977), while in Pilipchuky et al. (1999) a nonlinear Duffing oscillator under multiplicative impulsive excitation is studied through a non-smooth temporal transformation. Finally a typical case of engineering interest in which parametric impulsive inputs arise is related to the rocking motion of rigid blocks under base excitation. Impulsive Dirac's delta forces can in fact be used to model energy dissipation occurring during the impact

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of the block with the ground (Dimentberg et al., 1993, Cai et al., 1995, Prieto et al., 2004).

Further it is worth noting that a completely different field of engineering interest in which parametric impulses appear is related to the study of systems under parametric Poissonian white noise processes. A Poissonian white noise process is in fact represented by a train of Dirac's deltas of random amplitude occurring at random time instants, and it is frequently used to model a wide range of phenomena in stochastic dynamics (Tung, 1967, Roberts, 1966, Lin, 1963, Merchant, 1964, Liepmann, 1952). Due to this characteristic, many papers have been then devoted to the study of systems under multiplicative Poisson white noise input (Caddemi and Di Paola, 1995, Di Matteo et al., 2014, Di Paola and Pirrotta, 2004, Er et al., 2009, Ibrahim, 1985, Iwankiewicz, 2003, Papadimitriou et al., 1999, Pirrotta, 2005, 2007, Proppe, 2002, Sun et al., 2013, Zheng and Zhu, 2010), which thus require the solution of differential equations with parametric Dirac's delta forces input.

Framed into this class of systems under (deterministic) multiplicative impulsive input, solution in terms of the jump can exactly be obtained in the particular case of the so-called quasi-linear systems (or bilinear or simply linear systems), see Zheng-Rong (1987) and Pilipchuk (1996). For more complicated cases, when the amplitude of the instantaneous impulse depends on a general nonlinear function of the response, the jump cannot be evaluated using the theory of distributions, (see e.g. Kech and Teodoresku (1978)). In this context, the exact jump prediction has been performed in Caddemi and Di Paola (1997) by means of a numerical series, for any kind of nonlinear system. The only limitation is that the nonlinear function of the response, that multiplies the Dirac's delta, belongs to the class C_∞. In Di Paola and Pirrotta (1999) the same problem has been solved leading to an explicit form of the jump involving the amplitude of the impulse and the response immediately before the impulse.

Here the problem is reconsidered and a closed form solution, obtained as limit of the series proposed in Di Paola and Pirrotta (1999), has been introduced for two general nonlinear functions belonging to the classes of antisymmetric and symmetric nonlinearities.

Several numerical applications have been performed to assess the validity of the proposed formulation. Specifically two systems, whose response is known in closed form, have been considered as benchmark. Finally a nonlinear single degree of freedom system under both external and parametric excitation has been proposed as a last example in order to show the versatility and reliability of the formulation.

2. Jumps for external and multiplicative impulses

Let the differential equation of a generic dynamical system be given in the form

$$\begin{cases} \dot{x}(t) = f(x,t) + Y_k\, g(x,t)\,\delta(t-t_k) \\ x(0) = x_0 \end{cases}; \quad t_k > 0 \quad \text{(1a, b)}$$

where $f(x,t)$ and $g(x,t)$ are nonlinear functions of the response $x(t)$, Y_k is the amplitude of the Dirac's delta $\delta(t-t_k)$ at the time instant t_k and x_0 is the assigned initial condition. Note that as $g(x,t)=1$ the impulse is external otherwise the impulse is parametric.

Solution of equation (1a) may be obtained subdividing the time axis in three parts, according to the following steps:

(i) Solve equation $\dot{x}(t) = f(x,t)$, $\forall t < t_k$, with initial condition x_0, find the solution immediately before the impulse occurrence. Such a value can be labeled as $x(t_k^-) = \bar{x}$, where the apex $-$ in t_k stands for immediately before the impulse.
(ii) Evaluate the jump $J(t_k)$ due to the Dirac's delta, so that $x(t_k^+) = \bar{x} + J(t_k)$, where the apex $+$ in t_k means immediately after the impulse.
(iii) Evaluate the response after the impulse by solving the differential equation $\dot{x}(t) = f(x,t)$, $\forall t > t_k$ assuming as initial condition the value $x(t_k^+)$.

Problems *i.* and *iii.* are trivial and may be easily solved step-by-step or in some cases the solution is known in closed form, for instance when $f(x,t)$ is a linear function or $f(x,t)$ is such that $\dot{x}(t) = f(x,t)$ belongs to the class of Bernoulli equation.

Finding the jump for the external Dirac's delta, i.e. $g(x,t)=1$, is also trivial since in this case the jump is simply $J(t_k) = Y_k$.

On the other hand, when $g(x,t)$ is a nonlinear function, finding the jump is not a trivial task. In some previous papers (Di Paola and Pirrotta, 1999) the jump for the case of parametric impulse and $g(x,t) \in C_\infty$ (the class of ∞-times differentiable functions) has been given in a series expansion

$$J(t_k) = x(t_k^+) - x(t_k^-) = \sum_{j=1}^{\infty} Y_k^j \frac{g^{(j)}\left(x(t_k^-), t_k\right)}{j!} \quad (2)$$

where $g^{(j)}(x,t)$ is evaluated in recursive form as follows

$$g^{(j)}(x,t) = \frac{\partial g^{(j-1)}(x,t)}{\partial x} g^{(1)}(x,t); \; g^{(1)}(x,t) = g(x,t) \quad (3)$$

Note that an alternative expression of equation (2) has been found by Sun et al. (2013) which confirms the goodness of equation (2) and gives the jump prediction in a different series expansion.

Here another way of finding the jump is proposed which gives the jump prediction in analytical form.

In order to find $x\left(t_k^+\right)$ once $x\left(t_k^-\right)$ is already known, as customary it can be assumed that the Dirac's delta is a window function with finite duration ε and with an amplitude Y_k/ε, so that the total area of the impulse is preserved. This situation is depicted in Figure 1. Note that the correct value of $x\left(t_k^+\right)$ is obtained as the limit when $\varepsilon \to 0$.

Calling $z(\tau)$ the solution of the following differential equation

$$\begin{cases} \dot{z}(\tau) = f(z, t_k + \tau) + \dfrac{Y_k}{\varepsilon}\, g(z, t_k + \tau) \\ z(0) = x\left(t_k^-\right) \end{cases} \tag{4a, b}$$

at the limit as $\varepsilon \to 0$ then $z(\varepsilon) \to x\left(t_k^-\right)$. Equation (4a) is an ordinary differential equation whose exact solution is in general unknown, unless equation (4a) is a Bernoulli equation.

On the other hand, considering that the contribution of the first term at the right hand side of equation (4a) is of order $f(z, t)\,\varepsilon$ and it is only necessary to evaluate the value $z(\varepsilon)$ when $\varepsilon \to 0$, then at the limit as $\varepsilon \to 0$ that term is an infinitesimal of the first order with respect to the finite jump. Therefore the first term at the right hand side of equation (4a) may be neglected and the following simpler differential equation has to be solved to capture the value of $x\left(t_k^+\right)$

$$\begin{cases} \dot{z}(\tau) = \dfrac{Y_k}{\varepsilon}\, g(z, t_k + \tau) \\ z(0) = x\left(t_k^-\right) \end{cases} \tag{5a, b}$$

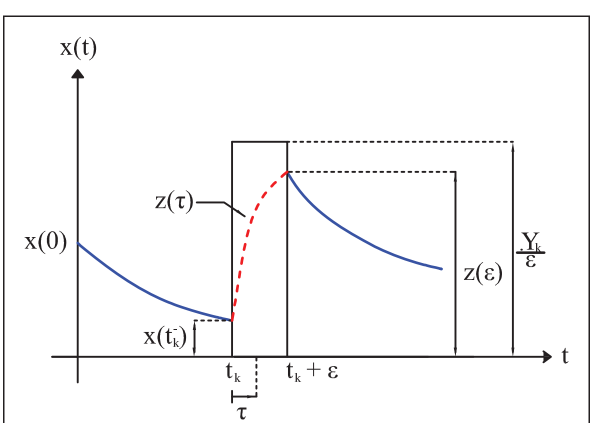

Figure 1. Response to the window function of duration ε and amplitude Y_k/ε.

For the ensuing derivations an analytical form of $z(\varepsilon)$ is needed, so it is necessary to confine the attention to some quite general form of antisymmetric and symmetric nonlinearities, in the form

$$g(z, t) = |z|^{\alpha}\mathrm{sgn}(z);\ \ \alpha \in \mathbb{R}^+ \tag{6a}$$

$$g(z, t) = |z|^{\alpha};\ \ \alpha \in \mathbb{R}^+ \tag{6b}$$

By solving equation (5a) with $g(z, t)$ represented in equation (6a) solution for the linear case ($\alpha = 1$) as well as for nonlinearity of the type $\mathrm{sgn}(\cdot)$ ($\alpha = 0$), cubic ($\alpha = 3$) and so on $\left(|z|^{2\alpha+1}\mathrm{sgn}(z) = z^{2\alpha+1};\ \alpha = 0, 1, 3, \ldots\right)$ may be found. Further by knowing the solution of equation (5a) with the nonlinear parametric function $g(z, t)$ expressed in equation (6b), the case of external input ($\alpha = 0$) is solved, and for ($\alpha = 2, 4, \ldots$) the cases of the type $|z|^{2\alpha} = z^{2\alpha}$.

