Identification and Estimation of Auction Models with Dual Risk-Averse Bidders

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Sources of risk aversion

- Theory can generate risk aversion in two main ways:

  1. Curvature of the von Neumann-Morgenstern utility function

  2. Non-linear probability weighting

- The empirical literature has focused on expected utility (EU) models

- Expected utility (EU) models incorporate 1. but not 2.

Expected utility vs. dual utility

- Dual utility (DU) models incorporate 2. but not 1.
  - Risk aversion is wholly attributable to probability weighting
  - Constant marginal utility of wealth

- Why dual utility?
  - Consistent with profit maximization (Yaari, 1987)
  - Better captures industry risk management practices (e.g., VaR, CVaR)
  - ...
  - (As we will see) improved tractability

This paper: estimate an auction model with dual risk aversion

1. Characterize BNE of first- and second-price auctions
2. Dual counterpart to classic identification results (GPV09)
3. Estimation framework for parallel FPA/SPA bidding
4. Application to USFS timber auctions

Related literature

1. Empirical models of auctions that relax risk-neutrality
   - **EU models:** Bajari and Hortacsu, 2005; Guerre Perrigne Vuong, 2009; Lu and Perrigne, 2008; Campo et al, 2011; Fang and Tang, 2014; Bolotnyy and Vasserman, 2023; Chen, Gentry, Li, Lu, 2025; Luo and Takahashi, Forthcoming. **Non-EU models:** Aryal et al, 2018

2. Empirical models of rank-dependent utility
   - **Experiments:** Goeree, Holt, Palfrey, 2002; Armantier and Treich, 2009. **Consumer choice:** Ciccheti and Dubin, 1994; Barseghyan et al., 2013; Barseghyan et al., 2016

3. Mechanism design with constant risk aversion
   - Gershkov, Moldovanu, Strack, Zhang (2022, 2023a, 2023b)

1. Theoretical framework

2. Identification

3. Estimation

4. Application: USFS Timber Sales

5. Conclusion

Preferences

- x is a random variable on $[0, \bar{x}]$ drawn from $H(\cdot)$

- Begin with a **rank-dependent utility** model (Quiggan, 1982),

$$U(x) = \int_0^{\bar{x}} u(s) \cdot g'(1 - H(s)) \cdot dH(s) = \int_0^{\bar{x}} u'(s) \cdot g(1 - H(s))\, ds$$

  - $u(\cdot)$ is a standard vN-M utility function, while $g : [0,1] \to [0,1]$ is a cdf
  - Three special cases:

	$u(\cdot)$	$g(\cdot)$
Risk-neutrality	Identity	Identity
EU risk-aversion	Concave	Identity
Dual risk-aversion	**Identity**	**Convex**

Auction game: overview

- A risk-neutral seller is endowed with a single indivisible good
- I symmetric potential bidders with independent private values $v_i \sim F\left(\cdot|I\right)$
- All bidders are symmetrically equipped with some convex distortion $g\left(\cdot\right)$
- We characterize monotone, symmetric BNE of the SPA and FPA

Optimal bidding in the FPA

- Optimal bidding problem for a bidder with value v

$$\max_b \quad \{v - b\} \cdot g\left(1 - P\{b \text{ loses}\}\right)$$

- The equilibrium bid function satisfies:

$$\beta'(v) = \{v - \beta(v)\} \cdot (1 - I) \cdot \frac{f(v|I)}{F(v|I)} \cdot \left\{ \frac{g'\left(F(v|I)^{I-1}\right) \cdot F(v|I)^{I-1}}{g\left(F(v|I)^{I-1}\right)} \right\}$$

  - Under risk-neutrality, term in brackets collapses to one
  - This differential equation admits a closed form solution (unlike EU case)

