Kinematic and dynamic modeling of the 3-RPS parallel manipulator

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Abstract—*This paper presents the kinematic and dynamic modeling of a three degrees of freedom 3-RPS parallel robot. The orientation and position degrees of freedom of the plate-form of this robot are coupled, that leads to complicated kinematic and dynamic models. We propose to exploit the architectural characteristics of the mechanism to give a closed form solution to these problems.*

Keywords: parallel robot, Jacobian matrix, kinematic modeling, dynamic modeling.

I. Introduction

The Jacobian matrix of a general 6-DOF manipulators is a (6×6) matrix, relating the end effector screw vector (linear and angular velocities) to the six active joint velocities. However, it is not straight forward to deal with the Jacobian matrix of limited degrees of freedom parallel manipulators with coupled position and orientation degrees of freedom.

This paper focuses on a 3-DOF parallel robot proposed for the first time by [1] and was since then the object of several research works. For example [2] indicated various applications of the mechanism, as a part of a six degrees of freedom parallel robot, [3] studied a robot ARTISAN whose end effector is a 3-RPS parallel micro-robot, and [4] Studied a device of compensation of force based on this mechanism. In [5] this robot was studied as the wrist of a hybrid robot of surgery.

Generally speaking, the Jacobian matrix of a 3-RPS manipulator is a (6×3) rectangular matrix, this imposes difficulties on the forward and inverse velocity solutions and consequently on the dynamic modeling of the structure. Very few works dealt with this problem, which was avoided most time by considering this structure as a part of a 6-DOF hybrid robot such as in [3]. From these few works we find [6] and [7] who chose to resolve the kinematic problem by using the screws theory. And [8] proposed a method based on equations of constraints obtained from the structural characteristics of the mechanism.

This paper aims to formulate the kinematic and dynamic equations of this mechanism in closed form. In the general configurations of the mechanism there are six degrees of freedom (position and orientation) parameters for the moving platform. However, only three of them are independent. This paper presents a method to obtain three constraint equations, which are consistent with the kinematic structure of the robot. These equations determine the relationships between the independent and dependent degrees of freedom of the platform. The inverse and direct dynamic models of the robot are established using a generalisation of the method proposed by the authors in [11].

II. Description of the 3-RPS robot

Figure 1 shows a 3-DOF parallel robot composed of a moving triangular platform connected to a fixed base with the same shape by three (RPS) extendable legs. The extremities of each leg are fitted with a 1-DOF revolute joint (R) at the base and a 3-DOF spherical joint (S) at the platform. The lengths of the legs are actuated using prismatic joints (P). The three axes of the revolute joints at the base are arranged in 120 degrees and the axis of every one is parallel to the opposite segment of the triangular base.

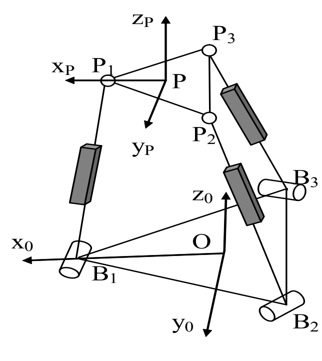

Figure 1. The 3-RPS robot

Assuming that B_i is the point connecting leg i to the base, representing the intersecting point of the axis of the revolute joint and the leg axis, and P_i is the point connecting leg i to the platform, representing the center of the spherical joint. The frame Σ_0 is defined fixed with the base, its origin is the point O, center of the triangle $B_1B_2B_3$, and frame Σ_P is fixed with the mobile platform with the origin P is located at the center of the triangle

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$P_1P_2P_3$. We place these frames as shown in Figure 2.

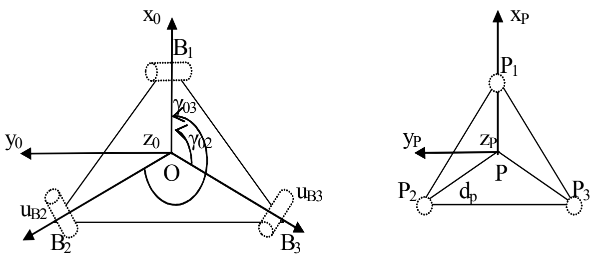

Figure 2. Base frame Σ_0 and platform frame Σ_P

The notations of Khalil and Kleinfinger [9], are used to describe the geometry of the tree structure composed of the base and the legs. The definition of the local link frames of leg i are given in Figure 3, while the geometric parameters are given in Table 1, where:

- a(j) denotes the frame antecedent to frame j,
- $\mu(j) = 1$ if joint j is active and $\mu(j) = 0$ if it is passive,
- σ (j) = 1 if joint j is prismatic and σ (j) = 0 if it is revolute,

The parameters $(\gamma_j, b_j, \alpha_j, d_j, \theta_j, r_j)$ determine the location of Σ_j with respect to its antecedent Σ_i.

