Automation and the Future of Work: Assessing the Role of Labor Flexibility

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JOB MARKET PAPER

This Version: October 8, 2020

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Abstract

We study the economic incentives for automation when labor and machines are perfect substitutes. Labor may still be employed in production, even when it is a costlier input than robots on a productivity-adjusted basis. This occurs if firms face uninsurable idiosyncratic risk, adjusting the stock of machines is costly, and workers can be hired and fired quickly enough. Even though labor survives, jobs become less stable, as workers are hired in short-lived bursts to cope with shocks. We calibrate a general equilibrium, multi-industry version of our model to match data on robot adoption in US manufacturing sectors, and use it to compute the employment and labor share consequences of progress in automation technology. A fall in the relative price of robots leads to relatively few jobs losses, while reductions in adjustment costs, or improvements in relative robot productivity, can be far more disruptive. The model-implied semi-elasticity of aggregate employment to robot penetration ranges between 0.01% and 0.1%, depending on the underlying source of increased robot adoption. Adding reduced-form hiring and firing costs to our benchmark model reveals that the scare of automation is justified when regulations impose substantial rigidity on employment relations.

Keywords: Automation, Investment, Capital-Labor Substitution, Factor Shares, Tasks, Technological Change.

JEL Classification: O33, J23, E22, E23, E24.

We thank Daron Acemoglu, George-Marios Angeletos, David Autor, Martin Beraja, Ricardo Caballero, Tommaso Monacelli, Pascual Restrepo, Alp Simsek, Iván Werning, Joel Flynn, Clémence Idoux, Nicola Rosaia, Charles Serfaty, Martina Uccioli, and Macro Lunch participants at MIT Economics for helpful discussions and suggestions. We are grateful to Daron Acemoglu and Pascual Restrepo for sharing the IFR-SIC crosswalk used for our quantitative analysis.

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1 Introduction

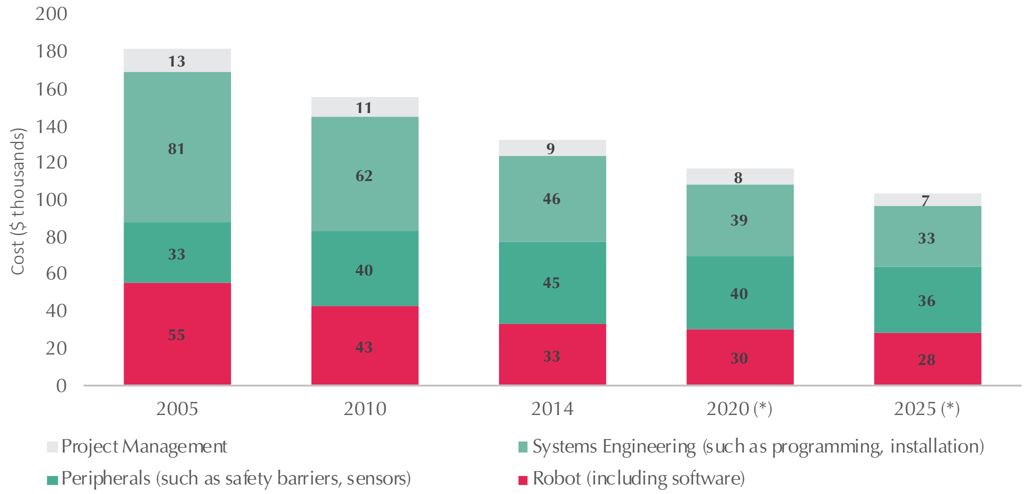
Over the last few years, the progress in robotics, software engineering, and AI, coupled with the secular decline in the labor share of output, has sparked a discussion on whether machines will progressively replace humans in performing tasks (Acemoglu and Restrepo, 2018c; Autor, 2015; Autor and Salomons, 2018; Berg et al., 2018; Graetz and Michaels, 2018; Sachs and Kotlikoff, 2012). In economic terms, the full displacement of humans requires perfect substitution between robots and workers within tasks; otherwise human labor will always be needed. In this paper, we investigate the long-run consequences of automation under this deliberately extreme assumption, to provide an upper bound for potential employment losses.

The recent literature has stressed a number of reasons why robots and human workers might not be perfect substitutes. Autor (2015) discusses the ability of workers to perform the multiple and differentiated tasks that typically constitute a job. Acemoglu and Restrepo (2018b,c,d) highlight the importance of human comparative advantage in carrying out specific tasks, a point that is also made by Berg et al. (2018) and Graetz and Michaels (2018). Finally, there might be some non-routine occupations, or categories of workers that are poised to benefit from automation (see again Berg et al. (2018) and Sachs and Kotlikoff (2012); Sachs et al. (2015)). While we acknowledge the importance of these mechanisms, we abstract from them to focus on tasks where workers and robots can indeed be perfect substitutes. In this sense, we describe a worst-case scenario for workers employed in low-skilled, routine occupations.

Our baseline model follows the spirit of the task framework proposed by Acemoglu and Restrepo (2018b), and focuses on a firm choosing whether to automate a single task where humans and robots are perfect substitutes. We describe how human labor can survive thanks to its *flexibility as an input in production*, which we identify as its distinctive comparative advantage.[1] Labor survives alongside robots under three main assumptions that set our framework apart from the existing literature. First, we introduce demand shocks by assuming that firms face uninsurable idiosyncratic risk. Shocks create a source of demand for flexible inputs that can readily adjust to a volatile environment. Second, we model robots as having capital-like features that impair their rapid deployment in production, in contrast to the standard assumption of a rental market that allows for immediate adjustments of the robot stock. Finally, we assume that employees can be hired and fired more easily than robots can be bought, installed, and sold. Accordingly, firms respond to positive shocks using the more flexible factor, but generally employ *both* workers and machines. Notably, firms employ both workers and machines even if labor and robots are perfect substitutes, and even if robots are cheaper on a productivity-adjusted basis. Our model therefore presents a context where imperfect long-run sub-

[1] Tesla's recent history is a real-world example of the mechanism we describe. After months of unsuccessful attempts at scaling up the production of the Tesla Model 3 by radical automation, Elon Musk tweeted: "*Yes, excessive automation at Tesla was a mistake. To be precise, my mistake. Humans are underrated.*" Installing and adapting robots to the various tasks turned out to be harder than expected, pushing the company to meet its demand backlog by hiring thousands of (human) workers. See the Forbes coverage in Muller (2018).

Figure 1: Total System Cost of a Typical Spot-Welding Robot in the U.S. Automotive Industry



Authors' elaboration on data from Sirkin et al. (2015). Values are expressed in nominal terms. Asterisks denote BCG projections for the components of interest.

stitutability between robots and workers *arises endogenously*. This result relies on the joint presence of adjustment costs and revenue risk, as neither of these ingredients in isolation would produce a finite long-run elasticity of substitution between robot and workers. Our model also shows that the survival of production-line employment comes at the cost of reduced job stability; simulated time series from our model show that, in an automated world, labor is only hired in short-lived "bursts" to cope with sudden increases in the desired production scale.

Our modeling choices are informed by data on robot costs. Figure 1 reports data from Sirkin et al. (2015) detailing the main cost items for the setup of a spot-welding robot in the U.S. automotive industry. The figure highlights three main facts that feature in our model and calibration. First, in line with the broader evidence in International Federation of Robotics (2017) and Korus (2019), the purchase price of robots has been trending down, and it is projected to keep doing so in the future. Second, purchase costs represent but a small fraction of the total cost of a robotic system, which is mostly made up of installation-related costs, reflected by the adjustment costs in our model. Third, the nature of these adjustment costs, and particularly those related to programming and "peripherals", suggest that robot systems have a firm-specific component that might affect their redeployment to different contexts.

In order to gauge the quantitative implications of our theoretical findings, we develop a multi-industry, general equilibrium version of our baseline model, which we calibrate to match data on the adoption of robots between 2010 and 2014. This exercise reveals that, in line with the evidence in Figure 1, robot adjustment costs might indeed be sizable. This result stems from the low aggregate elasticity of robot penetration to purchase prices observed in the data. Under our calibration, we establish that even a dramatic reduction in the relative price of robots causes only a modest fall in aggregate employment, with changes in the technical substitutability and flexibility of robots posing a more substantial threat. Notably, general equilibrium effects mitigate employment losses, as they imply a fall in the equilibrium wage. In line with quantitative findings from the empirical literature (Acemoglu and Restrepo, 2018d), the semi-elasticity of aggregate employment to robot penetration implied by our calibrated model ranges between 0.01% and 0.1%, depending on the underlying source of increased robot adoption.

We show that the mechanism of the baseline model is robust to using linear or fixed costs of adjustment. This allows us to work with a more tractable model without loss of generality in our conclusions. To clarify how the comparative advantage of humans arises, and to obtain analytical results, our benchmark model features frictionless hiring and firing. We depart from this assumption in an extension. Labor adjustment costs, which we interpret as reduced-form market frictions, dampen the flexibility comparative advantage, and increase the long-run displacement of production-line workers. Our model suggests that removing strict employment protection measures could be an effective policy to safeguard unskilled jobs in the long run, counter to what intuition might suggest. However, we also show that more rigid employment regulations can slow down the substitution of workers with machines, depending on the speed of transition to lower robot prices. Our findings highlight that policymakers face a trade-off between long-run and short-run employment outcomes. The rigid-labor extension also shows that when the transition to lower robot prices is gradual, stricter labor market regulations can induce firms to anticipate the adoption of robots to smooth out workforce adjustment costs. This result speaks to empirical evidence suggesting that higher unionization is associated with higher robot penetration at the current stage (Acemoglu and Restrepo, 2018a).

Section 2 develops a simple model to build intuition. Section 3 extends the analysis to an infinite horizon. Section 4 presents the multi-sector, general equilibrium model and our calibration. Section 5 illustrates the robustness to alternative cost specifications, and the extension featuring labor market rigidity. Section 6 concludes.

1.1 Related Literature

Our work relates to two distinct strands of literature. The first deals with automation and its long-run impact on employment and factor shares. The second relates to modeling investment with risk and imperfect reversibility.

The main contributions in the automation strand can be classified into two groups, based on the nature of technological change. In particular, a number of papers view automation as a form of factor-augmenting innovation. Furthest from our framework is Bessen (2019), who models technological progress in automation as labor-augmenting. Other papers, like Sachs and Kotlikoff (2012), Sachs et al. (2015), Nordhaus (2015), and Berg et al. (2018) see innovation in robotics as a form of capital-augmenting technology. Progress in automation can then replace workers according to the elasticity of substitution between labor and capital. Specific modeling assumptions in most of these papers put technological bounds to the degree of substitutability. Bessen (2019) prevents substitution altogether; Sachs and Kotlikoff (2012) assume that robot capital is complementary to old generation workers in an OLG framework; Berg et al. (2018) assume that robots enter in a CES aggregate with workers, and in most scenarios they assume that the elasticity of substitution is finite. In other instances, this literature focuses on scenarios where full substitution is possible: in Sachs et al. (2015), the whole economy can instantly convert to full automation, and adopt a technology linear in robot capital; Nordhaus (2015) assumes that the productivity of capital can grow without bounds relative to labor, until a singularity is reached and production technology asymptotes to a linear technology in information capital; Berg et al. (2018) consider a scenario where robots are perfectly substitutable with humans in all tasks, and also find that labor can end up being fully displaced. All the factor-augmenting strand either imposes explicit limits on the extent of automation through technological assumptions, or otherwise concludes in favor of a long-run demise of labor, with a factor share falling to zero.

Our stance is decidedly closer to the task-based approach adopted by Acemoglu and Restrepo (2018b,c) and Graetz and Michaels (2018), in that we posit that labor and robots are perfect substitutes within each task. We believe that a model attempting to understand the consequences of new technologies should highlight their nature as labor-replacing. As pointed out by Acemoglu and Restrepo (2018b), "the expansion of the set of tasks that can be produced by machines [...] always reduces the labor share and it reduces labor demand and the equilibrium wage unless the productivity gains from automation are sufficiently large." Acemoglu and Restrepo (2018c) assume that some tasks are inherently human and study the effects of automation on factor shares along a growth path where such tasks can be created. The existence of a balanced growth path relies on a race between the expansion of automated versus non-automated tasks, where robots cannot be profitably employed. By contrast, Graetz and Michaels (2018) develop a model where machines can only be used in an exogenously determined subset of tasks. Unlike these frameworks, we choose to focus on a single task where labor can be fully substituted away, thereby excluding any technological limit to automation.

Regardless of the specific view of automation, the theoretical literature described above has two common features. First, all these contributions assume a rental market for robots. Second, firms operate in a deterministic environment. These assumptions allow for bang-bang solutions in favor of either robots or labor when perfect substitution is possible. We believe that robots constitute a form of capital investment, characterized by substantial rigidity and reallocation frictions, and accordingly connect to the long and rich literature on investment by adding risk and adjustment costs that hinder robot reallocation. The seminal contributions by Abel (1983), Pindyck (1988, 1991), Caballero (1991), and Abel and Eberly (1996, 1997) have highlighted the crucial role of uncertainty and rigidity in preventing a quick adjustment of capital to its desired optimal scale, contrary to the neoclassical framework (Jorgenson, 1963). It is outside the scope of this paper to review this literature extensively, but it is important to recall some key contributions on the sign of uncertainty on investment. An excellent summary of the above literature, and a resolution of the long controversy on this topic, is provided by Caballero (1991), who shows that the stochasticity of returns depresses investment when the returns function features enough concavity to discourage the firm from investing in a relatively fixed factor, even in the face of increased upside risk. Indeed, with decreasing returns, and even if the profit function is convex in prices, the firm has an optimal scale for each price and suffers a relevant penalty from being mis-sized. This result provides fundamental theoretical guidance for our findings on the effect of risk on automation, since we adopt a decreasing-returns-to-scale production function.

Our theoretical model is closest to Abel and Eberly (1996), in that we assume decreasing returns to scale, perfectly flexible labor, and solve our model in continuous time. However, we depart from the standard investment literature by adopting a production function that features perfect substitutability of labor and capital. Our main contribution consists in bridging the literature on automation with that on risky investment under imperfect reversibility, thereby capturing the distinctive characteristics of robots as a labor-substituting form of capital. Other fundamental references are Dixit and Pindyck (1994) and Stokey (2009), which operate in a similar framework and provide important theoretical foundations for many of our results. Finally, we rely on Achdou et al. (2017, 2014) for our numerical solution method.

2 Two-Period Model

We start by presenting a simple two-period model to clarify the mechanism at play. This simple model can be seen as a snapshot of the more complex, fully-dynamic problem presented in the next section. In both periods, there is a continuum measure one of firms endowed with the same production technology. In period 0, each firm starts with a given robot stock R_0, and it observes an idiosyncratic revenue-shifting shock z_0, which we interpret as a productivity shock or a preference shock for differentiated goods produced by the individual firms. Then, each firm produces and invests for the following period, solving

$$V_0(R_0, z_0) = \max_{R_1 \geq 0} \Pi(R_0, z_0) - p_R(R_1 - R_0) - \frac{\psi_R}{2}(R_1 - R_0)^2 + \beta \mathbb{E}[V_1(R_1, z_1) \mid z_0],$$

where the expectation is taken over the values of a stochastic revenue shifter $z_1 \sim F(z_1 \mid z_0)$, whose distribution is independent of R_0. The firm can buy or sell robots at a list price p_R. Regardless of whether it buys or sells robots, the firm faces a convex cost for any non-zero level of investment. In addition to analytical tractability, we focus on convex costs for two reasons. First, when buying robots, we envision that the firm will have to adjust the working environment or the production process to make it navigable by automated machines. This might involve sizable costs that are more than proportional to the upfront investment (Sirkin et al., 2015). Second, it is also reasonable to think that a seller will have to face similar adjustment costs to dismantle a robot system in place, related for example to the removal of barriers and sensors from the workspace.[2,3]

In period 1, the stochastic revenue shifter z_1 realizes, the firm produces and the world ends. Thus,

$$V_1(R,z) = \Pi(R_1, z_1).$$

For both periods, we define

$$\Pi(R,z) \equiv \max_{L\geq 0, 0\leq u\leq 1} Q(L,R,u,z) - wL - muR,$$

as the *operating profit function*. We let u denote the utilization rate of the existing robot stock, m the flow cost of robots, and w the flow wage. Both these flow rates are constant and known at the beginning of time 0. We now discuss the properties of the operating profit function, which is an important building block of our model.

2.1 The Operating Profit Function

The production function is assumed to be:

$$Q(L,R,u,z) = z\left(\Gamma L + (1-\Gamma)\, uR\right)^{\theta},$$

where $\Gamma = \text{MRTS}_{LR}/(1+\text{MRTS}_{LR})$, a normalized marginal rate of technical substitution that controls the fixed rate at which robots and labor can be technologically substituted for each other. In this sense, Γ captures how easily a task can be automated. An important point to note is that labor and robots are *perfect substitutes*. We make this extreme assumption to abstract from forms of labor-capital complementarity that the literature has already shown act as a backstop to full automation. It is worth noting here that the substitutability assumption sets robots apart from traditional capital, which is usually taken as a complement to labor. By contrast, we see automation as enabling *perfect* substitution, and complete displacement of, e.g., low-skilled manufacturing workers.

The parameter $\theta \in (0,1)$ captures *decreasing returns to scale*, which can be motivated by a downward sloping demand curve for the differentiated product of the firm, or by the presence of a fixed factor of production. Note that this unmodeled factor could encompass workers that are complemented by robots, as well as more traditional complementary capital. Finally, decreasing returns to scale allow for a well-defined notion of desired firm size, which plays a crucial role in our focus

[2] In practice, it is not uncommon for firms to rent advanced or complex machines. Accordingly, R_t could be interpreted as the size of leasing obligations that a firm has, which often involve penalties for early termination.

[3] Our results are not qualitatively changed if we assume irreversible investment coupled with convex adjustment costs for scaling up.

on idiosyncratic revenue volatility. The parameter Ω, which we will refer to as *flow labor savings*, denotes the productivity-adjusted flow cost savings from using robots instead of labor to produce a unit of output:

$$\Omega \equiv \frac{1-\Gamma}{\Gamma} w - m.$$

We assume that $\Omega > 0$ to make the problem interesting, as there cannot be any automation if the less flexible factor has larger flow costs. Under this assumption, as stated formally in Proposition 1, our model features full automation of the task if the firm's revenues are deterministic and there are no adjustment costs. The solution of the static problem, detailed in the appendix, yields

$$\Pi(R,z) = \begin{cases} (1-\theta)\, pz\,(1-\Gamma)^{\theta}\, \bar{R}(z)^{\theta} + \left(\frac{(1-\Gamma)w}{\Gamma} - m\right) R & R \leq \bar{R}(z) \\ pz(1-\Gamma)^{\theta} R^{\theta} - mR & \hat{R}(z) \geq R > \bar{R}(z) \\ pz(1-\Gamma)^{\theta} \left(\hat{R}(z)\right)^{\theta} - m\hat{R}(z) & \hat{R}(z) < R, \end{cases}$$

where

$$\bar{R}(z) \equiv \frac{1}{1-\Gamma} \left(\frac{pz\theta\Gamma}{w} \right)^{\frac{1}{1-\theta}}, \qquad \hat{R}(z) \equiv \frac{1}{1-\Gamma} \left(\frac{pz\theta(1-\Gamma)}{m} \right)^{\frac{1}{1-\theta}}$$

are the *full automation cutoff* and the *optimal rental-market scale*, respectively. Note that decreasing returns to scale imply that the firm has a desired size associated with each value of the revenue shifter z. If a rental market for robots existed, the firm could readily adjust its robot stock, and, given that flow labor savings are positive, it would choose $\hat{R}(z)$ and use no labor at all. However, we assume that robots are a state variable, therefore the firm will expand (using labor) or shrink (using utilization) to get as close as possible to its desired size. We also define $\hat{z}(R)$ and $\bar{z}(R)$ as the inverses of the above functions. If the revenue shock is small enough, $z < \hat{z}(R)$, the firm will have a robot stock large enough to produce at the optimal size without hiring labor, and it will use the utilization margin to operate below capacity and achieve $\Pi\left(\hat{R}(z), z\right)$. If instead the realization of z is large enough relative to the installed robot stock, $z \geq \bar{z}(R)$, the firm will adjust using labor. Given z, the firms uses labor to adjust if the installed robot stock is lower than the full automation cutoff $\bar{R}(z)$. Indeed, decreasing returns to scale cause the marginal product of labor,

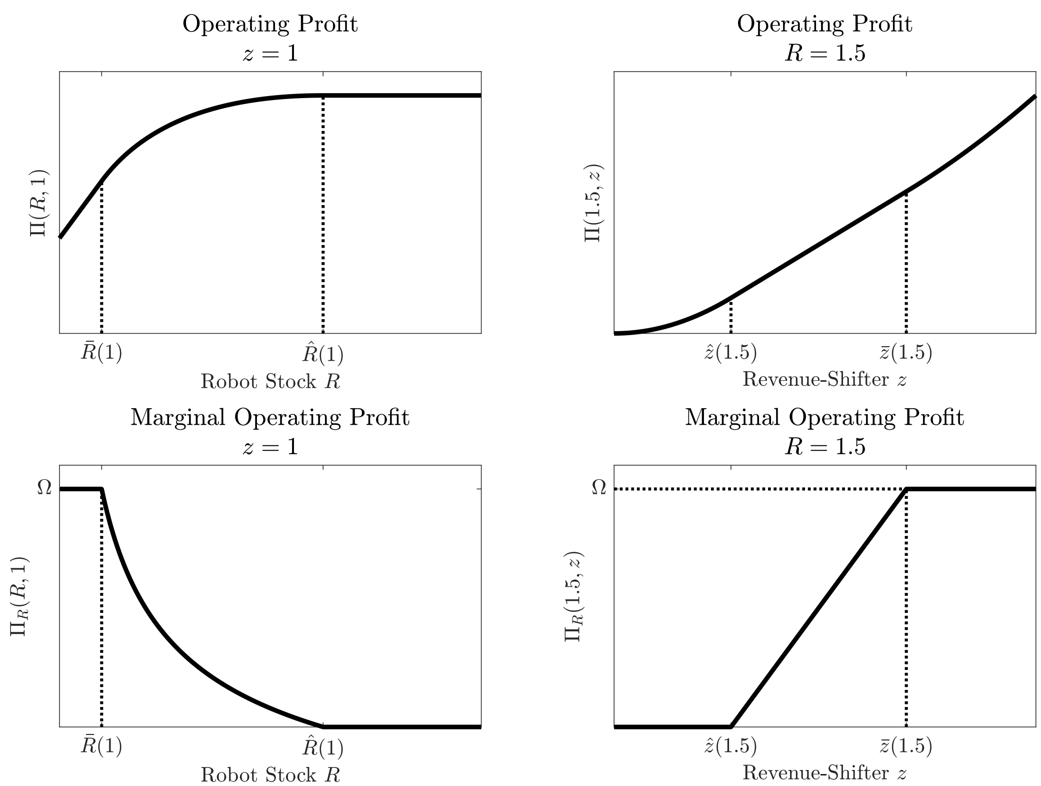
$$\mathrm{MP}_L(L,R) \equiv \theta pz\Gamma(\Gamma L + (1-\Gamma)R)^{\theta-1},$$

to be decreasing in R, and to drop below the wage rate w for any $L > 0$ if $R > \bar{R}(z)$. Static labor maximization implies that the firm will hire labor to achieve revenues $pz(1-\Gamma)^{\theta}\, \bar{R}(z)^{\theta}$. In this region, the optimal labor policy is given by

$$L^{\star}(R,z) = \frac{1-\Gamma}{\Gamma} \left[\bar{R}(z) - R \right].$$

Finally, for intermediate realizations of z, the firm fully utilizes its installed robot stock without hir-

Figure 2: Operating Profit and Marginal Operating Profit.



ing any labor. We sketch the solution in Figure 2.

