Small Open Economy Model

Anna Catarina Tavella e Matheus Franciscão

2022-xx-xx

1 - Problem of the consumer (for reference)

$$max\ \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t) = max\ E_0 \sum_{t=0}^{\infty} \beta^t U\left(\left[(1-\alpha)^{\frac{1}{\eta}}(C_{H,t})^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}}(C_{F,t})^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}, N_t\right)$$

subject to the budget constraint (specified below), where

$$C_{H,t} \equiv \left(\int_0^1 C_{H,t}(j)^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}},\ C_{F,t} \equiv \left(\int_0^1 (C_{i,t})^{\frac{\gamma-1}{\gamma}} di\right)^{\frac{\gamma}{\gamma-1}},\ C_{i,t} \equiv \left(\int_0^1 C_{i,t}(j)^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}}$$

and $C_t \equiv \left[(1-\alpha)^{\frac{1}{\eta}}(C_{H,t})^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}}(C_{F,t})^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}$

Substituting, we get

$$max\ \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U\left(\left[(1-\alpha)^{\frac{1}{\eta}}\left[\left(\int_0^1 C_{H,t}(j)^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}}\right]^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}}\left[\left(\int_0^1 (C_{F,t})^{\frac{\gamma-1}{\gamma}} di\right)^{\frac{\gamma}{\gamma-1}}\right]^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}, N_t\right) =$$

$$max\ \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U\left(\left[(1-\alpha)^{\frac{1}{\eta}}\left[\left(\int_0^1 C_{H,t}(j)^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}}\right]^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}}\left[\left(\int_0^1 \left(\left(\int_0^1 C_{i,t}(j)^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}}\right)^{\frac{\gamma-1}{\gamma}} di\right)^{\frac{\gamma}{\gamma-1}}\right]^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}, N_t\right)$$

subject to the budget constraint:

$$\int_0^1 P_{H,t}(j)C_{H,t}(j)dj + \int_0^1\int_0^1 P_{i,t}(j)C_{i,t}(j)dj\ di + \mathbb{E}_t\{Q_{t,t+1}D_{t+1}\} \leq D_t + W_tN_t + Tt$$

$$\mathcal{L} = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t U\left(\left[(1-\alpha)^{\frac{1}{\eta}}\left[\left(\int_0^1 C_{H,t}(j)^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}}\right]^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}}\left[\left(\int_0^1 \left(\left(\int_0^1 C_{i,t}(j)^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}}\right)^{\frac{\gamma-1}{\gamma}} di\right)^{\frac{\gamma}{\gamma-1}}\right]^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}, N_t\right)$$

$$+\lambda_t\left(D_t + W_tN_t + Tt - \int_0^1 P_{H,t}(j)C_{H,t}(j)dj - \int_0^1\int_0^1 P_{i,t}(j)C_{i,t}(j)dj\ di - \mathbb{E}_t\{Q_{t,t+1}D_{t+1}\}\right)$$

2 - Finding the demand function for each specific good

$$C_{H,t}(j) = \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\varepsilon} C_{H,t},\ C_{i,t}(j) = \left(\frac{P_{i,t}(j)}{P_{i,t}}\right)^{-\varepsilon} C_{i,t} \text{ and } C_{i,t} = \left(\frac{P_{i,t}}{P_{F,t}}\right)^{-\gamma} C_{F,t}$$

It's easier by calculating the MRS (marginal rate of substitution) between $C_{H,t}(j)$ and $C_{H,t}$, as by the optimal allocation, it has to be the rate of prices in every period of time (otherwise the consumer could by a little less of the product with relative higher price and buy another with relative lower price, increasing his utility).

$$\frac{\partial U(C_t, N_t)}{\partial C_{H,t}(j)} = U_c(C_t, N_t)\frac{\eta}{1-\eta}\left(C_t^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}-1}(1-\alpha)^{\frac{1}{\eta}}\frac{\eta-1}{\eta}(C_{H,t})^{-\frac{1}{\eta}}\frac{\varepsilon}{\varepsilon-1}\left(C_{H,t}^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}-1}\int_0^1 \frac{\varepsilon-1}{\varepsilon} C_{H,t}(j)^{-\frac{1}{\varepsilon}} dj$$

After simplifying, we get

$$\frac{\partial U(C_t,N_t)}{\partial C_{H,t}(j)} = U_c(C_t,N_t)(1-\alpha)^{\frac{1}{\eta}} C_t^{\frac{1}{\eta}} (C_{H,t})^{-\frac{1}{\eta}} C_{H,t}^{\frac{1}{\varepsilon}} \int_0^1 C_{H,t}(j)^{-\frac{1}{\varepsilon}} dj = U_c(C_t,N_t)\left[(1-\alpha)\frac{C_t}{C_{H,t}}\right]^{\frac{1}{\eta}} \int_0^1 \left[\frac{C_{H,t}}{C_{H,t}(j)}\right]^{\frac{1}{\varepsilon}} dj$$

