Annex - Detailed derivation of all equations of paper “Monetary Policy and Exchange Rate Volatility in a Small Open Economy”

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1 - Problem of the consumer

Equation (1) - The representative household

$$\max_{C_t,N_t} \quad E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t)$$

This is the problem of the representative consumer in each economy, where C_t represents the consumption basket and N_t denotes the hours of labour.

Equation (2) - Consumption index

$$C_t = \left[(1-\alpha)^{\frac{1}{\eta}} (C_{H,t})^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}} (C_{F,t})^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}$$

This is the composite consumption index, which the representative consumer maximizes. Substituting in equation (1), the representative consumer problem becomes

$$\max_{C_t,N_t} \quad E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t) = \max_{C_{H,t},C_{F,t},N_t} \quad E_0 \sum_{t=0}^{\infty} \beta^t U\left(\left[(1-\alpha)^{\frac{1}{\eta}} (C_{H,t})^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}} (C_{F,t})^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}, N_t\right).$$

subject to the budget constraint specified in equation (3) below, where

$$C_{H,t} \equiv \left(\int_0^1 C_{H,t}(j)^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}}, \; C_{F,t} \equiv \left(\int_0^1 (C_{i,t})^{\frac{\gamma-1}{\gamma}} di\right)^{\frac{\gamma}{\gamma-1}}, \; C_{i,t} \equiv \left(\int_0^1 C_{i,t}(j)^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}}$$

Substituting the expressions above and using equations (1) and (2) the representative consumer problem can be written as:

$$\max_{C_{H,t}(j),C_{i,t},N_t} \quad E_0 \sum_{t=0}^{\infty} \beta^t U\left(\left[(1-\alpha)^{\frac{1}{\eta}}\left[\left(\int_0^1 C_{H,t}(j)^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}}\right]^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}}\left[\left(\int_0^1 (C_{i,t})^{\frac{\gamma-1}{\gamma}} di\right)^{\frac{\gamma}{\gamma-1}}\right]^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}, N_t\right) =$$

$$\max_{C_{H,t}(j),C_{i,t}(j),N_t}$$

$$E_0 \textstyle\sum_{t=0}^{\infty} \beta^t U\left(\left[(1-\alpha)^{\frac{1}{\eta}}\left[\left(\int_0^1 C_{H,t}(j)^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}}\right]^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}}\left[\left(\int_0^1 \left(\left(\int_0^1 C_{i,t}(j)^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}}\right)^{\frac{\gamma-1}{\gamma}} di\right)^{\frac{\gamma}{\gamma-1}}\right]^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}, N_t\right)$$

Equation (3) - Consumer budget constraint

$$\int_0^1 P_{H,t}(j) C_{H,t}(j) dj + \int_0^1 \int_0^1 P_{i,t}(j) C_{i,t}(j) dj \; di + \mathbb{E}_t[Q_{t,t+1} D_{t+1}] \leq D_t + W_t N_t + T_t$$

The representative households maximizes consumption subject to the budget constrain above. The first integral represents the budget share spent in domestic items. The double integral is the share spent in imported goods: the sum of all items from all countries bought, in domestic currency. The third term from the RHS is the budget share allocated to assets due in t+1. The RHS incorporates the sum of all payments received from assets summarized in D_t (including shares of firms and other assets traded in an international market), the labour income and transfers from/to the government.

Equations (4) - Demand functions for each specific good

$$C_{H,t}(j) = \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\varepsilon} C_{H,t} \text{ and } C_{i,t}(j) = \left(\frac{P_{i,t}(j)}{P_{i,t}}\right)^{-\varepsilon} C_{i,t}$$

To get the demand functions first we write the Lagrangean:

$$\mathcal{L} = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t U \left(\left[(1-\alpha)^{\frac{1}{\eta}} \left[\left(\int_0^1 C_{H,t}(j)^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}} \right]^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}} \left[\left(\int_0^1 \left(\left(\int_0^1 C_{i,t}(j)^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}} \right)^{\frac{\gamma-1}{\gamma}} di \right)^{\frac{\gamma}{\gamma-1}} \right]^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}}, N_t \right)$$

$$+\lambda_t \left(D_t + W_t N_t + Tt - \int_0^1 P_{H,t}(j) C_{H,t}(j) dj - \int_0^1 \int_0^1 P_{i,t}(j) C_{i,t}(j) dj \; di - \mathbb{E}_t\{Q_{t,t+1} D_{t+1}\} \right)$$

and calculate the MRS (marginal rate of substitution) between $C_{H,t}(j)$ and $C_{H,t}$, as by the optimal allocation, it has to be the rate of prices in every period of time (otherwise the consumer could by a little less of the product with relative higher price and buy another with relative lower price, increasing his utility).

$$\frac{\partial U(C_t, N_t)}{\partial C_{H,t}(j)} = U_c(C_t, N_t) \frac{\eta}{1-\eta} \left(C_t^{\frac{\eta-1}{\eta}} \right)^{\frac{\eta}{\eta-1}-1} (1-\alpha)^{\frac{1}{\eta}} \frac{\eta-1}{\eta} (C_{H,t})^{-\frac{1}{\eta}} \frac{\varepsilon}{\varepsilon-1} \left(C_{H,t}^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}-1} \int_0^1 \frac{\varepsilon-1}{\varepsilon} C_{H,t}(j)^{-\frac{1}{\varepsilon}} dj$$

After simplifying, we get

$$\frac{\partial U(C_t, N_t)}{\partial C_{H,t}(j)} = U_c(C_t, N_t)(1-\alpha)^{\frac{1}{\eta}} C_t^{\frac{1}{\eta}} (C_{H,t})^{-\frac{1}{\eta}} C_{H,t}^{\frac{1}{\varepsilon}} \int_0^1 C_{H,t}(j)^{-\frac{1}{\varepsilon}} dj = U_c(C_t, N_t) \left[(1-\alpha) \frac{C_t}{C_{H,t}} \right]^{\frac{1}{\eta}} \int_0^1 \left[\frac{C_{H,t}}{C_{H,t}(j)} \right]^{\frac{1}{\varepsilon}} dj$$

Similarly,

$$\frac{\partial U(C_t, N_t)}{\partial C_{H,t}} = U_c(C_t, N_t) \frac{\eta}{1-\eta} \left(C_t^{\frac{\eta-1}{\eta}} \right)^{\frac{\eta}{\eta-1}-1} (1-\alpha)^{\frac{1}{\eta}} \frac{\eta-1}{\eta} (C_{H,t})^{-\frac{1}{\eta}}$$

$$\frac{\partial U(C_t, N_t)}{\partial C_{H,t}} = U_c(C_t, N_t)(1-\alpha)^{\frac{1}{\eta}} C_t^{\frac{1}{\eta}} (C_{H,t})^{-\frac{1}{\eta}} = U_c(C_t, N_t) \left[(1-\alpha) \frac{C_t}{C_{H,t}} \right]^{\frac{1}{\eta}}$$

$$\frac{\frac{\partial U(C_t, N_t)}{\partial C_{H,t}(j)}}{\frac{\partial U(C_t, N_t)}{\partial C_{H,t}}} = \frac{U_c(C_t, N_t) \left[(1-\alpha) \frac{C_t}{C_{H,t}} \right]^{\frac{1}{\eta}} \int_0^1 \left[\frac{C_{H,t}}{C_{H,t}(j)} \right]^{\frac{1}{\varepsilon}} dj}{U_c(C_t, N_t) \left[(1-\alpha) \frac{C_t}{C_{H,t}} \right]^{\frac{1}{\eta}}} = \frac{\int_0^1 P_{H,t}(j) dj}{P_{H,t}}$$

After simplifying again, the expression is almost the demand function we want.

$$\int_0^1 \left[\frac{C_{H,t}}{C_{H,t}(j)} \right]^{\frac{1}{\varepsilon}} dj = \int_0^1 \frac{P_{H,t}(j)}{P_{H,t}} dj$$

As the interval of both integrals are the same and the variable being integrated is also the same, what is inside the integral in both sides have also to be the same. So,

$$\left[\frac{C_{H,t}}{C_{H,t}(j)} \right]^{\frac{1}{\varepsilon}} = \frac{P_{H,t}(j)}{P_{H,t}} \Rightarrow \left[\frac{C_{H,t}(j)}{C_{H,t}} \right]^{-\frac{1}{\varepsilon}} = \frac{P_{H,t}(j)}{P_{H,t}} \Rightarrow C_{H,t}(j) = \left[\frac{P_{H,t}(j)}{P_{H,t}} \right]^{-\varepsilon} C_{H,t}$$

Calculating now the MRS (marginal rate of substitution) between $C_{i,t}(j)$ and $C_{i,t}$, which is also equal the rate of the prices.

$$\frac{\partial U(C_t, N_t)}{\partial C_{i,t}(j)} =$$

$$U_c(C_t, N_t) \frac{\eta}{1-\eta} \left(C_t^{\frac{\eta-1}{\eta}} \right)^{\frac{\eta}{\eta-1}-1} \alpha^{\frac{1}{\eta}} \frac{\eta-1}{\eta} (C_{F,t})^{-\frac{1}{\eta}} \frac{\gamma}{\gamma-1} \left(C_{H,t}^{\frac{\gamma-1}{\gamma}} \right)^{\frac{\gamma}{\gamma-1}-1} \int_0^1 \frac{\gamma-1}{\gamma} C_{i,t}^{-\frac{1}{\gamma}} \left[\frac{\varepsilon}{\varepsilon-1} \left(C_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}-1} \int_0^1 \frac{\varepsilon-1}{\varepsilon} C_{i,t}(j)^{-\frac{1}{\varepsilon}} dj \right] di$$

After simplifying, we get

$$\frac{\partial U(C_t,N_t)}{\partial C_{i,t}(j)} = U_c(C_t,N_t)\alpha^{\frac{1}{\eta}} C_t^{\frac{1}{\eta}} (C_{F,t})^{-\frac{1}{\eta}} C_{F,t}^{\frac{1}{\gamma}} \int_0^1 C_{i,t}^{-\frac{1}{\gamma}} \left[C_{i,t}^{\frac{1}{\varepsilon}} \int_0^1 C_{i,t}(j)^{-\frac{1}{\varepsilon}} dj \right] di$$

$$\frac{\partial U(C_t,N_t)}{\partial C_{i,t}(j)} = U_c(C_t,N_t) \left[\alpha \frac{C_t}{C_{F,t}}\right]^{\frac{1}{\eta}} \int_0^1 \left[\frac{C_{F,t}}{C_{i,t}}\right]^{\frac{1}{\gamma}} \int_0^1 \left[\frac{C_{i,t}}{C_{i,t}(j)}\right]^{\frac{1}{\varepsilon}} dj\ di$$

$$\frac{\partial U(C_t,N_t)}{\partial C_{i,t}} = U_c(C_t,N_t) \frac{\eta}{1-\eta} \left(C_t^{\frac{\eta-1}{\eta}} \right)^{\frac{\eta}{\eta-1}-1} \alpha^{\frac{1}{\eta}} \frac{\eta-1}{\eta} (C_{F,t})^{-\frac{1}{\eta}} \frac{\gamma}{\gamma-1} \left(C_{H,t}^{\frac{\gamma-1}{\gamma}} \right)^{\frac{\gamma}{\gamma-1}-1} \int_0^1 \frac{\gamma-1}{\gamma} C_{i,t}^{-\frac{1}{\gamma}} di$$

$$\frac{\partial U(C_t,N_t)}{\partial C_{i,t}} = U_c(C_t,N_t) \left[\alpha \frac{C_t}{C_{F,t}}\right]^{\frac{1}{\eta}} \int_0^1 \left[\frac{C_{F,t}}{C_{i,t}}\right]^{\frac{1}{\gamma}} di$$

Calculating the MRS we have

$$\frac{\frac{\partial U(C_t,N_t)}{\partial C_{i,t}(j)}}{\frac{\partial U(C_t,N_t)}{\partial C_{i,t}}} = \frac{U_c(C_t,N_t) \left[\alpha \frac{C_t}{C_{F,t}}\right]^{\frac{1}{\eta}} \int_0^1 \left[\frac{C_{F,t}}{C_{i,t}}\right]^{\frac{1}{\gamma}} \int_0^1 \left[\frac{C_{i,t}}{C_{i,t}(j)}\right]^{\frac{1}{\varepsilon}} dj\ di}{U_c(C_t,N_t) \left[\alpha \frac{C_t}{C_{F,t}}\right]^{\frac{1}{\eta}} \int_0^1 \left[\frac{C_{F,t}}{C_{i,t}}\right]^{\frac{1}{\gamma}} di} = \frac{\int_0^1 P_{i,t}(j) dj}{P_{i,t}}$$

As before, we can simplify again. Also, as there's a continuum of firms, we can consider that the price of each product $P_{i,t}(j)$ is only correlated with its specific demand function and not with the demand function of other in its country or another country, it follows that each specific price is uncorrelated with $C_{F,t}$ and $C_{i,t}$. Also, as each firm is very small, we can consider that it has negligible influence on the aggregate index price of its country ($P_{i,t}$). With these independence assumption, the joint distribution is equal to the product of the marginal distributions.

