FinOpt-Workflow

January 27, 2026

1 FinOpt: Complete Workflow Validation

2 Configuration: Seasonality and Contribution

[2]:
```python
# --- Seasonality pattern for variable income (12 months, Jan-Dec) ---

# Structure: months = ["Jan", "Feb", "Mar", "Apr", "May", "Jun",
#                       "Jul", "Aug", "Sep", "Oct", "Nov", "Dec"]

seasonality_variable = [0.00, 0.00, 0.00, 0.6, 1, 1.16,
                        1, 1.10, 0.50, 0.90, 0.85, 1]

monthly_contrib_fixed = [0.45, 0.45, 0.45, 0.45, 0.45, 0.45,
                         0.45, 0.45, 0.45, 0.45, 0.45, 0.3]

monthly_contrib_variable = [1.0] * 12

monthly_contribution ={"fixed": monthly_contrib_fixed, "variable":␣
 ↪monthly_contrib_variable}
```

3 Initialize FinancialModel

[3]:
```python
# IncomeModel Instantiation
income = IncomeModel(
    fixed=FixedIncome(
        base=1_488_000.0,       # CLP/month
        annual_growth=0.03,     # 3% nominal annual growth
        salary_raises={
            date(2026, 4, 1): 400_000,  # +400k in April 2026
            date(2027, 6, 1): 400_000,   # +400k in Jun 2027
            date(2028, 6, 1): 300_000   # +300k in Jun 2028
        },
        name="fixed"
    ),
    variable=VariableIncome(
        base=60_000.0,         # Base variable income
        seasonality=seasonality_variable,
```

```python
        sigma=0.10,              # 10% monthly noise
        floor=0.0,               # No negative income
        cap=400_000.0,           # Maximum 400k/month
        annual_growth=0.0,       # No growth in variable
        name="variable"
    ),
    monthly_contribution = monthly_contribution
)

# --- Account configuration ---
accounts = [
    Account.from_annual(
        name="Vivienda",
        display_name="Cuenta Ahorro Vivienda (BE)",
        annual_return=0.025,
        annual_volatility=0.01,
        initial_wealth=1600000
    ),
    Account.from_annual(
        name="Clooney",
        display_name="Conservative Clooney (Fintual)",
        annual_return=0.07,
        annual_volatility=0.08,
        initial_wealth=744747
    ),
    Account.from_annual(
        name="VOO",
        display_name="VOO S&P500 ETF",
        annual_return=0.11,
        annual_volatility=0.12,
        initial_wealth=900000
    )
]

# --- Correlation matrix ---
# Clooney portfolio have S&P500 positive correlation (rho = 0.3)

correlation_dict={("Clooney", "VOO"): 0.3}

# --- Initialize FinancialModel ---
model = FinancialModel(income, accounts, default_correlation = correlation_dict)
model
```

[3]:

```
FinancialModel(M=3, accounts=['Vivienda', 'Clooney', 'VOO'], cache=enabled)
```

4 Simulation parameters

[4]:
```
# --- Simulation parameters ---
n_sims = 500
months = 26
start_date = date(2025, 11, 1)
```

5 Income Module

Total monthly income at time t is composed of fixed and variable parts:

$$Y_t = y_t^{\text{fixed}} + Y_t^{\text{variable}}$$

5.1 Fixed Income

The fixed component, y_t^{fixed}, reflects a baseline salary subject to compounded annual growth g and scheduled raises d_k, Δ_k (e.g., promotions or tenure milestones):

$$y_t^{\text{fixed}} = \text{current\_salary}(t) \cdot (1+m)^{\Delta t}$$

where $m = (1+g)^{1/12} - 1$ is the **monthly compounded rate**, and Δt represents time since the last raise.

5.2 Variable Income

The variable component, Y_t^{variable}, models irregular income sources (e.g., freelance work or bonuses) with:

- **Seasonality**: $s \in \mathbb{R}^{12}$ (multiplicative monthly factors),
- **Noise**: $\epsilon_t \sim \mathcal{N}(0, \sigma^2)$ (Gaussian shocks),
- **Growth**: same compounded rate m applied to a base income level,
- **Boundaries**: optional floor and cap constraints.

The underlying stochastic projection is:

$$\tilde{Y}_t = \max(\text{floor},\ \mu_t(1+\epsilon_t)), \quad \text{where } \mu_t = \text{base} \cdot (1+m)^t \cdot s_{(t \bmod 12)}$$

Then, guardrails are applied as:

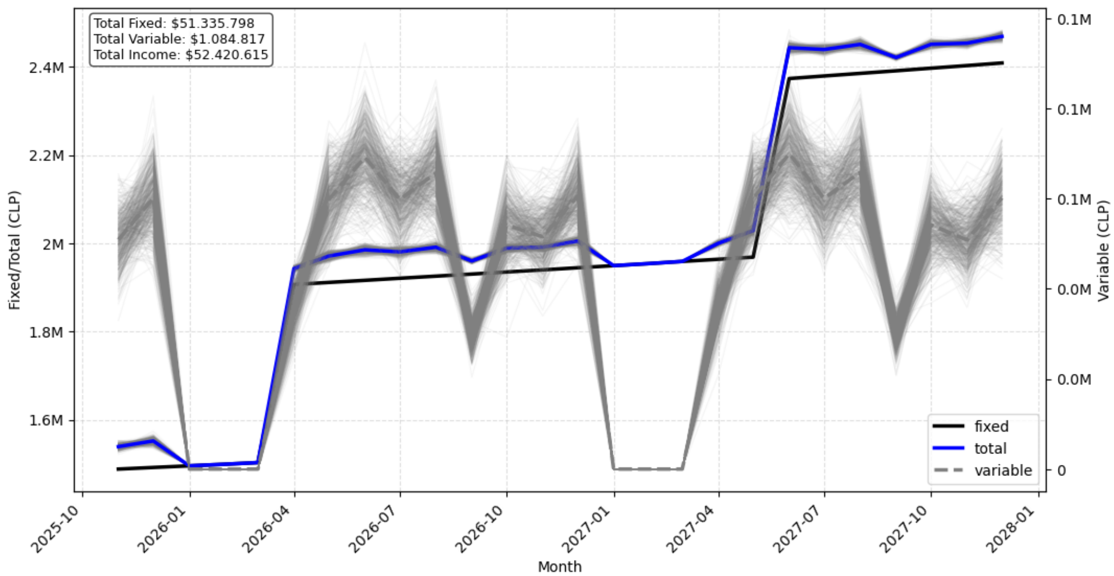
$$Y_t^{\text{variable}} = \begin{cases} 0 & \text{if } \tilde{Y}_t < 0 \\ \tilde{Y}_t & \text{if } 0 \le \tilde{Y}_t \le \text{cap} \\ \text{cap} & \text{if } \tilde{Y}_t > \text{cap} \end{cases}$$

Note: In expectation (ignoring noise truncation), $\mathbb{E}[Y_t] = y_t^{\text{fixed}} + \mu_t$

5.3 Income Projection

Dual-axis plot with: - **Left axis**: Fixed income (deterministic) + Total income - **Right axis**: Variable income (stochastic with trajectories) - **Trajectories**: Individual noise realizations (n=300) - **Confidence band**: 95% CI for variable income

[5]:
```
# Income Projection Simulation
model.plot(mode='income', T=months, start=start_date)
```