Similarly,

$$\frac{\partial U(C_t,N_t)}{\partial C_{H,t}} = U_c(C_t,N_t)\frac{\eta}{1-\eta}\left(C_t^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}-1}(1-\alpha)^{\frac{1}{\eta}}\frac{\eta-1}{\eta}(C_{H,t})^{-\frac{1}{\eta}} = U_c(C_t,N_t)(1-\alpha)^{\frac{1}{\eta}} C_t^{\frac{1}{\eta}} (C_{H,t})^{-\frac{1}{\eta}} = U_c(C_t,N_t)\left[(1-\alpha)\frac{C_t}{C_{H,t}}\right]^{\frac{1}{\eta}}$$

$$\frac{\frac{\partial U(C_t,N_t)}{\partial C_{H,t}(j)}}{\frac{\partial U(C_t,N_t)}{\partial C_{H,t}}} = \frac{U_c(C_t,N_t)\left[(1-\alpha)\frac{C_t}{C_{H,t}}\right]^{\frac{1}{\eta}} \int_0^1 \left[\frac{C_{H,t}}{C_{H,t}(j)}\right]^{\frac{1}{\varepsilon}} dj}{U_c(C_t,N_t)\left[(1-\alpha)\frac{C_t}{C_{H,t}}\right]^{\frac{1}{\eta}}} = \frac{\int_0^1 P_{H,t}(j)dj}{P_{H,t}}$$

After simplifying again, the expression is almost the demand function we want.

$$\int_0^1 \left[\frac{C_{H,t}}{C_{H,t}(j)}\right]^{\frac{1}{\varepsilon}} dj = \int_0^1 \frac{P_{H,t}(j)}{P_{H,t}} dj$$

As the interval of both integrals are the same and the variable being integrated is also the same, what is inside the integral in both sides have also to be the same. So,

$$\left[\frac{C_{H,t}}{C_{H,t}(j)}\right]^{\frac{1}{\varepsilon}} = \frac{P_{H,t}(j)}{P_{H,t}} \Rightarrow \left[\frac{C_{H,t}(j)}{C_{H,t}}\right]^{-\frac{1}{\varepsilon}} = \frac{P_{H,t}(j)}{P_{H,t}} \Rightarrow C_{H,t}(j) = \left[\frac{P_{H,t}(j)}{P_{H,t}}\right]^{-\varepsilon} C_{H,t}$$

Calculating now the MRS (marginal rate of substitution) between $C_{i,t}(j)$ and $C_{i,t}$, which is also equal the rate of the prices.

$$\frac{\partial U(C_t,N_t)}{\partial C_{i,t}(j)} =$$

$$U_c(C_t,N_t)\frac{\eta}{1-\eta}\left(C_t^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}-1}\alpha^{\frac{1}{\eta}}\frac{\eta-1}{\eta}(C_{F,t})^{-\frac{1}{\eta}}\frac{\gamma}{\gamma-1}\left(C_{H,t}^{\frac{\gamma-1}{\gamma}}\right)^{\frac{\gamma}{\gamma-1}-1}\int_0^1 \frac{\gamma-1}{\gamma} C_{i,t}^{-\frac{1}{\gamma}}\left[\frac{\varepsilon}{\varepsilon-1}\left(C_{i,t}^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}-1}\int_0^1 \frac{\varepsilon-1}{\varepsilon} C_{i,t}(j)^{-\frac{1}{\varepsilon}} dj\right] di$$

After simplifying, we get

$$\frac{\partial U(C_t,N_t)}{\partial C_{i,t}(j)} = U_c(C_t,N_t)\alpha^{\frac{1}{\eta}} C_t^{\frac{1}{\eta}} (C_{F,t})^{-\frac{1}{\eta}} C_{F,t}^{\frac{1}{\gamma}} \int_0^1 C_{i,t}^{-\frac{1}{\gamma}}\left[C_{i,t}^{\frac{1}{\varepsilon}}\int_0^1 C_{i,t}(j)^{-\frac{1}{\varepsilon}} dj\right] di$$

$$\frac{\partial U(C_t,N_t)}{\partial C_{i,t}(j)} = U_c(C_t,N_t)\left[\alpha\frac{C_t}{C_{F,t}}\right]^{\frac{1}{\eta}} \int_0^1 \left[\frac{C_{F,t}}{C_{i,t}}\right]^{\frac{1}{\gamma}} \int_0^1 \left[\frac{C_{i,t}}{C_{i,t}(j)}\right]^{\frac{1}{\varepsilon}} dj\ di$$

$$\frac{\partial U(C_t,N_t)}{\partial C_{i,t}} = U_c(C_t,N_t)\frac{\eta}{1-\eta}\left(C_t^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}-1}\alpha^{\frac{1}{\eta}}\frac{\eta-1}{\eta}(C_{F,t})^{-\frac{1}{\eta}}\frac{\gamma}{\gamma-1}\left(C_{H,t}^{\frac{\gamma-1}{\gamma}}\right)^{\frac{\gamma}{\gamma-1}-1}\int_0^1 \frac{\gamma-1}{\gamma} C_{i,t}^{-\frac{1}{\gamma}} di$$

$$\frac{\partial U(C_t,N_t)}{\partial C_{i,t}} = U_c(C_t,N_t)\left[\alpha\frac{C_t}{C_{F,t}}\right]^{\frac{1}{\eta}} \int_0^1 \left[\frac{C_{F,t}}{C_{i,t}}\right]^{\frac{1}{\gamma}} di$$