Equilibrium

Proposition

1. *In a SPA with reserve price r, the optimal strategy is* $\beta(v_i) = v_i$ *if* $v_i \geq r$.
2. *In a FPA with reserve price r, the optimal strategy is to bid:*

$$\beta(v_i) = v_i - \frac{\int_r^{v_i} g\left(F^{I-1}(s)\right) ds}{g\left(F^{I-1}(v_i)\right)}$$

if $v_i \geq r$. *Moreover,* $\beta(v_i)$ *is greater than the risk-neutral optimal bid.*

- In **Li, Lu, O'Keefe (2025)**, we characterize optimal reserve prices

Summary

- $[g, F]$ is not identified in first price auction data alone
  - Analogous to non-identification of $[u, F]$ in the EU model (GPV09)

- We consider two identification strategies:
  1. (Exogenous) variation in I
  2. Parallel second-price auction data

Non-identification

- Let $\mathbf{Q}(b_1, \ldots, b_I|I) \equiv \Pi_{i \leq I} Q(b_i|I)$ denote the joint distribution of FPA bids
- If $[g, F]$ rationalizes $\mathbf{Q}(\cdot|I)$, then so does $[\tilde{g}, \tilde{F}]$
  - $\tilde{g}(\alpha) = g(\alpha)^\delta$ for $\delta > 1$
  - $\tilde{F}(\cdot|I)$ is the distribution of $\tilde{\xi}(b)$, where:

$$\tilde{\xi}(b) = b + \frac{1}{I-1} \cdot \frac{Q(b|I)}{q(b|I)} \left\{ \frac{Q^{I-1}(b|I) \cdot \tilde{g}'\left(Q^{I-1}(b|I)\right)}{\tilde{g}\left(Q^{I-1}(b|I)\right)} \right\}^{-1}$$

Identification with exogenous participation

- Suppose $F(\cdot|I) = F(\cdot) \Rightarrow v(\cdot|I) = v(\cdot)$ for all $I \in \mathcal{I}$

- FOC for bidder with α-quantile valuation:

$$b(\alpha|I) = v(\alpha) - b'_R(\alpha|I) \cdot z\left(\alpha^{I-1}\right)$$

  - $b_R(\cdot|I)$ is the quantile function of $Q_R(\cdot|I) \equiv Q(\cdot|I)^{I-1}$
  - $z(\cdot) = g(\cdot)/g'(\cdot)$

- **Compatibility conditions:** because $v(\alpha)$ does not vary with I,

$$b(\alpha|I_1) + b'_R(\alpha|I_1) \cdot z\left(\alpha^{I_1-1}\right) = b(\alpha|I_2) + b'_R(\alpha|I_2) \cdot z\left(\alpha^{I_2-1}\right)$$

Identification with exogenous participation

- Suppose $F(\cdot|I) = F(\cdot) \Rightarrow v(\cdot|I) = v(\cdot)$ for all $I \in \mathcal{I}$
- FOC for bidder with α-quantile valuation:

$$v(\alpha) = b(\alpha|I) + b_R'(\alpha|I) \cdot z\left(\alpha^{I-1}\right)$$

  - $b_R(\cdot|I)$ is the quantile function of $Q_R(\cdot|I) \equiv Q(\cdot|I)^{I-1}$
  - $z(\cdot) = g(\cdot)/g'(\cdot)$

- **Compatibility conditions:** because $v(\alpha)$ does not vary with I,

$$z(\alpha) = \frac{b\left(\alpha^{1/(I_1-1)}|I_2\right) - b\left(\alpha^{1/(I_1-1)}|I_1\right)}{b_R'\left(\alpha^{1/(I_1-1)}|I_1\right)} + \frac{b_R'\left(\alpha^{1/(I_1-1)}|I_2\right)}{b_R'\left(\alpha^{1/(I_1-1)}|I_1\right)} \cdot z\left(\alpha^{(I_2-1)/(I_1-1)}\right)$$

Identification with exogenous participation (cont)