$$\int_0^1 \int_0^1 \left[\frac{C_{F,t}}{C_{i,t}}\right]^{\frac{1}{\gamma}} \left[\frac{C_{i,t}}{C_{i,t}(j)}\right]^{\frac{1}{\varepsilon}} dj\ di = \int_0^1 \left[\frac{C_{F,t}}{C_{i,t}}\right]^{\frac{1}{\gamma}} di \int_0^1 \frac{P_{i,t}(j)}{P_{i,t}} dj = \int_0^1 \int_0^1 \frac{P_{i,t}(j)}{P_{i,t}} \left[\frac{C_{F,t}}{C_{i,t}}\right]^{\frac{1}{\gamma}} dj\ di$$

Now, as before, the integrand in both sides needs to be the same. The we get the second demand equation.

$$\left[\frac{C_{F,t}}{C_{i,t}}\right]^{\frac{1}{\gamma}} \left[\frac{C_{i,t}}{C_{i,t}(j)}\right]^{\frac{1}{\varepsilon}} = \frac{P_{i,t}(j)}{P_{i,t}} \left[\frac{C_{F,t}}{C_{i,t}}\right]^{\frac{1}{\gamma}} \Rightarrow \left[\frac{C_{i,t}(j)}{C_{i,t}}\right]^{-\frac{1}{\varepsilon}} = \frac{P_{i,t}(j)}{P_{i,t}} \Rightarrow C_{i,t}(j) = \left[\frac{P_{i,t}(j)}{P_{i,t}}\right]^{-\varepsilon} C_{i,t}$$

Equation (5) - Demand for foreign goods from each country

$$C_{i,t} = \left(\frac{P_{i,t}}{P_{F,t}}\right)^{-\gamma} C_{F,t}$$

To find the aggregate demand for each country, in terms of total foreign demand, we proceed by calculating the MRS between the aggregate consumption for the country and the aggregate consumption of foreign goods, which the optimal allocation resulting from the rate between the prices, as before.

$$\frac{\partial U(C_t,N_t)}{\partial C_{F,t}} = U_c(C_t,N_t) \frac{\eta}{1-\eta} \left(C_t^{\frac{\eta-1}{\eta}} \right)^{\frac{\eta}{\eta-1}-1} \alpha^{\frac{1}{\eta}} \frac{\eta-1}{\eta} (C_{F,t})^{-\frac{1}{\eta}} = U_c(C_t,N_t) \left[\alpha \frac{C_t}{C_{F,t}}\right]^{\frac{1}{\eta}}$$

$$\frac{\frac{\partial U(C_t,N_t)}{\partial C_{i,t}}}{\frac{\partial U(C_t,N_t)}{\partial C_{F,t}}} = \frac{U_c(C_t,N_t) \left[\alpha \frac{C_t}{C_{F,t}}\right]^{\frac{1}{\eta}} \int_0^1 \left[\frac{C_{F,t}}{C_{i,t}}\right]^{\frac{1}{\gamma}} di}{U_c(C_t,N_t) \left[\alpha \frac{C_t}{C_{F,t}}\right]^{\frac{1}{\eta}}} = \frac{\int_0^1 P_{i,t} di}{P_{F,t}}$$

As $P_{F,t}$ doesn't depend on a specific i, we can put it inside the integral. Then we get again two integrans which have to be the same for the equality to hold.

$$\left[\frac{C_{F,t}}{C_{i,t}}\right]^{\frac{1}{\gamma}} = \frac{P_{i,t}}{P_{F,t}} \Rightarrow \left[\frac{C_{i,t}}{C_{F,t}}\right]^{-\frac{1}{\gamma}} = \frac{P_{i,t}}{P_{F,t}} \quad \Rightarrow \quad C_{i,t} = \left[\frac{P_{i,t}}{P_{F,t}}\right]^{-\gamma} C_{F,t}$$

Equations (6) - Optimal share between the domestic and imported goods

$C_{H,t} = (1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t$ and $C_{F,t} = \alpha\left(\frac{P_{F,t}}{P_t}\right)^{-\eta} C_t$

Now we will calculate the MRS between the domestic products and the total consumption, which has to be equal to the rate of prices. After, we will do the same for the foreign products.

$$\frac{\frac{\partial U(C_t,N_t)}{\partial C_{H,t}}}{\frac{\partial U(C_t,N_t)}{\partial C_t}} = \frac{U_c(C_t,N_t)\left[(1-\alpha)\frac{C_t}{C_{H,t}}\right]^{\frac{1}{\eta}}}{U_c(C_t,N_t)} = \frac{P_{H,t}}{P_t} \;\Rightarrow\; (1-\alpha)\frac{C_t}{C_{H,t}} = \left(\frac{P_{H,t}}{P_t}\right)^{\eta} \;\Rightarrow\; C_{H,t} = (1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t$$

$$\frac{\frac{\partial U(C_t,N_t)}{\partial C_{F,t}}}{\frac{\partial U(C_t,N_t)}{\partial C_t}} = \frac{U_c(C_t,N_t)\left[\alpha\frac{C_t}{C_{F,t}}\right]^{\frac{1}{\eta}}}{U_c(C_t,N_t)} = \frac{P_{F,t}}{P_t} \;\Rightarrow\; \alpha\frac{C_t}{C_{F,t}} = \left(\frac{P_{F,t}}{P_t}\right)^{\eta} \;\Rightarrow\; C_{F,t} = \alpha\left(\frac{P_{F,t}}{P_t}\right)^{-\eta} C_t$$

Equation (7) - Aggregated expenditure

$P_tC_t + \mathbb{E}_t[Q_{t,t+1}D_{t+1}] \leq D_t + W_tN_t + T_t$

Now that we have the demand functions

$C_{H,t}(j) = \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\varepsilon} C_{H,t}$;

$C_{i,t}(j) = \left(\frac{P_{i,t}(j)}{P_{i,t}}\right)^{-\varepsilon} C_{i,t}$; and $C_{i,t} = \left(\frac{P_{i,t}}{P_{F,t}}\right)^{-\gamma} C_{F,t}$

let's prove that

$\int_0^1 P_{H,t}(j)C_{H,t}(j)dj = P_{H,t}C_{H,t}$, $\int_0^1 P_{i,t}(j)C_{i,t}(j)dj = P_{i,t}C_{i,t}$ and $\int_0^1 P_{i,t}C_{i,t}dj = P_{F,t}C_{F,t}$

using the definition of the price indexes:

$P_{H,t} \equiv \left(\int_0^1 P_{H,t}(j)^{1-\varepsilon}dj\right)^{\frac{1}{1-\varepsilon}}$, $P_{i,t} \equiv \left(\int_0^1 P_{i,t}(j)^{1-\varepsilon}dj\right)^{\frac{1}{1-\varepsilon}}$ and $P_{F,t} \equiv \left(\int_0^1 P_{i,t}^{1-\gamma}di\right)^{\frac{1}{1-\gamma}}$

$$\int_0^1 P_{H,t}(j)C_{H,t}(j)dj = \int_0^1 P_{H,t}(j)\left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\varepsilon} C_{H,t}dj = \frac{C_{H,t}}{P_{H,t}^{-\varepsilon}}\int_0^1 P_{H,t}(j)^{1-\varepsilon}dj = \frac{C_{H,t}}{P_{H,t}^{-\varepsilon}}P_{H,t}^{1-\varepsilon} = P_{H,t}C_{H,t}$$

$$\int_0^1 P_{i,t}(j)C_{i,t}(j)dj = \int_0^1 P_{i,t}(j)\left(\frac{P_{i,t}(j)}{P_{i,t}}\right)^{-\varepsilon} C_{i,t}dj = \frac{C_{i,t}}{P_{i,t}^{-\varepsilon}}\int_0^1 P_{i,t}(j)^{1-\varepsilon}dj = \frac{C_{i,t}}{P_{i,t}^{-\varepsilon}}P_{i,t}^{1-\varepsilon} = P_{i,t}C_{i,t}$$

$$\int_0^1 P_{i,t}C_{i,t}di = \int_0^1 P_{i,t}\left(\frac{P_{i,t}}{P_{F,t}}\right)^{-\gamma} C_{F,t}di = \frac{C_{F,t}}{P_{F,t}^{-\gamma}}\int_0^1 P_{F,t}^{1-\gamma}dj = \frac{C_{F,t}}{P_{F,t}^{-\varepsilon}}P_{F,t}^{1-\varepsilon} = P_{F,t}C_{F,t}$$

With this aggregation, the budget constraint can be simplified

$$\int_0^1 P_{H,t}(j)C_{H,t}(j)dj + \int_0^1\int_0^1 P_{i,t}(j)C_{i,t}(j)dj\ di + \mathbb{E}_t\{Q_{t,t+1}D_{t+1}\} \leq D_t + W_tN_t + T_t$$

$$P_{H,t}C_{H,t} + \int_0^1 P_{i,t}C_{i,t}di = P_{H,t}C_{H,t} + P_{F,t}C_{F,t} \leq D_t + W_tN_t + Tt - \mathbb{E}_t\{Q_{t,t+1}D_{t+1}\}$$

As the total consumption expenditure by the representative consumer is with the domestic produts or foreig products, the budget constraint becomes:

$P_tC_t \leq D_t + W_tN_t + T_t - \mathbb{E}_t\{Q_{t,t+1}D_{t+1}\}$

Equation (8) - Intratemporal substitution between consumption and labor

$$C_t^\sigma N_t^\varphi = \frac{W_t}{P_t}$$

Now we arrived at a standard problem for the representative consumer. Considering the functional form for the utility function as $U(C_t, N_t) = \dfrac{C_t^{1-\sigma}}{1-\sigma} - \dfrac{N_t^{1+\varphi}}{1+\varphi}$,

$$\max_{C_t,N_t} \; E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t) = \max_{C_t,N_t} \; E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} \right)$$

subject to $D_t + W_t N_t + Tt - \mathbb{E}_t[Q_{t,t+1} D_{t+1}] - P_t C_t = 0$, as an optimal condition (the constraint is binding, otherwise the household could consume a little more with the same budget and would not be optimizing).

Now we can write the Lagrangean

$$\mathcal{L} = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left[\left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} \right) + \lambda_t \left(D_t + W_t N_t + Tt - Q_{t,t+1} D_{t+1} - P_t C_t \right) \right]$$

with first order conditions (FOCs):

(C_t): $\beta^t U_C(C_t, N_t) = \beta^t C_t^{-\sigma} = \beta^t \lambda_t P_t \quad \Rightarrow \quad C_t^{-\sigma} = \lambda_t P_t$

(N_t): $-\beta^t U_n(C_t, N_t) = \beta^t N_t^{\varphi} = \beta^t \lambda_t W_t \quad \Rightarrow \quad N_t^{\varphi} = \lambda_t W_t$

(D_{t+1}): $\beta^t \lambda_t Q_{t,t+1} = \beta^{t+1} \lambda_{t+1} \quad \Rightarrow \quad \dfrac{\lambda_{t+1}}{\lambda_t} = \dfrac{Q_{t,t+1}}{\beta}$

Dividing (N_t) FOC by (C_t) FOC, we have the standard equation of intratemporal substitution between consumption and leisure $-\dfrac{U_C(C_t, N_t)}{U_N(C_t, N_t)} = C_t^\sigma N_t^\varphi = \dfrac{W_t}{P_t}$

Equation (9) - Euler equation

$$\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} = Q_{t,t+1}$$

Using the FOCs calculated before, we can advance one period for the consumption FOC, we have $\mathbb{E}_t[C_{t+1}^{-\sigma}] = \mathbb{E}_t[\lambda_{t+1} P_{t+1}]$

Dividing the consumption FOC in t+1 by the equation in t and substituting by the $\mathbb{E}_t[\lambda_{t+1}]/\lambda_t$ in the D_{t+1} FOC, we get the Euler equation

$$\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} = \frac{\lambda_{t+1} P_{t+1}}{\lambda_t P_t} \quad \Rightarrow \quad \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} = \frac{Q_{t,t+1}}{\beta} \quad \Rightarrow \quad \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} = Q_{t,t+1}$$

Equation (10) - Euler equation with gross returns

$$\beta R_t \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \right] = 1$$

As $Q_{t,t+1}$, is the price of a a riskless one-period discount bond in domestic currency with gross return R_t, we can substitute $R_t = \dfrac{1}{\mathbb{E}_t[Q_{t,t+1}]}$, we can take expectations in both sides to get the equation (10)

$$\mathbb{E}_t \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \right] = \mathbb{E}_t[Q_{t,t+1}] \quad \Rightarrow \quad \beta \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \right] = \frac{1}{R_t} \quad \Rightarrow \quad \beta R_t \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \right] = 1$$

Equation (11) - Log-linearized Euler equation

$c_t = \mathbb{E}_t\left[c_{t+1}\right] - \dfrac{1}{\sigma}\left(r_t - \mathbb{E}_t\left[\pi_{t+1}\right] - \rho\right)$

Log-linearizing $C_t^{\sigma} N_t^{\varphi} = \dfrac{W_t}{P_t}$ is straight forward: $w_t - p_t = \sigma c_t + \varphi n_t$

To log-linerize the Euler equation, we'll use the Taylor expansion: $f(x) \approx f(x_0) + f'(x_o)(x - x_0)$. When expanding the exponential function around 0, we get $e^x = e^0 + e^0(x-0) = 1 + x$

$\mathbb{E}_t\left[\exp\left(\ln\left[\beta R_t \left(\frac{C_{t+1}}{C_t}\right)^{-\sigma} \frac{P_t}{P_{t+1}}\right]\right)\right] = \mathbb{E}_t\left[\exp\left(\ln(\beta) + r_t - \sigma(c_{t+1} - c_t) + p_t - p_{t+1}\right)\right] = 1 \quad \Rightarrow$