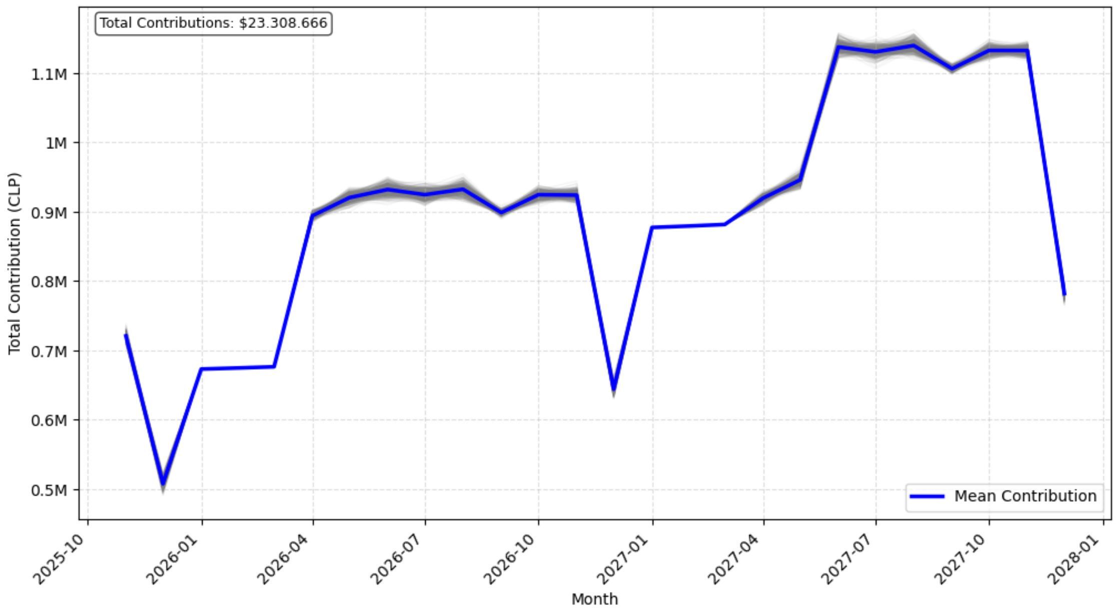
5.4 Contribution Projection

A fraction of income is allocated each month through calendar-rotating schedules:

$$A_t = \alpha^f_{(t \bmod 12)} \cdot y_t^{\text{fixed}} + \alpha^v_{(t \bmod 12)} \cdot Y_t^{\text{variable}}$$

where $\alpha^f, \alpha^v \in [0,1]^{12}$ control the fixed and variable contribution rates by applying the 12-month fractional arrays to projected incomes, rotated according to `start` date and repeated cyclically for horizons > 12 months.

[6]:
```
# Contribution Projection Simulation
model.plot(mode='contributions', T=months, start=start_date)
```



6 Return Module

6.1 Multi-Account Return Model

For M accounts with correlated returns:

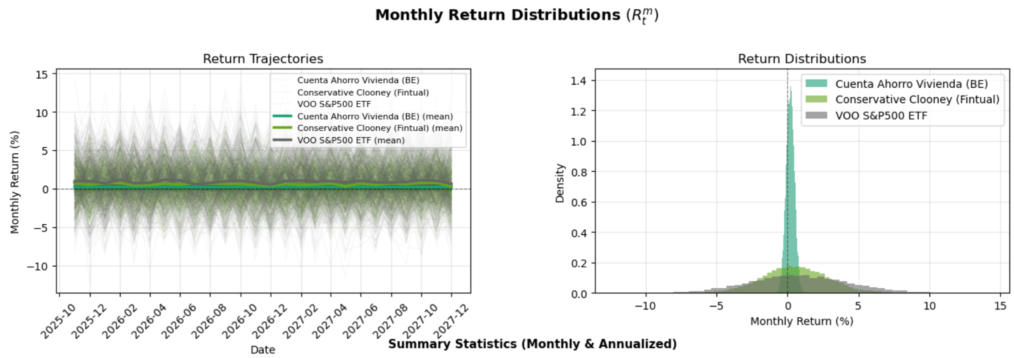
$$1 + R_t^m \sim \text{LogNormal}(\mu_{\log}^m, \Sigma)$$

where $\Sigma = D \cdot \rho \cdot D$ is the covariance matrix: - $D = \text{diag}(\sigma_{\log}^1, \ldots, \sigma_{\log}^M)$ - $\rho \in \mathbb{R}^{M \times M}$ is the correlation matrix (symmetric, PSD)

6.2 Monthly Return Distribution (Marginal Analysis)

Visualizes IID monthly returns across both accounts with 4 panels: - **Trajectories**: Individual paths for each account - **Histograms**: Marginal distributions (overlaid) - **Statistics**: Mean, std, quantiles per account

[7]:
```
model.plot(mode = 'returns', T = months, start=start_date)
```



Summary Statistics (Monthly & Annualized)

Account	Mean	Std	Median	IQR (Q25-Q75)	Min	Max
Cuenta Ahorro Vivienda (BE)	0.21%	0.29%	0.21%	[0.01%, 0.40%]	-0.94%	1.28%
↳ *Annualized*	*2.50%*	*1.00%*	*2.49%*	*[0.13%, 4.94%]*	—	—
Conservative Clooney (Fintua	0.58%	2.29%	0.55%	[-0.96%, 2.11%]	-7.50%	9.38%
↳ *Annualized*	*7.16%*	*7.94%*	*6.83%*	*[-10.89%, 28.46%]*	—	—
VOO S&P500 ETF	0.88%	3.45%	0.81%	[-1.46%, 3.19%]	-12.25%	14.29%
↳ *Annualized*	*11.08%*	*11.97%*	*10.17%*	*[-16.15%, 45.75%]*	—	—

T=26 months | n_sims=500 | seed=None

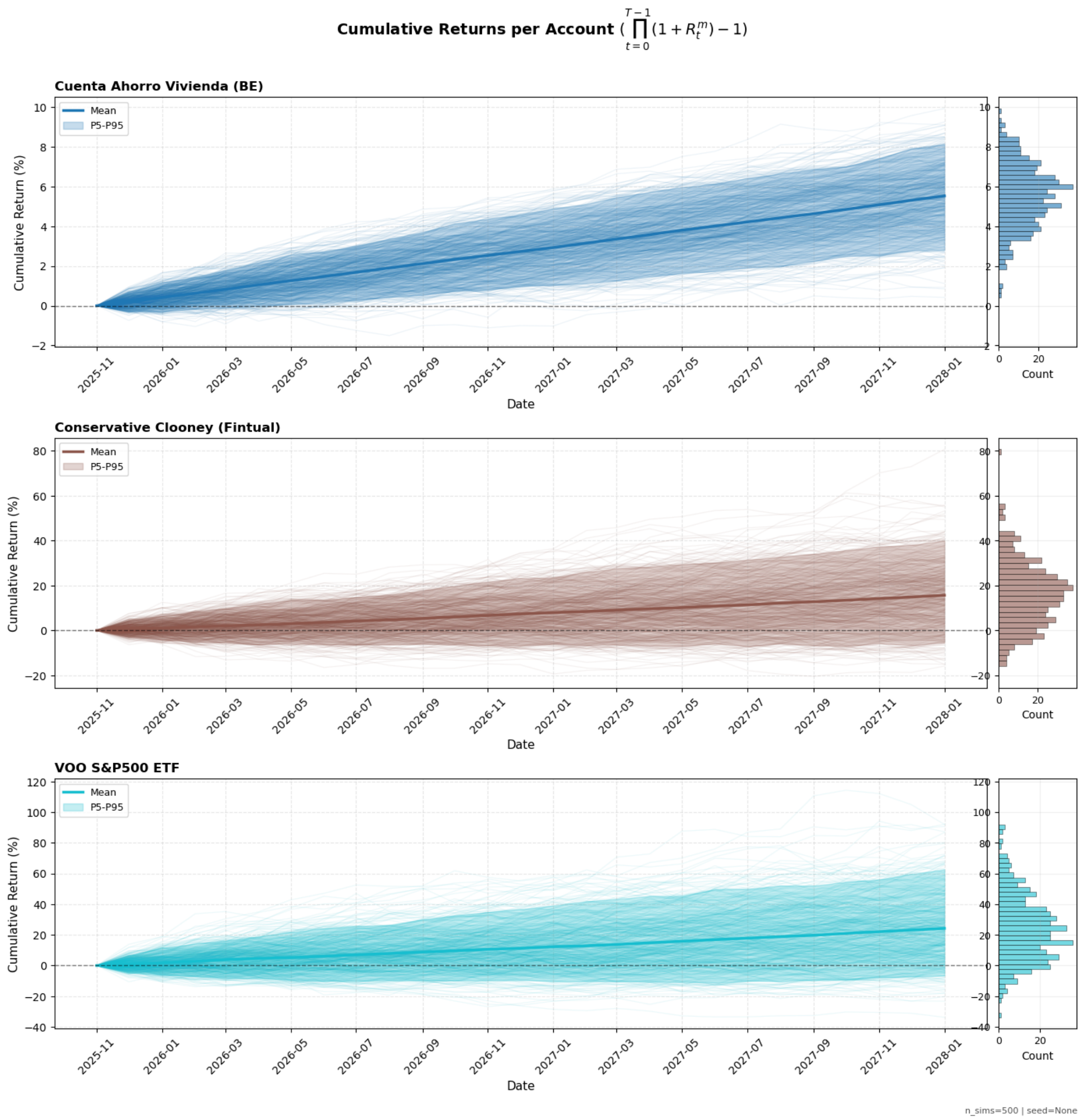
6.3 Cumulative Returns per Account

For M accounts with correlated returns:

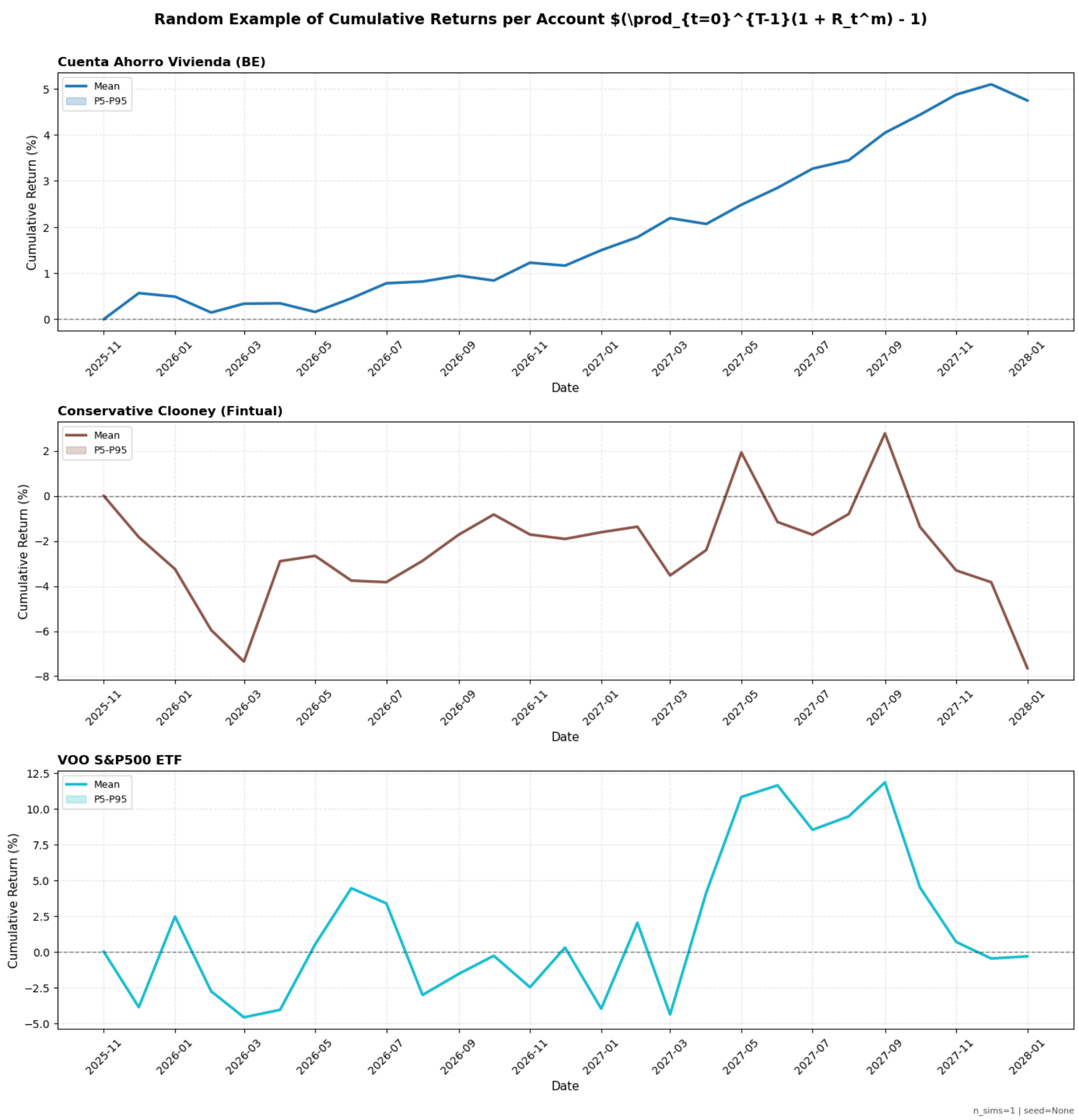
$$R_{\text{cm}}^m(T) = \prod_{t=0}^{T-1}(1 + R_t^m) - 1$$

Cross-sectional correlation persists through time but does not compound.

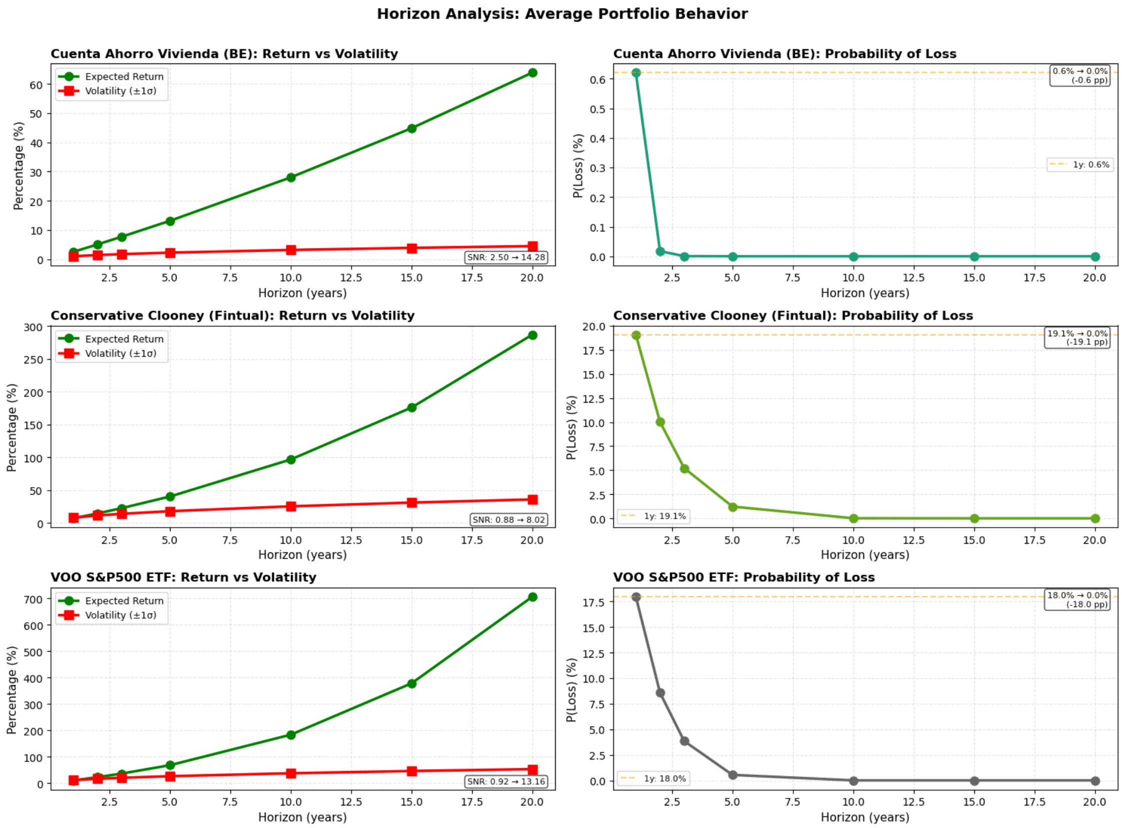
[8]:
```
model.plot(mode = 'returns_cumulative', T = months, start=start_date)
```




```
[9]: model.plot(mode = 'returns_cumulative', T = months, start=start_date, n_sims=1,
     title= r'Random Example of Cumulative Returns per Account
     $(\prod_{t=0}^{T-1}(1 + R_t^m) - 1)')
```


Random Example of Cumulative Returns per Account $(\prod_{t=0}^{T-1}(1 + R_t^m) - 1)
Cuenta Ahorro Vivienda (BE)
Conservative Clooney (Fintual)
VOO S&P500 ETF
Mean
P5-P95
Cumulative Return (%)
Date
2025-11
2026-01
2026-03
2026-05
2026-07
2026-09
2026-11
2027-01
2027-03
2027-05
2027-07
2027-09
2027-11
2028-01
n_sims=1 | seed=None

6.4 Horizon Analysis: Time Diversification by Account



7 Portfolio Module

7.1 Wealth Projection Under Allocation Policy

Recursive dynamics (without withdrawals):

$$W_{t+1}^m = \left(W_t^m + A_t x_t^m\right)\left(1 + R_t^m\right)$$

Recursive dynamics (with withdrawals):

$$W_{t+1}^m = \left(W_t^m + A_t x_t^m - D_t^m\right)\left(1 + R_t^m\right)$$

We define $A_t^m = A_t \cdot x_t^m$ which is the contribution allocated to account m via policy $X = \{x_t^m\}_{t,m}$, and D_t^m is the withdrawal from account m in month t.