The above solution implies that, for a given revenue shock z, marginal operating profits are constant at Ω for $R \leq \bar{R}(z)$. In this region, each additional unit of robots reduces the number of labor hires, $((1-\Gamma)/\Gamma)(\bar{R}(z) - R)$, needed to achieve the same optimal level of revenues, $p z (1-\Gamma)^{\theta} \bar{R}(z)^{\theta}$. In the intermediate region, marginal operating profits decrease in R. Finally, when $R \geq \hat{R}(z)$, marginal operating profits are 0, as the firm turns off any additional robot received in order to achieve the optimal scale $\hat{R}(z)$. The following Lemma summarizes the properties of the operating profit function. All omitted proofs are reported in the appendices.

Lemma 1 (**Properties of** $\Pi(R,z)$)**.** *If labor savings $\Omega > 0$, the operating profit function, $\Pi(R,z)$, is weakly increasing in R for all z and in z for all R, weakly concave in R for all z, and weakly convex in z for all R. Moreover, the marginal operating profit, $\Pi_R(R,z)$, is bounded from above by Ω and from below by 0, weakly decreasing in R, and weakly increasing in z.*

2.2 Properties of the Solution

The solution of the investment problem at time 0 is characterized by the following first order condition for the optimal robot stock $R_1^\star$:

$$\psi_R(R_1^\star - R_0) + p_R = \beta \mathbb{E}\left[\Pi_R\left(R_1^\star, z_1\right) \mid z_0\right],$$

with the usual interpretation that marginal costs and expected discounted benefits are equalized.[4] The following propositions clarify our main mechanism.

Proposition 1 (**Conditions for Full Automation in Period 1**). *Suppose that $p_R < \beta\Omega$, and that there is no uncertainty regarding the revenue shock z, that is, $z_t = z^d$ for all t. Then, if $\psi_R = 0$, the task is entirely produced by robots in period 1, and no labor is employed.*

Proof. If $\psi_R = 0$ and the price level is constant at z^d, the first order condition for the choice of R_1 reduces to

$$p_R = \beta \Pi_R \left(R_1^\star, z^d\right).$$

By Lemma 1 the right hand side is decreasing in R_1 and equal to $\beta\Omega$ if and only if $R_1 = \bar{R}(z^d)$, the full automation threshold. It immediately follows that, if $p_R < \beta\Omega$ then $R_1^\star > \bar{R}(z^d)$, so the task is fully automated. □

This proposition illustrates that, with labor savings and without risk or adjustment costs, our model features full automation of the task. As the firm knows which output level it will need to produce tomorrow and forever, it can just invest in a stock of capital that is sufficient to fully replace labor, thus saving on this expensive factor. The following proposition shows that this result collapses once idiosyncratic risk and adjustment costs are present. In this instance, the rigidity of the robot stock in responding to shocks makes hiring perfectly flexible labor profitable.

Proposition 2 (**No Aggregate Full Automation in Period 1**). *Suppose that $F(z_1 \mid z_0)$ has unbounded support on $[0,\infty)$. Denote by $L^\star(R,z)$ the choice of labor that maximizes operating profits given a robot stock R and a revenue-shifter z. Then, for each $p_R \geq 0$, $\psi_R > 0$,*

$$L_1^d \equiv \int_0^\infty L^\star(R_1, z_1)\mathrm{d}G_1(R_1, z_1) > 0.$$

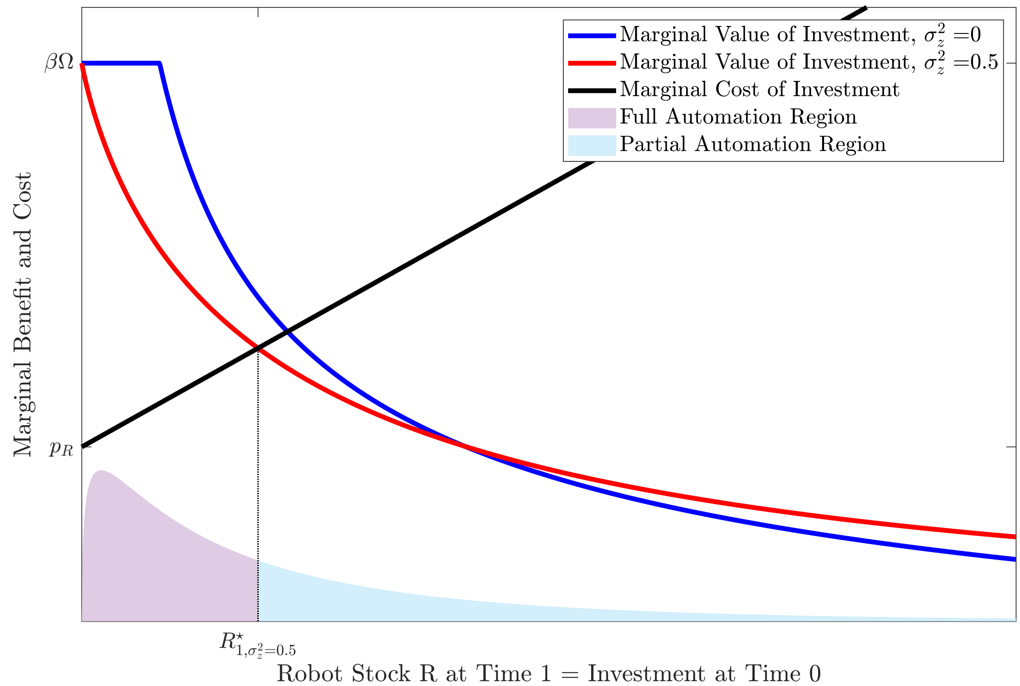
Given its installed robot stock R_1, each firm can always receive a revenue-shifting shock $z > \bar{z}(R_1)$ large enough to induce it to hire labor. Therefore, the aggregate labor demand in period 1 will be strictly positive. Figure 3 depicts the optimal choice of the firm, together with the distribution of the full automation cutoffs $\bar{R}(z)$, assuming that all firms start with the same value of the revenue shock z_0, and no robot stocks, $R_0 = 0$.[5] At time 0, all firms choose to install the same amount of robots $R_1^\star$, as they share the same value for the expected marginal value of robots.

Under the assumption that the firm is unable to adjust the robot stock in time for production, stochastic demand realizations create a strong incentive to hire labor in the face of a sudden increase in the desired production level, which drives the result in Proposition 2. In particular, after installing

[4] To make the problem interesting, we assume that $p_R < \beta\Omega$, a sufficient condition to avoid an uninteresting complementary slackness condition to ensure a non-negative robot stock, $R_1^\star$.

[5] For all these plots, we assume that $z_0 = 1$ and that $z_1 \mid z_0 \sim \Gamma(\kappa,\lambda)$, with parameters chosen so that $\mathbb{E}(z_1) = 1$, $\sigma_z^2 \equiv \mathbb{V}(z_1) = .5$.

Figure 3: Simple Model Solution, with $\bar{R}(z)$ Distribution.



a robot stock, each firm receives a stochastic revenue shock z, which determines a distribution of full-automation cutoffs $\bar{R}(z)$. As a result, all firms receiving low shocks z such that $\bar{R}(z) \leq R_1^\star$ will be fully automated, while all firms with high enough z so that $\bar{R}(z) > R_1^\star$ will hire some labor. Therefore, given the same starting z_0 and R_0 the realization of z will split firms into two groups depending on whether they are fully or partially automated. The distribution of $\bar{R}(z)$ is also shown in Figure 3, where we highlight the masses corresponding to the two groups.

3 Infinite-Horizon Model

We now extend our simple model to an infinite-horizon setting. There are three main reasons for this extension. First and foremost, the two-period setting does not allow firms to adjust over time, as in that framework capital is essentially set in advance. The infinite-horizon setting allows us to clarify that the survival of labor is a long-run result and does not hinge on the putty-clay feature of the two-period model. Firms will adjust towards a *desired* level of robots that does not involve full automation. Second, the infinite-horizon model allows us to interpret ψ_R as a parameter capturing the rigidity of the machine stock in responding to shocks. Finally, the model presented in this section features a stochastic revenue process that can be mapped more easily to the data, contrary to the two period model that just considers permanent shocks.

Time is continuous and lasts forever, and there is no aggregate uncertainty. There is a measure-one continuum of firms, each producing a homogeneous good with the production function Q, and realizing a flow instantaneous operating profit $\Pi(R_t, z_t)$ as described in the previous section. The firm's problem is

$$\begin{aligned}
\max_{I:\mathbf{\Omega}\times[0,\infty)\to\mathbb{R}} \quad & \mathbb{E}_0 \int_0^\infty \exp\{-\rho t\}\left(\Pi(R_t, z_t) - p_R I_t - \frac{\psi_R}{2}(I_t)^2\right)\mathrm{d}t \\
s.t. \quad & \mathrm{d}R_t = (I_t - \delta R_t)\mathrm{d}t, \\
& \mathrm{d}z_t = \mu(z_t)\mathrm{d}t + \sigma(z_t)\mathrm{d}W_t, \\
& R_0, z_0 \quad \text{given.}
\end{aligned}$$

The only source of risk in the model arises from the *idiosyncratic* revenue shock z_t, a standard Itô process. Denote by $\left(\mathbf{\Omega}, \mathscr{B}, (\mathscr{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P}\right)$ the associated filtered probability space, where $\mathscr{Z} \subseteq \mathbb{R}^+$. We assume that the stochastic process has the FOSD property as defined below.[6]

Definition 1. *Denoting as $F(z_{t+s}|z_t)$ the conditional distribution of the revenue shifter at horizon $t+s$, given a starting point $z(t) = z$, we say that the stochastic process for z has the* FOSD property *if*

$$z_t' > z_t \quad \Rightarrow F\left(z_{t+s}|z_t'\right) \succeq_{FOSD} F(z_{t+s}|z_t).$$

We take the output of the firm as the numeraire. In this section we solve the firm's problem for a given wage rate w, discount factor ρ and robot purchase price p_R. In Section 4, we embed the firm's problem in a model where the wage rate is determined in general equilibrium. We assume that the robot stock depreciates at rate δ, capturing physical depreciation or technological obsolescence. Finally, investment is subject to a quadratic adjustment cost, parameterized by ψ_R. This adjustment cost captures installation and configuration costs, as well as a potential discount when selling robots.

To make the problem interesting, we assume throughout that the present discounted value of labor savings is larger than the purchase price of robots,

$$p_R < \frac{\Omega}{\rho + \delta}.$$

Following Dixit and Pindyck (1994), we can write the problem of the firm recursively as[7]

$$\rho V(R,z) = \max_{I\in\mathbb{R}} \quad \Pi(R,z) - p_R I - \frac{\psi_R}{2} I^2 + (I - \delta R) V_R(R,z) + \mu(z) V_z(R,z) + \frac{\sigma^2(z)}{2} V_{zz}(R,z).$$

[6]Whenever possible, in the interest of generality we also prove our results for a continuous-time Markov Chain, which we will hereafter refer to as CTMC. In this case, $\mathrm{d}z_t$ follows a Poisson jump process defined over a finite set $\mathscr{Z} \subset \mathbb{R}_+$ with N_z levels $z_1 < \cdots < z_N$.

[7]In the case of a CTMC, the HJB equation reads

$$\rho V(R,z_i) = \max_{I\in\mathbb{R}} \quad \Pi(R,z_i) - p_R I - \frac{\psi_R}{2} I^2 + (I - \delta R) V_R(R,z_i) + \sum_{j\neq i} \lambda_{ij}\left[V\left(R,z_j\right) - V(R,z_i)\right].$$

Here λ_{ij} denotes the arrival rate of a shock taking the revenue process from state z_i to state z_j.

While our results do not rely on the specification of the diffusion process, we shall focus on a Cox-Ingersoll-Ross (CIR) process for the revenue-shifter, defined by

$$\mathrm{d}z_t = -\theta_z(z_t - z^d)\mathrm{d}t + \sigma\sqrt{z_t}\mathrm{d}W.$$

Here, $\mathrm{d}W$ is a Wiener process, z^d is the unconditional mean of z, θ_z controls the rate of mean-reversion towards the long run average, and σ scales the instantaneous variance. We choose this process because it is stationary, and because it allows us to control both the long-run variance and the rate of mean reversion independently.[8]

3.1 The Investment Policy Function

Regardless of the stochastic process for z, the optimal investment choice is given by the first order condition

$$p_R + \psi_R I^\star(R,z) = V_R(R,z),$$

with the usual interpretation that investment is set to equalize the marginal benefit from holding an additional unit of capital to the net marginal cost.

We now formulate a useful lemma summarizing properties of the value function that will be fundamental for the characterization of the policy functions and aggregates.[9]

Lemma 2 (**Properties of the Value Function**). *The value function $V(R,z)$ is increasing in R for all z, and concave in R for all z. $V_R(R,z)$ is bounded from above by $\frac{\Omega}{\rho+\delta}$, and weakly decreasing in w. Moreover, if the stochastic process has the FOSD property as in Definition 1, then $V(R,z)$ is increasing in z, and $V_R(R,z)$ is nondecreasing in z.*

Many of the properties of the value function are standard. A more interesting point pertains to the marginal value of a unit of robots. In our model, the labor substitution margin generates a natural upper bound on the marginal operating profits. As a result, the marginal present discounted value of a unit of robots will never exceed $\Omega/(\rho+\delta)$, the PDV of labor savings, regardless of how high the realization of z is. We use Lemmas 1 and 2 to derive the properties of the investment policy stated in the following proposition.

Proposition 3 (**Properties of the Optimal Investment Policy**). *Suppose that the stochastic process for z satisfies the FOSD property in Definition 1. Then, the optimal investment policy $I^\star(R,z)$ is non-*

[8]Recall that a CIR diffusion process that mean reverts to z^d admits a stationary distribution which is a Gamma with parameters $\alpha_z = \frac{2\theta_z z^d}{\sigma^2}$ and $\beta_z = \frac{\alpha_z}{z^d}$, so that the unconditional variance of z is $\frac{z^d}{2\theta_z}\sigma^2$. Furthermore, it admits a closed form for the conditional distribution at horizon $t+s$, for any $s>0$, which is given by a non-central Chi-square distribution with $\frac{4\theta_z z^d}{\sigma^2}$ degrees of freedom and non-centrality parameter given by $\frac{4\theta_z \exp\{-\theta_z s\}}{\sigma^2(1-\exp\{-\theta_z s\})} z(t)$. This also implies that the conditional distribution exhibits the FOSD property as in Definition 1.

[9]Whenever we focus on derivatives of the value or policy functions with respect to the revenue shock z, the results only apply to a diffusion.

increasing in R for all z, non-decreasing in z for all R, non-decreasing in w for all (R,z), and bounded from above by

$$\frac{1}{\psi_R}\left[\frac{\Omega}{\rho+\delta}-p_R\right].$$

Proof. Optimal investment satisfies the first order condition

$$I^\star(R,z)=\frac{1}{\psi_R}\left[V_R(R,z)-p_R\right].$$

It follows that $I^\star(R,z)$ inherits the properties of $V_R(R,z)$. By Lemma 2,the properties in the statement follow. □

These intuitive properties are extremely important to the no-full-automation result detailed below. The direct implications of Proposition 3 are that a high enough purchase price, or rigidity, of robots, will go a long way towards safeguarding human labor in the long-run. More importantly, we will show that given *any* purchase price $p_R \geq 0$, and for any diffusion process satisfying the FOSD property detailed above, there always exists a finite value of the adjustment cost parameter $\psi_R > 0$ such that the stationary distribution of firms $G(R,z)$ does not involve full automation.

We now characterize the stationary distribution of firms. We start by defining a *desired stochastic steady state.*

Definition 2 (Desired Stochastic Steady State). *For any price z, we define the* desired stochastic steady state *as the stock of robots $R^\star(z)$ such that optimal investment just covers depreciation,*

$$I^\star(R^\star(z),z)=\delta R^\star(z).$$

$R^\star(z)$ is the stock of robots that the firm would optimally choose if it received the same revenue shock z forever, while operating in a stochastic environment. In this sense, $R^\star(z)$ is the *desired* steady state level of robots at the running profitability level z. Since $I^\star(R,z)$ is non-increasing in R, for any z, optimal net investment is negative if $R > R^\star(z)$ and positive otherwise. Proposition 3 gives us the following corollary.

Corollary 1 (Properties of $R^\star(z)$). *Under the assumptions of Proposition 3, $R^\star(z)$ is non-decreasing in z, and non-decreasing in w for all z. If $\delta > 0$, then $R^\star(z)$ is bounded from above by*

$$R^\star_{\max}\equiv\frac{1}{\delta\psi_R}\left[\frac{\Omega}{\rho+\delta}-p_R\right],\ \forall z$$

and tends to 0 as $\psi_R\to\infty$.[10]

[10] Note that if $\delta=0$, there is no limiting value of $R^\star(z)$, as investment becomes identically 0 for all z and R as $\psi_R\to\infty$, so any robot stock is equally desirable as a steady state. By contrast, when depreciation is strictly positive, the fact that investment tends to 0 as $\psi_R\to\infty$ implies that in the long-run no positive robot stock is ever maintained.

Proof. Follows directly from Proposition 3, evaluating investment at $R^\star$ and using Definition 2. $\square$

The fact that $R^\star(z)$ is non-decreasing in w implies that the aggregate demand for labor is downward sloping. Moreover, the formula for the robot stock upper bound clarifies the effect of many key parameters on the extent of automation. In particular, the bound is strictly decreasing in ψ_R, δ, ρ, and p_R. As robots become less flexible, more prone to obsolescence, or simply more costly, the investment opportunity they provide becomes less attractive for any given PDV of labor savings.

3.2 The Stationary Distribution and the Extent of Automation

Given a stationary process for z, the solution of the model gives rise to a stationary distribution of firms $G(R,z,\infty)$ that solves the relevant Kolmogorov Forward Equation. As a first step, we state the following proposition that applies to the case of a degenerate distribution for the revenue-shifter.

Proposition 4 (**Steady State of the Deterministic Model**)**.** *Consider the firm's problem with* $z_t = z^d$, $\forall t$*. The model has a unique steady state level of robots* $R^\star(z^d)$*. If either* $\psi_R = 0$ *or* $\delta = 0$*, the unique steady state features full automation. If* $\delta > 0$*, there exists a finite value* $\tilde{\psi}_R > 0$ *such that partial automation obtains for* $\psi_R \geq \tilde{\psi}_R$ *and full automation occurs if* $\psi_R < \tilde{\psi}_R$*.*

The intuition is similar to the two-period model, with the only added complication arising from depreciation and the infinite horizon. Recall that partial automation in this setting is a byproduct of the need to pay adjustment costs on depreciating capital. This clarifies why automation is always complete when either δ or ψ_R is zero.[11] Note that the full-automation result carries over to a setting where the firm faces a risky environment, but no adjustment costs.[12] We now characterize the stationary distribution in the general case.

Proposition 5 (**Bounds of the Stationary Distribution**)**.** *Given a diffusion or CTMC for* z *that admits a stationary distribution with support* $\mathscr{Z}$ *such that* $\inf \mathscr{Z} = z_1$ *and* $\sup \mathscr{Z} = z_N$*, the stationary distribution* $G(R,z)$ *has support* $[R^\star(z_1), R^\star(z_N)] \times \mathscr{Z}$*. If* $\delta > 0$*, then the stationary distribution has support* $[R^\star(z_1), R^\star_{\max}] \times \mathscr{Z}$*.*

Proposition 5 states that the stationary distribution cannot feature any firms above $R^\star(z_N)$ or below $R^\star(z_1)$, as implied by the definition of desired stochastic steady states. If robots become obsolete or break down with positive probability, then labor substitution generates a natural upper

[11] Recall that we assumed that $p_R < \Omega/(\rho+\delta)$. If we had not made this assumption, automation would be either full or nil depending on whether $p_R < \Omega/(\rho+\delta)$ or $p_R > \Omega/(\rho+\delta)$.

[12] In this case the firm can just set the robot stock to its desired frictionless level by selling or buying the relevant amount of robots. To see that this entails full automation, note that in this case the optimal robot stock in each period solves

$$V_R(R^\star(z), z) = p_R < \frac{\Omega}{\rho+\delta}, \quad \forall z$$

due to our main working assumption. This immediately implies that $R^\star(z) > \bar{R}(z)$ for all z.

bound to the installation of robots, following the logic in Corollary 1, so that the upper bound of the robot distribution can be tightened to the minimum between $R^{\star}(z_N)$ and $R^{\star}_{\max}$. This important result paves the way for partial automation in the long-run distribution if z_N is high enough. Indeed, the full automation cutoff, $\bar{R}(z)$, is increasing and convex in z, while the maximum desired steady state level of robots is invariant to z. Thus, all firms receiving shocks high enough will not want to obtain capital stocks that are above the relevant $\bar{R}(z)$ and will end up partially automated. Proposition 6 makes this point formally. It uses our characterization of $R^{\star}(z)$ to extend Proposition 2 to an infinite-horizon setting, and to a (potentially) bounded support for the revenue-shock distribution. We begin by stating our definition of the *upper bound to full automation.*

Definition 3 (**Full automation bound**). *Given a stationary distribution for z with CDF, $F(z)$, we define an* upper bound to full automation *as the mass of firms $F(\check{z})$, where $\check{z}$ is the revenue-shifter level such that $\check{z} = \bar{z}\left(R^{\star}_{\max}\right)$.*

When $\delta > 0$, by definition of $R^{\star}_{\max}$ and $\bar{z}(R)$

$$\check{z} \equiv \left[\frac{1-\Gamma}{\delta\psi_R}\left(\frac{\Omega}{\rho+\delta} - p_R\right)\right]^{1-\theta} \frac{w}{p\theta\Gamma}.$$

Clearly, $\check{z}$ is unaffected by the properties of the stochastic process. As a result, once the effect of key parameters of the process for z on the stationary distribution $F(\cdot)$ is known, it is immediate to characterize their effect on the upper bound to full automation, $F(\check{z})$. For example, in the case of the CIR diffusion process, a mean-preserving spread of the stationary distribution lowers $F(z)$ for all z. As a result, since $\check{z}$ is not affected by the properties of $F(\cdot)$, the upper bound to full automation falls.

