Small Open Economy Model

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1 - Problem of the consumer (for reference)

$$max\ \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t) = max\ E_0 \sum_{t=0}^{\infty} \beta^t U\left(\left[(1-\alpha)^{\frac{1}{\eta}}(C_{H,t})^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}}(C_{F,t})^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}, N_t\right)$$

subject to the budget constraint (specified below), where

$$C_{H,t} \equiv \left(\int_0^1 C_{H,t}(j)^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}},\ C_{F,t} \equiv \left(\int_0^1 (C_{i,t})^{\frac{\gamma-1}{\gamma}} di\right)^{\frac{\gamma}{\gamma-1}},\ C_{i,t} \equiv \left(\int_0^1 C_{i,t}(j)^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}}$$

and $C_t \equiv \left[(1-\alpha)^{\frac{1}{\eta}}(C_{H,t})^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}}(C_{F,t})^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}$

Substituting, we get

$$max\ \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U\left(\left[(1-\alpha)^{\frac{1}{\eta}}\left[\left(\int_0^1 C_{H,t}(j)^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}}\right]^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}}\left[\left(\int_0^1 (C_{F,t})^{\frac{\gamma-1}{\gamma}} di\right)^{\frac{\gamma}{\gamma-1}}\right]^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}, N_t\right) =$$

$$max\ \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U\left(\left[(1-\alpha)^{\frac{1}{\eta}}\left[\left(\int_0^1 C_{H,t}(j)^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}}\right]^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}}\left[\left(\int_0^1 \left(\left(\int_0^1 C_{i,t}(j)^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}}\right)^{\frac{\gamma-1}{\gamma}} di\right)^{\frac{\gamma}{\gamma-1}}\right]^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}, N_t\right)$$

subject to the budget constraint:

$$\int_0^1 P_{H,t}(j)C_{H,t}(j)dj + \int_0^1\int_0^1 P_{i,t}(j)C_{i,t}(j)dj\ di + \mathbb{E}_t\{Q_{t,t+1}D_{t+1}\} \leq D_t + W_tN_t + Tt$$

$$\mathcal{L} = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t U\left(\left[(1-\alpha)^{\frac{1}{\eta}}\left[\left(\int_0^1 C_{H,t}(j)^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}}\right]^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}}\left[\left(\int_0^1 \left(\left(\int_0^1 C_{i,t}(j)^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}}\right)^{\frac{\gamma-1}{\gamma}} di\right)^{\frac{\gamma}{\gamma-1}}\right]^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}, N_t\right)$$

$$+\lambda_t\left(D_t + W_tN_t + Tt - \int_0^1 P_{H,t}(j)C_{H,t}(j)dj - \int_0^1\int_0^1 P_{i,t}(j)C_{i,t}(j)dj\ di - \mathbb{E}_t\{Q_{t,t+1}D_{t+1}\}\right)$$

2 - Finding the demand function for each specific good

$C_{H,t}(j) = \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\varepsilon} C_{H,t}$, $C_{i,t}(j) = \left(\frac{P_{i,t}(j)}{P_{i,t}}\right)^{-\varepsilon} C_{i,t}$ and $C_{i,t} = \left(\frac{P_{i,t}}{P_{F,t}}\right)^{-\gamma} C_{F,t}$

It's easier by calculating the MRS (marginal rate of substitution) between $C_{H,t}(j)$ and $C_{H,t}$, as by the optimal allocation, it has to be the rate of prices in every period of time (otherwise the consumer could by a little less of the product with relative higher price and buy another with relative lower price, increasing his utility).

$$\frac{\partial U(C_t, N_t)}{\partial C_{H,t}(j)} = U_c(C_t, N_t)\frac{\eta}{1-\eta}\left(C_t^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}-1}(1-\alpha)^{\frac{1}{\eta}}\frac{\eta-1}{\eta}(C_{H,t})^{-\frac{1}{\eta}}\frac{\varepsilon}{\varepsilon-1}\left(C_{H,t}^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}-1}\int_0^1 \frac{\varepsilon-1}{\varepsilon}C_{H,t}(j)^{-\frac{1}{\varepsilon}}dj$$

After simplifying, we get

$$\frac{\partial U(C_t,N_t)}{\partial C_{H,t}(j)} = U_c(C_t,N_t)(1-\alpha)^{\frac{1}{\eta}} C_t^{\frac{1}{\eta}} (C_{H,t})^{-\frac{1}{\eta}} C_{H,t}^{\frac{1}{\varepsilon}} \int_0^1 C_{H,t}(j)^{-\frac{1}{\varepsilon}} dj = U_c(C_t,N_t)\left[(1-\alpha)\frac{C_t}{C_{H,t}}\right]^{\frac{1}{\eta}} \int_0^1 \left[\frac{C_{H,t}}{C_{H,t}(j)}\right]^{\frac{1}{\varepsilon}} dj$$

Similarly,

$$\frac{\partial U(C_t,N_t)}{\partial C_{H,t}} = U_c(C_t,N_t)\frac{\eta}{1-\eta}\left(C_t^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}-1}(1-\alpha)^{\frac{1}{\eta}}\frac{\eta-1}{\eta}(C_{H,t})^{-\frac{1}{\eta}} = U_c(C_t,N_t)(1-\alpha)^{\frac{1}{\eta}} C_t^{\frac{1}{\eta}} (C_{H,t})^{-\frac{1}{\eta}} = U_c(C_t,N_t)\left[(1-\alpha)\frac{C_t}{C_{H,t}}\right]^{\frac{1}{\eta}}$$

$$\frac{\frac{\partial U(C_t,N_t)}{\partial C_{H,t}(j)}}{\frac{\partial U(C_t,N_t)}{\partial C_{H,t}}} = \frac{U_c(C_t,N_t)\left[(1-\alpha)\frac{C_t}{C_{H,t}}\right]^{\frac{1}{\eta}}\int_0^1\left[\frac{C_{H,t}}{C_{H,t}(j)}\right]^{\frac{1}{\varepsilon}}dj}{U_c(C_t,N_t)\left[(1-\alpha)\frac{C_t}{C_{H,t}}\right]^{\frac{1}{\eta}}} = \frac{\int_0^1 P_{H,t}(j)dj}{P_{H,t}}$$

After simplifying again, the expression is almost the demand function we want.

$$\int_0^1 \left[\frac{C_{H,t}}{C_{H,t}(j)}\right]^{\frac{1}{\varepsilon}} dj = \int_0^1 \frac{P_{H,t}(j)}{P_{H,t}} dj$$

As the interval of both integrals are the same and the variable being integrated is also the same, what is inside the integral in both sides have also to be the same. So,

$$\left[\frac{C_{H,t}}{C_{H,t}(j)}\right]^{\frac{1}{\varepsilon}} = \frac{P_{H,t}(j)}{P_{H,t}} \Rightarrow \left[\frac{C_{H,t}(j)}{C_{H,t}}\right]^{-\frac{1}{\varepsilon}} = \frac{P_{H,t}(j)}{P_{H,t}} \Rightarrow C_{H,t}(j) = \left[\frac{P_{H,t}(j)}{P_{H,t}}\right]^{-\varepsilon} C_{H,t}$$

Calculating now the MRS (marginal rate of substitution) between $C_{i,t}(j)$ and $C_{i,t}$, which is also equal the rate of the prices.

$$\frac{\partial U(C_t,N_t)}{\partial C_{i,t}(j)} =$$

$$U_c(C_t,N_t)\frac{\eta}{1-\eta}\left(C_t^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}-1}\alpha^{\frac{1}{\eta}}\frac{\eta-1}{\eta}(C_{F,t})^{-\frac{1}{\eta}}\frac{\gamma}{\gamma-1}\left(C_{H,t}^{\frac{\gamma-1}{\gamma}}\right)^{\frac{\gamma}{\gamma-1}-1}\int_0^1\frac{\gamma-1}{\gamma}C_{i,t}^{-\frac{1}{\gamma}}\left[\frac{\varepsilon}{\varepsilon-1}\left(C_{i,t}^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}-1}\int_0^1\frac{\varepsilon-1}{\varepsilon}C_{i,t}(j)^{-\frac{1}{\varepsilon}}dj\right]di$$

After simplifying, we get

$$\frac{\partial U(C_t,N_t)}{\partial C_{i,t}(j)} = U_c(C_t,N_t)\alpha^{\frac{1}{\eta}}C_t^{\frac{1}{\eta}}(C_{F,t})^{-\frac{1}{\eta}}C_{F,t}^{\frac{1}{\gamma}}\int_0^1 C_{i,t}^{-\frac{1}{\gamma}}\left[C_{i,t}^{\frac{1}{\varepsilon}}\int_0^1 C_{i,t}(j)^{-\frac{1}{\varepsilon}}dj\right]di$$

$$\frac{\partial U(C_t,N_t)}{\partial C_{i,t}(j)} = U_c(C_t,N_t)\left[\alpha\frac{C_t}{C_{F,t}}\right]^{\frac{1}{\eta}}\int_0^1\left[\frac{C_{F,t}}{C_{i,t}}\right]^{\frac{1}{\gamma}}\int_0^1\left[\frac{C_{i,t}}{C_{i,t}(j)}\right]^{\frac{1}{\varepsilon}}dj\ di$$

$$\frac{\partial U(C_t,N_t)}{\partial C_{i,t}} = U_c(C_t,N_t)\frac{\eta}{1-\eta}\left(C_t^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}-1}\alpha^{\frac{1}{\eta}}\frac{\eta-1}{\eta}(C_{F,t})^{-\frac{1}{\eta}}\frac{\gamma}{\gamma-1}\left(C_{H,t}^{\frac{\gamma-1}{\gamma}}\right)^{\frac{\gamma}{\gamma-1}-1}\int_0^1\frac{\gamma-1}{\gamma}C_{i,t}^{-\frac{1}{\gamma}}di$$

$$\frac{\partial U(C_t,N_t)}{\partial C_{i,t}} = U_c(C_t,N_t)\left[\alpha\frac{C_t}{C_{F,t}}\right]^{\frac{1}{\eta}}\int_0^1\left[\frac{C_{F,t}}{C_{i,t}}\right]^{\frac{1}{\gamma}}di$$

Calculating the MRS we have

$$\frac{\frac{\partial U(C_t,N_t)}{\partial C_{i,t}(j)}}{\frac{\partial U(C_t,N_t)}{\partial C_{i,t}}} = \frac{U_c(C_t,N_t)\left[\alpha\frac{C_t}{C_{F,t}}\right]^{\frac{1}{\eta}}\int_0^1\left[\frac{C_{F,t}}{C_{i,t}}\right]^{\frac{1}{\gamma}}\int_0^1\left[\frac{C_{i,t}}{C_{i,t}(j)}\right]^{\frac{1}{\varepsilon}}dj\ di}{U_c(C_t,N_t)\left[\alpha\frac{C_t}{C_{F,t}}\right]^{\frac{1}{\eta}}\int_0^1\left[\frac{C_{F,t}}{C_{i,t}}\right]^{\frac{1}{\gamma}}di} = \frac{\int_0^1 P_{i,t}(j)dj}{P_{i,t}}$$

As before, we can simplify again. Also, as there's a continuum of firms, we can consider that the price of each product $P_{i,t}(j)$ is only correlated with its specific demand function and not with the demand function of other in its country or another country, it follows that each specific price is uncorrelated with $C_{F,t}$ and $C_{i,t}$. Also, as each firm is very small, we can consider that it has negligible influence on the aggregate index price of its country ($P_{i,t}$). Whith these independence assumption, the joint distribution is equal to the product of the marginal distributions.

$$\int_0^1\int_0^1\left[\frac{C_{F,t}}{C_{i,t}}\right]^{\frac{1}{\gamma}}\left[\frac{C_{i,t}}{C_{i,t}(j)}\right]^{\frac{1}{\varepsilon}}dj\ di = \int_0^1\left[\frac{C_{F,t}}{C_{i,t}}\right]^{\frac{1}{\gamma}}di\int_0^1\frac{P_{i,t}(j)}{P_{i,t}}dj = \int_0^1\int_0^1\frac{P_{i,t}(j)}{P_{i,t}}\left[\frac{C_{F,t}}{C_{i,t}}\right]^{\frac{1}{\gamma}}dj\ di$$

Now, as before, the integrand in both sides needs to be the same. The we get the second demand equation.

$$\left[\frac{C_{F,t}}{C_{i,t}}\right]^{\frac{1}{\gamma}}\left[\frac{C_{i,t}}{C_{i,t}(j)}\right]^{\frac{1}{\varepsilon}} = \frac{P_{i,t}(j)}{P_{i,t}}\left[\frac{C_{F,t}}{C_{i,t}}\right]^{\frac{1}{\gamma}} \Rightarrow \left[\frac{C_{i,t}(j)}{C_{i,t}}\right]^{-\frac{1}{\varepsilon}} = \frac{P_{i,t}(j)}{P_{i,t}} \Rightarrow C_{i,t}(j) = \left[\frac{P_{i,t}(j)}{P_{i,t}}\right]^{-\varepsilon} C_{i,t}$$

To find the aggregate demand for each country, in terms of total foreign demand, we proceed by calculating the MRS between the aggregate consumption for the country and the aggregate consumption of foreign goods, which the optimal allocation resulting from the rate between the prices, as before.

$$\frac{\partial U(C_t,N_t)}{\partial C_{F,t}} = U_c(C_t,N_t)\frac{\eta}{1-\eta}\left(C_t^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}-1}\alpha^{\frac{1}{\eta}}\frac{\eta-1}{\eta}\left(C_{F,t}\right)^{-\frac{1}{\eta}} = U_c(C_t,N_t)\left[\alpha\frac{C_t}{C_{F,t}}\right]^{\frac{1}{\eta}}$$

$$\frac{\frac{\partial U(C_t,N_t)}{\partial C_{i,t}}}{\frac{\partial U(C_t,N_t)}{\partial C_{F,t}}} = \frac{U_c(C_t,N_t)\left[\alpha\frac{C_t}{C_{F,t}}\right]^{\frac{1}{\eta}}\int_0^1\left[\frac{C_{F,t}}{C_{i,t}}\right]^{\frac{1}{\gamma}}di}{U_c(C_t,N_t)\left[\alpha\frac{C_t}{C_{F,t}}\right]^{\frac{1}{\eta}}} = \frac{\int_0^1 P_{i,t}di}{P_{F,t}}$$

As $P_{F,t}$ doesn't depend on a specific i, we can put it inside the integral. Then we get again two integrans which have to be the same for the equality to hold.

$$\left[\frac{C_{F,t}}{C_{i,t}}\right]^{\frac{1}{\gamma}} = \frac{P_{i,t}}{P_{F,t}} \Rightarrow \left[\frac{C_{i,t}}{C_{F,t}}\right]^{-\frac{1}{\gamma}} = \frac{P_{i,t}}{P_{F,t}} \quad\Rightarrow\quad C_{i,t} = \left[\frac{P_{i,t}}{P_{F,t}}\right]^{-\gamma}C_{F,t}$$

3 - Aggregating the expenditure

Now that we have the demand functions

$C_{H,t}(j) = \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\varepsilon}C_{H,t}$, $C_{i,t}(j) = \left(\frac{P_{i,t}(j)}{P_{i,t}}\right)^{-\varepsilon}C_{i,t}$ and $C_{i,t} = \left(\frac{P_{i,t}}{P_{F,t}}\right)^{-\gamma}C_{F,t}$

let's prove that

$\int_0^1 P_{H,t}(j)C_{H,t}(j)dj = P_{H,t}C_{H,t}$, $\int_0^1 P_{i,t}(j)C_{i,t}(j)dj = P_{i,t}C_{i,t}$ and $\int_0^1 P_{i,t}C_{i,t}dj = P_{F,t}C_{F,t}$

using the definition of the price indexes:

$P_{H,t} \equiv \left(\int_0^1 P_{H,t}(j)^{1-\varepsilon}dj\right)^{\frac{1}{1-\varepsilon}}$, $P_{i,t} \equiv \left(\int_0^1 P_{i,t}(j)^{1-\varepsilon}dj\right)^{\frac{1}{1-\varepsilon}}$ and $P_{F,t} \equiv \left(\int_0^1 P_{i,t}^{1-\gamma}di\right)^{\frac{1}{1-\gamma}}$

$$\int_0^1 P_{H,t}(j)C_{H,t}(j)dj = \int_0^1 P_{H,t}(j)\left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\varepsilon}C_{H,t}dj = \frac{C_{H,t}}{P_{H,t}^{-\varepsilon}}\int_0^1 P_{H,t}(j)^{1-\varepsilon}dj = \frac{C_{H,t}}{P_{H,t}^{-\varepsilon}}P_{H,t}^{1-\varepsilon} = P_{H,t}C_{H,t}$$

$$\int_0^1 P_{i,t}(j)C_{i,t}(j)dj = \int_0^1 P_{i,t}(j)\left(\frac{P_{i,t}(j)}{P_{i,t}}\right)^{-\varepsilon}C_{i,t}dj = \frac{C_{i,t}}{P_{i,t}^{-\varepsilon}}\int_0^1 P_{i,t}(j)^{1-\varepsilon}dj = \frac{C_{i,t}}{P_{i,t}^{-\varepsilon}}P_{i,t}^{1-\varepsilon} = P_{i,t}C_{i,t}$$

$$\int_0^1 P_{i,t}C_{i,t}di = \int_0^1 P_{i,t}\left(\frac{P_{i,t}}{P_{F,t}}\right)^{-\gamma}C_{F,t}di = \frac{C_{F,t}}{P_{F,t}^{-\gamma}}\int_0^1 P_{F,t}^{1-\gamma}dj = \frac{C_{F,t}}{P_{F,t}^{-\varepsilon}}P_{F,t}^{1-\varepsilon} = P_{F,t}C_{F,t}$$

