Hidden Markov Models and Gaussian Mixture Models

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Automatic Speech Recognition— ASR Lectures 4&5
28/31 January 2013

Overview

HMMs and GMMs

- Key models and algorithms for HMM acoustic models
- Gaussians
- GMMs: Gaussian mixture models
- HMMs: Hidden Markov models
- HMM algorithms
  - Likelihood computation (forward algorithm)
  - Most probable state sequence (Viterbi algorithm)
  - Estimting the parameters (EM algorithm)

Fundamental Equation of Statistical Speech Recognition

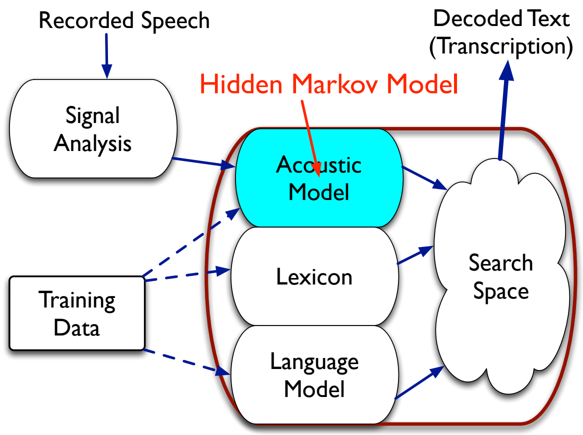
If $\mathbf{X}$ is the sequence of acoustic feature vectors (observations) and $\mathbf{W}$ denotes a word sequence, the most likely word sequence $\mathbf{W}^*$ is given by

$$\mathbf{W}^* = \arg\max_{\mathbf{W}} P(\mathbf{W} \mid \mathbf{X})$$

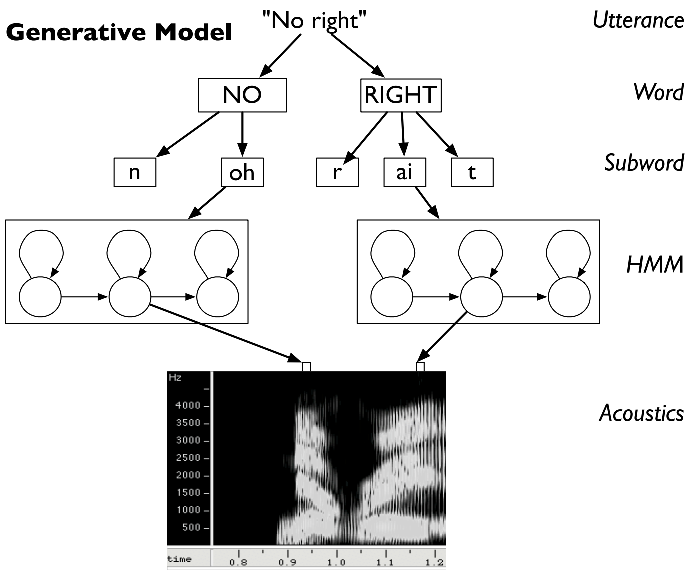
Applying Bayes' Theorem:

$$\begin{aligned} P(\mathbf{W} \mid \mathbf{X}) &= \frac{p(\mathbf{X} \mid \mathbf{W})P(\mathbf{W})}{p(\mathbf{X})} \\ &\propto p(\mathbf{X} \mid \mathbf{W})P(\mathbf{W}) \\ \mathbf{W}^* &= \arg\max_{\mathbf{W}} \underbrace{p(\mathbf{X} \mid \mathbf{W})}_{\text{Acoustic model}} \underbrace{P(\mathbf{W})}_{\text{Language model}} \end{aligned}$$

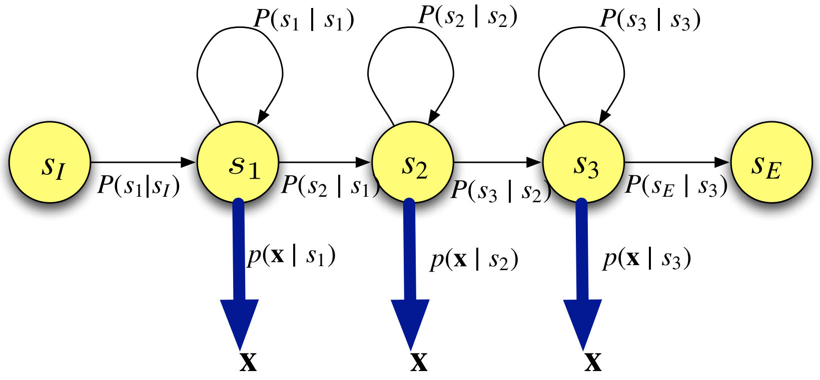
Acoustic Modelling



Hierarchical modelling of speech



Acoustic Model: Continuous Density HMM

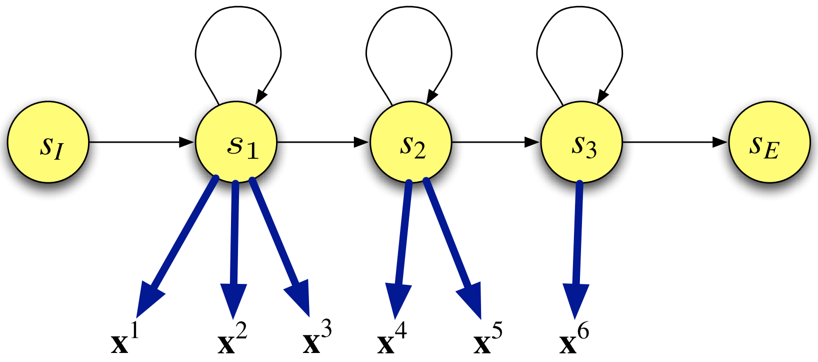


Probabilistic finite state automaton

Paramaters $\boldsymbol{\lambda}$:

- Transition probabilities: $a_{kj} = P(s_j \mid s_k)$
- Output probability density function: $b_j(\mathbf{x}) = p(\mathbf{x} \mid s_j)$

Acoustic Model: Continuous Density HMM



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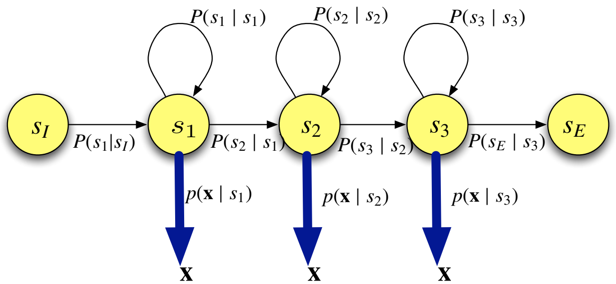
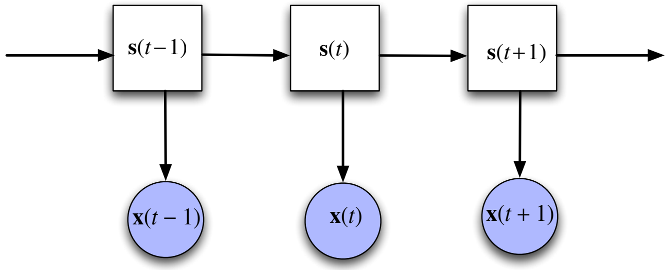
HMM Assumptions



1. **Observation independence** An acoustic observation $\mathbf{x}$ is conditionally independent of all other observations given the state that generated it
2. **Markov process** A state is conditionally independent of all other states given the previous state