We are now ready to state and prove our main theoretical result.

Proposition 6 (**Conditions for Long-Run Partial Automation**). *Consider a non-degenerate diffusion or CTMC for z admitting a stationary distribution on $\mathscr{Z}$ such that* $\inf \mathscr{Z} = z_1$ *and* $\sup \mathscr{Z} = z_N$, $z_1 \geq 0$. *Define $G\left(\tilde{R}, \tilde{z}, \infty\right) \equiv \mathbb{P}\left\{R \leq \tilde{R}, z \leq \tilde{z}\right\}$, the CDF of the stationary distribution. For any purchase price of robots p_R, as long as $\delta > 0$,*[13] *there exist a finite value of the adjustment cost parameter ψ_R such that the stationary distribution $g(R,z)$ does not feature full automation. In particular, for $\check{z}$ as in Definition 3,*

$$\int \mathbb{1}\left\{(R,z) : R \geq \bar{R}(z) \wedge z \geq \check{z}\right\} \mathrm{d}G(R,z) = 0.$$

If $z_N = \infty$, any $\psi_R > 0$ implies a stationary distribution that does not feature full automation.

Proof. By Proposition 5, if $\delta > 0$, the stationary distribution has support bounded from above by $R^{\star}_{\max}$. To prove the statement, we need to show that there exists a finite value of ψ_R such that $\bar{R}(z_N) >$

[13] Note that the case with $\delta = 0$ is uninteresting, as, given an initial distribution $G(R,z,0)$, as $\psi_R \to \infty$, the stationary distribution $G(R,z,\infty)$ will become closer and closer to the initial distribution.

$R^{\star}_{\max}$. Since $\bar{R}(z)$ is increasing in z, this amounts to setting ψ_R such that $\check{z} < z_N$. if the process has unbounded support, the statement is trivial as $\check{z} < \infty$ for any $\psi_R > 0$. Otherwise, using the statement in Definition 3, we can set

$$\psi_R > \left(\frac{w}{p\theta\Gamma z_N}\right)^{\frac{1}{1-\theta}} \left[\frac{1-\Gamma}{\delta}\left(\frac{\Omega}{\rho+\delta} - p_R\right)\right].$$

□

This proposition highlights labor's comparative advantage thanks to its flexibility. If robot reallocation involves sufficiently important frictions, captured by the parameter ψ_R, human labor will be saved *in the long run*. Importantly, this result obtains for any level of p_R, including 0, and for finite values of ψ_R. If frictions are large, there is no price of robots low enough to sustain full automation. The intuition for this result hinges on the existence of the upper bound $R^{\star}_{\max}$ to the desired robot stocks. Summing up the mechanics of our model, this bound arises naturally from combining the perfect substitutability of robots and labor with rigidities in the adjustment of the robot stock and decreasing returns to scale. Indeed, these features create a cap to marginal returns to machines and investment, as in Proposition 3. The economic intuition is that robots are most productive when the revenue shock is high relative to the existing stock, or else the firm would just under-utilize the robots it already has. When the firm is thus undersized, it finds it optimal to hire labor to achieve its desired scale, so that the marginal value of a unit of robots is exactly given by labor savings. This generates the upper bound $R^{\star}_{\max}$ to the desired robot stock that the firm wants to maintain, as these capped marginal values are traded off with increasing marginal costs arising from the upkeep of the machines.

While we can analytically prove the existence of bounds to full automation, we are unable to provide comparative statics results for objects of interest like the aggregate stock of robots, or labor demand, at this level of generality, with the exception of the effects of higher flow wages on aggregate labor. We can only be certain of the effects of parameters of interest on $R^{\star}_{\max}$, but not on the shape of the whole $R^{\star}(z)$ schedule. It follows that, while Corollary 1 provides some guidance as to what happens to the *maximum* stock of robots in the economy, it cannot shed light to what happens to the *average* stock of robots. We provide a numerical example to clarify the workings of our model and display its time series properties, and later characterize numerically the comparative statics of labor demand.

3.3 An Illustrative Numerical Example

In this section, we provide an illustrative numerical example to clarify the definitions and propositions stated above, and a simulated time series of the variables of interest. The following figures are produced using the parameter values described in Appendix C.1, which are chosen to highlight the features of our model, and generally differ from those we later use for the full calibration. We solve the model following the procedure in Achdou et al. (2014, 2017).

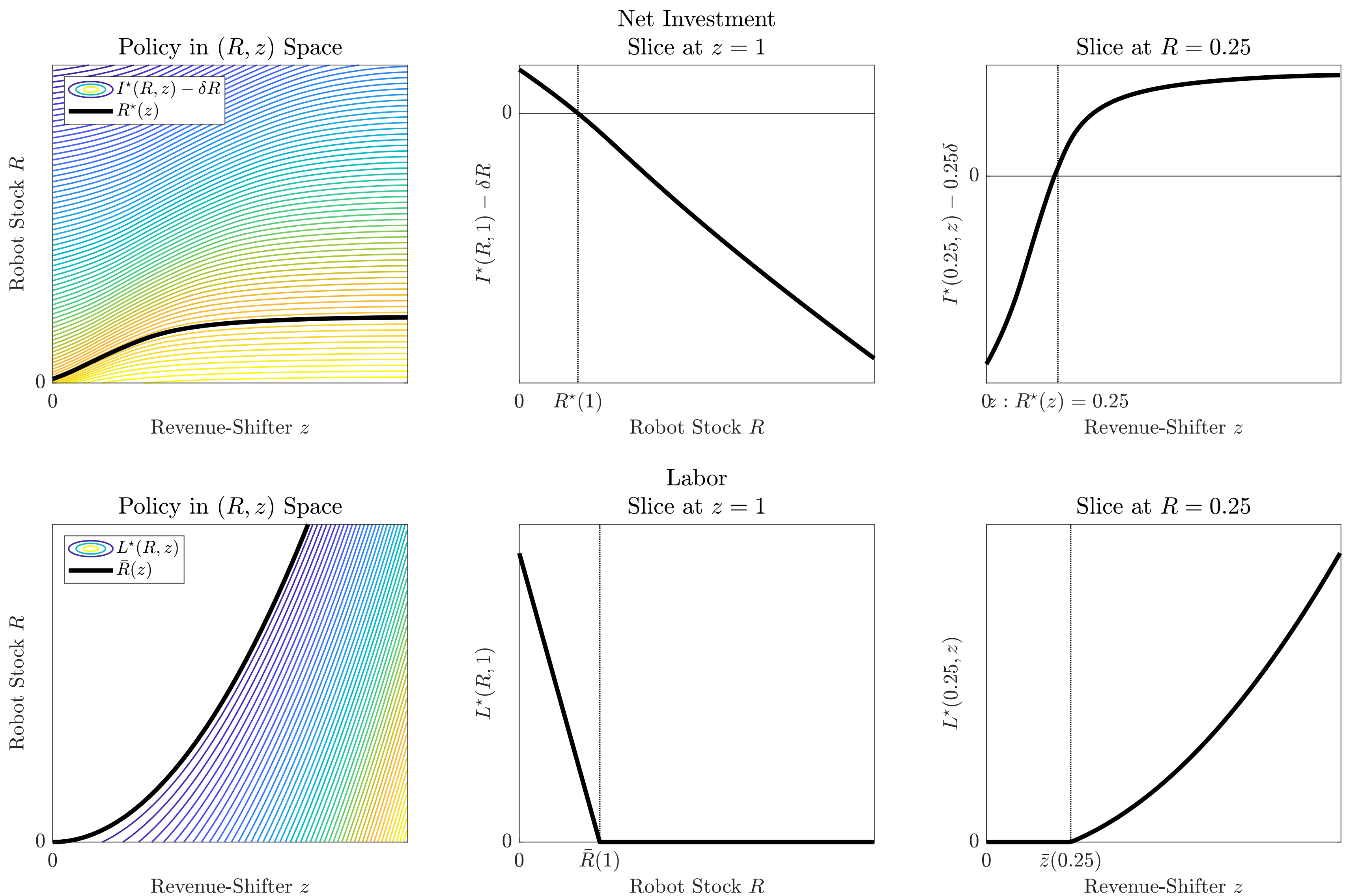
Figure 4 shows the policy functions for investment net of depreciation and labor in a stochastic environment, illustrating the results on investment from Proposition 3. The leftmost panels display the contour lines of the policies in the (z,R) space, highlighting the locus of points corresponding to $R^\star(z)$ for net investment and the full-automation threshold $\bar{R}(z)$ for labor, while the center and right panels show slices of the policy at specific z and R. Focusing first on investment, we can see that, for any fixed z, net investment is negative above $R^\star(z)$ and positive below. Moreover, the figure shows that $R^\star(z)$ is increasing in z and flattens out as z increases. Looking at the central panel, we can see that net investment is indeed decreasing in R, while the rightmost panel shows that it is increasing in z, convex for low values of z, and concave otherwise. For a given z, labor is decreasing in R and falls to 0 at $\bar{R}(z)$. Furthermore, as a function z, the labor policy is increasing and convex in z. This last point comes from the convexity of the marginal product of labor in z.

Figure 5 displays the dynamic properties of our model. Panel 5a shows the net investment policy function together with the schedules for optimal static capital, $\hat{R}(z)$, the full automation threshold, $\bar{R}(z)$, the desired stochastic steady states $R^\star(z)$ and the locus $\mathbb{E}[\mathrm{d}z/\mathrm{d}t]=0$. The graph summarizes the dynamics in the (z,R) space. If the stochastic process for z reverted towards the mean deterministically, the intersection of the solid lines would be the unique attractive steady state of the model. The arrows represent the drift of R—determined by the optimal investment policy–and z in the state space, and scaled proportionally to the drift size. Starting from any point (z,R), mean-reversion pushes the price back to the unconditional mean z^d, while the firm's optimal investment policy moves the robot stock towards the corresponding $R^\star(z)$.

Panel 5b displays simulated time series for the main variables of interest, starting from a zero robot stock and a revenue shock equal to its conditional mean. Consistent with the presence of adjustment costs, the path for the robot stock is substantially smoother than the stochastic revenue process, while labor is highly volatile in response to shocks, falling to zero when the revenue shifter is low relative to the installed robot stock. Interestingly, we find a mildly positive correlation between the robot stock and the labor series. To understand this result, consider the persistence of the stochastic process and the relative downside risk in higher states. When a large shock hits, robots are fixed, and labor is hired to meet heightened demand for the firm's product. At the same time, the firm chooses to increase its robot stock, as shocks are persistent and will fade away only after some time. However, firms do not want to scale up their robot ownership substantially, as they are concerned that they will not be able to fully utilize them. These two facts explain the mild positive correlation that we find, and highlight a potential concern for the empirical literature trying to estimate capital-labor complementarity from firm-level data. This time series highlights that, when automation is present, labor is used essentially to pick up the slack left by a relatively rigid robot stock, and survives only thanks to its flexibility in the face of revenue shocks. Through the lenses of our model, future low-skilled, routine jobs will be increasingly characterized by temporary employment coupled with higher turnover rates. Our model is therefore equipped to speak to the evidence that exposure to automation is associated with more "non-traditional" work arrangements, featuring a higher degree of hours volatility (Rutledge et al., 2019).

Figure 6 shows the stationary distribution together with the marginal distributions over R and z, and the schedules $\bar{R}(z)$ and $R^{\star}(z)$. The graph depicts the determination of the threshold $\check{z}$ as the revenue shock corresponding to the intersection of $R^{\star}_{\max}$ with the $\bar{R}(z)$ schedule. The marginal distribution over z is highlighted to show the area corresponding to the full automation upper bound and its complement, the partial automation lower bound. These bounds are very conservative, as the whole mass of firms that are below the $\bar{R}(z)$ schedule and to the left of $\check{z}$ are actually hiring workers and therefore *not* fully automated.

Figure 4: Policy Functions



The calibration used to produce this and the following illustrative figures is reported in Appendix C.1.

Figure 5: Stochastic Dynamics

(a) Net Investment Policy Function: Optimal Drifts.

(b) Simulated Time Series.

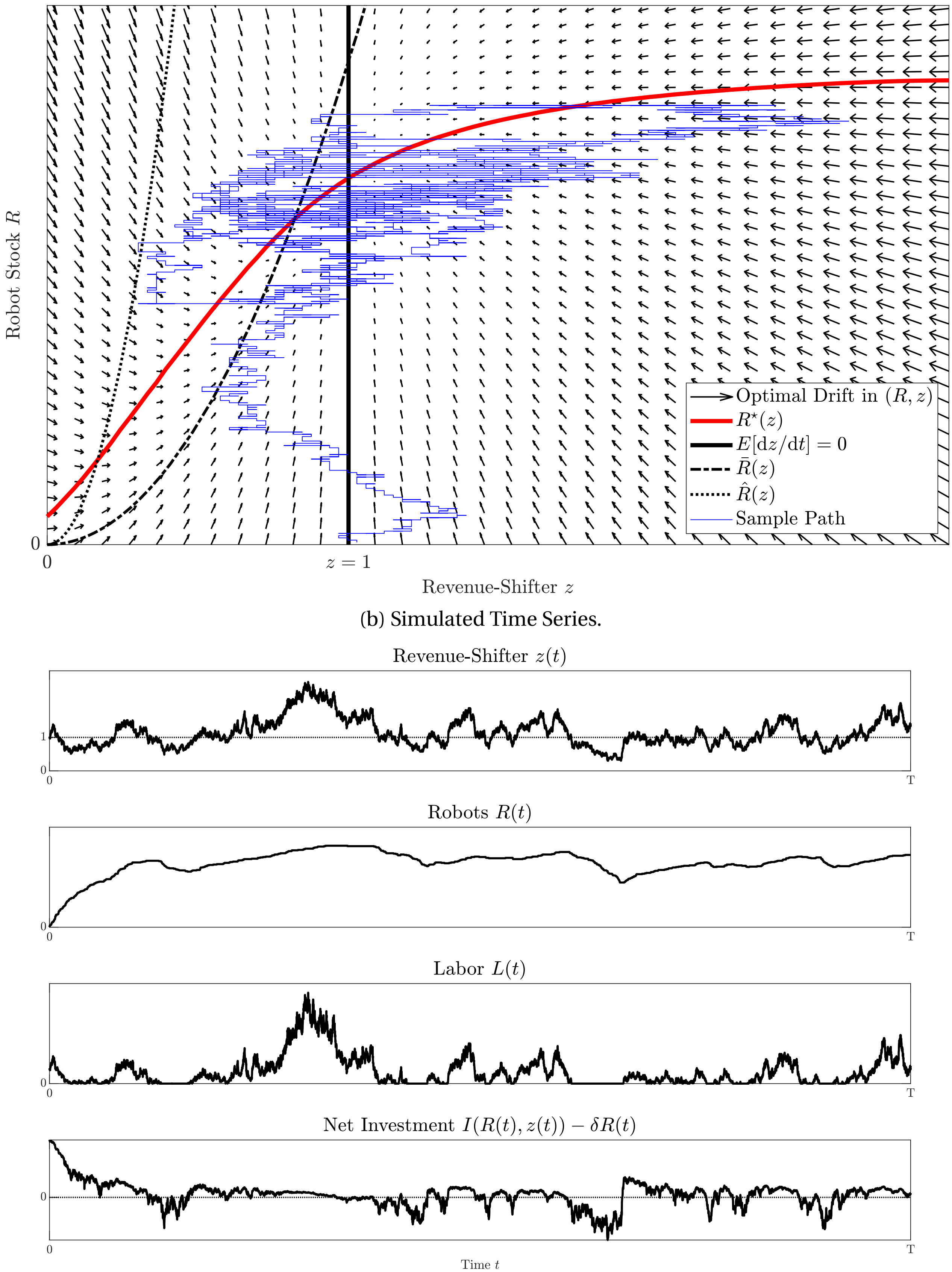
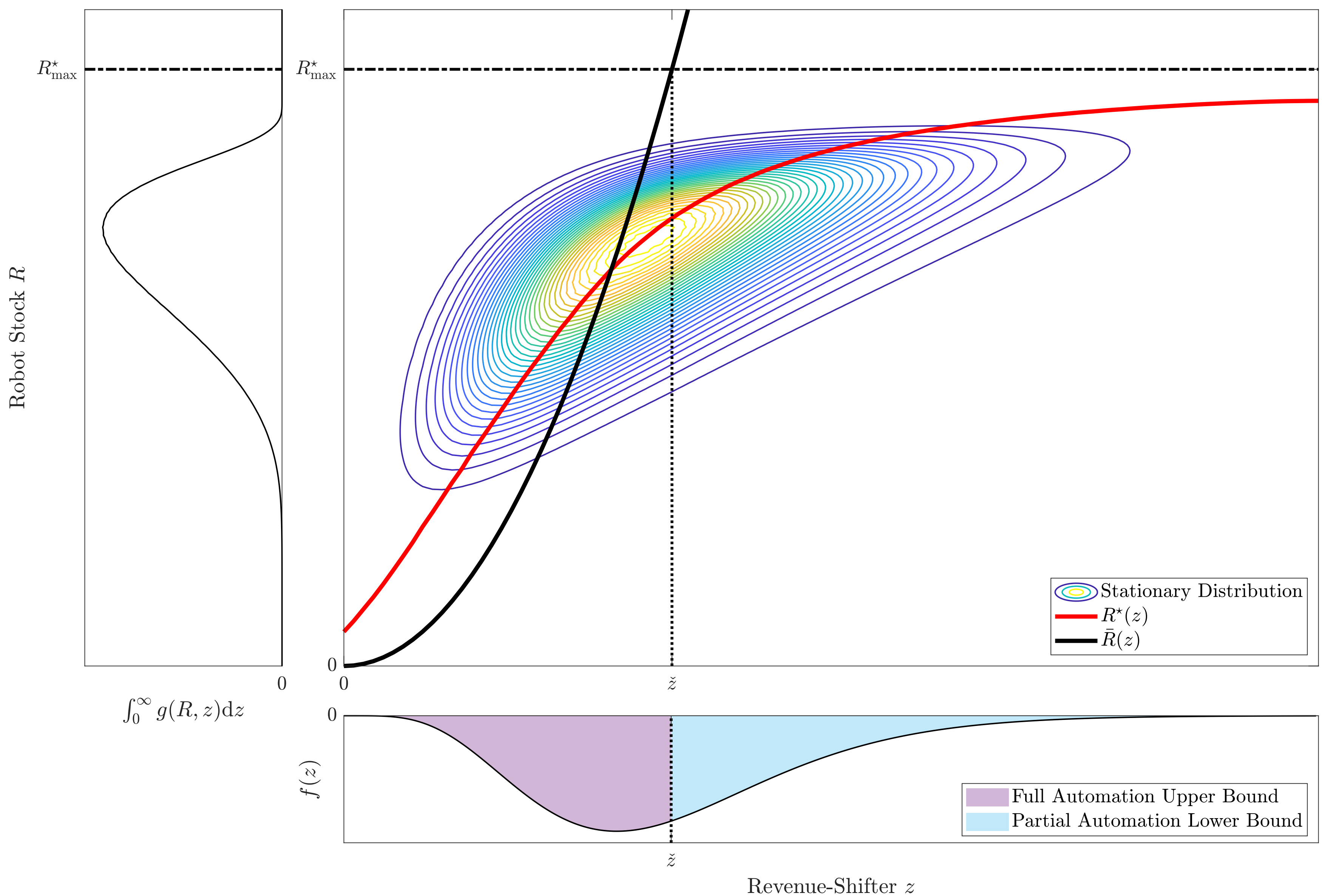


Figure 6: Stationary Distribution and Marginals over z and R, with Automation Bounds.



3.4 Aggregate Labor Demand Comparative Statics

In this section we characterize the aggregate labor demand and explore numerically its comparative statics in the main parameters of interest. In Section 4.1, we build a general equilibrium model, and we consider a special case where we can analytically characterize the effects of changes in p_R and ψ_R on aggregate labor and the robot stock in general equilibrium. We also verify numerically that the sign of the comparative statics holds in our calibration.

Given a stationary distribution $G(R,z)$, aggregate labor demand is given by

$$L^d(w) \equiv \int L^\star(R,z)\mathrm{d}G(R,z),$$

where $L^\star(R,z)$ is the optimal individual demand of labor for a firm at (R,z),

$$L^\star(R,z) = \frac{1-\Gamma}{\Gamma}\left[\bar{R}(z) - R\right]\mathbf{1}_{\{R\leq\bar{R}(z)\}}.$$

The following proposition summarizes the properties of the labor demand schedule $L^d(w)$.