With this aggregation, the budget constraint can be simplified

$$\int_0^1 P_{H,t}(j)C_{H,t}(j)dj + \int_0^1\int_0^1 P_{i,t}(j)C_{i,t}(j)dj\ di + \mathbb{E}_t\{Q_{t,t+1}D_{t+1}\} \le D_t + W_tN_t + Tt$$

$$P_{H,t}C_{H,t} + \int_0^1 P_{i,t}C_{i,t}di = P_{H,t}C_{H,t} + P_{F,t}C_{F,t} \le D_t + W_tN_t + Tt - \mathbb{E}_t\{Q_{t,t+1}D_{t+1}\}$$

As the total consumption expenditure by the representative consumer is with the domestic produts or foreig products, the budget constraint becomes:

$P_tC_t \le D_t + W_tN_t + T_t - \mathbb{E}_t\{Q_{t,t+1}D_{t+1}\}$

4 - Finding the optimal share between the domestic and imported goods

Now we will calculate the MRS between the domestic products and the total consumption, which has to be equal to the rate of prices. After, we will do the same for the foreign products.

$$\frac{\frac{\partial U(C_t,N_t)}{\partial C_{H,t}}}{\frac{\partial U(C_t,N_t)}{\partial C_t}} = \frac{U_c(C_t,N_t)\left[(1-\alpha)\frac{C_t}{C_{H,t}}\right]^{\frac{1}{\eta}}}{U_c(C_t,N_t)} = \frac{P_{H,t}}{P_t} \quad\Rightarrow\quad (1-\alpha)\frac{C_t}{C_{H,t}} = \left(\frac{P_{H,t}}{P_t}\right)^{\eta} \quad\Rightarrow\quad C_{H,t} = (1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta}C_t$$

$$\frac{\dfrac{\partial U(C_t,N_t)}{\partial C_{F,t}}}{\dfrac{\partial U(C_t,N_t)}{\partial C_t}} = \frac{U_c(C_t,N_t)\left[\alpha\dfrac{C_t}{C_{F,t}}\right]^{\frac{1}{\eta}}}{U_c(C_t,N_t)} = \frac{P_{F,t}}{P_t} \quad \Rightarrow \quad \alpha\frac{C_t}{C_{F,t}} = \left(\frac{P_{F,t}}{P_t}\right)^{\eta} \quad \Rightarrow \quad C_{F,t} = \alpha\left(\frac{P_{F,t}}{P_t}\right)^{-\eta} C_t$$

5 - Standard problem of the representative consumer

Now we arrive at a standard problem for the representative consumer

$$\underset{C_t,N_t,D_{t+1}}{max} \ \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t,N_t) = \underset{C_t,N_t}{max} \ E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi}\right)$$

subject to $D_t + W_t N_t + Tt - \mathbb{E}_t\{Q_{t,t+1}D_{t+1}\} - P_t C_t = 0$, as an optimal condition.

using the separable utility function specified as $U(C_t,N_t) = \dfrac{C_t^{1-\sigma}}{1-\sigma} - \dfrac{N_t^{1+\varphi}}{1+\varphi}$

$$\mathcal{L} = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left[\left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi}\right) + \lambda_t \left(D_t + W_t N_t + Tt - Q_{t,t+1}D_{t+1} - P_t C_t\right)\right]$$

with first order conditions (FOCs):

(C_t): $\beta^t U_C(C_t,N_t) = \beta^t C_t^{-\sigma} = \beta^t \lambda_t P_t \quad \Rightarrow \quad C_t^{-\sigma} = \lambda_t P_t$

(N_t): $-\beta^t U_n(C_t,N_t) = \beta^t N_t^{\varphi} = \beta^t \lambda_t W_t \quad \Rightarrow \quad N_t^{\varphi} = \lambda_t W_t$

(D_{t+1}): $\beta^t \lambda_t Q_{t,t+1} = \beta^{t+1}\mathbb{E}[\lambda_{t+1}] \quad \Rightarrow \quad \dfrac{\mathbb{E}[\lambda_{t+1}]}{\lambda_t} = \dfrac{Q_{t,t+1}}{\beta}$

Dividing (N_t) FOC by (C_t) FOC, we have the standard equation of intratemporal substitution between consumption and leisure $-\dfrac{U_C(C_t,N_t)}{U_N(C_t,N_t)} = C_t^{\sigma} N_t^{\varphi} = \dfrac{W_t}{P_t}$

Advancing one period for the consumption FOC, we have $\mathbb{E}[C_{t+1}^{-\sigma}] = \mathbb{E}[\lambda_{t+1}P_{t+1}]$

As the model will be log-linearized and an approximation of the first order will be used to solve it, we can ignore the Jensen's inequality where there is an expectation operator. Up to first order approximation, $\mathbb{E}[xy] \approx \mathbb{E}[x]\mathbb{E}[y]$.

Dividing the consumption FOC in t+1 by the equation in t and substituting by the $\mathbb{E}[\lambda_{t+1}]/\lambda_t$ in the D_{t+1} FOC, we get the Euler equation

$$\mathbb{E}\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\right] = \mathbb{E}\left[\frac{\lambda_{t+1}P_{t+1}}{\lambda_t P_t}\right] \quad \Rightarrow \quad \mathbb{E}\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\right] = \frac{\mathbb{E}[Q_{t,t+1}]}{\beta} \quad \Rightarrow \quad \beta R_t \mathbb{E}\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\right] = 1$$

as $Q_{t,t+1}$, is the price of a a riskless one-period discount bond in domestic currency with gross return R_t.

Log-linearizing $C_t^{\sigma}N_t^{\varphi} = \dfrac{W_t}{P_t}$ is straight forward: $w_t - p_t = \sigma c_t + \varphi n_t$

To log-linerize the Euler equation, we'll use the Taylor expansion: $f(x) \approx f(x_0) + f'(x_o)(x - x_0)$. When expanding the exponential function around 0, we get $e^x = e^0 + e^0(x-0) = 1 + x$

$$\mathbb{E}\left[\exp\left(\ln\left[\beta R_t \left(\tfrac{C_{t+1}}{C_t}\right)^{-\sigma}\tfrac{P_t}{P_{t+1}}\right]\right)\right] = \mathbb{E}\left[\exp\left(\ln(\beta) + r_t - \sigma(c_{t+1} - c_t) + p_t - p_{t+1}\right)\right] = 1 \quad \Rightarrow$$

$$1 + \ln(\beta) + r_t - \sigma(\mathbb{E}\left[c_{t+1}\right] - c_t) - \mathbb{E}\left[\pi_{t+1}\right] = 1 \quad \Rightarrow \quad \sigma c_t = \sigma\mathbb{E}\left[c_{t+1}\right] + \mathbb{E}\left[\pi_{t+1}\right] - r_t - \ln(\beta) \Rightarrow \quad c_t = \mathbb{E}\left[c_{t+1}\right] - \frac{1}{\sigma}\left(r_t - \mathbb{E}\left[\pi_{t+1}\right] - \rho\right)$$

as $\rho \equiv \frac{1-\beta}{\beta} \approx -\ln(\beta)$

6 - Terms of trade

Let's log-linearize the expression for the bilateral terms of trade $S_t \equiv \dfrac{P_{F,t}}{P_{H,t}} = \left(\displaystyle\int_0^1 S_{i,t}^{1-\gamma}di\right)^{\frac{1}{1-\gamma}} \quad \Rightarrow \quad S_t^{1-\gamma} = \displaystyle\int_0^1 S_{i,t}^{1-\gamma}di$

$$\exp\left(\ln\left[S_t^{1-\gamma}\right]\right) = \int_0^1 \exp\left(\ln\left[S_{i,t}^{1-\gamma}\right]\right)di \quad \Rightarrow \quad \exp[(1-\gamma)s_t] = \int_0^1 \exp[(1-\gamma)s_{i,t}]di$$

Applying the exponential Taylor expansion ($e^x = 1 + x$) in both sides, we get $1 + (1+\gamma)s_t = 1 + (1+\gamma)\displaystyle\int_0^1 s_{i,t}di \quad \Rightarrow \quad s_t = \displaystyle\int_0^1 s_{i,t}di$

To log-linearize the CPI formula, considering that it is a symmetric steady-state, we have $P_{H,t} = P_{F,t} = P_t$. Now, taking logs in both sides and using the Taylor expansion for a vector of two variables we have $f(x,y) \approx f(x_0,y_0) + \left.\frac{\partial f(x,y)}{\partial x}\right|_{x_0,y_0}(x-x_0) + \left.\frac{\partial f(x,y)}{\partial y}\right|_{x_0,y_0}(y-y_0)$ So, $(P_t)^{\eta-1} = (1-\alpha)(P_{H,t})^{1-\eta} + \alpha(P_{F,t})^{1-\eta}$

By the CPI definition, we have $P_t \equiv \left[(1-\alpha)(P_{H,t})^{1-\eta} + \alpha(P_{F,t})^{1-\eta}\right]^{\frac{1}{1-\eta}} \quad \Rightarrow \quad (P_t)^{1-\eta} = (1-\alpha)(P_{H,t})^{1-\eta} + \alpha(P_{F,t})^{1-\eta}$

Taking logs, we have $(1-\eta)\ln(P_t) = \ln\left[(1-\alpha)(P_{H,t})^{1-\eta} + \alpha(P_{F,t})^{1-\eta}\right] = f(P_{H,t}, P_{F,t})$

Applying the Taylor expansion on the right side using $x = P_{H,t}$, $y = P_{F,t}$ and $x_0 = y_0 = P_t$,

$$(1-\eta)p_t \approx \ln\left[(1-\alpha)(P_t)^{1-\eta} + \alpha(P_t)^{1-\eta}\right] + \frac{(1-\alpha)(1-\eta)P_t^{-\eta}}{P_t^{1-\eta}}(P_{H,t}-P_t) + \frac{\alpha(1-\eta)P_t^{-\eta}}{P_t^{1-\eta}}(P_{F,t}-P_t)$$

$$(1-\eta)p_t = \ln(P_t^{1-\eta}) + (1-\alpha)(1-\eta)\frac{P_{H,t}-P_t}{P_t} + \alpha(1-\eta)\frac{P_{F,t}-P_t}{P_t}.$$

$p_t \approx p_t + (1-\alpha)[\ln(P_{H,t}) - \ln(P_t)] + \alpha[\ln(P_{F,t}) - \ln(P_t)] \quad \Rightarrow \quad p_t = (1-\alpha)p_{H,t} + \alpha\, p_{F,t}$, as defined in the paper.

As $s_t \equiv p_{F,t} - p_{H,t}$, $p_t = (1-\alpha)P_{H,t} + \alpha(s_t + p_{H,t}) = p_{H,t} + \alpha s_t$

The domestic inflation rate is defined as $\pi_{H,t} \equiv p_{H,t} - p_{H,t-1}$, taking the difference between the equation between t and t-1, we have $p_t - p_{t-1} = p_{H,t} - p_{H,t-1} + \alpha(s_t - s_{t-1}) \quad \Rightarrow \quad \pi_t = \pi_{H,t} + \alpha\Delta s_t$.

Assuming that the law of one price is valid in all times (the same goods produced in different countries have the same price when converting to the domestic currency, using the nominal interest rate), we have $P_{i,t}(j) = \mathcal{E}_{i,t}P_{i,t}^i(j)$ for all $i,j \in [0,1]$.

As $P_{i,t}^i \equiv \left(\int_0^1 P_{i,t}^i(j)^{1-\varepsilon}dj\right)^{\frac{1}{1-\varepsilon}}$, we have $\mathcal{E}_{i,t}P_{i,t}^i = \mathcal{E}_{i,t}\left(\int_0^1 P_{i,t}^i(j)^{1-\varepsilon}dj\right)^{\frac{1}{1-\varepsilon}} = \left((\mathcal{E}_{i,t})^{1-\varepsilon}\int_0^1 P_{i,t}^i(j)^{1-\varepsilon}dj\right)^{\frac{1}{1-\varepsilon}}$

$$= \left(\int_0^1 \left(\mathcal{E}_{i,t}P_{i,t}^i(j)\right)^{1-\varepsilon}dj\right)^{\frac{1}{1-\varepsilon}} = \left(\int_0^1 P_{i,t}(j)^{1-\varepsilon}dj\right)^{\frac{1}{1-\varepsilon}} = P_{i,t}$$

Now, we have $P_{F,t} = \left(\int_0^1 P_{i,t}^{1-\gamma}di\right)^{\frac{1}{1-\gamma}} = \left(\int_0^1 \left(\mathcal{E}_{i,t}P_{i,t}^i\right)^{1-\gamma}di\right)^{\frac{1}{1-\gamma}} \quad \Rightarrow \quad P_{F,t}^{1-\gamma} = \int_0^1 \left(\mathcal{E}_{i,t}P_{i,t}^i\right)^{1-\gamma}di$

Log-linearizing the last expression, we get $\exp\left[(1-\gamma)\ln P_{F,t}\right] = \int_0^1 \exp\left[(1-\gamma)\ln\left(\mathcal{E}_{i,t}P_{i,t}^i\right)\right]di$

$$\Rightarrow \quad 1 + (1-\gamma)p_{F,t} = \int_0^1 \left[1 + (1-\gamma)(e_{i,t} + p_{i,t}^i)\right]di \quad \Rightarrow \quad p_{F,t} = \int_0^1 (e_{i,t} + p_{i,t}^i)di = e_t + p_t^*,$$

where $e_t \equiv \int_0^1 e_{i,t}\,di$ and $p_t^* \equiv \int_0^1 p_{i,t}^i\,di$. Also, we have that $s_t = p_{F,t} - p_{H,t} = e_t + p_t^* - p_{H,t}$.

Defining the bilateral real exchange rate $\mathcal{Q}_{i,t} \equiv \frac{\mathcal{E}_{i,t}P_{i,t}}{P_t}$ and the (log) effective real exchange rate $q_t \equiv \int_0^1 q_{i,t}di$ we have $q_t = \int_0^1 \ln\left(\frac{\mathcal{E}_{i,t}P_{i,t}}{P_t}\right)di = \int_0^1 (e_{i,t} + p_{i,t} - p_t)\,di = e_t + p_t^* - p_t = s_t + p_{H,t} - (p_{H,t} + \alpha s_t) = (1-\alpha)s_t$

7 - International risk sharing

The problem of the representative household in any country is the same, as the economies are all equal. There is, any country has an Euler equation like $\beta\mathbb{E}\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\right] = Q_{t,t+1}$.

The condition for the clearing of the international market is that $Q_{t,t+1}$ is unique. So the price converted to a common current has to be the same. So, for every foreign country, the Euler equation becomes

$$\beta\mathbb{E}\left[\left(\frac{C_{t+1}^i}{C_t^i}\right)^{-\sigma}\frac{P_t^i\mathcal{E}_t^i}{P_{t+1}^i\mathcal{E}_{t+1}^i}\right] = Q_{t,t+1}.$$

Combining both equations, using the definition of the real exchange rate $\mathcal{Q}_{i,t} \equiv \frac{\mathcal{E}_{i,t}P_{i,t}}{P_t}$ and solving for C_t, we have

$$\beta\mathbb{E}\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\right]=\beta\mathbb{E}\left[\left(\frac{C^i_{t+1}}{C^i_t}\right)^{-\sigma}\frac{P^i_t\mathcal{E}^i_t}{P^i_{t+1}\mathcal{E}^i_{t+1}}\right]\ \Rightarrow\ \mathbb{E}\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{1}{P_{t+1}}\right]=\mathbb{E}\left[\left(\frac{C^i_{t+1}}{C^i_t}\right)^{-\sigma}\frac{\mathcal{Q}_{i,t}}{P^i_{t+1}\mathcal{E}^i_{t+1}}\right]$$

$$\Rightarrow\ (C_t)^\sigma=\mathbb{E}\left[{C_{t+1}}^\sigma\left(C^i_{t+1}\right)^{-\sigma}\frac{P_{t+1}}{P^i_{t+1}\mathcal{E}^i_{t+1}}\right]\mathcal{Q}_{i,t}(C^i_t)^\sigma\ \Rightarrow\ C_t=\mathbb{E}\left[\frac{C_{t+1}}{C^i_{t+1}}(\mathcal{Q}_{i,t+1})^{-\frac{1}{\sigma}}\right]C^i_t\mathcal{Q}^{\frac{1}{\sigma}}_{i,t}\ \Rightarrow\ C_t=\vartheta_iC^i_t\mathcal{Q}^{\frac{1}{\sigma}}_{i,t},$$

where $\vartheta_i=\mathbb{E}\left[\frac{C_{t+1}}{C^i_{t+1}}(\mathcal{Q}_{i,t+1})^{-\frac{1}{\sigma}}\right]$ is a constant and generally will depend on initial relative net asset positions. Assuming identical conditions for all economies, the net asset position for all of the is zero. In this case, $\vartheta_i=\vartheta=1$ for all i. As the symmetric foresight steady-state in this condition is shown in the appendix A.

