Sampling uncertainty in coordinate measurement data analysis

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There are a number of important software related issues in coordinate metrology. After measurement data are collected in the form of position vectors, the data analysis software must derive the necessary geometric information from the point set, and uncertainty plays an important role in the analysis. When extreme fit approaches ($L_{-\infty}$ norm estimation approaches) are employed for form error evaluation, the uncertainty is closely related to the sampling process used to gather the data. The measurement points are a subset of the true surface, and, consequently, the extreme fit result differs from the true value. In this paper, we investigate the functional relationship between the uncertainty in an extreme fit and the number of points measured. Two major issues are addressed in this paper. The first addresses and identifies the parameters that affect the functional relationship. The second develops a methodology to apply this relationship to the sampling of measurement points.   © 1998 Elsevier Science Inc.

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Introduction

Three-dimensional (3-D) metrology has brought a significant change in dimensional measurement. Compared to traditional comparison methods (*e.g.*, comparing to a surface plate), 3-D measurement yields more comprehensive and, unfortunately complex, information about the real part geometry. Three-dimensional metrology generates surface coordinates of a measured feature instead of measuring the geometric dimensions. Thus, the measurement output is a collection of digitized surface points. With these surface points, more details about surface variation can be observed, and various sections of different geometric features can be measured in a single process. Coordinate measuring machines (CMMs) are pres-

ently widely employed as 3-D measurement devices.

After measurement data are collected by a 3-D measuring machine, an independent numerical analysis must be performed. Because the measurement data are a set of points, the raw data must be interpreted in a parametric form by an appropriate analysis procedure. This is necessary to obtain geometric parameters of part variation or to make a tolerance conformance decision. Thus, the measurement data analysis for 3-D metrology is a software procedure that is separate and distinct form the measurement data-collecting procedure, which is a hardware procedure. The data analysis function has become one of the key components in 3-D metrology, and it is the focus of this paper.

For the evaluation of a form error, such as flatness as defined in ASME standard Y14.5,[1,2] a minimum zone must be established from the measurement data. The minimum zone is a geometric value that defines the least possible height of the

point set on a plane. Minimum zone evaluation algorithms have been reported in various studies.[3-10] The problem is formulated as an extreme fit evaluation that searches for the extreme points that characterize the zone. The extreme fit refers to fitting using $L_{-\infty}$ norm estimation. The studies show that the extreme fit yields more accurate fitting results because: (1) it yields smaller zone value than the zone evaluated by a least-squares fit; and (2) it is more consistent with the standard definition and is a numerical realization of physical fittings. Also, some concern has been expressed regarding interpreting the results of the least-squares fit, because it is a statistical estimation rather than an exact solution. Thus, the extreme fit is becoming a more widely accepted fitting method. However, as indicated several studies,[11-13] such an extreme fit has limitations in accuracy. Part of the uncertainty is because of the numerical instability in the actual implementation of the fit. Also, the plug-in nature of the extreme fit itself causes uncertainty. A more detailed discussion related to the uncertainty is addressed later in this paper.

The important fact is that the uncertainty in the extreme fit is closely related to the sampling of measurement points. This leads to the issue of uncertainty evaluation and its relationship to sampling. The sampling problem can be broken into two components, the location of the measurement points and the sample size. The measurement point allocation problem is related to the type of error or deformation to be characterized by the measurement. Typically, a uniform distribution over the measurement region is practiced; however, a nonuniform distribution may be more effective for characterizing specific systematic deformations. This measurement point allocation problem is significant, but not the target of this research. In this paper, we limit the scope of the sampling to the sample size only. For flatness evaluation, where deviations from nominal are considered to be a stochastic deformation, the probability that the error can occur for a unit area is considered uniform; thus, a uniform distribution of measurement points is typically preferred. It is obvious that an evaluation with more measurement points yields more accurate results. Unfortunately, this advantage is countered by the fact that the number of points is directly related to the measurement time and the data-processing cost. Thus, to achieve efficient and accurate measurement results, guidelines for choosing the sample size must be established.

The objective of this research is to investigate how the sample size is related to uncertainty in the extreme fit evaluation. We examine the functional (mathematical) relationship between the sample size and the uncertainty. Also we examine key parameters that affect to the uncertainty in the extreme fit evaluation. We also discuss using the results of these analyses to develop a sampling strategy in inspection planning stages. Furthermore, a model for a generalized form error distribution is given along with two different approaches to derive the functional relationship (analytic approximation model and numerical model). We employ order statistics for the analytic model and a neural network for the numerical model. The theoretical concepts developed in this paper are then applied to a flatness example.

Related work

Recognition of the sampling problem can be found in Hocken et. al.[12] and Caskey et. al.,[14] where extensive numerical experiments on data-fitting algorithms were conducted to investigate sampling issues. They tested least-squares fit, min-max fit, and minimum zone evaluations on various geometric primitives, such as line, plane, circle, sphere, and cylinder. They generated simulated measurement data embedded with a characteristic form error, such as tri-lobbing error for a circular feature. Then, they ran multiple sampling and fitting procedures and determined the variation of the evaluated parameters. The results were reported for the cases of a line, plane, circle, sphere, and cylinder. With their results, they concluded that variation is substantially large with the sampling size in current practice.