Calculating the MRS we have

$$\frac{\frac{\partial U(C_t,N_t)}{\partial C_{i,t}(j)}}{\frac{\partial U(C_t,N_t)}{\partial C_{i,t}}} = \frac{U_c(C_t,N_t)\left[\alpha\frac{C_t}{C_{F,t}}\right]^{\frac{1}{\eta}} \int_0^1 \left[\frac{C_{F,t}}{C_{i,t}}\right]^{\frac{1}{\gamma}} \int_0^1 \left[\frac{C_{i,t}}{C_{i,t}(j)}\right]^{\frac{1}{\varepsilon}} dj\ di}{U_c(C_t,N_t)\left[\alpha\frac{C_t}{C_{F,t}}\right]^{\frac{1}{\eta}} \int_0^1 \left[\frac{C_{F,t}}{C_{i,t}}\right]^{\frac{1}{\gamma}} di} = \frac{\int_0^1 P_{i,t}(j)dj}{P_{i,t}}$$

As before, we can simplify again. Also, as there's a continuum of firms, we can consider that the price of each product $P_{i,t}(j)$ is only correlated with its specific demand function and not with the demand function of other in its country or another country, it follows that each specific price is uncorrelated with $C_{F,t}$ and $C_{i,t}$. Also, as each firm is very small, we can consider that it has negligible influence on the aggregate index price of its country ($P_{i,t}$). Whith these independence assumption, the joint distribution is equal to the product of the marginal distributions.

$$\int_0^1 \int_0^1 \left[\frac{C_{F,t}}{C_{i,t}}\right]^{\frac{1}{\gamma}} \left[\frac{C_{i,t}}{C_{i,t}(j)}\right]^{\frac{1}{\varepsilon}} dj\ di = \int_0^1 \left[\frac{C_{F,t}}{C_{i,t}}\right]^{\frac{1}{\gamma}} di \int_0^1 \frac{P_{i,t}(j)}{P_{i,t}} dj = \int_0^1 \int_0^1 \frac{P_{i,t}(j)}{P_{i,t}} \left[\frac{C_{F,t}}{C_{i,t}}\right]^{\frac{1}{\gamma}} dj\ di$$

Now, as before, the integrand in both sides needs to be the same. The we get the second demand equation.

$$\left[\frac{C_{F,t}}{C_{i,t}}\right]^{\frac{1}{\gamma}} \left[\frac{C_{i,t}}{C_{i,t}(j)}\right]^{\frac{1}{\varepsilon}} = \frac{P_{i,t}(j)}{P_{i,t}} \left[\frac{C_{F,t}}{C_{i,t}}\right]^{\frac{1}{\gamma}} \Rightarrow \left[\frac{C_{i,t}(j)}{C_{i,t}}\right]^{-\frac{1}{\varepsilon}} = \frac{P_{i,t}(j)}{P_{i,t}} \Rightarrow C_{i,t}(j) = \left[\frac{P_{i,t}(j)}{P_{i,t}}\right]^{-\varepsilon} C_{i,t}$$

To find the aggregate demand for each country, in terms of total foreign demand, we proceed by calculating the MRS between the aggregate consumption for the country and the aggregate consumption of foreign goods, which the optimal allocation resulting from the rate between the prices, as before.

$$\frac{\partial U(C_t,N_t)}{\partial C_{F,t}} = U_c(C_t,N_t)\frac{\eta}{1-\eta}\left(C_t^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}-1}\alpha^{\frac{1}{\eta}}\frac{\eta-1}{\eta}\left(C_{F,t}\right)^{-\frac{1}{\eta}} = U_c(C_t,N_t)\left[\alpha\frac{C_t}{C_{F,t}}\right]^{\frac{1}{\eta}}$$

$$\frac{\frac{\partial U(C_t,N_t)}{\partial C_{i,t}}}{\frac{\partial U(C_t,N_t)}{\partial C_{F,t}}} = \frac{U_c(C_t,N_t)\left[\alpha\frac{C_t}{C_{F,t}}\right]^{\frac{1}{\eta}}\int_0^1\left[\frac{C_{F,t}}{C_{i,t}}\right]^{\frac{1}{\gamma}}di}{U_c(C_t,N_t)\left[\alpha\frac{C_t}{C_{F,t}}\right]^{\frac{1}{\eta}}} = \frac{\int_0^1 P_{i,t}di}{P_{F,t}}$$

As $P_{F,t}$ doesn't depend on a specific i, we can put it inside the integral. Then we get again two integrans which have to be the same for the equality to hold.

$$\left[\frac{C_{F,t}}{C_{i,t}}\right]^{\frac{1}{\gamma}} = \frac{P_{i,t}}{P_{F,t}} \Rightarrow \left[\frac{C_{i,t}}{C_{F,t}}\right]^{-\frac{1}{\gamma}} = \frac{P_{i,t}}{P_{F,t}} \quad\Rightarrow\quad C_{i,t} = \left[\frac{P_{i,t}}{P_{F,t}}\right]^{-\gamma}C_{F,t}$$

3 - Aggregating the expenditure

Now that we have the demand functions

$$C_{H,t}(j) = \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\varepsilon}C_{H,t},\ C_{i,t}(j) = \left(\frac{P_{i,t}(j)}{P_{i,t}}\right)^{-\varepsilon}C_{i,t} \text{ and } C_{i,t} = \left(\frac{P_{i,t}}{P_{F,t}}\right)^{-\gamma}C_{F,t}$$

let's prove that

$$\int_0^1 P_{H,t}(j)C_{H,t}(j)dj = P_{H,t}C_{H,t}\ ,\ \int_0^1 P_{i,t}(j)C_{i,t}(j)dj = P_{i,t}C_{i,t} \text{ and } \int_0^1 P_{i,t}C_{i,t}dj = P_{F,t}C_{F,t}$$

using the definition of the price indexes:

$$P_{H,t} \equiv \left(\int_0^1 P_{H,t}(j)^{1-\varepsilon}dj\right)^{\frac{1}{1-\varepsilon}},\ P_{i,t} \equiv \left(\int_0^1 P_{i,t}(j)^{1-\varepsilon}dj\right)^{\frac{1}{1-\varepsilon}} \text{ and } P_{F,t} \equiv \left(\int_0^1 P_{i,t}^{1-\gamma}di\right)^{\frac{1}{1-\gamma}}$$

$$\int_0^1 P_{H,t}(j)C_{H,t}(j)dj = \int_0^1 P_{H,t}(j)\left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\varepsilon}C_{H,t}dj = \frac{C_{H,t}}{P_{H,t}^{-\varepsilon}}\int_0^1 P_{H,t}(j)^{1-\varepsilon}dj = \frac{C_{H,t}}{P_{H,t}^{-\varepsilon}}P_{H,t}^{1-\varepsilon} = P_{H,t}C_{H,t}$$

$$\int_0^1 P_{i,t}(j)C_{i,t}(j)dj = \int_0^1 P_{i,t}(j)\left(\frac{P_{i,t}(j)}{P_{i,t}}\right)^{-\varepsilon}C_{i,t}dj = \frac{C_{i,t}}{P_{i,t}^{-\varepsilon}}\int_0^1 P_{i,t}(j)^{1-\varepsilon}dj = \frac{C_{i,t}}{P_{i,t}^{-\varepsilon}}P_{i,t}^{1-\varepsilon} = P_{i,t}C_{i,t}$$

$$\int_0^1 P_{i,t}C_{i,t}di = \int_0^1 P_{i,t}\left(\frac{P_{i,t}}{P_{F,t}}\right)^{-\gamma}C_{F,t}di = \frac{C_{F,t}}{P_{F,t}^{-\gamma}}\int_0^1 P_{F,t}^{1-\gamma}dj = \frac{C_{F,t}}{P_{F,t}^{-\varepsilon}}P_{F,t}^{1-\varepsilon} = P_{F,t}C_{F,t}$$

With this aggregatio, the budget constraint can be simplified

$$\int_0^1 P_{H,t}(j)C_{H,t}(j)dj + \int_0^1\int_0^1 P_{i,t}(j)C_{i,t}(j)dj\ di + \mathbb{E}_t\{Q_{t,t+1}D_{t+1}\} \leq D_t + W_tN_t + Tt$$

$$P_{H,t}C_{H,t} + \int_0^1 P_{i,t}C_{i,t}di = P_{H,t}C_{H,t} + P_{F,t}C_{F,t} \leq D_t + W_tN_t + Tt - \mathbb{E}_t\{Q_{t,t+1}D_{t+1}\}$$

As the total consumption expenditure by the representative consumer is with the domestic produts or foreig products, the budget constraint becomes:

$$P_tC_t \leq D_t + W_tN_t + Tt - \mathbb{E}_t\{Q_{t,t+1}D_{t+1}\}$$

4 - Finding the optimal share between the domestic and imported goods

Now we will calculate the MRS between the domestic products and the total consumption, which has to be equal to the rate of prices. After, we will do the same for the foreign products.

$$\frac{\frac{\partial U(C_t,N_t)}{\partial C_{H,t}}}{\frac{\partial U(C_t,N_t)}{\partial C_t}} = \frac{U_c(C_t,N_t)\left[(1-\alpha)\frac{C_t}{C_{H,t}}\right]^{\frac{1}{\eta}}}{U_c(C_t,N_t)} = \frac{P_{H,t}}{P_t} \quad\Rightarrow\quad (1-\alpha)\frac{C_t}{C_{H,t}} = \left(\frac{P_{H,t}}{P_t}\right)^{\eta} \quad\Rightarrow\quad C_{H,t} = (1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta}C_t$$

$$\frac{\frac{\partial U(C_t,N_t)}{\partial C_{F,t}}}{\frac{\partial U(C_t,N_t)}{\partial C_t}} = \frac{U_c(C_t,N_t)\left[\alpha\frac{C_t}{C_{F,t}}\right]^{\frac{1}{\eta}}}{U_c(C_t,N_t)} = \frac{P_{F,t}}{P_t} \quad \Rightarrow \quad \alpha\frac{C_t}{C_{F,t}} = \left(\frac{P_{F,t}}{P_t}\right)^{\eta} \quad \Rightarrow \quad C_{F,t} = \alpha\left(\frac{P_{F,t}}{P_t}\right)^{-\eta} C_t$$

5 - Standard problem of the representative consumer

Now we arrive at a standard problem for the representative consumer

$$\underset{C_t,N_t,D_{t+1}}{max}\ \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t U(C_t,N_t) = \underset{C_t,N_t}{max}\ E_0\sum_{t=0}^{\infty}\beta^t\left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi}\right)$$

subject to $D_t + W_tN_t + Tt - \mathbb{E}_t\{Q_{t,t+1}D_{t+1}\} - P_tC_t = 0$, as an optimal condition.

using the separable utility function specified as $U(C_t,N_t) = \dfrac{C_t^{1-\sigma}}{1-\sigma} - \dfrac{N_t^{1+\varphi}}{1+\varphi}$

$$\mathcal{L} = \mathbb{E}_t\sum_{t=0}^{\infty}\beta^t\left[\left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi}\right) + \lambda_t\left(D_t + W_tN_t + Tt - Q_{t,t+1}D_{t+1} - P_tC_t\right)\right]$$

with first order conditions (FOCs):