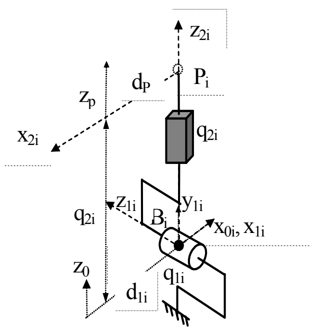

Figure 3. Link frames of one leg

j_i	$a(j_i)$	μ_{ji}	σ_{ji}	γ_{ji}	b_{ji}	α_{ji}	d_{ji}	θ_{ji}	r_{ji}
1_1	0	0	0	0	0	$-\pi/2$	d_1	q_{11}	0
2_1	1_1	1	1	0	0	$\pi/2$	0	0	q_{21}
1_2	0	0	0	$4\pi/3$	0	$-\pi/2$	d_1	q_{12}	0
2_2	1_2	1	1	0	0	$\pi/2$	0	0	q_{22}
1_3	0	0	0	$2\pi/3$	0	$-\pi/2$	d_1	q_{13}	0
2_3	1_3	1	1	0	0	$\pi/2$	0	0	q_{23}

Table 1. Geometric parameters of the equivalent tree structure of the base and the legs

We notice that the coordinates of the point P_i in Σ_0 can be obtained in terms of the two joint variables $\mathbf{q_i} = \begin{bmatrix} q_{1i} & q_{2i} \end{bmatrix}^T$. These coordinates lie in the (x-y) plan of frame Σ_{1i}

III. Inverse geometric model of a leg

The prismatic joint variables q_{2i} are calculated as:

$$q_{2i} = \sqrt{\left({}^0\mathbf{P_i}(1) - {}^0\mathbf{B_i}(1)\right)^2 + \left({}^0\mathbf{P_i}(2) - {}^0\mathbf{B_i}(2)\right)^2 + \left({}^0\mathbf{P_i}(3) - {}^0\mathbf{B_i}(3)\right)^2} \quad (1)$$

The revolute joint variable at the base q_{1i} is calculated as:

$$q_{1i} = \text{atan2}({}^0\mathbf{P_i}(2), {}^0\mathbf{P_i}(1)) \quad (2)$$

Noting that the points B_i coordinates in frame Σ_0 are constant and given as:

$${}^0\mathbf{B_1} = \begin{bmatrix} d_1 \\ 0 \\ 0 \end{bmatrix}, \; {}^0\mathbf{B_2} = \begin{bmatrix} -\frac{1}{2}d_1 \\ \frac{\sqrt{3}}{2}d_1 \\ 0 \end{bmatrix}, \; {}^0\mathbf{B_3} = \begin{bmatrix} -\frac{1}{2}d_1 \\ -\frac{\sqrt{3}}{2}d_1 \\ 0 \end{bmatrix} \quad (3)$$

and the points P_i coordinates in frame Σ_0 are obtained as:

$$\begin{bmatrix} {}^0\mathbf{P_i} \\ 1 \end{bmatrix} = {}^0\mathbf{T_P} \begin{bmatrix} {}^P\mathbf{P_i} \\ 1 \end{bmatrix} \quad (4)$$

where ${}^P\mathbf{P_i}$ is the same as ${}^0\mathbf{B_i}$ after replacing d_1 by d_P. and ${}^0\mathbf{T_P}$ is the homogenous transformation matrix from the base frame Σ_0 to frame Σ_P, that can be represented as:

$${}^0\mathbf{T_P} = \begin{bmatrix} \mathbf{s} & \mathbf{n} & \mathbf{a} & \mathbf{P} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (5)$$

The orientation vectors of the platform $\mathbf{s}, \mathbf{n}, \mathbf{a}$ can be expressed as a function of the coordinates of the origin P, this problem was treated in [4] and [10], it has 8 solutions among which 4 only are acceptable.

IV. Kinematic modeling

The Jacobian matrix of the leg i is the matrix giving the terminal velocity of the leg , equivalent to the velocity of point P_i, in terms of its joint velocities:

$$\mathbf{v_i} = \mathbf{J_i}\dot{\mathbf{q}}_\mathbf{i} \quad (6)$$

Calculating $\mathbf{J_i}$ in the frame 1i gives [12]:

$${}^{1i}\mathbf{J_i} = \begin{bmatrix} {}^{1i}\mathbf{a_{1i}} \times {}^{1i}\mathbf{B_iP_i} & {}^{1i}\mathbf{a_{2i}} \end{bmatrix} \quad (7)$$

where $\mathbf{a_{1i}}$ is the unit vector along the first revolute joint axis, $\mathbf{a_{2i}}$ is the unit vector along the prismatic joint axis and $\mathbf{B_iP_i}$ is the vector from B_i to P_i of each leg.

Finally the Jacobian matrix of leg i is obtained as:

$${}^{1i}\mathbf{J_i} = \begin{bmatrix} -q_{2i} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad (8)$$

Since the motion of the terminal point of each leg is in the plan $x_{1i}y_{1i}$, thus the velocity of the terminal point of leg i is given as:

$$\begin{bmatrix} {}^{1i}v_{xi} \\ {}^{1i}v_{yi} \end{bmatrix} = \begin{bmatrix} -q_{2i} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{q}_{1i} \\ \dot{q}_{2i} \end{bmatrix} \quad (9)$$

So:

$$\mathbf{{}^{1i}v_i} = \mathbf{{}^{1i}J_{di}\dot{q}_i} \quad (10)$$

$\mathbf{{}^{1i}J_{di}}$ is the reduced Jacobian matrix of leg i. Its inverse is given as:

$$\mathbf{{}^{1i}J_{di}^{-1}} = \begin{bmatrix} \frac{1}{-q_{2i}} & 0 \\ 0 & 1 \end{bmatrix} \quad (11)$$