The international market clearing implies that the total goods produced in a country is consumed by domestically or it's exported. The integral represents the sum of the demand for products of the economy analysed by foreign countries. In a case with economies not with measure zero, we need to exclude the economy analysed from the integral do avoid double counting.

$$Y=C_H+C_i=(1-\alpha)\left(\frac{P_H}{P}\right)^{-\eta}C+\int_0^1\left(\frac{P^i_i}{P_F}\right)^{-\gamma}C_Fdi=(1-\alpha)\left(\frac{P_H}{P}\right)^{-\eta}C+\alpha\int_0^1\left(\frac{P^i_i}{P^i_F}\right)^{-\gamma}\left(\frac{P^i_F}{P^i}\right)^{-\eta}C^idi$$

$$Y=(1-\alpha)\left(\frac{P_H}{P}\right)^{-\eta}C+\alpha\int_0^1\left(\frac{P^i_F}{P^i_i}\right)^{\gamma}\left(\frac{P^i_F}{P^i}\right)^{-\eta}C^idi=(1-\alpha)\left(\frac{P_H}{P}\right)^{-\eta}C+\alpha\int_0^1\left(\frac{\mathcal{E}_iP^i_F}{P_H}\right)^{\gamma}\left(\frac{P^i_F}{P^i}\right)^{-\eta}C^idi,$$

where P^i_i is the price in the domestic economy converted to the currency of country i, or $P^i_i=\frac{P_i}{\mathcal{E}_i}=\frac{P_H}{\mathcal{E}_i}$, as the goods have the same price in the international market, after converting to the same currency. After simplifying, we have

$$Y=\left(\frac{P_H}{P}\right)^{-\eta}\left[(1-\alpha)C+\alpha\int_0^1\left(\frac{\mathcal{E}_iP^i_F}{P_H}\right)^{\gamma-\eta}\left(\frac{\mathcal{E}_iP_i}{P}\right)^{\eta}C^idi\right]=\left(\frac{P_H}{P}\right)^{-\eta}\left[(1-\alpha)C+\alpha\int_0^1\left(\frac{\mathcal{E}_iP^i_F}{P_H}\right)^{\gamma-\eta}Q^\eta_iC^idi\right]$$

Considering that $P=\left[(1-\alpha)\left(P_H\right)^{1-\eta}+\alpha\left(P_F\right)^{1-\eta}\right]^{\frac{1}{1-\eta}}$ in the steady-state, $P^{1-\eta}=(1-\alpha)\left(P_H\right)^{1-\eta}+\alpha\left(P_F\right)^{1-\eta}$, as $\mathcal{S}_i\equiv\frac{P_i}{P_H}$. So,

$$\left(\frac{P}{P_H}\right)^{1-\eta}=(1-\alpha)+\alpha\left(\frac{P_F}{P_H}\right)^{1-\eta}=(1-\alpha)+\alpha\mathcal{S}_i^{1-\eta}\ \Rightarrow\ \frac{P}{P_H}=\left[(1-\alpha)+\alpha\mathcal{S}^{1-\eta}\right]^{\frac{1}{1-\eta}}=\left[(1-\alpha)+\alpha\int_0^1(\mathcal{S}_i)^{1-\eta}di\right]^{\frac{1}{1-\eta}}\equiv h(\mathcal{S})$$

Defining $\mathcal{S}^i=\frac{\mathcal{E}_iP^i_F}{P_i}$ and using the fact that $C^i=C\mathcal{Q}^{-\frac{1}{\sigma}}$ as $\vartheta_i=1$ in a symmetric steady-state, we have

$$Y=h(\mathcal{S})^\eta C\left[(1-\alpha)+\alpha\int_0^1\left(\frac{\mathcal{E}_iP^i_F}{P_i}\frac{P_i}{P_H}\right)^{\gamma-\eta}Q_i^{\eta-\frac{1}{\sigma}}di\right]=h(\mathcal{S})^\eta C\left[(1-\alpha)+\alpha\int_0^1\left(\mathcal{S}^i\frac{P_F}{P_H}\right)^{\gamma-\eta}Q_i^{\eta-\frac{1}{\sigma}}di\right]$$

As we will work with a first order approximation, the equality below is valid.

$$Y=h(\mathcal{S})^\eta C\left[(1-\alpha)+\alpha\int_0^1\left(\mathcal{S}^i\mathcal{S}_i\right)^{\gamma-\eta}Q_i^{\eta-\frac{1}{\sigma}}di\right]=h(\mathcal{S})^\eta C\left[(1-\alpha)+\alpha\int_0^1(\mathcal{S}^i)^\gamma di\int_0^1(\mathcal{S}_i)^{-\eta}di\int_0^1Q_i^{\eta-\frac{1}{\sigma}}di\right]$$

$$Y=h(\mathcal{S})^\eta C\left[(1-\alpha)+\alpha\mathcal{S}^{-\eta}\int_0^1\left(\frac{P^i_F}{P_H}\right)^{\gamma}di\int_0^1\left(\frac{\mathcal{E}_iP^i_F}{P}\right)^{\eta-\frac{1}{\sigma}}di\right],$$

as if $\mathcal{S}^{1-\gamma}=\int_0^1\mathcal{S}^{1-\gamma}di$, we can substitute variables $-\eta=1-\gamma$ and we get the result.

$$Y=h(\mathcal{S})^\eta C\left[(1-\alpha)+\alpha\mathcal{S}^{-\eta}\left(\frac{1}{P_H}\right)^{\gamma}\int_0^1\left(P^i_F\right)^\gamma di\int_0^1\left(\frac{\mathcal{E}_iP^i_F}{P}\right)^{\eta-\frac{1}{\sigma}}di\right]$$

$$Y=h(\mathcal{S})^\eta C\left[(1-\alpha)+\alpha\mathcal{S}^{-\eta}\left(\frac{1}{P_H}\right)^{\gamma}(P^*)^\gamma\int_0^1\left(\frac{\mathcal{E}_iP^i_F}{P_H}\frac{P_H}{P}\right)^{\eta-\frac{1}{\sigma}}di\right],$$

using the fact that $\left(P^i_F\right)^{1-\gamma}=\int_0^1\left(P^i_i\right)^{1-\gamma}di$ and using P* for the international price index of imported goods.

$$Y=h(\mathcal{S})^\eta C\left[(1-\alpha)+\alpha\mathcal{S}^{-\eta}\left(\frac{P^*}{P_H}\right)^{\gamma}\int_0^1\left(\frac{\mathcal{S}^i}{h(\mathcal{S})}\right)^{\eta-\frac{1}{\sigma}}di\right]=h(\mathcal{S})^\eta C\left[(1-\alpha)+\alpha\mathcal{S}^{-\eta}\mathcal{S}^\gamma\left(\frac{1}{h(\mathcal{S})}\right)^{\eta-\frac{1}{\sigma}}\int_0^1(\mathcal{S}^i)^{\eta-\frac{1}{\sigma}}di\right]$$

$$Y = h(\mathcal{S})^\eta C\left[(1-\alpha) + \alpha\mathcal{S}^{\gamma-\eta}\left(\frac{1}{h(\mathcal{S})}\right)^{\eta-\frac{1}{\sigma}}\mathcal{S}^{\eta-\frac{1}{\sigma}}\right] = h(\mathcal{S})^\eta C\left[(1-\alpha) + \alpha\mathcal{S}^{\gamma-\eta}\left(\frac{\mathcal{S}}{h(\mathcal{S})}\right)^{\eta-\frac{1}{\sigma}}\right],$$

which yields the result. $Y = h(\mathcal{S})^\eta C\left[(1-\alpha) + \alpha\mathcal{S}^{\gamma-\eta}q(\mathcal{S})^{\eta-\frac{1}{\sigma}}\right]$, where $\mathcal{Q} = \dfrac{\mathcal{S}}{h(\mathcal{S})} \equiv q(\mathcal{S})$

Substituting $C = C^* q(\mathcal{S})^{\frac{1}{\sigma}}$ in the expression above, we have

$$Y = (1-\alpha)h(\mathcal{S})^\eta C + \alpha h(\mathcal{S})^\eta\mathcal{S}^{\gamma-\eta}q(\mathcal{S})^{\eta-\frac{1}{\sigma}} = (1-\alpha)h(\mathcal{S})^\eta C^* q(\mathcal{S})^{\frac{1}{\sigma}} + \alpha h(\mathcal{S})^\eta\mathcal{S}^{\gamma-\eta}q(\mathcal{S})^{\eta-\frac{1}{\sigma}}C^* q(\mathcal{S})^{\frac{1}{\sigma}}$$

Imposing market clearing $C^* = Y^*$, we have

$$Y = \left[(1-\alpha)h(\mathcal{S})^\eta q(\mathcal{S})^{\frac{1}{\sigma}} + \alpha\mathcal{S}^{\gamma-\eta}h(\mathcal{S})^\eta q(\mathcal{S})^\eta\right]Y^* = \left[(1-\alpha)h(\mathcal{S})^\eta q(\mathcal{S})^{\frac{1}{\sigma}} + \alpha\mathcal{S}^{\gamma}h(\mathcal{S})^{-\eta}q(\mathcal{S})^{-\eta}h(\mathcal{S})^\eta q(\mathcal{S})^\eta\right]Y^*$$

$$Y = \left[(1-\alpha)h(\mathcal{S})^\eta q(\mathcal{S})^{\frac{1}{\sigma}} + \alpha\mathcal{S}^{\gamma}\right]Y^* = \left[(1-\alpha)h(\mathcal{S})^\eta q(\mathcal{S})^{\frac{1}{\sigma}} + \alpha q(\mathcal{S})^{\gamma}h(\mathcal{S})^{\gamma}\right]Y^* \equiv v(\mathcal{S})Y^*,$$

where $v(\mathcal{S}) > 0$, $v'(\mathcal{S}) > 0$ and $v(1) = 1$

The clearing of labour market in steady-state implies (the derivation of the two equations below are in the firm's equations)

$$C^\sigma\left(\frac{Y}{A}\right)^\varphi = \frac{W}{P};$$

$$C^\sigma\left(\frac{Y}{A}\right)^\varphi = \frac{W}{P}\quad MC_t = \frac{W_t(1-\tau)}{P_{H,t}A_t}\quad\Rightarrow\quad MC = \frac{W(1-\tau)}{P_H A}$$

From the Taylor expansion in the price-setting problem of the firm, we have

$$\sum_{k=0}^{\infty}(\beta\theta)^k E_t\left\{1 - \frac{\varepsilon}{\varepsilon-1}MC\right\} = 0\quad\Rightarrow\quad MC = 1 - \frac{1}{\varepsilon}$$

$$MC = \frac{W(1-\tau)}{P_H A} = 1 - \frac{1}{\varepsilon}\quad\Rightarrow\quad\frac{W}{P} = A\frac{1-\frac{1}{\varepsilon}}{1-\tau}\frac{P_H}{P} = A\frac{1-\frac{1}{\varepsilon}}{1-\tau}\frac{1}{h(\mathcal{S})}$$

$$C^\sigma\left(\frac{Y}{A}\right)^\varphi = A\frac{1-\frac{1}{\varepsilon}}{1-\tau}\frac{1}{h(\mathcal{S})}\quad\Rightarrow\quad\left(C^*\mathcal{Q}^{\frac{1}{\sigma}}\right)^{\frac{\sigma}{\varphi}}\left(\frac{Y}{A}\right) = \left(A\frac{1-\frac{1}{\varepsilon}}{1-\tau}\frac{1}{h(\mathcal{S})}\right)^{\frac{1}{\varphi}}$$

$$Y = A^{1+\frac{1}{\varphi}}\left(\frac{1-\frac{1}{\varepsilon}}{(1-\tau)(C^*)^\sigma}\frac{1}{h(\mathcal{S})\mathcal{Q}}\right)^{\frac{1}{\varphi}} = A^{\frac{1+\varphi}{\varphi}}\left(\frac{1-\frac{1}{\varepsilon}}{(1-\tau)(Y^*)^\sigma\mathcal{S}}\right)^{\frac{1}{\varphi}}$$

Substituting $Y = Y^* = A^{\frac{1+\varphi}{\sigma+\varphi}}\left(\dfrac{1-\frac{1}{\varepsilon}}{1-\tau}\right)^{\frac{1}{\sigma+\varphi}}$ and solving for $\mathcal{S}$, we have

$$A^{\frac{1+\varphi}{\sigma+\varphi}}\left(\frac{1-\frac{1}{\varepsilon}}{1-\tau}\right)^{\frac{1}{\sigma+\varphi}} = A^{\frac{1+\varphi}{\varphi}}\left(\frac{1-\frac{1}{\varepsilon}}{(1-\tau)(Y^*)^\sigma\mathcal{S}}\right)^{\frac{1}{\varphi}}\quad\Rightarrow\quad A^{\frac{\varphi(1+\varphi)}{\sigma+\varphi}}\left(\frac{1-\frac{1}{\varepsilon}}{1-\tau}\right)^{\frac{\varphi}{\sigma+\varphi}} = A^{1+\varphi}\frac{1-\frac{1}{\varepsilon}}{(1-\tau)(Y^*)^\sigma\mathcal{S}}$$

$$A^{\frac{\varphi+\varphi^2-\sigma-\varphi-\sigma\varphi-\varphi^2}{\sigma+\varphi}}\left(\frac{1-\frac{1}{\varepsilon}}{1-\tau}\right)^{\frac{\varphi-\sigma-\varphi}{\sigma+\varphi}} = \frac{1}{(Y^*)^\sigma\mathcal{S}}\quad\Rightarrow\quad A^{\frac{-\sigma(1+\varphi)}{\sigma+\varphi}}\left(\frac{1-\frac{1}{\varepsilon}}{1-\tau}\right)^{\frac{-\sigma}{\sigma+\varphi}} = \frac{1}{\left[A^{\frac{1+\varphi}{\sigma+\varphi}}\left(\frac{1-\frac{1}{\varepsilon}}{1-\tau}\right)^{\frac{1}{\sigma+\varphi}}\right]^\sigma\mathcal{S}}$$

$$A^{-\frac{1+\varphi}{\sigma+\varphi}}\left(\frac{1-\frac{1}{\varepsilon}}{1-\tau}\right)^{-\frac{1}{\sigma+\varphi}} = \frac{1}{\left[A^{\frac{1+\varphi}{\sigma+\varphi}}\left(\frac{1-\frac{1}{\varepsilon}}{1-\tau}\right)^{\frac{1}{\sigma+\varphi}}\right]\mathcal{S}^{\frac{1}{\sigma}}}\quad\Rightarrow\quad A^{-(1+\varphi)}\left(\frac{1-\frac{1}{\varepsilon}}{1-\tau}\right)^{-1} = \frac{1}{\left[A^{1+\varphi}\left(\frac{1-\frac{1}{\varepsilon}}{1-\tau}\right)\right]\mathcal{S}^{\frac{\sigma+\varphi}{\sigma}}}\quad\Rightarrow\quad\mathcal{S}^{\frac{\sigma+\varphi}{\sigma}} = 1,$$

which gives the result $\mathcal{S} = 1$, which in turn implies $\mathcal{S}_i = 1$ for all i (purchasing parity holds).

8 - Uncovered interest parity and the terms of trade

As $\mathbb{E}\left[\mathcal{Q}_{t,t+1}R_t^i\dfrac{\mathcal{E}_{i,t+1}}{\mathcal{E}_{i,t}}\right] = 1$ and $\mathbb{E}\left[\mathcal{Q}_{t,t+1}R_t\right] = 1$, we have that

$$\mathbb{E}\left[\mathcal{Q}_{t,t+1}R_t^i\frac{\mathcal{E}_{i,t+1}}{\mathcal{E}_{i,t}}\right] = \mathbb{E}\left[\mathcal{Q}_{t,t+1}R_t\right]\quad\Rightarrow\quad\mathbb{E}\left[\mathcal{Q}_{t,t+1}\left(R_t - R_t^i\left[\frac{\mathcal{E}_{i,t+1}}{\mathcal{E}_{i,t}}\right]\right)\right] = 0$$

But to log-linearize it's better to do both sides separately.

$$\mathbb{E}\left[\exp\left(\ln\left[\mathcal{Q}_{t,t+1}R_t^i\frac{\mathcal{E}_{i,t+1}}{\mathcal{E}_{i,t}}\right]\right)\right]\approx$$

$$\mathbb{E}\left[1+\ln\frac{QR^i\mathcal{E}_i}{\mathcal{E}_i}+\frac{1}{\mathcal{Q}R^i}R^i\frac{\mathcal{E}_i}{\mathcal{E}_i}(\mathcal{Q}_{t,t+1}-\mathcal{Q})+\frac{1}{QR^i}\mathcal{Q}\frac{\mathcal{E}_i}{\mathcal{E}_i}(R_t^i-R^i)-\frac{1}{\mathcal{Q}R^i}\mathcal{Q}\frac{R^i}{\mathcal{E}_i}(\mathcal{E}_{i,t+1}-\mathcal{E}_i)+\frac{1}{\mathcal{Q}R^i}\mathcal{Q}\frac{R^i\mathcal{E}_i}{\mathcal{E}_i^2}(\mathcal{E}_{i,t}-\mathcal{E}_i)\right]$$

$$=1+\ln(\mathcal{Q}R^i)+\mathbb{E}\left[\hat{q}_t+\hat{r_t}^i-\hat{e}_{i,t+1}+\hat{e}_{i,t}\right]$$

$$\mathbb{E}\left[\exp\left(\ln\left[\mathcal{Q}_{t,t+1}R_t\right]\right)\right]\approx\mathbb{E}\left[1+\ln(\mathcal{Q}R)+\frac{1}{\mathcal{Q}R}R(\mathcal{Q}_{t,t+1}-Q)+\frac{1}{\mathcal{Q}R}\mathcal{Q}(R_t-R)\right]$$

$$1+\ln(\mathcal{Q}R^i)+\mathbb{E}\left[\hat{q}_t+\hat{r}_t^i-\hat{e}_{i,t+1}+\hat{e}_{i,t}\right]=1+\ln(\mathcal{Q}R)+\mathbb{E}\left[\hat{q}_t+\hat{r}_t\right]\quad\Rightarrow\quad\hat{r}^i-\mathbb{E}\left[e_{i,t+1}-e_{i,t}\right]=\hat{r}\quad\Rightarrow\quad r_t^i-r_t=\mathbb{E}_t[\Delta e_{i,t+1}]$$