Closed-form representation:

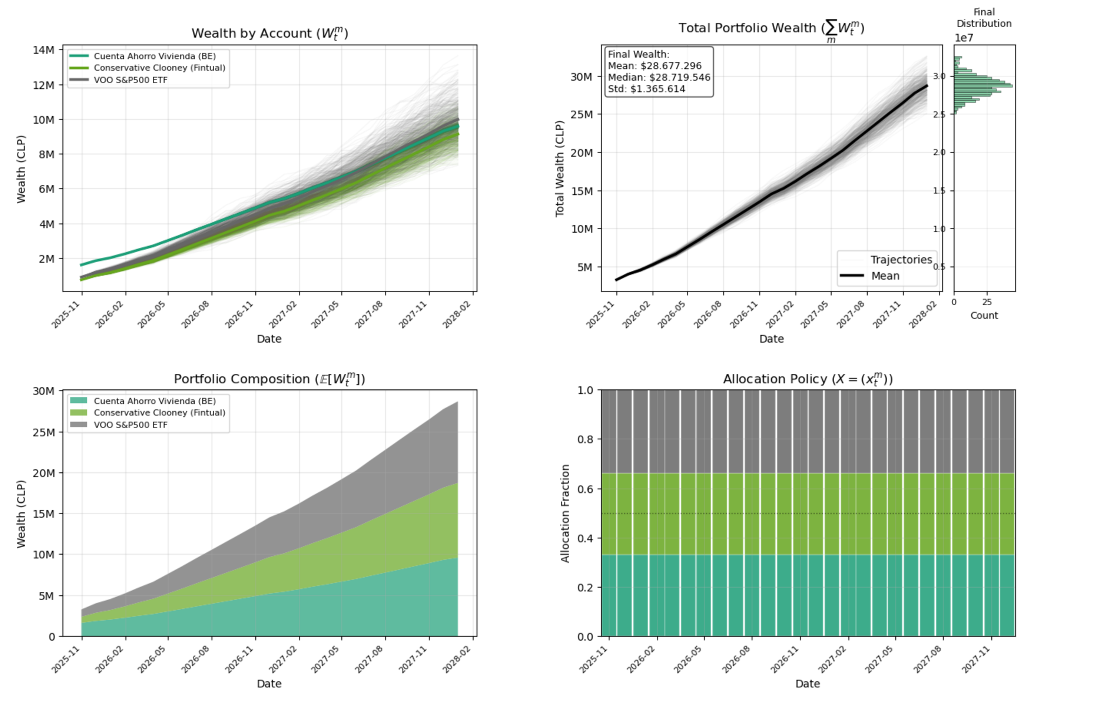
$$W_t^m = W_0^m \, F_{0,t}^m + \sum_{s=0}^{t-1} \left(A_s \, x_s^m - D_s^m\right) F_{s,t}^m$$

with accumulation factor $F_{s,t}^m = \prod_{r=s}^{t-1}(1 + R_r^m)$.

Key insight: $W_t^m(X)$ is **linear affine in policy** X → analytical gradients enable convex optimization. Since D is a parameter (not a decision variable), convexity is preserved.

What to observe: - Top-left: wealth per account with Monte Carlo trajectories - Top-right: total wealth + final distribution histogram - Bottom-left: portfolio composition over time - Bottom-right: allocation policy heatmap

```
/home/mlioi/fin-opt/src/portfolio.py:1249: UserWarning: This figure includes
Axes that are not compatible with tight_layout, so results might be incorrect.
  fig.tight_layout(rect=[0, 0.01, 1, 0.96 if title else 1])
```



8 Withdrawal Module

8.1 Scheduled Cash Outflows

The withdrawal module models planned cash outflows (retiros) from investment accounts. These represent purchases, emergency expenses, or periodic distributions that reduce portfolio wealth.

Wealth dynamics with withdrawals:

$$W_{t+1}^m = (W_t^m + A_t \cdot x_t^m - D_t^m)(1 + R_t^m)$$

where D_t^m is the withdrawal from account m in month t. Withdrawals occur at the **START** of the month (before returns are applied), meaning the withdrawn amount does not earn returns that month.

Affine representation (critical for convex optimization):

$$W_t^m = W_0^m \cdot F_{0,t}^m + \sum_{s=0}^{t-1}(A_s \cdot x_s^m - D_s^m) \cdot F_{s,t}^m$$

Key insight: D is a **PARAMETER** (not a decision variable), so the representation remains affine in X, preserving convexity for CVaR optimization.

8.2 Withdrawal Types

1. **WithdrawalEvent**: Single scheduled withdrawal at a specific date
   - Deterministic amount (same across all scenarios)
   - Calendar-aware date resolution
2. **StochasticWithdrawal**: Withdrawal with uncertainty
   - Samples from truncated Gaussian: $D \sim \mathcal{N}(\mu, \sigma^2)$ with floor/cap
   - Models variable expenses (medical costs, emergency repairs)
3. **WithdrawalModel**: Facade combining scheduled + stochastic withdrawals

8.3 Withdrawal Schedule Configuration

[12]:
```python
# --- Define Withdrawal Schedule ---

# Scheduled withdrawals (deterministic)
scheduled_withdrawals = WithdrawalSchedule(events=[
    # Housing down payment from savings account
    WithdrawalEvent(
        account="Vivienda",
        amount=2_500_000,
        date=date(2026, 12, 1),
        description="Pie departamento"
    ),
    # Vacation from conservative fund
    WithdrawalEvent(
        account="Clooney",
        amount=800_000,
        date=date(2026, 6, 1),
        description="Vacaciones invierno"
    ),
])

# Stochastic withdrawals (with uncertainty)
stochastic_withdrawals = [
    StochasticWithdrawal(
        account="Clooney",
        base_amount=300_000,
        sigma=100_000,
        date=date(2026, 9, 1),
        floor=100_000,
```

```
        cap=600_000,
        seed=42
    ),
]

# Combined withdrawal model
withdrawal_model = WithdrawalModel(
    scheduled=scheduled_withdrawals,
    stochastic=stochastic_withdrawals
)

# Display withdrawal summary
print("=" * 70)
print("WITHDRAWAL SCHEDULE SUMMARY")
print("=" * 70)

print("\nScheduled Withdrawals (Deterministic):")
for event in scheduled_withdrawals.events:
    print(f"  - {event.date.strftime('%Y-%m')}: ${event.amount:,.0f} from␣
 ↪{event.account}")
    if event.description:
        print(f"    Purpose: {event.description}")

print("\nStochastic Withdrawals (Variable):")
for w in stochastic_withdrawals:
    timing = f"month {w.month}" if w.month is not None else w.date.
 ↪strftime('%Y-%m')
    print(f"  - {timing}: ${w.base_amount:,.0f} +/- ${w.sigma:,.0f} from {w.
 ↪account}")
    print(f"    Range: [${w.floor:,.0f}, ${w.cap:,.0f}]" if w.cap else f"   ␣
 ↪Floor: ${w.floor:,.0f}")

# Expected totals by account
expected_totals = withdrawal_model.total_expected(accounts)
print("\nExpected Total Withdrawals by Account:")
for acc_name, total in expected_totals.items():
    if total > 0:
        print(f"  - {acc_name}: ${total:,.0f}")

print("\n" + "=" * 70)
```

```
======================================================================
WITHDRAWAL SCHEDULE SUMMARY
======================================================================

Scheduled Withdrawals (Deterministic):
  - 2026-12: $2,500,000 from Vivienda
```

```
    Purpose: Pie departamento
  - 2026-06: $800,000 from Clooney
    Purpose: Vacaciones invierno

Stochastic Withdrawals (Variable):
  - 2026-09: $300,000 +/- $100,000 from Clooney
    Range: [$100,000, $600,000]

Expected Total Withdrawals by Account:
  - Vivienda: $2,500,000
  - Clooney: $1,100,000

======================================================================
```