HMM Assumptions



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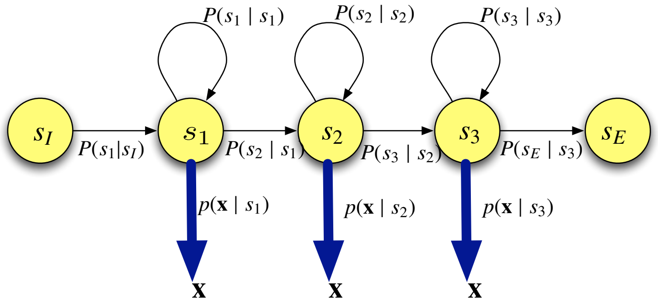
HMM OUTPUT DISTRIBUTION

Output distribution



Single multivariate Gaussian with mean $\boldsymbol{\mu}^j$, covariance matrix $\boldsymbol{\Sigma}^j$:

$$b_j(\mathbf{x}) = p(\mathbf{x} \mid s_j) = \mathcal{N}(\mathbf{x}; \boldsymbol{\mu}^j, \boldsymbol{\Sigma}^j)$$

M-component Gaussian mixture model:

$$b_j(\mathbf{x}) = p(\mathbf{x} \mid s_j) = \sum_{m=1}^{M} c_{jm} \mathcal{N}(\mathbf{x}; \boldsymbol{\mu}^{jm}, \boldsymbol{\Sigma}^{jm})$$

Background: cdf

Consider a real valued random variable X

- Cumulative distribution function (cdf) $F(x)$ for X:

$$F(x) = P(X \leq x)$$

- To obtain the probability of falling in an interval we can do the following:

$$\begin{aligned} P(a < X \leq b) &= P(X \leq b) - P(X \leq a) \\ &= F(b) - F(a) \end{aligned}$$

Background: pdf

- The rate of change of the cdf gives us the *probability density function* (pdf), $p(x)$:

$$p(x) = \frac{d}{dX} F(x) = F'(x)$$
$$F(x) = \int_{-\infty}^{x} p(x) dx$$

- $p(x)$ is **not** the probability that X has value x. But the pdf is proportional to the probability that X lies in a small interval centred on x.
- Notation: p for pdf, P for probability

The Gaussian distribution (univariate)

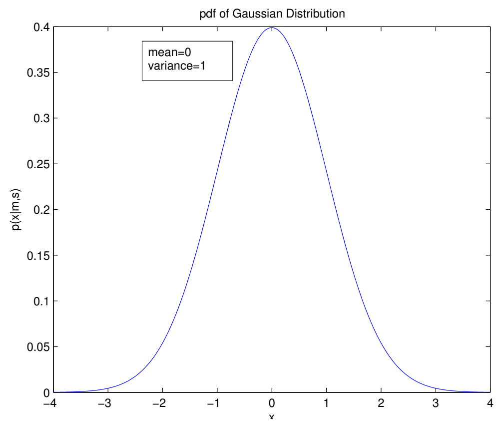
- The Gaussian (or Normal) distribution is the most common (and easily analysed) continuous distribution
- It is also a reasonable model in many situations (the famous "bell curve")
- If a (scalar) variable has a Gaussian distribution, then it has a probability density function with this form:

$$p(x|\mu, \sigma^2) = N(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

- The Gaussian is described by two parameters:
  - the mean μ (location)
  - the variance σ^2 (dispersion)

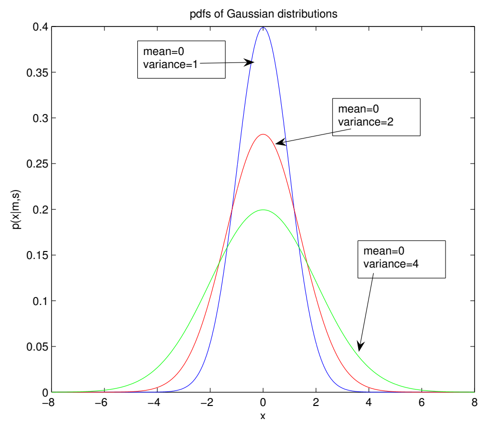
Plot of Gaussian distribution

- Gaussians have the same shape, with the location controlled by the mean, and the spread controlled by the variance
- One-dimensional Gaussian with zero mean and unit variance ($\mu = 0$, $\sigma^2 = 1$):



Properties of the Gaussian distribution

$$N(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$



Parameter estimation

- Estimate mean and variance parameters of a Gaussian from data $x^1, x^2, \ldots, x^n$
- Use sample mean and sample variance estimates:

$$\mu = \frac{1}{n}\sum_{i=1}^{n} x^i \qquad \text{(sample mean)}$$

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} (x^i - \mu)^2 \qquad \text{(sample variance)}$$

Exercise

Consider the log likelihood of a set of N data points $\{x^1, \ldots, x^N\}$ being generated by a Gaussian with mean μ and variance σ^2:

$$L = \ln p(\{x^1, \ldots, x^n\} \mid \mu, \sigma^2) = -\frac{1}{2}\sum_{n=1}^{N}\left(\frac{(x_n - \mu)^2}{\sigma^2} - \ln\sigma^2 - \ln(2\pi)\right)$$

$$= -\frac{1}{2\sigma^2}\sum_{n=1}^{N}(x_n - \mu)^2 - \frac{N}{2}\ln\sigma^2 - \frac{N}{2}\ln(2\pi)$$

By maximising the the log likelihood function with respect to μ show that the maximum likelihood estimate for the mean is indeed the sample mean:

$$\mu_{ML} = \frac{1}{N}\sum_{n=1}^{N} x_n.$$

The multidimensional Gaussian distribution

- The d-dimensional vector $\mathbf{x}$ is multivariate Gaussian if it has a probability density function of the following form:

$$p(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{d/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})\right)$$

  The pdf is parameterized by the mean vector $\boldsymbol{\mu}$ and the covariance matrix $\boldsymbol{\Sigma}$.
- The 1-dimensional Gaussian is a special case of this pdf
- The argument to the exponential $0.5(\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})$ is referred to as a *quadratic form*.

Covariance matrix

- The mean vector $\boldsymbol{\mu}$ is the expectation of $\mathbf{x}$:

$$\boldsymbol{\mu} = E[\mathbf{x}]$$

- The covariance matrix $\boldsymbol{\Sigma}$ is the expectation of the deviation of $\mathbf{x}$ from the mean:

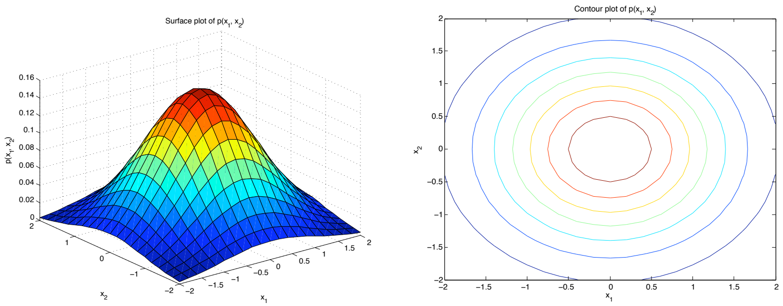
$$\boldsymbol{\Sigma} = E[(\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})^T]$$

- $\boldsymbol{\Sigma}$ is a $d \times d$ symmetric matrix:

$$\Sigma_{ij} = E[(x_i - \mu_i)(x_j - \mu_j)] = E[(x_j - \mu_j)(x_i - \mu_i)] = \Sigma_{ji}$$

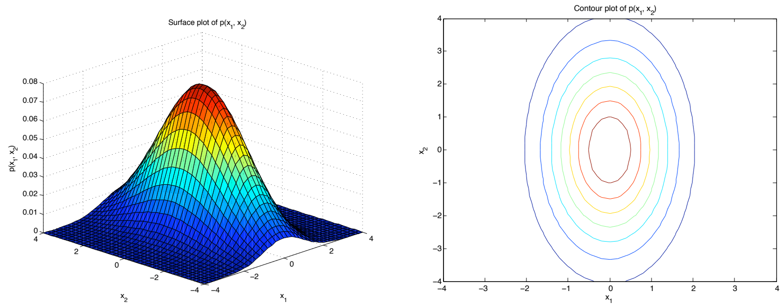
- The sign of the covariance helps to determine the relationship between two components:
  - If x_j is large when x_i is large, then $(x_j - \mu_j)(x_i - \mu_i)$ will tend to be positive;
  - If x_j is small when x_i is large, then $(x_j - \mu_j)(x_i - \mu_i)$ will tend to be negative.