The aggregation comes from the FOC. The uncovered interest rate parity allow households to invest both in domestic and foreign assets: B_t, B_t^*. The budget constraint can be written as

$$P_t+Q_{t,t+1}D_{t+1}+Q_{t,t+1}^*\mathcal{E}_{t+1}D_{t+1}^*\leq D_t+\mathcal{E}_tD_t^*+W_tN_t+T_t$$

an the lagrangean becomes

$$\mathcal{L}=\mathbb{E}_t\sum_{t=0}^{\infty}\beta^t\left[\left(\frac{C_t^{1-\sigma}}{1-\sigma}-\frac{N_t^{1+\varphi}}{1+\varphi}\right)+\lambda_t\left(\mathcal{E}_tD_t^*+W_tN_t+T_t-Q_{t,t+1}^*\mathcal{E}_tD_{t+1}^*-Q_{t,t+1}D_{t+1}-P_tC_t\right)\right]$$

The FOCs are:

$$(C_t)\ C_t^{-\sigma}=\lambda_tP_t\quad\Rightarrow\quad\mathbb{E}[C_{t+1}^{-\sigma}]=\mathbb{E}[\lambda_{t+1}P_{t+1}]\quad\Rightarrow\quad\mathbb{E}\left[\frac{\lambda_{t+1}}{\lambda_t}\right]=\mathbb{E}\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\right]$$

$$(D_{t+1}^*)\ \beta\ \mathbb{E}[\lambda_{t+1}\mathcal{E}_{t+1}]=\lambda_tQ_{t,t+1}^*\mathcal{E}_t\quad\Rightarrow\quad\mathbb{E}\left[\frac{\lambda_{t+1}}{\lambda_t}\right]=\mathbb{E}\left[\frac{Q_{t,t+1}^*\mathcal{E}_t}{\beta\mathcal{E}_{t+1}}\right]$$

$$(D_{t+1}^*)\ \beta\ \mathbb{E}[\lambda_{t+1}]=\lambda_tQ_{t,t+1}\quad\Rightarrow\quad\mathbb{E}\left[\frac{\lambda_{t+1}}{\lambda_t}\right]=\mathbb{E}\left[\frac{Q_{t,t+1}}{\beta}\right]$$

$$(N_t)\ N_t^{\varphi}=\lambda_tW_t$$

Combining the consumption FOC and the foreign bond's FOC we have

$$\mathbb{E}\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\right]=\mathbb{E}\left[\frac{Q_{t,t+1}^*\mathcal{E}_t}{\beta\mathcal{E}_{t+1}}\right]\quad\Rightarrow\quad\beta\ \mathbb{E}\left[\frac{1}{Q_{t,t+1}^*}\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right]=1$$

Doing the same steps for the domestic bonds, we have

$$\mathbb{E}\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\right]=\mathbb{E}\left[\frac{Q_{t,t+1}}{\beta}\right]\quad\Rightarrow\quad\beta\ \mathbb{E}\left[\frac{1}{Q_{t,t+1}}\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\right]=1$$

Dividing the equation for foreign bonds by the equation for domestic bonds, we have

$$\mathbb{E}\left[\frac{Q_{t,t+1}}{Q_{t,t+1}^*}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right]=1\quad\Rightarrow\quad\frac{Q_{t,t+1}}{Q_{t,t+1}^*}=\mathbb{E}\left[\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right]\quad\Rightarrow\quad\ln(Q_{t,t+1})-\ln(Q_{t,t+1}^*)\approx\mathbb{E}[\ln\mathcal{E}_{t+1}-\ln\mathcal{E}_t]\quad\Rightarrow\quad r_t-r_t^*=\mathbb{E}[\Delta e_{t+1}]$$

From the definition of the log terms of trade, we have $s_t=e_t+p_t^*-p_{H,t}$ and $\mathbb{E}[s_{t+1}]=\mathbb{E}[e_{t+1}+p_{t+1}^*-p_{H,t+1}]$. Subtracting the first by the second, we get

$s_t-\mathbb{E}[s_{t+1}]=e_t+p_t^*-p_{H,t}-\mathbb{E}[e_{t+1}]-\mathbb{E}[p_{t+1}^*]-\mathbb{E}[p_{H,t+1}]\quad\Rightarrow\quad s_t=\mathbb{E}[s_{t+1}]-\mathbb{E}[\Delta e_{t+1}]-\mathbb{E}[\pi_{t+1}^*]-\mathbb{E}[\pi_{H,t+1}]=r_t^*-\mathbb{E}[\pi_{t+1}^*]-(r_t-\mathbb{E}[\pi_{H,t+1}])+\mathbb{E}[s_{t+1}]$

9 Firms

$$N_t\equiv\int_0^1N_t(j)dj=\int_0^1\frac{Y_t(j)}{A_t}dj=\int_0^1\frac{Y_t(j)}{A_t}\frac{Y_t}{Y_t}dj=\int_0^1\frac{Y_t(j)}{Y_t}dj\frac{Y_t}{A_t}=\frac{Y_tZ_t}{A_t},\text{ where }Z_t\equiv\int_0^1\frac{Y_t(j)}{Y_t}dj$$

The marginal cost is the same for all firms because of constant returns to scale. In this case, the quantity produced is determined by the cost function. For each additional worker, the firm has to pay W_t and produces $\frac{Y_t(j)}{A_t}$ and receives P_t by product.

So, abstracting from the subsidy, the marginal cost is $MC_t=\frac{W_t}{P_{H,t}A_t}$ If we include the subsidy, it's as the firm pays a smaller salary. So, the marginal cost becomes $MC_t=\frac{W_t(1-\tau)}{P_{H,t}A_t}$

Combining the consumption FOC and the labor FOC int he steady-state, we have

$$\frac{C^{-\sigma}}{P}=\lambda=N^{\varphi}=\left(\frac{Y}{A}\right)^{\varphi}\quad\Rightarrow\quad C^{\sigma}\left(\frac{Y}{A}\right)^{\varphi}=\frac{W}{P}$$

Also, we know that $Y_t^{\frac{\varepsilon-1}{\varepsilon}}=\int_0^1 Y_t(j)^{\frac{\varepsilon-1}{\varepsilon}}dj$, so $Y_{t+k}^{\frac{\varepsilon-1}{\varepsilon}}=\int_0^1 Y_{t+k}(j)^{\frac{\varepsilon-1}{\varepsilon}}dj$ and

$$Y_{t+k}(j)\leq\left(\frac{\bar{P}_{H,t}}{P_{H,t+k}}\right)^{-\varepsilon}\left(C_{H,t+k}+\int_0^1 C^i_{H,t+k}di\right)$$

Then maximizing, the restriction above is binding (otherwise part of the production would have been wasted, which contradicts the hypothesis of optimization when the costs are not null). The total demand is

$$Y_{t+k}^{\frac{\varepsilon-1}{\varepsilon}}=\int_0^1\left[\left(\frac{\bar{P}_{H,t}}{P_{H,t+k}}\right)^{-\varepsilon}\left(C_{H,t+k}+\int_0^1 C^i_{H,t+k}di\right)\right]^{\frac{\varepsilon-1}{\varepsilon}}dj$$

As nothing in the RHS of the above expression depends on (j), we can take everything out of the integral.

$$Y_{t+k}^{\frac{\varepsilon-1}{\varepsilon}}=\left[\left(\frac{\bar{P}_{H,t}}{P_{H,t+k}}\right)^{-\varepsilon}\left(C_{H,t+k}+\int_0^1 C^i_{H,t+k}di\right)\right]^{\frac{\varepsilon-1}{\varepsilon}}\int_0^1 dj \text{ and}$$

$$Y_{t+k}=\left(\frac{\bar{P}_{H,t}}{P_{H,t+k}}\right)^{-\varepsilon}\left(C_{H,t+k}+\int_0^1 C^i_{H,t+k}di\right)$$

The problem of the firm (price setting) is:

$$\max_{\bar{P}_{H,t}}\sum_{k=0}^{\infty}\theta^k E_t\left\{\beta^k\left(\frac{C_{H,t+k}}{C_{H,t}}\right)^{-\sigma}\left(\frac{P_{H,t}}{P_{H,t+k}}\right)\left[\left(\frac{\bar{P}_{H,t}}{P_{H,t+k}}\right)^{-\varepsilon}\left(C_{H,t+k}+\int_0^1 C^i_{H,t+k}di\right)(\bar{P}_{H,t}-MC^n_{t+k})\right]\right\}$$

$$\max_{\bar{P}_{H,t}}\sum_{k=0}^{\infty}\theta^k E_t\left\{\beta^k\left(\frac{C_{H,t+k}}{C_{H,t}}\right)^{-\sigma}\left(C_{H,t+k}+\int_0^1 C^i_{H,t+k}di\right)\left(\frac{P_{H,t}}{P_{H,t+k}}\right)\left[\frac{\bar{P}^{1-\varepsilon}_{H,t}}{P^{-\varepsilon}_{H,t+k}}-\left(\frac{\bar{P}_{H,t}}{P_{H,t+k}}\right)^{-\varepsilon}MC^n_{t+k})\right]\right\}$$

, which yields, after deriving

$$\max_{\bar{P}_{H,t}}\sum_{k=0}^{\infty}\theta^k E_t\left\{\beta^k\left(\frac{C_{H,t+k}}{C_{H,t}}\right)^{-\sigma}\left(C_{H,t+k}+\int_0^1 C^i_{H,t+k}di\right)\left(\frac{P_{H,t}}{P_{H,t+k}}\right)\left[(1-\varepsilon)\frac{\bar{P}^{-\varepsilon}_{H,t}}{P^{-\varepsilon}_{H,t+k}}+\varepsilon\frac{\bar{P}^{-\varepsilon-1}_{H,t}}{P^{-\varepsilon}_{H,t+k}}MC^n_{t+k})\right]\right\}=0$$

Under flexible prices,

$$(1-\varepsilon)+\varepsilon\frac{MC^n_t}{\bar{P}_{H,t}}=0\quad\Rightarrow\quad\bar{P}_{H,t}=\frac{\varepsilon}{\varepsilon-1}MC^n_t$$

From appendix B, $\max_{\bar{P}_{H,t}}\sum_{k=0}^{\infty}\theta^k E_t\left\{Q_{t,t+k}Y_{t+k}\left[\bar{P}_{H,t}-MC^n_{t+k}\right]\right\}$ subject to the demand constraints

$$Y_{t+k}\leq\left(\frac{\bar{P}_{H,t}}{P_{H,t+k}}\right)^{-\varepsilon}\left(C_{H,t+k}+\int_0^1 C^i_{H,t+k}di\right)$$

Because there's monopolistic competition, the demand depends on the price set. Substituting the restriction in the maximization problem we have

$$\max_{\bar{P}_{H,t}}\sum_{k=0}^{\infty}\theta^k E_t\left\{Q_{t,t+k}\left[\left(\frac{\bar{P}_{H,t}(j)}{P_{H,t+k}(j)}\right)^{-\varepsilon}\left(C_{H,t+k}+\int_0^1 C^i_{H,t+k}di\right)\right]\left[\bar{P}_{H,t}-MC^n_{t+k}\right]\right\}$$

Calculating the CPO, we have:

$$\sum_{k=0}^{\infty}\theta^k E_t\left\{Q_{t,t+k}\left[(-\varepsilon)\left(\frac{\bar{P}^{-\varepsilon-1}_{H,t}}{P^{-\varepsilon}_{H,t+k}}\right)\left(C_{H,t+k}+\int_0^1 C^i_{H,t+k}di\right)\left[\bar{P}_{H,t}-MC^n_{t+k}\right]\right]+Y_{t+k}\right\}=0$$

$$\sum_{k=0}^{\infty}\theta^k E_t\left\{Q_{t,t+k}\left[\frac{(-\varepsilon)}{\bar{P}_{H,t}}\left(\frac{\bar{P}_{H,t}}{P_{H,t+k}}\right)^{-\varepsilon}\left(C_{H,t+k}+\int_0^1 C^i_{H,t+k}di\right)\left[\bar{P}_{H,t}-MC^n_{t+k}\right]\right]+Y_{t+k}\right\}=0$$

$$\sum_{k=0}^{\infty}\theta^k E_t\left\{Q_{t,t+k}\left[\frac{(-\varepsilon)}{\bar{P}_{H,t}}Y_{t+k}\left[\bar{P}_{H,t}-MC^n_{t+k}\right]\right]+Y_{t+k}\right\}=0$$

As ε and $\bar{P}_{H,t}$ don't depend on k and the expression is equal zero, we can do the following operation

$$\sum_{k=0}^{\infty}\theta^k E_t\left\{Q_{t,t+k}\left[Y_{t+k}\left[-\varepsilon+\varepsilon\frac{MC^n_{t+k}}{\bar{P}_{H,t}}\right]\right]+Y_{t+k}\right\}=0 \quad\Rightarrow\quad \sum_{k=0}^{\infty}\theta^k E_t\left\{Q_{t,t+k}Y_{t+k}\left[(1-\varepsilon)+\varepsilon\frac{MC^n_{t+k}}{\bar{P}_{H,t}}\right]\frac{\bar{P}_{H,t}}{1-\varepsilon}\right\}=0$$

$$\sum_{k=0}^{\infty}\theta^k E_t\left\{Q_{t,t+k}Y_{t+k}\left[\bar{P}_{H,t}-\frac{\varepsilon}{\varepsilon-1}MC^n_{t+k}\right]\right\}=0$$

Using the fact that $Q_{t,t+k}=\beta^k\left(\frac{C_{t+k}}{C_t}\right)^{-\sigma}\frac{P_t}{P_{t+k}}$, we have

$$\sum_{k=0}^{\infty}\theta^k E_t\left\{\beta^k\left(\frac{C_{t+k}}{C_t}\right)^{-\sigma}\frac{P_t}{P_{t+k}}Y_{t+k}\left[\bar{P}_{H,t}-\frac{\varepsilon}{\varepsilon-1}MC^n_{t+k}\right]\right\}=0$$

As P_t and C_t doesn't depend on k, we can put it out of summation and ignore (as the expression equals zero)

$$\sum_{k=0}^{\infty}(\beta\theta)^k E_t\left\{C^{-\sigma}_{t+k}P^{-1}_{t+k}Y_{t+k}\left[\bar{P}_{H,t}-\frac{\varepsilon}{\varepsilon-1}MC^n_{t+k}\right]\right\}=0$$

$$\sum_{k=0}^{\infty}(\beta\theta)^k E_t\left\{C^{-\sigma}_{t+k}Y_{t+k}\frac{P_{H,t-1}}{P_{t+k}}\left[\frac{\bar{P}_{H,t}}{P_{H,t-1}}-\frac{\varepsilon}{\varepsilon-1}\frac{P_{H,t+k}}{P_{H,t-1}}\frac{MC^n_{t+k}}{P_{H,t+k}}\right]\right\}=0$$

$$\sum_{k=0}^{\infty}(\beta\theta)^k E_t\left\{C^{-\sigma}_{t+k}Y_{t+k}\frac{P_{H,t-1}}{P_{t+k}}\left[\frac{\bar{P}_{H,t}}{P_{H,t-1}}-\frac{\varepsilon}{\varepsilon-1}\Pi^H_{t-1,t+k}MC_{t+k}\right]\right\}=0 \text{ where } \Pi^H_{t-1,t+k}=\frac{P_{H,t+k}}{P_{H,t-1}} \text{ and } MC_{t+k}=\frac{MC^n_{t+k}}{P_{H,t+k}}$$

As we will use a first-order approximation, we can ignore the Jensen's inequality. To make the multivariate Taylor expansion, we can use $f(x+\Delta x)\approx f(x)+\Delta x^T\nabla f|_x(\Delta x)$, where x is the vector of variables in the steady-state.

$$\sum_{k=0}^{\infty}(\beta\theta)^k E_t\left\{C^{-\sigma}_{t+k}Y_{t+k}\frac{P_{H,t-1}}{P_{t+k}}\left[\frac{\bar{P}_{H,t}}{P_{H,t-1}}-\frac{\varepsilon}{\varepsilon-1}\Pi^H_{t-1,t+k}MC_{t+k}\right]\right\}=\sum_{k=0}^{\infty}(\beta\theta)^k E_t\left\{C^{-\sigma}Y\frac{P_H}{P}\left[\frac{\bar{P}_H}{P_H}-\frac{\varepsilon}{\varepsilon-1}MC\right]\right\}$$

$$+\sum_{k=0}^{\infty}(\beta\theta)^k E_t\left\{-\sigma C^{-\sigma-1}Y\frac{P_H}{P}\left[1-\frac{\varepsilon}{\varepsilon-1}MC\right](C_{t+k}-C)\right\}+\sum_{k=0}^{\infty}(\beta\theta)^k E_t\left\{C^{-\sigma}\frac{P_H}{P}\left[1-\frac{\varepsilon}{\varepsilon-1}MC\right](Y_{t+k}-Y)\right\}$$

$$+\sum_{k=0}^{\infty}(\beta\theta)^k E_t\left\{\left(C^{-\sigma}Y\frac{1}{P}\left[1-\frac{\varepsilon}{\varepsilon-1}MC\right]+C^{-\sigma}Y\frac{P_H}{P}\left[-\frac{1}{P_H}\right]\right)(P_{H,t-1}-P_H)\right\}$$

$$+\sum_{k=0}^{\infty}(\beta\theta)^k E_t\left\{-C^{-\sigma}Y\frac{P_H}{P^2}\left[1-\frac{\varepsilon}{\varepsilon-1}MC\right](P_{t+k}-P)\right\}+\sum_{k=0}^{\infty}(\beta\theta)^k E_t\left\{C^{-\sigma}Y\frac{P_H}{P}\left[\frac{1}{P_H}\right](\bar{P}_{H,t}-P_H)\right\}$$

$$+\sum_{k=0}^{\infty}(\beta\theta)^k E_t\left\{C^{-\sigma}Y\frac{P_H}{P}\left[-\frac{\varepsilon}{\varepsilon-1}MC\right](\Pi^H_{t-1,t+k}-\Pi^H)\right\}+\sum_{k=0}^{\infty}(\beta\theta)^k E_t\left\{C^{-\sigma}Y\frac{P_H}{P}\left[-\frac{\varepsilon}{\varepsilon-1}\Pi^H\right](MC_{t+k}-MC)\right\}=0$$