Note that in the aforementioned cases in which $\alpha \in \mathbb{N}^+$, equation (5a) belongs to the class of Bernoulli equations, thus an exact solution is known in closed form. On the other hand as $\alpha \in \mathbb{R}^+$ the solution of equation (5a) is not known in analytical form to the best of authors knowledge and then it will be herein presented.

In order to understand the problem at hands firstly the simple case for ($\alpha = 1$) in equation (6a) can be solved. In this case equation (5a) is simply a linear equation whose solution is given as

$$z(\tau) = x\left(t_k^-\right)e^{\frac{Y_k}{\varepsilon}\tau} \tag{7}$$

and then $z(\varepsilon) = x\left(t_k^+\right)$ is given as

$$x\left(t_k^+\right) = x\left(t_k^-\right)e^{Y_k};\qquad J(t_k) = x\left(t_k^-\right)\left(e^{Y_k} - 1\right) \tag{8a, b}$$

Equation (8) shows that the solution at the end of the interval is independent of ε and then $\lim_{\tau\to\varepsilon}\ z(\tau) = x\left(t_k^+\right) = x\left(t_k^-\right)e^{Y_k}$. This is a very relevant aspect that will be demonstrated for all other forms of nonlinearities (equations (6)). Specifically as ε is a finite quantity, the evolution of $z(\tau)$ is strongly dependent on the chosen value of ε, but at the end of the interval ε the value $z(\varepsilon)$ only depends on the total area of the window function; thus it returns the jump also for a Dirac's delta impulse. Of course, as expected, equation (8) exactly coalesces to the jump evaluated by equation (2) for the case considered. In fact, since for the case $g(x) = x$, $g^{(j)}(x) = x\ \forall j$ and then the jump evaluated in equation (2) is the classical Taylor series expansion of the jump evaluated by means of equation (8b), provided the Taylor series converges. Further from equation (8) it can be clearly realized that the jump in the case of parametric impulse may be drastically different from the case of external impulse in which the jump is simply Y_k.

It is worth noting that implicitly a window function $\psi(Y_k,\tau)=\frac{Y_k}{\varepsilon}[U(\tau)-U(\tau-\varepsilon)]$, where $U(\bullet)$ is the unit step function, of duration ε and constant amplitude Y_k/ε in the interval $[0,\ \varepsilon]$ (Figure 1) has been assumed at the right hand side of equation (5a). In Appendix A it will be shown that, not only the jump does not depend on the chosen value of ε, but further it does not even depend on the shape of the function $\psi(Y_k,\tau)$, provided the area in the interval is preserved.

3. Antisymmetric nonlinearities

Consider the more general case of nonlinear parametric function in equation (6a). Taking into account equation (3), the following relations are obtained

$$g^{(1)}(z,t)=|z|^{\alpha}\mathrm{sgn}(z) \tag{9a}$$

$$g^{(2)}(z,t)=\alpha|z|^{2\alpha-1}\mathrm{sgn}(z) \tag{9b}$$

$$g^{(3)}(z,t)=\alpha(2\alpha-1)|z|^{3\alpha-2}\mathrm{sgn}(z) \tag{9c}$$

$$g^{(4)}(z,t)=\alpha(2\alpha-1)(3\alpha-2)|z|^{4\alpha-3}\mathrm{sgn}(z) \tag{9d}$$

Then, generalizing

$$g^{(j)}(z,t)=|z|^{j\alpha-(j-1)}\left[\prod_{k=2}^{j}(k-1)\alpha-(k-2)\right]\mathrm{sgn}(z) \tag{10}$$

Exact solution of the system equations (5) with $g(z,t)$ represented in equation (6a), can be found by evaluating the limit of the series in equation (2) with equation (10), giving

$$z(\tau)=\left[\left|x\left(t_k^-\right)\right|^{1-\alpha}+\frac{Y_k\tau}{\varepsilon}(1-\alpha)\right]^{\frac{1}{1-\alpha}}\mathrm{sgn}\left(x\left(t_k^-\right)\right) \tag{11}$$

Note that at the limit as $\alpha\to 1$, $z(\tau)$ coalesces with equation (7). Further, by assuming equation (11) as the exact response, its derivative with respect to τ exactly coalesces with $(Y_k/\varepsilon)\left|z(\tau)\right|^{\alpha}\mathrm{sgn}(z)$, as expected, then $z(\tau)$ in equation (11) is the exact response during the interval $0<\tau<\varepsilon$.

From equation (11) it can be easily shown that $\lim_{\tau\to\varepsilon} z(\tau)=x\left(t_k^+\right)$ is independent on ε and then also in this case it may be asserted that the jump only depends on the area of the impulse. The jump is then given as

$$\begin{aligned}J(t_k)&=\lim_{\varepsilon\to 0} z(\varepsilon)-x\left(t_k^-\right)\\&=\left[\left|x\left(t_k^-\right)\right|^{1-\alpha}+Y_k(1-\alpha)\right]^{\frac{1}{1-\alpha}}\mathrm{sgn}\left(x\left(t_k^-\right)\right)-x\left(t_k^-\right)\end{aligned} \tag{12}$$

Equation (12) needs of some deeper insight. First of all the factor in square brackets has to be positive; otherwise, unless $\alpha=1-1/n,\ n\in\mathbb{N}$, the jump becomes imaginary and this is in contrast with the physics of the problem at hands. This is because equation (12) gives the exact solution with some limitations, specifically:

$$Y_k\geq-\frac{\left|x\left(t_k^-\right)\right|^{1-\alpha}}{1-\alpha};\quad 0<\alpha\leq 1 \tag{13a}$$

$$Y_k<\frac{\left|x\left(t_k^-\right)\right|^{1-\alpha}}{|1-\alpha|};\quad 1<\alpha<\infty \tag{13b}$$

Consider the case $0<\alpha\leq 1$. As reported in equation (13a) Y_k will be greater than a certain value in order to have a jump belonging to the real axis. However an inspection of equation (11) reveals that there is a certain value of τ into the interval $0<\tau<\varepsilon$, say $\bar{\tau}$, such that $z(\bar{\tau})=0$, that is

$$\bar{\tau}=-\frac{\varepsilon}{Y_k(1-\alpha)}\left|x\left(t_k^-\right)\right|^{1-\alpha} \tag{14}$$

This situation is depicted in Figure 2.

From $\bar{\tau}$ up to ε the differential equation (5a) becomes

$$\begin{cases}\dot{z}(\tau-\bar{\tau})=\dfrac{Y_k}{\varepsilon}g(z,t_k+\tau-\bar{\tau})\\ z(\bar{\tau})=0\end{cases};\quad \tau>\bar{\tau} \tag{15a, b}$$

Therefore, since the system is autonomous with zero initial condition, the solution is zero from $\bar{\tau}$ to $t_k+\varepsilon$. This means that as the inequality in equation (13a) is not fulfilled then the jump is simply $J(t_k)=-x\left(t_k^-\right)$.

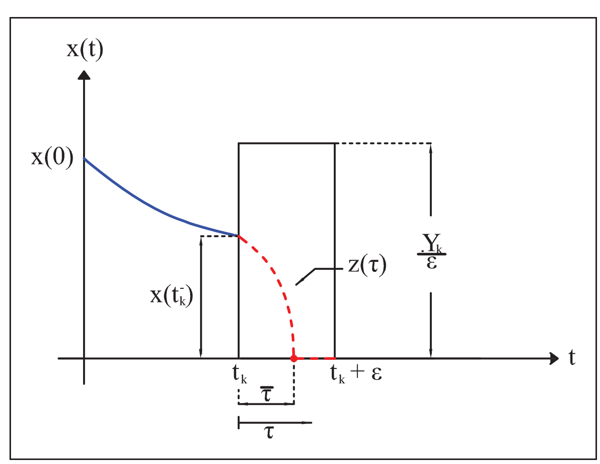

Figure 2. Situation in which inequalities in Equations (13a) are not fulfilled.

Further if $Y_k = -\left|x\left(t_k^-\right)\right|^{1-\alpha}/(1-\alpha)$ then the zero of $z(\tau)$ happens at $\tau = \varepsilon$, and after this time instant the response is zero.

Consider now the case $1 < \alpha < \infty$. When a parametric impulse appears then, as $Y_k > \left|x\left(t_k^-\right)\right|^{1-\alpha}/|1-\alpha|$, the response in ε is divergent and an instantaneous instability is experienced. This may be seen also by equation (2) since the series is divergent. On the other hand for $\alpha = 3$, equation (5a) is a Bernoulli equation and equation (11) exactly coalesces with the corresponding exact solution that is

$$z(\tau) = \frac{x\left(t_k^-\right)}{\sqrt{1 - \frac{2\tau}{\varepsilon} Y_k x\left(t_k^-\right)^2}} \tag{16}$$

and equation (16) (or equation (12) evaluated as $\alpha \to 3$) for $Y_k > \left|x\left(t_k^-\right)\right|^{1-\alpha}/|1-\alpha|$ as $\tau \to \varepsilon$ gives an infinite value.