$1 + \ln(\beta) + r_t - \sigma(\mathbb{E}_t\left[c_{t+1}\right] - c_t) - \mathbb{E}_t\left[\pi_{t+1}\right] = 1 \quad \Rightarrow \quad \sigma c_t = \sigma \mathbb{E}_t\left[c_{t+1}\right] + \mathbb{E}_t\left[\pi_{t+1}\right] - r_t - \ln(\beta)$

Rearranging, we get

$\Rightarrow \quad c_t = \mathbb{E}_t\left[c_{t+1}\right] - \dfrac{1}{\sigma}\left(r_t - \mathbb{E}_t\left[\pi_{t+1}\right] - \rho\right)$ as $\rho \equiv \dfrac{1-\beta}{\beta} \approx -\ln(\beta)$

2 - External Sector

Equation (12) - Log-linear effective terms of trade

$s_t = \displaystyle\int_0^1 s_{i,t} di$

Let's log-linearize the expression for the bilateral terms of trade $S_t \equiv \dfrac{P_{F,t}}{P_{H,t}} = \left(\displaystyle\int_0^1 S_{i,t}^{1-\gamma} di\right)^{\frac{1}{1-\gamma}} \quad \Rightarrow \quad S_t^{1-\gamma} = \displaystyle\int_0^1 S_{i,t}^{1-\gamma} di$

$\exp\left(\ln\left[S_t^{1-\gamma}\right]\right) = \displaystyle\int_0^1 \exp\left(\ln\left[S_{i,t}^{1-\gamma}\right]\right) di \quad \Rightarrow \quad \exp[(1-\gamma)s_t] = \displaystyle\int_0^1 \exp[(1-\gamma)s_{i,t}] di$

Applying the exponential Taylor expansion ($e^x = 1 + x$) in both sides, we get $1 + (1+\gamma)s_t = 1 + (1+\gamma)\displaystyle\int_0^1 s_{i,t} di \quad \Rightarrow \quad s_t = \displaystyle\int_0^1 s_{i,t} di$

Equation (13) - Log-linear CPI around a symmetric steady state satisfying PPP

$p_t = p_{H,t} + \alpha s_t$

To log-linearize the CPI formula, considering that it is a symmetric steady-state, we have $P_{H,t} = P_{F,t} = P_t$. Now, taking logs in both sides and using the Taylor expansion for a vector of two variables we have $f(x,y) \approx f(x_0,y_0) + \left.\dfrac{\partial f(x,y)}{\partial x}\right|_{x_0,y_0}(x - x_0) + \left.\dfrac{\partial f(x,y)}{\partial y}\right|_{x_0,y_0}(y - y_0)$ So, $(P_t)^{\eta-1} = (1-\alpha)(P_{H,t})^{1-\eta} + \alpha(P_{F,t})^{1-\eta}$

By the CPI definition, we have $P_t \equiv \left[(1-\alpha)(P_{H,t})^{1-\eta} + \alpha(P_{F,t})^{1-\eta}\right]^{\frac{1}{1-\eta}} \quad \Rightarrow \quad (P_t)^{1-\eta} = (1-\alpha)(P_{H,t})^{1-\eta} + \alpha(P_{F,t})^{1-\eta}$

Taking logs, we have $(1-\eta)\ln(P_t) = \ln\left[(1-\alpha)(P_{H,t})^{1-\eta} + \alpha(P_{F,t})^{1-\eta}\right] = f(P_{H,t}, P_{F,t})$

Applying the Taylor expansion on the right side using $x = P_{H,t}$, $y = P_{F,t}$ and $x_0 = y_0 = P_t$,

$(1-\eta)p_t \approx \ln\left[(1-\alpha)(P_t)^{1-\eta} + \alpha(P_t)^{1-\eta}\right] + \dfrac{(1-\alpha)(1-\eta)P_t^{-\eta}}{P_t^{1-\eta}}(P_{H,t} - P_t) + \dfrac{\alpha(1-\eta)P_t^{-\eta}}{P_t^{1-\eta}}(P_{F,t} - P_t)$

$(1-\eta)p_t = \ln(P_t^{1-\eta}) + (1-\alpha)(1-\eta)\dfrac{P_{H,t} - P_t}{P_t} + \alpha(1-\eta)\dfrac{P_{F,t} - P_t}{P_t}.$

$p_t \approx p_t + (1-\alpha)[\ln(P_{H,t}) - \ln(P_t)] + \alpha[\ln(P_{F,t}) - \ln(P_t)] \quad \Rightarrow \quad p_t = (1-\alpha)p_{H,t} + \alpha\ p_{F,t}$, as defined in the paper.

As $s_t \equiv p_{F,t} - p_{H,t}$, $p_t = (1-\alpha)P_{H,t} + \alpha(s_t + p_{H,t}) = p_{H,t} + \alpha s_t$

Equation (14) - Domestic inflation

$\pi_t = \pi_{H,t} + \alpha \Delta s_t$

The domestic inflation rate is defined as $\pi_{H,t} \equiv p_{H,t} - p_{H,t-1}$, taking the difference between the equation between t and t-1, we have $p_t - p_{t-1} = p_{H,t} - p_{H,t-1} + \alpha(s_t - s_{t-1}) \quad \Rightarrow \quad \pi_t = \pi_{H,t} + \alpha \Delta s_t$.

Equation (15) - Log-linear terms of trade

$s_t = \mathbb{E}_t + p_t^* - p_{H,t}$

Assuming that the law of one price is valid in all times (the same goods produced in different countries have the same price when converting to the domestic currency, using the nominal interest rate), we have $P_{i,t}(j) = \mathcal{E}_{i,t} P_{i,t}^i(j)$ for all $i, j \in [0,1]$.

As $P_{i,t}^i \equiv \left(\int_0^1 P_{i,t}^i(j)^{1-\varepsilon} dj \right)^{\frac{1}{1-\varepsilon}}$, we have $\mathcal{E}_{i,t} P_{i,t}^i = \mathcal{E}_{i,t} \left(\int_0^1 P_{i,t}^i(j)^{1-\varepsilon} dj \right)^{\frac{1}{1-\varepsilon}} = \left((\mathcal{E}_{i,t})^{1-\varepsilon} \int_0^1 P_{i,t}^i(j)^{1-\varepsilon} dj \right)^{\frac{1}{1-\varepsilon}}$

$= \left(\int_0^1 \left(\mathcal{E}_{i,t} P_{i,t}^i(j) \right)^{1-\varepsilon} dj \right)^{\frac{1}{1-\varepsilon}} = \left(\int_0^1 P_{i,t}(j)^{1-\varepsilon} dj \right)^{\frac{1}{1-\varepsilon}} = P_{i,t}$

Now, we have $P_{F,t} = \left(\int_0^1 P_{i,t}^{1-\gamma} di \right)^{\frac{1}{1-\gamma}} = \left(\int_0^1 \left(\mathcal{E}_{i,t} P_{i,t}^i \right)^{1-\gamma} di \right)^{\frac{1}{1-\gamma}} \quad \Rightarrow \quad P_{F,t}^{1-\gamma} = \int_0^1 \left(\mathcal{E}_{i,t} P_{i,t}^i \right)^{1-\gamma} di$

Log-linearizing the last expression, we get $\exp\left[(1-\gamma)\ln P_{F,t}\right] = \int_0^1 \exp\left[(1-\gamma)\ln\left(\mathcal{E}_{i,t} P_{i,t}^i\right)\right] di$

$\Rightarrow \quad 1 + (1-\gamma) p_{F,t} = \int_0^1 \left[1 + (1-\gamma)(e_{i,t} + p_{i,t}^i)\right] di \quad \Rightarrow \quad p_{F,t} = \int_0^1 (e_{i,t} + p_{i,t}^i) di = \mathbb{E}_t + p_t^*,$

where $\mathbb{E}_t \equiv \int_0^1 e_{i,t} \, di$ and $p_t^* \equiv \int_0^1 p_{i,t}^i \, di$. Also, we have that $s_t = p_{F,t} - p_{H,t} = \mathbb{E}_t + p_t^* - p_{H,t}$.

Defining the bilateral real exchange rate $\mathcal{Q}_{i,t} \equiv \dfrac{\mathcal{E}_{i,t} P_{i,t}}{P_t}$ and the (log) effective real exchange rate $q_t \equiv \int_0^1 q_{i,t} di$ we have $q_t = \int_0^1 \ln\left(\dfrac{\mathcal{E}_{i,t} P_{i,t}}{P_t}\right) di = \int_0^1 (e_{i,t} + p_{i,t} - p_t) \, di = \mathbb{E}_t + p_t^* - p_t = s_t + p_{H,t} - (p_{H,t} + \alpha s_t) = (1-\alpha) s_t$

Equation (16) - International risk sharing

$\beta \left(\dfrac{C_{t+1}^i}{C_t^i} \right)^{-\sigma} \left(\dfrac{P_t^i}{P_{t+1}^i} \right) \left(\dfrac{\mathcal{E}_t^i}{\mathcal{E}_{t+1}^i} \right) = Q_{t,t+1}$

The problem of the representative household in any country is the same, as the economies are all equal. There is, any country has an Euler equation like $\beta \left(\dfrac{C_{t+1}}{C_t} \right)^{-\sigma} \dfrac{P_t}{P_{t+1}} = Q_{t,t+1}$.

The condition for the clearing of the international market is that $Q_{t,t+1}$ is unique. So the price converted to a common current has to be the same. So, for every foreign country, the Euler equation becomes

$\beta \left(\dfrac{C_{t+1}^i}{C_t^i} \right)^{-\sigma} \dfrac{P_t^i \mathcal{E}_t^i}{P_{t+1}^i \mathcal{E}_{t+1}^i} = Q_{t,t+1}$.

Equation (17) - Consumption and the bilateral real exchange rate

$C_t = \vartheta_i C_t^i \mathcal{Q}_{i,t}^{\frac{1}{\sigma}}$

Combining both equations, using the definition of the real exchange rate $\mathcal{Q}_{i,t} \equiv \dfrac{\mathcal{E}_{i,t} P_{i,t}}{P_t}$ and solving for C_t, we have

$$\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{P_t}{P_{t+1}}=\beta\left(\frac{C^i_{t+1}}{C^i_t}\right)^{-\sigma}\frac{P^i_t\mathcal{E}^i_t}{P^i_{t+1}\mathcal{E}^i_{t+1}}\quad\Rightarrow\quad\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{1}{P_{t+1}}=\left(\frac{C^i_{t+1}}{C^i_t}\right)^{-\sigma}\frac{\mathcal{Q}_{i,t}}{P^i_{t+1}\mathcal{E}^i_{t+1}}$$

$$\Rightarrow\quad (C_t)^{\sigma}={C_{t+1}}^{\sigma}\left(C^i_{t+1}\right)^{-\sigma}\frac{P_{t+1}}{P^i_{t+1}\mathcal{E}^i_{t+1}}\mathcal{Q}_{i,t}(C^i_t)^{\sigma}\quad\Rightarrow\quad C_t=\frac{C_{t+1}}{C^i_{t+1}}(\mathcal{Q}_{i,t+1})^{-\frac{1}{\sigma}}C^i_t\mathcal{Q}_{i,t}^{\frac{1}{\sigma}}\quad\Rightarrow\quad C_t=\vartheta_iC^i_t\mathcal{Q}_{i,t}^{\frac{1}{\sigma}},$$

where $\vartheta_i=\dfrac{C_{t+1}}{C^i_{t+1}}(\mathcal{Q}_{i,t+1})^{-\frac{1}{\sigma}}$ is a constant and generally will depend on initial relative net asset positions.

Equations (A.1) and (A.2) - Domestic production and external production

$$Y=\left[(1-\alpha)h(\mathcal{S})^{\eta}q(\mathcal{S})^{\frac{1}{\sigma}}+\alpha q(\mathcal{S})^{\gamma}h(\mathcal{S})^{\gamma}\right]Y^*\equiv v(\mathcal{S})Y^*$$

Assuming identical conditions for all economies, the net asset position for all of the is zero. In this case, $\vartheta_i=\vartheta=1$ for all i. As the symmetric foresight steady-state in this condition is shown in the appendix A.