On the other hand, the terminal velocity of leg i can be calculated as a function of the screw of the platform ${}^{\mathbf{0}}V_{\mathbf{p}}$ as:

$$\begin{bmatrix} {}^{1i}v_{xi} \\ {}^{1i}v_{yi} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{{}^{1i}R_0} & -\mathbf{{}^{1i}R_0}\,\mathbf{{}^{0}\hat{PP}_i} \end{bmatrix} {}^{\mathbf{0}}V_{\mathbf{p}} \quad (12)$$

where ${}^{\mathbf{0}}v_{\mathbf{p}}$ is the velocity of the platform, it is composed of the linear and angular velocities: ${}^{\mathbf{0}}V_{\mathbf{p}} = \begin{bmatrix} \mathbf{{}^{0}v_p^T} & \mathbf{{}^{0}\omega_p^T} \end{bmatrix}^T$, $\mathbf{{}^{1i}R_0}$ is the orientation matrix of the base frame with respect to frame Σ_{1i}:

$$\mathbf{{}^{1i}R_0} = \begin{bmatrix} C\gamma_{1i}Cq_{1i} & S\gamma_{1i}Cq_{1i} & -Sq_{1i} \\ -C\gamma_{1i}Sq_{1i} & -S\gamma_{1i}Sq_{1i} & -Cq_{1i} \\ -S\gamma_{1i} & C\gamma_{1i} & 0 \end{bmatrix},$$

and $\mathbf{{}^{0}\hat{PP}_i}$ designates the (3×3) skew matrix associated with the vector $\mathbf{{}^{0}PP_i}$.

By using only rows (1 and 2) of relation (12), we obtain:

$$\begin{bmatrix} {}^{1i}v_{xi} \\ {}^{1i}v_{yi} \end{bmatrix} = \mathbf{{}^{1i}J_{vi}}\,{}^{\mathbf{0}}V_{\mathbf{p}} \quad (13)$$

where, the matrix $\mathbf{{}^{1i}J_{vi}}$ is of dimension (2×6) and can be written as:

$$\mathbf{{}^{1i}J_{vi}} = \begin{bmatrix} \mathbf{{}^{1i}R_0}(1,:) & -\left(\mathbf{{}^{1i}R_0}\,\mathbf{{}^{0}\hat{PP}_i}\right)(1,:) \\ \mathbf{{}^{1i}R_0}(2,:) & -\left(\mathbf{{}^{1i}R_0}\,\mathbf{{}^{0}\hat{PP}_i}\right)(2,:) \end{bmatrix} \quad (14)$$

with $\mathbf{A}(i,:)$ denotes the ith row of matrix $\mathbf{A}$.

Each row of the inverse kinematic Jacobian matrix of the robot is obtained using the inverse kinamatic model of a leg i and the relation (13) as:

$$\dot{q}_{2i} = \mathbf{{}^{1i}J_{di}^{-1}}(2,:)\,\mathbf{{}^{1i}J_{vi}}\,{}^{\mathbf{0}}V_{\mathbf{p}} \quad (15)$$

Calculating $\dot{q}_{2i}$ for i =1, 2, 3 using (15) we obtain the three rows of the complete inverse kinematic Jacobian matrix of the robot as:

$$\mathbf{\dot{q}_a} = \mathbf{{}^{0}J_P^{-1}}\,{}^{\mathbf{0}}V_{\mathbf{P}} \quad (16)$$

with:

$$\mathbf{{}^{0}J_P^{-1}} = \begin{bmatrix} \mathbf{{}^{11}J_{d1}^{-1}}(2,:)\,\mathbf{{}^{11}J_{v1}} \\ \mathbf{{}^{12}J_{d2}^{-1}}(2,:)\,\mathbf{{}^{12}J_{v2}} \\ \mathbf{{}^{13}J_{d3}^{-1}}(2,:)\,\mathbf{{}^{13}J_{v3}} \end{bmatrix} \quad (17)$$

The calculation of $\dot{q}_{2i}$ is given in terms of the linear and the angular velocities of the platform $\mathbf{{}^{0}v_p}, \mathbf{{}^{0}\omega_p}$. The inverse Jacobian matrix $\mathbf{{}^{0}J_P^{-1}}$ is of dimension (3×6) thus not invertible. To resolve this problem we have to express the angular velocity of the platform as a function of the linear velocity to obtain a (3×3) matrix. To do that we make use of the constraint defined by the third row of relation (12) for i =1 to 3. We obtain then the following equations:

$$\begin{aligned} {}^{11}v_{z1} &= 0 = \mathbf{{}^{11}R_0}(3,:)\,\mathbf{{}^{0}v_p} - \left(\mathbf{{}^{11}R_0}\,\mathbf{{}^{0}\hat{PP}_1}\right)(3,:)\,\mathbf{{}^{0}\omega_p} \\ {}^{12}v_{z2} &= 0 = \mathbf{{}^{12}R_0}(3,:)\,\mathbf{{}^{0}v_p} - \left(\mathbf{{}^{12}R_0}\,\mathbf{{}^{0}\hat{PP}_2}\right)(3,:)\,\mathbf{{}^{0}\omega_p} \\ {}^{13}v_{z3} &= 0 = \mathbf{{}^{13}R_0}(3,:)\,\mathbf{{}^{0}v_p} - \left(\mathbf{{}^{13}R_0}\,\mathbf{{}^{0}\hat{PP}_3}\right)(3,:)\,\mathbf{{}^{0}\omega_p} \end{aligned} \quad (18)$$