In Figure 3 the three solutions equation (16), equation (11) and equation (2) are contrasted for two values of the intensity of Y_k, one that fulfills the limitation equation (13b) and the other one with a value of intensity greater than $\left|x\left(t_k^-\right)\right|^{-2}/2$, and two different values of ε.

Inspection of Figure 3 reveals that:

(i) The various solutions obtained through equations (2), (11) and (16), perfectly matches.
(ii) The value of $z(\tau)$ at the end of the interval is independent on the value of ε selected.
(iii) For a value of Y_k that does not fulfill the limitation equation (13b), the value at the end of the interval is ∞.

Note that in Appendix B the evolution of $z(\tau)$, considering different initial conditions and different values of Y_k, will be analyze in deeper details, for the nonlinear parametric function in equation (6a).

4. Symmetric nonlinearities

Consider now the case of nonlinear parametric function in equation (6b). Taking into account equation (3), the following relations are obtained

$$g^{(1)}(z,t) = |z|^{\alpha} \tag{17a}$$

$$g^{(2)}(z,t) = \alpha|z|^{2\alpha-1}\mathrm{sgn}(z) \tag{17b}$$

$$g^{(3)}(z,t) = \alpha(2\alpha-1)|z|^{3\alpha-2} \tag{17c}$$

$$g^{(4)}(z,t) = \alpha(2\alpha-1)(3\alpha-2)|z|^{4\alpha-3}\mathrm{sgn}(z) \tag{17d}$$

Then, generalizing

$$g^{(j)}(z,t) = |z|^{j\alpha-(j-1)}\left[\prod_{k=2}^{j}(k-1)\alpha-(k-2)\right]\mathrm{sgn}(z)^{j+1} \tag{18}$$

Following the procedure adopted in the previous section, exact solution of the system equations (5) with $g(z,t)$ represented in equation (6b), can be found by evaluating the limit of the series in equation (2) with equation (18), giving

$$z(\tau) = \left[\left|x\left(t_k^-\right)\right|^{1-\alpha} + \frac{Y_k\tau}{\varepsilon}(1-\alpha)\mathrm{sgn}\left(x\left(t_k^-\right)\right)\right]^{\frac{1}{1-\alpha}}\mathrm{sgn}\left(x\left(t_k^-\right)\right) \tag{19}$$

Derivative of equation (19) with respect to τ yields $(Y_k/\varepsilon)\left|z(\tau)\right|^{\alpha}$, thus it can be asserted that equation

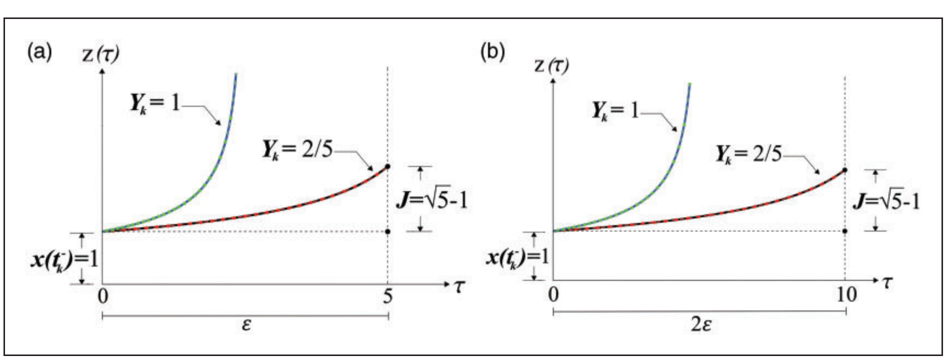

Figure 3. Solution of $z(\tau)$ for $\alpha = 3$ and two different values of Y_k. Exact solution equation (16) – Continuous line; equation (11) – Dashed dotted line: (a) $\varepsilon = 5$, (b) $2\varepsilon = 10$.

(19) is the exact solution. Also in this case $z(\varepsilon)$ is independent of ε and then $z(\varepsilon)$ is the value $x(t_k^+)$ for the parametric impulse of the kind $|x|^\alpha \delta(t-t_k)$. The jump is then given as

$$\begin{aligned} J(t_k) &= \lim_{\varepsilon \to 0} z(\varepsilon) - x(t_k^-) \\ &= \left[\left| x(t_k^-) \right|^{1-\alpha} + Y_k (1-\alpha) \operatorname{sgn}\left(x(t_k^-)\right) \right]^{\frac{1}{1-\alpha}} \\ &\quad \times \operatorname{sgn}\left(x(t_k^-)\right) - x(t_k^-) \end{aligned} \tag{20}$$

Note that equation (20) remains valid under the limitations

$$\begin{cases} Y_k \geq -\dfrac{\left|x(t_k^-)\right|^{1-\alpha}}{1-\alpha} & \forall\, x(t_k^-) > 0 \\ Y_k \leq \dfrac{\left|x(t_k^-)\right|^{1-\alpha}}{1-\alpha} & \forall\, x(t_k^-) < 0 \end{cases} ; \quad 0 < \alpha \leq 1 \tag{21a}$$

$$\begin{cases} Y_k \leq \dfrac{\left|x(t_k^-)\right|^{1-\alpha}}{\alpha - 1} & \forall\, x(t_k^-) > 0 \\ Y_k \geq -\dfrac{\left|x(t_k^-)\right|^{1-\alpha}}{\alpha - 1} & \forall\, x(t_k^-) < 0 \end{cases} ; \quad 1 < \alpha < \infty \tag{21b}$$

Also in this case if the limitation in equation (21a) is not fulfilled then it exist a certain value of τ into the interval $0 < \tau < \varepsilon$, say $\bar{\tau}$, such that $z(\bar{\tau}) = 0$. As for the case of equation (13a), once $z(\tau)$ reaches zero, it remains zero up to $\tau = \varepsilon$. It follows that the jump in this case is simply $J(t_k) = -x(t_k^-)$.

The second limitation equation (21b) also provides as $\tau \to \varepsilon$ an infinite jump, and then this is an effective limitation in the sense that if equation (21b) is not fulfilled the system experiences an instantaneous instability.

For $\alpha = 2$ the solution of the Bernoulli equation

$$\begin{cases} \dot{z}(\tau) = \dfrac{Y_k}{\varepsilon} \left| z(\tau) \right|^2 = \dfrac{Y_k}{\varepsilon} z(\tau)^2 \\ z(\tau) = x(t_k^-) \end{cases} \tag{22a, b}$$

is already known and is given as

$$z(\tau) = \frac{x(t_k^-)}{1 - \frac{Y_k \tau}{\varepsilon} x(t_k^-)} \tag{23}$$

and this solution coalesces with equation (19).

Note that in Appendix B the evolution of $z(\tau)$, for different initial conditions and different values of Y_k, will be analyze in deeper details, also for the nonlinear parametric function in equation (6b).

5. Extension to multi-degree-of-freedom systems

For sake of completeness, in this section the extension of the previous concepts to multi-degree-of-freedom (MDOF) nonlinear systems, is outlined.

Consider a dynamical system, enforced by a parametric impulsive input, whose equation of motion can be cast in the form

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, t) + Y_k\, \mathbf{g}(\mathbf{y}, t) \delta(t - t_k), \quad \mathbf{y}(0) = \mathbf{y_0} \tag{24}$$

where $\mathbf{y}$ is a n-vector of state variables, $\mathbf{f}(\mathbf{y}, t)$ and $\mathbf{g}(\mathbf{y}, t)$ are nonlinear differentiable n-vector functions of $\mathbf{y}$ and t.

Following the procedure reported in Di Paola and Pirrotta (1999), for every $t \leq t_k^-$ and $t \geq t_k^+$ solution of equation (24) can easily be reached solving the simpler differential equations

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, t), \quad \mathbf{y}(0) = \mathbf{y_0}, \quad \forall t \leq t_k^- \tag{25a}$$

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, t), \quad \mathbf{y}(t_k^+), \quad \forall t \geq t_k^+ \tag{25b}$$

while the jump can be evaluated in the form

$$\mathbf{J}(t_k) = \sum_{j=1}^{\infty} \frac{Y_k^j}{j!} \mathbf{g}^{(j)}\left(\mathbf{y}(t_k^-), t_k\right) \tag{26}$$

where $\mathbf{g}^{(j)}$ can be evaluated in the recursive form as

$$\mathbf{g}^{(j)}(\mathbf{y}, t) = \left(\nabla \mathbf{g}^{(j-1)}(\mathbf{y}, t)\right) \mathbf{g}^{(1)}(\mathbf{y}, t) \tag{27}$$

in which $\nabla \mathbf{g}^{(j-1)}(\mathbf{y}, t)$ is the gradient operator, given as

$$\nabla \mathbf{g}^{(j-1)} = \begin{bmatrix} \dfrac{\partial g_1^{(j-1)}}{\partial y_1} & \dfrac{\partial g_1^{(j-1)}}{\partial y_2} & \dots & \dfrac{\partial g_1^{(j-1)}}{\partial y_n} \\ \dfrac{\partial g_2^{(j-1)}}{\partial y_1} & \dfrac{\partial g_2^{(j-1)}}{\partial y_2} & \dots & \dfrac{\partial g_2^{(j-1)}}{\partial y_n} \\ \vdots & \vdots & \dots & \vdots \\ \dfrac{\partial g_n^{(j-1)}}{\partial y_1} & \dfrac{\partial g_n^{(j-1)}}{\partial y_2} & \dots & \dfrac{\partial g_n^{(j-1)}}{\partial y_n} \end{bmatrix} \tag{28}$$

where $\mathbf{g}_s^{(j-1)}(\mathbf{y}, t)$ is the s-th component of the vector $\mathbf{g}^{(j-1)}(\mathbf{y}, t)$.