Spherical Gaussian



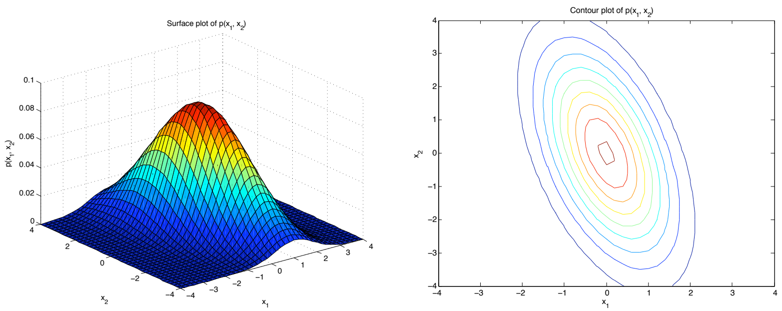
$$\boldsymbol{\mu} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \rho_{12} = 0$$

Diagonal Covariance Gaussian



$$\boldsymbol{\mu} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \qquad \rho_{12} = 0$$

Full covariance Gaussian



$$\boldsymbol{\mu} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix} \qquad \rho_{12} = -0.5$$

Parameter estimation

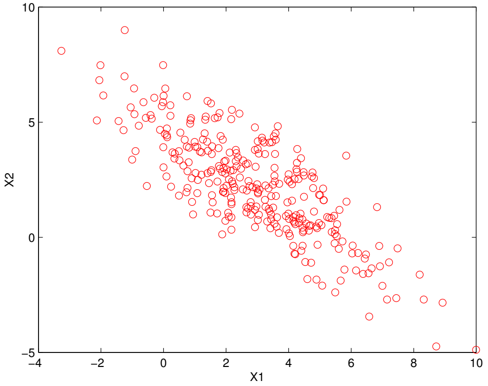
- It is possible to show that the mean vector $\hat{\boldsymbol{\mu}}$ and covariance matrix $\hat{\boldsymbol{\Sigma}}$ that maximize the likelihood of the training data are given by:

$$\hat{\boldsymbol{\mu}} = \frac{1}{N} \sum_{n=1}^{N} \mathbf{x}^n$$

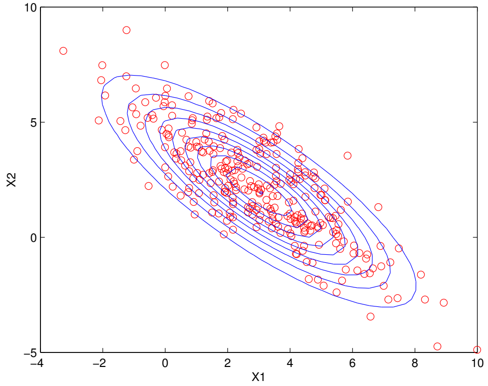
$$\hat{\boldsymbol{\Sigma}} = \frac{1}{N} \sum_{n=1}^{N} (\mathbf{x}^n - \hat{\boldsymbol{\mu}})(\mathbf{x}^n - \hat{\boldsymbol{\mu}})^T$$

- The mean of the distribution is estimated by the sample mean and the covariance by the sample covariance

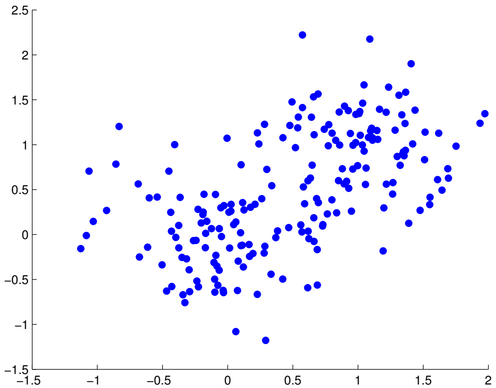
Example data



Maximum likelihood fit to a Gaussian

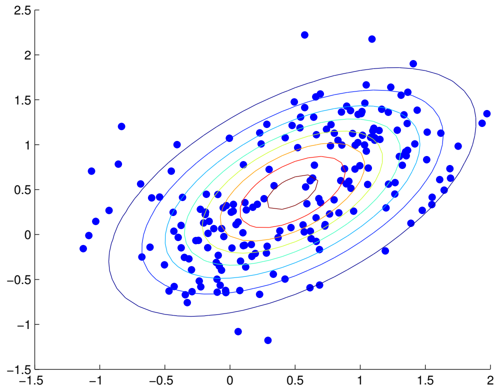


Data in clusters (example 1)



$$\boldsymbol{\mu}_1 = [0 \quad 0]^T \qquad \boldsymbol{\mu}_2 = [1 \quad 1]^T \qquad \boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2 = 0.2\mathbf{I}$$

Example 1 fit by a Gaussian

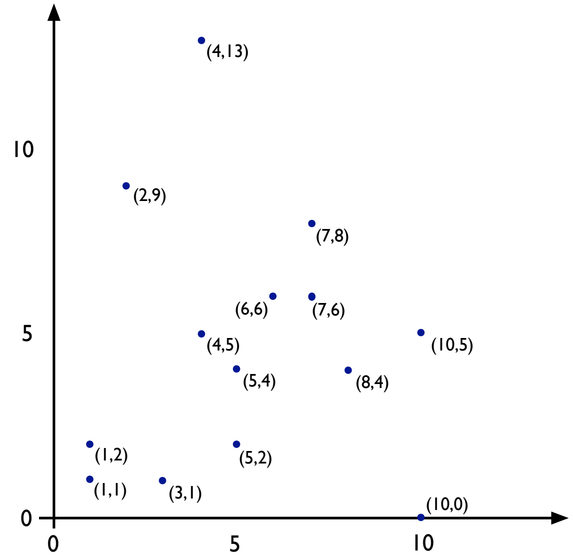


$$\boldsymbol{\mu}_1 = [0 \quad 0]^T \qquad \boldsymbol{\mu}_2 = [1 \quad 1]^T \qquad \boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2 = 0.2\mathbf{I}$$

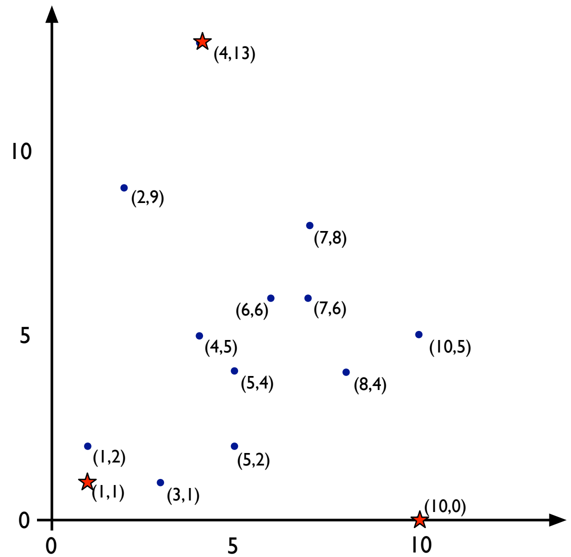
k-means clustering

- k-means is an automatic procedure for clustering unlabelled data
- Requires a prespecified number of clusters
- Clustering algorithm chooses a set of clusters with the minimum within-cluster variance
- Guaranteed to converge (eventually)
- Clustering solution is dependent on the initialisation