As $C^{-\sigma},Y,P_H,P$ don't depend on k, we have

$$\sum_{k=0}^{\infty}(\beta\theta)^k E_t\left\{1-\frac{\varepsilon}{\varepsilon-1}MC\right\}+\sum_{k=0}^{\infty}(\beta\theta)^k E_t\left\{-\sigma\left[1-\frac{\varepsilon}{\varepsilon-1}MC\right]\hat{c}_{t+k}\right\}+\sum_{k=0}^{\infty}(\beta\theta)^k E_t\left\{\left[1-\frac{\varepsilon}{\varepsilon-1}MC\right]\hat{y}_{t+k}\right\}$$

$$+\sum_{k=0}^{\infty}(\beta\theta)^k E_t\left\{\left(-\frac{\varepsilon}{\varepsilon-1}MC\right)\hat{p}_{H,t-1}\right\}+\sum_{k=0}^{\infty}(\beta\theta)^k E_t\left\{-\left[1-\frac{\varepsilon}{\varepsilon-1}MC\right](\hat{p}_{t+k})\right\}+\sum_{k=0}^{\infty}(\beta\theta)^k E_t\left\{(\hat{\bar{p}}_{H,t})\right\}$$

$$+\sum_{k=0}^{\infty}(\beta\theta)^k E_t\left\{\left[-\frac{\varepsilon}{\varepsilon-1}MC\right]\pi_{t-1,t+k}\right\}+\sum_{k=0}^{\infty}(\beta\theta)^k E_t\left\{MC\left[-\frac{\varepsilon}{\varepsilon-1}\right]\widehat{mc}_{t+k}\right\}=0$$

$$\sum_{k=0}^{\infty}(\beta\theta)^k E_t\left\{\left(1-\frac{\varepsilon}{\varepsilon-1}MC\right)(1-\sigma\hat{c}_{t+k}+\hat{y}_{t+k}+\hat{p}_{H,t-1}-\hat{p}_{t+k}+\pi_{t-1,t+k}+\widehat{mc}_{t+k})-\hat{p}_{H,t-1}+\hat{\bar{p}}_{H,t}-\pi_{t-1,t+k}-\widehat{mc}_{t+k}\right\}$$

As the Taylor approximation is around zero (we are assuming regularity conditions to all functions),

$$\sum_{k=0}^{\infty}(\beta\theta)^k E_t\left\{1-\frac{\varepsilon}{\varepsilon-1}MC\right\}=0$$, we have

$$\sum_{k=0}^{\infty}(\beta\theta)^k E_t\left\{-\hat{p}_{H,t-1}+\hat{\bar{p}}_{H,t}-\pi_{t-1,t+k}-\widehat{mc}_{t+k}\right\}=0$$

$$\frac{1}{1-\beta\theta}(\bar{p}_{H,t}-p_H)+\sum_{k=0}^{\infty}(\beta\theta)^k E_t\left\{-(p_{H,t-1}-p_H)-(p_{H,t+k}-p_{H,t-1})-\widehat{mc}_{t+k}\right\}=0$$

$$\frac{1}{1-\beta\theta}(\bar{p}_{H,t}-p_{H,t-1})+\sum_{k=0}^{\infty}(\beta\theta)^k E_t\left\{-p_{H,t+k}+p_{H,t}-p_{H,t}+p_{H,t-1}-\widehat{mc}_{t+k}\right\}=0$$

$$\bar{p}_{H,t}=p_{H,t-1}+\pi_{H,t}+\sum_{k=0}^{\infty}(\beta\theta)^k E_t\{\pi_{H,t+k}\}-\beta\theta\sum_{k=0}^{\infty}(\beta\theta)^k E_t\{\pi_{H,t+k}\}+(1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k E_t\{\widehat{mc}_{t+k}\}$$

$$\bar{p}_{H,t}=p_{H,t-1}+\sum_{k=0}^{\infty}(\beta\theta)^k E_t\{\pi_{H,t+k}\}+(1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k E_t\{\widehat{mc}_{t+k}\}$$

Rational expectations imply that the difference between the actual inflation and the future expectations is the steady-state inflation, which is zero. This expression can also be written as (it's easy to see when trying to iterate forward).

$$\bar{p}_{H,t}-p_{H,t-1}=\beta\theta E_t\{\bar{p}_{H,t+1}-p_{H,t}+\pi_{H,t}+(1-\beta\theta)\widehat{mc}_t\}$$

Substituting $\widehat{mc}_t=mc_t^n-p_{H,t}+\mu$, we have

$$\bar{p}_{H,t}-p_{H,t-1}=\beta\theta E_t\{\bar{p}_{H,t+1}-p_{H,t}+\pi_{H,t}+(1-\beta\theta)(mc_t^n-p_{H,t}+\mu)\}$$

$$\bar{p}_{H,t}-p_{H,t-1}=E_t\{\beta\theta\bar{p}_{H,t+1}-\beta\theta p_{H,t}+p_{H,t}-p_{H,t-1}+mc_t^n-p_{H,t}+\mu-\beta\theta mc_t^n+\beta\theta p_{H,t}-\beta\theta\mu\}$$

$$\bar{p}_{H,t}=E_t\{\beta\theta\bar{p}_{H,t+1}+mc_t^n+\mu-\beta\theta mc_t^n-\beta\theta\mu\}$$

which can also be written as $p_{H,t}=\mu+(1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k E_t\{mc_{t+k}^n\}$

The price-setting equation is $P_H\equiv\left[\theta(P_{H,t-1})^{1-\varepsilon}+(1-\theta)(\bar{P}_{H,t})^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}$

Log-linearizing the equation above around the steady-state with the Taylor expansion gives

$$P_{H,t}\approx[\theta(P_H)^{1-\varepsilon}+(1-\theta)(P_H)^{1-\varepsilon}]^{\frac{1}{1-\varepsilon}}+\frac{1}{1-\varepsilon}\left[(P_H)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}-1}\theta(1-\varepsilon)P_H^{-\varepsilon}(P_{H,t-1}-P_H)$$

$$+\frac{1}{1-\varepsilon}\left[(P_H)^{1-\varepsilon}\right]^{\frac{\varepsilon}{1-\varepsilon}}(1-\theta)(1-\varepsilon)P_H^{-\varepsilon}(\bar{P}_{H,t}-P_H)$$

$$P_{H,t}\approx P_H+\theta(P_H)^{\varepsilon}(P_H)^{-\varepsilon}(P_{H,t-1}-P_H)+(1-\theta)(P_H)^{\varepsilon}(P_H)^{-\varepsilon}(\bar{P}_{H,t}-P_H)=P_H+\theta P_{H,t-1}-\theta P_H+\bar{P}_{H,t}-P_H-\theta\bar{P}_{H,t}+\theta P_H$$

$$P_{H,t}=\theta P_{H,t-1}+(1-\theta)\bar{P}_{H,t}\quad\Rightarrow\quad\frac{P_{H,t}-P_{H,t-1}}{P_H}=\frac{\theta P_{H,t-1}-P_{H,t-1}+(1-\theta)\bar{P}_{H,t}}{P_H}\quad\Rightarrow\quad\pi_{H,t}=(1-\theta)(\bar{p}_{H,t}-p_{H,t-1})$$

Substituting $\bar{p}_{H,t}-p_{H,t-1}$ we have

$$\pi_{H,t}=(1-\theta)(\beta\theta E_t\{\bar{p}_{H,t+1}-p_{H,t}+\pi_{H,t}+(1-\beta\theta)\widehat{mc}_t\})=(1-\theta)\left(\beta\theta E_t\left\{\frac{\pi_{H,t+1}}{1-\theta}+\pi_{H,t}+(1-\beta\theta)\widehat{mc}_t\right\}\right)$$

$$\theta\pi_{H,t}=\beta\theta E_t\left\{\pi_{H,t+1}\right\}+(1-\theta)(1-\beta\theta)\widehat{mc}_t\quad\Rightarrow\quad\pi_{H,t}=\beta E_t\left\{\pi_{H,t+1}\right\}+\frac{(1-\theta)(1-\beta\theta)}{\theta}\widehat{mc}_t$$

which gives $\pi_{H,t}=\beta E_t\left\{\pi_{H,t+1}\right\}+\lambda\widehat{mc}_t$ where $\lambda=\dfrac{(1-\theta)(1-\beta\theta)}{\theta}$

10 - Equilibrium

Good market clearing in the representative small open economy ("home") requires that the total produced inside the small economy is consumed either by the households of this economy or imported (from the small economy, subscript H) from households of any other country (superscript i).

$$Y_t(j)=C_{H,t}(j)+\int_0^1 C_{H,t}^i(j)di=\left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\varepsilon}C_{H,t}+\int_0^1\left(\frac{P_{H,t}^i(j)}{P_{H,t}^i}\right)^{-\varepsilon}C_{H,t}^i di$$

$$Y_t(j) = \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\varepsilon}(1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta}C_t + \int_0^1 \left(\frac{P^i_{H,t}(j)}{P^i_{H,t}}\right)^{-\varepsilon}\left(\frac{P^i_{H,t}}{P^i_{F,t}}\right)^{-\gamma} C^i_{F,t}di$$

$$Y_t(j) = \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\varepsilon}(1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta}C_t + \int_0^1 \left(\frac{P^i_{H,t}(j)}{P^i_{H,t}}\right)^{-\varepsilon}\left(\frac{P_{H,t}}{\mathcal{E}_{i,t}P^i_{F,t}}\right)^{-\gamma}\alpha\left(\frac{P^i_{F,t}}{P^i_t}\right)^{-\eta} C^i_t di$$

Assuming symmetric preferences across countries, the home country will sell the variety of good j for the same price, independently of which country is buying, which implies $\frac{P^i_{H,t}(j)}{P^i_{H,t}} = \frac{P_{H,t}(j)}{P_{H,t}}$

Thus, $Y_t(j) = \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\varepsilon}\left[(1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta}C_t + \alpha\int_0^1 \left(\frac{P_{H,t}}{\mathcal{E}_{i,t}P^i_{F,t}}\right)^{-\gamma}\left(\frac{P^i_{F,t}}{P^i_t}\right)^{-\eta} C^i_t di\right]$

Plugging on the definition of aggregate domestic output $Y_t^{\frac{\varepsilon-1}{\varepsilon}} = \int_0^1 Y_t(j)^{\frac{\varepsilon-1}{\varepsilon}}dj$ we have

$$Y_t^{\frac{\varepsilon-1}{\varepsilon}} = \int_0^1 \left(\left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\varepsilon}\left[(1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta}C_t + \alpha\int_0^1 \left(\frac{P_{H,t}}{\mathcal{E}_{i,t}P^i_{F,t}}\right)^{-\gamma}\left(\frac{P^i_{F,t}}{P^i_t}\right)^{-\eta} C^i_t di\right]\right)^{\frac{\varepsilon-1}{\varepsilon}} dj$$

$$Y_t^{\frac{\varepsilon-1}{\varepsilon}} = \left[(1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta}C_t + \alpha\int_0^1 \left(\frac{P_{H,t}}{\mathcal{E}_{i,t}P^i_{F,t}}\right)^{-\gamma}\left(\frac{P^i_{F,t}}{P^i_t}\right)^{-\eta} C^i_t di\right]^{\frac{\varepsilon-1}{\varepsilon}}\left(\frac{1}{P_{H,t}}\right)^{1-\varepsilon}\int_0^1 P_{H,t}(j)^{1-\varepsilon}dj$$

By the definition of price index, $P_H^{1-\varepsilon} = \int_0^1 P_{H,t}(j)^{1-\varepsilon}dj$, we have

$$Y_t^{\frac{\varepsilon-1}{\varepsilon}} = \left[(1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta}C_t + \alpha\int_0^1 \left(\frac{P_{H,t}}{\mathcal{E}_{i,t}P^i_{F,t}}\right)^{-\gamma}\left(\frac{P^i_{F,t}}{P^i_t}\right)^{-\eta} C^i_t di\right]^{\frac{\varepsilon-1}{\varepsilon}}\left(\frac{1}{P_{H,t}}\right)^{1-\varepsilon}P_H^{1-\varepsilon}$$

$$Y_t^{\frac{\varepsilon-1}{\varepsilon}} = \left[(1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta}C_t + \alpha\int_0^1 \left(\frac{P_{H,t}}{\mathcal{E}_{i,t}P^i_{F,t}}\right)^{-\gamma}\left(\frac{P^i_{F,t}}{P^i_t}\right)^{-\eta} C^i_t di\right]^{\frac{\varepsilon-1}{\varepsilon}}$$

$$Y_t = (1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta}C_t + \alpha\int_0^1 \left(\frac{P_{H,t}}{\mathcal{E}_{i,t}P^i_{F,t}}\right)^{-\gamma}\left(\frac{P^i_{F,t}}{P^i_t}\right)^{-\eta} C^i_t di$$

Using the fact that $\mathcal{Q}_{i,t} = \frac{\mathcal{E}_{i,t}P^i_t}{P_t}$

$$Y_t = (1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta}C_t + \alpha\int_0^1 \left(\frac{\mathcal{E}_{i,t}P^i_{F,t}}{P_{H,t}}\right)^{\gamma}(P^i_{F,t})^{-\eta}\left(\frac{\mathcal{Q}_{i,t}P_t}{\mathcal{E}_{i,t}}\right)^{\eta} C^i_t di = \left(\frac{P_{H,t}}{P_t}\right)^{-\eta}\left[(1-\alpha)C_t + \alpha\int_0^1 \left(\frac{\mathcal{E}_{i,t}P^i_{F,t}}{P_{H,t}}\right)^{\gamma-\eta}\mathcal{Q}^{\eta}C^i_t di\right]$$

Considering that $\mathcal{S}_{i,t} = \frac{P_{i,t}}{P_{H,t}}$, $\mathcal{S}^i_t = \frac{\mathcal{E}_{i,t}P^i_{F,t}}{P_{i,t}}$, and $C^i_t = C_t\mathcal{Q}^{-\frac{1}{\sigma}}$ we have

$$Y_t = \left(\frac{P_{H,t}}{P_t}\right)^{-\eta}\left[(1-\alpha)C_t + \alpha\int_0^1 \left(\frac{P_{i,t}}{P_{H,t}}\frac{\mathcal{E}_{i,t}P^i_{F,t}}{P_{i,t}}\right)^{\gamma-\eta}\mathcal{Q}^{\eta}\mathcal{Q}^{-\frac{1}{\sigma}}C_t di\right] = \left(\frac{P_{H,t}}{P_t}\right)^{-\eta}C_t\left[(1-\alpha) + \alpha\int_0^1 \left(\mathcal{S}_{i,t}\mathcal{S}^i_t\right)^{\gamma-\eta}\mathcal{Q}^{\eta-\frac{1}{\sigma}}di\right]$$

If $\sigma = \eta = \gamma = 1$, then

$$Y_t = \left(\frac{P_{H,t}}{P_t}\right)^{-1}C_t\left[(1-\alpha) + \alpha\int_0^1 \left(\mathcal{S}_{i,t}\mathcal{S}^i_t\right)^{0}\mathcal{Q}^0 di\right] = \frac{P_t}{P_{H,t}}C_t = C_t\mathcal{S}^{\alpha}_t$$

as if $\eta = 1$, the CPI takes the form $P_t = (P_{H,t})^{1-\alpha}(P_{F,t})^{\alpha}$ implying $\frac{P_t}{P_{H,t}} = \left(\frac{P_{F,t}}{P_{H,t}}\right)^{\alpha} = \mathcal{S}^{\alpha}$