By developing (18), we obtain:

$$\mathbf{B}\begin{bmatrix} {}^{0}v_{px} \\ {}^{0}v_{py} \\ {}^{0}v_{pz} \end{bmatrix} = \mathbf{A}\begin{bmatrix} {}^{0}\omega_{px} \\ {}^{0}\omega_{py} \\ {}^{0}\omega_{pz} \end{bmatrix} \quad (19)$$

where:

$$\mathbf{A} = \begin{bmatrix} {}^{0}PP_{1z} & 0 & -{}^{0}PP_{1x} \\ -\frac{\sqrt{3}}{2}\,{}^{0}PP_{2z} & -\frac{1}{2}\,{}^{0}PP_{2z} & \left(\frac{1}{2}\,{}^{0}PP_{2y} + \frac{\sqrt{3}}{2}\,{}^{0}PP_{2x}\right) \\ \frac{\sqrt{3}}{2}\,{}^{0}PP_{3z} & -\frac{1}{2}\,{}^{0}PP_{3z} & \left(\frac{1}{2}\,{}^{0}PP_{3y} - \frac{\sqrt{3}}{2}\,{}^{0}PP_{3x}\right) \end{bmatrix} \quad (20)$$

and:

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & -\sqrt{3}/2 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \end{bmatrix} \quad (21)$$

we deduce that v_{zp} is an independent component of $\mathbf{v_p}$, the degrees of freedom of the robot could be 2 rotations and the translations along z, or 2 translations of which one is along z and 1 rotation, or three translations which is the case treated in this paper.

From (19), we can express ${}^{0}\boldsymbol{\omega}_{\mathbf{P}}$ as a function of ${}^{0}\mathbf{v}_{\mathbf{P}}$:

$$ {}^{0}\boldsymbol{\omega}_{\mathbf{P}} = \mathbf{C}\,{}^{0}\mathbf{v}_{\mathbf{P}} \tag{22} $$

where:

$$ \mathbf{C} = \mathbf{A}^{-1}\mathbf{B} \tag{23} $$

By combining ${}^{0}\boldsymbol{\omega}_{\mathbf{P}}$ with $\mathbf{v}_{\mathbf{P}}$, we obtain the screw vector $\mathbb{V}_{\mathbf{P}}$ only as a function of the linear velocity $\mathbf{v}_{\mathbf{P}}$:

$$ {}^{0}\mathbb{V}_{\mathbf{P}} = {}^{0}\mathrm{a}_{\mathbf{r}}\,{}^{0}\mathbf{v}_{\mathbf{P}} \tag{24} $$

where ${}^{0}\mathrm{a}_{\mathbf{r}}$ is the (6×3) matrix written as:

$$ \mathrm{a}_{\mathbf{r}} = \begin{bmatrix} \mathbf{I}_3 \\ \mathbf{C} \end{bmatrix} \tag{25} $$

Substituting (24) in (16), we obtain:

$$ \dot{\mathbf{q}}_{\mathbf{a}} = {}^{0}\mathbf{J}_{\mathbf{r}}^{-1}\,{}^{0}\mathbf{v}_{\mathbf{P}} \tag{26} $$

where ${}^{0}\mathbf{J}_{\mathbf{r}}^{-1} = {}^{0}\mathbf{J}_{\mathbf{P}}^{-1}\,{}^{0}\mathrm{a}_{\mathbf{r}}$.

The direct kinematic model is obtained from (26):

$$ {}^{0}\mathbf{v}_{\mathbf{P}} = {}^{0}\mathbf{J}_{\mathbf{r}}\dot{\mathbf{q}}_{\mathbf{a}} \tag{27} $$

Using (22) and (27), we obtain:

$$ {}^{0}\boldsymbol{\omega}_{\mathbf{P}} = \mathbf{C}\,{}^{0}\mathbf{J}_{\mathbf{r}}\dot{\mathbf{q}}_{\mathbf{a}} \tag{28} $$

Finally, the complete Jacobian matrix of the robot of dimension (6×3) is obtained as:

$$ {}^{0}\mathbf{J}_{\mathbf{P}} = \begin{bmatrix} {}^{0}\mathbf{J}_{\mathbf{r}} \\ \mathbf{C}\,{}^{0}\mathbf{J}_{\mathbf{r}} \end{bmatrix} = {}^{0}\mathrm{a}_{\mathbf{r}}\,{}^{0}\mathbf{J}_{\mathbf{r}} \tag{29} $$

such that:

$$ {}^{0}\mathbb{V}_{\mathbf{P}} = {}^{0}\mathbf{J}_{\mathbf{P}}\dot{\mathbf{q}}_{\mathbf{a}} \tag{30} $$

