Limit of functions and sequences

Math 100 Vantage College

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Limit of a function

Definition (Limit of a function)

We say that as x goes to a, the limit of $f(x)$ is L and write

$$\lim_{x \to a} f(x) = L,$$

when the value of $f(x)$ is arbitrarily close to L provided that x is sufficiently close to (but not equal to) a.

Remarks:

- It has nothing to do with $f(a)$, even if it does NOT exist.
- Taking the limit as $x \to a$ means that $x \neq a$.
- $L \in \mathbb{R}$. And neither $+\infty$ nor $-\infty$ is a number.

Special cases of limits

- A special case when the limit DNE:

$$\lim_{x \to a} f(x) = \infty \text{ or } -\infty.$$

- Limit at infinity:

$$\lim_{x \to \infty} f(x) = L \text{ or } \lim_{x \to -\infty} f(x) = L.$$

- One-sided limits:

$$\lim_{x \to a-} f(x) = L, \quad \lim_{x \to a+} f(x) = L.$$

Computing limits

Assume that $\lim_{x\to a} f(x) = F$ and $\lim_{x\to a} g(x) = G$. $\alpha, \beta \in \mathbb{R}$. Then

- $\lim_{x\to a}(\alpha f(x) + \beta g(x)) = \alpha F + \beta G$.
- $\lim_{x\to a} f(x)g(x) = FG$.
- $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{F}{G}$, if $G \neq 0$.
- $\lim_{x\to a}(f(x))^n = F^n$.
- $\lim_{x\to a}(f(x))^{1/n} = F^{1/n}$, when the n-th root is well defined.

Computing limits

- You can freely do the above arithmetics, provided that after doing this the result exists!
- These rules also apply to the cases when f or g goes to infinity, whenever it makes sense.
- Exponentials dominate powers; factorials dominate exponentials.

$$\lim_{x\to\infty} \frac{x^r}{a^x} = 0,$$

$$\lim_{n\to\infty} \frac{a^n}{n!} = 0. \ r > 0, \ a > 1.$$

Useful theorems

Theorem

$$\lim_{x\to a} f(x) = L \iff \lim_{x\to a-} f(x) = \lim_{x\to a+} f(x) = L.$$

The RHS says three things:

- The left limit exists.
- The right limit exists.
- And they are equal to each other.

Useful theorems

Theorem (Squeeze theorem)

Let $a \in \mathbb{R}$ and f, g, h be three functions such that

$$f(x) \leq g(x) \leq h(x),$$

for all x in an interval around a, except possibly at $x = a$ itself. If

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L,$$

then also

$$\lim_{x \to a} g(x) = L.$$

Remarks:

- Again there is no business of $f(a), g(a), h(a)$.
- It also applies to sequences, which are special cases of functions.

Useful theorems

Theorem (Bounded monotone convergence)

$(a_n)_{n\in\mathbb{N}}$ is a sequence s.t.

- a_n is increasing. ($a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$.)
- a_n is bounded above. ($a_n < M$ for all $n \in \mathbb{N}$ and some $M \in \mathbb{R}$.)

Then $(a_n)_{n\in\mathbb{N}}$ converges.