Note that, the outlined procedure can be used to determine the response of a very wide class of dynamical systems. In this regard, as in the previous sections, it is advantageous to focus the attention on two general form of the nonlinear parametric functions $\mathbf{g}(\mathbf{y}, t)$.

For instance consider now, for simplicity sake, a single-degree-of-freedom (SDOF) system whose equation of motion can be written as

$$\ddot{x}+f(x,\dot{x},t)=Y_k\,|\dot{x}|^{\alpha}\mathrm{sgn}(\dot{x})\,\delta(t-t_k);\quad t_k>0 \tag{29a}$$

or else

$$\ddot{x}+f(x,\dot{x},t)=Y_k\,|\dot{x}|^{\alpha}\delta(t-t_k);\quad t_k>0 \tag{29b}$$

Equations (29) can be cast in the form of equation (24) where $\mathbf{y}=\begin{bmatrix}y_1 & y_2\end{bmatrix}^{\mathbf{T}}$ is the state variables vector and the parametric functions $\mathbf{g}(\mathbf{y},t)$ is given as

$$\mathbf{g}(\mathbf{y},t)=\begin{bmatrix}0 & |y_2|^{\alpha}\mathrm{sgn}(y_2)\end{bmatrix}^{\mathbf{T}};\quad \alpha\in\mathbb{R}^+ \tag{30a}$$

$$\mathbf{g}(\mathbf{y},t)=\begin{bmatrix}0 & |y_2|^{\alpha}\end{bmatrix}^{\mathbf{T}};\quad \alpha\in\mathbb{R}^+ \tag{30b}$$

for equation (29a) and (29b) respectively.

For every $t\le t_k^-$ and $t\ge t_k^+$ solution of equations (29) can easily reached solving the simpler system in equations (25), while in t_k the value of the jump has to be evaluated.

If $\mathbf{g}(\mathbf{y},t)$ is given as in equation (30a), then equation (26) restitutes the value of the jump in closed form as

$$\mathbf{J}(t_k)=\left[0\quad\left[\left|y_2\left(t_k^-\right)\right|^{1-\alpha}+Y_k(1-\alpha)\right]^{\frac{1}{1-\alpha}}\mathrm{sgn}\left(y_2\left(t_k^-\right)\right)-y_2\left(t_k^-\right)\right]^{\mathbf{T}} \tag{31}$$

On the other hand, if $\mathbf{g}(\mathbf{y},t)$ is given as in equation (30b), then the value of the jump can be evaluated in closed form through equation (26) as

$$\mathbf{J}(t_k)=\begin{bmatrix}0 & \left[\left|y_2\left(t_k^-\right)\right|^{1-\alpha}+Y_k(1-\alpha)\mathrm{sgn}\left(y_2\left(t_k^-\right)\right)\right]^{\frac{1}{1-\alpha}} \\ & \times\mathrm{sgn}\left(y_2\left(t_k^-\right)\right)-y_2\left(t_k^-\right)\end{bmatrix}^{\mathbf{T}} \tag{32}$$

Thus, for the two cases considered, the system response can be easily evaluated through equations (25), in which $\mathbf{y}\left(t_k^+\right)$ is given as

$$\mathbf{y}\left(t_k^+\right)=\mathbf{y}\left(t_k^-\right)+\mathbf{J}(t_k) \tag{33}$$

6. Numerical examples

In order to show the accuracy of the proposed solution, in this section two benchmark examples are considered for the half oscillator case equations (1). Further, as far as SDOF nonlinear systems are concerned, solution for a Duffing oscillator under parametric input is presented.

It is worth remarking that, as detailed in Pirrotta (2005), numerical methods (for example in Mathematica environment), could fail in finding the correct value of the jump, and hence of the response itself, of systems under parametric impulsive input. In this case in fact numerical simulations should be treated with care, and particular procedures, such as those reported in Pirrotta (2005) and Di Paola and Pirrotta (1999), should be implemented to determine the correct response of the system.

On this base, in this section two systems whose correct solution is known in closed form are considered as benchmark.

6.1. Benchmark for an antisymmetric nonlinear parametric function

Let the equation of motion be given in the form

$$\begin{cases}\dot{x}(t)=-a\,x(t)\ln(x(t))+Y_k\,x(t)\,\delta(t-t_k)\\ x(0)=x_0\end{cases};\quad t_k>0 \tag{34a, b}$$

This is a nonlinear equation with a linear multiplicative term, thus it belongs to the class of systems with $g(x,t)=|x|^{\alpha}\mathrm{sgn}(x)$ for $\alpha=1$.

Following the procedure described in Section 3, the value immediately before the Dirac's delta is $x\left(t_k^-\right)=x_0^{\exp(-a\,t_k)}$, while the jump evaluated by means of equation (12) at the limit for $\alpha\to 1$ is given as

$$J(t_k)=x\left(t_k^-\right)[\exp(Y_k)-1] \tag{35}$$

It is worth stressing that equation (34a) can be exactly solved, considering the following invertible nonlinear transformation

$$y=\ln(x);\quad x=\exp(y) \tag{36}$$

Since $\dot{x}=\exp(y)\dot{y}$, then substituting equation (36) in equations (34) yields

$$\begin{cases}\dot{y}(t)=-a\,y(t)+Y_k\,\delta(t-t_k)\\ y(0)=\ln(x_0)\end{cases};\quad t_k>0 \tag{37a, b}$$

Equation (37a) is a linear differential equation under an external input, hence the solution of this system can be obtained using the classical differential calculus as

$$y(t)=y(0)\exp(-at),\quad \forall t<t_k \tag{38}$$

while the jump is simply given by $J(t_k) = Y_k$, and then the response after the impulse is given in the form

$$y(t) = y(0)\exp(-at) + Y_k \exp(-a(t - t_k)), \quad \forall t > t_k \tag{39}$$

Finally through the transformation equation (36) the response of the original system equation (34) can be restituted using equations (38) and (39).

In Figure 4 the solution obtained through the non-linear transformation equation (36) vis-a-vis solution of the original system equation (34) with the jump in equation (35) is depicted, for the parameters $a = 1$, $t_k = 2s$, and two different values of Y_k (positive and negative).

As shown in the previous figure, solution obtained through the two procedures perfectly matches and equation (35) exactly restitutes the value of the jump.

6.2. Benchmark for a symmetric nonlinear parametric function

Let the equation of motion be given in the form

$$\begin{cases} \dot{x}(t) = -a\,x(t) + Y_k\,x(t)^2\,\delta(t - t_k) \\ x(0) = x_0 \end{cases}; \quad t_k > 0 \tag{40a, b}$$

Note that this equation belongs to the class of systems with $g(x, t) = |x|^\alpha$ for $\alpha = 2$.

Following the procedure described in section 4, the value immediately before the Dirac's delta is $x\left(t_k^-\right) = x_0 \exp(-a\,t_k)$, while the jump evaluated by means of equation (20) is given as

$$J(t_k) = \frac{Y_k\,x\left(t_k^-\right)^2}{1 - Y_k\,x\left(t_k^-\right)} \tag{41}$$

It is worth stressing that equation (40a) is a Bernoulli equation, thus it can be exactly solved. Specifically, consider the following invertible nonlinear transformation

$$y = x^{-1}; \quad x = y^{-1} \tag{42}$$

Since $\dot{x} = -y^{-2}\dot{y}$, then substituting equation (42) in equations (40) yields

$$\begin{cases} \dot{y}(t) = a\,y(t) - Y_k\,\delta(t - t_k) \\ y(0) = x_0^{-1} \end{cases}; \quad t_k > 0 \tag{43a, b}$$

Equation (43) is a linear differential equation under an external input, hence the solution of this system is

$$y(t) = y(0)\exp(at), \quad \forall t < t_k \tag{44}$$

while the jump is simply given by $J(t_k) = -Y_k$, and then the response after the impulse is given in the form

$$y(t) = y(0)\exp(at) - Y_k \exp(a(t - t_k)), \quad \forall t > t_k \tag{45}$$

Finally through the transformation equation (42) the response of the original system equations (40) can be restituted using equations (44) and (45).

In Figure 5 comparison among the solution obtained through the nonlinear transformation equation (42) and solution of the original system equations (40) with the jump in equation (41) is reported, for the parameters $a = 1$; $x_0 = 1$; $t_k = 2s$, and two different values of Y_k (positive and negative).