(C_t): $\beta^tC_t^{-\sigma} = \beta^t\lambda_tP_t \quad \Rightarrow \quad C_t^{-\sigma} = \lambda_tP_t$

(N_t): $\beta^tN_t^{\varphi} = \beta^t\lambda_tW_t \quad \Rightarrow \quad N_t^{\varphi} = \lambda_tP_t$

(D_{t+1}): $\beta^t\lambda_tQ_{t,t+1} = \beta^{t+1}\mathbb{E}[\lambda_{t+1}] \quad \Rightarrow \quad \dfrac{\mathbb{E}[\lambda_{t+1}]}{\lambda_t} = \dfrac{Q_{t,t+1}}{\beta}$

Dividing (N_t) FOC by (C_t) FOC, we have the standard equation of intratemporal substitution between consumption and leisure $C_t^{\sigma}N_t^{\varphi} = \dfrac{W_t}{P_t}$

Advancing one period for the consumption FOC, we have $\mathbb{E}[C_{t+1}^{-\sigma}] = \mathbb{E}[\lambda_{t+1}P_{t+1}]$

As the model will be log-linearized and an approximation of the first order will be used to solve it, we can ignore the Jensen's inequality where there is an expectation operator. Up to first order approximation, $\mathbb{E}[xy] \approx \mathbb{E}[x]\mathbb{E}[y]$.

Dividing the consumption FOC in t+1 by the equation in t and substituting by the $\mathbb{E}[\lambda_{t+1}]/\lambda_t$ in the D_{t+1} FOC, we get the Euler equation

$$\mathbb{E}\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\right] = \mathbb{E}\left[\frac{\lambda_{t+1}P_{t+1}}{\lambda_tP_t}\right] \quad \Rightarrow \quad \mathbb{E}\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\right] = \frac{\mathbb{E}[Q_{t,t+1}]}{\beta} \quad \Rightarrow \quad \beta R_t\mathbb{E}\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\right] = 1$$

as $Q_{t,t+1}$, is the price of a a riskless one-period discount bond in domestic currency with gross return R_t.

6 - Log linearization

Log-linearizing $C_t^{\sigma}N_t^{\varphi} = \dfrac{W_t}{P_t}$ is straight forward: $w_t - p_t = \sigma c_t + \varphi n_t$

To log-linerize the Euler equation, we'll use the Taylor expansion: $f(x) \approx f(x_0) + f'(x_o)(x - x_0)$. When expanding the exponential function around 0, we get $e^x = e^0 + e^0(x-0) = 1 + x$

$$\mathbb{E}\left[\exp\left(\ln\left[\beta R_t\left(\tfrac{C_{t+1}}{C_t}\right)^{-\sigma}\tfrac{P_t}{P_{t+1}}\right]\right)\right] = \mathbb{E}\left[\exp\left(\ln(\beta) + r_t - \sigma(c_{t+1} - c_t) + p_t - p_{t+1}\right)\right] = 1 \quad \Rightarrow$$

$$1 + \ln(\beta) + r_t - \sigma(\mathbb{E}\left[c_{t+1}\right] - c_t) - \mathbb{E}\left[\pi_{t+1}\right] = 1 \quad \Rightarrow \quad \sigma c_t = \sigma\mathbb{E}\left[c_{t+1}\right] + \mathbb{E}\left[\pi_{t+1}\right] - r_t - \ln(\beta) \Rightarrow \quad c_t = \mathbb{E}\left[c_{t+1}\right] - \frac{1}{\sigma}\left(r_t - \mathbb{E}\left[\pi_{t+1}\right] - \rho\right)$$

as $\rho \equiv \frac{1-\beta}{\beta} \approx -\ln(\beta)$

7 - Terms of trade

Let's log-linearize the expression for the bilateral terms of trade $S_t \equiv \dfrac{P_{F,t}}{P_{H,t}} = \left(\int_0^1 S_{i,t}^{1-\gamma}di\right)^{\frac{1}{1-\gamma}} \quad \Rightarrow \quad S_t^{1-\gamma} = \int_0^1 S_{i,t}^{1-\gamma}di$

$$\exp\left(\ln\left[S_t^{1-\gamma}\right]\right) = \int_0^1 \exp\left(\ln\left[S_{i,t}^{1-\gamma}\right]\right)di \quad \Rightarrow \quad \exp[(1-\gamma)s_t] = \int_0^1 \exp[(1-\gamma)s_{i,t}]di$$

Applying the exponential Taylor expansion ($e^x = 1+x$) in both sides, we get $1+(1+\gamma)s_t = 1+(1+\gamma)\int_0^1 s_{i,t}di \quad\Rightarrow\quad s_t = \int_0^1 s_{i,t}di$

To log-linearize the CPI formula, considering that it is a symmetric steady-state, we have $P_{H,t} = P_{F,t} = P_t$. Now, taking logs in both sides and using the Taylor expansion for a vector of two variables we have $f(x,y) \approx f(x_0,y_0) + \left.\frac{\partial f(x,y)}{\partial x}\right|_{x_0,y_0}(x-x_0) + \left.\frac{\partial f(x,y)}{\partial y}\right|_{x_0,y_0}(y-y_0)$ So, $(P_t)^{\eta-1} = (1-\alpha)(P_{H,t})^{1-\eta} + \alpha(P_{F,t})^{1-\eta}$

