# Coupled Van Der Pol benchmark

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## 1 Introduction

The Van der Pol model is considered a classical prototype of a self-excited oscillator. It has been used to model oscillations in a wide variety of applications such as biological rhythms, heartbeats, chemical oscillations, electrical circuits and circadian rhythms. The study of coupled oscillators provides information on emergent properties of the coupled system, such as synchronization, clustering, oscillation death, oscillation modes and stability [1], [2].

In ARCH-COMP 2019 we considered a single Van der Pol oscillator. In this note we propose the natural extension of the model to several coupled oscillators. In [1], [2], a system of N coupled Van der Pol oscillators in a ring configuration is presented, see Fig. 1. Reachability results for the case of $N = 2$ oscillator have been presented in [3].

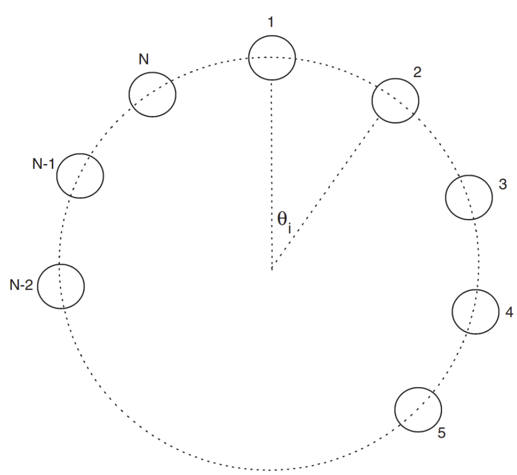


Figure 1: Ring of coupled Van der Pol oscillators

In the rest of this note we consider the case of $N = 2$ oscillators with coupling constant $b > 0$ and stiffness constant $\mu > 0$. The dynamics is given by the following differential equations:

$$\begin{cases} x_1' &= y_1 \\ y_1' &= \mu(1 - x_1^2)y_1 - x_1 + b(x_2 - x_1) \\ x_2' &= y_2 \\ y_2' &= \mu(1 - x_2^2)y_2 - x_2 + b(x_1 - x_2) \end{cases} \tag{1}$$

## 2 Reachability settings

We fix the coupling constant to $b = 1$. Recall that the higher the value of $\mu$, the problem is more stiff. We propose two scenarios:

1. Case $\mu = 1$ and we set the initial condition $x_1(0) \in [1.25, 1.55]$, $y_1(0) \in [2.35, 2.45]$, $x_2(0) \in [1.25, 1.55]$, $y_2(0) \in [2.35, 2.45]$. The unsafe set is given by $y \geq \alpha$ for a time horizon of $[0, 7]$.

2. Case $\mu = 2$ and we set the initial condition $x_1(0) \in [1.55, 1.85]$, $y_1(0) \in [2.35, 2.45]$, $x_2(0) \in [1.55, 1.85]$, $y_2(0) \in [2.35, 2.45]$. The unsafe set is given by $y \geq \beta$ for a time horizon of $[0, 8]$.

Note that in the second scenario, the time horizon is $T = 8.0$ is used because 7.0 is not sufficient for the oscillator to make a complete loop. We suggest using the following values for the safety constraints:

- For the $\mu = 1$ case, use $\alpha = 2.75$. This is the same value used in ARCH-COMP 2019 for one oscillator.

- For the $\mu = 2$ case, use $\beta = 4.05$. However, it would be interesting to know if a tool can prove whether the tighter requirement $\beta = 4.0$ is met (or use a value smaller than 4.05).

# 3 Preliminary results

See Figs. 2 and 3 for the cases $\mu = 1$ and $\mu = 2$ respectively.

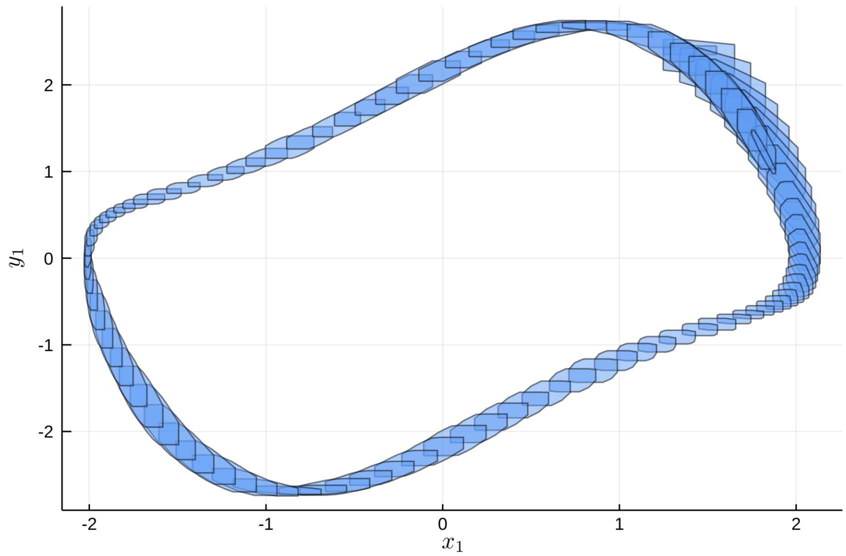


Figure 2: Case $\mu = 1$.

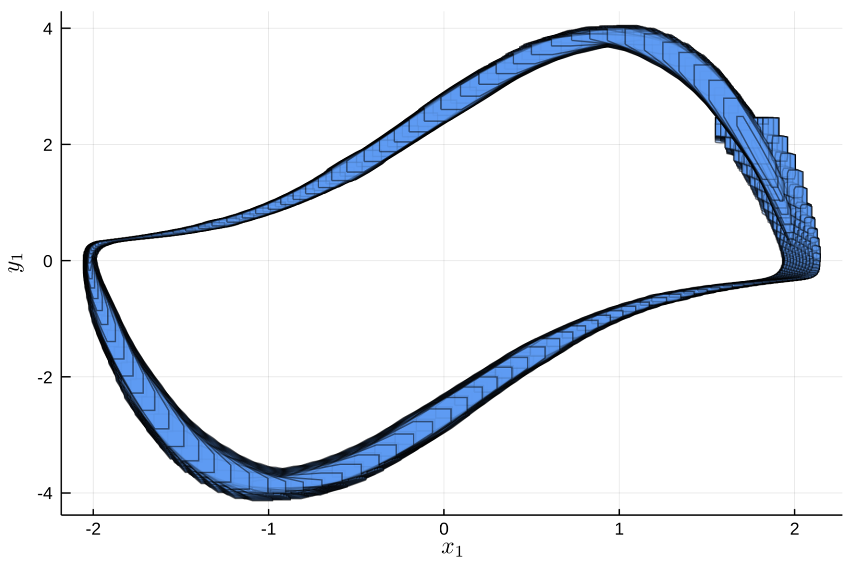


Figure 3: Case $\mu = 2$.

# References


1. Barron, Miguel Angel. Stability of a ring of coupled van der Pol oscillators with non-uniform distribution of the coupling parameter. Journal of applied research and technology 14.1 (2016): 62-66.

2. Barron, Miguel A., and Mihir Sen. Dynamic behavior of a large ring of coupled self-excited oscillators. Journal of Computational and Nonlinear Dynamics 8.3 (2013).

3. Coupled Van der Pol oscillator. HyPro benchmarks. The Flow* model file is available for download in this link.