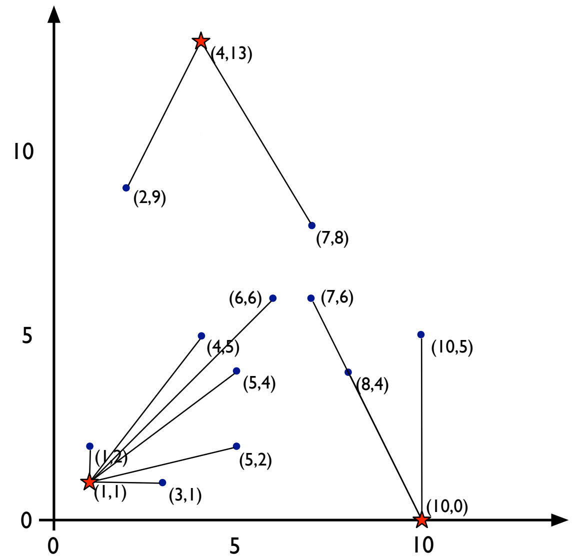
k-means example: data set



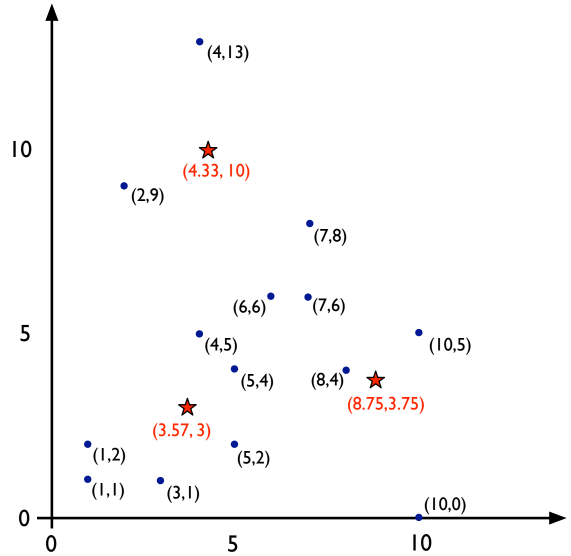
k-means example: initialization



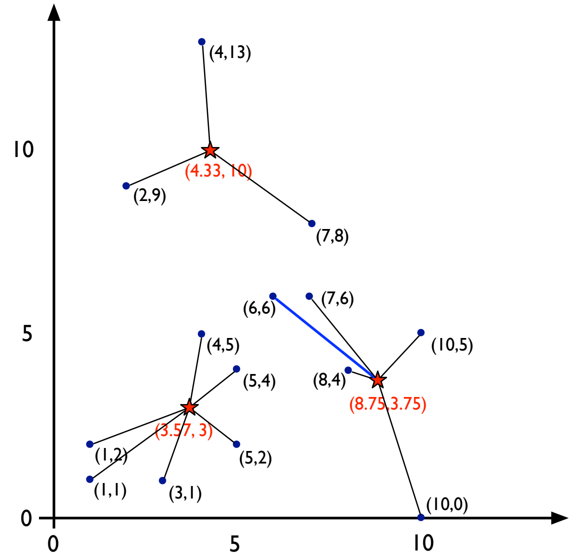
k-means example: iteration 1 (assign points to clusters)



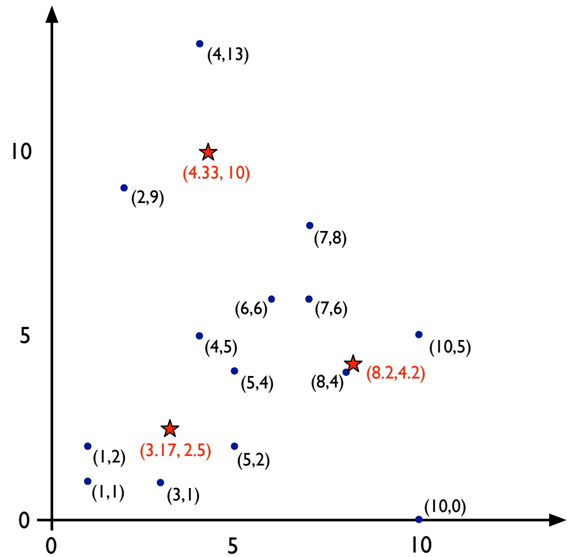
k-means example: iteration 1 (recompute centres)



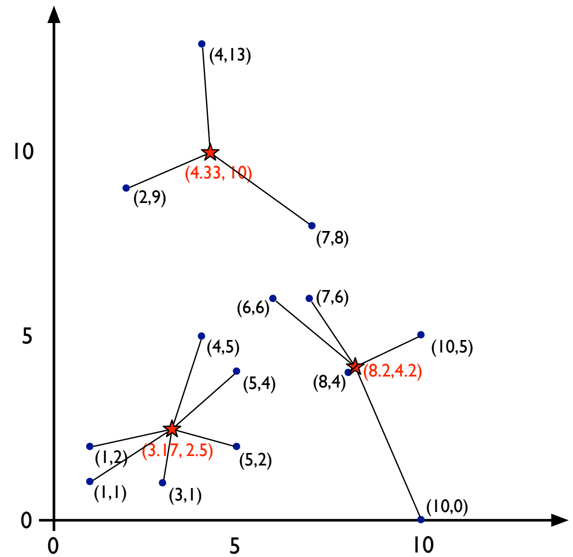
k-means example: iteration 2 (assign points to clusters)



k-means example: iteration 2 (recompute centres)



k-means example: iteration 3 (assign points to clusters)



No changes, so converged

Mixture model

- A more flexible form of density estimation is made up of a linear combination of component densities:

$$p(\mathbf{x}) = \sum_{j=1}^{M} p(\mathbf{x}|j)P(j)$$

- This is called a *mixture model* or a *mixture density*
- $p(\mathbf{x}|j)$: component densities
- $P(j)$: mixing parameters
- Generative model:
  1. Choose a mixture component based on $P(j)$
  2. Generate a data point $\mathbf{x}$ from the chosen component using $p(\mathbf{x}|j)$

Component occupation probability

- We can apply Bayes' theorem:

$$P(j|\mathbf{x}) = \frac{p(\mathbf{x}|j)P(j)}{p(\mathbf{x})} = \frac{p(\mathbf{x}|j)P(j)}{\sum_{j=1}^{M} p(\mathbf{x}|j)P(j)}$$

- The posterior probabilities $P(j|\mathbf{x})$ give the probability that component j was responsible for generating data point $\mathbf{x}$
- The $P(j|\mathbf{x})$s are called the *component occupation probabilities* (or sometimes called the *responsibilities*)
- Since they are posterior probabilities:

$$\sum_{j=1}^{M} P(j|\mathbf{x}) = 1$$

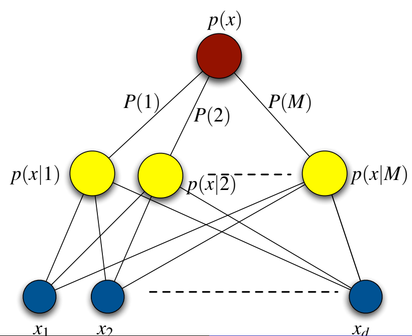
Parameter estimation

- *If* we knew which mixture component was responsible for a data point:
  - we would be able to assign each point unambiguously to a mixture component
  - and we could estimate the mean for each component Gaussian as the sample mean (just like k-means clustering)
  - and we could estimate the covariance as the sample covariance
- *But* we don't know which mixture component a data point comes from...
- Maybe we could use the component occupation probabilities $P(j|\mathbf{x})$?