8.4 Withdrawal Array Visualization

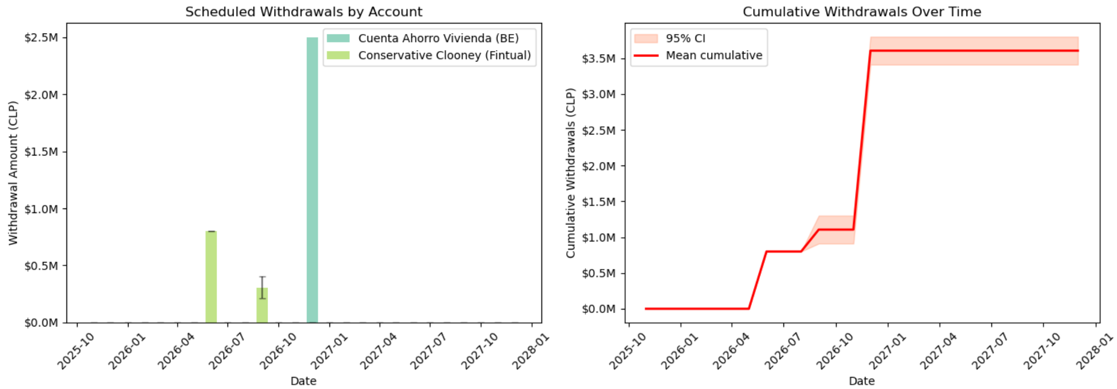
Convert the withdrawal schedule to a numerical array for simulation. The array has shape `(n_sims, T, M)` where stochastic withdrawals vary across scenarios.

[13]:
```python
# Generate withdrawal array for visualization
D_array = withdrawal_model.to_array(
    T=months,
    start_date=start_date,
    accounts=accounts,
    n_sims=n_sims,
    seed=42
)

print(f"Withdrawal array shape: {D_array.shape}")
print(f"  - n_sims: {D_array.shape[0]}")
print(f"  - T (months): {D_array.shape[1]}")
print(f"  - M (accounts): {D_array.shape[2]}")

# Create visualization
import matplotlib.pyplot as plt

fig, axes = plt.subplots(1, 2, figsize=(14, 5))

# Left: Withdrawal timeline by account (mean + std)
ax1 = axes[0]
dates = pd.date_range(start=start_date, periods=months, freq='MS')
colors = plt.cm.Set2(np.linspace(0, 1, len(accounts)))

for m, (acc, color) in enumerate(zip(accounts, colors)):
    D_account = D_array[:, :, m]
    mean_withdrawal = D_account.mean(axis=0)
    std_withdrawal = D_account.std(axis=0)
```

```
    # Only plot if there are withdrawals
    if mean_withdrawal.sum() > 0:
        ax1.bar(dates, mean_withdrawal, width=20, alpha=0.7, label=acc.display_name, color=color)
        # Add error bars for stochastic withdrawals
        if std_withdrawal.max() > 0:
            ax1.errorbar(dates, mean_withdrawal, yerr=std_withdrawal,
                        fmt='none', color='black', alpha=0.5, capsize=3)

ax1.set_xlabel('Date')
ax1.set_ylabel('Withdrawal Amount (CLP)')
ax1.set_title('Scheduled Withdrawals by Account')
ax1.legend(loc='upper right')
ax1.yaxis.set_major_formatter(plt.FuncFormatter(lambda x, _: f'${x/1e6:.1f}M'))
ax1.tick_params(axis='x', rotation=45)

# Right: Cumulative withdrawals over time
ax2 = axes[1]
cumulative_total = D_array.sum(axis=2).cumsum(axis=1)  # Sum across accounts, cumsum over time
mean_cumulative = cumulative_total.mean(axis=0)
std_cumulative = cumulative_total.std(axis=0)

ax2.fill_between(dates,
                 mean_cumulative - 2*std_cumulative,
                 mean_cumulative + 2*std_cumulative,
                 alpha=0.3, color='coral', label='95% CI')
ax2.plot(dates, mean_cumulative, 'r-', linewidth=2, label='Mean cumulative')

ax2.set_xlabel('Date')
ax2.set_ylabel('Cumulative Withdrawals (CLP)')
ax2.set_title('Cumulative Withdrawals Over Time')
ax2.legend()
ax2.yaxis.set_major_formatter(plt.FuncFormatter(lambda x, _: f'${x/1e6:.1f}M'))
ax2.tick_params(axis='x', rotation=45)

plt.tight_layout()
plt.show()

# Summary statistics
print("\n" + "=" * 70)
print("WITHDRAWAL STATISTICS")
print("=" * 70)
total_withdrawals = D_array.sum(axis=(1, 2))  # Sum over time and accounts per sim
print(f"Total withdrawals per scenario:")
print(f"  Mean: ${total_withdrawals.mean():,.0f}")
```

```python
print(f"  Std:  ${total_withdrawals.std():,.0f}")
print(f"  Min:  ${total_withdrawals.min():,.0f}")
print(f"  Max:  ${total_withdrawals.max():,.0f}")
```

```
Withdrawal array shape: (500, 26, 3)
  - n_sims: 500
  - T (months): 26
  - M (accounts): 3
```




```
======================================================================
WITHDRAWAL STATISTICS
======================================================================
Total withdrawals per scenario:
  Mean: $3,606,248
  Std:  $97,575
  Min:  $3,400,000
  Max:  $3,847,709
```

8.5 Wealth Projection with Withdrawals

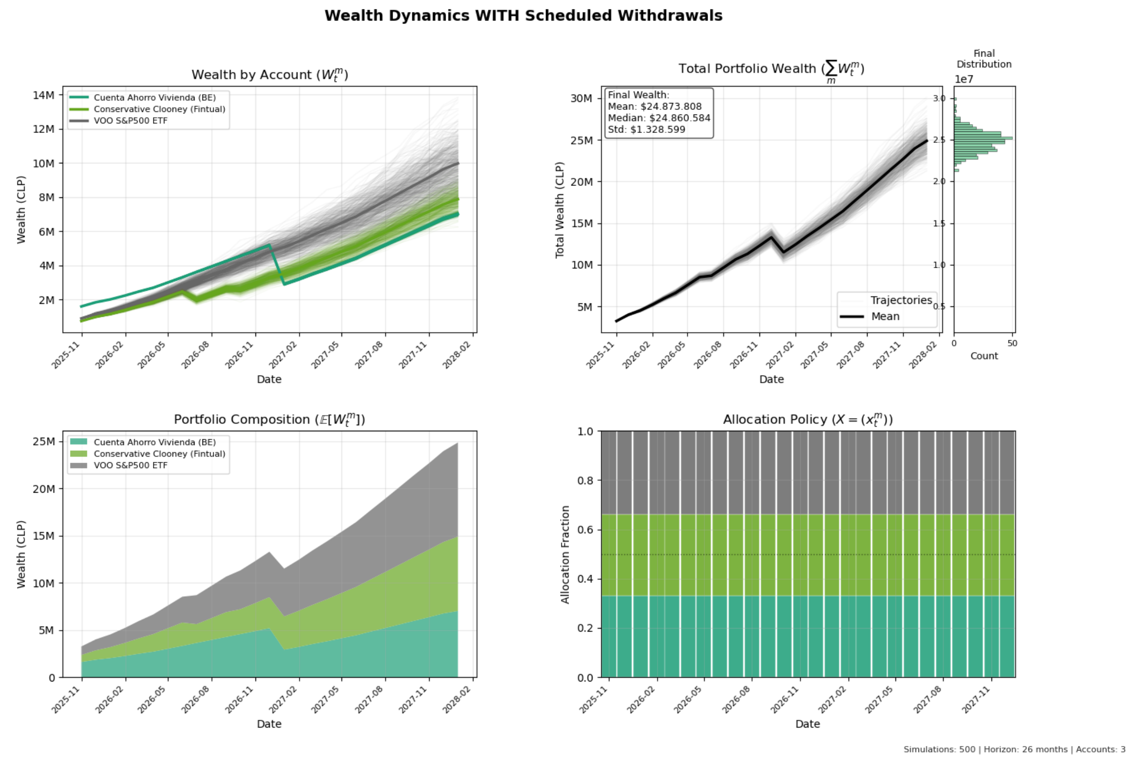
Compare wealth trajectories with and without scheduled withdrawals using the same allocation policy.