The international market clearing implies that the total goods produced in a country is consumed by domestically or it's exported. The integral represents the sum of the demand for products of the economy analysed by foreign countries. In a case with economies not with measure zero, we need to exclude the economy analysed from the integral do avoid double counting.

$$Y=C_H+C_i=(1-\alpha)\left(\frac{P_H}{P}\right)^{-\eta}C+\int_0^1\left(\frac{P^i_i}{P_F}\right)^{-\gamma}C_Fdi=(1-\alpha)\left(\frac{P_H}{P}\right)^{-\eta}C+\alpha\int_0^1\left(\frac{P^i_i}{P^i_F}\right)^{-\gamma}\left(\frac{P^i_F}{P^i}\right)^{-\eta}C^idi$$

$$Y=(1-\alpha)\left(\frac{P_H}{P}\right)^{-\eta}C+\alpha\int_0^1\left(\frac{P^i_F}{P^i_i}\right)^{\gamma}\left(\frac{P^i_F}{P^i}\right)^{-\eta}C^idi=(1-\alpha)\left(\frac{P_H}{P}\right)^{-\eta}C+\alpha\int_0^1\left(\frac{\mathcal{E}_iP^i_F}{P_H}\right)^{\gamma}\left(\frac{P^i_F}{P^i}\right)^{-\eta}C^idi,$$

where P^i_i is the price in the domestic economy converted to the currency of country i, or $P^i_i=\dfrac{P_i}{\mathcal{E}_i}=\dfrac{P_H}{\mathcal{E}_i}$, as the goods have the same price in the international market, after converting to the same currency. After simplifying, we have

$$Y=\left(\frac{P_H}{P}\right)^{-\eta}\left[(1-\alpha)C+\alpha\int_0^1\left(\frac{\mathcal{E}_iP^i_F}{P_H}\right)^{\gamma-\eta}\left(\frac{\mathcal{E}_iP_i}{P}\right)^{\eta}C^idi\right]=\left(\frac{P_H}{P}\right)^{-\eta}\left[(1-\alpha)C+\alpha\int_0^1\left(\frac{\mathcal{E}_iP^i_F}{P_H}\right)^{\gamma-\eta}Q_i^{\eta}C^idi\right]$$

Considering that $P=\left[(1-\alpha)\left(P_H\right)^{1-\eta}+\alpha\left(P_F\right)^{1-\eta}\right]^{\frac{1}{1-\eta}}$ in the steady-state, $P^{1-\eta}=(1-\alpha)\left(P_H\right)^{1-\eta}+\alpha\left(P_F\right)^{1-\eta}$, as $\mathcal{S}_i\equiv\dfrac{P_i}{P_H}$. So,

$$\left(\frac{P}{P_H}\right)^{1-\eta}=(1-\alpha)+\alpha\left(\frac{P_F}{P_H}\right)^{1-\eta}=(1-\alpha)+\alpha\mathcal{S}_i^{1-\eta}\quad\Rightarrow\quad\frac{P}{P_H}=\left[(1-\alpha)+\alpha\mathcal{S}^{1-\eta}\right]^{\frac{1}{1-\eta}}=\left[(1-\alpha)+\alpha\int_0^1(\mathcal{S}_i)^{1-\eta}di\right]^{\frac{1}{1-\eta}}\equiv h(\mathcal{S})$$

Defining $\mathcal{S}^i=\dfrac{\mathcal{E}_iP^i_F}{P_i}$ and using the fact that $C^i=C\mathcal{Q}^{-\frac{1}{\sigma}}$ as $\vartheta_i=1$ in a symmetric steady-state, we have

$$Y=h(\mathcal{S})^{\eta}C\left[(1-\alpha)+\alpha\int_0^1\left(\frac{\mathcal{E}_iP^i_F}{P_i}\frac{P_i}{P_H}\right)^{\gamma-\eta}Q_i^{\eta-\frac{1}{\sigma}}di\right]=h(\mathcal{S})^{\eta}C\left[(1-\alpha)+\alpha\int_0^1\left(\mathcal{S}^i\frac{P_F}{P_H}\right)^{\gamma-\eta}Q_i^{\eta-\frac{1}{\sigma}}di\right]$$

As we will work with a first order approximation, the equality below is valid.

$$Y=h(\mathcal{S})^{\eta}C\left[(1-\alpha)+\alpha\int_0^1\left(\mathcal{S}^i\mathcal{S}_i\right)^{\gamma-\eta}Q_i^{\eta-\frac{1}{\sigma}}di\right]=h(\mathcal{S})^{\eta}C\left[(1-\alpha)+\alpha\int_0^1(\mathcal{S}^i)^{\gamma}di\int_0^1(\mathcal{S}_i)^{-\eta}di\int_0^1Q_i^{\eta-\frac{1}{\sigma}}di\right]$$

$$Y=h(\mathcal{S})^{\eta}C\left[(1-\alpha)+\alpha\mathcal{S}^{-\eta}\int_0^1\left(\frac{P^i_F}{P_H}\right)^{\gamma}di\int_0^1\left(\frac{\mathcal{E}_iP^i_F}{P}\right)^{\eta-\frac{1}{\sigma}}di\right],$$

as if $\mathcal{S}^{1-\gamma}=\displaystyle\int_0^1\mathcal{S}^{1-\gamma}di$, we can substitute variables $-\eta=1-\gamma$ and we get the result.

$$Y=h(\mathcal{S})^{\eta}C\left[(1-\alpha)+\alpha\mathcal{S}^{-\eta}\left(\frac{1}{P_H}\right)^{\gamma}\int_0^1\left(P^i_F\right)^{\gamma}di\int_0^1\left(\frac{\mathcal{E}_iP^i_F}{P}\right)^{\eta-\frac{1}{\sigma}}di\right]$$

$$Y=h(\mathcal{S})^{\eta}C\left[(1-\alpha)+\alpha\mathcal{S}^{-\eta}\left(\frac{1}{P_H}\right)^{\gamma}(P^*)^{\gamma}\int_0^1\left(\frac{\mathcal{E}_iP^i_F}{P_H}\frac{P_H}{P}\right)^{\eta-\frac{1}{\sigma}}di\right],$$

using the fact that $\left(P_F^i\right)^{1-\gamma} = \int_0^1 \left(P_i^i\right)^{1-\gamma} di$ and using P* for the international price index of imported goods.

$$Y = h(\mathcal{S})^\eta C \left[(1-\alpha) + \alpha \mathcal{S}^{-\eta} \left(\frac{P^*}{P_H}\right)^\gamma \int_0^1 \left(\frac{\mathcal{S}^i}{h(\mathcal{S})}\right)^{\eta - \frac{1}{\sigma}} di\right] = h(\mathcal{S})^\eta C \left[(1-\alpha) + \alpha \mathcal{S}^{-\eta} \mathcal{S}^\gamma \left(\frac{1}{h(\mathcal{S})}\right)^{\eta - \frac{1}{\sigma}} \int_0^1 (\mathcal{S}^i)^{\eta - \frac{1}{\sigma}} di\right]$$

$$Y = h(\mathcal{S})^\eta C \left[(1-\alpha) + \alpha \mathcal{S}^{\gamma-\eta} \left(\frac{1}{h(\mathcal{S})}\right)^{\eta - \frac{1}{\sigma}} \mathcal{S}^{\eta - \frac{1}{\sigma}}\right] = h(\mathcal{S})^\eta C \left[(1-\alpha) + \alpha \mathcal{S}^{\gamma-\eta} \left(\frac{\mathcal{S}}{h(\mathcal{S})}\right)^{\eta - \frac{1}{\sigma}}\right],$$

which yields the result. $Y = h(\mathcal{S})^\eta C \left[(1-\alpha) + \alpha \mathcal{S}^{\gamma-\eta} q(\mathcal{S})^{\eta - \frac{1}{\sigma}}\right]$, where $\mathcal{Q} = \dfrac{\mathcal{S}}{h(\mathcal{S})} \equiv q(\mathcal{S})$

Substituting $C = C^* q(\mathcal{S})^{\frac{1}{\sigma}}$ in the expression above, we have

$$Y = (1-\alpha) h(\mathcal{S})^\eta C + \alpha h(\mathcal{S})^\eta \mathcal{S}^{\gamma-\eta} q(\mathcal{S})^{\eta - \frac{1}{\sigma}} = (1-\alpha) h(\mathcal{S})^\eta C^* q(\mathcal{S})^{\frac{1}{\sigma}} + \alpha h(\mathcal{S})^\eta \mathcal{S}^{\gamma-\eta} q(\mathcal{S})^{\eta - \frac{1}{\sigma}} C^* q(\mathcal{S})^{\frac{1}{\sigma}}$$

Imposing market clearing $C^* = Y^*$, we have

$$Y = \left[(1-\alpha) h(\mathcal{S})^\eta q(\mathcal{S})^{\frac{1}{\sigma}} + \alpha \mathcal{S}^{\gamma-\eta} h(\mathcal{S})^\eta q(\mathcal{S})^\eta\right] Y^* = \left[(1-\alpha) h(\mathcal{S})^\eta q(\mathcal{S})^{\frac{1}{\sigma}} + \alpha \mathcal{S}^\gamma h(\mathcal{S})^{-\eta} q(\mathcal{S})^{-\eta} h(\mathcal{S})^\eta q(\mathcal{S})^\eta\right] Y^*$$

$$Y = \left[(1-\alpha) h(\mathcal{S})^\eta q(\mathcal{S})^{\frac{1}{\sigma}} + \alpha \mathcal{S}^\gamma\right] Y^* = \left[(1-\alpha) h(\mathcal{S})^\eta q(\mathcal{S})^{\frac{1}{\sigma}} + \alpha q(\mathcal{S})^\gamma h(\mathcal{S})^\gamma\right] Y^* \equiv v(\mathcal{S}) Y^*,$$

where $v(\mathcal{S}) > 0$, $v'(\mathcal{S}) > 0$ and $v(1) = 1$

Equation (A.3) - Production in the steady-state

$$Y = A^{\frac{1+\varphi}{\varphi}} \left(\frac{1 - \frac{1}{\varepsilon}}{(1-\tau)(Y^*)^\sigma \mathcal{S}}\right)^{\frac{1}{\varphi}}$$

The clearing of labour market in steady-state implies (the derivation of the two equations below are in the firm's equations)

$$C^\sigma \left(\frac{Y}{A}\right)^\varphi = \frac{W}{P}$$

$$MC_t = \frac{W_t(1-\tau)}{P_{H,t} A_t} \quad \Rightarrow \quad MC = \frac{W(1-\tau)}{P_H A}$$

From the Taylor expansion in the price-setting problem of the firm, we have

$$\sum_{k=0}^\infty (\beta\theta)^k \mathbb{E}_t \left\{1 - \frac{\varepsilon}{\varepsilon - 1} MC\right\} = 0 \quad \Rightarrow \quad MC = 1 - \frac{1}{\varepsilon}$$

$$MC = \frac{W(1-\tau)}{P_H A} = 1 - \frac{1}{\varepsilon} \quad \Rightarrow \quad \frac{W}{P} = A \frac{1 - \frac{1}{\varepsilon}}{1-\tau} \frac{P_H}{P} = A \frac{1 - \frac{1}{\varepsilon}}{1-\tau} \frac{1}{h(\mathcal{S})}$$

$$C^\sigma \left(\frac{Y}{A}\right)^\varphi = A \frac{1 - \frac{1}{\varepsilon}}{1-\tau} \frac{1}{h(\mathcal{S})} \quad \Rightarrow \quad \left(C^* \mathcal{Q}^{\frac{1}{\sigma}}\right)^{\frac{\sigma}{\varphi}} \left(\frac{Y}{A}\right) = \left(A \frac{1 - \frac{1}{\varepsilon}}{1-\tau} \frac{1}{h(\mathcal{S})}\right)^{\frac{1}{\varphi}}$$

$$Y = A^{1+\frac{1}{\varphi}} \left(\frac{1 - \frac{1}{\varepsilon}}{(1-\tau)(C^*)^\sigma} \frac{1}{h(\mathcal{S})\mathcal{Q}}\right)^{\frac{1}{\varphi}} = A^{\frac{1+\varphi}{\varphi}} \left(\frac{1 - \frac{1}{\varepsilon}}{(1-\tau)(Y^*)^\sigma \mathcal{S}}\right)^{\frac{1}{\varphi}}$$

Substituting $Y = Y^* = A^{\frac{1+\varphi}{\sigma+\varphi}} \left(\dfrac{1 - \frac{1}{\varepsilon}}{1-\tau}\right)^{\frac{1}{\sigma+\varphi}}$ and solving for $\mathcal{S}$, we have

$$A^{\frac{1+\varphi}{\sigma+\varphi}} \left(\frac{1 - \frac{1}{\varepsilon}}{1-\tau}\right)^{\frac{1}{\sigma+\varphi}} = A^{\frac{1+\varphi}{\varphi}} \left(\frac{1 - \frac{1}{\varepsilon}}{(1-\tau)(Y^*)^\sigma \mathcal{S}}\right)^{\frac{1}{\varphi}} \quad \Rightarrow \quad A^{\frac{\varphi(1+\varphi)}{\sigma+\varphi}} \left(\frac{1 - \frac{1}{\varepsilon}}{1-\tau}\right)^{\frac{\varphi}{\sigma+\varphi}} = A^{1+\varphi} \frac{1 - \frac{1}{\varepsilon}}{(1-\tau)(Y^*)^\sigma \mathcal{S}}$$

$$A^{\frac{\varphi+\varphi^2-\sigma-\varphi-\sigma\varphi-\varphi^2}{\sigma+\varphi}} \left(\frac{1 - \frac{1}{\varepsilon}}{1-\tau}\right)^{\frac{\varphi-\sigma-\varphi}{\sigma+\varphi}} = \frac{1}{(Y^*)^\sigma \mathcal{S}} \quad \Rightarrow \quad A^{\frac{-\sigma(1+\varphi)}{\sigma+\varphi}} \left(\frac{1 - \frac{1}{\varepsilon}}{1-\tau}\right)^{\frac{-\sigma}{\sigma+\varphi}} = \frac{1}{\left[A^{\frac{1+\varphi}{\sigma+\varphi}} \left(\frac{1-\frac{1}{\varepsilon}}{1-\tau}\right)^{\frac{1}{\sigma+\varphi}}\right]^\sigma \mathcal{S}}$$

$$A^{-\frac{1+\varphi}{\sigma+\varphi}}\left(\frac{1-\frac{1}{\varepsilon}}{1-\tau}\right)^{-\frac{1}{\sigma+\varphi}} = \frac{1}{\left[A^{\frac{1+\varphi}{\sigma+\varphi}}\left(\frac{1-\frac{1}{\varepsilon}}{1-\tau}\right)^{\frac{1}{\sigma+\varphi}}\right]\mathcal{S}^{\frac{1}{\sigma}}} \quad\Rightarrow\quad A^{-(1+\varphi)}\left(\frac{1-\frac{1}{\varepsilon}}{1-\tau}\right)^{-1} = \frac{1}{\left[A^{1+\varphi}\left(\frac{1-\frac{1}{\varepsilon}}{1-\tau}\right)\right]\mathcal{S}^{\frac{\sigma+\varphi}{\sigma}}} \quad\Rightarrow\quad \mathcal{S}^{\frac{\sigma+\varphi}{\sigma}} = 1,$$

which gives the result $\mathcal{S} = 1$, which in turn implies $\mathcal{S}_i = 1$ for all i (purchasing parity holds).