As in the previous case, Figure 5 show a perfect match between the solution obtained through the two procedures, thus equation (41) exactly restitutes the value of the jump.

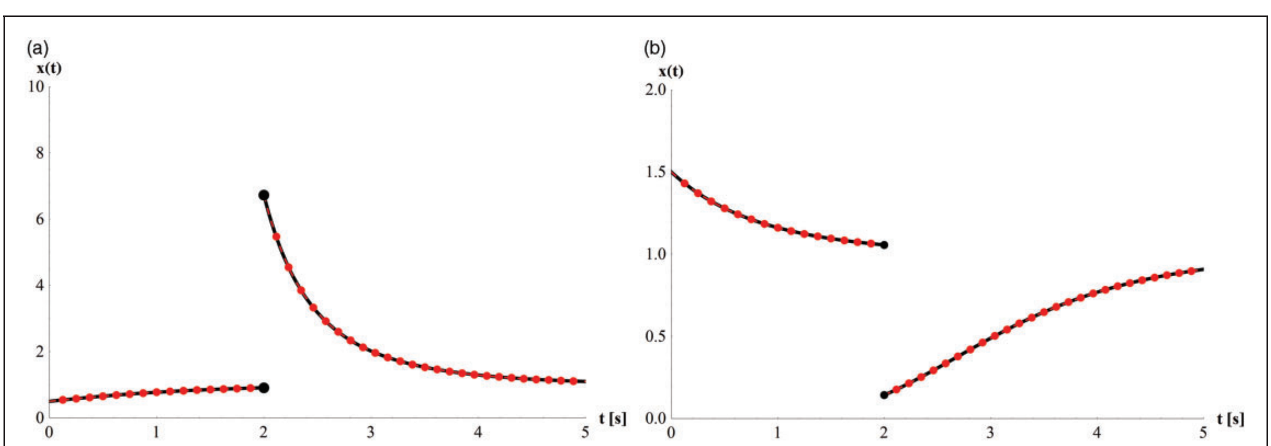

Figure 4. Solution of $x(t)$ for two different values of Y_k. Original system equations (34) – Continuous line; equations (36), (38) and (39) – Dashed dotted line: (a) $x_0 = 1/2$, $Y_k = 2$, (b) $x_0 = 3/2$, $Y_k = -2$.

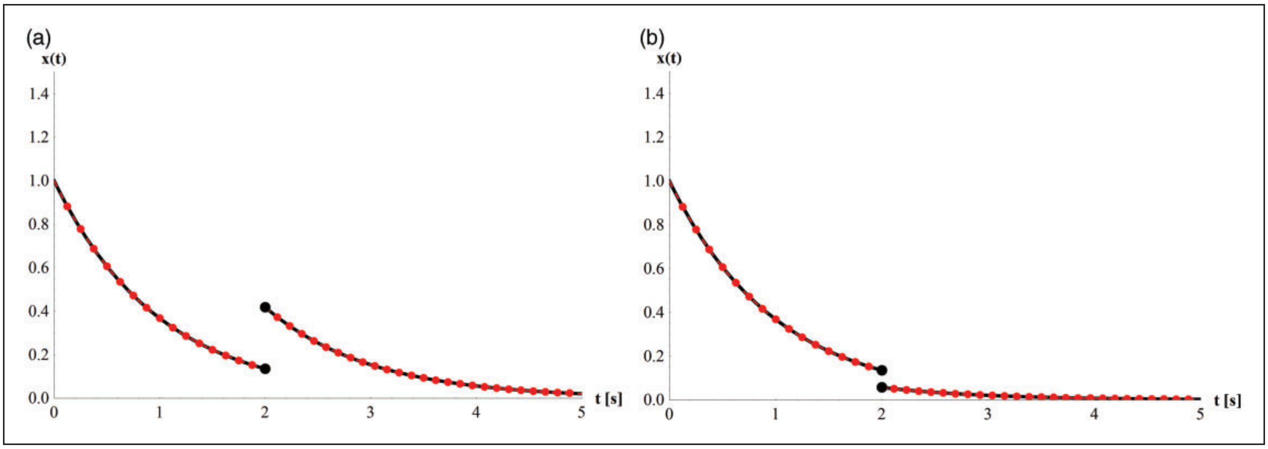

Figure 5. Solution of $x(t)$ for two different values of Y_k. Original system equations (40) – Continuous line; equations (42), (44) and (45) – Dashed dotted line: (a) $Y_k = 5$, (b) $Y_k = -10$.

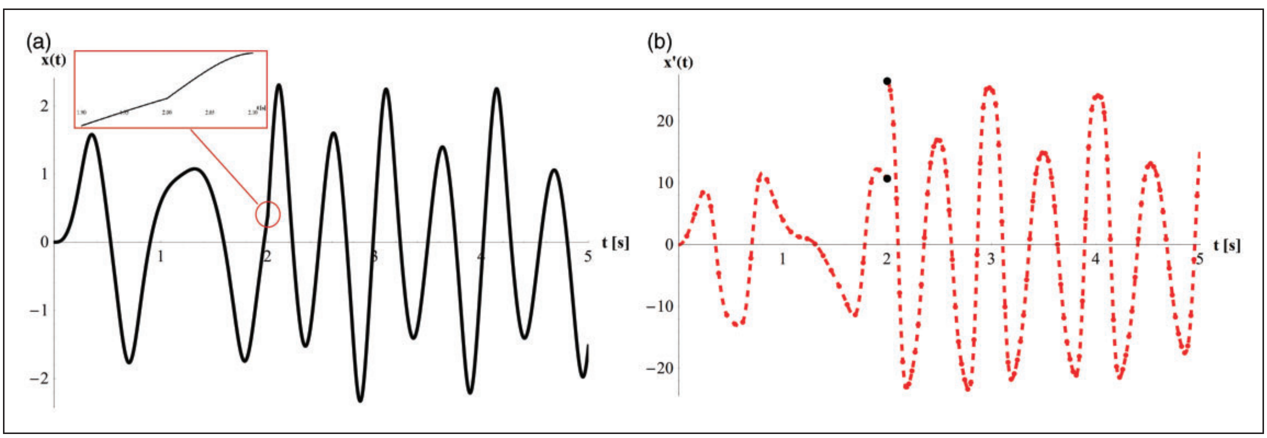

Figure 6. Response of the system equations (45): (a) Displacement $x(t)$, (b) Velocity $\dot{x}(t)$.

6.3. Duffing oscillator

As a third example, the case of a Duffing oscillator under both an external forcing function and a parametric impulse is addressed. The equation of motion of the aforementioned system can be given in the form

$$\begin{cases} \ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2\left(x + \beta x^3\right) \\ \quad = \omega_0^2 r_0 \sin(\omega_0 t) + Y_k\,|\dot{x}|^\alpha \mathrm{sgn}(\dot{x})\,\delta(t - t_k)\,; \qquad t_k > 0 \\ x(0) = \dot{x}(0) = 0 \end{cases} \tag{46a, b}$$

Following the procedure described in section 5, this equation can be recast in the form of equation (24), where $\mathbf{y} = \begin{bmatrix} x & \dot{x} \end{bmatrix}^{\mathbf{T}} = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^{\mathbf{T}}$ is the state variable vector, $\mathbf{g}(\mathbf{y}, t)$ is given as in equation (29a) while $\mathbf{f}(\mathbf{y}, t)$ is given as

$$\mathbf{f}(\mathbf{y}, t) = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -\omega_0^2\beta y_1^3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\omega_0^2 r_0 \sin(\omega_0 t) \tag{47}$$

For every $t \le t_k^-$ and $t \ge t_k^+$ solution of equations (46) can be easily obtained solving the simpler differential equations equations (25), through any numerical method available or for instance using the procedure described in Di Paola and Pirrotta (1999). Once the response in $t = t_k^-$ is evaluated, equations (31) and (33) restitute the jump in t_k and the response of the system $\mathbf{y}\left(t_k^+\right)$.

In Figure 6 the response of the system in terms of both displacement $x(t)$ (Figure 6a) and velocity $\dot{x}(t)$ (Figure 6b) is reported. Analysis has been performed using the following parameters for the system

$\zeta = 0.01$, $\beta = 1$, $r_0 = 2$, $\omega_0 = 2\pi$ and $Y_k = 1$, $\alpha = 3/2$, $t_k = 2s$ for the parametric input.

From this figure, the jump in the velocity $\dot{x}(t)$ (Figure 6b) is evident. Of course, since the chosen nonlinear parametric vector function $\mathbf{g}(\mathbf{y}, t)$ depends on the velocity only, no jump in the response displacement $x(t)$ occurs, while a sudden change of the inclination is evident in $t_k = 2s$.

7. Conclusions

In this paper the response of nonlinear systems under external and multiplicative impulses has been analyzed. Specifically, based on previous published results, prediction of the jump exhibited at each Dirac's delta occurrence has been obtained in closed form for two general classes of multiplicative nonlinear functions. Further, depending on the Dirac's delta amplitude, analyses show that the system response may exhibit an instantaneous instability or converge instantaneously to zero at the impulse occurrence. Finally extension to MDOF systems has been presented.