By the CPI definition, we have $P_t \equiv \left[(1-\alpha)(P_{H,t})^{1-\eta} + \alpha(P_{F,t})^{1-\eta}\right]^{\frac{1}{1-\eta}} \quad\Rightarrow\quad (P_t)^{1-\eta} = (1-\alpha)(P_{H,t})^{1-\eta} + \alpha(P_{F,t})^{1-\eta}$

Taking logs, we have $(1-\eta)\ln(P_t) = \ln\left[(1-\alpha)(P_{H,t})^{1-\eta} + \alpha(P_{F,t})^{1-\eta}\right] = f(P_{H,t}, P_{F,t})$

Applying the Taylor expansion on the right side using $x = P_{H,t}$, $y = P_{F,t}$ and $x_0 = y_0 = P_t$,

$$(1-\eta)p_t \approx \ln\left[(1-\alpha)(P_t)^{1-\eta} + \alpha(P_t)^{1-\eta}\right] + \frac{(1-\alpha)(1-\eta)P_t^{-\eta}}{P_t^{1-\eta}}(P_{H,t}-P_t) + \frac{\alpha(1-\eta)P_t^{-\eta}}{P_t^{1-\eta}}(P_{F,t}-P_t)$$

$$(1-\eta)p_t = \ln(P_t^{1-\eta}) + (1-\alpha)(1-\eta)\frac{P_{H,t}-P_t}{P_t} + \alpha(1-\eta)\frac{P_{F,t}-P_t}{P_t}.$$

$p_t \approx p_t + (1-\alpha)[\ln(P_{H,t}) - \ln(P_t)] + \alpha[\ln(P_{F,t}) - \ln(P_t)] \quad\Rightarrow\quad p_t = (1-\alpha)p_{H,t} + \alpha\, p_{F,t}$, as defined in the paper.

As $s_t \equiv p_{F,t} - p_{H,t}$, $p_t = (1-\alpha)P_{H,t} + \alpha(s_t + p_{H,t}) = p_{H,t} + \alpha s_t$

The domestic inflation rate is defined as $\pi_{H,t} \equiv p_{H,t} - p_{H,t-1}$, taking the difference between the equation between t and t-1, we have $p_t - p_{t-1} = p_{H,t} - p_{H,t-1} + \alpha(s_t - s_{t-1}) \quad\Rightarrow\quad \pi_t = \pi_{H,t} + \alpha\Delta s_t$.

Assuming that the law of one price is valid in all times (the same goods produced in different countries have the same price when converting to the domestic currency, using the nominal interest rate), we have $P_{i,t}^i(j) = \mathcal{E}_{i,t}P_{i,t}^i(j)$ for all $i,j \in [0,1]$.

As $P_{i,t}^i \equiv \left(\int_0^1 P_{i,t}^i(j)^{1-\varepsilon}dj\right)^{\frac{1}{1-\varepsilon}}$, we have $\mathcal{E}_{i,t}P_{i,t}^i = \mathcal{E}_{i,t}\left(\int_0^1 P_{i,t}^i(j)^{1-\varepsilon}dj\right)^{\frac{1}{1-\varepsilon}} = \left((\mathcal{E}_{i,t})^{1-\varepsilon}\int_0^1 P_{i,t}^i(j)^{1-\varepsilon}dj\right)^{\frac{1}{1-\varepsilon}}$

$$= \left(\int_0^1 \left(\mathcal{E}_{i,t}P_{i,t}^i(j)\right)^{1-\varepsilon}dj\right)^{\frac{1}{1-\varepsilon}} = \left(\int_0^1 P_{i,t}(j)^{1-\varepsilon}dj\right)^{\frac{1}{1-\varepsilon}} = P_{i,t}$$

Now, we have $P_{F,t} = \left(\int_0^1 P_{i,t}^{1-\gamma}di\right)^{\frac{1}{1-\gamma}} = \left(\int_0^1 \left(\mathcal{E}_{i,t}P_{i,t}^i\right)^{1-\gamma}di\right)^{\frac{1}{1-\gamma}} \quad\Rightarrow\quad P_{F,t}^{1-\gamma} = \int_0^1 \left(\mathcal{E}_{i,t}P_{i,t}^i\right)^{1-\gamma}di$

Log-linearizing the last expression, we get $\exp\left[(1-\gamma)\ln P_{F,t}\right] = \int_0^1 \exp\left[(1-\gamma)\ln\left(\mathcal{E}_{i,t}P_{i,t}^i\right)\right]di$

$$\Rightarrow\quad 1 + (1-\gamma)p_{F,t} = \int_0^1 \left[1 + (1-\gamma)(e_{i,t} + p_{i,t}^i)\right]di \quad\Rightarrow\quad p_{F,t} = \int_0^1 (e_{i,t} + p_{i,t}^i)di = e_t + p_t^*,$$

where $e_t \equiv \int_0^1 e_{i,t}\,di$ and $p_t^* \equiv \int_0^1 p_{i,t}^i\,di$. Also, we have that $s_t = p_{F,t} - p_{H,t} = e_t + p_t^* - p_{H,t}$.

Defining the bilateral real exchange rate $\mathcal{Q}_{i,t} \equiv \frac{\mathcal{E}_{i,t}P_{i,t}}{P_t}$ and the (log) effective real exchange rate $q_t \equiv \int_0^1 q_{i,t}di$ we have $q_t = \int_0^1 \ln\left(\frac{\mathcal{E}_{i,t}P_{i,t}}{P_t}\right)di = \int_0^1 (e_{i,t} + p_{i,t} - p_t)\,di = e_t + p_t^* - p_t = s_t + p_{H,t} - (p_{H,t} + \alpha s_t) = (1-\alpha)s_t$

8 - International risk sharing

The problem of the representative household in any country is the same, as the economies are all equal. There is, any country has an Euler equation like $\beta\mathbb{E}\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\right] = Q_{t,t+1}$.