V. The second order kinematic model of leg i

The second order kinematic model of leg i gives the joint accelerations of leg i as a function of the acceleration of the terminal point of leg i ${}^{1i}\dot{\mathrm{v}}_{\mathrm{i}}$, it is obtained as:

$$ \ddot{\mathbf{q}}_{\mathbf{i}} = {}^{1i}\mathbf{J}_{\mathbf{di}}^{-1}\left({}^{1i}\dot{\mathrm{v}}_{\mathrm{i}} - {}^{1i}\dot{\mathbf{J}}_{\mathbf{di}}\dot{\mathbf{q}}_{\mathbf{i}}\right) \tag{31} $$

where ${}^{1i}\dot{\mathbf{J}}_{\mathbf{di}}\dot{\mathbf{q}}_{\mathbf{i}}$ is obtained as:

$$ {}^{1i}\dot{\mathbf{J}}_{\mathbf{di}}\dot{\mathbf{q}}_{\mathbf{i}} = \begin{bmatrix} -\dot{\mathrm{q}}_{2i}\dot{\mathrm{q}}_{1i} \\ 0 \end{bmatrix} \tag{32} $$

and ${}^{1i}\dot{\mathrm{v}}_{\mathrm{i}}$ can be expressed as a function of ${}^{0}\dot{\mathrm{v}}_{\mathbf{P}}$ as:

$$ {}^{1i}\dot{\mathrm{v}}_{\mathrm{i}} = {}^{1i}\mathbf{R}_0\left(\begin{bmatrix} \mathbf{I}_{d3} & -{}^{0}\hat{\mathbf{PP}}_{\mathrm{i}} \end{bmatrix}{}^{0}\dot{\mathbb{V}}_{\mathbf{P}} + {}^{0}\boldsymbol{\omega}_{\mathbf{P}} \times \left({}^{0}\boldsymbol{\omega}_{\mathbf{P}} \times {}^{0}\,\mathbf{PP}_{\mathrm{i}}\right)\right) \tag{33} $$

where ${}^{0}\dot{\mathbb{V}}_{\mathbf{P}}$ is obtained by differentiating (24) as:

$$ {}^{0}\dot{\mathbb{V}}_{\mathbf{P}} = {}^{0}\mathrm{a}_{\mathbf{r}}\,{}^{0}\dot{\mathbf{v}}_{\mathbf{P}} + {}^{0}\dot{\mathrm{a}}_{\mathbf{r}}\,{}^{0}\mathbf{v}_{\mathbf{P}} \tag{34} $$

To obtain $\dot{\mathrm{a}}_{\mathbf{r}}$ we proceed as the following:

by differentiating equation (19) we obtain:

$$ \mathbf{B}\,{}^{0}\dot{\mathbf{v}}_{\mathbf{p}} = \mathbf{A}\,{}^{0}\dot{\boldsymbol{\omega}}_{\mathbf{p}} + \dot{\mathbf{A}}\,{}^{0}\boldsymbol{\omega}_{\mathbf{p}} \tag{35} $$

where $\dot{\mathbf{A}}$ is the derivative of (20):

$$ \dot{\mathbf{A}} = \begin{bmatrix} {}^{0}\dot{\mathrm{P}}\mathrm{P}_{1z} & 0 & -{}^{0}\dot{\mathrm{P}}\mathrm{P}_{1x} \\ -\frac{\sqrt{3}}{2}\,{}^{0}\dot{\mathrm{P}}\mathrm{P}_{2z} & -\frac{1}{2}\,{}^{0}\dot{\mathrm{P}}\mathrm{P}_{2z} & \left(\frac{1}{2}\,{}^{0}\dot{\mathrm{P}}\mathrm{P}_{2y} + \frac{\sqrt{3}}{2}\,{}^{0}\dot{\mathrm{P}}\mathrm{P}_{2x}\right) \\ \frac{\sqrt{3}}{2}\,{}^{0}\dot{\mathrm{P}}\mathrm{P}_{3z} & -\frac{1}{2}\,{}^{0}\dot{\mathrm{P}}\mathrm{P}_{3z} & \left(\frac{1}{2}\,{}^{0}\dot{\mathrm{P}}\mathrm{P}_{3y} - \frac{\sqrt{3}}{2}\,{}^{0}\dot{\mathrm{P}}\mathrm{P}_{3x}\right) \end{bmatrix} \tag{36} $$

with:

$$ {}^{0}\dot{\mathbf{P}}\mathbf{P}_{\mathrm{i}} = {}^{0}\boldsymbol{\omega}_{\mathbf{P}} \times {}^{0}\mathbf{PP}_{\mathrm{i}} \tag{37} $$

from (35) and (22), we obtain:

$$ {}^{0}\dot{\boldsymbol{\omega}}_{\mathbf{p}} = \mathbf{C}\,{}^{0}\dot{\mathbf{v}}_{\mathbf{p}} - \mathbf{A}^{-1}\dot{\mathbf{A}}\mathbf{C}\,{}^{0}\mathbf{v}_{\mathbf{p}} \tag{38} $$

By combining ${}^{0}\dot{\boldsymbol{\omega}}_{\mathbf{p}}$ with ${}^{0}\dot{\mathbf{v}}_{\mathbf{P}}$, and identifying to (34), we obtain:

$$ \dot{\mathrm{a}}_{\mathbf{r}} = \begin{bmatrix} \mathbf{0}_{3\times3} \\ -\mathbf{A}^{-1}\dot{\mathbf{A}}\mathbf{C} \end{bmatrix} \tag{39} $$