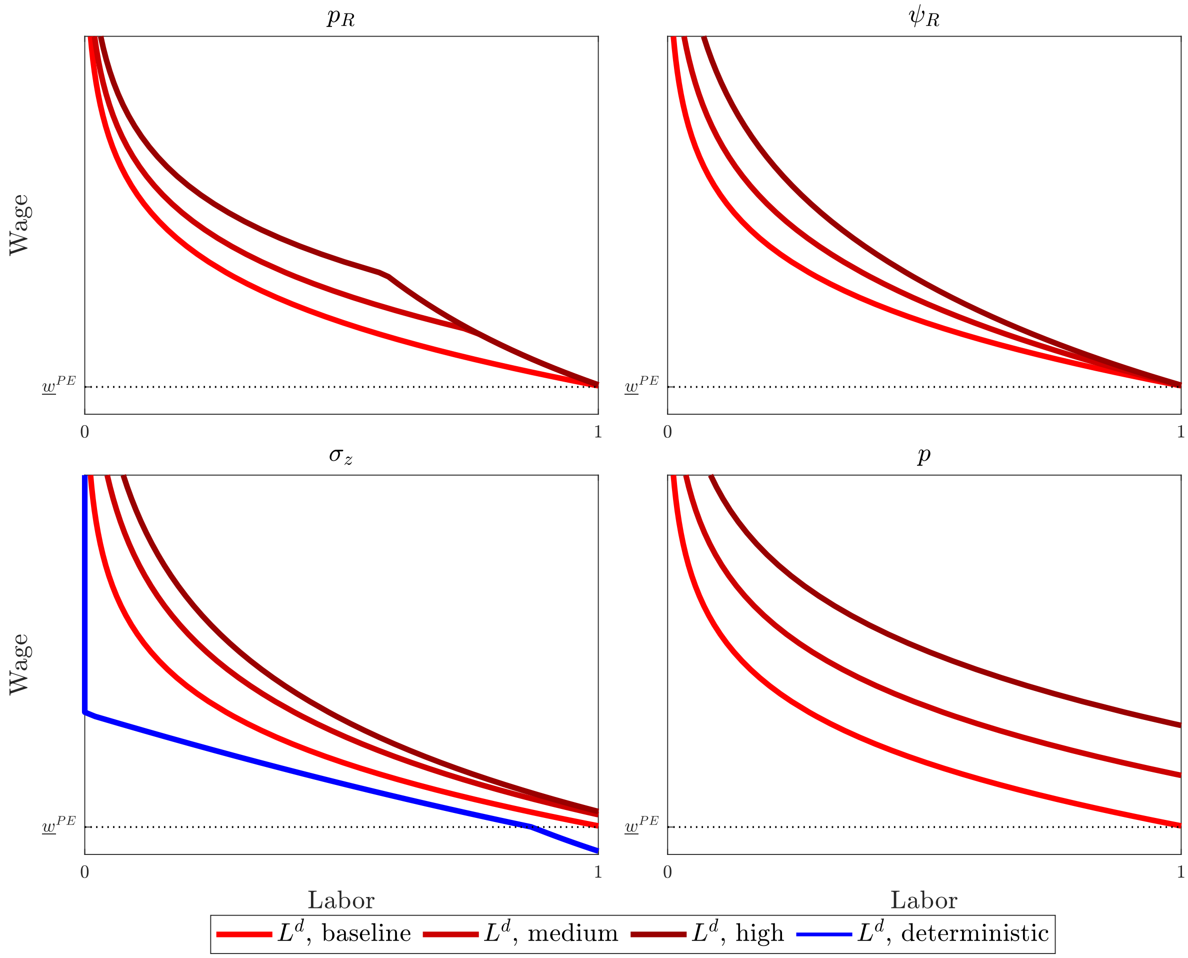
Proposition 7 (**Aggregate Labor Demand**)**.** *The aggregate labor demand is non-increasing in w, with $\lim_{w\to\infty} L^d(w) = 0$ and $\lim_{w\to 0} L^d(w) = \infty$.*

The proposition ensures that labor demand behaves as one would expect. The overall effect on labor demand stems from combining the individual reduction in demand with a rightward shift in the distribution of robots at any given price. Given a distribution $G(R,z)$, a higher wage prompts each individual firm to hire less labor as the full-automation cutoffs $\bar{R}(z)$—the highest scale optimally attained hiring labor—shift down. In addition, a higher wage increases labor savings, thus raising the marginal returns to robots and prompting higher investment for each (R,z). This increases the desired robot stocks $R^\star(z)$ for each z, implying that the stationary distribution concentrates more mass at higher robot stocks for any given price. On average, more firms find themselves with robot stocks above their full-automation cutoffs relative to the previous scenario.

We verify numerically that labor demand is increasing in p_R, ψ_R, in the unconditional variance of the stochastic process for z, and in the product price p, as shown in Figure 7. As robots become harder to adjust or more costly to buy, labor demand shifts up, as firms choose to accumulate less of a rigid and expensive factor, relying instead on flexible, cheap labor. Similarly, as the unconditional variance increases, not only will firms prefer a more flexible factor regardless of their revenue shock, but they will also more frequently be exposed to large z's, requiring massive labor hiring. Finally, a higher price unambiguously raises labor demand.

Figure 7 also plots the interesting benchmark provided by the deterministic labor demand for $z^d = 1$, the average of the revenue shocks in all stochastic scenarios. In particular, we can see that in the deterministic framework, labor demand is identically 0 for high enough values of the wage,

Figure 7: Comparative Statics on $L^d(w)$ for p_R, ψ_R, σ_p and p.



L^d, baseline plots the same calibration for all panels: $p_R = 1.5, \psi_R = 20, \sigma_p = .4$, $\theta_p = .05$. The darker lines corresponding to L^d, medium and L^d, high show an increase in the parameter of interest in each case. Panel p_R: medium $p_R = 2$; high $p_R = 2.5$. Panel ψ_R: medium $\psi_R = 50$; high $\psi_R = 100$. Panel σ_p: medium $\sigma_p = .6$; high $\sigma_p = .8$. Panel p: medium $p = 1.25$; high $p = 1.50$.

where the firm finds full automation optimal. This contrasts starkly with the stochastic case, where labor is only asymptotically zero. Indeed, it will be impossible for all firms to be fully automated in all states, as this would require an infinite robot stock. When the wage is low enough, firms do not install any robots in the deterministic case. In particular this occurs when the wage is below the threshold:

$$\underline{w}^{PE} \text{ s.t. } p_R > \frac{\frac{1-\Gamma}{\Gamma}\underline{w}^{PE} - m}{\rho + \delta}.$$

Here the wage is low enough that the present discounted value of labor savings falls below the purchase price of robots. This generates the kink in labor demand that can be observed across all calibrations. Intuitively, the elasticity of labor demand falls discretely as robots stop being a viable alternative to labor.

4 Calibrated Model

In this section, we develop a version of our model that we can calibrate to the data to gauge the quantitative implications of developments in robotic technology. Our focus is on the relative price of robots, adjustment costs, and the relative productivity of robots and workers. As documented in Korus (2019); Sirkin et al. (2015) robot prices have been going down steadily in recent years, and are projected to continue doing so in the foreseeable future. We use the calibrated model to assess the labor market consequences of a continuing downward trend in these variables, or a potential increase in the relative productivity of robots.

4.1 Equilibrium with Multiple Sectors

In order to map our model to the data, we consider each task as a sector, so that parameters can be intended as sector-level aggregates and can be set to target sectoral characteristics. Accordingly, the revenues of each firm i in sector s can be written as

$$p_s Q_s(z_i, L_i, R_i, u_i) \equiv p_s z_i \left(\Gamma_s L_i + (1-\Gamma_s)\, u_i R_i\right)^{\theta_s}.$$

The parameter Γ_s can be interpreted as the average relative MRTS across all the tasks that are aggregated to produce the output of sector s, which commands a price of p_s. The production technology and the stochastic process have the same parameters for all firms in the same sector, although the value of the stochastic shock will vary across firms at each instant t. The investment problem of the firm is unchanged, and we assume that all firms in all sectors face the same adjustment cost schedule and robot prices. While this assumption is restrictive, it is suitable for our calibration exercise as we use data for robotic arms in all sectors, thus focusing on the same type of robot capital.

We leave robot producers outside of our general equilibrium model, so the robot price is exogenous. We do this for a number of reasons. First, we want to focus on relatively unskilled production-line employees, who are most directly threatened by automation. We believe that these workers would have a hard time reallocating to a robot-producing sector, which would likely require a higher-skilled workforce. Second, we want to look at the worst-case scenario for low-skilled workers, and we believe that this is best captured by an economy where these individuals cannot be easily reallocated to the robot-producing sector. This scenario is consistent with our other modeling choices that exclude a built-in backstop to full automation, which would arise naturally if our workers had to be employed in the robot-producing sector. Finally, we want to conduct comparative statics with respect to a fall in the relative robot price, which can be most directly achieved by determining the robot price exogenously.[14]

[14] In a previous version of our general equilibrium model, we experimented with a perfectly elastic supply of robots arising from a robot-producing sector operating a technology linear in labor, supplied by the same workers employed in the other sectors in the economy. In that version, the robot price is endogenized as the equilibrium wage over the

In our equilibrium model, we are mainly interested in determining wages and relative prices. There is no aggregate uncertainty. Labor is homogeneous across sectors so there is a single labor market. Labor demand is the same as in the previous section, but now we have to sum over the N sectors composing our economy:

$$L^d(w,\mathbf{p}) = \sum_{s=1}^{N} \int_0^\infty L_s\left(w, p_s, R, z\right) dG_s(R,z).$$

Here $\mathbf{p}$ is the vector collecting all the p_s sectoral prices, and the dependence of labor on w is explicit. Finally note that both the individual labor demand, L_s, and the stationary distribution, G_s, vary across sectors as a result of different stochastic processes, and different prices, which will generate sectoral heterogeneity in the relevant cutoffs $\bar{R}$ and $\hat{R}$.

The equilibrium wage is determined by crossing the labor demand with an isoelastic labor supply,

$$L^s(w) = \left(\frac{w}{\chi}\right)^{\varphi},$$

where φ denotes the Frisch labor supply. This supply schedule arises from a representative household endowed with GHH preferences over consumption and labor, where the final good is the numeraire. For simplicity, and since we are not concerned with the equilibrium interest rate, we assume that households are hand-to-mouth and own a differentiated portfolio of all the firms in the economy. As a result, they receive all the profits in the economy in addition to labor income.

Households consume a final good that is the CES aggregate of the Y_s goods produced by the various sectors,

$$Y \equiv A^F \left(\sum_{s=1}^{N} \xi_s Y_s^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}},$$

where A_F is an aggregate productivity term and σ denotes the elasticity of substitution across industries. The firm cost minimization problem determines the demand for each intermediate good Y_s as

$$Y_s^D = \left(\frac{\xi_s}{p_s}\right)^{\sigma} \frac{Y}{A^F}.$$

The costs associated with robot maintenance, purchase and adjustment are rebated to the household. In essence, these goods and services are produced using the same aggregate of intermediates as the final good. The supply of each intermediate good is given by

$$Y_s^S = \int Q_s(z,L,R,u)dG_s(R,z).$$

productivity of the robot sector. In that model, improvements in robot technology can push up the equilibrium wage (as more workers are demanded to produce robots) thus accelerating the demise of labor in manufacturing sectors, as workers quickly transition to the robot sector. However, this effect is partially reversed if productivity is increased enough and less workers are demanded by the robot sector.

An equilibrium is then prices $w, \{p_s\}$ and quantities $L, Y, \{Y_s\}$ such that the labor market and all goods markets clear. We provide a formal definition, together with the full system of equations, in Appendix D.

4.2 The Partial Automation Limit

We cannot provide general analytical results to characterize the system. However, we can obtain closed-form expressions for the main aggregates and characterize some comparative statics in the special case where no firm is fully automated. This case is empirically relevant, as no firms are currently fully automated, and our calibration suggests that the data on robot penetration can only be matched by a scenario where almost all firms are partially automated. In our model, this scenario occurs when a sector has $\check{z}_s \to 0$, which implies that the upper bound to full automation is given by $F_s(\check{z}_s) \to 0$. In this limit case, all firms within each sector are partially automated and choose to install an amount of robots arbitrarily close to[15]

$$R^\star_{\max,s} = \frac{1}{\psi_R \delta}\left(\frac{\frac{1-\Gamma_s}{\Gamma_s} w - m}{\rho+\delta} - p_R \right). \tag{1}$$

Therefore, while not varying at the level of the individual firm, the installed stock of robots varies across sectors according to the relative MRTS parametrized by Γ_s. To understand this result, recall that $\check{z}$ represents the highest revenue shock that implies full automation for a firm installing $R^\star_{\max,s}$. Thus, requiring that $F_s(\check{z}_s) \to 0$ means that there is essentially no downside to installing the highest desired robot stock, as the firm will almost always utilize it fully to save on labor costs. Equivalently, the highest desired robot stock is small enough that the firm finds itself almost always partially automated. In this setting, we can show the following result.

Proposition 8 (**Comparative Statics in General Equilibrium**). *Consider the general equilibrium model with $F_s(\check{z}_s) \to 0$, $\theta_s = \theta$ for all s, and elasticity of substitution across industries $\sigma = 1$. In the neighborhood of an equilibrium supported by prices $(w, \mathbf{p})$, aggregate equilibrium labor is increasing in m, p_R and ψ_R, and aggregate robot penetration—defined as the ratio of the aggregate robot stock R to aggregate labor L—is decreasing in m, p_R and ψ_R.*

Proposition 8 shows that price effects do not overturn partial equilibrium decisions by firms when p_R or ψ_R are changed. The assumption that decreasing returns are the same across all sec-

[15]The reader might wonder how $\check{z}_s \to 0$ without having $R^\star_{\max,s} \to 0$. In the partial automation scenario,

$$\check{z}_s = \left[\frac{1-\Gamma_s}{\Gamma_s} \frac{R^\star_{\max,s}}{L_s} \int z^{\frac{1}{1-\theta_s}} \, \mathrm{d}F(z) \right]^{1-\theta_s}.$$

Therefore, $\check{z}_s \to 0$ if the ratio of $R^\star_{\max,s}$ to sectoral labor is low enough.

tors allows us to exclude reallocation effects from changes in the equilibrium wage, regardless of heterogeneity in the relative productivity of robots. This also results also from the Cobb-Douglas assumption for the final good aggregator, which prevents reallocation in expenditure shares.

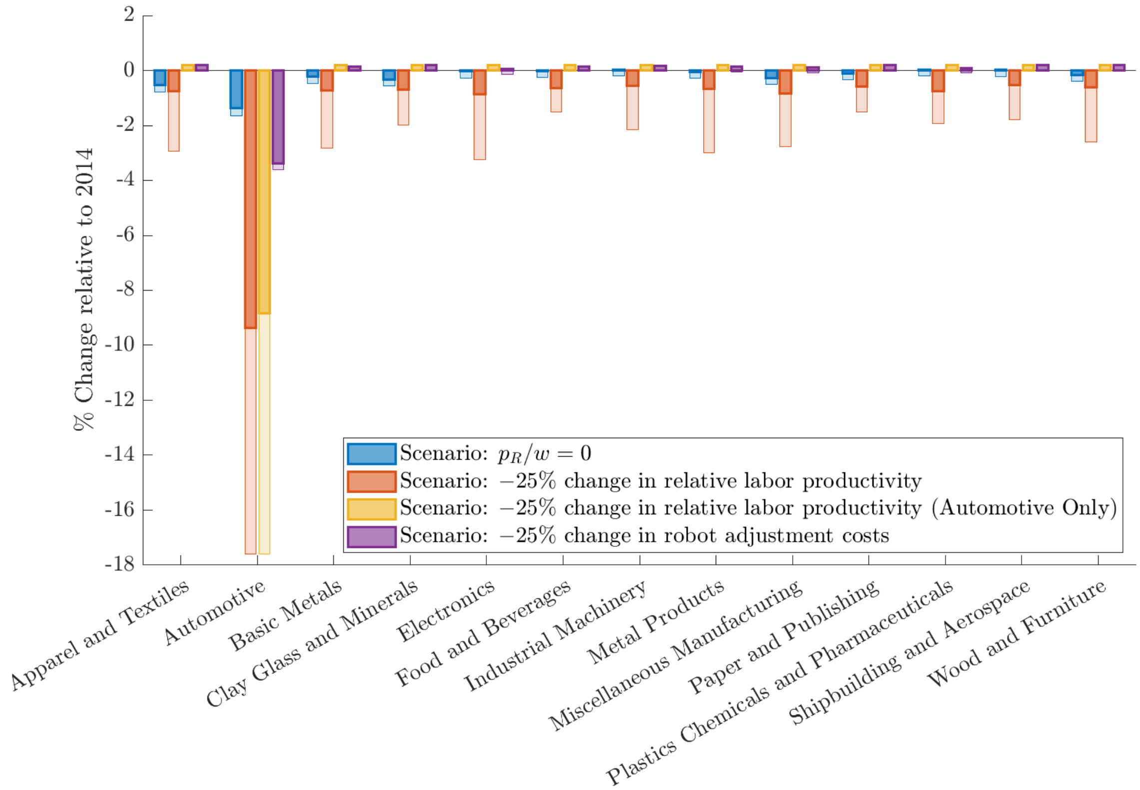
4.3 Data and Calibration Strategy

We use data on average robot penetration taken from Acemoglu and Restrepo (2018d). These data pertain to the 13 manufacturing sectors defined by the International Federation of Robotics (IFR). The values for 2014 provide a target for R_s/L_s in each sector of our model. We construct the relative price of robots in 2010 and 2014, using data from Korus (2019) and average wages in manufacturing production occupations from the Occupational Employment Statistics, that we use to fix p_R/w.[16] We use NBER-CES data (Becker et al., 2016) to obtain plausible values for the scale parameter θ by targeting the average production workers' share of value added before 1980 in each sector, before robots became ubiquitous (Graetz and Michaels, 2018). Compustat data give us estimates of sector-specific revenue shock processes (Capital IQ Compustat, 2019). Following Oberfield and Raval (2014), we set the elasticity of substitution across industry sectors to $\sigma = 1$. This conveniently makes the final good a Cobb-Douglas aggregate and allows us to calibrate the ξ_s as the share of total manufacturing value added for each IFR sector in 2014. We obtain these shares from BEA data referring to the various sectors considered by IFR, which we match to two-digit SIC codes following Acemoglu and Restrepo (2018d). In our baseline calibration, we set the elasticity of labor supply to $\varphi = 1$. We target a total aggregate labor of $L = 1$ in 2014, and set the productivity parameters Γ_s to match the number of robots per employee in each sector. Our baseline calibration also sets ψ_R to match the increase in aggregate robot penetration observed between 2010 and 2014, given the observed fall in the relative price of robots. Finally, we set $m = 0$ so that all steady state maintenance costs are subsumed in the adjustment cost paid to replenish depreciation. We report a detailed description of our data, strategy, and numerical algorithm in Appendix C.

Our model requires a large value of adjustment costs ψ_R in order to match the existing robot penetration and, more importantly, the low aggregate elasticity of robot penetration to purchase prices that we observe in the data. Note that this fact could not be explained by a version of our model featuring no risk and adjustment costs, as Proposition 4 shows. According to the calibrated model, the current data on robot penetration are matched by the partial-automation scenario described in Section 4.2. Almost all firms *within each sector s* choose to install approximately the same level of robots, $R^{\star}_{\max,s}$, as specified in Equation 1. The robot stock still varies across sectors with the technological parameters Γ_s. The stock of robots observed in the data in each sector is small relative to the full automation level $\bar{R}(z)$ for most values of the revenue shock z. Importantly, this result arises naturally in an equilibrium that features the (low) robot penetrations observed in the data.

[16] The wage is determined in equilibrium, so we force the ratio of robot prices to the flow wage to be fixed at the desired level.

Figure 8: Total Sectoral Employment



Note: the full length of the bars gives the partial-equilibrium effect, while darker bars highlight total employment effects in general equilibrium.

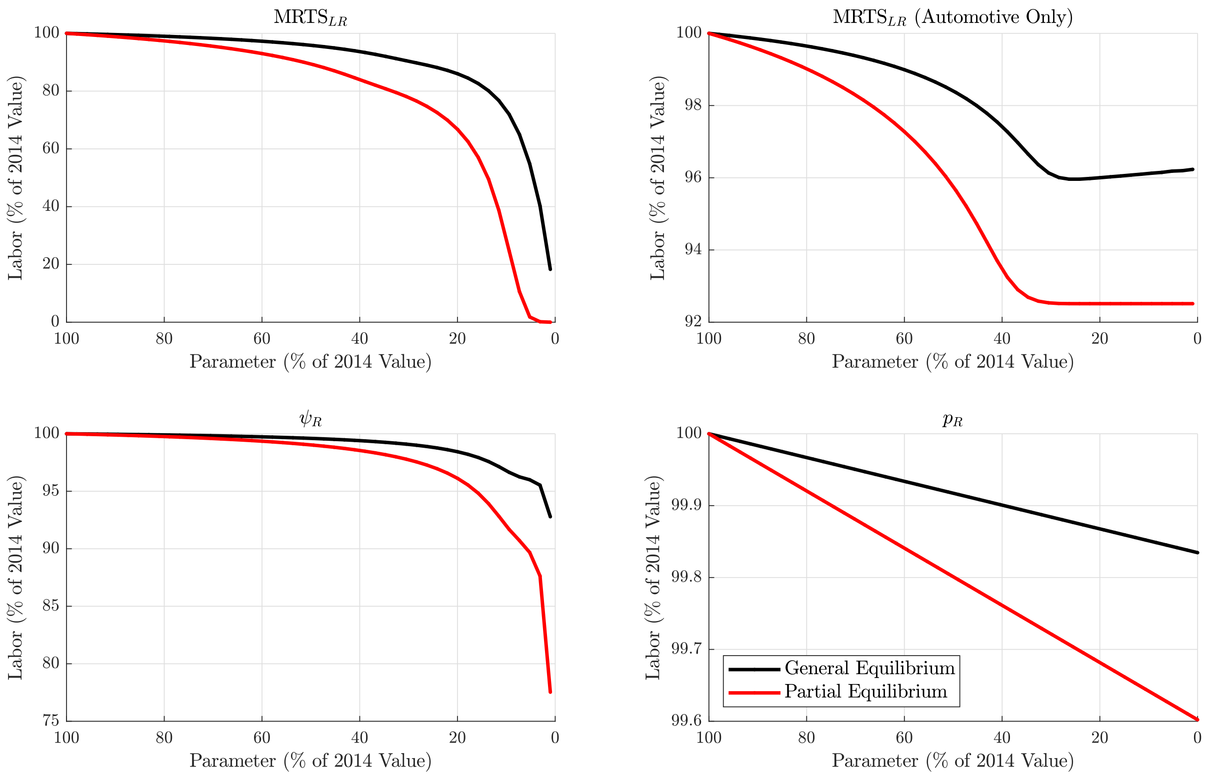
In Figure 8, we quantify the labor market consequences of four scenarios, relative to the calibrated 2014 equilibrium. First, we focus on a reduction of the purchase price of robots all the way down to $p_R = 0$. Second, we analyze a fall of 25% in the MRTS_{LR}, i.e., an increase in relative robot productivity across all sectors. Third, we examine a fall of 25% in the MRTS_{LR} limited to the automotive sector. Finally, we consider a fall in adjustment costs, and thus in steady state maintenance costs, of 25%. *Ceteris paribus*, a robot price reduction yields limited employment effects in the calibrated model. This result is hardly surprising, since the value of ψ_R that we find is high relative to the equilibrium wage, suggesting that robots are in fact relatively rigid (Sirkin et al., 2015). Consistent with the intuitions in our model, the only way to generate a high robot penetration is to have a low adjustment cost parameter and/or a low MRTS_{LR}. For instance, the higher penetration of robots in the automotive sector is interpreted by our model as a low MRTS_{LR}. Figure 8 also shows how labor is threatened by technological improvements that render robots relatively more productive, or that reduce their installation and resale costs. While we present the results for changes of $p_R, \psi_R, \text{MRTS}_{LR}$ in isolation, it is crucial to note that changes in one variable have knock-on effects on the other variables. In particular, a fall in ψ_R or MRTS_{LR} increases the elasticity of aggregate labor to the relative price of robots.