Gaussian mixture model

- The most important mixture model is the *Gaussian Mixture Model* (GMM), where the component densities are Gaussians
- Consider a GMM, where each component Gaussian $N_j(\mathbf{x}; \boldsymbol{\mu}_j, \sigma_j^2)$ has mean $\boldsymbol{\mu}_j$ and a spherical covariance $\boldsymbol{\Sigma} = \sigma^2\mathbf{I}$

$$p(\mathbf{x}) = \sum_{j=1}^{P} P(j)p(\mathbf{x}|j) = \sum_{j=1}^{P} P(j)N_j(\mathbf{x}; \boldsymbol{\mu}_j, \sigma_j^2)$$



GMM Parameter estimation when we know which component generated the data

- Define the indicator variable $z_{jn} = 1$ if component j generated component $\mathbf{x}^n$ (and 0 otherwise)
- If z_{jn} wasn't hidden then we could count the number of observed data points generated by j:

$$N_j = \sum_{n=1}^{N} z_{jn}$$

- And estimate the mean, variance and mixing parameters as:

$$\hat{\boldsymbol{\mu}}_j = \frac{\sum_n z_{jn}\mathbf{x}^n}{N_j}$$

$$\hat{\sigma}_j^2 = \frac{\sum_n z_{jn}||\mathbf{x}^n - \boldsymbol{\mu}_k||^2}{N_j}$$

$$\hat{P}(j) = \frac{1}{N}\sum_n z_{jn} = \frac{N_j}{N}$$

Soft assignment

- Estimate "soft counts" based on the component occupation probabilities $P(j|\mathbf{x}^n)$:

$$N_j^* = \sum_{n=1}^{N} P(j|\mathbf{x}^n)$$

- We can imagine assigning data points to component j weighted by the component occupation probability $P(j|\mathbf{x}^n)$
- So we could imagine estimating the mean, variance and prior probabilities as:

$$\hat{\boldsymbol{\mu}}_j = \frac{\sum_n P(j|\mathbf{x}^n)\mathbf{x}^n}{\sum_n P(j|\mathbf{x}^n)} = \frac{\sum_n P(j|\mathbf{x}^n)\mathbf{x}^n}{N_j^*}$$

$$\hat{\sigma}_j^2 = \frac{\sum_n P(j|\mathbf{x}^n)||\mathbf{x}^n - \boldsymbol{\mu}_k||^2}{\sum_n P(j|\mathbf{x}^n)} = \frac{\sum_n P(j|\mathbf{x}^n)||\mathbf{x}^n - \boldsymbol{\mu}_k||^2}{N_j^*}$$

$$\hat{P}(j) = \frac{1}{N}\sum_n P(j|\mathbf{x}^n) = \frac{N_j^*}{N}$$

EM algorithm

- *Problem!* Recall that:

$$P(j|\mathbf{x}) = \frac{p(\mathbf{x}|j)P(j)}{p(\mathbf{x})}$$

  We need to know $p(\mathbf{x}|j)$ and $P(j)$ to estimate the parameters of $p(\mathbf{x}|j)$ and to estimate $P(j)$....
- Solution: an iterative algorithm where each iteration has two parts:
  - Compute the component occupation probabilities $P(j|\mathbf{x})$ using the current estimates of the GMM parameters (means, variances, mixing parameters) (E-step)
  - Computer the GMM parameters using the current estimates of the component occupation probabilities (M-step)
- Starting from some initialization (e.g. using k-means for the means) these steps are alternated until convergence
- This is called the *EM Algorithm* and can be shown to maximize the likelihood

Maximum likelihood parameter estimation

- The likelihood of a data set $\mathbf{X} = \{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\}$ is given by:

$$\mathcal{L} = \prod_{n=1}^{N} p(\mathbf{x}^n) = \prod_{n=1}^{N}\sum_{j=1}^{M} p(\mathbf{x}^n|j)P(j)$$

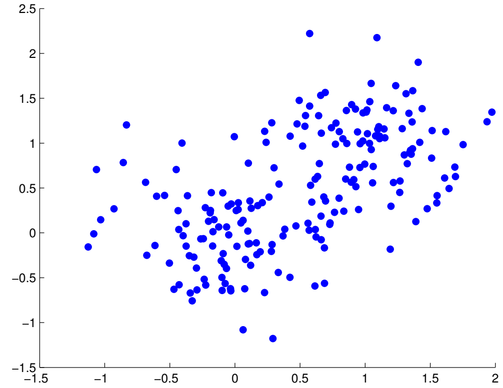
- We can regard the *negative log likelihood* as an error function:

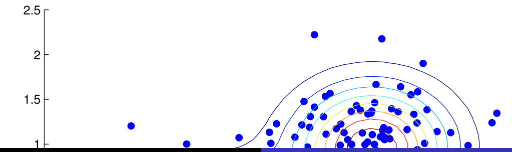
$$E = -\ln\mathcal{L} = -\sum_{n=1}^{N}\ln p(\mathbf{x}^n)$$

$$= -\sum_{n=1}^{N}\ln\left(\sum_{j=1}^{M} p(\mathbf{x}^n|j)P(j)\right)$$

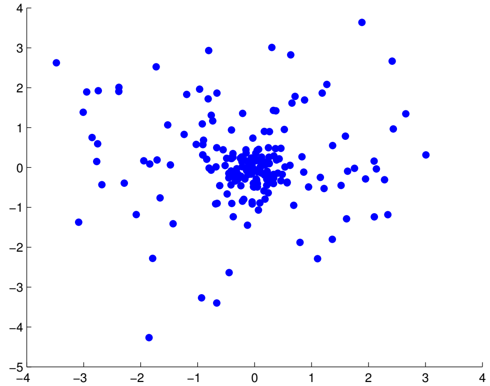
- Considering the derivatives of E with respect to the parameters, gives expressions like the previous slide

Example 1 fit using a GMM



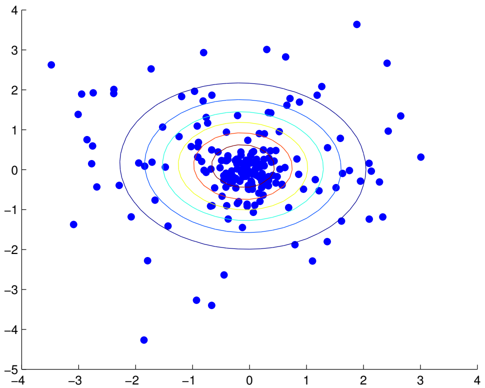


Peakily distributed data (Example 2)



$$\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2 = [0 \quad 0]^T \qquad \boldsymbol{\Sigma}_1 = 0.1\mathbf{I} \qquad \boldsymbol{\Sigma}_2 = 2\mathbf{I}$$