Equation (18) - Log-linearized relation between consumption and terms of trade

$$c_t = c_t^* + \frac{1}{\sigma}q_t = c_t^* + \left(\frac{1-\alpha}{\sigma}\right)$$

Considering $\vartheta_i = \vartheta = 1$, we have $C_t = \vartheta_i C_t^i \mathcal{Q}_{i,t}^{\frac{1}{\sigma}} = C_t^i \mathcal{Q}_{i,t}^{\frac{1}{\sigma}}$.

Log-linearizing, we get $c_t = c_t^i + \dfrac{1}{\sigma}q_t$.

Integrating both sides, we have $\displaystyle\int_0^1 c_t di = \int_0^1 c_t^i di + \frac{1}{\sigma}\int_0^1 q_t di \quad\Rightarrow\quad c_t = c_t^* + \frac{1}{\sigma}q_t = c_t^* + \frac{(1-\alpha)s_t}{\sigma}$,

where $c_t^* \equiv \int_0^1 c_t di$

Equation (19) - Uncovered interest parity

$$r_t^i - r_t = \mathbb{E}_t[\Delta e_{i,t+1}]$$

As $\mathbb{E}_t\left[\mathcal{Q}_{t,t+1}R_t^i\dfrac{\mathcal{E}_{i,t+1}}{\mathcal{E}_{i,t}}\right] = 1$ and $\mathbb{E}_t\left[\mathcal{Q}_{t,t+1}R_t\right] = 1$, we have that

$$\mathbb{E}_t\left[\mathcal{Q}_{t,t+1}R_t^i\frac{\mathcal{E}_{i,t+1}}{\mathcal{E}_{i,t}}\right] = \mathbb{E}_t\left[\mathcal{Q}_{t,t+1}R_t\right] \quad\Rightarrow\quad \mathbb{E}_t\left[\mathcal{Q}_{t,t+1}\left(R_t - R_t^i\left[\frac{\mathcal{E}_{i,t+1}}{\mathcal{E}_{i,t}}\right]\right)\right] = 0$$

But to log-linearize it's better to do both sides separately.

$$\mathbb{E}_t\left[\exp\left(\ln\left[\mathcal{Q}_{t,t+1}R_t^i\frac{\mathcal{E}_{i,t+1}}{\mathcal{E}_{i,t}}\right]\right)\right] \approx$$

$$\mathbb{E}_t\left[1 + \ln\frac{\mathcal{Q}R^i\mathcal{E}_i}{\mathcal{E}_i} + \frac{1}{\mathcal{Q}R^i}R^i\frac{\mathcal{E}_i}{\mathcal{E}_i}(\mathcal{Q}_{t,t+1}-\mathcal{Q}) + \frac{1}{\mathcal{Q}R^i}\mathcal{Q}\frac{\mathcal{E}_i}{\mathcal{E}_i}(R_t^i - R^i) - \frac{1}{\mathcal{Q}R^i}\mathcal{Q}\frac{R^i}{\mathcal{E}_i}(\mathcal{E}_{i,t+1}-\mathcal{E}_i) + \frac{1}{\mathcal{Q}R^i}\mathcal{Q}\frac{R^i\mathcal{E}_i}{\mathcal{E}_i^2}(\mathcal{E}_{i,t}-\mathcal{E}_i)\right]$$

$$= 1 + \ln(\mathcal{Q}R^i) + \mathbb{E}_t\left[\hat{q}_t + \hat{r}_t{}^i - \hat{e}_{i,t+1} + \hat{e}_{i,t}\right]$$

$$\mathbb{E}_t\left[\exp\left(\ln\left[\mathcal{Q}_{t,t+1}R_t\right]\right)\right] \approx \mathbb{E}_t\left[1 + \ln(\mathcal{Q}R) + \frac{1}{\mathcal{Q}R}R(\mathcal{Q}_{t,t+1}-Q) + \frac{1}{\mathcal{Q}R}\mathcal{Q}(R_t - R)\right]$$

$$1 + \ln(\mathcal{Q}R^i) + \mathbb{E}_t\left[\hat{q}_t + \hat{r}_t^i - \hat{e}_{i,t+1} + \hat{e}_{i,t}\right] = 1 + \ln(\mathcal{Q}R) + \mathbb{E}_t\left[\hat{q}_t + \hat{r}_t\right] \quad\Rightarrow\quad \hat{r}^i - \mathbb{E}_t\left[e_{i,t+1} - e_{i,t}\right] = \hat{r} \quad\Rightarrow\quad r_t^i - r_t = \mathbb{E}_t[\Delta e_{i,t+1}]$$

Equation (20) - Terms of trade

$$s_t = r_t^* - \mathbb{E}_t[\pi_{t+1}^*] - (r_t - \mathbb{E}_t[\pi_{H,t+1}]) + \mathbb{E}_t[s_{t+1}]$$

The aggregation comes from the FOC. The uncovered interest rate parity allow households to invest both in domestic and foreign assets: B_t, B_t^*. The budget constraint can be written as

$$P_t + Q_{t,t+1}D_{t+1} + Q_{t,t+1}^*\mathcal{E}_{t+1}D_{t+1}^* \leq D_t + \mathcal{E}_t D_t^* + W_t N_t + T_t$$

an the Lagrangean becomes

$$\mathcal{L} = \mathbb{E}_t\sum_{t=0}^{\infty}\beta^t\left[\left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi}\right) + \lambda_t\left(\mathcal{E}_t D_t^* + W_t N_t + T_t - Q_{t,t+1}^*\mathcal{E}_t D_{t+1}^* - Q_{t,t+1}D_{t+1} - P_t C_t\right)\right]$$

The FOCs are:

(C_t) $C_t^{-\sigma} = \lambda_t P_t \quad \Rightarrow \quad \mathbb{E}_t[C_{t+1}^{-\sigma}] = \mathbb{E}_t[\lambda_{t+1} P_{t+1}] \quad \Rightarrow \quad \mathbb{E}_t\left[\frac{\lambda_{t+1}}{\lambda_t}\right] = \mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma} \frac{P_t}{P_{t+1}}\right]$

(D_{t+1}^*) $\beta\ \mathbb{E}_t[\lambda_{t+1}\mathcal{E}_{t+1}] = \lambda_t Q_{t,t+1}^* \mathcal{E}_t \quad \Rightarrow \quad \mathbb{E}_t\left[\frac{\lambda_{t+1}}{\lambda_t}\right] = \mathbb{E}_t\left[\frac{Q_{t,t+1}^*\mathcal{E}_t}{\beta\mathcal{E}_{t+1}}\right]$

(D_t^*) $\beta\ \mathbb{E}_t[\lambda_{t+1}] = \lambda_t Q_{t,t+1} \quad \Rightarrow \quad \mathbb{E}_t\left[\frac{\lambda_{t+1}}{\lambda_t}\right] = \mathbb{E}_t\left[\frac{Q_{t,t+1}}{\beta}\right]$

(N_t) $N_t^{\varphi} = \lambda_t W_t$

Combining the consumption FOC and the foreign bond's FOC we have

$$\mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma} \frac{P_t}{P_{t+1}}\right] = \mathbb{E}_t\left[\frac{Q_{t,t+1}^*\mathcal{E}_t}{\beta\mathcal{E}_{t+1}}\right] \quad \Rightarrow \quad \beta\ \mathbb{E}_t\left[\frac{1}{Q_{t,t+1}^*}\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma} \frac{P_t}{P_{t+1}}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right] = 1$$

Doing the same steps for the domestic bonds, we have

$$\mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma} \frac{P_t}{P_{t+1}}\right] = \mathbb{E}_t\left[\frac{Q_{t,t+1}}{\beta}\right] \quad \Rightarrow \quad \beta\ \mathbb{E}_t\left[\frac{1}{Q_{t,t+1}}\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma} \frac{P_t}{P_{t+1}}\right] = 1$$

Dividing the equation for foreign bonds by the equation for domestic bonds, we have

$$\mathbb{E}_t\left[\frac{Q_{t,t+1}}{Q_{t,t+1}^*}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right] = 1 \quad \Rightarrow \quad \frac{Q_{t,t+1}}{Q_{t,t+1}^*} = \mathbb{E}_t\left[\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right] \quad \Rightarrow \quad \ln(Q_{t,t+1}) - \ln(Q_{t,t+1}^*) \approx \mathbb{E}_t[\ln\mathcal{E}_{t+1} - \ln\mathcal{E}_t] \quad \Rightarrow \quad r_t - r_t^* = \mathbb{E}_t[\Delta e_{t+1}]$$

From the definition of the log terms of trade, we have $s_t = \mathbb{E}_t + p_t^* - p_{H,t}$ and $\mathbb{E}_t[s_{t+1}] = \mathbb{E}_t[e_{t+1} + p_{t+1}^* - p_{H,t+1}]$. Subtracting the first by the second, we get

$$s_t - \mathbb{E}_t[s_{t+1}] = \mathbb{E}_t + p_t^* - p_{H,t} - \mathbb{E}_t[e_{t+1}] - \mathbb{E}_t[p_{t+1}^*] - \mathbb{E}_t[p_{H,t+1}]$$

$$s_t = \mathbb{E}_t[s_{t+1}] - \mathbb{E}_t[\Delta e_{t+1}] - \mathbb{E}_t[\pi_{t+1}^*] - \mathbb{E}_t[\pi_{H,t+1}] = r_t^* - \mathbb{E}_t[\pi_{t+1}^*] - (r_t - \mathbb{E}_t[\pi_{H,t+1}]) + \mathbb{E}_t[s_{t+1}]$$

Equation (21) - Solving for terms of trade

$$s_t = \mathbb{E}_t\left[\sum_{k=0}^{\infty}[\big(r_{t+t}^* - \pi_{t+k+1}^*\big) - (r_{t+t} - \pi_{t+k+1})]\right]$$

Solving forward aquation (20), we have

$$s_t = r_t^* - \mathbb{E}_t[\pi_{t+1}^*] - (r_t - \mathbb{E}_t[\pi_{H,t+1}]) + \mathbb{E}_t[r_{t+1}^* - \mathbb{E}_t[\pi_{t+2}^*] - (r_{t+1} - \mathbb{E}_t[\pi_{H,t+2}]) + \mathbb{E}_t[s_{t+2}]]$$

$$s_t = r_t^* - \mathbb{E}_t[\pi_{t+1}^*] - (r_t - \mathbb{E}_t[\pi_{H,t+1}]) + \mathbb{E}_t[r_{t+1}^* - \mathbb{E}_t[\pi_{t+2}^*] - (r_{t+1} - \mathbb{E}_t[\pi_{H,t+2}]) + \mathbb{E}_t[r_{t+2}^* - \mathbb{E}_t[\pi_{t+3}^*] - (r_{t+2} - \mathbb{E}_t[\pi_{H,t+3}]) + \mathbb{E}_t[s_{t+3}]]]$$

$$s_t = \mathbb{E}_t\left[\sum_{k=0}^{\infty}[\big(r_{t+t}^* - \pi_{t+k+1}^*\big) - (r_{t+t} - \pi_{t+k+1})]\right]$$

3 - Firms

Equation (22) - Log-linearized production function

$y_t = a_t + n_t$

A representative firm has a technology with constant returns:

$Y_t(j) = A_t N_t(j)$, where $a_t = \ln(A_t)$ follows the AR(1) process $a_t = \rho_a a_{t-1} + \varepsilon_t$. Aggregating across firms and log-linearizing we get that:

$$\int_0^1 Y_t(j)dj = \int_0^1 A_t N_t(j)dj \Rightarrow Y_t = A_t N_t \Rightarrow y_t = a_t + n_t$$

The technology defined above leads to a real marginal cost that does not depend on the firm or the produced quantity:

$MC_t = \frac{\partial}{\partial Y_t(j)} \frac{Y_t(j)}{A_t} \frac{W_t(1-\tau)}{P_{H,t}} = \frac{W_t(1-\tau)}{P_{H,t}A_t}$ where τ is an employment subsidy. In this case, the subsidy is defined as the necessary for the production at the efficient level (perfect competition).