Weckenmann et. al.,[15,16] also addressed sampling issues for coordinate metrology. They recognized that extreme fit is more suitable for functionality evaluation, but it yields variations that are sensitive to sampling. They investigated the effect of sample point location and sample size on a circular feature with tri-lobbing error and a linear feature with an undulation form error. From their numerical experiments, they showed the uncertainty of the form error evaluation, represented as the dispersion of the evaluated value, decreases with increasing sample size.

Mestre and Abou-Kandil[17] approached the problem from a different direction. Recognizing the problem with current extreme fit evaluations that ignore the variation in unmeasured sites, they proposed a new method for the flatness evaluation. Employing Bayesian prediction theory, they derived the confidence interval of the surface variation in unmeasured sites. Then, the minimum zone was determined with respect to the derived

Figure 1 True value versus estimated value by measured data

true flatness

plug-in estimation of flatness
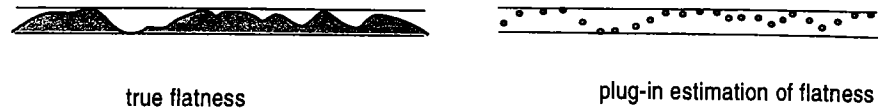
confidence interval. In so doing, they could overcome the plug-in estimation error of extreme fit evaluation, and the ensuing estimated minimum zone was closer to the true flatness value.

As discussed above, the sampling uncertainty problem in coordinate metrology has been recognized in previous studies, but a formal evaluation method has not been reported, and the results are limited to a specific case. Thus, it is impossible to predict the uncertainty for a given sampling condition or to predict the number of the samples . required for a given accuracy level. However, it is necessary to provide a more formal analysis of the uncertainty so that more strategic measurement planning can be achieved. To do so, the uncertainty must be precisely defined, and its mathematical relationship with the measurement condition must be examined.

Extreme fit and uncertainty

See Equation (1). A typical form of the extreme problem is represented as follows:

$$\min M(\theta) = \max\{\text{distance}[\,p_i,\, S(\theta)\,]\} \qquad (1)$$

where S is the fitted surface model, θ represents the model parameters, and p_i are measured points. Because the function $M(\theta)$ is nondifferentiable, it must be determined by an iterative search algorithm. If the fitting geometry is properly defined, there is a unique solution that satisfies the

optimality condition and is constrained by the extreme points. Thus, the fitting geometry is defined by a few extreme points in the measured point set.

The uncertainty associated with extreme fit comes from the plug-in nature of the extreme fit. The plug-in estimation is the evaluation of a parameter with a finite number of samples regarding them as the entire population. The extreme fit geometry is determined from a few extreme points. However, the extreme points in the measurement data that contribute to the extreme fit geometry are not necessarily the extreme points in the true surface profile. As shown in *Figure 1*, the true flatness is defined as the minimum zone over the true surface profile. However, the flatness is evaluated only from the measured points, and the extreme points in the measurement data are samples of the true surface points. Thus, the true flatness value and the flatness evaluated by measured points are not equivalent, because the measurement data fail to capture the true extreme points.

Depending upon the data sampling strategy, the evaluated extreme fit values will vary. *Figure 2* represents the flatness value evaluated from a surface with different test sets. Ninety-five test sets are taken from a simulated surface. The surface has noise amplitude that is uniform over the area with a uniform noise distribution superimposed upon the measurement. For each test set, the flatness is
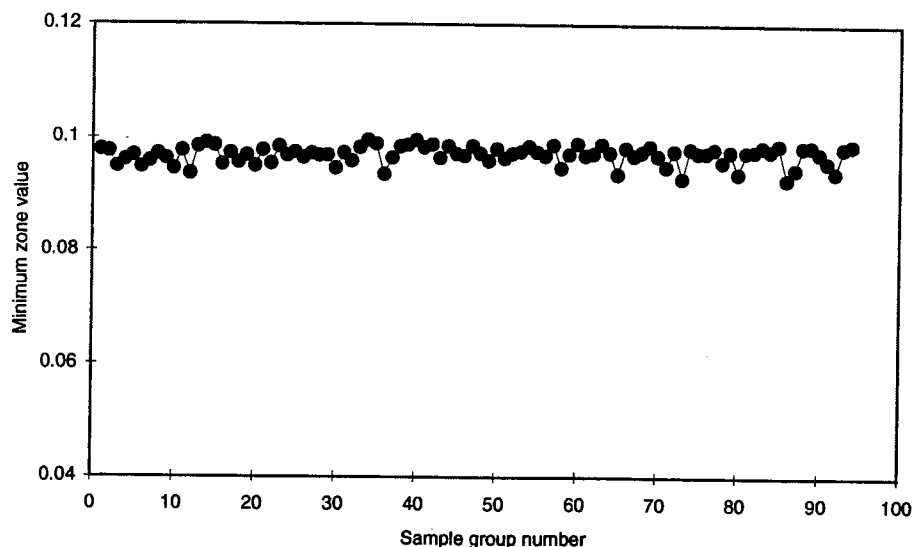
Figure 2 Variation of a minimum zone value with respect to different samples
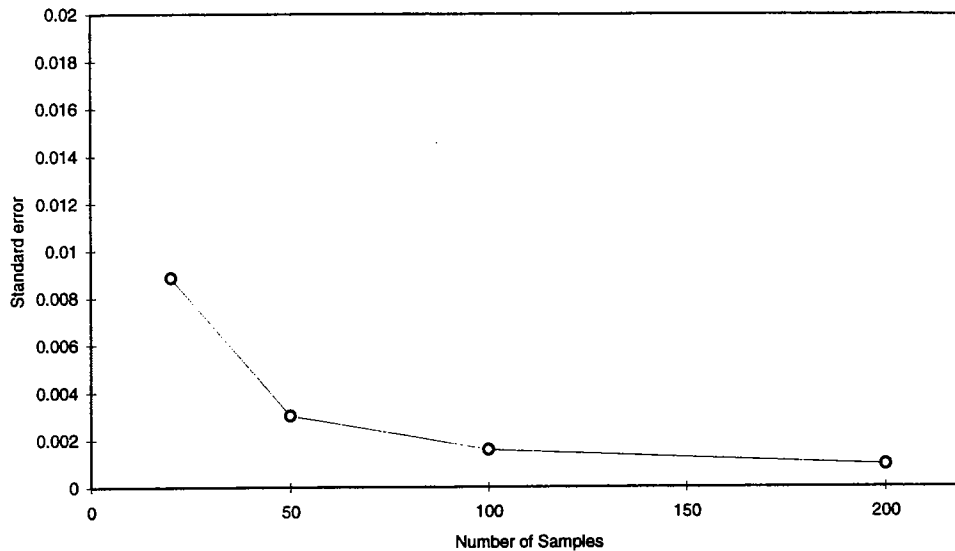
Figure 3 The standard error versus number of sample