Log-linearizing the equation (25) using the Taylor expansion around the symmetric steady-state, we have

$$Y_t \approx Y + (-\eta)\left(P_H\right)^{-\eta-1}\left(\frac{1}{P}\right)^{-\eta}C\left[(1-\alpha) + \alpha\int_0^1 \left(\mathcal{S}_i\mathcal{S}^i\right)^{\gamma-\eta}\mathcal{Q}_i^{\eta-\frac{1}{\sigma}}di\right]\left(P_{H,t} - P_H\right)$$

$$+\eta\left(P_H\right)^{-\eta}\left(\frac{1}{P}\right)^{-\eta+1} C\left[(1-\alpha)+\alpha \int_0^1\left(\mathcal{S}_i \mathcal{S}^i\right)^{\gamma-\eta} \mathcal{Q}_i^{\eta-\frac{1}{\sigma}} di\right](P_t-P)+\left(\frac{P_H}{P}\right)^{-\eta}\left[(1-\alpha)+\alpha \int_0^1\left(\mathcal{S}_i \mathcal{S}^i\right)^{\gamma-\eta} \mathcal{Q}_i^{\eta-\frac{1}{\sigma}} di\right](C_t-C)$$

$$+\left(\frac{P_H}{P}\right)^{-\eta} C\left[\alpha \int_0^1(\gamma-\eta)\left(\mathcal{S}_i\right)^{\gamma-\eta-1}\left(\mathcal{S}^i\right)^{\gamma-\eta} \mathcal{Q}_i^{\eta-\frac{1}{\sigma}}(\mathcal{S}_{i,t}-\mathcal{S}_i) di\right]+\left(\frac{P_H}{P}\right)^{-\eta} C\left[\alpha \int_0^1(\gamma-\eta)\left(\mathcal{S}_i\right)^{\gamma-\eta}\left(\mathcal{S}^i\right)^{\gamma-\eta-1} \mathcal{Q}_i^{\eta-\frac{1}{\sigma}}(\mathcal{S}_t^i-\mathcal{S}^i) di\right]$$

$$+\left(\frac{P_H}{P}\right)^{-\eta} C\left[\alpha \int_0^1\left(\eta-\frac{1}{\sigma}\right)\left(\mathcal{S}_i \mathcal{S}^i\right)^{\gamma-\eta} \mathcal{Q}_i^{\eta-\frac{1}{\sigma}-1}(\mathcal{Q}_{i,t}-\mathcal{Q}_i) di\right]$$

As shown in appendix A (and above, in the international risk sharing section), in a symmetric steady-state $\mathcal{Q}_i=\mathcal{S}_i=\mathcal{S}^i=1$ for all i (purchasing parity holds). Thus

$$Y_t-Y=\left(\frac{P_H}{P}\right)^{-\eta} C\left[-\eta \frac{P_{H,t}-P_H}{P_H}+\eta \frac{P_t-P}{P}+\frac{C_t-C}{C}\right]$$

$$+\left(\frac{P_H}{P}\right)^{-\eta} C\left[\alpha \int_0^1(\gamma-\eta)\left[\frac{\mathcal{S}_t^i-\mathcal{S}^i}{\mathcal{S}^i}+\frac{\mathcal{S}_{i,t}-\mathcal{S}_i}{\mathcal{S}_i}\right]+\left(\eta-\frac{1}{\sigma}\right) \frac{\mathcal{Q}_{i,t}-\mathcal{Q}_i}{\mathcal{Q}_i} di\right]$$

As $\left(\frac{P}{P_H}\right)^{1-\eta}=(1-\alpha)+\alpha \int_o^1\left(\mathcal{S}_i\right)^{1-\eta} di=1$ (from international risk sharing section) in a symmetric steady-state and $C=Y$ in the international market clearing, we have

$$\frac{Y_t-Y}{Y}=-\eta \hat{p}_{H,t}+\eta \hat{p}_t+\hat{c}_t+\left[\alpha(\gamma-\eta) \int_0^1 s_t^i di+\alpha(\gamma-\eta) \int_0^1 s_{i,t} di+\alpha\left(\eta-\frac{1}{\sigma}\right) \int_0^1 q_{i,t} di\right]$$

Considering that $p_t-p_{H,t}=\alpha s_t$ and recalling that $\int_0^1 s_t^i di=0$, $s_t=\int_0^1 s_{i,t} di$ and $q_t \equiv \int_0^1 q_{i,t} di$, we have

$$y_t-y=\eta \alpha s_t+c_t-c+\alpha(\gamma-\eta) s_t+\alpha\left(\eta-\frac{1}{\sigma}\right) q_{i,t} \quad \Rightarrow \quad y_t=c_t+\alpha \gamma s_t+\alpha\left(\eta-\frac{1}{\sigma}\right) q_{i,t}$$

As $q_t=(1-\alpha) s_t$, derived in the section domestic inflation and CPI inflation, we have

$$y_t=c_t+\alpha \gamma s_t+\alpha\left(\eta-\frac{1}{\sigma}\right)(1-\alpha) s_t=c_t+\frac{\alpha}{\sigma}[\sigma \gamma+(\sigma \eta-1)(1-\alpha)] s_t=c_t+\frac{\alpha \omega}{\sigma} s_t$$

where $\omega \equiv \sigma \gamma+(\sigma \eta-1)(1-\alpha)$

As this equation holds for all countries, from a generic country we have

$$y_t^i=c_t^i+\frac{\alpha \omega}{\sigma} s_t^i$$

Aggregating over all countries we have the world market clearing condition. The equatily follows from the fact that $\int_0^1 s_t^i di=0$

$$y_t^* \equiv \int_0^1 y_t^i di=\int_0^1 c_t^i di \equiv c_t^*$$

As $c_t=c_t^*+\left(\frac{1-\alpha}{\sigma}\right) s_t$, we have that

$$y_t-\frac{\alpha \omega}{\sigma} s_t=c_t^*+\left(\frac{1-\alpha}{\sigma}\right) s_t \quad \Rightarrow \quad y_t=y_t^*+\left(\frac{1-\alpha}{\sigma}+\frac{\alpha \omega}{\sigma}\right) s_t \quad \Rightarrow \quad y_t=y_t^*+\frac{1}{\sigma_\alpha} s_t$$

where $\sigma_\alpha \equiv \frac{\sigma}{(1-\alpha)+\alpha \omega}$

Recalling that $c_t=E_t\{c_{t+1}\}-\frac{1}{\sigma}(r_t-E_t\{\pi_{t+1}\}-\rho)$, we have

$$y_t-\frac{\alpha \omega}{\sigma} s_t=E_t\{c_{t+1}\}-\frac{1}{\sigma}(r_t-E_t\{\pi_{t+1}\}-\rho) \quad \Rightarrow \quad y_t=\frac{\alpha \omega}{\sigma} s_t+E_t\{y_{t+1}\}-\frac{\alpha \omega}{\sigma} E_t\{s_{t+1}\}-\frac{1}{\sigma}(r_t-E_t\{\pi_{t+1}\}-\rho)$$

$$y_t=E_t\{y_{t+1}\}-\frac{1}{\sigma}(r_t-E_t\{\pi_{t+1}\}-\rho)-\frac{\alpha \omega}{\sigma} E_t\{\Delta s_{t+1}\}$$

As $\pi_t=\pi_{H,t}+\alpha \Delta s_t$, we have

$$y_t=E_t\{y_{t+1}\}-\frac{1}{\sigma}(r_t-E_t\{\pi_{H,t+1}\}-\alpha E_t\{\Delta s_{t+1}\}-\rho)-\frac{\alpha \omega}{\sigma} E_t\{\Delta s_{t+1}\}=E_t\{y_{t+1}\}-\frac{1}{\sigma}(r_t-E_t\{\pi_{H,t+1}\}-\rho)-\frac{\alpha(\omega-1)}{\sigma} E_t\{\Delta s_{t+1}\}$$

$$y_t=E_t\{y_{t+1}\}-\frac{1}{\sigma}(r_t-E_t\{\pi_{H,t+1}\}-\rho)-\frac{\alpha \Theta}{\sigma} E_t\{\Delta s_{t+1}\},$$

where $\Theta = \omega - 1 = \sigma\gamma + (\sigma\eta - 1)(1-\alpha) - 1 = (\sigma\gamma - 1) + (\sigma\eta - 1)(1-\alpha)$

As $\Delta s_t = \sigma_\alpha(\Delta y_t - \Delta y_t^*)$

$$y_t = E_t\{y_{t+1}\} - \frac{1}{\sigma}(r_t - E_t\{\pi_{H,t+1}\} - \rho) - \frac{\alpha\Theta}{\sigma}E_t\{\sigma_\alpha(\Delta y_{t+1} - \Delta y_{t+1}^*)\}$$

$$\sigma y_t = \sigma E_t\{y_{t+1}\} - (r_t - E_t\{\pi_{H,t+1}\} - \rho) - \alpha\Theta\sigma_\alpha(E_t\{\Delta y_{t+1}\} - E_t\{\Delta y_{t+1}^*\})$$

$$\sigma y_t = \sigma E_t\{y_{t+1}\} - (r_t - E_t\{\pi_{H,t+1}\} - \rho) - \alpha(\omega - 1)\frac{\sigma}{(1-\alpha)+\alpha\omega}(E_t\{y_{t+1}\} - y_t - E_t\{\Delta y_{t+1}^*\})$$

$$\frac{\sigma y_t - \alpha\sigma y_t + \sigma\alpha\omega y_t - \sigma\alpha\omega y_t + \alpha\sigma y_t}{(1-\alpha)+\alpha\omega} = \frac{\sigma E_t\{y_{t+1}\} - \alpha\sigma E_t\{y_{t+1}\} + \sigma\alpha\omega E_t\{y_{t+1}\} - \sigma\alpha\omega E_t\{y_{t+1}\} + \alpha\sigma E_t\{y_{t+1}\}}{(1-\alpha)+\alpha\omega}$$

$$-(r_t - E_t\{\pi_{H,t+1}\} - \rho) + \alpha(\omega - 1)\frac{\sigma}{(1-\alpha)+\alpha\omega}E_t\{\Delta y_{t+1}^*\}$$

$$\frac{\sigma y_t}{(1-\alpha)+\alpha\omega} = \frac{\sigma E_t\{y_{t+1}\}}{(1-\alpha)+\alpha\omega} - (r_t - E_t\{\pi_{H,t+1}\} - \rho) + \alpha\Theta\sigma_\alpha E_t\{\Delta y_{t+1}^*\}$$

$$\sigma_\alpha y_t = \sigma_\alpha E_t\{y_{t+1}\} - (r_t - E_t\{\pi_{H,t+1}\} - \rho) + \alpha\Theta\sigma_\alpha E_t\{\Delta y_{t+1}^*\} \quad \Rightarrow \quad y_t = E_t\{y_{t+1}\} - \frac{1}{\sigma_\alpha}(r_t - E_t\{\pi_{H,t+1}\} - \rho) + \alpha\Theta E_t\{\Delta y_{t+1}^*\}$$

Trade balance

Defining net exports in terms of domestic output $nx_t \equiv \frac{1}{Y}\left(Y_t - \frac{P_t}{P_{H,t}}C_t\right)$, defined as a fraction of steady-state output. If $\sigma = \eta = \gamma = 1$,

$$Y_t = \left(\frac{P_{H,t}}{P_t}\right)^{-1} C_t \left[(1-\alpha) + \alpha\int_0^1 \left(\mathcal{S}_{i,t}\mathcal{S}_t^i\right)^0 \mathcal{Q}^0 di\right] = \frac{P_t}{P_{H,t}}C_t = C_t\mathcal{S}_t^\alpha$$

$Y_t = \frac{P_t}{P_{H,t}}C_t \quad \Rightarrow \quad Y_t P_{H,t} = P_t C_t$, implying a balanced trade.

Log-linearizing $Y_t = C_t\mathcal{S}_t^\alpha$, we have $y_t = c_t + \alpha s_t$

A first-order approximation for nx_t is (noting that nx is zero in the steady-state), recalling that in the steady-state, $P_H = P$ and $Y = C$

$$nx_t = \frac{1}{Y}\left((Y_t - Y) - \frac{C}{P}(P_t - P) + \frac{C}{P}(P_{H,t} - P) - (C_t - C)\right) = y_t - p_t + p_{H,t} - c_t$$

As $p_t - p_{H,t} = \alpha s_t$ and $y_t = c_t + \frac{\alpha\omega}{\sigma}$, we have $nx_t = y_t - c_t - \alpha s_t = \frac{\alpha\omega}{\sigma}s_t - \alpha s_t \quad \Rightarrow \quad nx_t = \alpha\left(\frac{\omega}{\sigma} - 1\right)s_t$

In this model, nx_t iz zero if $\frac{\omega}{\sigma} - 1 = 0$, or $\frac{\sigma\gamma + (\sigma\eta - 1)(1-\alpha)}{\sigma} = 1$, which means that $\sigma(\gamma - 1) + (1-\alpha)(\sigma\eta - 1) = 0$

Marginal cost and inflation dynamics in the small open economy

The two relations below were derived in the section "Firms"

$\pi_{H,t} = \beta E_t\{\pi_{H,t+1}\} + \lambda\widehat{mc}_t$ where $\lambda \equiv \frac{(1-\beta\theta)(1-\theta)}{\theta}$

The log-linearized equation of marginal cost is $mc_t = -\nu + w_t - p_{H,t} - a_t$ where $\nu \equiv -\ln(1-\tau)$ and τ is the subsidy. Thus, $mc_t = -\nu + (w_t - p_t) - (p_{H,t} - p_t) - a_t$

From the log-linearized FOC, we have $w_t - p_t = \sigma c_t + \varphi n_t$

Thus, $mc_t = -\nu + \sigma c_t + \varphi n_t + \alpha s_t - a_t$

Using $c_t = c_t^* + \left(\frac{1-\alpha}{\sigma}\right)s_t$ and $y_t = a_t + n_t$, we have

$$mc_t = -\nu + \sigma\left[c_t^* + \left(\frac{1-\alpha}{\sigma}\right)s_t\right] + \varphi(y_t - a_t) + \alpha s_t - a_t = -\nu + \sigma c_t^* + (1-\alpha)s_t + \varphi(y_t - a_t) + \alpha s_t - a_t$$

As the world consumption is equal to its production,

$$mc_t = -\nu + \sigma y_t^* + \varphi y_t + s_t - (1+\varphi)a_t$$

Using $y_t = y_t^* + \frac{1}{\sigma_\alpha} s_t$ we can substitute for s_t in the expression above

$mc_t = -\nu + \sigma y_t^* + \varphi y_t + \sigma_\alpha(y_t - y_t^*) - (1+\varphi)a_t \quad \Rightarrow \quad mc_t = -\nu + (\sigma_\alpha + \varphi)y_t + (\sigma - \sigma_\alpha)y_t^* - (1+\varphi)a_t$

Equilibrium dynamics

The output gap is defined as $x_t \equiv y_t - \bar{y}_t$

To find The domestic natural level of output we impose $mc_t = -\mu$ and solving for domestic output as $y_t = \bar{y}_t$

$-\mu = -\nu + (\sigma_\alpha + \varphi)\bar{y}_t + (\sigma - \sigma_\alpha)y_t^* - (1+\varphi)a_t \quad \Rightarrow \quad (\sigma_\alpha + \varphi)\bar{y}_t = \nu - \mu + (1+\varphi)a_t - (\sigma - \sigma_\alpha)y_t^*$

$\bar{y}_t = \frac{\nu - \mu}{\sigma_\alpha + \varphi} + \frac{1+\varphi}{\sigma_\alpha + \varphi}a_t - \frac{\sigma - \sigma_\alpha}{\sigma_\alpha + \varphi}y_t^*$

As $\Theta = (\sigma\gamma - 1) + (\sigma\eta - 1)(1-\alpha)$, $\omega = \sigma\gamma + (\sigma\eta - 1)(1-\alpha) = \Theta + 1$ $\sigma_\alpha = \frac{\sigma}{(1-\alpha) + \alpha\omega}$, we have

$\sigma - \sigma_\alpha = \sigma - \frac{\sigma}{(1-\alpha)+\alpha\omega} = \frac{-\alpha\sigma + \alpha\omega\sigma}{(1-\alpha)+\alpha\omega}$

Let's verify that $\sigma - \sigma_\alpha = -\alpha\Theta\sigma_\alpha$

$\alpha\Theta\sigma_\alpha = \alpha(\omega - 1)\frac{\sigma}{(1-\alpha)+\alpha\omega} = \frac{\alpha\sigma\omega - \alpha\sigma}{(1-\alpha)+\alpha\omega} = \sigma - \sigma_\alpha$

Thus, $\bar{y}_t = \frac{\nu - \mu}{\sigma_\alpha + \varphi} + \frac{1+\varphi}{\sigma_\alpha + \varphi}a_t - \frac{\alpha\Theta\sigma_\alpha}{\sigma_\alpha + \varphi}y_t^* \quad \Rightarrow \quad \bar{y}_t = \Omega + \Gamma a_t + \alpha\Psi y_t^*$ where

$\Omega = \frac{\nu - \mu}{\sigma_\alpha + \varphi}$, $\Gamma = \frac{1+\varphi}{\sigma_\alpha + \varphi} > 0$ as $\sigma_\alpha > 0$ and $\Psi = -\frac{\Theta\sigma_\alpha}{\sigma_\alpha + \varphi}$

Substituting y_t in $mc_t = -\nu + (\sigma_\alpha + \varphi)y_t + (\sigma - \sigma_\alpha)y_t^* - (1+\varphi)a_t$, we have $mc_t = -\nu + (\sigma_\alpha + \varphi)(x_t + \bar{y}_t) + (\sigma - \sigma_\alpha)y_t^* - (1+\varphi)a_t$

$mc_t = -\nu + (\sigma_\alpha + \varphi)\left(x_t + \frac{\nu - \mu}{\sigma_\alpha + \varphi} + \frac{1+\varphi}{\sigma_\alpha + \varphi}a_t - \frac{\alpha\Theta\sigma_\alpha}{\sigma_\alpha + \varphi}y_t^*\right) + (\sigma - \sigma_\alpha)y_t^* - (1+\varphi)a_t = -\nu + (\sigma_\alpha + \varphi)x_t + \nu - \mu - \alpha\Theta\sigma_\alpha y_t^* + (\sigma - \sigma_\alpha)y_t^*$

$mc_t - (-\mu) = (\sigma_\alpha + \varphi)x_t - \mu - \alpha\left[\omega - 1\right]\frac{\sigma}{(1-\alpha)+\alpha\omega}y_t^* + \frac{-\alpha\sigma + \alpha\omega\sigma}{(1-\alpha)+\alpha\omega}y_t^* + \mu$