- K recursive applications gives

$$z(\alpha) = \overbrace{\sum_{0\le k\le K}\left\{\frac{b(\alpha_k|I_2) - b(\alpha_k|I_1)}{b'_R(\alpha_k|I_1)} \cdot \prod_{l=0}^{k-1}\frac{b'_R(\alpha_l|I_2)}{b'_R(\alpha_l|I_1)}\right\}}^{\tilde{z}_K(\alpha)} + \overbrace{z(\alpha_K)\cdot\prod_{l=0}^{K-1}\frac{b'_R(\alpha_l|I_2)}{b'_R(\alpha_l|I_1)}}^{R_K(\alpha)}$$

where $\alpha_k = \alpha_{k-1}^{(I_2-1)/(I_1-1)}$ and $\alpha_0 = \alpha^{1/(I_1-1)}$

- Because $\tilde{z}_K(\alpha)$ is observable, $z(\alpha)$ is identified if $\lim_{K\to\infty} R_K(\alpha) = 0$

- Then $g(\alpha) = \exp\left\{-\int_\alpha^1 [z(s)]^{-1}\, ds\right\}$ is identified from $g(1) = 1$

Identification with second-price auction data

- In some settings, second-price and first-price auctions are held in parallel

- Re-arranging the quantile FOC:

$$z(\alpha) = \left\{ b_R' \left(\alpha^{\frac{1}{I-1}} | I \right) \right\}^{-1} \cdot \left[v \left(\alpha^{\frac{1}{I-1}} | I \right) - b \left(\alpha^{\frac{1}{I-1}} | I \right) \right]$$

- $\beta(v) = v \Rightarrow v(\cdot|I) \equiv F^{-1}(\cdot|I)$ is identified from the winning bids in SPA

Can the data distinguish EU and DU?

- With FPA bids alone, **no**

- Suppose I varies exogenously **and** we have SPA bids. Then:

1. The dual utility (DU) and expected utility (EU) models are testable

2. The rank-dependent utility model $[u, g, F]$ itself is not identified

    - Without a restriction, $u(\cdot)/u'(\cdot)$ and $g(\cdot)/g'(\cdot)$ only known up to scale

Estimation with SPA winning bids

- By integrating over quantiles (Liu Lou 2017, Luo Wan 2018), we obtain:

$$g(\alpha) = \exp\left(- \int_{b\left(\alpha^{1/(I-1)}|I\right)}^{b(1|I)} \left[\frac{1}{\psi(u|I) - u} \right] \cdot du \right)$$

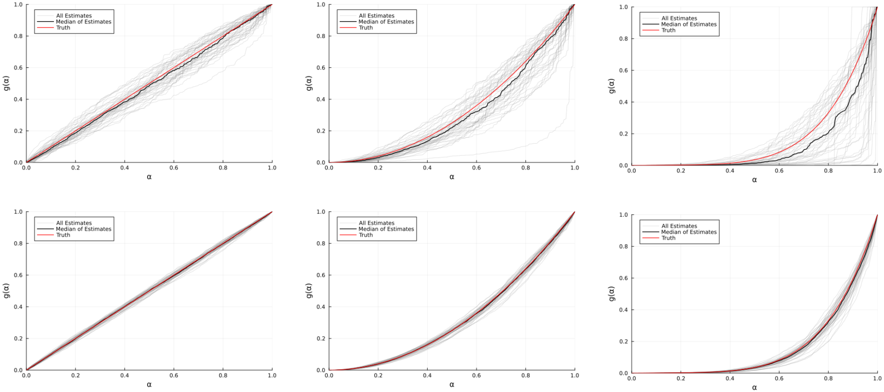
  - $\psi(\cdot|I) \equiv F^{-1}(Q(\cdot|I)|I)$ is the statistical mapping from bids to values

- A **tuning-parameter free** estimator $\hat{g}(\cdot)$ can be constructed from bid ECDFs

- Can impose shape restrictions via post-processing (Chen et al 2021) or by constrained sieve estimation (Chen, 2007) e.g., with convex splines (Meyer, 2008)

Simulated point estimates, $n \in \{100, 1000\}$

Model 1: $g(x) = x$ | **Model 2:** $g(x) = x^2$ | **Model 3:** $g(x) = \frac{x + x^5}{1 + x}$



Note: simulations assume $I = 3$, $v \sim \text{LogNormal}(1, 1)$ truncated at $\bar{v} = 10$.