Example 2 fit by a Gaussian



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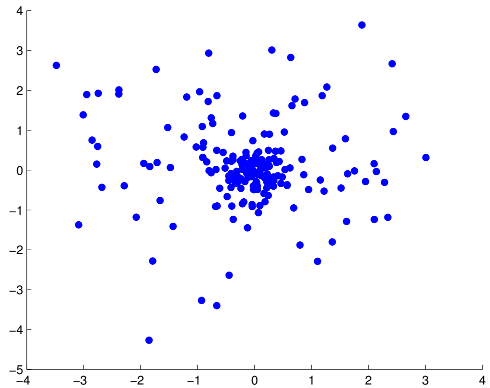
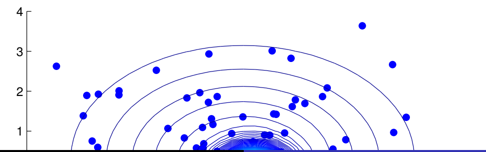
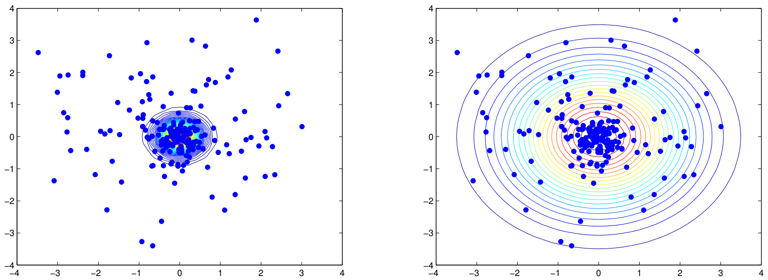
Example 2 fit by a GMM





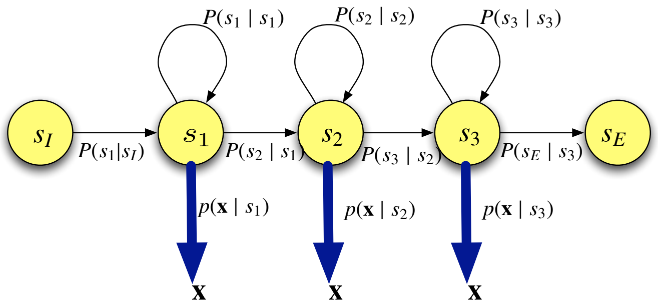
Example 2: component Gaussians



Comments on GMMs

- GMMs trained using the EM algorithm are able to self organize to fit a data set
- Individual components take responsibility for parts of the data set (probabilistically)
- Soft assignment to components not hard assignment — "soft clustering"
- GMMs scale very well, e.g.: large speech recognition systems can have 30,000 GMMs, each with 32 components: sometimes 1 million Gaussian components!! And the parameters all estimated from (a lot of) data by EM

Back to HMMs...



Output distribution:

- Single multivariate Gaussian with mean $\boldsymbol{\mu}^j$, covariance matrix $\boldsymbol{\Sigma}^j$:
$$b_j(\mathbf{x}) = p(\mathbf{x} \mid s_j) = \mathcal{N}(\mathbf{x}; \boldsymbol{\mu}^j, \boldsymbol{\Sigma}^j)$$
- M-component Gaussian mixture model:
$$b_j(\mathbf{x}) = p(\mathbf{x} \mid s_j) = \sum_{m=1}^{M} c_{jm}\mathcal{N}(\mathbf{x}; \boldsymbol{\mu}^{jm}, \boldsymbol{\Sigma}^{jm})$$

The three problems of HMMs

Working with HMMs requires the solution of three problems:

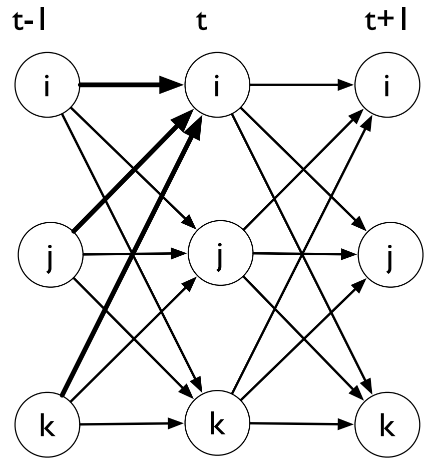
1. **Likelihood** Determine the overall likelihood of an observation sequence $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_t, \ldots, \mathbf{x}_T)$ being generated by an HMM
2. **Decoding** Given an observation sequence and an HMM, determine the most probable hidden state sequence
3. **Training** Given an observation sequence and an HMM, learn the best HMM parameters $\boldsymbol{\lambda} = \{\{a_{jk}\}, \{b_j()\}\}$

1. Likelihood: The Forward algorithm

- Goal: determine $p(\mathbf{X} \mid \boldsymbol{\lambda})$
- Sum over all possible state sequences $s_1 s_2 \ldots s_T$ that could result in the observation sequence X
- Rather than enumerating each sequence, compute the probabilities recursively (exploiting the Markov assumption)

Recursive algorithms on HMMs

Visualize the problem as a *state-time trellis*



1. Likelihood: The Forward algorithm

- Goal: determine $p(\mathbf{X} \mid \boldsymbol{\lambda})$
- Sum over all possible state sequences $s_1 s_2 \ldots s_T$ that could result in the observation sequence X
- Rather than enumerating each sequence, compute the probabilities recursively (exploiting the Markov assumption)
- *Forward probability*, $\alpha_t(s_j)$: the probability of observing the observation sequence $\mathbf{x}_1 \ldots \mathbf{x}_t$ and being in state s_j at time t:

$$\alpha_t(s_j) = p(\mathbf{x}_1, \ldots, \mathbf{x}_t, S(t) = s_j \mid \boldsymbol{\lambda})$$

1. Likelihood: The Forward recursion

- Initialization

$$\alpha_0(s_I) = 1$$
$$\alpha_0(s_j) = 0 \qquad \text{if } s_j \neq s_I$$

- Recursion

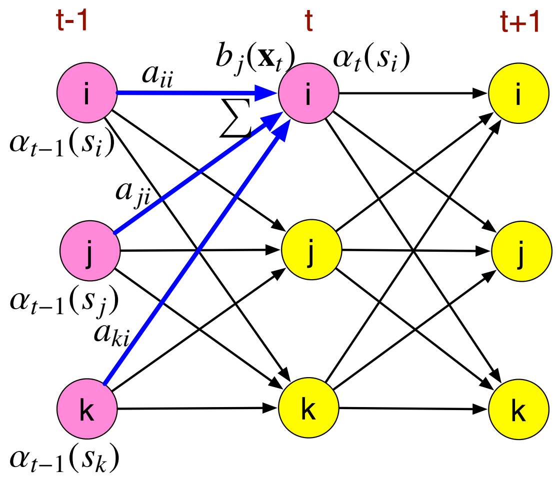
$$\alpha_t(s_j) = \sum_{i=1}^{N} \alpha_{t-1}(s_i) a_{ij} b_j(\mathbf{x}_t)$$

- Termination

$$p(\mathbf{X} \mid \boldsymbol{\lambda}) = \alpha_T(s_E) = \sum_{i=1}^{N} \alpha_T(s_i) a_{iE}$$

1. Likelihood: Forward Recursion

$$\alpha_t(s_j) = p(\mathbf{x}_1, \ldots, \mathbf{x}_t, S(t) = s_j \mid \boldsymbol{\lambda})$$



Viterbi approximation

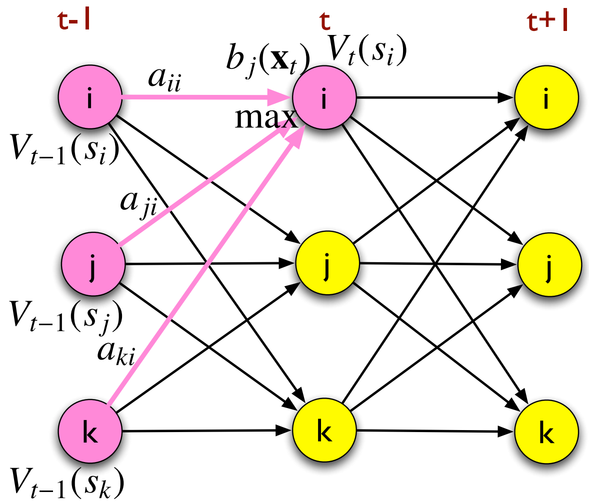
- Instead of summing over all possible state sequences, just consider the most likely
- Achieve this by changing the summation to a maximisation in the recursion:

$$V_t(s_j) = \max_i V_{t-1}(s_i) a_{ij} b_j(\mathbf{x}_t)$$

- Changing the recursion in this way gives the likelihood of the most probable path
- We need to keep track of the states that make up this path by keeping a sequence of *backpointers* to enable a Viterbi *backtrace*: the backpointer for each state at each time indicates the previous state on the most probable path