Log-linearizing this expression, and setting $\nu \equiv -\ln(1-\tau)$ we get: $mc_t(j) = -\nu + w_t - p_{H,t} - a_t$

Also, $N_t \equiv \int_0^1 N_t(j)dj = \int_0^1 \frac{Y_t(j)}{A_t}dj = \int_0^1 \frac{Y_t(j)}{A_t}\frac{Y_t}{Y_t}dj = \int_0^1 \frac{Y_t(j)}{Y_t}dj\frac{Y_t}{A_t} = \frac{Y_t Z_t}{A_t}$, where $Z_t \equiv \int_0^1 \frac{Y_t(j)}{Y_t}dj$

Equation (B.1) Firm price-setting

$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left\{ Q_{t,t+k} Y_{t+k} \left[\overline{P}_{H,t} - \frac{\varepsilon}{\varepsilon - 1} MC_{t+k}^n \right] \right\} = 0$$

If the firm can adjust price at time t, it will set its price at $\overline{P}_{H,t}(j)$, which maximizes the present value of its future profit:

$$\max_{\overline{P}_{H,t}(j)} \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left[Q_{t,t+k}[Y_{t+k}(j)(\overline{P}_{H,t}(j) - MC_{t+k}P_{H,t+k})] \right]$$

Also, we know that $Y_t^{\frac{\varepsilon-1}{\varepsilon}} = \int_0^1 Y_t(j)^{\frac{\varepsilon-1}{\varepsilon}}dj$, so $Y_{t+k}^{\frac{\varepsilon-1}{\varepsilon}} = \int_0^1 Y_{t+k}(j)^{\frac{\varepsilon-1}{\varepsilon}}dj$ and

$$Y_{t+k}(j) \le \left(\frac{\overline{P}_{H,t}}{P_{H,t+k}}\right)^{-\varepsilon} \left(C_{H,t+k} + \int_0^1 C_{H,t+k}^i di\right)$$

Then maximizing, the restriction above is binding (otherwise part of the production would have been wasted, which contradicts the hypothesis of optimization when the costs are not null). The total demand is

$$Y_{t+k}^{\frac{\varepsilon-1}{\varepsilon}} = \int_0^1 \left[\left(\frac{\overline{P}_{H,t}}{P_{H,t+k}}\right)^{-\varepsilon} \left(C_{H,t+k} + \int_0^1 C_{H,t+k}^i di\right) \right]^{\frac{\varepsilon-1}{\varepsilon}} dj$$

As nothing in the RHS of the above expression depends on (j), we can take everything out of the integral.

$$Y_{t+k}^{\frac{\varepsilon-1}{\varepsilon}} = \left[\left(\frac{\overline{P}_{H,t}}{P_{H,t+k}}\right)^{-\varepsilon} \left(C_{H,t+k} + \int_0^1 C_{H,t+k}^i di\right) \right]^{\frac{\varepsilon-1}{\varepsilon}} \int_0^1 dj \text{ and}$$

$$Y_{t+k} = \left(\frac{\overline{P}_{H,t}}{P_{H,t+k}}\right)^{-\varepsilon} \left(C_{H,t+k} + \int_0^1 C_{H,t+k}^i di\right)$$

The price setting problem of the firm becomes then:

$$\max_{\overline{P}_{H,t}(j)} \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left[Q_{t,t+k}[Y_{t+k}(j)(\overline{P}_{H,t}(j) - MC_{t+k}P_{H,t+k})] \right]$$

subject to $\left(\frac{\overline{P}_{H,t}}{P_{H,t+k}}\right)^{-\varepsilon} \left(C_{H,t+k} + \int_0^1 C_{H,t+k}^i di\right) - Y_{t+k}(j) = 0$

Substituting the restriction in the maximization problem, we get an unrestricted maximization problem.

$$\max_{\overline{P}_{H,t}} \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left\{ Q_{t,t+k} \left[\left(\frac{\overline{P}_{H,t}}{P_{H,t+k}}\right)^{-\varepsilon} \left(C_{H,t+k} + \int_0^1 C_{H,t+k}^i di\right) (\overline{P}_{H,t} - MC_{t+k}^n) \right] \right\}$$

$$= \max_{\overline{P}_{H,t}} \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left\{ Q_{t,t+k} \left(C_{H,t+k} + \int_0^1 C_{H,t+k}^i di\right) \left[\frac{\overline{P}_{H,t}^{1-\varepsilon}}{P_{H,t+k}^{-\varepsilon}} - \left(\frac{\overline{P}_{H,t}}{P_{H,t+k}}\right)^{-\varepsilon} MC_{t+k}^n \right] \right\},$$

which yields, after deriving with respect to $\overline{P}_{H,t}$

$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left\{ Q_{t,t+k} \left(C_{H,t+k} + \int_0^1 C_{H,t+k}^i di\right) \left[(1-\varepsilon)\frac{\overline{P}_{H,t}^{-\varepsilon}}{P_{H,t+k}^{-\varepsilon}} + \varepsilon \frac{\overline{P}_{H,t}^{-\varepsilon-1}}{P_{H,t+k}^{-\varepsilon}} MC_{t+k}^n \right] \right\} = 0$$

Under flexible prices, $\overline{P}_{H,t} = P_{H,t}$ and $\overline{P}_{H,t+k} = P_{H,t+k}$

$$(1-\varepsilon)+\varepsilon\frac{\overline{MC}_t^n}{\overline{P}_{H,t}}=0 \quad\Rightarrow\quad \overline{P}_{H,t}=\frac{\varepsilon}{\varepsilon-1}\overline{MC}_t^n$$

Substituting back to $Q_{t,t+k}$ and rearranging, observing that $Y_{t+k}=\left(\frac{\overline{P}_{H,t}}{P_{H,t+k}}\right)^{-\varepsilon}\left(C_{H,t+k}+\int_0^1 C^i_{H,t+k}di\right)$, we get

$$\sum_{k=0}^{\infty}\theta^k\mathbb{E}_t\left\{Q_{t,t+k}\left[(-\varepsilon)\left(\frac{\overline{P}_{H,t}^{-\varepsilon-1}}{P_{H,t+k}^{-\varepsilon}}\right)\left(C_{H,t+k}+\int_0^1 C^i_{H,t+k}di\right)\left[\overline{P}_{H,t}-MC^n_{t+k}\right]+Y_{t+k}\right]\right\}=0$$

$$\sum_{k=0}^{\infty}\theta^k\mathbb{E}_t\left\{Q_{t,t+k}\left[\frac{(-\varepsilon)}{\overline{P}_{H,t}}\left(\frac{\overline{P}_{H,t}}{P_{H,t+k}}\right)^{-\varepsilon}\left(C_{H,t+k}+\int_0^1 C^i_{H,t+k}di\right)\left[\overline{P}_{H,t}-MC^n_{t+k}\right]+Y_{t+k}\right]\right\}=0$$

$$\sum_{k=0}^{\infty}\theta^k\mathbb{E}_t\left\{Q_{t,t+k}\left[\frac{(-\varepsilon)}{\overline{P}_{H,t}}Y_{t+k}\left[\overline{P}_{H,t}-MC^n_{t+k}\right]+Y_{t+k}\right]\right\}=0$$

As ε and $\overline{P}_{H,t}$ don't depend on k and the expression is equal zero, we can do the following operation

$$\sum_{k=0}^{\infty}\theta^k\mathbb{E}_t\left\{Q_{t,t+k}\left[Y_{t+k}\left[-\varepsilon+\varepsilon\frac{MC^n_{t+k}}{\overline{P}_{H,t}}\right]\right]+Y_{t+k}\right\}=0 \quad\Rightarrow\quad \sum_{k=0}^{\infty}\theta^k\mathbb{E}_t\left\{Q_{t,t+k}Y_{t+k}\left[(1-\varepsilon)+\varepsilon\frac{MC^n_{t+k}}{\overline{P}_{H,t}}\right]\frac{\overline{P}_{H,t}}{1-\varepsilon}\right\}=0$$

$$\sum_{k=0}^{\infty}\theta^k\mathbb{E}_t\left\{Q_{t,t+k}Y_{t+k}\left[\overline{P}_{H,t}-\frac{\varepsilon}{\varepsilon-1}MC^n_{t+k}\right]\right\}=0$$

Equation (B.2) - log-linearization of price setting

$\overline{p}_{H,t}-p_{H,t-1}=\beta\theta\mathbb{E}_t\{\overline{p}_{H,t+1}-p_{H,t}\}+\pi_{H,t}+(1-\beta\theta)\widehat{mc}_t$

Using the fact that $Q_{t,t+k}=\beta^k\left(\frac{C_{t+k}}{C_t}\right)^{-\sigma}\frac{P_t}{P_{t+k}}$, we have

$$\sum_{k=0}^{\infty}\theta^k\mathbb{E}_t\left\{\beta^k\left(\frac{C_{t+k}}{C_t}\right)^{-\sigma}\frac{P_t}{P_{t+k}}Y_{t+k}\left[\overline{P}_{H,t}-\frac{\varepsilon}{\varepsilon-1}MC^n_{t+k}\right]\right\}=0$$

As P_t and C_t doesn't depend on k, we can put it out of summation and ignore (as the expression equals zero)

$$\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t\left\{C_{t+k}^{-\sigma}P_{t+k}^{-1}Y_{t+k}\left[\overline{P}_{H,t}-\frac{\varepsilon}{\varepsilon-1}MC^n_{t+k}\right]\right\}=0$$

$$\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t\left\{C_{t+k}^{-\sigma}Y_{t+k}\frac{P_{H,t-1}}{P_{t+k}}\left[\frac{\overline{P}_{H,t}}{P_{H,t-1}}-\frac{\varepsilon}{\varepsilon-1}\frac{P_{H,t+k}}{P_{H,t-1}}\frac{MC^n_{t+k}}{P_{H,t+k}}\right]\right\}=0$$

$$\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t\left\{C_{t+k}^{-\sigma}Y_{t+k}\frac{P_{H,t-1}}{P_{t+k}}\left[\frac{\overline{P}_{H,t}}{P_{H,t-1}}-\frac{\varepsilon}{\varepsilon-1}\Pi^H_{t-1,t+k}MC_{t+k}\right]\right\}=0 \text{ where } \Pi^H_{t-1,t+k}=\frac{P_{H,t+k}}{P_{H,t-1}} \text{ and } MC_{t+k}=\frac{MC^n_{t+k}}{P_{H,t+k}}$$

As we will use a first-order approximation, we can ignore the Jensen's inequality. To make the multivariate Taylor expansion, we can use $f(x+\Delta x)\approx f(x)+\Delta x^T\nabla f|_x(\Delta x)$, where x is the vector of variables in the steady-state.

$$\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t\left\{C_{t+k}^{-\sigma}Y_{t+k}\frac{P_{H,t-1}}{P_{t+k}}\left[\frac{\overline{P}_{H,t}}{P_{H,t-1}}-\frac{\varepsilon}{\varepsilon-1}\Pi^H_{t-1,t+k}MC_{t+k}\right]\right\}=\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t\left\{C^{-\sigma}Y\frac{P_H}{P}\left[\frac{\overline{P}_H}{P_H}-\frac{\varepsilon}{\varepsilon-1}MC\right]\right\}$$

$$+\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t\left\{-\sigma C^{-\sigma-1}Y\frac{P_H}{P}\left[1-\frac{\varepsilon}{\varepsilon-1}MC\right](C_{t+k}-C)\right\}+\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t\left\{C^{-\sigma}\frac{P_H}{P}\left[1-\frac{\varepsilon}{\varepsilon-1}MC\right](Y_{t+k}-Y)\right\}$$

$$+\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t\left\{\left(C^{-\sigma}Y\frac{1}{P}\left[1-\frac{\varepsilon}{\varepsilon-1}MC\right]+C^{-\sigma}Y\frac{P_H}{P}\left[-\frac{1}{P_H}\right]\right)(P_{H,t-1}-P_H)\right\}$$

$$+\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t\left\{-C^{-\sigma}Y\frac{P_H}{P^2}\left[1-\frac{\varepsilon}{\varepsilon-1}MC\right](P_{t+k}-P)\right\}+\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t\left\{C^{-\sigma}Y\frac{P_H}{P}\left[\frac{1}{P_H}\right](\overline{P}_{H,t}-P_H)\right\}$$

$$+\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t\left\{C^{-\sigma}Y\frac{P_H}{P}\left[-\frac{\varepsilon}{\varepsilon-1}MC\right](\Pi^H_{t-1,t+k}-\Pi^H)\right\}+\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t\left\{C^{-\sigma}Y\frac{P_H}{P}\left[-\frac{\varepsilon}{\varepsilon-1}\Pi^H\right](MC_{t+k}-MC)\right\}=0$$

As $C^{-\sigma}, Y, P_H, P$ don't depend on k, we have

$$\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t\left\{1-\frac{\varepsilon}{\varepsilon-1}MC\right\}+\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t\left\{-\sigma\left[1-\frac{\varepsilon}{\varepsilon-1}MC\right]\hat{c}_{t+k}\right\}+\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t\left\{\left[1-\frac{\varepsilon}{\varepsilon-1}MC\right]\hat{y}_{t+k}\right\}$$

$$+\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t\left\{\left(-\frac{\varepsilon}{\varepsilon-1}MC\right)\hat{p}_{H,t-1}\right\}+\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t\left\{-\left[1-\frac{\varepsilon}{\varepsilon-1}MC\right](\hat{p}_{t+k})\right\}+\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t\left\{(\hat{\overline{p}}_{H,t})\right\}$$

$$+\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t\left\{\left[-\frac{\varepsilon}{\varepsilon-1}MC\right]\pi_{t-1,t+k}\right\}+\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t\left\{MC\left[-\frac{\varepsilon}{\varepsilon-1}\right]\widehat{mc}_{t+k}\right\}=0$$

$$\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t\left\{\left(1-\frac{\varepsilon}{\varepsilon-1}MC\right)(1-\sigma\hat{c}_{t+k}+\hat{y}_{t+k}+\hat{p}_{H,t-1}-\hat{p}_{t+k}+\pi_{t-1,t+k}+\widehat{mc}_{t+k})-\hat{p}_{H,t-1}+\hat{\overline{p}}_{H,t}-\pi_{t-1,t+k}-\widehat{mc}_{t+k}\right\}$$