VI. Inverse dynamic model

The inverse dynamic model of the robot gives the active joint torques as a function of the platform trajectory. It is obtained by the sum of the dynamics of the platform $\mathbb{F}_{\mathbf{P}}$ calculated as a function of the Cartesian variables of the platform, and the dynamics of the legs $\mathbf{H}_{\mathrm{i}}$ calculated in terms of the joint variables of the legs, after projecting them on the active joint axes [11]:

$$ \boldsymbol{\Gamma} = \mathbf{J}_{\mathbf{P}}^{\mathbf{T}}\mathbb{F}_{\mathbf{P}} + \sum_{\mathbf{i=1}}^{3}\left(\frac{\partial\dot{\mathbf{q}}_{\mathbf{i}}}{\partial\dot{\mathbf{q}}_{\mathbf{a}}}\right)^{\mathbf{T}}\mathbf{H}_{\mathrm{i}} \tag{40} $$

by considering (9), (13), (16) and (24), this relation can be rewritten as:

$$ \boldsymbol{\Gamma} = \mathbf{J}_{\mathbf{r}}^{\mathbf{T}}\mathrm{a}_{\mathbf{r}}^{\mathbf{T}}\left(\mathbb{F}_{\mathbf{P}} + \sum_{\mathbf{i=1}}^{3}\mathbf{J}_{\mathbf{vi}}^{\mathbf{T}}\mathbf{J}_{\mathbf{di}}^{-\mathbf{T}}\mathbf{H}_{\mathbf{i}}\right) \tag{41} $$

where $\mathbb{F}_{\mathbf{P}}$ is calculated by the following Newton-Euler equation [12]:

$$ \mathbb{F}_{\mathbf{P}} = \mathbb{J}_{\mathbf{P}}\begin{bmatrix} \dot{\mathbf{v}}_{\mathbf{P}} - \mathbf{g} \\ \dot{\boldsymbol{\omega}}_{\mathbf{P}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\omega}_{\mathbf{P}} \times \left(\boldsymbol{\omega}_{\mathbf{P}} \times \mathbf{MS}_{\mathbf{P}}\right) \\ \boldsymbol{\omega}_{\mathbf{P}} \times \left(\mathbf{I}_{\mathbf{P}}\boldsymbol{\omega}_{\mathbf{P}}\right) \end{bmatrix} \tag{42} $$

$\mathbf{g}$ acceleration of gravity,

$\mathbf{I}_{\mathbf{P}}$ (3×3) inertia matrix of the platform around the origin of the platform frame Σ_P,

$\mathbf{MS}_{\mathbf{P}}$ (3×1) vector of first moments of the platform around the origin of the platform frame Σ_P:

$$\mathbf{MS_P} = \begin{bmatrix} MX_P & MY_P & MZ_P \end{bmatrix}^T,$$

$\mathbb{J}_\mathbf{P}$ (6×6) spatial inertia matrix of the platform:

$$\mathbb{J}_\mathbf{P} = \begin{bmatrix} M_P \mathbf{I_3} & -\hat{\mathbf{MS}}_\mathbf{P} \\ \hat{\mathbf{MS}}_\mathbf{P} & \mathbf{I_P} \end{bmatrix}$$

M_P mass of the platform, $\mathbf{I_3}$ (3×3) identity matrix

$\hat{\mathbf{MS}}_\mathbf{P}$ designates the (3×3) skew matrix associated with the vector $\mathbf{MS_P}$,

$\mathbf{H_i}$ represent the inverse dynamic model of a leg i.

$\mathbf{H_i}$ is the dynamic model of two degrees of freedom serial structure. Many methods can be used to calculate it [13]-[17], one can use the method with which he is familiar. To reduce the computational cost, the recursive Newton-Euler algorithm using base inertial parameters and customized symbolic methods can be used [16], [18]. In Appendix we present the model of one leg as obtained by the software SYMORO+ [19]) .

From the equation (40) and since the active variables are independent, the expression of the inverse dynamic model can be reduced to the following form:

$$\mathbf{\Gamma} = \begin{bmatrix} H_{21} \\ H_{22} \\ H_{23} \end{bmatrix} + \mathbf{J_r^T a_r^T} \left(\mathbb{F}_\mathbf{P} + \begin{bmatrix} \mathbf{J}_{v1}^T \mathbf{J}_{d1}^{-T}(:,1) & \mathbf{J}_{v2}^T \mathbf{J}_{d2}^{-T}(:,1) & \mathbf{J}_{v3}^T \mathbf{J}_{d3}^{-T}(:,1) \end{bmatrix} \begin{bmatrix} H_{11} \\ H_{12} \\ H_{13} \end{bmatrix} \right) \tag{43}$$

where $(:,1)$ denotes the first column of matrix $\mathbf{J}_{v2}^T \mathbf{J}_{2d}^{-T}$, H_{1i} represents the passive joint torques of leg i, and H_{2i} represent the active joints forces of leg i. The symbolic expressions of the elements used in this equation are given in Appendix. The total number of operations to calculate the inverse dynamic model are about 200 additions and 300 multiplications (without taking into account the calculation of direct geometric model).