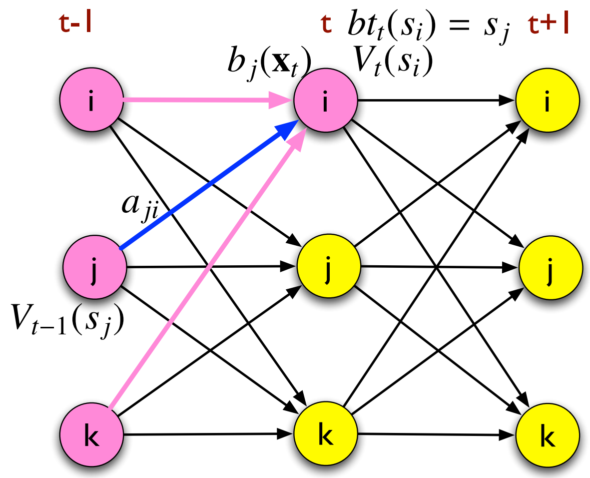
Viterbi Recursion

Likelihood of the most probable path



Viterbi Recursion

Backpointers to the previous state on the most probable path



2. Decoding: The Viterbi algorithm

- Initialization

$$V_0(s_I) = 1$$
$$V_0(s_j) = 0 \qquad \text{if } s_j \neq s_I$$
$$bt_0(s_j) = 0$$

- Recursion

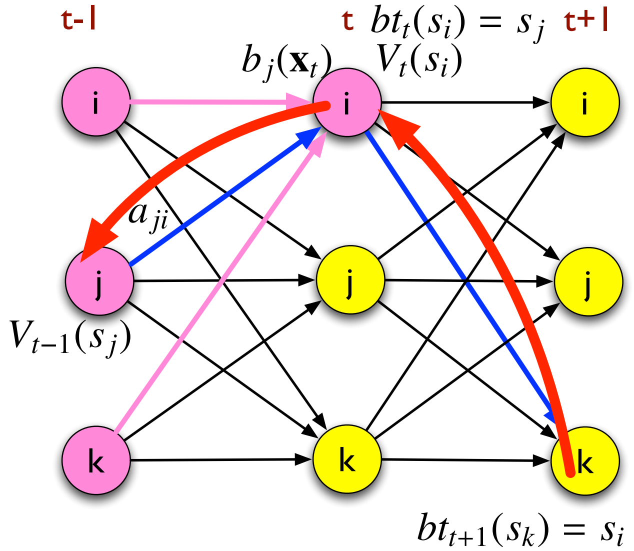
$$V_t(s_j) = \max_{i=1}^{N} V_{t-1}(s_i) a_{ij} b_j(\mathbf{x}_t)$$
$$bt_t(s_j) = \arg\max_{i=1}^{N} V_{t-1}(s_i) a_{ij} b_j(\mathbf{x}_t)$$

- Termination

$$P^* = V_T(s_E) = \max_{i=1}^{N} V_T(s_i) a_{iE}$$
$$s_T^* = bt_T(q_E) = \arg\max_{i=1}^{N} V_T(s_i) a_{iE}$$

Viterbi Backtrace

Backtrace to find the state sequence of the most probable path



3. Training: Forward-Backward algorithm

- Goal: Efficiently estimate the parameters of an HMM $\boldsymbol{\lambda}$ from an observation sequence
- Assume single Gaussian output probability distribution

$$b_j(\mathbf{x}) = p(\mathbf{x} \mid s_j) = \mathcal{N}(\mathbf{x}; \boldsymbol{\mu}^j, \boldsymbol{\Sigma}^j)$$

- Parameters $\boldsymbol{\lambda}$:
  - Transition probabilities a_{ij}:

$$\sum_j a_{ij} = 1$$

  - Gaussian parameters for state s_j: mean vector $\boldsymbol{\mu}^j$; covariance matrix $\boldsymbol{\Sigma}^j$

Viterbi Training

- If we knew the state-time alignment, then each observation feature vector could be assigned to a specific state
- A state-time alignment can be obtained using the most probable path obtained by Viterbi decoding
- Maximum likelihood estimate of a_{ij}, if $C(s_i \rightarrow s_j)$ is the count of transitions from s_i to s_j

$$\hat{a}_{ij} = \frac{C(s_i \rightarrow s_j)}{\sum_k C(s_i \rightarrow s_k)}$$

- Likewise if Z_j is the set of observed acoustic feature vectors assigned to state j, we can use the standard maximum likelihood estimates for the mean and the covariance:

$$\hat{\boldsymbol{\mu}}^j = \frac{\sum_{x \in Z_j} x}{|Z_j|}$$
$$\hat{\boldsymbol{\Sigma}}^j = \frac{\sum_{x \in Z_j} (x - \hat{\boldsymbol{\mu}}^j)(x - \hat{\boldsymbol{\mu}}^j)^T}{|Z_j|}$$

EM Algorithm

- Viterbi training is an approximation—we would like to consider *all* possible paths
- In this case rather than having a hard state-time alignment we estimate a probability
- *State occupation probability*: The probability $\gamma_t(s_j)$ of occupying state s_j at time t given the sequence of observations.
  Compare with component occupation probability in a GMM
- We can use this for an iterative algorithm for HMM training: the EM algorithm
- Each iteration has two steps:
  - E-step: estimate the state occupation probabilities (Expectation)
  - M-step: re-estimate the HMM parameters based on the estimated state occupation probabilities (Maximisation)

Backward probabilities

- To estimate the state occupation probabilities it is useful to define (recursively) another set of probabilities—the *Backward* probabilities

$$\beta_t(s_j) = p(\mathbf{x}_{t+1}, \mathbf{x}_{t+2}, \mathbf{x}_T \mid S(t) = s_j, \boldsymbol{\lambda})$$

  The probability of future observations given a the HMM is in state s_j at time t
- These can be recursively computed (going backwards in time)
  - Initialisation

$$\beta_T(s_i) = a_{iE}$$

  - Recursion

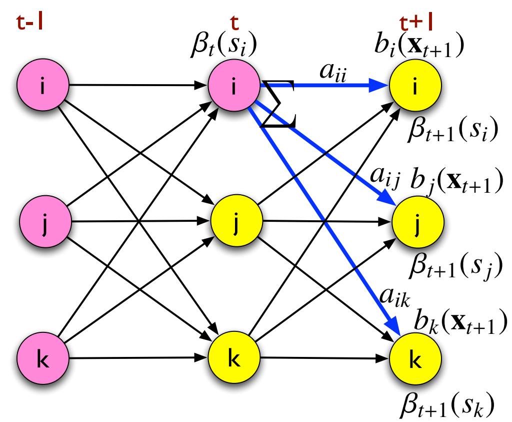
$$\beta_t(s_i) = \sum_{j=1}^{N} a_{ij} b_j(\mathbf{x}_{t+1}) \beta_{t+1}(s_j)$$