evaluated by a minimum zone evaluation method, and the flatness value is plotted with respect to the test set number. Although the samples are taken from the same surface, the evaluated flatness value varies with different test sets.

This variation is attributable to the plug-in evaluation of the minimum zone evaluation, and it is the sampling uncertainty. Thus, when an extreme fit value is evaluated, the value has the standard deviation related to the sampling process. The standard deviation can be found only from an infinite number of test sets and evaluations. To alleviate this problem, we use standard error as an estimate of the standard deviation. The standard error can be evaluated from the estimated standard deviation of the finite number of test sets and evaluations. The standard error represents the uncertainty in the extreme fit evaluation. Therefore, we refer to the uncertainty as the "standard error" in this paper.

The next issue that must be addressed is the relationship between uncertainty and sample size. Clearly, as more datapoints are used for evaluating the fitting surface, the extreme fit surface becomes closer to the true value, because larger number of measurement points yields a more comprehensive sampling of the true surface. *Figure 3* shows the relation between the sample size and the uncertainty. The same flatness example is tested with different numbers of measured points. From the same surface, one hundred different test sets are randomly taken with a given number of measured points, and the flatness is evaluated for each test set. As tested in a previous example, the test sets yield different flatness values. Then, the standard error of the flatness value is derived. The procedure has been repeated for the cases of different numbers of measured points, and the results are shown in *Figure 3*. Intuitively, the standard error decreases with respect to the sample size. However, the metrologist requires more than intuition; rather, we must mathematically model such a relationship so that a prediction can be made. To do so, we also need to model the other parameters that may affect to the uncertainty. One of the important parameters is the distribution of form error, as discussed in the following section.

Distribution of form error

Form error is surface variation caused by imperfect manufacturing. The form error of the surface is mapped into the measured points, and it can be represented as the distribution of the probability density with respect to a fixed coordinate reference frame (*Figure 4*). The probability density distribution is referred to as form error distribution in this paper. To investigate the surface form error
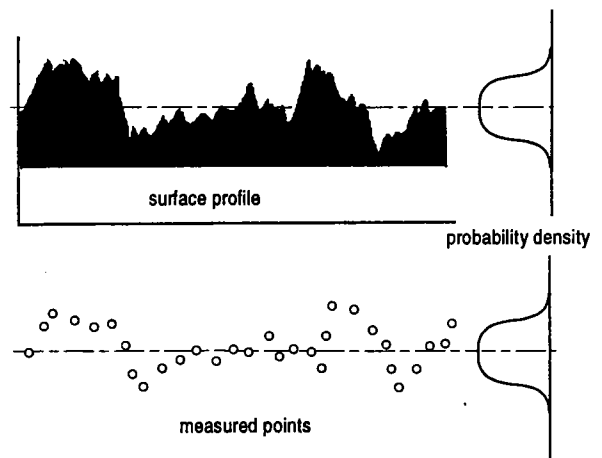
Figure 4 Form error distribution

probability density
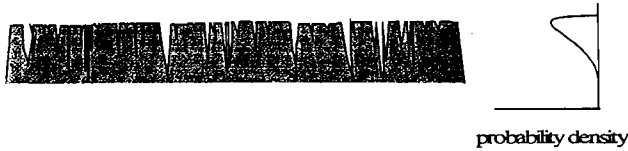
Figure 5 Typical non-normal engineering surface

effect on uncertainty, the form error distribution must be modeled. A normal distribution is a typical assumption in a statistical situation. The majority of random distributions with a large number of data, that are encountered in the real world, are close to the normal distribution. Even for engineering surfaces, form errors are often assumed to be normally distributed, but the actual distribution may seriously depart from normal distribution (*Figure 5*). The normal distribution assumption is appropriate when form error is affected by various small-scale, random, and independent effects. It is also assumed that mechanical properties must be uniform and stable. Actual engineering surfaces undergo various nonuniform errors caused by tool wear, variation of stiffness, heat treatment deformation, etc. It is difficult to model such form errors with a normal distribution, because the normal distribution is symmetric and has only two degrees of freedom.