The condition for the clearing of the international market is that $Q_{t,t+1}$ is unique. So the price converted to a common current has to be the same. So, for every foreign country, the Euler equation becomes

$$\beta\mathbb{E}\left[\left(\frac{C_{t+1}^i}{C_t^i}\right)^{-\sigma}\frac{P_t^i\mathcal{E}_t^i}{P_{t+1}^i\mathcal{E}_{t+1}^i}\right] = Q_{t,t+1}.$$

Combining both equations, using the definition of the real exchange rate $\mathcal{Q}_{i,t} \equiv \dfrac{\mathcal{E}_{i,t}P_{i,t}}{P_t}$ and solving for C_t, we have

$$\beta\mathbb{E}\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\right]=\beta\mathbb{E}\left[\left(\frac{C^i_{t+1}}{C^i_t}\right)^{-\sigma}\frac{P^i_t\mathcal{E}^i_t}{P^i_{t+1}\mathcal{E}^i_{t+1}}\right] \;\Rightarrow\; \mathbb{E}\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{1}{P_{t+1}}\right]=\mathbb{E}\left[\left(\frac{C^i_{t+1}}{C^i_t}\right)^{-\sigma}\frac{\mathcal{Q}_{i,t}}{P^i_{t+1}\mathcal{E}^i_{t+1}}\right]$$

$$\Rightarrow\;(C_t)^\sigma=\mathbb{E}\left[{C_{t+1}}^\sigma\left(C^i_{t+1}\right)^{-\sigma}\frac{P_{t+1}}{P^i_{t+1}\mathcal{E}^i_{t+1}}\right]\mathcal{Q}_{i,t}(C^i_t)^\sigma \;\Rightarrow\; C_t=\mathbb{E}\left[\frac{C_{t+1}}{C^i_{t+1}}(\mathcal{Q}_{i,t+1})^{-\frac{1}{\sigma}}\right]C^i_t\mathcal{Q}^{\frac{1}{\sigma}}_{i,t} \;\Rightarrow\; C_t=\vartheta_iC^i_t\mathcal{Q}^{\frac{1}{\sigma}}_{i,t},$$

where $\vartheta_i=\mathbb{E}\left[\frac{C_{t+1}}{C^i_{t+1}}(\mathcal{Q}_{i,t+1})^{-\frac{1}{\sigma}}\right]$ is a constant and generally will depend on initial relative net asset positions. Assuming identical conditions for all economies, the net asset position for all of the is zero. In this case, $\vartheta_i=\vartheta=1$ for all i. As the symmetric foresight steady-state in this condition is shown in the appendix A.

The international market clearing implies that the total goods produced in a country is consumed by domestically or it's exported. The integral represents the sum of the demand for products of the economy analysed by foreign countries. In a case with economies not with measure zero, we need to exclude the economy analysed from the integral do avoid double counting.

$$Y=C_H+C_i=(1-\alpha)\left(\frac{P_H}{P}\right)^{-\eta}C+\int_0^1\left(\frac{P^i_i}{P_F}\right)^{-\gamma}C_Fdi=(1-\alpha)\left(\frac{P_H}{P}\right)^{-\eta}C+\alpha\int_0^1\left(\frac{P^i_i}{P^i_F}\right)^{-\gamma}\left(\frac{P^i_F}{P^i}\right)^{-\eta}C^idi$$

$$Y=(1-\alpha)\left(\frac{P_H}{P}\right)^{-\eta}C+\alpha\int_0^1\left(\frac{P^i_F}{P^i_i}\right)^{\gamma}\left(\frac{P^i_F}{P^i}\right)^{-\eta}C^idi=(1-\alpha)\left(\frac{P_H}{P}\right)^{-\eta}C+\alpha\int_0^1\left(\frac{\mathcal{E}_iP^i_F}{P_H}\right)^{\gamma}\left(\frac{P^i_F}{P^i}\right)^{-\eta}C^idi,$$

where P^i_i is the price in the domestic economy converted to the currency of country i, or $P^i_i=\dfrac{P_i}{\mathcal{E}_i}=\dfrac{P_H}{\mathcal{E}_i}$, as the goods have the same price in the international market, after converting to the same currency. After simplifying, we have

$$Y=\left(\frac{P_H}{P}\right)^{-\eta}\left[(1-\alpha)C+\alpha\int_0^1\left(\frac{\mathcal{E}_iP^i_F}{P_H}\right)^{\gamma-\eta}\left(\frac{\mathcal{E}_iP_i}{P}\right)^{\eta}C^idi\right]=\left(\frac{P_H}{P}\right)^{-\eta}\left[(1-\alpha)C+\alpha\int_0^1\left(\frac{\mathcal{E}_iP^i_F}{P_H}\right)^{\gamma-\eta}Q^\eta_iC^idi\right]$$