  - Termination

$$p(\mathbf{X} \mid \boldsymbol{\lambda}) = \beta_0(s_I) = \sum_{j=1}^{N} a_{Ij} b_j(\mathbf{x}_1) \beta_1(s_j) = \alpha_T(s_E)$$

Backward Recursion

$$\beta_t(s_j) = p(\mathbf{x}_{t+1}, \mathbf{x}_{t+2}, \mathbf{x}_T \mid S(t) = s_j, \boldsymbol{\lambda})$$



State Occupation Probability

- The **state occupation probability** $\gamma_t(s_j)$ is the probability of occupying state s_j at time t given the sequence of observations
- Express in terms of the forward and backward probabilities:

$$\gamma_t(s_j) = P(S(t) = s_j \mid \mathbf{X}, \boldsymbol{\lambda}) = \frac{1}{\alpha_T(s_E)} \alpha_t(j) \beta_t(j)$$

  recalling that $p(\mathbf{X}|\boldsymbol{\lambda}) = \alpha_T(s_E)$
- Since

$$\begin{aligned}
\alpha_t(s_j)\beta_t(s_j) &= p(\mathbf{x}_1, \ldots, \mathbf{x}_t, S(t) = s_j \mid \boldsymbol{\lambda}) \\
&\quad p(\mathbf{x}_{t+1}, \mathbf{x}_{t+2}, \mathbf{x}_T \mid S(t) = s_j, \boldsymbol{\lambda}) \\
&= p(\mathbf{x}_1, \ldots, \mathbf{x}_t, \mathbf{x}_{t+1}, \mathbf{x}_{t+2}, \ldots, \mathbf{x}_T, S(t) = s_j \mid \boldsymbol{\lambda}) \\
&= p(\mathbf{X}, S(t) = s_j \mid \boldsymbol{\lambda})
\end{aligned}$$

$$P(S(t) = s_j \mid \mathbf{X}, \boldsymbol{\lambda}) = \frac{p(\mathbf{X}, S(t) = s_j \mid \boldsymbol{\lambda})}{p(\mathbf{X}|\boldsymbol{\lambda})}$$

Re-estimation of Gaussian parameters

- The sum of state occupation probabilities through time for a state, may be regarded as a "soft" count
- We can use this "soft" alignment to re-estimate the HMM parameters:

$$\hat{\boldsymbol{\mu}}^j = \frac{\sum_{t=1}^{T} \gamma_t(s_j) x_t}{\sum_{t=1}^{T} \gamma_t(s_j)}$$

$$\hat{\boldsymbol{\Sigma}}^j = \frac{\sum_{t=1}^{T} \gamma_t(s_j)(x_t - \hat{\boldsymbol{\mu}}^j)(x - \hat{\boldsymbol{\mu}}^j)^T}{\sum_{t=1}^{T} \gamma_t(s_j)}$$

Re-estimation of transition probabilities

- Similarly to the state occupation probability, we can estimate $\xi_t(s_i, s_j)$, the probability of being in s_i at time t and s_j at $t+1$, given the observations:

$$\begin{aligned}\xi_t(s_i, s_j) &= P(S(t) = s_i, S(t+1) = s_j \mid \mathbf{X}, \boldsymbol{\lambda}) \\ &= \frac{P(S(t) = s_i, S(t+1) = s_j, \mathbf{X} \mid \boldsymbol{\lambda})}{p(\mathbf{X}|\Lambda)} \\ &= \frac{\alpha_t(s_i) a_{ij} b_j(\mathbf{x}_{t+1}) \beta_{t+1}(s_j)}{\alpha_T(s_E)}\end{aligned}$$

- We can use this to re-estimate the transition probabilities

$$\hat{a}_{ij} = \frac{\sum_{t=1}^{T} \xi_t(s_i, s_j)}{\sum_{k=1}^{N} \sum_{t=1}^{T} \xi_t(s_i, s_k)}$$

Pulling it all together

- Iterative estimation of HMM parameters using the EM algorithm. At each iteration
  - E step For all time-state pairs
    1. Recursively compute the forward probabilities $\alpha_t(s_j)$ and backward probabilities $\beta_t(j)$
    2. Compute the state occupation probabilities $\gamma_t(s_j)$ and $\xi_t(s_i, s_j)$
  - M step Based on the estimated state occupation probabilities re-estimate the HMM parameters: mean vectors $\boldsymbol{\mu}^j$, covariance matrices $\boldsymbol{\Sigma}^j$ and transition probabilities a_{ij}
- The application of the EM algorithm to HMM training is sometimes called the Forward-Backward algorithm

Extension to a corpus of utterances

- We usually train from a large corpus of R utterances
- If $\mathbf{x}_t^r$ is the tth frame of the rth utterance $\mathbf{X}^r$ then we can compute the probabilities $\alpha_t^r(j)$, $\beta_t^r(j)$, $\gamma_t^r(s_j)$ and $\xi_t^r(s_i, s_j)$ as before
- The re-estimates are as before, except we must sum over the R utterances, eg:

$$\hat{\boldsymbol{\mu}}^j = \frac{\sum_{r=1}^{R} \sum_{t=1}^{T} \gamma_t^r(s_j) x_t^r}{\sum_{r=1}^{R} \sum_{t=1}^{T} \gamma_t^r(s_j)}$$

Extension to Gaussian mixture model (GMM)

- The assumption of a Gaussian distribution at each state is very strong; in practice the acoustic feature vectors associated with a state may be strongly non-Gaussian
- In this case an M-component Gaussian mixture model is an appropriate density function:

$$b_j(\mathbf{x}) = p(\mathbf{x} \mid s_j) = \sum_{m=1}^{M} c_{jm} \mathcal{N}(\mathbf{x}; \boldsymbol{\mu}^{jm}, \boldsymbol{\Sigma}^{jm})$$

  Given enough components, this family of functions can model any distribution.
- Train using the EM algorithm, in which the component estimation probabilities are estimated in the E-step

EM training of HMM/GMM

- Rather than estimating the state-time alignment, we estimate the component/state-time alignment, and component-state occupation probabilities $\gamma_t(s_j, m)$: the probability of occupying mixture component m of state s_j at time t
- We can thus re-estimate the mean of mixture component m of state s_j as follows

$$\hat{\boldsymbol{\mu}}^{jm} = \frac{\sum_{t=1}^{T} \gamma_t(s_j, m) x_t}{\sum_{t=1}^{T} \gamma_t(s_j, m)}$$

  And likewise for the covariance matrices (mixture models often use diagonal covariance matrices)
- The mixture coefficients are re-estimated in a similar way to transition probabilities:

$$\hat{c}_{jm} = \frac{\sum_{t=1}^{T} \gamma_t(s_j, m)}{\sum_{\ell=1}^{M} \sum_{t=1}^{T} \gamma_t(s_j, \ell)}$$

Doing the computation

- The forward, backward and Viterbi recursions result in a long sequence of probabilities being multiplied
- This can cause floating point *underflow* problems
- In practice computations are performed in the log domain (in which multiplies become adds)
- Working in the log domain also avoids needing to perform the exponentiation when computing Gaussians

Summary: HMMs

- HMMs provide a generative model for statistical speech recognition
- Three key problems
  1. Computing the overall likelihood: the Forward algorithm
  2. Decoding the most likely state sequence: the Viterbi algorithm
  3. Estimating the most likely parameters: the EM (Forward-Backward) algorithm
- Solutions to these problems are tractable due to the two key HMM assumptions
  1. Conditional independence of observations given the current state
  2. Markov assumption on the states

References: HMMs

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- Renals and Hain (2010). "Speech Recognition", *Computational Linguistics and Natural Language Processing Handbook*, Clark, Fox and Lappin (eds.), Blackwells.