[14]:
```python
# Simulate wealth WITH withdrawals
result_with_withdrawals = model.simulate(
    T=months,
    X=X_static,
    n_sims=n_sims,
    start=start_date,
    seed=42,
    withdrawals=withdrawal_model
)
```

```python
# Simulate wealth WITHOUT withdrawals (for comparison)
result_no_withdrawals = model.simulate(
    T=months,
    X=X_static,
    n_sims=n_sims,
    start=start_date,
    seed=42,
    withdrawals=None
)

# Compare total wealth at terminal date
print("=" * 70)
print("WEALTH COMPARISON: WITH vs WITHOUT WITHDRAWALS")
print("=" * 70)

W_with = result_with_withdrawals.total_wealth[:, -1]
W_without = result_no_withdrawals.total_wealth[:, -1]
delta = W_without - W_with

print(f"\nTerminal Wealth (T={months}):")
print(f"  Without withdrawals: ${W_without.mean():,.0f} (mean)")
print(f"  With withdrawals:    ${W_with.mean():,.0f} (mean)")
print(f"  Difference:          ${delta.mean():,.0f} (mean impact)")

print(f"\nExpected total withdrawals: ${total_withdrawals.mean():,.0f}")
print(f"Actual wealth impact:       ${delta.mean():,.0f}")
print(f"Return foregone:            ${delta.mean() - total_withdrawals.mean():,.
 ↪0f}")
print("  (withdrawals don't earn returns after extraction)")

# Plot comparison
model.plot(
    mode='wealth',
    result=result_with_withdrawals,
    X=X_static,
    title="Wealth Dynamics WITH Scheduled Withdrawals",
    show_trajectories=True
)
```

```
======================================================================
WEALTH COMPARISON: WITH vs WITHOUT WITHDRAWALS
======================================================================

Terminal Wealth (T=26):
  Without withdrawals: $28,661,280 (mean)
  With withdrawals:    $24,873,808 (mean)
  Difference:          $3,787,472 (mean impact)
```

```
Expected total withdrawals: $3,606,248
Actual wealth impact:       $3,787,472
Return foregone:            $181,224
  (withdrawals don't earn returns after extraction)
```



9 Goal-Driven Optimization

9.1 Problem Formulation

9.1.1 Non-Convex Chance Constraints

The **bilevel problem** seeks the minimum horizon T^* satisfying all goals:

$$\min_{T\in\mathbb{N}} \; T \; \max_{X\in\mathcal{F}_T} f(X)$$

where the **goal-feasible set** $\mathcal{F}_T$ contains all policies $X \in \mathcal{X}_T$ satisfying probabilistic constraints:

Intermediate goals (at fixed time $t < T$):

$$\mathbb{P}\big(W_t^m(X) \geq b_t^m\big) \geq 1 - \varepsilon_t^m, \quad \forall g \in \mathcal{G}_{\text{int}}$$

Terminal goals (at horizon T):

$$\mathbb{P}\big(W_T^m(X) \geq b^m\big) \geq 1 - \varepsilon^m, \quad \forall g \in \mathcal{G}_{\text{term}}$$

When scheduled withdrawals D_t^m are present, the optimizer adds **probabilistic feasibility constraints** ensuring sufficient wealth exists before each withdrawal:

Withdrawal Feasibility Constraints

$$\mathbb{P}(W_t^m \geq D_t^m) \geq 1 - \varepsilon_{\text{wd}}, \quad \forall t, m \text{ where } D_t^m > 0$$

These are reformulated using CVaR (same technique as goal constraints):

$$\text{CVaR}_{\varepsilon_{\text{wd}}}(D_t^m - W_t^m) \leq 0$$

with decision space (simplex):

$$\mathcal{X}_T = \left\{ X \in \mathbb{R}^{T \times M} : x_t^m \geq 0,\ \sum_{m=1}^{M} x_t^m = 1,\ \forall t = 0, \dots, T-1 \right\}$$

Challenge: Chance constraints involve indicator functions $\mathbb{1}[\cdot]$, which are discontinuous and non-convex. Standard approaches (MILP, sigmoid smoothing) either scale poorly or find local optima.

9.2 Convex Reformulation via CVaR

9.2.1 CVaR Reformulation: Convex Upper Bound

We replace each chance constraint with a **CVaR constraint** (Rockafellar & Uryasev, 2000):

$$\boxed{\mathbb{P}(W \geq b) \geq 1 - \varepsilon \quad \Longleftarrow \quad \text{CVaR}_\varepsilon(b - W) \leq 0}$$

where the **Conditional Value-at-Risk** of shortfall $L = b - W$ is:

$$\text{CVaR}_\varepsilon(L) = \text{VaR}_\varepsilon(L) + \frac{1}{\varepsilon}\mathbb{E}\big[(L - \text{VaR}_\varepsilon(L))_+\big]$$

Epigraphic formulation (convex, suitable for LP solvers):

$$\text{CVaR}_\varepsilon(L) = \min_{\gamma \in \mathbb{R}} \left\{ \gamma + \frac{1}{\varepsilon N} \sum_{i=1}^{N} [L^i - \gamma]_+ \right\}$$

Introducing auxiliary variables $z^i \geq [L^i - \gamma]_+$:

$$\begin{aligned} \text{CVaR}_\varepsilon(L) = \min_{\gamma, z} & \left\{ \gamma + \frac{1}{\varepsilon N} \sum_{i=1}^{N} z^i \right\} \\ \text{s.t.} \quad & z^i \geq L^i - \gamma, \quad \forall i \in [N] \\ & z^i \geq 0, \quad \forall i \in [N] \end{aligned}$$

Key property: If W^i is **affine in** X (as in our wealth dynamics), then $\text{CVaR}_\varepsilon(b - W)$ is **convex in** X.

9.2.2 Mathematical Relationship: Implication, Not Equivalence

Theorem (Rockafellar & Uryasev, 2000):

$$\text{CVaR}_\varepsilon(L) \le 0 \quad \Longrightarrow \quad \mathbb{P}(L \le 0) \ge 1 - \varepsilon$$

Proof sketch: CVaR averages the worst ε-tail of the distribution. If this mean is non-positive, then at least $(1-\varepsilon)$ of scenarios satisfy $L \le 0$.

The converse is NOT true: $\mathbb{P}(L \le 0) \ge 1 - \varepsilon$ does NOT imply CVaR ≤ 0 if the tail is heavy.

Interpretation: CVaR is a **conservative approximation** controlling both: 1. **Frequency** of violations (at most $\varepsilon \times 100\%$ scenarios fail) 2. **Severity** of violations (average loss in tail is non-positive)

The original chance constraint only controls frequency.

9.2.3 Convex Reformulated Problem - Multiple Objectives

Inner problem (for fixed T):

$$\begin{aligned}
\max_{X,\gamma,z} \quad & f(X) \\
\text{s.t.} \quad & \sum_{m=1}^{M} x_t^m = 1, \quad \forall t = 0, \dots, T-1 \quad \text{(simplex)} \\
& x_t^m \ge 0, \quad \forall t, m \quad \text{(non-negativity)} \\
& z_g^i \ge (b_g - W_{t_g}^{i,m_g}(X)) - \gamma_g, \quad \forall g, i \quad \text{(epigraph)} \\
& z_g^i \ge 0, \quad \forall g, i \\
& \gamma_g + \frac{1}{\varepsilon_g N} \sum_{i=1}^{N} z_g^i \le 0, \quad \forall g \quad \text{(CVaR constraint)}
\end{aligned}$$

where: - $W_t^{i,m}(X) = W_0^m F_{0,t,m}^i + \sum_{s=0}^{t-1} A_s^i \cdot x_s^m \cdot F_{s,t,m}^i$ **(affine in X)** - g indexes goals (both intermediate and terminal) - $t_g, m_g, b_g, \varepsilon_g$ are parameters of goal g - $f(X)$ is a convex objective function (see supported objectives below)

Global optimality guaranteed via convex programming (interior-point methods).