$\widehat{mc_t} = (\sigma_\alpha + \varphi)x_t + \frac{-\alpha\omega\sigma + \alpha\sigma}{(1-\alpha)+\alpha\omega}y_t^* + \frac{-\alpha\sigma + \alpha\omega\sigma}{(1-\alpha)+\alpha\omega}y_t^* \quad \Rightarrow \quad \widehat{mc_t} = (\sigma_\alpha + \varphi)x_t$

Substituting the equation above into $\pi_{H,t} = \beta E_t\{\pi_{H,t+1} + \lambda\widehat{mc}_t\}$, we have the a version of the New Keynesian Phillips Curve (NKPC)

$\pi_{H,t} = \beta E_t\{\pi_{H,t+1}\} + \lambda(\sigma_\alpha + \varphi)x_t = \beta E_t\{\pi_{H,t+1}\} + \kappa_\alpha x_t$, where $\kappa_\alpha \equiv \lambda(\sigma_\alpha + \varphi)$

Substituting y_t and $E_t\{y_{t+1}\}$ in $y_t = E_t\{y_{t+1}\} - \frac{1}{\sigma_\alpha}(r_t - E_t\{\pi_{H,t+1}\} - \rho) + \alpha\Theta E_t\{\Delta y_{t+1}^*\}$, we have

$$x_t + \Omega + \Gamma a_t + \alpha\Psi y_t^* = E_t\{x_{t+1} + \Omega + \Gamma a_{t+1} + \alpha\Psi y_{t+1}^*\} - \frac{1}{\sigma_\alpha}(r_t - E_t\{\pi_{H,t+1}\} - \rho) + \alpha\Theta E_t\{\Delta y_{t+1}^*\}$$

$x_t = E_t\{x_{t+1}\} + \Gamma(\rho_a a_t - a_t) + \alpha\Psi E_t\{\Delta y_{t+1}^*\} - \frac{1}{\sigma_\alpha}(r_t - E_t\{\pi_{H,t+1}\} - \rho) + \alpha\Theta E_t\{\Delta y_{t+1}^*\}$

$x_t = E_t\{x_{t+1}\} - \frac{1}{\sigma_\alpha}(r_t - E_t\{\pi_{H,t+1}\} - [\rho - \sigma_\alpha\Gamma(1-\rho_a)a_t + \alpha\sigma_\alpha(\Theta + \Psi)E_t\{\Delta y_{t+1}^*\}])$

$x_t = E_t\{x_{t+1}\} - \frac{1}{\sigma_\alpha}(r_t - E_t\{\pi_{H,t+1}\} - \overline{rr}_t)$ where $\overline{rr}_t \equiv \rho - \sigma_\alpha\Gamma(1-\rho_a)a_t + \alpha\sigma_\alpha(\Theta + \Psi)E_t\{\Delta y_{t+1}^*\}$ is the small open economy's natural rate of interest.

Optimal Monetary Policy: a special case

The problem of the central planner is:

$\max E_0 \sum_{i=0}^{\infty} \beta^t U(C_t, N_t)$ subject to $Y_t = A_t N_t$, $C_t = C_t^i \mathcal{Q}_{i,t}$ and $Y_t = \frac{P_t}{P_{H,t}}C_t$, as $\eta = \sigma = \gamma = 1$.

In this case, $Y_t = \frac{(P_{H,t})^{1-\alpha}(P_{F,t})^{\alpha}}{P_{H,t}}C_t = \frac{(P_{F,t})^{\alpha}}{(P_{H,t})^{\alpha}}C_t = \mathcal{S}_t^{\alpha}C_t \quad \Rightarrow \quad Y_t = \mathcal{S}_t^{\alpha}C_t \quad \Rightarrow \quad \mathcal{S}_t = \left(\frac{Y_t}{C_t}\right)^{\frac{1}{\alpha}}$. By equation 29, we have $y_t = y_t^* + s_t$ or $Y_t = Y_t^*\mathcal{S}_t = Y_t^*\left(\frac{Y_t}{C_t}\right)^{\frac{1}{\alpha}} \quad \Rightarrow \quad C_tY_t^{\alpha} = (Y_t^*)^{\alpha}Y_t \quad \Rightarrow \quad C_t = Y_t^{1-\alpha}(Y_t^*)^{\alpha}$.

The Central planner problem then becomes

$E_0 \max \sum_{i=0}^{\infty} \beta^t U(C_t, N_t)$ subject to $Y_t = A_tN_t$ and $C_t = Y_t^{1-\alpha}(Y_t^*)^{\alpha}$.

The Lagrangean can be written as

$$\mathcal{L} = E_0 \sum_{i=0}^{\infty} \beta^t \left\{U(C_t, N_t) + \Lambda_t(A_tN_t - Y_t) + \Phi_t\left(Y_t^{1-\alpha}(Y_t^*)^{\alpha} - C_t\right)\right\}$$

which yields the FOCS:

$(C_t)\ U_c(C_t, N_t) = \Phi_t$

$(N_t)\ U_N(C_t, N_t) = -\Lambda_tA_t$

$(Y_t)\ \Phi_t(1-\alpha)Y_t^{-\alpha}(Y_t^*)^{\alpha} = \Lambda_t$

$U_N(C_t, N_t) = -A_t\Phi_t(1-\alpha)Y_t^{-\alpha}(Y_t^*)^{\alpha} = -A_tU_c(C_t, N_t)(1-\alpha)Y_t^{-\alpha}(Y_t^*)^{\alpha}$

$-\frac{U_N(C_t, N_t)}{U_c(C_t, N_t)} = \frac{Y_t}{N_t}(1-\alpha)Y_t^{-\alpha}(Y_t^*)^{\alpha} = (1-\alpha)\frac{C_t}{N_t}$

Using the utility function $U(C_t, N_t) = \log(C_t) - \frac{N_t^{1+\varphi}}{1+\varphi}$ as $\sigma = 1$.

We have $U_C(C_t, N_t) = \frac{1}{C_t}$ and $U_N(C_t, N_t) = -N_t^{\varphi}$. Substituting in the FOCs relation, we have $N_t^{\varphi}C_t = (1-\alpha)\frac{C_t}{N_t} \quad \Rightarrow \quad N_t^{1+\varphi} = (1-\alpha) \quad \Rightarrow \quad N_t = N = (1-\alpha)^{\frac{1}{1+\varphi}}$, which is a constant employment.

We know that, under flexible prices, $\frac{\overline{MC}_t^n}{\overline{P}_{H,t}} = \overline{MC}_t = \frac{\varepsilon-1}{\varepsilon}$ (see section 9 for the derivation).

From the representative consumer FOCs (problem of item 5), we have the standard relation: $-\frac{U_C(\overline{C}_t, \overline{N}_t)}{U_N(\overline{C}_t, \overline{N}_t)} = \frac{\overline{W}_t}{\overline{P}_t}$. Thus,

The subsidy τ is chosen to achieve the optimal level of production if the prices were fully flexible. Thus, we have $\overline{MC}_t = \frac{\varepsilon-1}{\varepsilon} = \frac{\overline{W}_t(1-\tau)}{\overline{P}_{H,t}A_t} = -\frac{U_C(\overline{C}_t, \overline{N}_t)}{U_N(\overline{C}_t, \overline{N}_t)}\frac{\overline{P}_t(1-\tau)}{\overline{P}_{H,t}A_t}$.

As $\eta = 1$, $\overline{P}_t = (\overline{P}_{H,t})^{1-\alpha}(\overline{P}_{F,t})^{\alpha}$. Substituting,

$$\overline{MC}_t = -\frac{U_C(\overline{C}_t, \overline{N}_t)}{U_N(\overline{C}_t, \overline{N}_t)}\frac{(\overline{P}_{H,t})^{1-\alpha}(\overline{P}_{F,t})^{\alpha}(1-\tau)}{\overline{P}_{H,t}\overline{A}_t} = -\frac{U_C(\overline{C}_t, \overline{N}_t)}{U_N(\overline{C}_t, \overline{N}_t)}\left(\frac{\overline{P}_{F,t}}{\overline{P}_{H,t}}\right)^{\alpha}\frac{1-\tau}{\overline{A}_t} = -\frac{(1-\tau)}{\overline{A}_t}(\overline{\mathcal{S}}_i)^{\alpha}\frac{U_C(\overline{C}_t, \overline{N}_t)}{U_N(\overline{C}_t, \overline{N}_t)}$$

As $\sigma = 1$, $-\frac{U_C(\overline{C}_t, \overline{N}_t)}{U_N(\overline{C}_t, \overline{N}_t)} = \overline{C}_t\overline{N}_t^{\varphi} \quad \Rightarrow \quad \overline{MC}_t = \frac{(1-\tau)}{\overline{A}_t}(\overline{\mathcal{S}}_i)^{\alpha}\overline{C}_t\overline{N}_t^{\varphi}$

From equation 26 (equilibrium part), we know that $\overline{Y}_t = \overline{C}_t\overline{\mathcal{S}}_t^{\alpha} \quad \Rightarrow \quad \overline{\mathcal{S}}_t^{\alpha} = \frac{\overline{Y}_t}{\overline{C}_t}$. Thus, $\overline{MC}_t = \frac{(1-\tau)}{\overline{A}_t}\frac{\overline{Y}_t}{\overline{C}_t}\overline{C}_t\overline{N}_t^{\varphi}$

Substituting the technological constraint, $1 - \frac{1}{\varepsilon} = \overline{MC}_t = \frac{(1-\tau)}{\overline{A}_t}\overline{A}_t\overline{N}_t\overline{N}_t^{\varphi} = (1-\tau)\overline{N}_t^{1+\varphi} = (1-\tau)\left((1-\alpha)^{1+\varphi}\right)^{\frac{1}{1+\varphi}} = (1-\tau)(1-\alpha)$.

Hence, if τ is set to satisfy $(1-\tau)(1-\alpha) = 1 - \frac{1}{\varepsilon}$, we have also the log-linear form $\nu = \mu + \log(1-\alpha)$, where μ (defined earlier) is $\log(1-\tau)$ and the flexible price allocation is guaranteed.

Optimal policy Implementation

Solving forward equation (36), we have $\pi_{H,t} = \beta E_t\{\pi_{H,t+1}\} + \kappa_\alpha x_t = \beta E_t\{\beta E_t\{\pi_{H,t+2}\} + \kappa_\alpha x_{t+1}\} + \kappa_\alpha x_t = \beta^T E_t\{\pi_{H,T}\} + \sum_{j=t}^{T} \beta^{j-t}\kappa_\alpha x_j$

To stabilize inflation,

$\pi_{H,t} = \lim_{T\to\infty} \beta^T E_t\{\pi_{H,T}\} + \sum_{j=t}^{T} \beta^{j-t}\kappa_\alpha x_j$ which will stabilized ($\pi_{H,t} = 0$) only if the output gap is zero for every period. As $x_t \equiv y_t - \overline{y}_t \quad \Rightarrow \quad y_t = \overline{y}_t$.

In equation (37) we have $x_t = E_t\{x_{t+1}\} - \frac{1}{\sigma_\alpha}(r_t - E_t\{\pi_{H,t+1}\} - \overline{rr}_t)$. In this case, we have $E_t\{x_{t+1}\} = 0$, $E_t\{\pi_{H,t+1}\} = 0$, and $\sigma_\alpha = 1$, as

$$\sigma_\alpha \equiv \frac{\sigma}{(1-\alpha)+\alpha\omega} = \frac{1}{(1-\alpha)+\alpha(\sigma\gamma+(\sigma\eta-1)(1-\alpha))} = \frac{1}{(1-\alpha)+\alpha(1+(1-1)(1-\alpha))} = 1.$$

Thus, $r_t = \overline{rr}_t$.

Now we suppose that the Central Bank follows this rule when the economy is not in its steady-state: $r_t = \overline{rr}_t + \phi_\pi \pi_{H,t} + \phi_x x_t$.

After setting $r_t = \overline{rr}_t$ in a closed economy we have $x_t = E_t\{x_{t+1}\} - \frac{1}{\sigma_\alpha}(r_t - E_t\{\pi_{H,t+1}\} - \overline{rr}_t) = E_t\{x_{t+1}\} - \frac{1}{\sigma}(\overline{rr}_t - E_t\{\pi_{H,t+1}\} - \overline{rr}_t) = E_t\{x_{t+1}\} - \frac{1}{\sigma}E_t\{\pi_{H,t+1}\}$

We have now a system with 2 equations

$x_t = E_t\{x_{t+1}\} + \frac{1}{\sigma}E_t\{\pi_{H,t+1}\}$

$\pi_{H,t} = \beta E_t\{\pi_{H,t+1}\} + \kappa_\alpha x_t,$

which can be summarized as $\begin{bmatrix} 1 & 0 \\ -\kappa & 1 \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} 1 & \sigma^{-1} \\ 0 & \beta \end{bmatrix} \begin{bmatrix} E_t\{x_{t+1}\} \\ E_t\{\pi_{t+1}\} \end{bmatrix}$

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \kappa & 1 \end{bmatrix} \begin{bmatrix} 1 & \sigma^{-1} \\ 0 & \beta \end{bmatrix} \begin{bmatrix} E_t\{x_{t+1}\} \\ E_t\{\pi_{t+1}\} \end{bmatrix} = \begin{bmatrix} 1 & \sigma^{-1} \\ \kappa & \kappa\sigma^{-1}+\beta \end{bmatrix} \begin{bmatrix} E_t\{x_{t+1}\} \\ E_t\{\pi_{t+1}\} \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} 1 & \sigma^{-1} \\ \kappa & \beta+\kappa\sigma^{-1} \end{bmatrix} \begin{bmatrix} E_t\{x_{t+1}\} \\ E_t\{\pi_{t+1}\} \end{bmatrix}.$$

Defining $\mathbf{A_O} = \begin{bmatrix} 1 & \sigma^{-1} \\ \kappa & \beta+\kappa\sigma^{-1} \end{bmatrix}$, we can calculate its eigenvalues:

$$\begin{vmatrix} 1-\xi & \sigma^{-1} \\ \kappa & \beta+\kappa\sigma^{-1}-\xi \end{vmatrix} = (1-\xi)(\beta+\kappa\sigma^{-1}-\xi) - \kappa\sigma^{-1} = \beta+\kappa\sigma^{-1}-\xi-\beta\xi-\kappa\sigma^{-1}\xi+\xi^2-\kappa\sigma^{-1}$$

Now we have a quadratic function whose roots are the eigenvalues of the system $f(\xi) = \xi^2 - (1+\beta+\kappa\sigma^{-1})\xi + \beta$,

with the product of roots being $\beta < 1$ and the sum of the roots greater than 1 $1+\beta+\kappa\sigma^{-1}$.

We can see that if $f(\xi) > 0$ if $\xi \to \infty$, $f(0) = \beta > 0$ and $f(1) = 1^2 - (1+\beta+\kappa\sigma^{-1})1 + \beta = -\kappa\sigma^{-1} < 0$, as $\kappa = \lambda(\sigma+\varphi) = \frac{(1-\beta\theta)(1-\theta)}{\theta}(\sigma+\varphi) > 0$. Thus, we have one root between 0 and 1 and another one greater than 1 (from the intermediate value theorem). As there's one eigenvalue outside the unit root circle, there are infinite solution for this system, as both variables are forward looking.

Now if the Central bank has commits to the rule $r_t = \overline{rr}_t + \phi_\pi\pi_t + \phi_x x_t$, we have

$$x_t = E_t\{x_{t+1}\} - \frac{1}{\sigma}(\overline{rr}_t + \phi_\pi\pi_t + \phi_x x_t - E_t\{\pi_{t+1}\} - \overline{rr}_t) = -\phi_\pi\sigma^{-1} - \pi_t + \phi_x\sigma^{-1}x_t + E_t\{x_{t+1}\} + \sigma^{-1}E_t\{\pi_{t+1}\}$$

and the system becomes

$(1+\phi_x\sigma^{-1})x_t + \phi_\pi\sigma^{-1}\pi_t = E_t\{x_{t+1}\} + \sigma^{-1}E_t\{\pi_{t+1}\}$

$-\kappa x_t + \pi_{H,t} = \beta E_t\{\pi_{H,t+1}\},$

$$\begin{bmatrix} 1+\phi_x\sigma^{-1} & \phi_\pi\sigma^{-1} \\ -\kappa & 1 \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} 1 & \sigma^{-1} \\ 0 & \beta \end{bmatrix} \begin{bmatrix} E_t\{x_{t+1}\} \\ E_t\{\pi_{t+1}\} \end{bmatrix}$$

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \frac{1}{1+\phi_x\sigma^{-1}+\phi_\pi\sigma^{-1}\kappa} \begin{bmatrix} 1 & -\phi_\pi\sigma^{-1} \\ \kappa & 1+\phi_x\sigma^{-1} \end{bmatrix} \begin{bmatrix} 1 & \sigma^{-1} \\ 0 & \beta \end{bmatrix} \begin{bmatrix} E_t\{x_{t+1}\} \\ E_t\{\pi_{t+1}\} \end{bmatrix} = \frac{\sigma}{\sigma+\phi_x+\phi_\pi\kappa} \begin{bmatrix} 1 & \sigma^{-1}(1-\beta\phi_\pi) \\ \kappa & \sigma^{-1}(\kappa+\beta\sigma+\beta\phi_x) \end{bmatrix} \begin{bmatrix} E_t\{x_{t+1}\} \\ E_t\{\pi_{t+1}\} \end{bmatrix}$$

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \frac{1}{\sigma+\phi_x+\phi_\pi\kappa}\begin{bmatrix} \sigma & 1-\beta\phi_\pi \\ \kappa\sigma & \kappa+\beta(\sigma+\phi_x) \end{bmatrix}\begin{bmatrix} E_t\{x_{t+1}\} \\ E_t\{\pi_{t+1}\} \end{bmatrix} = \mathbf{A}_T \begin{bmatrix} E_t\{x_{t+1}\} \\ E_t\{\pi_{t+1}\} \end{bmatrix}, \text{ where}$$

$$\mathbf{A}_T \equiv \Omega \begin{bmatrix} \sigma & 1-\beta\phi_\pi \\ \kappa\sigma & \kappa+\beta(\sigma+\phi_x) \end{bmatrix} \text{ and } \Omega \equiv \frac{1}{\sigma+\phi_x+\phi_\pi\kappa}$$

If we restrict $\phi_\pi > 0$ and $\phi_x > 0$, $\Omega > 0$.