In this paper, a more appropriate model for the form error distribution, the beta distribution,[18] is employed. The beta distribution has a finite range, a property that better represents real material properties than the infinite tails of the normal distribution. The beta distribution has four degrees of freedom and can be used to represent a wide range of distributions from uniform distribution to bell-shaped distributions similar to normal distribution, as well as to model asymmetric distributions. Thus, the beta distribution is flexible enough to address the actual variations in real

surfaces. Furthermore, it has been shown that the beta distribution is more appropriate than the normal distribution for real surface form errors.[19]

See Equation (2), Equation (3), and Equation (4). The beta distribution can be formulated as follows. It is defined in a finite range a, b. The probability density function of a generalized beta distribution is defined with the exponents η and λ as follows.

$$f(x, \lambda, \eta, a, b)$$
$$= \frac{1}{(b-a) B(\lambda, \eta)} \left(\frac{x-a}{b-a}\right)^{\lambda-1} \left(1 - \frac{x-a}{b-a}\right)^{\eta-1}$$
$$0 < \lambda, \ 0 < \eta, \ a \leq x \leq b \qquad (2)$$

where $B(\lambda, \eta)$ is the beta function defined as

$$B(\lambda, \eta) = \int_0^1 z^{-1}(1-z)^{\eta-1} dz \qquad (3)$$

Substituting $a = 0$ and $b = 1$ in Equation (3), the unit beta distribution can be defined as follows.

$$f(z, \lambda, \eta) = \frac{1}{B(\lambda, \eta)} z^{\lambda-1}(1-z)^{\eta-1} \qquad (4)$$

$$0 < \lambda, \ 0 < \eta, \ 0 \leq z \leq 1$$

The function f represents the probability density of the surface area with respect to the height z, which represents the normalized surface form error level. The surface form error characteristic is defined by the beta-distribution exponents λ and η. When λ and η values are unity, the distribution is equal to a uniform distribution, and as λ and η values increase from unity, the distribution becomes similar to the normal distribution (*Figure 6*). λ and η are the weight of the distribution in each direction, respectively. As the value of λ becomes larger, the distribution skews toward 0, and
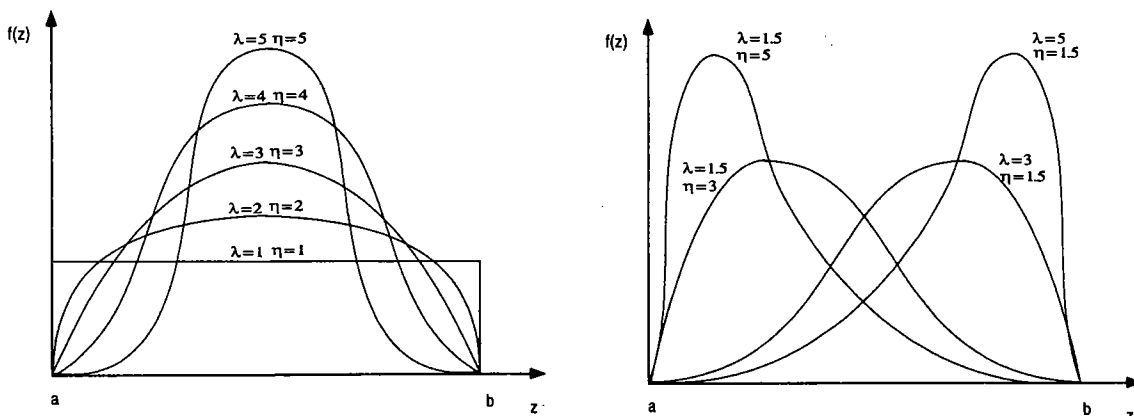
Figure 6 Beta distribution

as η values become large, the distribution skews toward 1. With the beta distribution, more general types of form errors, including nonsymmetric distribution, can be modeled.

Analytic approximation of the uncertainty function

The mathematical relationship between the uncertainty and the measured point characteristics can be represented by the standard error function in terms of the form error, sample size, and the geometry. Because an extreme fit is a nondifferentiable function, it must be solved by numerical iteration. Thus, the standard error of the flatness value can only be evaluated by a numerical simulation. An approximation is proposed and shown herein for the flatness example. The standard error of flatness evaluation represents the variance of the range of the maximum and the minimum points. We approximate it by the standard error of the range of the extreme points in a fixed geometric domain. The approximation does not consider the rigid body transformation; thus, the derived standard error may be smaller than the true error. The mathematical model for this approach is to add height variations from a known distribution to a plane and approximate the flatness from the extreme values of this added variation. Therefore, a fit is not made to the points, and the reference plane is fixed for all cases.

We employ order statistics to model the flatness evaluation. The order statistics provide statistical properties of ranked elements, including the maximum and the minimum. The extreme points can be modeled as the ranked elements in a certain geometric domain. The standard error of an extreme fit feature depends upon the variance of the extreme points when sampled from an unknown true surface. The variance of the extreme points is determined by the variance of order statistics elements.