9.2.4 Comparison: Original vs. CVaR

Observed conservativeness (empirical): - CVaR constraint: $\text{CVaR}_\varepsilon(L) \le 0$ - Resulting violation rate: typically $(0.5 - 0.8) \times \varepsilon$ (better than required)

Example (from our results): - Goal: $\mathbb{P}(W_T \ge 1\text{M}) \ge 80\%$ (i.e., $\varepsilon = 20\%$) - CVaR solution: violation rate = 9% (margin of 11%)

The 11% buffer is the "price" of convexity, buying us certified global optimality and numerical stability.

9.3 Goal Specification

9.4 Bilevel Optimization

The modified wealth dynamics in the optimizer:

$$W_t^m = W_0^m \cdot F_{0,t}^m + \sum_{s=0}^{t-1}(A_s \cdot x_s^m - D_s^m) \cdot F_{s,t}^m$$

9.4.1 Supported Convex Objectives

CVaROptimizer supports 7 convex objectives exploiting affine wealth structure: Consider $W_t^{i,m}(X) = W_0^m F_{0,t,m}^i + \sum_{s=0}^{t-1} A_s^i \cdot x_s^m \cdot F_{s,t,m}^i$ and $\mathbb{E}\left[\sum_{m=1}^M W_T^m\right] = \frac{1}{N}\sum_{i=1}^N \sum_{m=1}^M W_T^{i,m}(X)$.

1. risky — Growth maximization

$$\max_X \quad \mathbb{E}\left[\sum_{m=1}^M W_T^m\right]$$

2. balanced — Equal contribution across time

$$\max_X \quad -\sum_{t=1}^{T-1}\sum_{m=1}^M (x_t^m - x_{t-1}^m)^2$$

3. risky_turnover — Growth maximization with penalty over contributions across time

$$\max_X \quad \mathbb{E}\left[\sum_{m=1}^M W_T^m\right] - \sum_{t=1}^{T-1}\sum_{m=1}^M (x_t^m - x_{t-1}^m)^2$$

4. conservative — Minimize variance, chose conservative portfolio

$$\max_X \quad \mathbb{E}\left[\sum_{m=1}^M W_T^m\right] - \lambda \cdot \mathrm{Var}\left(\sum_{m=1}^M W_T^m\right)$$

[16]:
```
# --- Execute Optimization ---

print("\n" + "=" * 70)
print("BILEVEL OPTIMIZATION: COMPARING WITH/WITHOUT WITHDRAWALS")
print("=" * 70)

objective_params = {'lambda' : 50}
optimizer = CVaROptimizer(n_accounts=model.M, objective='balanced',␣
↪objective_params=objective_params)

# --- Optimization WITHOUT withdrawals ---
print("\n>>> Test 1: Optimizing WITHOUT withdrawals...")
opt_result_no_wd = model.optimize(
    goals=goals,
```

```
    optimizer=optimizer,
    T_max=120,
    n_sims=400,
    seed=42,
    verbose=True,
    withdrawals=None,  # No withdrawals
    solver='ECOS',
    max_iters=10000
)

# --- Optimization WITH withdrawals ---
print("\n>>> Test 2: Optimizing WITH withdrawals...")
opt_result = model.optimize(
    goals=goals,
    optimizer=optimizer,
    T_max=120,
    n_sims=400,
    seed=42,
    verbose=True,
    withdrawals=withdrawal_model,  # Include withdrawal constraints
    withdrawal_epsilon=0.05,       # 95% confidence for withdrawal feasibility
    solver='ECOS',
    max_iters=10000
)

# --- Comparison ---
print("\n" + "=" * 70)
print("OPTIMIZATION COMPARISON")
print("=" * 70)
print(f"\n  WITHOUT withdrawals:")
print(f"    T* = {opt_result_no_wd.T} months")
print(f"    Objective = {opt_result_no_wd.objective_value:.4f}")
print(f"\n  WITH withdrawals:")
print(f"    T* = {opt_result.T} months")
print(f"    Objective = {opt_result.objective_value:.4f}")
print(f"    Withdrawal constraints = {opt_result.diagnostics.
 ↪get('withdrawal_constraints', 0)}")

delta_T = opt_result.T - opt_result_no_wd.T
if delta_T > 0:
    print(f"\n  Impact: +{delta_T} month(s) required to fund withdrawals while␣
 ↪meeting goals")
elif delta_T == 0:
    print(f"\n  Impact: No change in optimal horizon (withdrawals absorbed␣
 ↪without delay)")
else:
    print(f"\n  Impact: {delta_T} month(s) - unusual, verify constraints")
```

```python
print("\n" + "=" * 70)
print("SELECTED RESULT: WITH WITHDRAWALS")
print("=" * 70)
print(opt_result.summary())
```

```
======================================================================
BILEVEL OPTIMIZATION: COMPARING WITH/WITHOUT WITHDRAWALS
======================================================================

>>> Test 1: Optimizing WITHOUT withdrawals…

=== GoalSeeker: BINARY search T   [12, 120] ===
[Iter 1] Binary search: testing T=66 (range=[12, 120])…

/home/mlioi/anaconda3/envs/finance/lib/python3.11/site-
packages/cvxpy/problems/problem.py:1539: UserWarning: Solution may be
inaccurate. Try another solver, adjusting the solver settings, or solve with
verbose=True for more information.
  warnings.warn(

      Feasible, obj=-0.00, time=0.533s

[Iter 2] Binary search: testing T=39 (range=[12, 66])…
      Feasible, obj=-0.00, time=0.260s

[Iter 3] Binary search: testing T=25 (range=[12, 39])…
      Feasible, obj=-0.00, time=0.086s

[Iter 4] Binary search: testing T=18 (range=[12, 25])…
      Feasible, obj=-0.02, time=0.122s

[Iter 5] Binary search: testing T=15 (range=[12, 18])…
      Infeasible, obj=0.00, time=0.244s

[Iter 6] Binary search: testing T=17 (range=[16, 18])…
      Feasible, obj=-0.16, time=0.098s

[Iter 7] Binary search: testing T=16 (range=[16, 17])…
      Feasible, obj=-1.39, time=6.851s

=== Optimal: T*=16 (binary search converged) ===


>>> Test 2: Optimizing WITH withdrawals…

=== GoalSeeker: BINARY search T   [12, 120] ===
[Iter 1] Binary search: testing T=66 (range=[12, 120])…
```

```
    Feasible, obj=-0.00, time=3.882s

[Iter 2] Binary search: testing T=39 (range=[12, 66])…
    Feasible, obj=-0.00, time=0.459s

[Iter 3] Binary search: testing T=25 (range=[12, 39])…
    Feasible, obj=-0.03, time=0.112s

[Iter 4] Binary search: testing T=18 (range=[12, 25])…
    Infeasible, obj=0.00, time=0.169s

[Iter 5] Binary search: testing T=22 (range=[19, 25])…
    Feasible, obj=-0.06, time=0.126s

[Iter 6] Binary search: testing T=20 (range=[19, 22])…
    Feasible, obj=-0.34, time=4.541s

[Iter 7] Binary search: testing T=19 (range=[19, 20])…
    Feasible, obj=-0.85, time=0.319s

=== Optimal: T*=19 (binary search converged) ===


======================================================================
OPTIMIZATION COMPARISON
======================================================================

  WITHOUT withdrawals:
    T* = 16 months
    Objective = -1.3931

  WITH withdrawals:
    T* = 19 months
    Objective = -0.8468
    Withdrawal constraints = 0

  Impact: +3 month(s) required to fund withdrawals while meeting goals

======================================================================
SELECTED RESULT: WITH WITHDRAWALS
======================================================================
OptimizationResult(
  Status:   Feasible
  Horizon: T=19 months
  Objective: -0.85
  Goals: 4 (2 intermediate, 2 terminal)
  Solve time: 0.319s
  Iterations: 0
```