Summary of simulated point estimates

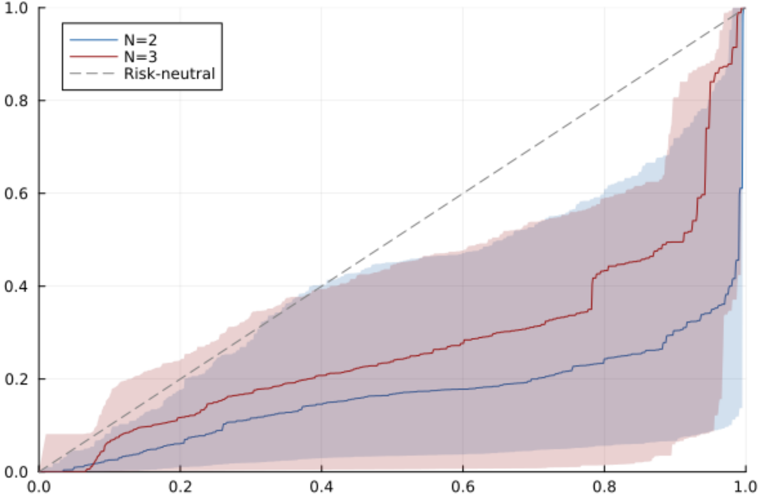
		$\|g(\cdot) - \hat{g}(\cdot)\|$		FS Test Rej. Rate	
		50%	95%	Affine	Convex
$n = 100$	Model 1	0.038	0.075	0.00	0.00
	Model 2	0.057	0.133	0.94	0.00
	Model 3	0.081	0.246	0.90	0.00
$n = 1000$	Model 1	0.009	0.022	0.00	0.00
	Model 2	0.013	0.031	1.00	0.00
	Model 3	0.018	0.053	1.00	0.00

Note: Based on 50 simulations; FS Test is Feng and Seo (2021) test.

1979 USFS Timber Sales

- In the late 1970s, the US Forest Service switched from SPA to FPA
- Non-binding reserve, pre-qualification of bidders $\Rightarrow I$ is observed
- We implement our estimator on 1979 auctions having $I \in \{2, 3\}$
- Prior EU-based work has found mixed evidence for risk aversion
  - In favor (Lu and Perrigne, 2008, Campo et al, 2011); against (Gimenes and Guerre, 2022)

Estimated distortion functions



Note: Reports pointwise bootstrap confidence bands.

Evidence of (dual) risk aversion

	FS Test p-value	
	Affine	Convex
$I = 2$	0.148	0.848
$I = 3$	0.136	0.888

Revenue comparison and optimal reserves

	SPA		FPA		Δ SPA to FPA	
	$I = 2$	$I = 3$	$I = 2$	$I = 3$	$I = 2$	$I = 3$
Optimal Reserve r^*	55,029	60,726	49,852	54,878	-5,177	-5,848
	(13,033)	(14,882)	(14,453)	(12,294)	(10,712)	(16,219)
Revenue ($r = r^*$)	42,708	54,565	46,163	63,207	3,455	8,641
	(2,663)	(2,946)	(2,476)	(5,584)	(1,408)	(4,586)
Revenue ($r = 0$)	35,773	48,909	41,563	61,774	5,790	12,865
	(1,474)	(2,394)	(1,778)	(5,876)	(1,820)	(5,510)

- Risk-neutral model: (1) overstates r^* by $\approx$10%; (2) understates FPA revenue

1. Theoretical framework

2. Identification

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Conclusion

- We study identification of auction models with dual risk averse bidders
- Appealing for applications in which bidders are profit-maximizing firms
- Application to timber sales (1) demonstrates plausibility of the dual utility model; (2) illustrates practical benefit of closed-form bidding strategies
- Next steps: quantification of the optimal mechanism? (Gershkov et al 2022)

Thank you!

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