Suppose the statistic x is distributed by a probability density function $f(x)$. Let us denote the cumulative probability distribution as $F(x)$. From the distribution, a random sample of n data are taken, and the sample data are denoted as x_i, $i = 1, \ldots n$. The random sample is a set of measured data in a certain geometric domain. Let us define x_m as the minimum of x_i, and x_M as the maximum of the x_i. See Equation (5). Then the range of x_i is the difference between the maximum and the minimum as

$$w = x_M - x_m, \qquad 0 < w < \infty \qquad (5)$$

If x_i is the normal directional variation for ith point of measurement data from a measured sur-

face, the range w is an approximation of the flatness value. The standard error of flatness evaluation can be derived from the variation of the range w. See Equation (6). From order statistics,[20] the joint density for x_m to be the minimum and for x_M to be the maximum is:

$$g(x_m, x_M) = n(n-1)[F(x_M) \\ - F(x_m)]^{n-2} f(x_m) f(x_M) \qquad (6)$$

See Equation (7). The probability density g can be expressed as a function of the range w and the minimum x_m as

$$g(x_m, w) = n(n-1)[F(x_m + w) \\ - F(x_m)]^{n-2} f(x_m) f(x_m + w) \qquad (7)$$

See Equation (8). The probability density for the range w can be derived as integrating Equation (7) with respect to x_m over all possible ranges as

$$h(w) = n(n-1) \int_{-\infty}^{\infty} [F(x_m + w) \\ - F(x_m)]^{n-2} f(x_m) f(x_m + w)\, dx_m \qquad (8)$$

Equation (8) represents the probability density of a variable w to be the range of the order statistics. Thus, it is the probability density of the flatness value when n datapoints are taken from a surface that has the probability density $f(x)$. See Equation (9) and Equation (10). Then, the mean of w is

$$\mu_w = E[w] = \int_0^{\infty} w \cdot h(w)\, dw \qquad (9)$$

and the variance is

$$\text{Var}(w) = E[(w - \mu_w)^2] \\ = \int_0^{\infty} (w - \mu_w)^2 \cdot h(w)\, dw \qquad (10)$$

From Equation (10), the standard error of flatness can be expressed as a function of the number of datapoints n. Let us consider the following example. See Equation (11). Suppose a surface is modeled by the beta distribution with the exponents η and λ unity, then the probability density of the noise x is a simple uniform distribution as

$$f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases} \qquad (11)$$

The functional relationship between the number of measurement points and the standard error of

the evaluation can be analytically modeled. See Equation (12). By Equation (8), the probability density of the range w can be determined as

$$h(w) = n(n-1) \int_0^{1-w} w^{n-2} dx_m$$
$$= n(n-1) w^{n-2}(1-w), \quad 0 < w < 1$$
(12)

See Equation (13), Equation (14), and Equation (15). Substituting Equation (12) into Equation (9) gives the mean of the range as

$$\mu_w = n(n-1) \int_0^1 w^{n-1}(1-w)\, dw$$
$$= n(n-1) \left(\frac{1}{n} - \frac{1}{n+1} \right)$$
$$= \frac{n-1}{n+1}$$
(13)

and the variance is given as

$$\text{Var}(w) = n(n-1) \int_0^1 \left(w - \frac{n-1}{n+1} \right)^2$$
$$\cdot w^{n-2} \cdot (1-w)\, dw$$
$$= \frac{2(n-1)}{(n+2)(n+1)^2}$$
(14)

Thus, the standard error of w can be expressed as

$$se(w) = \sqrt{\text{Var}(w)} = \sqrt{\frac{2(n-1)}{(n+2)(n+1)^2}}$$
(15)

Equation (15) represents the uncertainty of the flatness evaluation with respect to the sample size n, when the distribution is uniform. According to the formulation, for 100 points, the standard error of the flatness evaluation is 0.0138. Because the form error distribution is uniform as Equation (11), the standard error is 1.38% of the evaluated flatness value. If the standard error is to be smaller than 1%, the number of samples must be more than 140 points. This approximation yields optimistic results, because the rigid body transformation of the measured points is not considered.

For the general case, the formulation of the standard error is difficult to derive analytically. The terms in Equations (8), (9), and (10) are in integral form that is difficult to analytically formulate. For the general beta distribution as defined in Equation (4), analytical integration of Equations (8), (9), and (10) is not possible. Thus, the analytic formulation has limited applications. For a more general case, a numerical approach must be investigated.

Numerical approach

For a given measurement condition, the functional relationship between the standard error associated with evaluation and the measurement parameters must be derived. The standard error represents the uncertainty of the form error evaluation. Because such a functional relationship cannot be generally formulated as an analytic form, it must be estimated by an experimental approach. A neural network method is employed to model the functional relation. A neural network is trained to learn the pattern from a set of known data. After the network is trained, it is able to output the standard error for a new set of input data. The network represents the functional relationship between the input parameters and the output parameter. The advantage of this approach is that the explicit form of the function is not necessary. If the function is approximated by a multivariate regression, the explicit form of the function, such as a polynomial and the order of the polynomial must known beforehand. A neural network can represent a function without knowing the explicit form and is applicable to more general types of functions.

To build a network that represents the functional relationship, a training dataset must be provided. The training dataset includes various measurement conditions. In each case of the measurement, the possible standard error of the evaluated value must be provided. The standard error is evaluated by iterating the process of sampling from a given known surface and evaluating the extreme fit. Then, from repetitive evaluations of the extreme fit, the standard error of the evaluation is determined. The set of the measurement parameters, including the geometry, the form error, the number of samples, and the evaluated standard error, are used for network input and output. The network identifies the functional relationship between the parameters from the different training sets.

