Worksheet: Euler's method
Math 102 Section 102
Nov. 16, 2018

Use a computer. Work in a group.

(Exponential growth) Consider the exponential growth model, with per-capita growth rate $r = 1$ and initial condition $N_0 = N(0) = 1$:

$$\frac{dN}{dt} = N, \quad N(0) = 1.$$

which has solution $N(t) = e^t$

1. In the context of Euler's method, what is $f(N)$ for this model?

2. Let h be the step size. Determine the Euler iteration formula for the exponential growth model:

   $N_{n+1} =$

3. Suppose we want to approximate the solution to the exponential growth model from $t = 0$ to $t = 1$ using 5 steps. What is the step size h?

   $h =$

4. Fill in the following table.

Time t	Counter n	Approximate N_n	Exact $N(t) = e^t$	Error $\lvert N_n - N(t)\rvert$
0	0			
0.2	1			
0.4	2			
0.6	3			
0.8	4			
1	5			

5. Plot the approximate and exact solutions. *Hint: use a line chart (with markers for the approximate solution).*

6. What is the maximum error with 5 time steps? *Hint: use max(cell:range).* Repeat the calculation with 10, 20 and 40 time-steps. What happens to the maximum error as the time-step is doubled?

(Logistic growth and stability of Euler's method) The logistic equation,

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), \quad N(0) = N_0$$

has solution

$$N(t) = \frac{KN_0}{N_0 + (K - N_0)e^{-rt}}.$$

(solving the logistic equation is beyond the scope of Math 102).

1. With $r = 0.1$ and $K = 1000$, what is $f(N)$ in the context of Euler's Method?

2. Determine the Euler iteration formula for the exponential growth model:

   $N_{n+1} =$

3. $N_0 = 2$. Use a timestep of $h = 25$ to find an approximate solution up to time $t = 200$. What happens?

4. Use a timestep of $h = 1$ to find an approximate solution up to time $t = 200$. What happens?

5. What is the disadvantage of choosing a small timestep?

(Stability) Use Euler's Method to find a solution to

$$\frac{dy}{dt} = -3y, \quad y(0) = 1$$

with step size $h = 1$.

1. What's the analytical solution to this Initial Value Problem (IVP)? Does the solution exponentially grow or decay?

2. What happens to the approximate solution over time? Does the approximate solution exhibit behavior consistent with your prediction above?

3. What timestep size, h, should you use to ensure that the solution exhibits the correct qualitative behaviour?

Learn more. Euler's Method produces good approximations to solutions to some equations, provided that the timestep h is small enough. You can learn more about Euler's Method and related *Numerical Methods* at `https://en.wikipedia.org/wiki/Euler_method`.