Figure 8 allows a decomposition of employment effects into partial and general equilibrium effects. The full length of the bars shows the partial equilibrium effect, while the dark segment highlights total employment effects in general equilibrium. As a result, the lighter portion of the bars represents the attenuation effect that comes from accounting for general equilibrium price feedback. A finite labor supply elasticity occasions a fall in the wage that dampens the adverse effects on labor demand. This attenuation effect is starkest in the case of a fall in MRTS_{LR}; the total general-equilibrium effect is half of its partial-equilibrium counterpart. Further, there is a reallocation effect at play, which depends on the suitability of various sectors to automation. This is evident from comparing the automotive sector—currently featuring the highest relative robot productivity—to other sectors. For example, a fall in ψ_R reduces employment in automotive, while it leads to such a large fall in the equilibrium wage that other sectors hire more workers. Our general equilibrium framework also allows us to investigate the consequences of technological advances in a single sector. To this end, Figure 8 also displays the employment effect of a 25% fall in MRTS_{LR} in the automotive industry. Relative price and wage effects help labor survive. While there is still a substantial fall in the number of workers employed in the automotive sector, the reduced labor demand coming from this sector puts downward pressure on the real equilibrium wage, leading the other sectors to expand their employment.

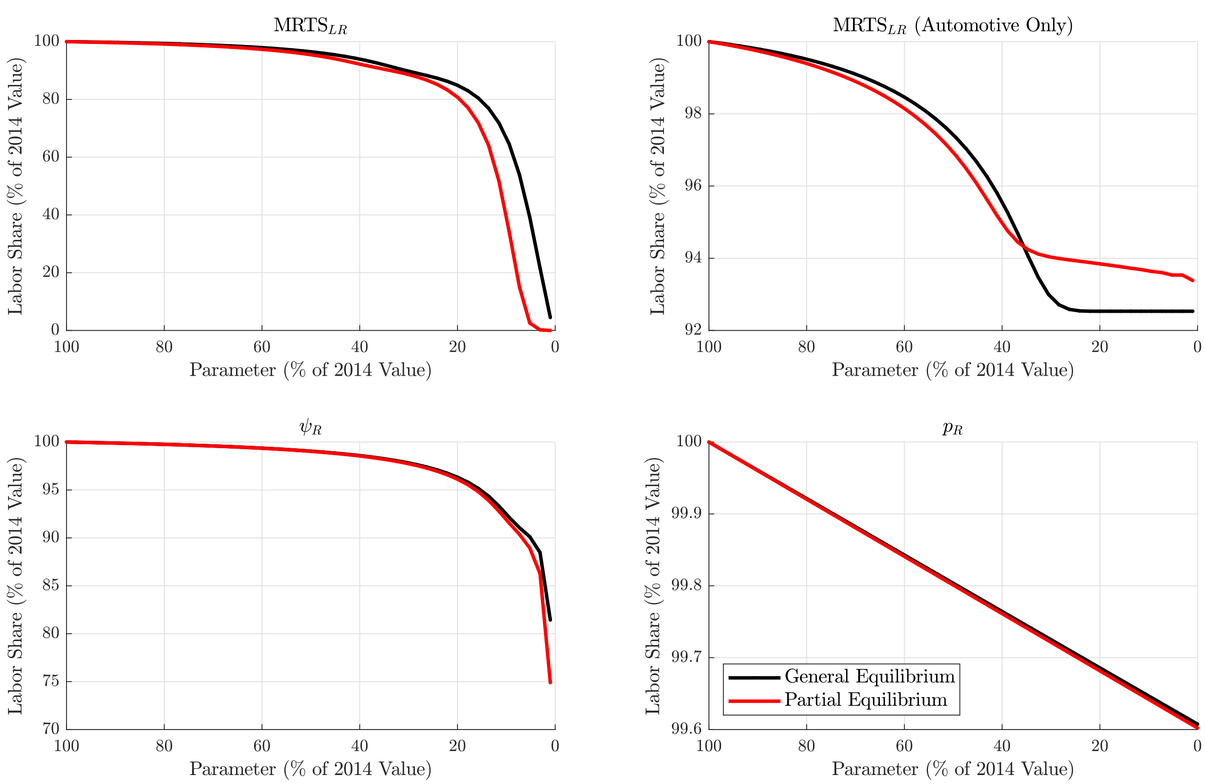
Figure 9 displays the aggregate employment and labor share consequences of smoothly varying the parameters of interest one at a time. Panel 9a shows that the employment effects are modest. The potential fall in MRTS_{LR} across all sectors poses the most relevant threat to the survival of labor. Nevertheless, a 70% reduction in relative labor productivity would cause a relatively modest 20% reduction in aggregate manufacturing employment. As a comparison, a 70% fall in ψ_R only causes about a 1% fall in aggregate employment, and an equivalent reduction in p_R reduces total labor by a mere 0.11%. The labor share consequences, reported in Figure 9b, are similar. Once again, there is substantial attenuation coming from general equilibrium price effects. This attenuation is apparent when technological advances are limited to just one sector, in this case the automotive sector. At fixed prices and wages, the aggregate employment fall coincides with the employment fall in the automotive industry. The flat portion of the curve reveals that a sufficiently large decrease in MRTS_{LR} completely wipes out human labor in automotive, which directly translates to a fall in aggregate employment that equals the initial number of employees in this sector. This effect is mitigated in general equilibrium by a substantial reallocation of labor to other sectors, encouraged by a fall in the equilibrium wage. This reallocation halves the total employment effect, but at the cost of a sharper reduction in the aggregate labor share compared to the partial equilibrium case, as can be seen in Figure 9b.

Figure 9: Comparative Statics on the Calibrated Model

(a) Aggregate Employment



(b) Labor Share



Our calibration compares favorably with estimates from Acemoglu and Restrepo (2018d), who find that an additional robot per thousand employees reduces the employment to population ratio in affected local labor markets by 0.2%. In the data generated by our model, one more robot per thousand employees is associated with an employment fall between 0.01%—if caused by a decline in p_R—and 0.1%—if caused by a decline in MRTS_{LR}. The fact that our numerical results on these non-targeted moments are similar to other estimates in the empirical literature supports our calibration results.

The results presented above suggest that, *ceteris paribus*, small changes in robotic technology other than the purchase price will have the largest effects in the automotive industry, with modest impacts on the remaining sectors. However, radical changes have the potential to affect all sectors dramatically. Finally, it is worth highlighting that our calibration only focused on industrial robots, namely mechanical arms, thereby ignoring the impact of other potentially relevant technologies.

5 Extensions and Robustness

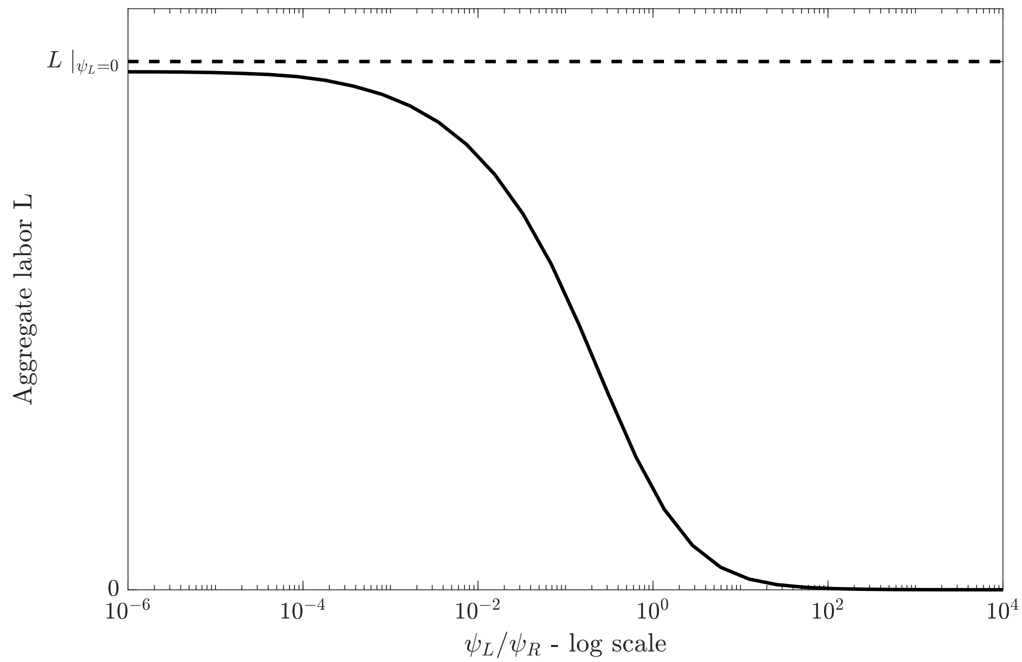
5.1 Introducing Labor Rigidity

Until now, we have assumed that labor can adjust instantly in response to shocks. In this section we relax that assumption, introducing quadratic costs on labor so that it, too, is a state variable in the firm's problem. We verify numerically that this model produces approximately the same results as our baseline model when labor adjustment costs are arbitrarily small. We show that the severity of labor market frictions—as captured by the ratio of labor to capital adjustment costs—is a key variable in determining the adverse effects of automation on employment. The higher the relative adjustment costs, the lower the comparative advantage of labor in responding to shocks, leading to quicker substitution away from labor. We thus provide additional substance to our claim that flexibility is the key source of labor's comparative advantage in our model. The immediate policy prescription is that a flexible labor market is the most effective backstop to long-run automation. This intuition is supported by evidence in Acemoglu and Restrepo (2018a) that unionization rates are correlated with the adoption of robots across OECD countries.

In what follows, we choose quadratic costs on labor, in keeping with the literature inaugurated by Sargent (1978). We choose quadratic costs for two main reasons. First, using the same type of adjustment costs for capital and labor immediately allows us to assess the relative rigidity of the two factors. Second, this type of cost can be easily managed by our solution method.[17] Formally, the

[17]Note that the additional state variable imposes a non-negligible computational burden on the algorithm, as now there are two endogenous state variables and the exogenous forcing term. This means that we need to be more parsimonious with the choice of approximating grids for the variables in our numerical scheme. We present this model with illustrative intent, as calibrating it to match the data points as in Section 4 would be infeasible.

Figure 10: Comparative Statics on ψ_L – Model with Labor Market Rigidities



firm's problem can be formulated recursively as

$$\rho V(R,L,z) = \max_{I_R\in\mathbb{R},I_L} \Pi(R,L,z) - p_R I_R - \frac{\psi_R}{2}(I_R)^2 + (I_R - \delta R)V_R(R,L,z) - \frac{\psi_L}{2}(I_L)^2 +$$
$$+ (I_L - sL)V_L(R,L,z) + \mu(z)V_z(R,z) + \frac{\sigma^2(z)}{2}V_{zz}(R,z).$$

Here ψ_L parametrizes labor adjustment costs and s is an exogenous separation rate. We augment the profit function to allow for under-utilization of labor:

$$\Pi(R,L,z) = \max_{0\le u_R,u_L\le 1} zp\left(\Gamma u_L L + (1-\Gamma)\,u_R R\right)^\theta - m\,u_R R - wL.$$

Note that we do not allow the firm to pay workers in proportion to the utilization rate, in order to better capture labor market rigidities. Accordingly, the only way to avoid paying workers is to fire them. The profit function reads

$$\Pi(R,L,z) \equiv \begin{cases} zp\left(\Gamma L + (1-\Gamma)R\right)^\theta - mR - wL & \Gamma L < (1-\Gamma)\left[\hat{R}(z) - R\right] \\ zp\left((1-\Gamma)\hat{R}(z)\right)^\theta - m\hat{R}(z) - \frac{\Gamma}{1-\Gamma}\Omega L & (1-\Gamma)\left[\hat{R}(z)-R\right] \le \Gamma L \le (1-\Gamma)\hat{R}(z) \\ zp(\Gamma L)^\theta - wL & (1-\Gamma)\hat{R}(z) < \Gamma L. \end{cases}$$

This solution entails full utilization of labor, regardless of the level of robots. By contrast, robots are turned off whenever full utilization would bring the firm above the desired scale $\hat{R}(z)$.

Figure 10 portrays aggregate labor demand as a function of relative adjustment costs $\frac{\psi_L}{\psi_R}$, keeping ψ_R fixed at the level from Section 3. We set the separation rate $s = \delta$, although our qualitative results are unaffected as long as $s > 0$. For reference, the dashed line plots aggregate labor demand for the model without adjustment costs. Labor demand is essentially the same as in the frictionless model for low ψ_L/ψ_R, and labor goes to 0 when the labor adjustment cost is sufficiently high relative to the robot adjustment cost. Intuitively, most of the fall in labor occurs before $\psi_L = \psi_R$, that is, where labor and robots are equally rigid. Some labor survives beyond this point, but this is just a byproduct of the quadratic costs specification, as the firm might prefer to adjust two factors by a little rather than a single factor by a lot in order to save on adjustment costs. Therefore, the above comparative statics might understate the adverse effects on labor stemming from higher adjustment costs.

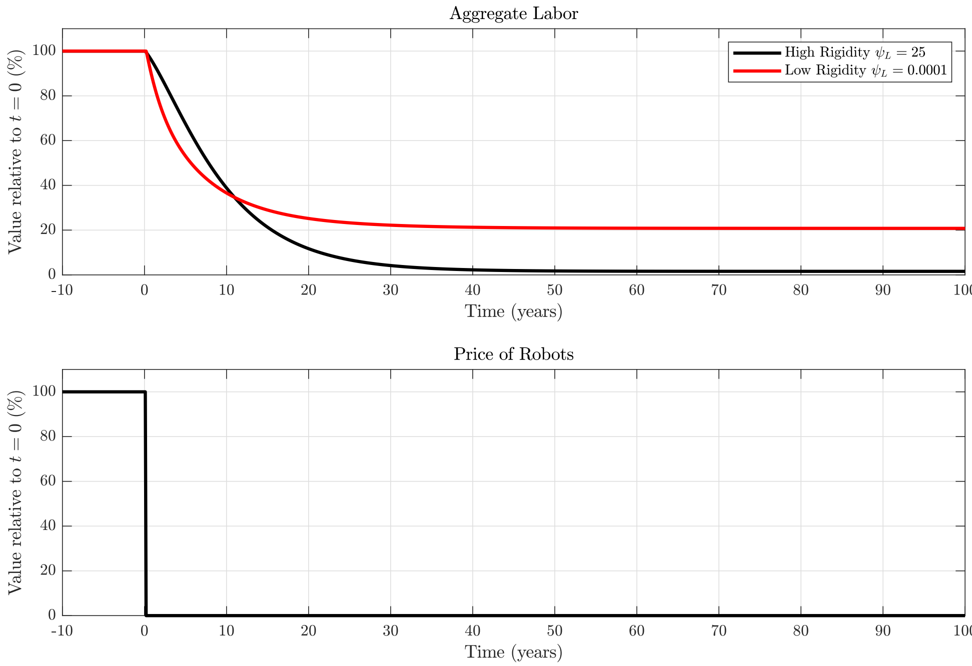
Arguably, our comparative statics on steady states do not make a strong case for a more flexible labor market. Indeed, existing labor protections could benefit labor by slowing the transition to a more automated production process. We perform a simple exercise where we fix the wage rate and postulate a perfectly anticipated fall in the price of robots all the way to zero. We perform this exercise to evaluate the worst-case scenario for workers. As highlighted in Section 4, general equilibrium feedbacks dampen the negative employment effects. We portray the results in Figure 11, which displays important differences between the transition featuring an immediate fall of the price of robots to 0, reported in the upper panel, and a more gradual one.

Starting from panel 11a, we see that the fall in labor is slower in the more rigid labor market. After about 10 years, though, the paths intersect. While the more flexible labor market limits employment losses to 80% of the original number, the more rigid market transitions to an almost fully automated scenario, consistent with the comparative statics highlighted above. These transition paths seem to provide some rationale for keeping existing protections in place. For instance, if we interpret the model as a sector of the economy, then the policymaker might prefer a slower adjustment to allow workers either to retire, or to have enough time to get the proper training to switch occupations. Similarly, employment-protection legislation (EPL) might be useful for transferring more surplus to workers, who one might argue are already disadvantaged given the trends in automation.

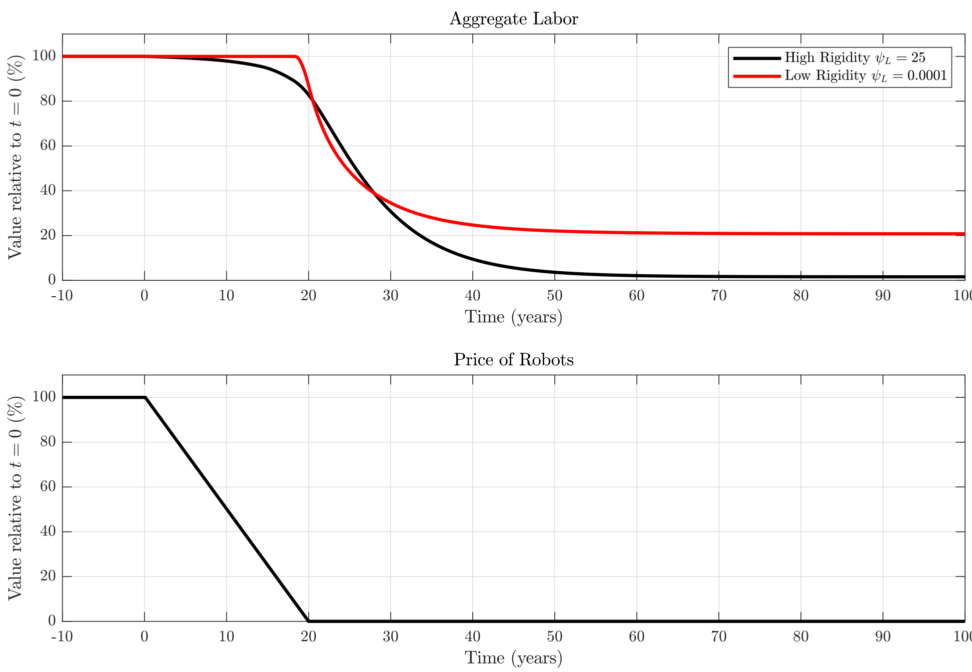
However, panel 11b reveals that EPL plays a much different role in a gradual transition. Here, the economy featuring larger labor market frictions starts adopting robots earlier relative to its low-friction counterpart. This can speak to the evidence in Acemoglu and Restrepo (2018a) that countries with higher unionization rates also have a higher robot adoption at the current world price for robots. The role of labor market frictions, while unambiguous in the long run, is therefore nuanced in the short term. In the long run, labor protections, far from benefiting workers, eliminate their distinctive comparative advantage. This effect is likely to be magnified in the real world, given that labor-substituting technological innovation responds to the incentives provided by a rigid labor market. However, in the short run and medium run they can contribute to a more gradual transition, depending on the pace of the improvement in automation technology.

Figure 11: Transition to Steady State with $p_R = 0$

(a) Immediate Fall to $p_R = 0$



(b) Gradual Transition



5.2 Robustness to Different Adjustment Costs Specifications

In this section, we explore the robustness of our findings in Section 3 to alternative adjustment costs specifications. We focus on linear adjustment costs, following Bentolila and Bertola (1990). Linear costs are interesting because they generate an inaction region, similar to fixed costs of adjustment

(Stokey, 2009). However, linear costs are closer to the evidence provided in Conway (2014), which highlights unit installation costs. We describe the solution algorithm we use for our numerical results in detail in Appendix E. The firm's problem is now

$$\rho V(R,z) = \max_{I\in\mathbb{R}} \quad \Pi(R,z) - \mathbb{1}\{I>0\}(\psi_+ + p_R)I \; - \mathbb{1}\{I<0\}(-\psi_- + p_R)I + \\ + (I-\delta R)V_R(R,z) + \mu(z)V_z(R,z) + \frac{\sigma^2(z)}{2}V_{zz}(R,z),$$

where ψ_+ and ψ_- denote linear adjustment costs for positive and negative investment. The F.O.C. for investment gives the following conditions on the value function derivative:

$$\begin{cases} V_R(R,z) = p_R - \psi_- & \text{if } I<0 \\ p_R - \psi_- < V_R(R,z) < p_R + \psi_+ & \text{if } I=0\,. \\ V_R(R,z) = p_R + \psi_+ & \text{if } I>0 \end{cases}$$

This implies that, for each z, the firm will be inactive for $R\in\left[R^\star_{\text{inv}}(z), R^\star_{\text{disinv}}(z)\right]$, where

$$V_R\left(R^\star_{\text{inv}}(z), z\right) = p_R + \psi_+ \quad \text{and} \quad V_R\left(R^\star_{\text{disinv}}(z), z\right) = p_R - \psi_-,$$

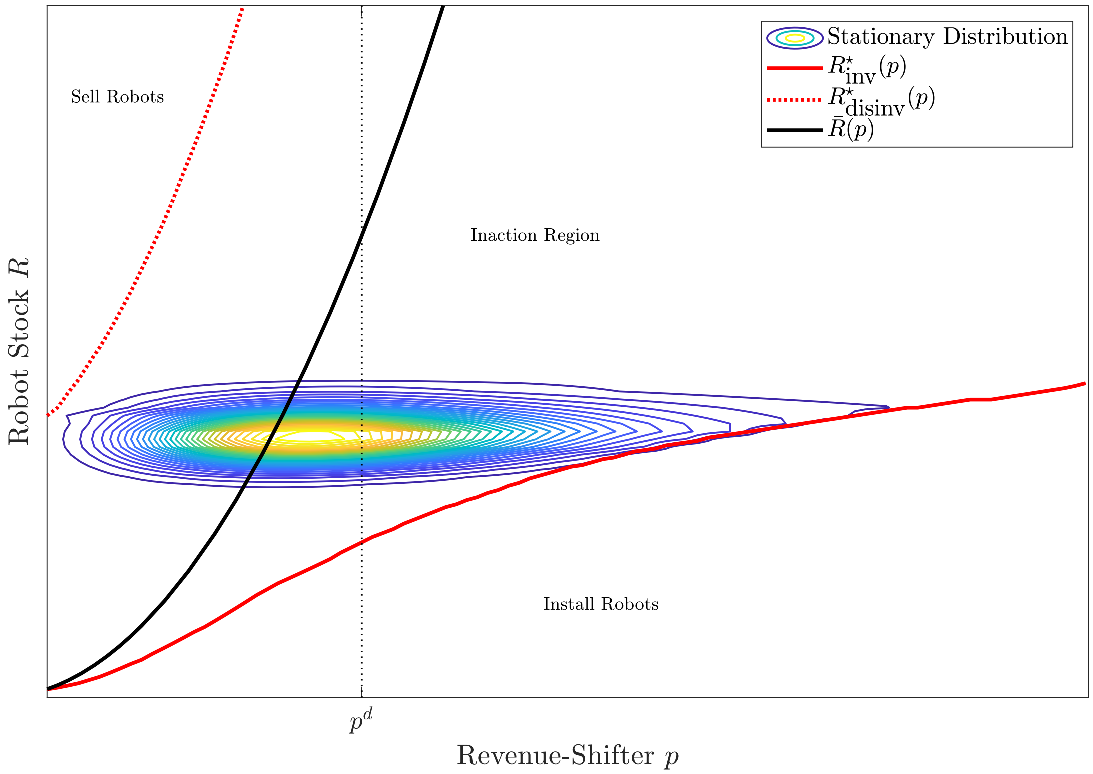
and the firm will adjust towards the investment/disinvestment cutoffs, $R^\star_{\text{inv}}(z)$ and $R^\star_{\text{disinv}}(z)$, if the revenue shock is such that the installed capital stock is outside of the inaction region. The inaction region is a closed interval because the value function inherits the weak concavity of the return function, so there will be unique cutoffs and $R^\star_{\text{inv}}(z) \le R^\star_{\text{disinv}}(z)$.