To demonstrate this approach, a flatness model is tested. The model is an L by L flat square represented by x-y plane (*Figure 7*). To simulate the form error of a real surface, noise with a beta distribution is added to the surface. The beta distribution is able to model a general type of stochastic error that occurs on engineering surfaces. The range of the form error is set as $a = 0$, $b = 0.1$ of the beta distribution. Thus, the ideal flatness value of the surface is 0.1. To generate training
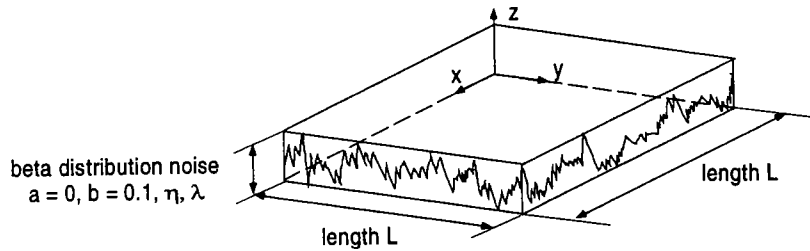
beta distribution noise
a = 0, b = 0.1, η, λ

Figure 7 Test model

datasets, flatness of the surface and the standard error of the flatness must be evaluated. A set of n points is randomly selected from the surface, then flatness is evaluated from the point set. The spatial distribution of the points on the surface is uniform, and the specific value of the surface profile at the sampled point is represented by the beta distribution. (The uniform point distribution may not be the best spatial distribution of points; however, it is frequently used in practice.) Flatness of the model surface is evaluated by the minimum zone evaluation. When a different set of n points are taken, the evaluated flatness value changes slightly. The standard error of the flatness is determined as the samples are repetitively resampled. The sampling uncertainty can then be represented by the standard error of the evaluated flatness value.

The parameters that define the model are the geometric parameter, length L, and the noise parameters η and λ of the beta distribution. A feedforward network with one hidden layer is used, as shown in *Figure 8*. The network consists of four inputs and an output. Inputs are model parameters and the sample size. The output is the standard error of flatness value.

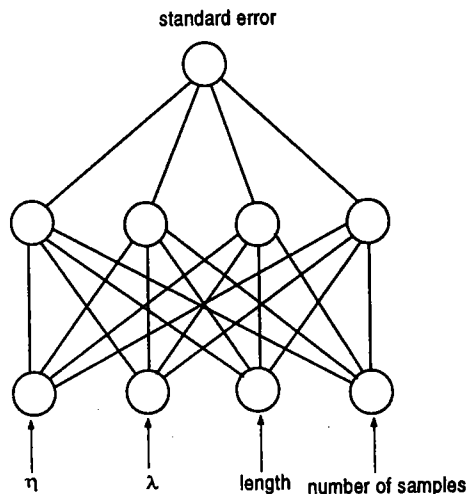The training data are generated, as shown in

Table 1. The ranges of the parameters for the training dataset are shown as the minimums and the maximums of the parameter values. Different parameter values, that increment from MIN to MAX, are tested. The beta distribution parameter spans from 1 to 5. With different λ and η values, the different cases of the surface form error are modeled. When λ and η are both unity, the beta distribution is equivalent to uniform distribution. In each case of the beta distribution, different number of measured data samples are taken. The numbers of samples are from 20 to 200, a typical CMM measurement size. The flatness is then evaluated. To determine the standard error of the evaluation, each case is iterated 100 times and from this iteration, the standard error of the flatness value is derived.
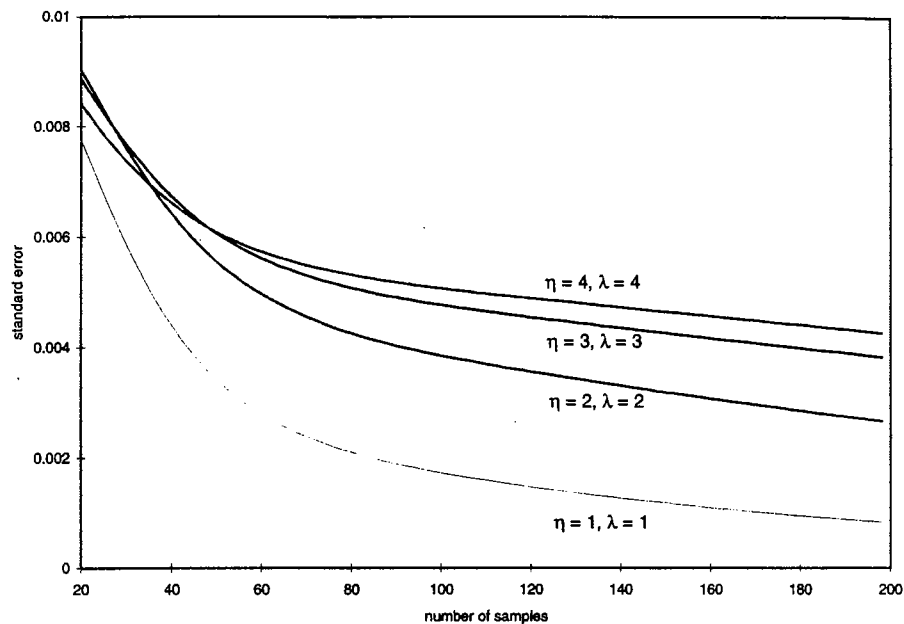
Results