As the Taylor approximation is around zero (we are assuming regularity conditions to all functions),

$\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t\left\{1-\frac{\varepsilon}{\varepsilon-1}MC\right\}=0$, we have

$$\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t\left\{-\hat{p}_{H,t-1}+\hat{\overline{p}}_{H,t}-\pi_{t-1,t+k}-\widehat{mc}_{t+k}\right\}=0$$

$$\frac{1}{1-\beta\theta}(\overline{p}_{H,t}-p_H)+\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t\left\{-(p_{H,t-1}-p_H)-(p_{H,t+k}-p_{H,t-1})-\widehat{mc}_{t+k}\right\}=0$$

$$\frac{1}{1-\beta\theta}(\overline{p}_{H,t}-p_{H,t-1})+\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t\left\{-p_{H,t+k}+p_{H,t}-p_{H,t}+p_{H,t-1}-\widehat{mc}_{t+k}\right\}=0$$

$$\overline{p}_{H,t}=p_{H,t-1}+\pi_{H,t}+\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t\{\pi_{H,t+k}\}-\beta\theta\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t\{\pi_{H,t+k}\}+(1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t\{\widehat{mc}_{t+k}\}$$

$$\overline{p}_{H,t}=p_{H,t-1}+\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t\{\pi_{H,t+k}\}+(1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t\{\widehat{mc}_{t+k}\}$$

Rational expectations imply that the difference between the actual inflation and the future expectations is the steady-state inflation, which is zero. This expression can also be written as (it's easy to see when trying to iterate forward).

$$\overline{p}_{H,t}-p_{H,t-1}=\beta\theta\mathbb{E}_t\{\overline{p}_{H,t+1}-p_{H,t}\}+\pi_{H,t}+(1-\beta\theta)\widehat{mc}_t$$

Equation (B.3) Price index

The price-setting equation is $P_H\equiv\left[\theta(P_{H,t-1})^{1-\varepsilon}+(1-\theta)(\overline{P}_{H,t})^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}$

Equation (23) - Expected nominal marginal costs

$$p_{H,t}=\mu+(1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t\{mc^n_{t+k}\}$$

Substituting $\widehat{mc}_t=mc^n_t-p_{H,t}+\mu$, we have

$$\overline{p}_{H,t}-p_{H,t-1}=\beta\theta\mathbb{E}_t\{\overline{p}_{H,t+1}-p_{H,t}+\pi_{H,t}+(1-\beta\theta)(mc^n_t-p_{H,t}+\mu)\}$$

$$\overline{p}_{H,t}-p_{H,t-1}=\mathbb{E}_t\{\beta\theta\overline{p}_{H,t+1}-\beta\theta p_{H,t}+p_{H,t}-p_{H,t-1}+mc^n_t-p_{H,t}+\mu-\beta\theta mc^n_t+\beta\theta p_{H,t}-\beta\theta\mu\}$$

$$\overline{p}_{H,t}=\mathbb{E}_t\{\beta\theta\overline{p}_{H,t+1}+mc^n_t+\mu-\beta\theta mc^n_t-\beta\theta\mu\}$$

which can also be written as $p_{H,t} = \mu + (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k \mathbb{E}_t\{mc^n_{t+k}\}$

4 - Equilibrium

Equation (24) - Demand for each firm in the small open economy

$$Y_t(j) = \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\varepsilon}\left[(1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t + \alpha\int_0^1 \left(\frac{P_{H,t}}{\mathcal{E}_{i,t}P^i_{F,t}}\right)^{-\gamma}\left(\frac{P^i_{F,t}}{P^i_t}\right)^{-\eta} C^i_t di\right]$$

Good market clearing in the representative small open economy ("home") requires that the total produced inside the small economy is consumed either by the households of this economy or imported (from the small economy, subscript H) from households of any other country (superscript i).

$$Y_t(j) = C_{H,t}(j) + \int_0^1 C^i_{H,t}(j)di = \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\varepsilon} C_{H,t} + \int_0^1 \left(\frac{P^i_{H,t}(j)}{P^i_{H,t}}\right)^{-\varepsilon} C^i_{H,t}di$$

$$Y_t(j) = \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\varepsilon}(1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t + \int_0^1 \left(\frac{P^i_{H,t}(j)}{P^i_{H,t}}\right)^{-\varepsilon}\left(\frac{P^i_{H,t}}{P^i_{F,t}}\right)^{-\gamma} C^i_{F,t}di$$

$$Y_t(j) = \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\varepsilon}(1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t + \int_0^1 \left(\frac{P^i_{H,t}(j)}{P^i_{H,t}}\right)^{-\varepsilon}\left(\frac{P_{H,t}}{\mathcal{E}_{i,t}P^i_{F,t}}\right)^{-\gamma}\alpha\left(\frac{P^i_{F,t}}{P^i_t}\right)^{-\eta} C^i_t di$$

Assuming symmetric preferences across countries, the home country will sell the variety of good j for the same price, independently of which country is buying, which implies $\frac{P^i_{H,t}(j)}{P^i_{H,t}} = \frac{P_{H,t}(j)}{P_{H,t}}$

Thus, $Y_t(j) = \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\varepsilon}\left[(1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t + \alpha\int_0^1 \left(\frac{P_{H,t}}{\mathcal{E}_{i,t}P^i_{F,t}}\right)^{-\gamma}\left(\frac{P^i_{F,t}}{P^i_t}\right)^{-\eta} C^i_t di\right]$

Equation (25) - Aggregate domestic output

$$Y_t = \left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t\left[(1-\alpha) + \alpha\int_0^1 \left(\mathcal{S}_{i,t}\mathcal{S}^i_t\right)^{\gamma-\eta}\mathcal{Q}_{i,t}^{\eta-\frac{1}{\sigma}} di\right]$$

Plugging on the definition of aggregate domestic output $Y_t^{\frac{\varepsilon-1}{\varepsilon}} = \int_0^1 Y_t(j)^{\frac{\varepsilon-1}{\varepsilon}} dj$ we have

$$Y_t^{\frac{\varepsilon-1}{\varepsilon}} = \int_0^1\left(\left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\varepsilon}\left[(1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t + \alpha\int_0^1 \left(\frac{P_{H,t}}{\mathcal{E}_{i,t}P^i_{F,t}}\right)^{-\gamma}\left(\frac{P^i_{F,t}}{P^i_t}\right)^{-\eta} C^i_t di\right]\right)^{\frac{\varepsilon-1}{\varepsilon}} dj$$

$$Y_t^{\frac{\varepsilon-1}{\varepsilon}} = \left[(1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t + \alpha\int_0^1 \left(\frac{P_{H,t}}{\mathcal{E}_{i,t}P^i_{F,t}}\right)^{-\gamma}\left(\frac{P^i_{F,t}}{P^i_t}\right)^{-\eta} C^i_t di\right]^{\frac{\varepsilon-1}{\varepsilon}}\left(\frac{1}{P_{H,t}}\right)^{1-\varepsilon}\int_0^1 P_{H,t}(j)^{1-\varepsilon}dj$$

By the definition of price index, $P_H^{1-\varepsilon} = \int_0^1 P_{H,t}(j)^{1-\varepsilon}dj$, we have

$$Y_t^{\frac{\varepsilon-1}{\varepsilon}} = \left[(1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t + \alpha\int_0^1 \left(\frac{P_{H,t}}{\mathcal{E}_{i,t}P^i_{F,t}}\right)^{-\gamma}\left(\frac{P^i_{F,t}}{P^i_t}\right)^{-\eta} C^i_t di\right]^{\frac{\varepsilon-1}{\varepsilon}}\left(\frac{1}{P_{H,t}}\right)^{1-\varepsilon} P_H^{1-\varepsilon}$$

$$Y_t^{\frac{\varepsilon-1}{\varepsilon}} = \left[(1-\alpha) \left(\frac{P_{H,t}}{P_t} \right)^{-\eta} C_t + \alpha \int_0^1 \left(\frac{P_{H,t}}{\mathcal{E}_{i,t} P_{F,t}^i} \right)^{-\gamma} \left(\frac{P_{F,t}^i}{P_t^i} \right)^{-\eta} C_t^i di \right]^{\frac{\varepsilon-1}{\varepsilon}}$$

$$Y_t = (1-\alpha) \left(\frac{P_{H,t}}{P_t} \right)^{-\eta} C_t + \alpha \int_0^1 \left(\frac{P_{H,t}}{\mathcal{E}_{i,t} P_{F,t}^i} \right)^{-\gamma} \left(\frac{P_{F,t}^i}{P_t^i} \right)^{-\eta} C_t^i di$$

Using the fact that $\mathcal{Q}_{i,t} = \dfrac{\mathcal{E}_{i,t} P_t^i}{P_t}$

$$Y_t = (1-\alpha) \left(\frac{P_{H,t}}{P_t} \right)^{-\eta} C_t + \alpha \int_0^1 \left(\frac{\mathcal{E}_{i,t} P_{F,t}^i}{P_{H,t}} \right)^{\gamma} \left(P_{F,t}^i \right)^{-\eta} \left(\frac{\mathcal{Q}_{i,t} P_t}{\mathcal{E}_{i,t}} \right)^{\eta} C_t^i di = \left(\frac{P_{H,t}}{P_t} \right)^{-\eta} \left[(1-\alpha) C_t + \alpha \int_0^1 \left(\frac{\mathcal{E}_{i,t} P_{F,t}^i}{P_{H,t}} \right)^{\gamma-\eta} \mathcal{Q}_{i,t}^{\eta} C_t^i di \right]$$

Considering that $\mathcal{S}_{i,t} = \dfrac{P_{i,t}}{P_{H,t}}$, $\mathcal{S}_t^i = \dfrac{\mathcal{E}_{i,t} P_{F,t}^i}{P_{i,t}}$, and $C_t^i = C_t \mathcal{Q}_{i,t}^{-\frac{1}{\sigma}}$ we have

$$Y_t = \left(\frac{P_{H,t}}{P_t} \right)^{-\eta} \left[(1-\alpha) C_t + \alpha \int_0^1 \left(\frac{P_{i,t}}{P_{H,t}} \frac{\mathcal{E}_{i,t} P_{F,t}^i}{P_{i,t}} \right)^{\gamma-\eta} \mathcal{Q}_{i,t}^{\eta} \mathcal{Q}_{i,t}^{-\frac{1}{\sigma}} C_t di \right] = \left(\frac{P_{H,t}}{P_t} \right)^{-\eta} C_t \left[(1-\alpha) + \alpha \int_0^1 \left(\mathcal{S}_{i,t} \mathcal{S}_t^i \right)^{\gamma-\eta} \mathcal{Q}_{i,t}^{\eta-\frac{1}{\sigma}} di \right]$$

Equation (26) - Small economy output in a particular case

$$Y_t = \mathcal{S}_t^{\alpha}$$

If $\sigma = \eta = \gamma = 1$, then

$$Y_t = \left(\frac{P_{H,t}}{P_t} \right)^{-1} C_t \left[(1-\alpha) + \alpha \int_0^1 \left(\mathcal{S}_{i,t} \mathcal{S}_t^i \right)^0 \mathcal{Q}_{i,t}^0 di \right] = \frac{P_t}{P_{H,t}} C_t = C_t \mathcal{S}_t^{\alpha}$$

as if $\eta = 1$, the CPI takes the form $P_t = (P_{H,t})^{1-\alpha} (P_{F,t})^{\alpha}$ implying $\dfrac{P_t}{P_{H,t}} = \left(\dfrac{P_{F,t}}{P_{H,t}} \right)^{\alpha} = \mathcal{S}^{\alpha}$

Equation (27) - Log-linearized output

$$y_t = c_t + \frac{\alpha\omega}{\sigma} s_t$$

Log-linearizing the equation (25) using the Taylor expansion around the symmetric steady-state, we have

$$Y_t \approx Y + (-\eta) (P_H)^{-\eta-1} \left(\frac{1}{P} \right)^{-\eta} C \left[(1-\alpha) + \alpha \int_0^1 \left(\mathcal{S}_i \mathcal{S}^i \right)^{\gamma-\eta} \mathcal{Q}_i^{\eta-\frac{1}{\sigma}} di \right] (P_{H,t} - P_H)$$

$$+\eta (P_H)^{-\eta} \left(\frac{1}{P} \right)^{-\eta+1} C \left[(1-\alpha) + \alpha \int_0^1 \left(\mathcal{S}_i \mathcal{S}^i \right)^{\gamma-\eta} \mathcal{Q}_i^{\eta-\frac{1}{\sigma}} di \right] (P_t - P) + \left(\frac{P_H}{P} \right)^{-\eta} \left[(1-\alpha) + \alpha \int_0^1 \left(\mathcal{S}_i \mathcal{S}^i \right)^{\gamma-\eta} \mathcal{Q}_i^{\eta-\frac{1}{\sigma}} di \right] (C_t - C)$$

$$+ \left(\frac{P_H}{P} \right)^{-\eta} C \left[\alpha \int_0^1 (\gamma-\eta) (\mathcal{S}_i)^{\gamma-\eta-1} \left(\mathcal{S}^i \right)^{\gamma-\eta} \mathcal{Q}_i^{\eta-\frac{1}{\sigma}} (\mathcal{S}_{i,t} - \mathcal{S}_i) di \right] + \left(\frac{P_H}{P} \right)^{-\eta} C \left[\alpha \int_0^1 (\gamma-\eta) (\mathcal{S}_i)^{\gamma-\eta} \left(\mathcal{S}^i \right)^{\gamma-\eta-1} \mathcal{Q}_i^{\eta-\frac{1}{\sigma}} (\mathcal{S}_t^i - \mathcal{S}^i) di \right]$$

$$+ \left(\frac{P_H}{P} \right)^{-\eta} C \left[\alpha \int_0^1 \left(\eta - \frac{1}{\sigma} \right) \left(\mathcal{S}_i \mathcal{S}^i \right)^{\gamma-\eta} \mathcal{Q}_i^{\eta-\frac{1}{\sigma}-1} (\mathcal{Q}_{i,t} - \mathcal{Q}_i) di \right]$$