Numerical results assess that the proposed closed form solution exactly predicts the jump for any nonlinear systems in which the parametric function belongs to the considered classes. However the same procedure can be applied to linear combinations of these functions and also to other nonlinear parametric functions.

It is worth remarking that classical numerical differential equation solvers are not generally able to accurately predict the response of systems under multiplicative impulsive input, since the jump cannot be generally correctly found through classical numerical schemes. Therefore the present formulation can be beneficial, since it allows for the exact determination of the response, keeping at minimum the computational cost.

Finally it has to be stressed that this formulation is also particularly useful for the stochastic response determination of systems under Poisson white noise parametric input. In fact, being the jump prediction derived in analytical form, the extension of schemes such as the Path Integral method to these kind of systems can be easily implemented.

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Appendix A

In this appendix some more details on the shape of the function $\psi(Y_k,\tau)$ in the interval $0<\tau<\varepsilon$ are presented, in order to show that the value of the jump depends only on the area of the parametric impulse, and not on the chosen $\psi(Y_k,\tau)$.

Consider the following nonlinear system

$$\begin{cases} \dot{z}(\tau)=\psi(Y_k,\tau)\,g(z,t_k+\tau) \\ z(0)=x\left(t_k^-\right) \end{cases} \tag{A1a, b}$$

which, as stated in section 2, reverts to the system in equations (5) if the function $\psi(Y_k,\tau)$ is a window of duration ε and amplitude Y_k/ε, that is $\psi(Y_k,\tau)=\frac{Y_k}{\varepsilon}[U(\tau)-U(\tau-\varepsilon)]$, where $U(\bullet)$ is the unit step function.

It is worth noting that, although for simplicity sake analyses in this paper have been carried out considering $\psi(Y_k,\tau)$ as a window, the value of the jumps in equations (12) and (20) are independent on the shape of $\psi(Y_k,\tau)$, provided that the total area of $\psi(Y_k,\tau)$ remains constant.

Let the parametric function in equation (A1a) be given as in equation (6a), that is $g(z,\tau)=|z|^{\alpha}\mathrm{sgn}(z)$, $\alpha\in\mathbb{R}^+$. As shown in section 3, equation (12) restitutes the exact value of the jump $J(t_k)$, if

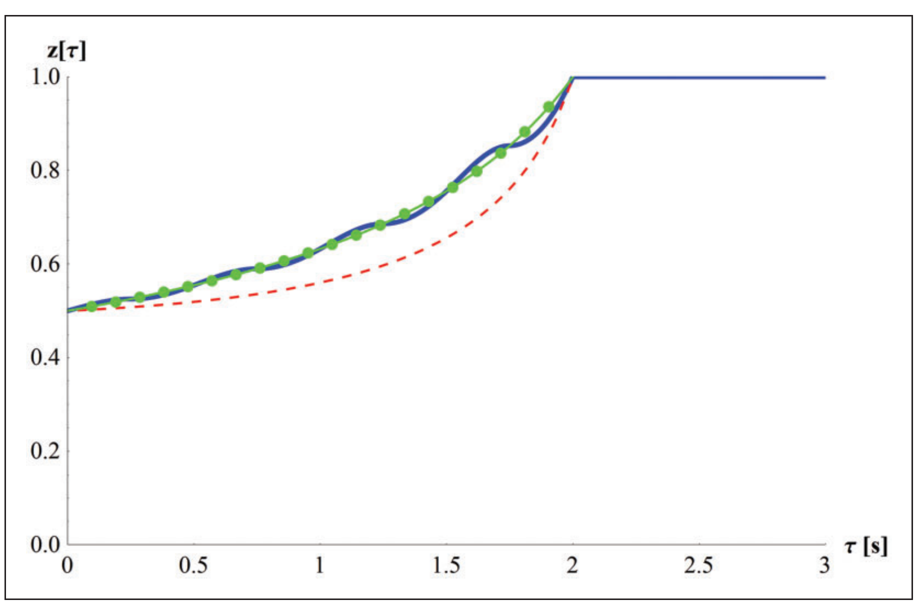

Figure A1. Solution of $z(\tau)$ Eqs. (A 1) for $\alpha=3$, $x_0=1/2$, $Y_k=3/2$ and $\varepsilon=2$.

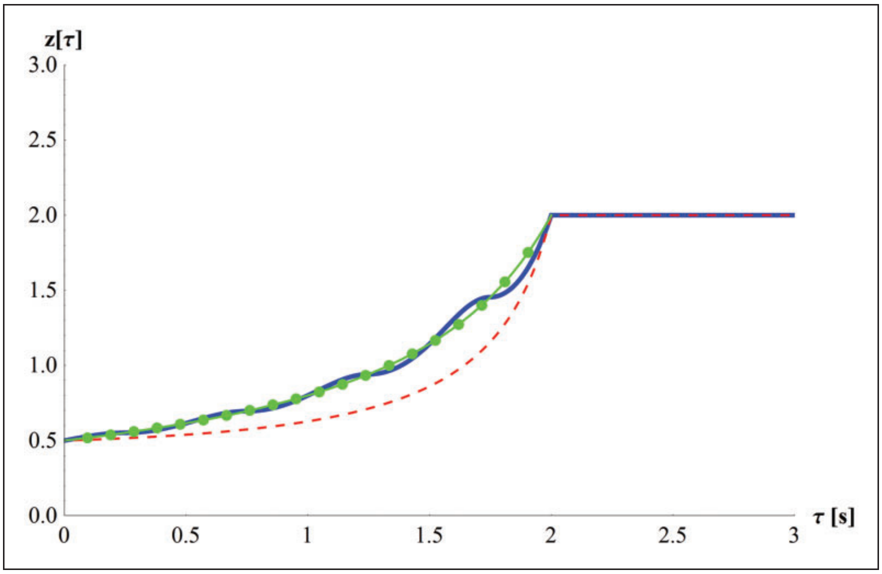

Figure A2. Solution of $z(\tau)$ Eqs. (A 1) for $\alpha = 2$, $x_0 = 1/2$, $Y_k = 3/2$ and $\varepsilon = 2$.

limitations in equations (13a,b) are fulfilled for $0 < \alpha \leq 1$ and $1 < \alpha < \infty$ respectively.

In order to demonstrate that $J(t_k)$ depends only on the area of the impulse, in Figure A1 the evolution of $z(\tau)$, with $g(z, \tau)$ given in equation (6a), is reported for different shapes of $\psi(Y_k, \tau)$. Specifically the response $z(\tau)$ considering three different functions $\psi(Y_k, \tau)$ is shown in Figure A1: $\psi(Y_k, \tau) = \varphi[U(\tau) - U(\tau - \varepsilon)]$ – dotted line; $\psi(Y_k, \tau) = \varphi\, \tau^3[U(\tau) - U(\tau - \varepsilon)]$ – dashed line; $\psi(Y_k, \tau) = \varphi\left|\cos(2\pi\, \tau)\right|[U(\tau) - U(\tau - \varepsilon)]$ – continuous line. Note that φ is a constant parameter evaluated in such a way that the area of $\psi(Y_k, \tau)$ in the interval $0 < \tau < \varepsilon$ is equal to Y_k.

As shown in this figure, although the evolution of $z(\tau)$ strongly depends on the function $\psi(Y_k, \tau)$, the value $z(\varepsilon)$ is the same for every $\psi(Y_k, \tau)$ considered, thus demonstrating that the value of the jump $J(t_k)$ is also independent on $\psi(Y_k, \tau)$.

Further, consider now the parametric function in equation (A1a) be given as in equation (6b), that is $g(z, \tau) = |z|^{\alpha}$, $\alpha \in \mathbb{R}^{+}$. In section 4, equation (20) returns the exact value of the jump $J(t_k)$, if limitations in equations (21a,b) are fulfilled for $0 < \alpha \leq 1$ and $1 < \alpha < \infty$ respectively.

In Figure A2 the evolution of $z(\tau)$, with $g(z, \tau)$ given in equation (6b), is reported for the same three different shapes of $\psi(Y_k, \tau)$ considered in Figure A1, that is: $\psi(Y_k, \tau) = \varphi[U(\tau) - U(\tau - \varepsilon)]$ – dotted line; $\psi(Y_k, \tau) = \varphi\, \tau^3[U(\tau) - U(\tau - \varepsilon)]$ – dashed line; $\psi(Y_k, \tau) = \varphi\left|\cos(2\pi\, \tau)\right|[U(\tau) - U(\tau - \varepsilon)]$ – continuous line. Of course also in this case the value $z(\varepsilon)$ is the same for every $\psi(Y_k, \tau)$ considered, even though the evolution of $z(\tau)$ is different in the time interval $0 < \tau < \varepsilon$.

Appendix B

In this Appendix the evolution of $z(\tau)$ is analyzed in more details. Specifically the response of systems in equations (5a,b) with nonlinear parametric functions as in equations (6a,b) is reported for different initial conditions and different values of Y_k.