The neural network was implemented using Stuttgart Neural Network Simulator,[21] version 4.1. The network was trained by the backpropagation method, using the momentum algorithm. A learning rate of 0.3 and the momentum term $m = 0.4$ were used. The network was trained in 50,000 cycles with the learning data. The square sum of error defined in Equation (13) is 0.3535, and the mean square error is 0.0026. Thus, the root-mean-square error is 0.0508. The approximated function has about 5% error in the output. The error is acceptable, considering the fact that the standard error contains a certain level of random noise, because it has been evaluated from a finite number of samples. Note that if the network were

standard error

η λ length   number of samples

Figure 8 Network model

Table 1 Training dataset for neural network

	Min value	Max value	Number of increments
η	1.	5.	5
λ	1.	5.	5
1	10.	1000.	3
Number of sample	20	200	4

Figure 9 Standard error with respect to different sizes of form error versus number of samples

trained to fit all of the outputs exactly, it might be overconstrained so that it generates unnecessary fluctuations.

Because the function is in the form of a neural network, the explicit mathematical form is not known, and the result must be presented in a numerical form. *Figure 9* is a plot of the function with respect to the sample size. The function has four input variables and an output variable, and, for the sake of visualization, four curves are drawn on *Figure 9*. The network function is able to yield the standard error value for any input variables. In *Figure 9*, the standard error monotonically decreases with respect to the sample size. For the distribution with $\lambda = 1$, $\eta = 1$, a uniform distribution with range 0.1, the standard error is below 1×10^{-3} with more than 170 points. Considering the magnitude of flatness (for $\lambda = 1$, $\eta = 1$), 0.1, the standard error is less than 1%. Compared to the analytic approximation, the experimental results yield a 20% higher standard error for the same number of points. This is mainly because of the simplification in the analytic approximation that ignored the rigid body transformation of the surface in the evaluation.

An interesting observation is that the standard error increases as the pair η, λ increases in beta distribution. As η and λ increase, the probability distribution is more dense at the center as the bell shape becomes steep, as shown in *Figure 6*. Thus, as η and λ increase, although the variance of the form error decreases, the standard error of flatness increases. This is consistent with the observation that the standard error of flatness evaluation from a surface with normally distributed

form error is larger than the standard error from a surface with uniformly distributed form error.

This phenomenon can be explained by the distribution of the maximum and minimum points. Although the maximum and minimum points for initial noise distribution may not be the extreme points determined by the minimum zone evaluation, the distribution of the maximum and the minimum points is closely related to the magnitude of the minimum zone. As the noise distribution approaches a uniform distribution, the maximum and the minimum points from n samples become fairly close to the limits. Thus, the range between the maximum and the minimum is more consistent. If the noise distribution is denser at the center, the average range between the maximum and the minimum is decreased but the variation of the range is increased. Thus, as η, λ increases, the standard error increases, as illustrated in *Figure 9*. As a result, the normal-like distribution yields larger standard error than the uniform distribution. This tendency is reversed as the number of samples decreases, as shown in *Figure 9*. If the number of samples is small, the samples are not fully distributed in the range. The discrepancy between extreme points of the sample and the true surface is larger. As the distribution becomes more uniform, the discrepancy increases. For steep bell-shaped distributions, the points are mostly distributed at the center, which has less sensitivity to the number of points. Thus, compared to uniform distribution, the variation of the range is small.

Figure 10 shows the trend of the standard error with respect to the skewness of the distribu-
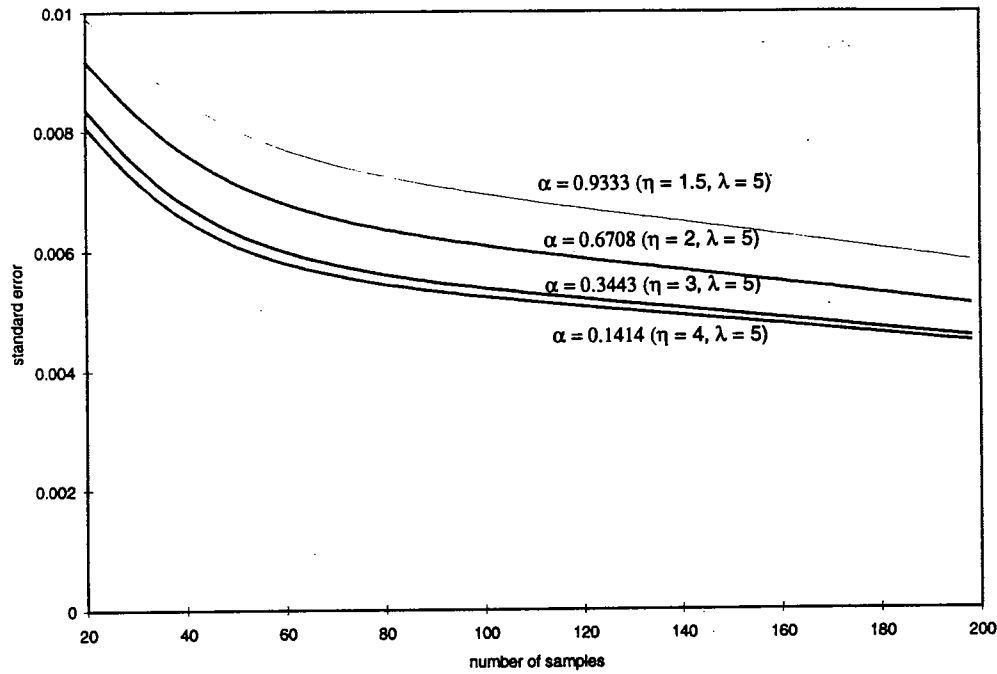
Figure 10 Standard error with respect to skewness α of distribution versus number of samples