This solution immediately implies that the stationary distribution will have positive density only inside the inaction region. Therefore, we can numerically verify a result that is analogous to Proposition 6. Given the other parameters and for any p_R, there exists a $\psi_+ > 0$ such that the stationary distribution does not feature full automation. The increase in ψ_+ shifts down the investment barrier $R^\star_{\text{inv}}(z)$ for all z, while $\bar{R}(z)$, the full automation threshold, is invariant to adjustment costs. As a result, it is possible to find a ψ_+ large enough that the investment barrier falls below $\bar{R}(z)$ for all z in a non-zero measure set of the support of $F(z)$. In Figure 12 we show an example of this instance, where it is clear that part of the stationary distribution falls below the $\bar{R}(z)$ schedule.

We sketch the solution of a fixed-cost variant of the model to show that it produces results that are similar to the linear specification we have analyzed thus far. Namely, a fixed cost model would also entail a desired level of capital to which firms want to adjust. Conditional on adjusting, a firm with a revenue shock z would like to set the robot stock to $R^\star_{\text{adj}}(z)$, given by

$$V_R\left(R^\star_{\text{adj}}(z), z\right) = p_R.$$

Figure 12: Stationary Distribution for the Linear Adjustment Costs Model



Given (R, z), a firm that does not adjust gets the value $V(R, z)$, defined recursively as

$$\rho V(R,z) = \Pi(R,z) - \delta R V_R(R,z) + \mu(z) V_z(R,z) + \frac{\sigma^2(z)}{2} V_{zz}(R,z).$$

It follows that, given a fixed cost of adjustment F, a firm chooses to adjust if and only if

$$V\left(R^\star_{\text{adj}}(z), z\right) \geq V(R,z) + p_R\left(R^\star_{\text{adj}}(z) - R\right) + F.$$

Given the properties of the value function, the right hand side of the above expression is concave in R, so in general there exist two robot stock cutoffs, defined as functions of z. Analogous to the case of linear adjustment costs, these functions provide the bounds for an inaction region where firms will just let installed robot stocks depreciate. The condition above clearly shows that a sufficiently high F will push the lower bound of the inaction region towards $R = 0$—and therefore below $\bar{R}(z)$—for any given finite z. As a result, a positive mass of firms will inevitably end up partially automated.

6 Conclusions and Future Work

We have shown that when robot capital involves substantial rigidity, and firms operate in a risky environment, labor can survive substantial innovations in automation if it is relatively more flexible.

This occurs even if there is perfect substitution between factors, and even if robots have a flow cost advantage. However, this comes at a cost: job stability. As the simulated time paths in Figure 5b show, labor is mostly hired in short-lived bursts to cope with demand peaks.

Moreover, our calibrated model has shown that the main threat to labor comes from significant improvements in robot productivity or reductions in robot reallocation frictions. By contrast, we do not envision an important role for falling list prices in generating employment losses.

Our main insight on human labor flexibility has important policy consequences. Labor can survive in the absence of barriers to human-robot substitution, and it can do so by being more flexible than robots. If this source of advantage is taken away—for example by overly rigid regulations—human labor may be eliminated even in the presence of revenue risk and adjustment costs for robots.

While we do not develop this point formally, our framework can speak to the supply side consequences of a host of robot taxes that have been recently proposed. Our calibration suggests that a tax on robot purchases, or one tied to ownership or utilization of robots, might not be enough to significantly alter the incentives for automation if robots become more productive and less costly to install. However, our results suggest that reducing the wage bill for firms, for instance by limiting social security contributions or income taxes on low-income individuals, might go a long way towards safeguarding unskilled and routine jobs.

Flexibility in performing tasks might be an inherently human ability that machines may never be able to reproduce. Our model can be easily extended to incorporate an *intensive* dimension of volatility, given by the stochasticity arising from the complexity of the environment in which robots have to operate. The simplest way to study this dimension in our framework is to focus on a robot-specific productivity shock, and move the scope of our analysis from tasks to jobs. In this setting, each job is an aggregate of tasks, which workers are asked to carry out stochastically. This could shed light on which jobs and sectors are inherently human, and which types of technical innovations could be most threatening for labor. In particular, we expect the survival of labor to be concentrated in supervision tasks to deal with breakdowns and unexpected requests.

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Appendix A Solution of the Static Revenue Maximization Problem

We assume that $\Omega > 0$. The static problem reads

$$\Pi(R,z) \equiv \max_{L\geq 0, 0\leq u\leq 1} z p \left(\Gamma L + (1-\Gamma) u R\right)^{\theta} - w L - m u R.$$

Solving for labor yields,

$$L^{\star}(R,z) = \begin{cases} \frac{1}{\Gamma}\left(\left(\frac{w}{pz\Gamma\theta}\right)^{\frac{1}{\theta-1}} - (1-\Gamma)u^{\star}R\right) & \text{if } uR \leq \left(\frac{w}{pz\Gamma\theta}\right)^{\frac{1}{\theta-1}} \frac{1}{1-\Gamma} \equiv \bar{R}(z) \\ 0 & \text{else} \end{cases}.$$

Thus,

$$\Pi(R,z) \equiv \begin{cases} (1-\theta) p z (1-\Gamma)^{\theta} \bar{R}(z)^{\theta} + \left(\frac{(1-\Gamma)w}{\Gamma} - m\right) u R & uR \leq \bar{R}(z) \\ pz(1-\Gamma)^{\theta} u^{\theta} R^{\theta} - m u R & uR > \bar{R}(z) \end{cases}.$$

Now, by the F.O.C. for u, we obtain $u^{\star}$ in the case of an interior solution,

$$\theta p A (\Gamma L^{\star} + (1-\Gamma) u^{\star} R)^{\theta-1} (1-\Gamma) R - m R = 0.$$

Plugging in optimal labor from above yields,

$$u^{\star}(R,z) = \frac{1}{(1-\Gamma)R}\left(\frac{m}{(1-\Gamma)\theta p A}\right)^{\frac{1}{\theta-1}} = \frac{\hat{R}(z)}{R}.$$

In particular, $u^{\star} = 1$ if

$$\bar{R}(z) \leq R \leq \hat{R}(z) = \left(\frac{pz\theta(1-\Gamma)^{\theta}}{m}\right)^{\frac{1}{1-\theta}},$$

the optimal static scale of the firm. Under our assumption that labor savings are strictly positive, i.e. $\Omega > 0$, it is straightforward to see that $u^{\star}(R,z) > 0$. We now show that in the case where both robots and labor are used, $u^{\star} = 1$. Indeed, for an internal solution in the first region, the F.O.C. requires

$$\frac{w}{\Gamma}(1-\Gamma)R - mR = 0,$$

i.e. no labor savings from having a (fully utilized) robot quantity R. Therefore, in the first region $u^{\star} = 1$ if labor savings are positive. Otherwise, the firm sets $u^{\star} = 0$. This static solution allows us to write the operating profit function as reported in the main text.

Appendix B Omitted Proofs

B.1 Two-Period Model

Proof of Lemma 1. The operating profit function reads:

$$\Pi(R,z)=\begin{cases}(1-\theta)\,pz\,(1-\Gamma)^{\theta}\,\bar{R}(z)^{\theta}+\left(\frac{(1-\Gamma)w}{\Gamma}-m\right)R & R\leq\bar{R}(z)\\ pz(1-\Gamma)^{\theta}R^{\theta}-mR & \hat{R}(z)\geq R>\bar{R}(z)\\ pz(1-\Gamma)^{\theta}\left(\hat{R}(z)\right)^{\theta}-m\hat{R}(z) & \hat{R}(z)<R\end{cases}$$

where:

$$\bar{R}(z)\equiv\frac{1}{1-\Gamma}\left(\frac{pz\theta\Gamma}{w}\right)^{\frac{1}{1-\theta}}\qquad\hat{R}(z)\equiv\frac{1}{1-\Gamma}\left(\frac{pz\theta(1-\Gamma)}{m}\right)^{\frac{1}{1-\theta}}$$

Define $\bar{z}(R),\hat{z}(R)$, as the inverse functions of $\bar{R}(z),\hat{R}(z)$, respectively. The derivative of the profit function with respect to R is given by,

$$\Pi_R(R,z)\equiv\begin{cases}\Omega\equiv\left(\frac{(1-\Gamma)w}{\Gamma}-m\right) & z>\bar{z}(R)\\ \theta zp(1-\Gamma)^{\theta}R^{\theta-1}-m & \bar{z}(R)\geq z>\hat{z}(R)\\ 0 & z\leq\hat{z}(R)\end{cases}.$$

Under the assumption of positive labor savings Ω, the above is non-negative $\forall z$. Furthermore, this derivative is decreasing in R. Indeed, both $\bar{z}(R)$ and $\hat{z}(R)$ are increasing in R, since they are the inverses of the functions $\bar{R}(z),\hat{R}(z)$, which are increasing in z. Moreover, its maximum is given by Ω (which in particular is also the value of the derivative when R is 0). Now, for a given $(\tilde{R},\tilde{z})$, consider an increase in z. If the initial $\tilde{z}$ is not at any of thresholds, taking derivatives shows that $\Pi_R(\tilde{R},\tilde{z})$ is strictly increasing in z if the $(\tilde{R},\tilde{z})$ are such that $\bar{z}(\tilde{R})>\tilde{z}>\hat{z}(\tilde{R})$. Otherwise, $\Pi_{R,z}(\tilde{R},\tilde{z})=0$. Finally, function is increasing in z if $\tilde{z}=\bar{z}(\tilde{R})$, or $\tilde{z}=\hat{z}(\tilde{R})$, since $\Omega\geq\theta zp(1-\Gamma)^{\theta}R^{\theta-1}-m\geq0$ for the values of R that imply full robot-stock utilization. It follows that $\Pi_R(R,z)$ is increasing in z. Following similar steps for R, shows that $\Pi_R(R,z)$ is decreasing in R.

The derivative of the operating profit function with respect to z reads

$$\Pi_z(R,z)\equiv\begin{cases}(1-\theta)\,p\,(1-\Gamma)^{\theta}\,\bar{R}(z)^{\theta}+(1-\theta)\,pz\,(1-\Gamma)^{\theta}\,\theta\bar{R}(z)^{\theta-1}\bar{R}_z(z) & z>\bar{z}(R)\\ p(1-\Gamma)^{\theta}R^{\theta} & \bar{z}(R)\geq z>\hat{z}(R)\\ p(1-\theta)\left(\hat{R}(z)\right)^{\theta}+pz(1-\theta)\theta\left(\hat{R}(z)\right)^{\theta-1}\hat{R}_z(z) & z\leq\hat{z}(R)\end{cases}.$$

Where the last line follows from replacing the definition of $\hat{R}(z)$ into $\Pi(R,z)$ for $\hat{R}(z)<R$, which

yields,

$$
\begin{aligned}
pz(1-\Gamma)^{\theta}\left(\hat{R}(z)\right)^{\theta}-m\hat{R}(z) &= pz(1-\Gamma)^{\theta}\left(\frac{1}{1-\Gamma}\left(\frac{pz\theta(1-\Gamma)}{m}\right)^{\frac{1}{1-\theta}}\right)^{\theta}-\frac{m}{1-\Gamma}\left(\frac{pz\theta(1-\Gamma)}{m}\right)^{\frac{1}{1-\theta}} \\
&= \left(\frac{pz\theta(1-\Gamma)}{m}\right)^{\frac{\theta}{1-\theta}}(pz-\theta pz) \\
&= pz(1-\theta)\hat{R}(z)^{\theta}.
\end{aligned}
$$

Further note that

$$
\hat{R}_z(z)\equiv\frac{\partial\hat{R}(z)}{z}=\frac{1}{1-\theta}z^{\frac{\theta}{1-\theta}}\frac{1}{1-\Gamma}\left(\frac{p\theta(1-\Gamma)}{m}\right)^{\frac{1}{1-\theta}}=\frac{1}{(1-\theta)z}\hat{R}(z)\geq 0,
$$

and similarly,

$$
\bar{R}_z(z)\equiv\frac{\partial\bar{R}(z)}{z}=\frac{1}{(1-\theta)z}\bar{R}(z)\geq 0.
$$

We can therefore rewrite

$$
\Pi_z(R,z)\equiv\begin{cases} p\,(1-\Gamma)^{\theta}\,\bar{R}(z)^{\theta} & z>\bar{z}(R) \\ p(1-\Gamma)^{\theta}R^{\theta} & \bar{z}(R)\geq z>\hat{z}(R) \\ p\,(1-\Gamma)^{\theta}\left(\hat{R}(z)\right)^{\theta} & z\leq\hat{z}(R)\end{cases}.
$$

It immediately follows that the derivative $\Pi_z(R,z)\geq 0$ for all (R,z). Moreover,

$$
\Pi_{zz}(R,z)\equiv\begin{cases} p\,(1-\Gamma)^{\theta}\,\theta\,\bar{R}(z)^{\theta-1}\bar{R}_z(z) & z>\bar{z}(R) \\ 0 & \bar{z}(R)\geq z>\hat{z}(R) \\ p\,(1-\Gamma)^{\theta}\,\theta\left(\hat{R}(z)\right)^{\theta-1}\hat{R}_z(z) & z\leq\hat{z}(R)\end{cases},
$$

is nonnegative, given the derivatives reported above. □

Proof of Proposition 2. First note that $\beta\mathbb{E}[\Pi_R(R_1,z_1)\,|\,z_0]$ is decreasing in R_1 for all z_0,z_1. In particular, given the definition above, it is continuous with limits given by,

$$
\beta\mathbb{E}[\Pi_R(R_1,z_1)\,|\,z_0]=\beta\Omega\quad\text{if } R_1\to 0,\qquad \beta\mathbb{E}[\Pi_R(R_1,z_1)\,|\,z_0]\to 0\quad\text{if } R_1\to\infty.
$$

The LHS of the FOC starts at $p_R-\psi_R R_0$ and goes to ∞ as $R_1\to\infty$. It follows that there is a unique solution $R_1^{\star}$ of the FOC. In the trivial case that $p_R>\beta\Omega+\psi_R R_0$, the firm sets $R_1=0$, and it only produces with labor in period $t=1$. When the solution for $R_1^{\star}$ is interior, the unbounded support ensures that, $\forall R_1^\star$, $\exists \bar{z}(R_1^\star)$ such that, $\forall z > \bar{z}(R_1^\star)$, $\bar{R}(z) > R_1^\star$, and therefore $L_1^\star(z) > 0$. It follows that

$$L_1^d \equiv \int_0^\infty L_1^\star(z) f(z) \mathrm{d}z > 0.$$

□

B.2 Continuous Time Model

Proof of Lemma 2. The properties cited are inherited by the instantaneous operating profit function $\Pi(R, z)$ (see e.g. Stokey (2009), p.233). Following Stokey (2009) (p.229), and considering a small time interval $\mathrm{d}t \to 0$, we can write an approximation for the value function as follows:

$$\begin{aligned} V(R_0, z_0) \approx\ & \Pi(R_0, z_0)\mathrm{d}t - p_R\left(I^\star(R_0, z_0)\right) - \frac{\psi_R}{2}\left(I^\star(R_0, z_0)\right)^2 + \\ & + \frac{1}{1+\rho \mathrm{d}t}\mathbb{E}\left[V(R_0 + \mathrm{d}R, z_0 + \mathrm{d}z)\right] \end{aligned} \tag{2}$$

Where:

$$\mathrm{d}R \equiv \left[I^\star(R_0, z_0) - \delta R_0\right]\mathrm{d}t,$$

is the drift associated to optimal investment $I^\star(R_0, z_0)$. Note that the above Equation 2 defines a contraction mapping $T_{\mathrm{d}t}(V)$ for all $\mathrm{d}t > 0$. We can therefore apply the results in Stokey et al. (1989) Corollary 1 (p.52). In particular, it is straightforward to verify the following properties of V that derive directly from the properties of $\Pi(R, z)$ summarized in Lemma 1:[18]

- $\Pi(R, z)$ increasing in R for all z implies that $V(R, z)$ is increasing in R for all z, $\Pi(R, z)$ weakly concave in R for all z implies that $V(R, z)$ is weakly concave in R for all z.[19]

- If the FOSD property is assumed, $\Pi(R, z)$ increasing in z for all R implies that $V(R, z)$ is increasing in z for all R, as it is immediate to see that $T_{\mathrm{d}t}$ will map a V increasing in z for all R

[18]These properties follow directly from Π since the Envelope Theorem ensures that the effects of changes in z_0 through the optimal investment $I^\star$ and the associated drift $\mathrm{d}R$ on the maximized value function V are second order.

[19]An alternative proof of concavity follows Dixit and Pindyck (1994), 1993, p.360. Consider two initial values of R, R_1, R_2 with associated optimal paths $\{R_{1,t}\}, \{R_{2,t}\}$ and investment policies $\{\Delta R_{1,t}\}, \{\Delta R_{2,t}\}$. Now consider the firm having initial capital stock:

$$\alpha R_1 + (1-\alpha) R_2, \quad \alpha \in [0,1]$$

Consider now the path $\{\alpha R_{1,t} + (1-\alpha) R_{2,t}\}$. This is clearly feasible, so V will have a value at least as large as the one obtained following this path. Following such path the firm obtains in each instant:

$$\begin{aligned} u_t\left(\{\alpha R_{1,t} + (1-\alpha) R_{2,t}\}, z_t\right) \equiv\ & \Pi\left(\alpha R_{1,t} + (1-\alpha) R_{2,t}, z_t\right) - p_R\left(\alpha \Delta R_{1,t} + (1-\alpha)\Delta R_{2,t}\right) + \\ & - \Psi\left(\alpha \Delta R_{1,t} + (1-\alpha)\Delta R_{2,t}\right) \end{aligned}$$

By concavity of Π and convexity of $\Psi(I) \equiv \frac{\psi_R}{2} I^2$:

$$u_t\left(\{\alpha R_{1,t} + (1-\alpha) R_{2,t}\}, z_t\right) \geq \alpha u_t\left(R_{1,t}, z_t\right) + (1-\alpha) u_t\left(R_{2,t}, z_t\right)$$

into a value function $V' = T(V)$ that has the same property. This is ensured by the definition of FOSD. For any $z_0'' > z_0'$, $\mathbb{E}\left[V(R_0 + \mathrm{d}R, z_0'' + \mathrm{d}z)\right] \geq \mathbb{E}\left[V(R_0 + \mathrm{d}R, z_0' + \mathrm{d}z)\right]$. The statement follows combining this fact with the fact that $\Pi(R, z)$ is increasing in z for all R.

To prove the properties of V_R, Consider the Envelope condition that can be obtained by differentiating the HJB in R side by side. Doing so yields:

$$\left(\rho + \delta\right) V_R(R,z) - \Pi_R(R,z) - \mu(z)\, V_{Rp}(R,z) - \frac{1}{2}\sigma^2(z)\, V_{Rpp}(R,z) - \dot{R}\, V_{RR}(R,z) = 0,$$

which can be rewritten using Itō's formula as:

$$\left(\rho + \delta\right) V_R(R,z) - \Pi_R(R,z) - \mathbb{E}\left[\frac{\mathrm{d}V_R(R,z)}{\mathrm{d}t}\right] = 0,$$

As for Equation (2), we can use an approximation for the derivative of the value function along the optimal path as:

$$V_R(R_0, z_0) \approx \Pi_R(R_0, z_0)\mathrm{d}t + \frac{1}{1 + (\rho + \delta)\mathrm{d}t}\mathbb{E}\left[V_R(R_0 + \mathrm{d}R, z_0 + \mathrm{d}z)\right]. \tag{3}$$

The RHS of Equation (3) also defines a contraction $T_{R,\mathrm{d}t}$ for any $\mathrm{d}t > 0$, as it satisfies the hypotheses of Blackwell's theorem.[20] To prove that $V_{RR} \leq 0$, consider a function V_R' that satisfies the property. Then it is immediate to see that $V_R'' = T_{R,\mathrm{d}t}(V_R')$ satisfies it as well. Indeed, the returns function $\Pi_R(R,z)$ has weakly negative derivative in R as well. It follows that the operator maps weakly concave value functions into weakly concave value functions. Since $T_{R,\mathrm{d}t}$ is a contraction mapping, we conclude that $V_{RR} \leq 0$. A similar reasoning shows that $V_{Rw} > 0$, and, if the FOSD property holds, that $V_{Rz} \geq 0$.

To prove the upper bound of $V_R(R,z)$, we rewrite the envelope condition using Lemma 1 in Ap-

---

This implies:

$$\begin{aligned}
E_0 \int_0^\infty e^{-\rho t} u_t\left(\left\{\alpha R_{1,t} + (1-\alpha) R_{2,t}\right\}, z_t\right)\mathrm{d}t &\geq \alpha E_0 \int_0^\infty e^{-\rho t} u_t\left(R_{1,t}, z_t\right)\mathrm{d}t + \\
&\quad + (1-\alpha) E_0 \int_0^\infty e^{-\rho t} u_t\left(R_{2,t}, z_t\right)\mathrm{d}t \\
E_0 \int_0^\infty e^{-\rho t} u_t\left(\left\{\alpha R_{1,t} + (1-\alpha) R_{2,t}\right\}, z_t\right)\mathrm{d}t &\geq \alpha V(R_1, z_0) + (1-\alpha) V(R_2, z_0) \\
V(\alpha R_1 + (1-\alpha) R_2, P) &\geq \alpha V(R_1, z_0) + (1-\alpha) V(R_2, z_0)
\end{aligned}$$

Proving the statement.