To satisfy the Blanchard and Khan conditions, we need that both eigenvalues are inside the unit circle.

$$|\mathbf{A}_T - \lambda\mathbf{I}| = 0$$

$$\left| \frac{1}{\sigma+\phi_x+\phi_\pi\kappa}\begin{bmatrix} \sigma & 1-\beta\phi_\pi \\ \kappa\sigma & \kappa+\beta(\sigma+\phi_x) \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} \frac{\sigma}{\sigma+\phi_x+\phi_\pi\kappa} - \lambda & \frac{1-\beta\phi_\pi}{\sigma+\phi_x+\phi_\pi\kappa} \\ \frac{\kappa\sigma}{\sigma+\phi_x+\phi_\pi\kappa} & \frac{\kappa+\beta(\sigma+\phi_x)}{\sigma+\phi_x+\phi_\pi\kappa} - \lambda \end{bmatrix} \right| = 0$$

$$\frac{\sigma\kappa+\beta\sigma(\sigma+\phi_y)}{(\sigma+\phi_x+\phi_\pi\kappa)^2} - \frac{\sigma+\kappa+\beta(\sigma+\phi_y)}{(\sigma+\phi_x+\phi_\pi\kappa)}\lambda + \lambda^2 - \frac{\sigma\kappa-\beta\phi_\pi\sigma\kappa}{(\sigma+\phi_x+\phi_\pi\kappa)^2} = \lambda^2 - \frac{\sigma+\kappa+\beta(\sigma+\phi_y)}{(\sigma+\phi_x+\phi_\pi\kappa)}\lambda + \frac{\sigma\beta(\sigma+\phi_y+\phi_\pi\kappa)}{(\sigma+\phi_x+\phi_\pi\kappa)^2} = 0$$

$$\lambda^2 - \frac{\sigma+\kappa+\beta(\sigma+\phi_y)}{\sigma+\phi_x+\phi_\pi\kappa}\lambda + \frac{\sigma\beta}{\sigma+\phi_x+\phi_\pi\kappa} = 0$$

LaSalle (1986) showed that both roots of the equation $x^2+bx+c=0$ are less than 1 if and only if $|c| < 1$ and $|b| < 1+c$. This comment was taken from Drago Bergholt notes.

We have that $\left|\frac{\sigma\beta}{\sigma+\phi_y+\phi_\pi\kappa}\right| < 1$

As $\sigma > 0$, $\beta > 0$, $\kappa > 0$, $\phi_\pi > 0$ and $\phi_y > 0$

$\frac{\sigma\beta}{\sigma+\phi_y+\phi_\pi\kappa} < 1 \quad\Rightarrow\quad \sigma\beta < \sigma+\phi_y+\phi_\pi\kappa \quad\Rightarrow\quad \sigma(\beta-1) < \phi_y+\phi_\pi\kappa$ This condition is always satisfied, as $\beta < 1$.

The second condition is $\left|\frac{\sigma+\kappa+\beta(\sigma+\phi_y)}{\sigma+\phi_y+\phi_\pi\kappa}\right| < 1 + \frac{\sigma\beta}{\sigma+\phi_y+\phi_\pi\kappa}$

$$\frac{\sigma+\kappa+\beta(\sigma+\phi_y)}{\sigma+\phi_y+\phi_\pi\kappa} < 1 + \frac{\sigma\beta}{\sigma+\phi_y+\phi_\pi\kappa}$$

$$\sigma+\kappa+\beta(\sigma+\phi_y) < \sigma+\phi_y+\phi_\pi\kappa+\sigma\beta \quad\Rightarrow\quad \kappa+\beta\phi_y < \phi_y+\phi_\pi\kappa \quad\Rightarrow\quad \kappa(\phi_\pi-1)+\phi_y(1-\beta) > 0$$

Macroeconomic implications

By equation (35) $\overline{y}_t = \Omega + \Gamma a_t - \alpha\Psi y_t^* = \frac{\nu-\mu}{\sigma_\alpha+\varphi} + \frac{1+\varphi}{\sigma_\alpha+\varphi}a_t - \alpha\frac{\Theta\sigma_\alpha}{\sigma_\alpha+\varphi}y_t^*$, we can see that a technological shock always increase the output level, as $\sigma_\alpha > 0$ and $\varphi > 0$.

Computing the natual level of the terms of trade, we have

$$\overline{s}_t = \sigma_\alpha(\overline{y}_t - y_t^*) = \sigma_\alpha\left(\frac{\nu-\mu}{\sigma_\alpha+\varphi} + \frac{1+\varphi}{\sigma_\alpha+\varphi}a_t - \alpha\frac{\Theta\sigma_\alpha}{\sigma_\alpha+\varphi}y_t^* - y_t^*\right) = \sigma_\alpha\left(\frac{\nu-\mu}{\sigma_\alpha+\varphi} + \frac{1+\varphi}{\sigma_\alpha+\varphi}a_t - \frac{\alpha\Theta\sigma_\alpha+\sigma_\alpha+\varphi}{\sigma_\alpha+\varphi}y_t^*\right)$$

$$\overline{s}_t = \sigma_\alpha\left(\frac{\nu-\mu}{\sigma_\alpha+\varphi} + \frac{1+\varphi}{\sigma_\alpha+\varphi}a_t - \frac{\alpha(\omega-1)\sigma_\alpha+\sigma_\alpha+\varphi}{\sigma_\alpha+\varphi}y_t^*\right) = \sigma_\alpha\left(\frac{\nu-\mu}{\sigma_\alpha+\varphi} + \frac{1+\varphi}{\sigma_\alpha+\varphi}a_t - \frac{(\alpha\omega-\alpha+1)\frac{\sigma}{1-\alpha+\alpha\omega}+\varphi}{\sigma_\alpha+\varphi}y_t^*\right)$$

$$\overline{s}_t = \sigma_\alpha\left(\frac{\nu-\mu}{\sigma_\alpha+\varphi} + \frac{1+\varphi}{\sigma_\alpha+\varphi}a_t - \frac{\sigma+\varphi}{\sigma_\alpha+\varphi}y_t^*\right) = \sigma_\alpha\Omega + \sigma_\alpha\Gamma a_t - \sigma_\alpha\Phi y_t^*, \text{ where } \Phi = \frac{\sigma+\varphi}{\sigma_\alpha+\varphi} > 0$$

As $\overline{p}_t = (1-\alpha)\overline{p}_{H,t} + \alpha\overline{p}_{F,t} = (1-\alpha)\overline{p}_{H,t} + \alpha(\overline{e}_t + p_t^*)$. As domestic prices are fully stabilized, $(1-\alpha)\overline{p}_{H,t}$ is a constant, so $\overline{p}_t$ is proportional to $\alpha(\overline{e}_t + p_t^*) = \alpha\overline{s}_t$

The welfare costs of deviations from the optimal

This section starts with the calculations of the appendix D. By the Taylor's rule, the second order approximation is: $f(x_t) = f(a) + f'(a)(x_t - a) + \frac{1}{2}f''(a)(x_t - a)^2 + \frac{1}{6}f'''(\widetilde{a})(x_t - a)^3$. We assume that the third term is small, as the deviations from the steady-state are assumed to be small (by the intermediate value theorem, $\widetilde{a}$ is between x_t and a.

We can take

$\frac{Y_t}{Y} = e^{\ln \frac{Y_t}{Y}} = 1 + \ln \frac{Y_t}{Y} + \frac{1}{2}\left(\ln \frac{Y_t}{Y}\right)^2 + \frac{1}{3!}\left(\ln \frac{Y_t}{Y}\right)^3 + ... = 1 + y_t + \frac{1}{2}(y_t)^2 + o(||a||^n)$, there a is the bound for the high order terms.

$$\frac{Y_t - Y}{Y} = 1 + y_t + \frac{y_t^2}{2} + o(||a||^n)$$

Combining equations (18): $c_t = c_t^* + \left(\frac{1-\alpha}{\sigma}\right)s_t$ and (29): $y_t = y_t^* + \frac{1}{\sigma_\alpha}s_t = y_t^* + \frac{1}{\sigma}s_t$ as $\omega = 1$, we have:

$c_t = c_t^* + \left(\frac{1-\alpha}{\sigma}\right)\sigma(y_t - y_t^*)$. As $c_t^* = y_t^*$ because of global market clearing, $c_t = y_t^* + y_t - y_t^* - \alpha y_t + \alpha y_t^* \quad \Rightarrow \quad c_t = (1-\alpha)y_t + \alpha y_t^*$.

As $x_t \equiv y_t - \overline{y}_t$, in the stabilized economy, $x_t = 0$ and $y_t = \overline{y}_t$. Thus, $\overline{c}_t = (1-\alpha)(0 - \overline{y}_t) + \alpha y_t^* = \alpha y_t^* - (1-\alpha)\overline{y}_t$

Substituting, we get: $c_t = (1-\alpha)(\overline{y}_t + x_t) + \alpha y_t^* = (1-\alpha)\overline{y}_t + \alpha y_t^* + (1-\alpha)x_t \quad \Rightarrow \quad c_t = \overline{c}_t + (1-\alpha)x_t$.

Expanding the log-deviation of the disutility of work, we have:

$$\left(\frac{N_t}{\overline{N}}\right)^{1+\varphi} = \exp[(1+\varphi)\widetilde{n}] = 1 + (1+\varphi)\widetilde{n}_t + \frac{1}{2}\widetilde{n}_t^2 + o(||a||^3) \quad \Rightarrow \quad N_t^{1+\varphi} = \overline{N}^{1+\varphi}\left(1 + (1+\varphi)\widetilde{n}_t + \frac{1}{2}\widetilde{n}_t^2 + o(||b||^3)\right)$$

$$\frac{N_t^{1+\varphi}}{1+\varphi} = \frac{\overline{N}^{1+\varphi}}{1+\varphi} + \overline{N}^{1+\varphi}\left[\widetilde{n}_t + \frac{1}{2}(1+\varphi)\widetilde{n}_t^2\right] + o(||a||^3)$$

Using the fact that $N_t = \left(\frac{Y_t}{A_t}\right)\int_0^1 \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} di$, $\int_0^1 \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} di = \frac{N_t A_t}{Y_t} \quad \Rightarrow \quad \log \int_0^1 \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} di = \log\left(\frac{N_t A_t}{Y_t}\right) = n_t + a_t - y_t$

If we define $z_t \equiv \log \int_0^1 \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} di$, then $z_t = n_t + a_t - y_t = n_t + a_t - (\overline{y}_t + x_t)$.

When prices are stabilized, $P_{H,t}(i) = P_{H,t}$ and $\overline{z}_t = 0$. Also, there are no productivity shocks, so $a_t = \overline{y}_t - \overline{n}_t$. Thus,

$z_t = \overline{n}_t + \widetilde{n}_t + \overline{y}_t - \overline{n}_t - (\overline{y}_t + x_t) \quad \Rightarrow \quad \widetilde{n}_t = z_t + x_t$

Lemma 1 (appendix D): The proof is there. There's just one passage that it took time to figure out what happened. From

From $E_i\{\widehat{p}_{H,t}(i)\} = \frac{(\varepsilon - 1)}{2}E_i\{\widehat{p}_{H,t}(i)^2\}$ and $\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} = 1 - \varepsilon\widehat{p}_{H,t}(i) + \frac{\varepsilon^2}{2}\widehat{p}_{H,t}(i)^2 + o(||a||^3)$

$$E_i\left[\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon}\right] = E_i\left[1 - \varepsilon\widehat{p}_{H,t}(i) + \frac{\varepsilon^2}{2}\widehat{p}_{H,t}(i)^2 + o(||a||^3)\right]$$

$\int_0^1 \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} di = 1 - \varepsilon e_i[\widehat{p}_{H,t}(i)] + \frac{\varepsilon^2}{2}E_i[\widehat{p}_{H,t}(i)^2] = 1 - \varepsilon\frac{(\varepsilon-1)}{2}E_i\{\widehat{p}_{H,t}(i)^2\} + \frac{\varepsilon^2}{2}E_i[\widehat{p}_{H,t}(i)^2] = 1 + \frac{\varepsilon}{2}E_i\{\widehat{p}_{H,t}(i)^2\} = 1 + \frac{\varepsilon}{2}var_i\{p_{H,t}(i)\}$

$$z_t = \log \int_0^1 \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} di = \log\left(1 + \frac{\varepsilon}{2}var_i\{p_{H,t}(i)\}\right) = \frac{\varepsilon}{2}var_i\{p_{H,t}(i)\} + o(||a||^3)$$

Rewriting the second-order approximation disutility of labor:

$$\frac{N_t^{1+\varphi}}{1+\varphi} = \frac{\overline{N}^{1+\varphi}}{1+\varphi} + \overline{N}^{1+\varphi}\left[x_t + z_t + \frac{1}{2}(1+\varphi)(x_t + z_t)^2\right] + o(||a||^3)$$

$$\frac{N_t^{1+\varphi}}{1+\varphi} = \frac{\overline{N}^{1+\varphi}}{1+\varphi} + \overline{N}^{1+\varphi}\left[x_t + z_t + \frac{1}{2}(1+\varphi)(x_t^2 + 2x_t z_t + z_t^2)\right] + o(||a||^3)$$

As z_t is a variance (second-order term), z_t^2 and $x_t z_t$ are terms of greater order, so they can be included in the remanescent terms $o(||a||^3)$. Thus, we get

$$\frac{N_t^{1+\varphi}}{1+\varphi} = \frac{\overline{N}^{1+\varphi}}{1+\varphi} + \overline{N}^{1+\varphi}\left[x_t + z_t + \frac{1}{2}(1+\varphi)x_t^2\right] + o(||a||^3).$$

Under the optimal subsidy assumption, from the consumer's FOC, we have that $\overline{N}_t^{1+\varphi} = (1-\alpha)$ (constant employment). Thus,

$$U(C_t, N_t) \equiv \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{\overline{N}^{1+\varphi}}{1+\varphi} - \overline{N}^{1+\varphi}\left[x_t + z_t + \frac{1}{2}(1+\varphi)x_t^2\right] + o(||a||^3)$$

$$U(C_t, N_t) = -\frac{(1-\alpha)}{1+\varphi} - (1-\alpha)\left[x_t + z_t + \frac{1}{2}(1+\varphi)x_t^2\right] + \frac{C_t^{1-\sigma}}{1-\sigma} + o(||a||^3)$$

$$U(C_t, N_t) = -(1-\alpha)\left[x_t + z_t + \frac{1}{2}(1+\varphi)x_t^2\right] + \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{(1-\alpha)}{1+\varphi}o(||a||^3) = -(1-\alpha)\left[z_t + \frac{1}{2}(1+\varphi)x_t^2\right] + t.i.p + o(||a||^3),$$ there t.i.p. denotes terms independent of policy and under the optimal policy, $x_t = 0$.

Lemma 2 from Woodford(2003): $\sum_{t=0}^{\infty}\beta^t var_i[p_{H,t}(i)] = \frac{1}{\lambda}\sum_{t=0}^{\infty}\beta^t \pi_{H,t}^2$, where $\lambda \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta}$

$$\mathbb{W} = \sum_{i=0}^{\infty}\beta^t U(C_t, N_t) = \sum_{i=0}^{\infty}\beta^t\left[-(1-\alpha)\left(z_t + \frac{1}{2}(1+\varphi)x_t^2\right) + t.i.p + o(||a||^3)\right]$$

$$\mathbb{W} = \sum_{i=0}^{\infty}\beta^t\left[-(1-\alpha)\left(\frac{\varepsilon}{2}var_i[p_{H,t}(i)] + o(||a||^3) + \frac{1}{2}(1+\varphi)x_t^2\right)\right] + t.i.p + o(||a||^3).$$

Using Lemma 2, we have:

$$\mathbb{W} = -\frac{(1-\alpha)}{2}\sum_{i=0}^{\infty}\beta^t\left[\frac{\varepsilon}{\lambda}\pi_{H,t}^2 + (1+\varphi)x_t^2\right] + t.i.p + o(||a||^3)$$

Now $\mathbb{E}_t\left[-\frac{(1-\alpha)}{2}\sum_{i=0}^{\infty}\beta^t\left[\frac{\varepsilon}{\lambda}\pi_{H,t}^2 + (1+\varphi)x_t^2\right] + t.i.p + o(||a||^3)\right] = -\frac{(1-\alpha)}{2}\sum_{i=0}^{\infty}\beta^t\left[\frac{\varepsilon}{\lambda}var(\pi_{H,t}) + (1+\varphi)var(x_t)\right]$ as $E[x_t] = E[\pi_t] = 0$

If $\beta \rightarrow 1$, any policy deviation in a period can be calculated as

$$\mathbb{V} = -\frac{(1-\alpha)}{2}\sum_{i=0}^{\infty}\beta^t\left[\frac{\varepsilon}{\lambda}var(\pi_{H,t}) + (1+\varphi)var(x_t)\right]$$