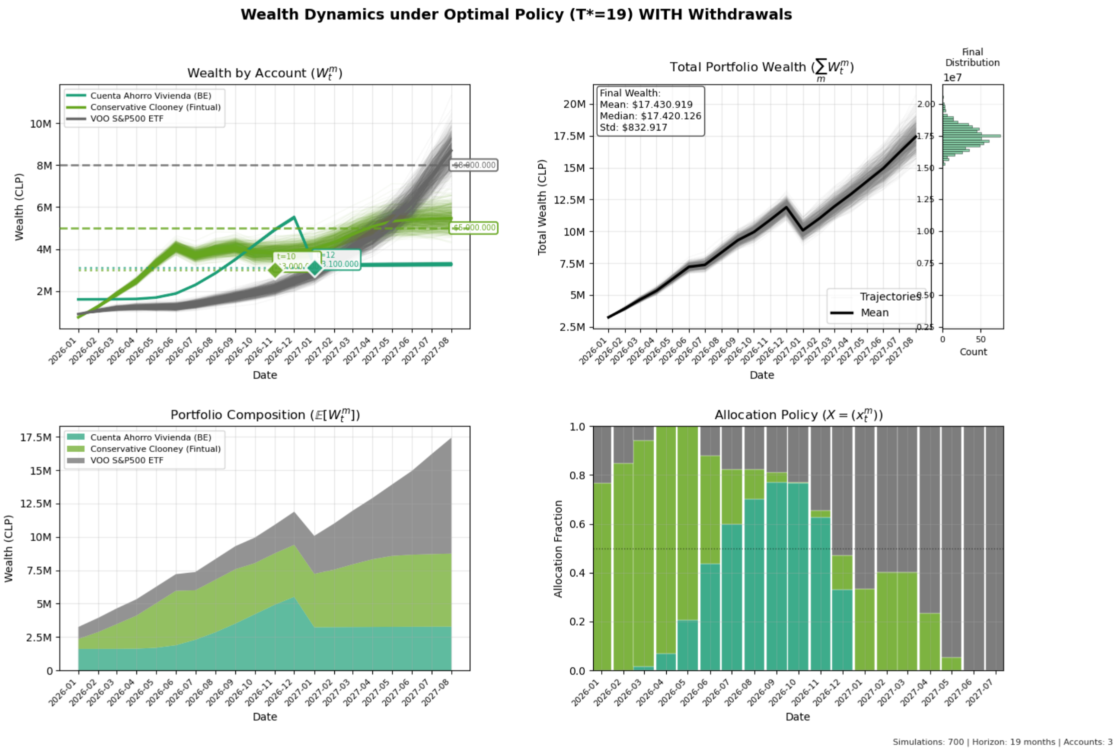
```
)
```

[17]:
```
optimizer.objective
```

[17]: 'balanced'

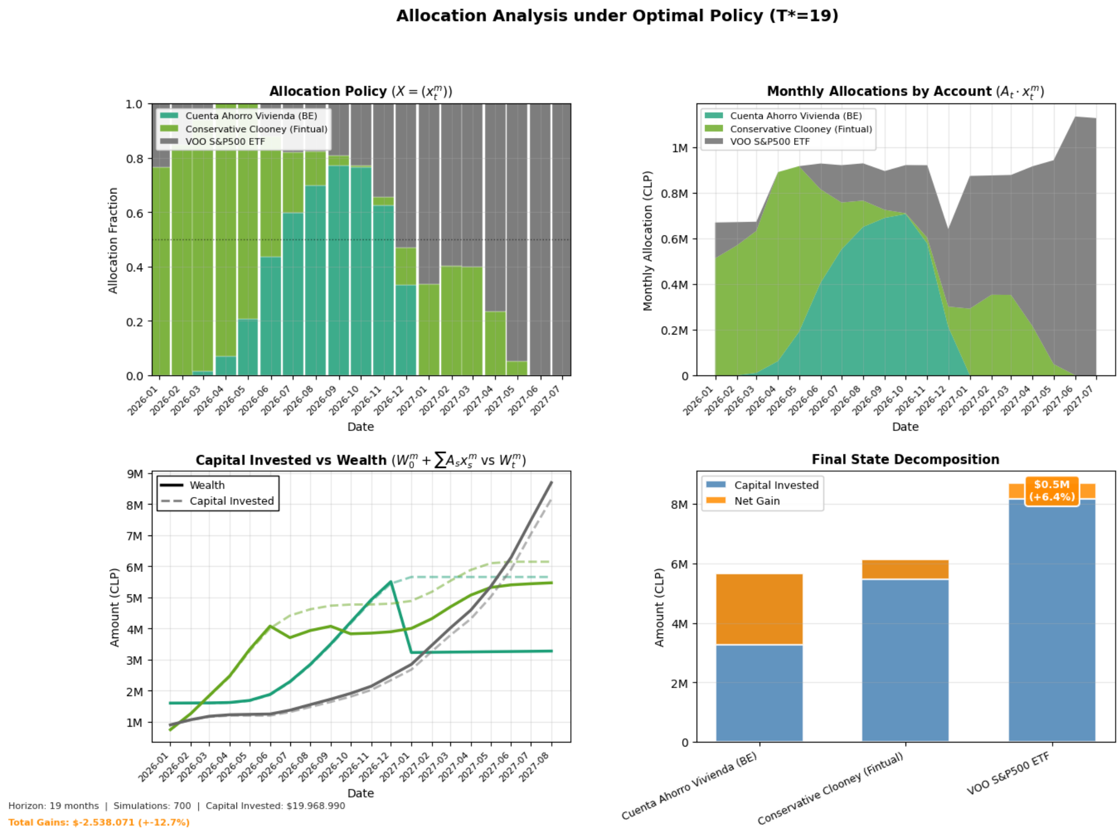
9.5 Optimal Policy Analysis

[18]:
```
# --- Simulate wealth under X* WITH withdrawals ---

# Generate fresh scenarios for validation (out-of-sample)
opt_sim = model.simulate_from_optimization(
    opt_result,
    n_sims=700,
    seed=999,  # Different seed from optimization
    withdrawals=withdrawal_model  # Include withdrawals in simulation
)

# Plot wealth dynamics
model.plot(
    mode='wealth',
    result=opt_sim,
    X=opt_result.X,
    title=f"Wealth Dynamics under Optimal Policy (T*={opt_result.T}) WITH␣
  ↪Withdrawals",
    show_trajectories=True,
    goals=goals
)
```



[22]:
```python
# Plot allocation analysis with investment gains
model.plot(
    mode='allocation',
    result=opt_sim,
    X=opt_result.X,
    title=f"Allocation Analysis under Optimal Policy (T*={opt_result.T})",
    show_trajectories=False
)
```

```
/home/mlioi/fin-opt/src/plotting.py:766: UserWarning: This figure includes Axes
that are not compatible with tight_layout, so results might be incorrect.
  fig.tight_layout(rect=[0, 0.05, 1, 0.97 if title else 0.99])
```



9.6 Goal Verification

9.6.1 In-Sample Verification

9.6.2 Out-of-Sample Verification

```
[NbConvertApp] Converting notebook FinOpt-Workflow.ipynb to pdf
[NbConvertApp] Support files will be in FinOpt-Workflow_files/
[NbConvertApp] Making directory ./FinOpt-Workflow_files
[NbConvertApp] Writing 90815 bytes to notebook.tex
[NbConvertApp] Building PDF
[NbConvertApp] Running xelatex 3 times: ['xelatex', 'notebook.tex', '-quiet']
[NbConvertApp] Running bibtex 1 time: ['bibtex', 'notebook']
[NbConvertApp] WARNING | bibtex had problems, most likely because there were no
citations
[NbConvertApp] PDF successfully created
[NbConvertApp] Writing 2440225 bytes to FinOpt-Workflow.pdf
```

```
[NbConvertApp] Converting notebook FinOpt-Workflow.ipynb to html
[NbConvertApp] WARNING | Alternative text is missing on 11 image(s).
[NbConvertApp] Writing 4004301 bytes to FinOpt-Workflow.html
```

[ ]:
```

```