[20]Monotonicity follows immediately from the fact that the operator is linear in V_R, while discounting is ensured by $(\rho + \delta)\mathrm{d}t > 0$ for all $\mathrm{d}t > 0$.

pendix B of Abel and Eberly (1993, p.22) to solve for $V_R(R,z)$ as[21]

$$V_R(R_0,z_0) = \mathbb{E}\left[\int_0^\infty e^{-(\rho+\delta)t}\Pi_R(R_t,z_t)\mathrm{d}t\right].$$

Using the bounds for $\Pi(R,z)$ in Lemma 1,

$$V_R(R_0,z_0) \leq \mathbb{E}\left[\int_0^\infty e^{-(\rho+\delta)t}\Omega\mathrm{d}t\right] = \frac{\Omega}{\rho+\delta}.$$

□

Proof of Proposition 4. By the proof of Proposition 3, we can write,

$$I^\star(R_t,z_t) = \frac{1}{\psi_R}\left(\int_t^\infty e^{-(\rho+\delta)(s-t)}\mathbb{E}_t\left\{\Pi_R\left(R_s^\star,z_s\right)\right\}\mathrm{d}s - p_R\right).$$

Removing the expectation and evaluating at the steady state we obtain, for $R^\star > 0$,

$$\delta R^\star(z^d) = \frac{1}{\psi_R}\left(\frac{\Pi_R(R^\star(z^d),z^d)}{\rho+\delta} - p_R\right).$$

First, note that, given z^d, we have

$$\Pi_R(R^\star(z^d),z^d) = \begin{cases} \min\left\{\Omega, \theta z^d p(1-\Gamma)^\theta R^\star(z^d)^{\theta-1} - m\right\} & R^\star(z^d) < \hat{R}(z^d) \\ 0 & R^\star(z^d) \geq \hat{R}(z^d). \end{cases}$$

Therefore, $R^\star(z^d) = 0$ if $p_R > \frac{\Omega}{\rho+\delta}$, regardless of the value of ψ_R, δ. If either $\psi_R, \delta = 0$, the FOC for investment gives

$$\frac{\Pi_R(R^\star(z^d),z^d)}{\rho+\delta} = p_R.$$

So if there are strictly positive labor savings, $0 < p_R < \frac{\Omega}{\rho+\delta}$—our main assumption—the optimal solution entails full automation, $R^\star(z^d) > \bar{R}(z^d)$.[22]

Turning to the case $\delta, \psi_R > 0$ and rearranging,

$$\psi_R\delta R^\star(z^d) + p_R = \frac{\Pi_R(R^\star(z^d),z^d)}{\rho+\delta}$$

the above equation has a unique solution as $\Pi(R,z)$ is weakly concave in R, so the RHS is strictly

[21] This part shows the case of a diffusion, but the same reasoning can be applied to a CTMC, by replacing the stochastic terms appropriately. Indeed, while the statement of Lemma 1 in Abel and Eberly (1993) considers a diffusion, all their passages can be directly applied to a CTMC, and to the case where the function of interest is multivariate, as in our case.

[22] When $\frac{\Omega}{\rho+\delta} = p_R$, the firm is indifferent between all values of the robot stock satisfying, $R^\star(z^d) \leq \bar{R}(z^d)$, so there is a continuum of steady state distributions that depend on the initial distribution of robot stocks.

increasing from p_R to ∞ and the LHS in weakly decreasing from $\frac{\Omega}{\rho+\delta}$ to 0. Since the revenue shock is fixed at z^d, this unique solution is the steady state of the model. Moreover, by Corollary 1 we have that $R^\star(z^d)$ is strictly decreasing in ψ_R and goes to 0 as ψ_R tends to infinity. It follows that there exists a finite $\tilde{\psi}_R$ such that $R^\star(z^d) < \bar{R}(z^d)$ for all $\psi_R > \tilde{\psi}_R$.

□

Proof of Proposition 5. For ease of notation, define $R_1^\star = R^\star(z_1)$, $R_N^\star = R^\star(z_N)$. The KFE for a Poisson process reads, defining $G\left(\bar{R}, z_i, t\right) = Pr\left(R_t \leq \bar{R}, z_t = z_i\right)$:

$$\frac{\partial}{\partial t}G(R_t, z_i, t) = -\frac{\partial R_t}{\partial t}\frac{\partial}{\partial R_t}G(R_t, z_i, t) - \sum_{j\neq i}\lambda_{ij}G(R_t, z_i, t) + \sum_{j\neq i}\lambda_{ji}G\left(R_t, z_j, t\right)$$

Integrating over z's yields:

$$\begin{aligned}\sum_i \frac{\partial}{\partial t}G(R_t, z_i, t) &= -\sum_i \frac{\partial R_t}{\partial t}\frac{\partial}{\partial R_t}G(R_t, z_i, t) + \\ &\quad + \sum_i\left\{-\sum_{j\neq i}\lambda_{ij}G(R_t, z_i, t) + \sum_{j\neq i}\lambda_{ji}G\left(R_t, z_j, t\right)\right\}\end{aligned}$$

At the stationary distribution it holds:

$$0 = -\sum_i \frac{\partial R}{\partial t}\frac{\partial}{\partial R}G(R, z_i, \infty)$$

since the last terms in parenthesis cancel out by definition of a stationary distribution. Note by definition of $R_N^\star$: $\frac{\partial R_t(R_N^\star, z_N)}{\partial t} = \left(I^\star(R_N^\star, z_N) - \delta R_N^\star\right) = 0$. By Corollary 1, investment is increasing in z and decreasing in R. It follows that

$$\frac{\partial R_t(R, z)}{\partial t} < 0 \quad \forall R > R_N^\star \wedge z \leq z_N$$

To avoid contradiction, in a stationary distribution we must have: $G(R, z, \infty) = 0 \quad \forall R > R_N^\star$. A similar argument can be made for $R_1^\star$ by flipping all inequalities. Combining the two arguments, $G(R, z_i, \infty) = 0 \quad \forall R \notin [R_1^\star, R_N^\star]$.

Now, consider a diffusion with KFE:

$$0 = -\frac{\partial\left[[I^\star(R, z) - \delta R]g(R, z, t)\right]}{\partial R} - \frac{\partial\left[\mu(z)g(R, z, t)\right]}{\partial z} + \frac{1}{2}\frac{\partial^2\left[\sigma^2(z)g(R, z, t)\right]}{\partial z^2}$$

Suppose $g(R, z, t) > 0$ for some $R < R_1^\star$ at all $z \in [z_1, z_N]$. Now, for all $R < R_1^\star$, and for all $z \in [z_1, z_N]$, the investment drift is strictly positive, by definition of $R_1^\star$. Therefore, the joint distribution over (R, z) will feature outflows in R for each revenue shock $z \in [z_1, z_N]$. Integrating over z, the marginal stationary distribution for R is positive below $R_1^\star$. Since $I^\star(R,z)-\delta R>0, \forall z, R<R_1^\star$,

$$-\mathrm{d}\big[[I^\star(R,z)-\delta R]g(R,z,t)\big]<0 \quad \forall z, R<R_1^\star,$$

directly contradicting the definition of stationary distribution. Assume now that $\delta>0$, then we can reason as above by flipping all inequalities for $R>R_{\max}^\star$. □

Proof of Proposition 7. By Proposition 3, investment is non-decreasing in w. As a result, the cutoffs $R^\star(z)$ are non-decreasing in w as well. From the Kolmogorov Forward Equation in Proposition 5, the stationary distribution entails weakly higher robot stocks for each z. Moreover, the cutoffs $\bar{R}(z)$ are strictly decreasing in w, which implies that individual labor demand is non-increasing in w for all (R,z) (strictly decreasing for firms with positive labor demand). Finally, individual labor demand is non-increasing in R for all z. Combining all these facts, the integral,

$$\int_{\mathscr{S}} L^\star(R,z)\mathrm{d}G(R,z),$$

is non-increasing in w. Note that as $w\to\infty$, the individual labor demand falls to 0. As $w\to 0$, we instead have that $\bar{R}(z)\to\infty$, implying that the individual labor policy tends to ∞ as well. □

B.3 General Equilibrium

Proof of Proposition 8. For ease of notation, define

$$E_s \equiv \int_0^\infty z^{\frac{1}{1-\theta_s}}\,dF_s(z).$$

We define an equilibrium as in appendix D.3. Under the assumptions that $\theta_s=\theta$ for all s, and $\sigma=1$, the system of equations pinning down the equilibrium prices, $(w,\mathbf{p})$ reads,

$$LM(w,\mathbf{p};\boldsymbol{\theta}) \equiv \sum_{s=1}^N \left\{ E_s\left(\frac{p_s\theta_s\Gamma_s^{\theta_s}}{w}\right)^{\frac{1}{1-\theta_s}} - \frac{1-\Gamma_s}{\Gamma_s}R_{\max,s}^\star\right\} - \left(\frac{w}{\chi}\right)^{\varphi} = 0,$$

$$IM_s(w,\mathbf{p};\boldsymbol{\theta}) \equiv \left[E_s\left(\frac{p_s\theta_s\Gamma_s}{w}\right)^{\frac{\theta_s}{1-\theta_s}}\right] - \xi_s\frac{\sum_j p_j^{\frac{1}{1-\theta_s}} E_j\left(\frac{\theta_j\Gamma_j}{w}\right)^{\frac{\theta_j}{1-\theta_j}}}{p_s} = 0 \qquad \forall s=1,\dots,N,$$

where the symbol $\boldsymbol{\theta}$ represents a generic parameter affecting the equilibrium on which we wish to perform the comparative statics. Here, the function $LM(\cdot)$ expresses excess demand for labor, while the functions $IM_s(\cdot)$'s give the excess demands for each intermediate good market. By Walras' law, we can omit the final good market. By the implicit function theorem, we can characterize the effect that a change in any of these parameters has on the equilibrium price system. To this end, premultiply the vector of direct effects of the parameters on the system of equation by the inverse of the Jacobian of the system with respect to prices. This yields,

$$\begin{bmatrix} \frac{dw}{d\boldsymbol{\theta}} \\ \frac{dp_1}{d\boldsymbol{\theta}} \\ \vdots \\ \frac{dp_N}{d\boldsymbol{\theta}} \end{bmatrix} = - \begin{bmatrix} \frac{dLM(w,\mathbf{p};\boldsymbol{\theta})}{dw} & \frac{dLM(w,\mathbf{p};\boldsymbol{\theta})}{dp_1} & \cdots & \frac{dLM(w,\mathbf{p};\boldsymbol{\theta})}{dp_N} \\ \frac{dIM_1(w,\mathbf{p};\boldsymbol{\theta})}{dw} & \frac{dIM_1(w,\mathbf{p};\boldsymbol{\theta})}{dp_1} & \cdots & \frac{dIM_1(w,\mathbf{p};\boldsymbol{\theta})}{dp_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{dIM_N(w,\mathbf{p};\boldsymbol{\theta})}{dw} & \frac{dIM_N(w,\mathbf{p};\boldsymbol{\theta})}{dp_1} & \cdots & \frac{dIM_N(w,\mathbf{p};\boldsymbol{\theta})}{dp_N} \end{bmatrix}^{-1} \begin{bmatrix} \frac{dLM(w,\mathbf{p};\boldsymbol{\theta})}{d\boldsymbol{\theta}} \\ \frac{dIM_1(w,\mathbf{p};\boldsymbol{\theta})}{d\boldsymbol{\theta}} \\ \vdots \\ \frac{dIM_N(w,\mathbf{p};\boldsymbol{\theta})}{d\boldsymbol{\theta}} \end{bmatrix}.$$

Starting from the first row of the Jacobian, we can see that,

$$\frac{dLM(w,\mathbf{p};\boldsymbol{\theta})}{dw} = -\sum_{s=1}^{N} \left\{ \frac{1}{1-\theta_s} E_s \left(\frac{p_s \theta_s \Gamma_s^{\theta_s}}{w^{\theta_s}} \right)^{\frac{1}{1-\theta_s}} + \frac{1-\Gamma_s}{\Gamma_s} \underbrace{\frac{dR^{\star}_{\max,s}}{dw}}_{>0} \right\} - \varphi\chi \left(\frac{w}{\chi} \right)^{\varphi-1} < 0,$$

$$\frac{dLM(w,\mathbf{p};\boldsymbol{\theta})}{dp_s} = \frac{1}{1-\theta_s} E_s \left(\frac{p_s^{\theta_s} \theta_s \Gamma_s^{\theta_s}}{w} \right)^{\frac{1}{1-\theta_s}} > 0 \qquad \forall s = 1, \ldots, N.$$

Next, considering the effect of the wage on the market clearing condition in each sector,

$$\frac{dIM_s(w,\mathbf{p};\boldsymbol{\theta})}{dw} = \frac{dY_s^S}{dw} - \frac{\xi_s}{p_s} \left[\sum_j \left\{ p_j \frac{dY_j^S}{dw} \right\} \right] \lesseqgtr 0.$$

However, note that the above is monotone in ξ_s. Plugging in the expressions for dY_s^S / dw, yields

$$\frac{dIM_s(w,\mathbf{p};\boldsymbol{\theta})}{dw} = \frac{1}{p_s} \left\{ -\frac{\theta_s}{1-\theta_s} w p_{si} \underbrace{\left(\frac{p_s \theta_s \Gamma_s}{w} \right)^{\frac{\theta_s}{1-\theta_s}} E_s}_{=Y_s^S} + \xi_s \sum_j p_j Y_j^S \frac{\theta_j}{1-\theta_j} w \right\}.$$

Setting the above to 0 and rearranging, delivers a threshold value $\bar{\xi}_s$,

$$\bar{\xi}_s = \frac{\frac{\theta_s}{1-\theta_s}}{\sum_j \xi_j \frac{\theta_j}{1-\theta_j}} \xi_s.$$

Note that if $\theta_s = \theta_j$ for all $s \neq j$, then $\bar{\xi}_s = \xi_s \ \forall s = 1, \ldots, N$. That is, under the assumption that decreasing returns to scale are equal across sectors, there are no effect on excess demand coming from the wage only. Finally,

$$\frac{dIM_s(w,\mathbf{p};\boldsymbol{\theta})}{dp_s} = (1-\xi_s) \frac{dY_s^S}{dp_s} + \frac{\xi_s}{p_s^2} \sum_{j \neq s} p_j Y_j^S > 0,$$

$$\frac{dIM_s(w,\mathbf{p};\boldsymbol{\theta})}{dp_j} = -\frac{\xi_s}{p_s} p_j \frac{dY_j^S}{dp_j} < 0.$$

Under the assumption that $\theta_s = \theta$ for all s, the sign pattern of the Jacobian is,

$$\text{sign}(J) = \begin{bmatrix} - & + & \cdots & \cdots & + \\ 0 & + & - & \cdots & - \\ \vdots & - & + & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & - \\ 0 & - & \cdots & - & + \end{bmatrix} = \begin{bmatrix} - & + \quad \cdots \quad \cdots \quad + \\ 0 & \\ \vdots & \\ \vdots & \underbrace{\begin{bmatrix} + & - & \cdots & - \\ - & + & \ddots & \vdots \\ \vdots & \ddots & \ddots & - \\ - & \cdots & - & + \end{bmatrix}}_{\equiv S} \\ 0 & \end{bmatrix}.$$

Note that m, p_R and ψ_R do not affect the excess demand for any of the intermediate goods, and they only affect the excess demand for labor. Thus,

$$\text{sign}\left(\begin{bmatrix} \frac{dLM(w,\mathbf{p};\boldsymbol{\theta})}{dp_R} \\ \frac{dIM_1(w,\mathbf{p};\boldsymbol{\theta})}{dp_R} \\ \vdots \\ \frac{dIM_N(w,\mathbf{p};\boldsymbol{\theta})}{dp_R} \end{bmatrix} \right) = \begin{bmatrix} + \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Therefore,

$$\frac{\mathrm{d}w}{\mathrm{d}p_R} = -(J^{-1})_{11} \frac{dLM(w,\mathbf{p};\boldsymbol{\theta})}{dp_R}.$$

Due to the structure of the matrix J,

$$(J^{-1})_{11} = \frac{1}{\det|J|} \det|S| = \frac{1}{J_{11}\det|S|} \det|S| = \frac{1}{J_{11}}.$$

We therefore have that,

$$\frac{\mathrm{d}w}{\mathrm{d}p_R} = -\frac{\frac{dLM(w,\mathbf{p};\boldsymbol{\theta})}{dp_R}}{\frac{dLM(w,\mathbf{p};\boldsymbol{\theta})}{dw}} > 0,$$

since,

$$\frac{dLM(w,\mathbf{p};\boldsymbol{\theta})}{dp_R} = \frac{1}{\psi_R \delta} \sum_{s=1}^{N} \left\{ \frac{1-\Gamma_s}{\Gamma_s} \right\} > 0.$$

The proof follows the same steps for ψ_R and m. Indeed,

$$\text{sign}\left(\begin{bmatrix} \frac{dLM(w,\mathbf{p};\boldsymbol{\theta})}{d\psi_R} \\ \frac{dIM_1(w,\mathbf{p};\boldsymbol{\theta})}{d\psi_R} \\ \vdots \\ \frac{dIM_N(w,\mathbf{p};\boldsymbol{\theta})}{d\psi_R} \end{bmatrix} \right) = \text{sign}\left(\begin{bmatrix} \frac{dLM(w,\mathbf{p};\boldsymbol{\theta})}{dm} \\ \frac{dIM_1(w,\mathbf{p};\boldsymbol{\theta})}{dm} \\ \vdots \\ \frac{dIM_N(w,\mathbf{p};\boldsymbol{\theta})}{dm} \end{bmatrix} \right) = \begin{bmatrix} + \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

and,

$$\frac{dLM(w,\mathbf{p};\boldsymbol{\theta})}{d\psi_R} = \frac{1}{\psi_R}\sum_{s=1}^{N}\left\{\frac{1-\Gamma_s}{\Gamma_s}\right\} R^*_{\max,s} > 0$$

$$\frac{dLM(w,\mathbf{p};\boldsymbol{\theta})}{dm} = \frac{1}{\psi_R\delta}\frac{1}{\rho+\delta}\sum_{s=1}^{N}\left\{\frac{1-\Gamma_s}{\Gamma_s}\right\} > 0.$$

Since there is no wealth effect on the labor supply, the comparative statics on aggregate employment can be carried out trivially by noting that the supply of labor is only an increasing function of the equilibrium wage. □

Appendix C Calibration

C.1 Illustrative Figures Calibration

The following table reports the detailed parametrization of the model we adopt for the illustrative figures in the main text.

Description	Variable/Parameter	Value
Wage	w	0.5
Robot Purchase Price	p_R	1.5
Normalized MRTS	Γ	0.5
Flow robot cost	m	0.2
Robot depreciation	δ	$\log(1+1/12)$
Discount factor	ρ	$\log(1+0.04)$
DRS parameter	θ	0.5
Robot adjustment cost	ψ_R	25
CIR mean-reversion parameter	θ_p	0.1
CIR long-run variance	σ_p	0.4
CIR log-run mean	z^d	1
Total factor productivity	p	1

C.2 Data Sources

The data sources that inform our calibration are the following:

- Compustat Daily Updates - Fundamentals Annual (1950-2019) for US firms accessed through the WRDS service. We use the series: "sale" (firm sales), "emp" (firm employment), "ppent" (net property, plant and equipment), "ppegt" (gross property, plant and equipment). We use "gvkey" identifiers and four-digit SIC codes to classify industries into the 19 sectors covered by the IFR statistics;

- Crosswalk between IFR and SIC sectors kindly provided by Daron Acemoglu and Pascual Restrepo;

- Data on robots per worker in 2010 and 2014 from the replication files for Acemoglu and Restrepo (2018d), retrieved at: `https://economics.mit.edu/faculty/acemoglu/data`. which aggregates IFR data by sector;

- We construct factor shares using the historical SIC-level data from the NBER-CES dataset, retrieved at: `http://www.nber.org/nberces/`.

- Fixed investment deflator (BEA): Implicit price deflator for fixed gross private investment deflator, seasonally adjusted quarterly and averaged annually. `https://fred.stlouisfed.org/series/A007RD3Q086SBEA`;

- GDP deflator by the BEA, seasonally adjusted quarterly and averaged annually. Retrieved at: https://fred.stlouisfed.org/series/GDPDEF;
- We use data from ARK investment and BCG to obtain a ballpark of the relative robot cost. In particular, Korus (2019) and Sirkin et al. (2015) contain time series for robot unit costs up to 2014;
- Average annual wage for production workers from the Occupation Employment Statistics database retrieved at: https://www.bls.gov/oes/tables.htm;
- Value added by industry from the BEA GDP-by-industry dataset. Retrieved at: https://apps.bea.gov/iTable/index_industry_gdpIndy.cfm

C.3 Calibrating the Parameters of the CIR Stochastic Process

The first step towards calibrating the model is estimating the parameters of the CIR process for z used throughout the paper. In order to do so, we choose to use Compustat firm-level TFP. We use the SIC codes to classify firms into the 13 IFR sectors with the help of the crosswalk mentioned above. We employ The following specification to recover TFP,

$$\log(\text{Sales}_{sit}) = \gamma_i + f_{ts} + \alpha_s \log(\text{Emp}_{sit}) + \beta_s \log(\text{K}_{si,t-1}) + \varepsilon_{ist}.$$

We include firm fixed effects, time-by-IFR-sector fixed effects and assume that the technological parameters α and β are the same for all firms within the same IFR sector. We deflate sales by the GDP deflator and capital stocks by the fixed investment deflator. Our real capital stock measure is built by perpetual inventory method, using as a starting point the total gross property plant and equipment ("ppegt") in the first year the firm appears in our panel, deflated by the fixed investment deflator for that year. We then add the net investment in property plant and equipment obtained using the differences of the series "ppent", deflated by the corresponding fixed investment deflator.[23]

The residuals of the above regression give an estimate of log firm-level TFP, from which we also remove a firm-level trend, in order to make the estimated series stationary. We then exponentiate the resulting series and we assume that all firms in the same IFR sector share the same CIR process parameters. We estimate the parameters of the CIR process by maximum likelihood, following the procedure in Wei et al. (2016), amended to account for the fact that all firms within each sector share the same process.[24] We estimate the parameters $\{a_s, b_s, \zeta_s\}$ for each sector s in the following CIR process,

$$\mathrm{d}z_{ts} = (a_s - b_s z_{ts})\mathrm{d}t + \zeta_s \sqrt{z_{ts}}\mathrm{d}W,$$

[23] We deal with the missing values for "ppent" by interpolating linearly using the nearest non-missing observations.

[24] This amounts to assuming that all firms are sampled in an i.i.d. fashion in each sector, so that we can pool the data of all firms in each sector.

which we then normalize to obtain a mean of 1 for each sector, consistently with our modeling choices. Thus, we calibrate θ_{ps} and σ_s as,

$$\theta_{ps} = \hat{b}_s, \qquad \sigma_s = \frac{\zeta_s}{\sqrt{\hat{a}_s/\hat{b}_s}}.$$

C.4 Calibrating the Parameter θ

In order to calibrate θ in a model-consistent way, we target the share of income going to production-line employees in the sectors of interest. Our model implies that this quantity is exactly equal to θ when aggregate robot penetration, defined as R/L, is zero. To obtain the relevant labor share, we use the NBER-CES data for 1958-2011, and compute the share of income going to production-line employees as the wage bill of production-line employees over value added. Once again, this quantity is computed by IFR sector, to which we map the SIC sectors by using our crosswalk. In order to purge the estimated series by cyclical fluctuations, we apply a HP filter with smoothing parameter, $\lambda = 6.25$, to our annual-frequency data, and keep the trend component.