Consider the system in equations (5), that is

$$\begin{cases} \dot{z}(\tau) = \dfrac{Y_k}{\varepsilon}\, g(z, t_k + \tau) \\ z(0) = x\left(t_k^{-}\right) \end{cases} \tag{B1a, b}$$

and let the parametric function be given as in equation (6a), that is $g(z, \tau) = |z|^{\alpha}\mathrm{sgn}(z)$, $\alpha \in \mathbb{R}^{+}$.

As demonstrated in section 3, equation (11) restitutes the exact evolution of the response $z(\tau)$, if limitations in equations (13a,b) are fulfilled for $0 < \alpha \leq 1$ and $1 < \alpha < \infty$ respectively.

For sake of completeness, in Figure B1 the evolution of $z(\tau)$, with $g(z, \tau)$ given in equation (6a) in which $1 < \alpha < \infty$, is reported for all the possible cases. In particular Figure B1a depicts the case of $z(0) > 0$ and $Y_k > 0$ while in Figure B1b the corresponding case of $z(0) < 0$ and $Y_k > 0$ is shown; further Figure B1c and Figure B1d reports the case of $Y_k < 0$ for $z(0) > 0$ and $z(0) < 0$ respectively. Note that in Figure B1a and Figure B1b the evolution of $z(\tau)$ is shown for two different values of Y_k, one that fulfils limitation in equation (13b) (continuous line) and one that does not fulfill equation (13b) (dashed line) for which $z(\tau)$ diverges.

Finally to further prove the validity of the proposed solution, response obtained in Mathematica

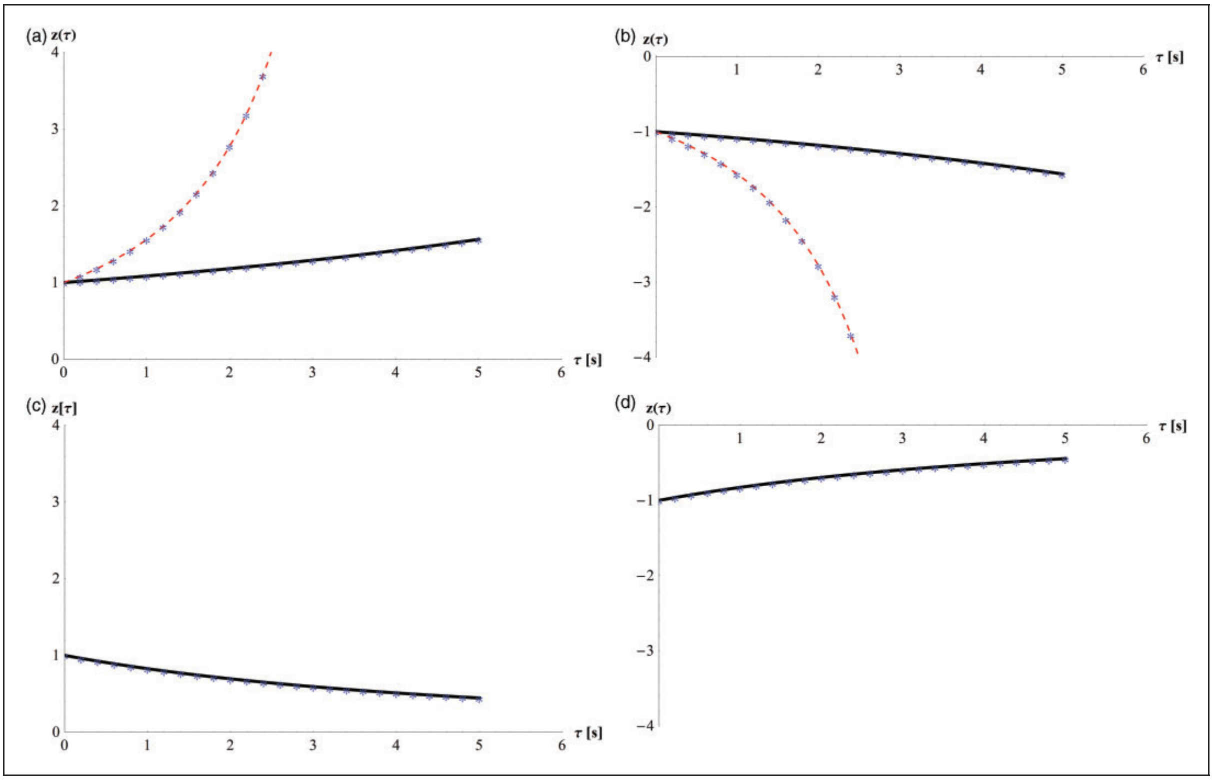

Figure B1. Solution of $z(\tau)$ Eqs. (11) for $\alpha = 3/2$ and $\varepsilon = 5$: (a) Continuous line $x_0 = 1,\ Y_k = 2/5$; dashed line $x_0 = 1,\ Y_k = 2$; (b) Continuous line $x_0 = -1,\ Y_k = 2/5$; dashed line $x_0 = -1,\ Y_k = 2$; (c) $x_0 = 1,\ Y_k = -1$; (d) $x_0 = -1,\ Y_k = -1$.

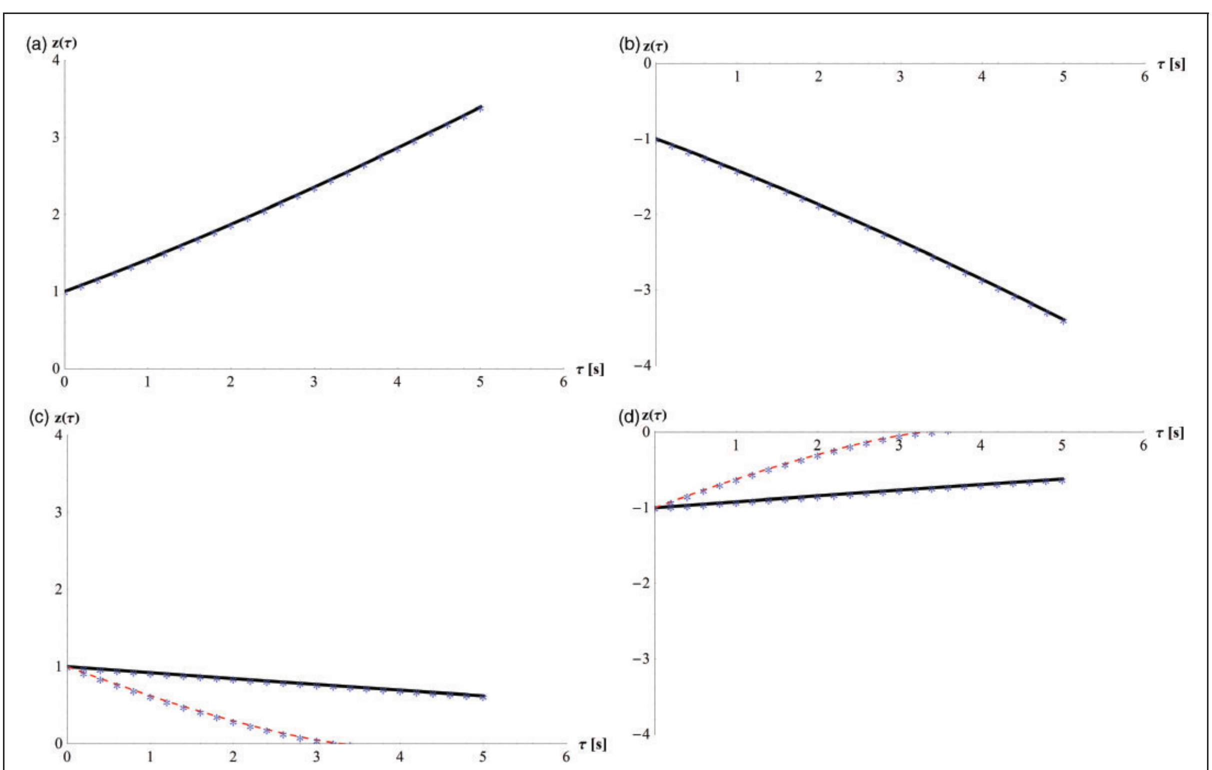

Figure B2. Solution of $z(\tau)$ equations (11) for $\alpha = 1/4$ and $\varepsilon = 5$: (a) $x_0 = 1,\ Y_k = 2$; (b) $x_0 = -1,\ Y_k = 2$; (c) Continuous line $x_0 = 1,\ Y_k = -2/5$; dashed line $x_0 = 1,\ Y_k = -2$; (d) Continuous line $x_0 = -1,\ Y_k = -2/5$; dashed line $x_0 = -1,\ Y_k = -2$.