tion. As the coefficient of skewness α increases, the standard error increases. Because the flatness value is theoretically symmetric with respect to the skewness direction, either in the positive direction or the negative direction, the sign of skewness should not affect the standard error of the flatness value. However, for actual measurement data, the sign of the skewness is important. A positive skewness value for a form error represents valleys in the surface, as shown in *Figure 5*. For a contacting surface, such as a bearing surface, the positive

skewness may be acceptable, because the valleys of a surface may have a minimal effect on the functionality of the part. Also the high spatial frequency characteristics cannot be measured by a relatively large probe. On the contrary, a negative skewness means spikes on the surface, which are critical deformations. The spikes are more apparently measured, and the standard error of the evaluation increases.

An interesting observation is the relation between the geometry of the model and the standard
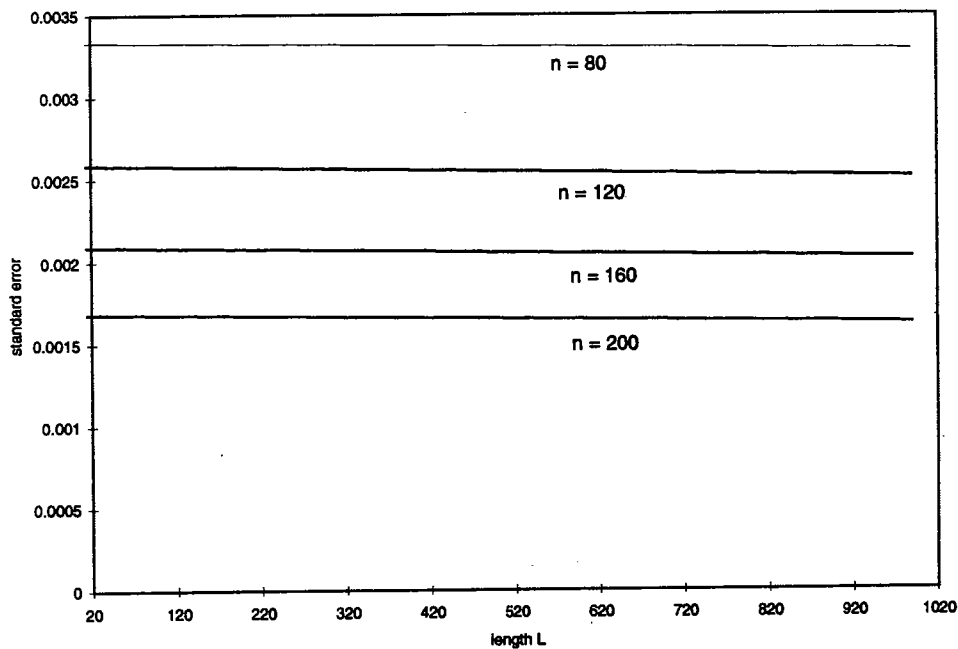
Figure 11 Standard error and the size of measurement region ($\eta = \lambda = 1.5$)

error. As shown in *Figure 11*, there is very weak correlation between the standard error and the geometry size. Throughout the range of the length used, from 20 to 1000, *Figure 11* shows almost consistent standard error. This result indicates the density of the measured points has little effect on the uncertainty. For a 200 number point case, the densities varied from 0.5 points per unit area to 2×10^{-4} points per unit area, yet they yield almost similar standard errors. Thus, it is the number of measurement points, not the density of the points, that is influential to the uncertainty.

Conclusions

The results show how the property of a measured point set affects the uncertainty of an extreme fit evaluation. The established numerical model represents how the standard error of the evaluation changes with respect to the number of measured points. The results indicate that the distribution of the form error is critical for the standard error as well as the number of points. When the distribution approaches a bell shape, which resembles a normal distribution, the standard error of the extreme fit evaluation tends to increase. When the skewness of the form error increases, the standard error also tends to increase. However, the size of measurement area has little effect on the standard error. Thus, the density of the measurement data is not critical for the uncertainty.

One factor that has not been considered in this paper is the distribution of measurement points on the measurement plane. The distribution of measurement points is regarded as uniform in this paper, because the flatness evaluation is a form error evaluation where the probability of such a variation being observed is relatively uniform compared to a systematic deformation characterization. Nonuniform distribution is more effective for a nonuniform variation, which is a systematic deformation. If the evaluation is to characterize a systematic deformation, a nonuniform distribution of measurement points should be considered.

For practical measurement situations, the number of measured points must be determined before the measurement. The results of this work, unfortunately, indicate that the uncertainty results depend significantly upon the distribution of the form error. This does not lend itself well to the determination of the number of measurement points needed to achieve a specified level of uncertainty. Because the distribution of form error is unknown before the measurement, it is difficult to predict the uncertainty of the evaluation based upon the number of points. Thus, the knowledge of the form distribution must be provided. Such a fact, unfortunately, indicates that dimensional inspection planning must be processed case by case.

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