We do not have data for robot penetration before the year 2004, so we cannot establish exactly when the robot penetration is sufficiently close to 0. However, the data reported in Acemoglu and Restrepo (2018d) suggests that most sectors had a reasonably low penetration of robots in 2004, with the exception of automotive. To calibrate θ we choose to take a mean of the HP-filtered series for the years 1956-1980. We choose 1980 as the final point for the time average as the 1980s saw a sharp increase in robot adoption in the US automotive industry.

C.5 Calibrating the Other Parameters

We calibrate our model annually choosing a required rate of return of 4%, which pins down the discount rate of the firm ρ. We then choose the parameters $\{\delta, p_R/w\}$in order to match targets from the evidence on robotic arms in years 2010 and 2014, which are the most recent data points on robots per worker that we can obtain from Acemoglu and Restrepo (2018d). Unless otherwise stated, the source for such figures is the International Federation of Robots. First, we set $\delta = 1/12$, to target an average service life of 12 years, as reported in International Federation of Robotics (2017). Next, we set the relative price of robots p_R/w in order to match the corresponding ratio in the data in 2010 and 2014. We obtain data on average annual wages for production employees (OES code 51-0000) in manufacturing from the OES and time series data for unit robot costs from Korus (2019). This gives:

$$\left(\frac{p_R}{w}\right)_{2010} = 1.4348, \qquad \left(\frac{p_R}{w}\right)_{2014} = 1.0209.$$

We set the elasticity of substitution between sectors at $\sigma = 1$, consistent with Oberfield and Raval (2014). As a result ξ_s represents the share of each sector in manufacturing value added, and we calibrate these parameters using the BEA GDP-by-industry data.

C.5.1 Calibrating Γ_s, ψ_R

We calibrate Γ_s to match the stock of robots per employees in 2014 across the various industries, and ψ_R/w to approximately match the percentage fall in the average stock of robots per employees between 2010 and 2014 weighted by 2014 value added shares, by solving the system of general equilibrium equations numerically. We implicitly assume that the economy is in steady state both in 2010 and in 2014. While this is obviously a strong assumption, any other choice would require taking a stance of the time required to transition to a new steady state, which will mechanically imply a value a ψ_R. Moreover, a transition in general equilibrium with 13 sectors would incur considerable numerical complications.

Our algorithm to solve for the 13 Γ_s's conditional on a value of ψ_R/w is given by a nest of two solvers.[25] The internal solver finds equilibrium prices and wages (14 variables) for any given set of sectoral Γ_s's, and the external solver looks for the Γ_s's (13 variables) that deliver the observed robot per workers across the IFR sectors, given the equilibrium found by the internal solver. This result in a high number of function evaluations at each solver step, that entails several thousands of calls to our basic routines that compute the stationary distributions in each sector. Nesting this procedure inside yet another solver to look for a ψ_R/w that matches exactly the fall in robot stocks would further raise the computational burden. Therefore, we choose to match the fall in robot stocks by using a closed-form approximation of equilibrium quantities that holds in the partial-automation limit of our model, as an exact match would be computationally prohibitive. Indeed, we verify that, for a wide range of chosen ψ_R/w, the low robot penetration observed in the data can only be matched by a model featuring almost no fully automated firms. As a result, aggregates can be approximated well by closed-form expressions. In particular, sectoral robot penetration reads:

$$\left(\frac{R}{L}\right)_s = \frac{R^{\star}_{\max,s}}{L_s}.$$

We use this result to obtain guesses for the Γ_i 's conditional on each value of ψ_R/w, by assuming that the share of labor in each sector is proportional to the share of manufacturing value added. Given our normalization for $L_{2014} = 1$, we obtain the following guesses:

$$\tilde{\Gamma}_s(\psi_R/w) = \frac{1}{1 + \frac{m}{w} + (\rho+\delta)(\delta\left(\frac{\psi_R}{w}\right)\left(\frac{R}{L}\right)_{s,2014}\xi_s + \left(\frac{p_R}{w}\right)_{2014})},$$

which only depends on other calibrated parameters and targets. We verify that these guesses are suf-

[25] We experimented solving for Γ_s's and prices in a single solver, but our choice presented here proved substantially more stable numerically, due to the nesting of the equilibrium price solver, that ensures that the Γ_s's are in each step consistent with equilibrium prices.

ficiently close to numerical solutions obtained solving the exact model for given ψ_R/w. Our strategy therefore consists in solving for a ψ_R/w that matches the percentage fall in the average robot stock in general equilibrium assuming that the true Γ_s's that match R/L for each sector in 2014 are given by our guesses $\tilde{\Gamma}_s$'s. Applying this method only misses our target percentage fall by 0.63%. The magnitude of the error corresponds to just 0.0236 of the observed percentage change.

Summing up, our calibration strategy for Γ_i's and ψ_R/w goes through the following steps:

1. Run solver for ψ_R/w:
   (a) Guess an initial value for ψ_R/w;
   (b) Compute guesses $\tilde{\Gamma}_s$ that match approximately $(R/L)_{s,2014}$ given ψ_R/w;
   (c) Find equilibrium prices and wage that clear markets in 2010 and 2014 setting $\Gamma_i = \tilde{\Gamma}_s$ and relative robot prices $(p_R/W)_{2010}, (p_R/W)_{2014}$;
   (d) Obtain equilibrium $(R/L)_{s,2010}, (R/L)_{s,2014}$, aggregate over sectors and compute the deviation between target and computed percentage increase in the robot stock;
   (e) Update ψ_R/w and go to step (b) until convergence.
2. Run solver for Γ_s's given ψ_R/w found in 1.
   (a) Guess an initial value for Γ_s's using the approximate formula;
   (b) Find equilibrium prices and wage that clear markets in 2014 given Γ_s and relative robot price $(p_R/W)_{2014}$;
   (c) Obtain equilibrium $(R/L)_{i,2014}$, and compute the deviation from the target;
   (d) Update the guess for Γ_s and go to step (b) until convergence.

Tables 1 and 2 summarize the parameters used for our calibration, together with the respective targets and sources.

Table 1: Parameters Common to all Sectors

Description	Variable	Value	Source/Target
Relative robot purchase price in 2010	$\left(\frac{p_R}{w}\right)_{2010}$	1.4348	Korus (2019) and OES
Relative robot purchase price in 2010	$\left(\frac{p_R}{w}\right)_{2014}$	1.0209	Korus (2019) and OES
Relative flow robot cost	$\left(\frac{m}{w}\right)$	0	Negligible energy costs of robots
Robot depreciation	δ	$\log(1+1/12)$	International Federation of Robotics (2017)
Discount factor	ρ	$\log(1+0.04)$	4% annual interest rate
Adjustment cost parameter	ψ_R	1278.96	2010-2014 fall in value-added weighted R/L
Final good productivity	A_F	1	Final good is the numéraire

Table 2: Targets and Calibrated Parameters

Sector	R/L 2010	R/L 2014	ξ	θ	Γ	σ_P	θ_P
Automotive	88	117	0.065	0.35	0.52	0.13	0.77
Electronics	10	13	0.15	0.28	0.77	0.19	0.72
Food and Beverages	4.9	6.2	0.12	0.21	0.85	0.13	0.47
Wood and Furniture	0.022	0.14	0.028	0.41	0.89	0.13	0.63
Miscellaneous	2.4	14	0.04	0.32	0.85	0.18	0.84
Basic Metals	5.4	7.2	0.031	0.38	0.87	0.12	0.67
Industrial Machinery	1.8	2.4	0.078	0.33	0.88	0.14	0.71
Metal Products	6.2	8.3	0.07	0.35	0.84	0.12	0.91
Clay Glass and Minerals	0.28	0.68	0.024	0.33	0.89	0.13	0.7
Paper and Publishing	0.0085	0.11	0.047	0.28	0.89	0.1	0.62
Plastics, *et cetera*	8	9.9	0.27	0.18	0.78	0.61	0.89
Apparel and Textiles	0.0081	0.045	0.014	0.43	0.89	0.098	0.54
Shipbuilding and Aerospace	0.15	0.54	0.068	0.32	0.89	0.13	0.77

Appendix D General Equilibrium

This Appendix provides additional details on the general equilibrium model described in Section 4.1. Throughout, time indexes are suppressed.

D.1 Goods Producers

We denote the final good consumed by agents in our economy by Y, that we take as the numéraire. The final good producer operates a CES production function that aggregates intermediate goods Y_s, at unit cost p_s. Under these assumptions, the static cost-minimization problem of the final good firm reads:

$$\max_{\{Y_s\}} \sum_{s=1}^{N} p_s Y_s$$
$$\text{s.t. } A^F \left(\sum_{s=1}^{N} \xi_s Y_s^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} = \bar{Y}.$$

here σ denotes the elasticity of substitution between goods from different intermediate sectors, and $\sum_{s=1}^{N} \xi_s = 1$. We then have that the demand for each intermediate good is given by:

$$Y_s^D = \left(\frac{\xi_s \tilde{P}}{p_s} \right)^{\sigma} \frac{Y}{A^F} = \left(\frac{\xi_s}{p_s} \right)^{\sigma} \frac{Y}{A^F}$$

where,

$$\tilde{P} = \left(\sum_{s=1}^{N} p_s^{1-\sigma} \xi_s^{\sigma} \right)^{\frac{1}{1-\sigma}} = 1,$$

is the ideal price index, which equals one due to our choice of numéraire.

The intermediate good supplied by each sector is an aggregate of the net output of the firms described in the main section:

$$Y_s^S = \int \{Q_s(z, L, R, u)\}\, dG_s(R, z) \equiv \int \left\{z\,(\Gamma_s L + (1-\Gamma_s)\, uR)^{\theta_s}\right\} dG_s(R, z).$$

In this context, we interpret the revenue-shifter shock as an idiosyncratic productivity shock faced by each firm. The solution of the intermediate firms' problem, as described in section 3, determines the labor demand coming from each sector s. The firm's problem also leads to an individual robot demand. The expenditures faced by the firm to purchase, maintain and adjust the robot stock are given by:

$$\Psi_s \equiv \int \left\{ mR + p_R I_s^\star(R, z) + \frac{\psi_R}{2}(I_i^\star(R, z))^2 \right\} \mathrm{d}G_s(R, z).$$

D.2 Households

The economy is populated by a measure-one mass of hand-to-mouth agents. The representative household also receives all profits and adjustment costs in the economy.[26] Accordingly, the household's problem reads,

$$\begin{aligned} &\max_{c_w, \ell} U(c_w, \ell) \\ &\text{s.t. } c_w = w\ell + \Pi + \Psi, \end{aligned}$$

where $\Psi = \sum_{s=1}^{N} \Psi_s$. Since we are not concerned with wealth effects on the labor supply, we shut down this channel assuming GHH preferences,

$$U(c_w, \ell) = \frac{\left(c_w - \chi \frac{\varphi}{1+\varphi} \ell^{1+\frac{1}{\varphi}}\right)^{1-\frac{1}{\sigma}} - 1}{1 - \frac{1}{\sigma}}.$$

A simple derivation gives the optimal labor supply of the household,

$$\ell^\star(w) = \left(\frac{w}{\chi p_y}\right)^{\varphi},$$

which, as usual for GHH preferences, only depends on the level of the real wage.

[26] We could equivalently have assumed that there is a sector producing all these goods/services which aggregates intermediaries in the same way as the final good. This equivalent formulation stresses that ultimately robots produced will generate income for somebody in the economy.

D.3 Equilibrium

The model is closed requiring equilibrium in the main markets. Labor market clearing requires,

$$\ell^{\star}(w) = L^d(w, \mathbf{p}),$$

where labor demand is given by the sum of sectoral labor demands,

$$L^d(w, \mathbf{p}) = \sum_{s=1}^{N} \int_0^{\infty} L_s\left(w, p_s, R, z\right) dG_s(R, z).$$

Final goods' market clearing requires,

$$Y = c_w.$$

The remaining market clearing conditions are simply given by equating demand and supply for intermediate goods,

$$Y_s^S(p_s) = Y_s^S(p_s) \qquad \forall s = 1, \ldots, N.$$

We define a *stationary equilibrium* in a N-sector economy as follows. Given a Markovian stochastic process for the productivity shock z_t for each sector s that admits a stationary distribution with CDF $F_s(z)$, , exogenous productivity parameter A^F and robot prices p_R, a stationary equilibrium is given by a set of CDF's $G_s(R, z)$ prices $\left\{w, \left\{p_s\right\}\right\}$, allocations $\{L_s, R, I, Y_s, Y, \ell, c_w\}$, firms' values $V_s(R, z)$, utilization choices u, satisfying

1. Individual optimal labor supply:
$$\ell^{\star}(w) = \left(\frac{w}{\chi}\right)^{\varphi};$$
2. Optimal workers' consumption:
$$c_w = w\ell + \Pi + \Psi;$$
3. Final goods' production function:
$$Y = A^F \left(\sum_{i=1}^{N} \xi_s Y_s^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}};$$
4. Perfect competition in the final goods' sector (price of the final good equals unit cost):
$$1 = \left(\sum_{s=1}^{N} p_s^{1-\sigma} \xi_s^{\sigma}\right)^{\frac{1}{1-\sigma}};$$

5. Cost minimization by final goods' producers:

$$Y_s^D = \left(\frac{\xi_s}{p_s}\right)^{\sigma} \frac{Y}{A^F} \qquad \forall s = 1, \dots, N;$$

6. Static profit optimization by firms for labor:

$$\begin{aligned} L_s^{\star}(R,z;w,p_s) &= \mathbf{1}\left\{ u_s^{\star}(R,z)R \leq \left(\frac{w}{p_s z \Gamma_s \theta_s}\right)^{\frac{1}{\theta_s - 1}} \frac{1}{1-\Gamma_s} \right\} \times \\ &\times \frac{1}{\Gamma_s}\left(\left(\frac{w}{p_s z \Gamma \theta}\right)^{\frac{1}{\theta_s-1}} - (1-\Gamma_s) u_s^{\star}(R,z)R\right) \qquad \forall s; \end{aligned}$$

7. Static profit optimization by firms for utilization (use only if positive labor savings, get as close as possible to desired size):

$$\begin{aligned} u_s^{\star}(R,z;w,p_s) &= \mathbf{1}\left\{ R \leq \left(\frac{p_s z \theta (1-\Gamma_s)^{\theta_s}}{m}\right)^{\frac{1}{1-\theta_s}} \right\} + \\ &+ \mathbf{1}\left\{ R > \left(\frac{p_s z \theta_s (1-\Gamma_s)^{\theta_s}}{m}\right)^{\frac{1}{1-\theta_s}} \right\} \times \\ &\times \frac{1}{(1-\Gamma_s)R}\left(\frac{m}{(1-\Gamma_s)\theta_s p_s z}\right)^{\frac{1}{\theta_s-1}} \qquad \forall s; \end{aligned}$$

8. Optimal individual investment by firms to solve the firms' problem given value $V(R,z)$:

$$I_s^{\star}(R,z;w,p_s) = \arg\max_I \Pi_s(R,z) - p_R I - \frac{\psi}{2}(I)^2 + E_s\left[\frac{dV_s(R,z)}{dt}\right] \qquad \forall s;$$

9. Value of an individual firm:

$$\rho V_s(R,z) = \Pi_s(R,z) - p_R I_s^{\star}(R,z) - \frac{\psi}{2}(I_s^{\star}(R,z))^2 + E_s\left[\frac{dV_s(R,z)}{dt}\right] \qquad \forall s;$$

10. Law of motion of individual robot stocks:

$$dR_s = \left[I_s^{\star}(R,z) - \delta R_s\right] dt \qquad \forall s;$$

11. Labor market equilibrium:

$$\sum_{i=1}^{N}\left[\int L_s^{\star}(R,z;w,p_s)\, dG_s(R,z)\right] - \ell^{\star}(w) = 0;$$

12. Intermediate goods' market clearing:

$$\int \left\{ z\left(\Gamma L_s^\star(R,z;w,p_s)+(1-\Gamma)u_s^\star(R,z)R\right)^\theta \right\} dG_s(R,z) - Y_s^D(p_s) = 0 \qquad \forall i;$$

13. Final goods' market clearing:

$$Y = c_w;$$

14. Kolmogorov Forward Equation for the stationary CDF of firms:

$$\frac{dG_s(R,z)}{dt} = 0 \qquad \forall s.$$

Appendix E Linear Costs Solution Algorithm

In this appendix, we describe the solution algorithm adopted to solve the model in Section 5.2. We follow a scheme similar to Achdou et al. (2017), making the adjustments that are required by the problem at hand.

Consider a discretization of the state space on a increasing grid for R, z with N_R and N_z points respectively. First, recall that the policy is to adjust immediately if outside of the inaction region, bringing the state variable all the way to the boundary, and to be inactive otherwise. This means that a conventional PDE defines the value function within the inaction region, while the value function outside of the inaction region is a simple linear function of the value function evaluated at the boundary. We seek the following matrix representation of the optimized HJB equation,

$$\rho \mathbf{v} = \mathbf{u} + A\mathbf{v}.$$

Where $\mathbf{v}$ is a vector of length $N_z \times N_R$, and A is a (sparse) matrix that we describe below. Consider first a case with just one value of z. Given an initial guess for the value function $V^0(R)$, we iterate on the following steps until convergence.

Step 1. Find cutoffs R_{inv} and R_{disinv}. We compute the forward and backward differences to approximate the derivative of the value function, dV^F, dV^B. By the concavity of the value function, we know that these two objects are decreasing in R. Therefore, we divide the state space by finding the first occurrence (starting from R_0, the smallest value on the grid) of an index i such that,

$$dV^F(R_i) < p_R + \psi_+.$$

This value for R_i gives us the first value of robots strictly inside the inaction region. Therefore we set $R^\star_{\text{inv}} \equiv R_{i-1}$. We proceed analogously to find the cutoff $R^\star_{\text{disinv}} \equiv R_j$, where j is defined as the index such that,

$$dV^B\left(R_j\right) < p_R - \psi_-.$$

Note that the above procedure imposes that the two cutoffs lie on the grid and that the inaction region contains at least one point. While this reduces the accuracy of the solution, the error in the computation of the cutoffs vanishes as the size of the grid for R increases. Moreover, this greatly improves the numerical stability of the algorithm. Given these cutoffs, we can define the inaction region consistently with the main text as: $\{R_{\text{inv}}, \ldots, R_{\text{disinv}}\}$. Investment will then be positive for all indexes i such that $R_i < R_{\text{inv}}$ and negative for all indexes j such that $R_j > R_{\text{disinv}}$. Inside the inaction region, the robot stock will depreciate at rate δ. Now we note that the optimal solution for investment entails,

$$V_R(R_i) = p_R + \psi_+ \qquad \forall i | R_i < R_{\text{inv}},$$

and,

$$V_R(R_j) = p_R - \psi_- \qquad \forall j | R_j > R_{\text{disinv}}.$$

Integrating, we immediately get,

$$V(R_i) = V(R_{\text{inv}}) - \left(\psi_+ + p_R\right)(R_{\text{inv}} - R_i), \qquad \forall i | R_i < R_{\text{inv}},$$

and,

$$V\left(R_j\right) = V(R_{\text{disinv}}) - \left(-\psi_- + p_R\right)\left(R_{\text{disinv}} - R_j\right), \qquad \forall j | R_j > R_{\text{disinv}}.$$

Step 2. By the above results, we can rewrite the above matrix representation as follows, denoting by i_{inv} the index such that $R_{\text{inv}} = R_{i_{\text{inv}}}$, and similarly for i_{disinv}:

$$\begin{bmatrix} V(R_0) \\ V(R_1) \\ \vdots \\ V\left(R_{i_{\text{inv}}-1}\right) \\ \rho V\left(R_{i_{\text{inv}}}\right) \\ \vdots \\ \rho V\left(R_{i_{\text{disinv}}}\right) \\ V\left(R_{i_{\text{disinv}}+1}\right) \\ \vdots \\ V(R_{\text{end}}) \end{bmatrix} = \begin{bmatrix} -\left(\psi_+ + p_R\right)\left(R_{i_{\text{inv}}} - R_0\right) \\ -\left(\psi_+ + p_R\right)\left(R_{i_{\text{inv}}} - R_1\right) \\ \vdots \\ -\left(\psi_+ + p_R\right)\left(R_{i_{\text{inv}}} - R_{i_{\text{inv}}-1}\right) \\ \Pi\left(R_{i_{\text{inv}}}\right) \\ \vdots \\ \Pi\left(R_{i_{\text{disinv}}}\right) \\ -\left(\psi_- - p_R\right)\left(R_{i_{\text{disinv}}+1} - R_{i_{\text{disinv}}}\right) \\ \vdots \\ -\left(\psi_- - p_R\right)\left(R_{\text{end}} - R_{i_{\text{disinv}}}\right) \end{bmatrix} +$$

$$+ \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \ldots & \delta R_{i_{\text{inv}}} & -\delta R_{i_{\text{inv}}} & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \vdots & \cdots & \delta R_{i_{\text{disinv}}} & -\delta R_{i_{\text{disinv}}} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} V(R_0) \\ V(R_1) \\ \vdots \\ V\left(R_{i_{\text{inv}}-1}\right) \\ V\left(R_{i_{\text{inv}}}\right) \\ \vdots \\ V\left(R_{i_{\text{disinv}}}\right) \\ V\left(R_{i_{\text{disinv}}+1}\right) \\ \vdots \\ V(R_{\text{end}}) \end{bmatrix}$$

We can then update the value function using either an iterative or implicit scheme as described in the appendix to Achdou et al. (2017).

Extending the problem to multiple price levels is trivial. The only difference is that the indexes denoting the inaction region will vary for each z. Moreover, the matrix described above becomes only one of the diagonal blocks in a bigger sparse matrix. Differently from the standard case, outflows from each block into other blocks are only allowed for states inside the inaction region within each block. Indeed, outside of these blocks, the linearity of the value function is ensured as described above, and there is no jump over z to be included in the matrix A, as control is instantaneous.

In order to compute the stationary distribution, we proceed as in Achdou et al. (2017), by using the adjoint of the matrix A, A^T, to iterate on an initial guess. The only difference is that now we have to ensure that any firm starting outside the inaction region will eventually abandon it. To do so, we just add -1 to the diagonal of all indexes outside the inaction region, ensuring that the rows of A sum to zero and that therefore A^T is indeed an infinitesimal generator. The resulting matrix will therefore eventually push all the mass outside the inaction region to relevant cutoffs.