Considering that $P=\left[(1-\alpha)\left(P_H\right)^{1-\eta}+\alpha\left(P_F\right)^{1-\eta}\right]^{\frac{1}{1-\eta}}$ in the steady-state, $P^{1-\eta}=(1-\alpha)\left(P_H\right)^{1-\eta}+\alpha\left(P_F\right)^{1-\eta}$, as $\mathcal{S}_i\equiv\dfrac{P_i}{P_H}$. So,

$$\left(\frac{P}{P_H}\right)^{1-\eta}=(1-\alpha)+\alpha\left(\frac{P_F}{P_H}\right)^{1-\eta}=(1-\alpha)+\alpha\mathcal{S}^{1-\eta}_i \;\Rightarrow\; \frac{P}{P_H}=\left[(1-\alpha)+\alpha\mathcal{S}^{1-\eta}\right]^{\frac{1}{1-\eta}}=\left[(1-\alpha)+\alpha\int_0^1(\mathcal{S}_i)^{1-\eta}di\right]^{\frac{1}{1-\eta}}\equiv h(\mathcal{S})$$

Defining $\mathcal{S}^i=\dfrac{\mathcal{E}_iP^i_F}{P_i}$ and using the fact that $C^i=C\mathcal{Q}^{-\frac{1}{\sigma}}$ as $\vartheta_i=1$ in a symmetric steady-state, we have

$$Y=h(\mathcal{S})^\eta C\left[(1-\alpha)+\alpha\int_0^1\left(\frac{\mathcal{E}_iP^i_F}{P_i}\frac{P_i}{P_H}\right)^{\gamma-\eta}Q^{\eta-\frac{1}{\sigma}}_idi\right]=h(\mathcal{S})^\eta C\left[(1-\alpha)+\alpha\int_0^1\left(\mathcal{S}^i\frac{P_F}{P_H}\right)^{\gamma-\eta}Q^{\eta-\frac{1}{\sigma}}_idi\right]$$

As we will work with a first order approximation, the equality below is valid.

$$Y=h(\mathcal{S})^\eta C\left[(1-\alpha)+\alpha\int_0^1\left(\mathcal{S}^i\mathcal{S}_i\right)^{\gamma-\eta}Q^{\eta-\frac{1}{\sigma}}_idi\right]=h(\mathcal{S})^\eta C\left[(1-\alpha)+\alpha\int_0^1(\mathcal{S}^i)^\gamma di\int_0^1(\mathcal{S}_i)^{-\eta}di\int_0^1Q^{\eta-\frac{1}{\sigma}}_idi\right]$$

$$Y=h(\mathcal{S})^\eta C\left[(1-\alpha)+\alpha\mathcal{S}^{-\eta}\int_0^1\left(\frac{P^i_F}{P_H}\right)^{\gamma}di\int_0^1\left(\frac{\mathcal{E}_iP^i_F}{P}\right)^{\eta-\frac{1}{\sigma}}di\right],$$

as if $\mathcal{S}^{1-\gamma}=\int_0^1\mathcal{S}^{\infty-\gamma}di$, we can substitute variables $-\eta=1-\gamma$ and we get the result.

$$Y=h(\mathcal{S})^\eta C\left[(1-\alpha)+\alpha\mathcal{S}^{-\eta}\left(\frac{1}{P_H}\right)^{\gamma}\int_0^1\left(P^i_F\right)^{\gamma}di\int_0^1\left(\frac{\mathcal{E}_iP^i_F}{P}\right)^{\eta-\frac{1}{\sigma}}di\right]$$

$$Y=h(\mathcal{S})^\eta C\left[(1-\alpha)+\alpha\mathcal{S}^{-\eta}\left(\tfrac{1}{P_H}\right)^{\gamma}(P^*)^\gamma\textstyle\int_0^1\left(\frac{\mathcal{E}_iP^i_F}{P_H}\frac{P_H}{P}\right)^{\eta-\frac{1}{\sigma}}di\right],$$

using the fact that $\left(P^i_F\right)^{1-\gamma}=\int_0^1\left(P^i_i\right)^{1-\gamma}di$ and using P* for the international price index of imported goods.

$$Y = h(\mathcal{S})^\eta C\left[(1-\alpha) + \alpha\mathcal{S}^{-\eta}\left(\frac{P^*}{P_H}\right)^\gamma \int_0^1 \left(\frac{\mathcal{S}^i}{h(\mathcal{S})}\right)^{\eta-\frac{1}{\sigma}} di\right] = h(\mathcal{S})^\eta C\left[(1-\alpha) + \alpha\mathcal{S}^{-\eta}\mathcal{S}^\gamma\left(\frac{1}{h(\mathcal{S})}\right)^{\eta-\frac{1}{\sigma}} \int_0^1 (\mathcal{S}^i)^{\eta-\frac{1}{\sigma}} di\right]$$

$$Y = h(\mathcal{S})^\eta C\left[(1-\alpha) + \alpha\mathcal{S}^{\gamma-\eta}\left(\frac{1}{h(\mathcal{S})}\right)^{\eta-\frac{1}{\sigma}} \mathcal{S}^{\eta-\frac{1}{\sigma}}\right] = h(\mathcal{S})^\eta C\left[(1-\alpha) + \alpha\mathcal{S}^{\gamma-\eta}\left(\frac{\mathcal{S}}{h(\mathcal{S})}\right)^{\eta-\frac{1}{\sigma}}\right],$$

which yields the result. $Y = h(\mathcal{S})^\eta C\left[(1-\alpha) + \alpha\mathcal{S}^{\gamma-\eta} q(\mathcal{S})^{\eta-\frac{1}{\sigma}}\right]$, where $\mathcal{Q} = \dfrac{\mathcal{S}}{h(\mathcal{S})} \equiv q(\mathcal{S})$