VII. Direct dynamic model

The state variables of parallel robot can be taken as the Cartesian position and velocity of the platform. Thus, the direct dynamic model of the robot gives the platform Cartesian acceleration as a function of the state variables and the input of the motorized torques/forces:

$${}^\mathbf{0}\dot{\mathbb{V}}_\mathbf{P} = \mathbf{f}\left({}^\mathbf{0}\mathbf{T_P}, \mathbb{V}_\mathbf{P}, \mathbf{\Gamma}\right) \tag{44}$$

Noting that the inverse dynamic model of leg i can be decomposed as:

$$\mathbf{H_i}\left(\mathbf{q_i}, \dot{\mathbf{q}}_\mathbf{i}, \ddot{\mathbf{q}}_\mathbf{i}\right) = \mathbf{A_i}\ddot{\mathbf{q}}_\mathbf{i} + \mathbf{h_i}\left(\mathbf{q_i}, \dot{\mathbf{q}}_\mathbf{i}\right) \tag{45}$$

with $\mathbf{A_i}$ is the inertia matrix of leg i, and $\mathbf{h_i}$ is the vector of the Coriolis, centrifugal and gravity torques or forces of leg i.

We can obtain the direct dynamic model from the inverse dynamic model by substituting (45), (31)-(33), (42) and (24) in (41).

Thus, the platform acceleration is obtained as:

$$\dot{\mathbf{v}}_\mathbf{P} = \mathbf{A}_\mathbf{robot}^{-1}\left(\mathbf{J}_\mathbf{r}^{-T}\mathbf{\Gamma} - \mathbf{h}_\mathbf{robot}\right) \tag{46}$$

with:

$$\mathbf{A}_\mathbf{robot} = \mathrm{a}_\mathbf{r}^\mathbf{T}\left(\mathbb{I}_\mathbf{P} + \sum_{i=1}^{3} \mathbf{J}_{vi}^\mathbf{T}\mathbf{A}_{Xi}\mathbf{J}_{vi} \right)\mathrm{a}_\mathbf{r} \tag{47}$$

$$\begin{aligned} \mathbf{h}_\mathbf{robot} = {} & \mathrm{a}_\mathbf{r}^\mathbf{T}\left(\mathbb{I}_\mathbf{P} + \sum_{i=1}^{3} \mathbf{J}_{vi}^\mathbf{T}\mathbf{A}_{Xi}\mathbf{J}_{vi} \right)\dot{\mathrm{a}}_\mathbf{r}\mathbf{v_P} + \\ & \mathrm{a}_\mathbf{r}^\mathbf{T}\left(\begin{bmatrix} \boldsymbol{\omega}_\mathbf{P} \times (\boldsymbol{\omega}_\mathbf{P} \times \mathbf{MS_P}) \\ \boldsymbol{\omega}_\mathbf{P} \times (\mathbf{I_P}\boldsymbol{\omega}_\mathbf{P}) \end{bmatrix} - \begin{bmatrix} M_P\mathbf{I_3} \\ \hat{\mathbf{MS}}_\mathbf{P} \end{bmatrix}\mathbf{g} \right) + \\ & \mathrm{a}_\mathbf{r}^\mathbf{T}\sum_{i=1}^{3}\left\{ \mathbf{J}_{vi}^\mathbf{T}\left(\mathbf{A}_{Xi}\left(\dot{\mathbf{J}}_{vi}\mathrm{a}_\mathrm{r}\mathbf{v_P} - \dot{\mathbf{J}}_\mathbf{i}\dot{\mathbf{q}}_\mathbf{i} \right) + \mathbf{h}_{Xi} \right) \right\} \end{aligned} \tag{48}$$

where $\mathbf{A}_{Xi} = \mathbf{J}_{di}^{-T}\mathbf{A_i}\mathbf{J}_{di}^{-1}$ and $\mathbf{h}_{Xi} = \mathbf{J}_{di}^{-T}\mathbf{h_i}$

VIII. Conclusion

This paper presents a novel solution for the kinematic and dynamic modeling of the 3-RPS parallel robot. This robot is characterized by a coupling between the 6 DOF of the platform. After presenting a (6×3) kinematic Jacobian matrix we developed a reduced (3×3) Jacobian matrix relating the linear velocity of the platform with respect to the three actuated joints. These Jacobian matrices are used in calculating the inverse and direct dynamic models of the robot. The inverse dynamic model has a reduced number of mathematical operations thanks to the use of the base inertial parameters and a customized Newton-Euler algorithm.

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Appendix: Calculation of some elements of Eq. 43

A-1-Calculation of the dynamic model of leg i

The base inertial parameters of leg i are:

ZZ1R, MX1, MY1, MX2, MZ2, M2.

where ZZ1R= YY2 + ZZ1,

The other parameters used on the dynamic model of leg i are: IA2, and G, represent respectively the motor's rotor inertia and the gravity. The dynamics of leg i is given as:

```
S1=sin(Q1i);
C1=cos(Q1i);
DV331=-QP1i* QP1i;
VP11=-G*S1;
VP21=-C1*G3;
No31=QDP1i*ZZ1R;
VSP12=Q2i*QDP1i + VP11;
VSP22=-(DV331*Q2i) + VP21;
VP12=2.*QP1i*QP2i + VSP12;
VP32=QDP2i - VSP22;
F12= DV331*MX2 + MZ2*QDP1i + M2*VP12;
F32= DV331*MZ2 - MX2.*QDP1i + M2*VP32;
N22=MZ2*VP12 - MX2*VP32;
N31=N22 + No31 + F12.*Q2i - MY1*VP11 + MX1*VP21;
H1i=N31 ;
H2i2=F32 + IA2*QDP2i ;
```

% Number of operations: 14 '+' or '-', 20 '*' or '/'. This number could be reduced more if the first link has symmetrical shape such that the first moments has one component different than zero.

where Qji, QPji, QDPji denotes respectively the position, velocity and acceleration of joint j of the leg i.