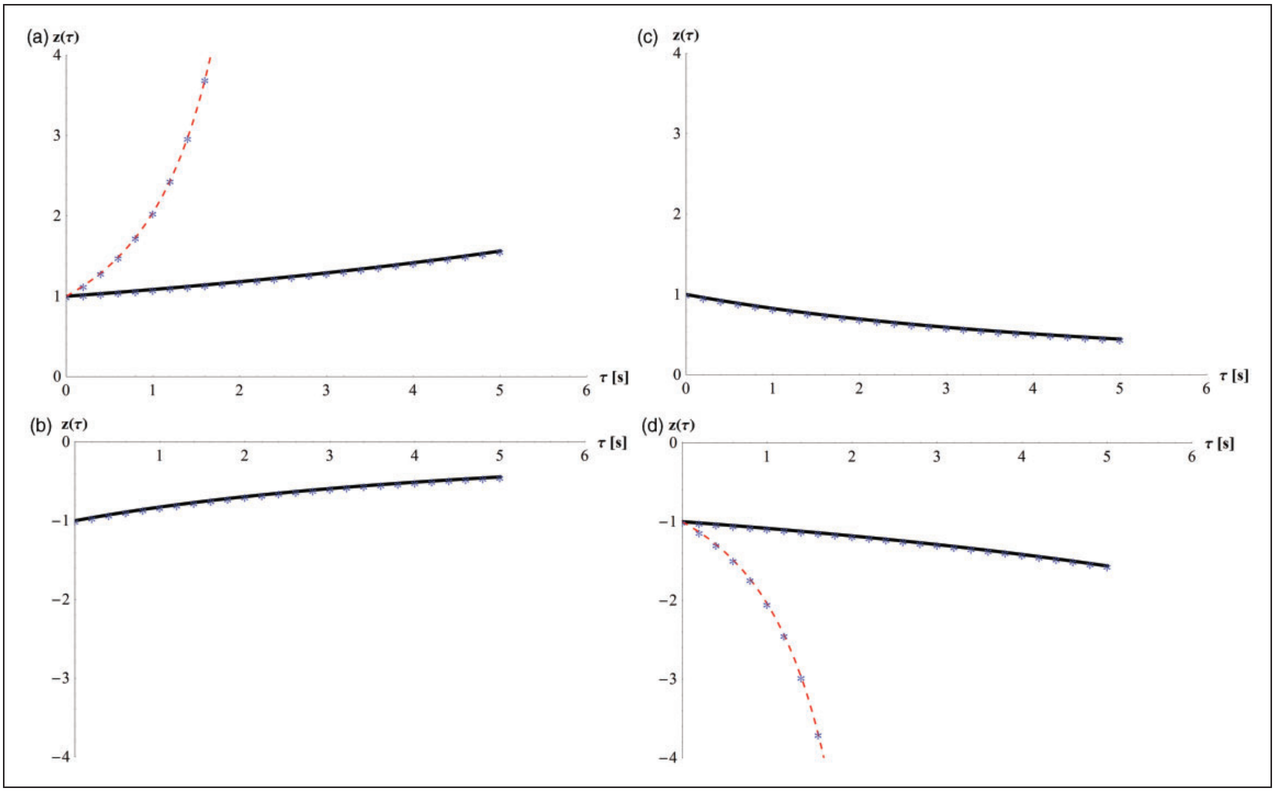

Figure B3. Solution of $z(\tau)$ equations (19) for $\alpha = 3/2$ and $\varepsilon = 5$: (a) Continuous line $x_0 = 1,\ Y_k = 2/5$; dashed line $x_0 = 1,\ Y_k = 3$; (b) $x_0 = -1,\ Y_k = 1$; (c) $x_0 = 1,\ Y_k = -1$; (d) Continuous line $x_0 = -1,\ Y_k = -2/5$; dashed line $x_0 = -1,\ Y_k = -3$.

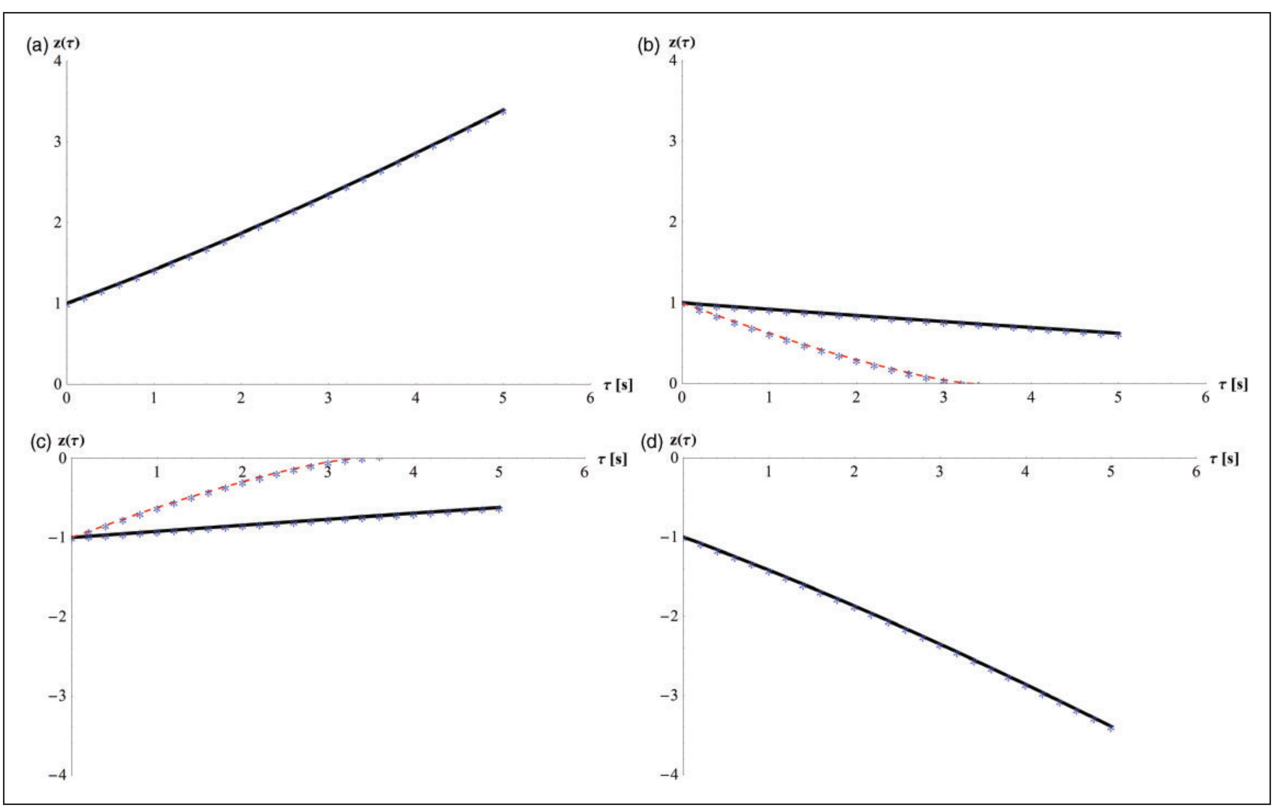

Figure B4. Solution of $z(\tau)$ equations (19) for $\alpha = 1/4$ and $\varepsilon = 5$: (a) $x_0 = 1,\ Y_k = 2$; (b) Continuous line $x_0 = 1,\ Y_k = -2/5$; dashed line $x_0 = 1,\ Y_k = -2$; (c) Continuous line $x_0 = -1,\ Y_k = 2/5$; dashed line $x_0 = -1,\ Y_k = 2$; (d) $x_0 = 1,\ Y_k = -2$.

environment through the numerical solution of equations (B1) with parametric function given in equation (6a) (blue stars), is also reported for all the cases considered.

The corresponding evolution of $z(\tau)$, with $g(z, \tau)$ given in equation (6a) for $0 < \alpha < 1$, is reported in Figure B2. Note that in Figure B2c and Figure B2d the evolution of $z(\tau)$ is shown for two different values of Y_k, one that fulfils limitation in equation (13a) (continuous line) and one that does not fulfill equation (13a) (dashed line) for which $z(\tau)$ goes to zero (equation (14)).

Consider now the system in equations (B1) with parametric function given in equation (6b), that is $g(z, \tau) = |z|^{\alpha}$, $\alpha \in \mathbb{R}^{+}$. As demonstrated in section 4, equation (19) restitutes the exact evolution of the response $z(\tau)$, if limitations in equations (21a,b) are fulfilled for $0 < \alpha \leq 1$ and $1 < \alpha < \infty$ respectively.

For this system Figures B3 depict all the possible cases of the evolution of $z(\tau)$, considering $1 < \alpha < \infty$. Once again in Figure B1a and Figure B1d the evolution of $z(\tau)$ is shown for two different values of Y_k, one that fulfils limitation in equation (21b) (black line) and one that does not fulfill equation (21b) (red dashed line) for which $z(\tau)$ diverges. Further, response obtained through the numerical solution of equations (B1) with parametric function given in equation (6b) in Mathematica environment (blue stars), is also reported for all the cases considered.

The analogous analyses for $0 < \alpha \leq 1$ are reported in Figures B4. Note that Figure B4b and Figure B4c the evolution of $z(\tau)$ is shown for two different values of Y_k, one that fulfils limitation in equation (21a) (continuous line) and one that does not fulfill equation (21a) (dashed line) for which there exists a certain value of τ into the interval $0 < \tau < \varepsilon$, say $\bar{\tau}$, such that $z(\bar{\tau}) = 0$.