A-2- Calculation of other matrices

$$
{}^{11}\mathbf{J}_{\mathbf{v}1}^{\mathrm{T}}\,{}^{11}\mathbf{J}_{\mathbf{d}1}^{\text{-T}}(:,1).H_{11} = \begin{bmatrix} \text{-}\mathrm{Cq}_{11} \\ 0 \\ \mathrm{Sq}_{11} \\ \mathrm{Sq}_{11}\mathrm{P}_{y1} \\ -\left(\mathrm{Cq}_{11}\mathrm{P}_{z1} + \mathrm{Sq}_{11}\mathrm{P}_{x1}\right) \\ \mathrm{Cq}_{11}\mathrm{P}_{y1} \end{bmatrix} H_{11}/\mathrm{q}_{21}
$$

$$
{}^{12}\mathbf{J}_{\mathbf{v}2}^{\mathrm{T}}\,{}^{12}\mathbf{J}_{\mathbf{d}2}^{\text{-T}}(:,1).H_{12} = \begin{bmatrix} \mathrm{Cq}_{12}/2 \\ -\sqrt{3}\mathrm{Cq}_{12}/2 \\ \mathrm{Sq}_{12} \\ \left(\sqrt{3}\mathrm{Cq}_{12}\mathrm{P}_{z2}/2 + \mathrm{Sq}_{12}\mathrm{P}_{y2}\right) \\ \left(\mathrm{Cq}_{12}\mathrm{P}_{z2}/2 - \mathrm{Sq}_{12}\mathrm{P}_{x2}\right) \\ -\left(\mathrm{Cq}_{12}\mathrm{P}_{y2} + \sqrt{3}\mathrm{Cq}_{12}\mathrm{P}_{x2}\right)/2 \end{bmatrix} H_{12}/\mathrm{q}_{22}
$$

$$
{}^{13}\mathbf{J}_{\mathbf{v}3}^{\mathrm{T}}\,{}^{13}\mathbf{J}_{\mathbf{d}3}^{\text{-T}}(:,1).H_{13} = \begin{bmatrix} \mathrm{Cq}_{13}/2 \\ \sqrt{3}\mathrm{Cq}_{13}/2 \\ \mathrm{Sq}_{13} \\ \left(-\sqrt{3}\mathrm{Cq}_{13}\mathrm{P}_{z3}/2 + \mathrm{Sq}_{13}\mathrm{P}_{y3}\right) \\ \left(\mathrm{Cq}_{13}\mathrm{P}_{z3}/2 - \mathrm{Sq}_{13}\mathrm{P}_{x3}\right) \\ -\left(\mathrm{Cq}_{13}\mathrm{P}_{y3} - \sqrt{3}\mathrm{Cq}_{13}\mathrm{P}_{x3}\right) \end{bmatrix} H_{13}/\mathrm{q}_{23}
$$

$$
{}^{0}\mathbf{J}_{\mathbf{P}}^{\text{-}T} =
$$

$$
\begin{bmatrix}
\text{-}\mathrm{Sq}_{l1} & \mathrm{Sq}_{l2}/2 & \mathrm{Sq}_{l3}/2 \\
0 & \text{-}\sqrt{3}\,\mathrm{Sq}_{l2}/2 & \sqrt{3}\,\mathrm{Sq}_{l3}/2 \\
-\mathrm{Cq}_{l1} & -\mathrm{Cq}_{l2} & -\mathrm{Cq}_{l3} \\
-\mathrm{Cq}_{l1}\mathrm{P}_{y1} & \sqrt{3}\,\mathrm{Sq}_{l2}\mathrm{P}_{z2}/2-\mathrm{Cq}_{l2}\mathrm{P}_{y2} & -\sqrt{3}\,\mathrm{Sq}_{l3}\mathrm{P}_{z3}/2-\mathrm{Cq}_{l3}\mathrm{P}_{y3} \\
\text{-}\mathrm{Sq}_{l1}\mathrm{P}_{z1}+\mathrm{Cq}_{l1}\mathrm{P}_{x1} & \mathrm{Sq}_{l2}\mathrm{P}_{z2}/2+\mathrm{Cq}_{l2}\mathrm{P}_{x2} & \mathrm{Sq}_{l3}\mathrm{P}_{z3}/2+\mathrm{Cq}_{l3}\mathrm{P}_{x3} \\
\mathrm{Sq}_{l1}\mathrm{P}_{y1} & \text{-}\mathrm{Sq}_{l2}\mathrm{P}_{y2}/2-\sqrt{3}\,\mathrm{Sq}_{l2}\mathrm{P}_{x2}/2 & \text{-}\mathrm{Sq}_{l3}\mathrm{P}_{y3}/2+\sqrt{3}\,\mathrm{Sq}_{l3}\mathrm{P}_{x3}/2
\end{bmatrix}
$$

where : in the previous matrices, we noted:

$$
{}^{0}\mathbf{PP}_{\mathrm{i}} = \left[\mathbf{P}_{\mathrm{xi}} \quad \mathbf{P}_{\mathrm{yi}} \quad \mathbf{P}_{\mathrm{zi}}\right]
$$