As shown in appendix A (and above, in the international risk sharing section), in a symmetric steady-state $\mathcal{Q}_i = \mathcal{S}_i = \mathcal{S}^i = 1$ for all i (purchasing parity holds). Thus

$$Y_t - Y = \left(\frac{P_H}{P} \right)^{-\eta} C \left[-\eta \frac{P_{H,t} - P_H}{P_H} + \eta \frac{P_t - P}{P} + \frac{C_t - C}{C} \right]$$

$$+ \left(\frac{P_H}{P} \right)^{-\eta} C \left[\alpha \int_0^1 (\gamma-\eta) \left[\frac{\mathcal{S}_t^i - \mathcal{S}^i}{\mathcal{S}^i} + \frac{\mathcal{S}_{i,t} - \mathcal{S}_i}{\mathcal{S}_i} \right] + \left(\eta - \frac{1}{\sigma} \right) \frac{\mathcal{Q}_{i,t} - \mathcal{Q}_i}{\mathcal{Q}_i} di \right]$$

As $\left(\frac{P}{P_H}\right)^{1-\eta} = (1-\alpha) + \alpha \int_o^1 (\mathcal{S}_i)^{1-\eta} di = 1$ (from international risk sharing section) in a symmetric steady-state and $C = Y$ in the international market clearing, we have

$$\frac{Y_t - Y}{Y} = -\eta \hat{p}_{H,t} + \eta \hat{p}_t + \hat{c}_t + \left[\alpha(\gamma - \eta) \int_0^1 s_t^i di + \alpha(\gamma - \eta) \int_0^1 s_{i,t} di + \alpha \left(\eta - \frac{1}{\sigma}\right) \int_0^1 q_{i,t} di\right]$$

Considering that $p_t - p_{H,t} = \alpha s_t$ and recalling that $\int_0^1 s_t^i di = 0$, $s_t = \int_0^1 s_{i,t} di$ and $q_t \equiv \int_0^1 q_{i,t} di$, we have

$$y_t - y = \eta \alpha s_t + c_t - c + \alpha(\gamma - \eta) s_t + \alpha \left(\eta - \frac{1}{\sigma}\right) q_{i,t} \quad \Rightarrow \quad y_t = c_t + \alpha \gamma s_t + \alpha \left(\eta - \frac{1}{\sigma}\right) q_{i,t}$$

As $q_t = (1-\alpha) s_t$, derived in the section domestic inflation and CPI inflation, we have

$$y_t = c_t + \alpha \gamma s_t + \alpha \left(\eta - \frac{1}{\sigma}\right)(1-\alpha) s_t = c_t + \frac{\alpha}{\sigma}\left[\sigma\gamma + (\sigma\eta - 1)(1-\alpha)\right] s_t = c_t + \frac{\alpha\omega}{\sigma} s_t$$

where $\omega \equiv \sigma\gamma + (\sigma\eta - 1)(1-\alpha)$

Equation (28) - International market clearing

$$y_t^* \equiv \int_0^1 y_t^i di = \int_0^1 c_t^i di \equiv c_t^*$$

The result comes directly from the definition above

Equation (29) - Relation between output and terms of trade

$$y_t = y_t^* + \frac{1}{\sigma_\alpha} s_t$$

As the equation (28) equation holds for all countries, from a generic country we have

$$y_t^i = c_t^i + \frac{\alpha\omega}{\sigma} s_t^i$$

Aggregating over all countries we have the world market clearing condition. The equatily follows from the fact that $\int_0^1 s_t^i di = 0$. International market clearing implies $y_t^* \equiv \int_0^1 y_t^i di = \int_0^1 c_t^i di \equiv c_t^*$.

As $c_t = c_t^* + \left(\frac{1-\alpha}{\sigma}\right) s_t$, we have that

$$y_t - \frac{\alpha\omega}{\sigma} s_t = c_t^* + \left(\frac{1-\alpha}{\sigma}\right) s_t \quad \Rightarrow \quad y_t = y_t^* + \left(\frac{1-\alpha}{\sigma} + \frac{\alpha\omega}{\sigma}\right) s_t \quad \Rightarrow \quad y_t = y_t^* + \frac{1}{\sigma_\alpha} s_t$$

where $\sigma_\alpha \equiv \frac{\sigma}{(1-\alpha) + \alpha\omega}$

Equation (30) - Log-linearized Euler equation

$$y_t = \mathbb{E}_t\{y_{t+1}\} - \frac{1}{\sigma_\alpha}(r_t - \mathbb{E}_t\{\pi_{H,t+1}\} - \rho) + \alpha\Theta\mathbb{E}_t\{\Delta y_{t+1}^*\}$$

Recalling that $c_t = \mathbb{E}_t\{c_{t+1}\} - \frac{1}{\sigma}(r_t - \mathbb{E}_t\{\pi_{t+1}\} - \rho)$, we have

$$y_t - \frac{\alpha\omega}{\sigma} s_t = \mathbb{E}_t\{c_{t+1}\} - \frac{1}{\sigma}(r_t - \mathbb{E}_t\{\pi_{t+1}\} - \rho) \quad \Rightarrow \quad y_t = \frac{\alpha\omega}{\sigma} s_t + \mathbb{E}_t\{y_{t+1}\} - \frac{\alpha\omega}{\sigma}\mathbb{E}_t\{s_{t+1}\} - \frac{1}{\sigma}(r_t - \mathbb{E}_t\{\pi_{t+1}\} - \rho)$$

$$y_t = \mathbb{E}_t\{y_{t+1}\} - \frac{1}{\sigma}(r_t - \mathbb{E}_t\{\pi_{t+1}\} - \rho) - \frac{\alpha\omega}{\sigma}\mathbb{E}_t\{\Delta s_{t+1}\}$$

As $\pi_t = \pi_{H,t} + \alpha \Delta s_t$, we have

$$y_t = \mathbb{E}_t\{y_{t+1}\} - \frac{1}{\sigma}(r_t - \mathbb{E}_t\{\pi_{H,t+1}\} - \alpha\mathbb{E}_t\{\Delta s_{t+1}\} - \rho) - \frac{\alpha\omega}{\sigma}\mathbb{E}_t\{\Delta s_{t+1}\} = \mathbb{E}_t\{y_{t+1}\} - \frac{1}{\sigma}(r_t - \mathbb{E}_t\{\pi_{H,t+1}\} - \rho) - \frac{\alpha(\omega-1)}{\sigma}\mathbb{E}_t\{\Delta s_{t+1}\}$$

$$y_t = \mathbb{E}_t\{y_{t+1}\} - \frac{1}{\sigma}(r_t - \mathbb{E}_t\{\pi_{H,t+1}\} - \rho) - \frac{\alpha\Theta}{\sigma}\mathbb{E}_t\{\Delta s_{t+1}\},$$

where $\Theta = \omega - 1 = \sigma\gamma + (\sigma\eta - 1)(1-\alpha) - 1 = (\sigma\gamma - 1) + (\sigma\eta - 1)(1-\alpha)$

As $\Delta s_t = \sigma_\alpha(\Delta y_t - \Delta y_t^*)$

$$y_t = \mathbb{E}_t\{y_{t+1}\} - \frac{1}{\sigma}(r_t - \mathbb{E}_t\{\pi_{H,t+1}\} - \rho) - \frac{\alpha\Theta}{\sigma}\mathbb{E}_t\{\sigma_\alpha(\Delta y_{t+1} - \Delta y_{t+1}^*)\}$$

$$\sigma y_t = \sigma\mathbb{E}_t\{y_{t+1}\} - (r_t - \mathbb{E}_t\{\pi_{H,t+1}\} - \rho) - \alpha\Theta\sigma_\alpha(\mathbb{E}_t\{\Delta y_{t+1}\} - \mathbb{E}_t\{\Delta y_{t+1}^*\})$$

$$\sigma y_t = \sigma\mathbb{E}_t\{y_{t+1}\} - (r_t - \mathbb{E}_t\{\pi_{H,t+1}\} - \rho) - \alpha(\omega-1)\frac{\sigma}{(1-\alpha)+\alpha\omega}(\mathbb{E}_t\{y_{t+1}\} - y_t - \mathbb{E}_t\{\Delta y_{t+1}^*\})$$

$$\frac{\sigma y_t - \alpha\sigma y_t + \sigma\alpha\omega y_t - \sigma\alpha\omega y_t + \alpha\sigma y_t}{(1-\alpha)+\alpha\omega} = \frac{\sigma\mathbb{E}_t\{y_{t+1}\} - \alpha\sigma\mathbb{E}_t\{y_{t+1}\} + \sigma\alpha\omega\mathbb{E}_t\{y_{t+1}\} - \sigma\alpha\omega\mathbb{E}_t\{y_{t+1}\} + \alpha\sigma\mathbb{E}_t\{y_{t+1}\}}{(1-\alpha)+\alpha\omega}$$

$$-(r_t - \mathbb{E}_t\{\pi_{H,t+1}\} - \rho) + \alpha(\omega-1)\frac{\sigma}{(1-\alpha)+\alpha\omega}\mathbb{E}_t\{\Delta y_{t+1}^*\}$$

$$\frac{\sigma y_t}{(1-\alpha)+\alpha\omega} = \frac{\sigma\mathbb{E}_t\{y_{t+1}\}}{(1-\alpha)+\alpha\omega} - (r_t - \mathbb{E}_t\{\pi_{H,t+1}\} - \rho) + \alpha\Theta\sigma_\alpha\mathbb{E}_t\{\Delta y_{t+1}^*\}$$

$$\sigma_\alpha y_t = \sigma_\alpha\mathbb{E}_t\{y_{t+1}\} - (r_t - \mathbb{E}_t\{\pi_{H,t+1}\} - \rho) + \alpha\Theta\sigma_\alpha\mathbb{E}_t\{\Delta y_{t+1}^*\} \quad \Rightarrow \quad y_t = \mathbb{E}_t\{y_{t+1}\} - \frac{1}{\sigma_\alpha}(r_t - \mathbb{E}_t\{\pi_{H,t+1}\} - \rho) + \alpha\Theta\mathbb{E}_t\{\Delta y_{t+1}^*\}$$

Equation (31) - The trade balance

$$nx_t = \alpha\left(\frac{\omega}{\sigma} - 1\right)s_t$$

Defining net exports in terms of domestic output $nx_t \equiv \frac{1}{Y}\left(Y_t - \frac{P_t}{P_{H,t}}C_t\right)$, defined as a fraction of steady-state output. If $\sigma = \eta = \gamma = 1$,

$$Y_t = \left(\frac{P_{H,t}}{P_t}\right)^{-1} C_t\left[(1-\alpha) + \alpha\int_0^1 \left(\mathcal{S}_{i,t}\mathcal{S}_t^i\right)^0 \mathcal{Q}^0 di\right] = \frac{P_t}{P_{H,t}}C_t = C_t\mathcal{S}_t^\alpha$$

$Y_t = \frac{P_t}{P_{H,t}}C_t \quad \Rightarrow \quad Y_tP_{H,t} = P_tC_t$, implying a balanced trade.

Log-linearizing $Y_t = C_t\mathcal{S}_t^\alpha$, we have $y_t = c_t + \alpha s_t$

A first-order approximation for nx_t is (noting that nx is zero in the steady-state), recalling that in the steady-state, $P_H = P$ and $Y = C$

$$nx_t = \frac{1}{Y}\left((Y_t - Y) - \frac{C}{P}(P_t - P) + \frac{C}{P}(P_{H,t} - P) - (C_t - C)\right) = y_t - p_t + p_{H,t} - c_t$$

As $p_t - p_{H,t} = \alpha s_t$ and $y_t = c_t + \frac{\alpha\omega}{\sigma}$, we have $nx_t = y_t - c_t - \alpha s_t = \frac{\alpha\omega}{\sigma}s_t - \alpha s_t \quad \Rightarrow \quad nx_t = \alpha\left(\frac{\omega}{\sigma} - 1\right)s_t$

In this model, nx_t iz zero if $\frac{\omega}{\sigma} - 1 = 0$, or $\frac{\sigma\gamma + (\sigma\eta - 1)(1-\alpha)}{\sigma} = 1$, which means that $\sigma(\gamma - 1) + (1-\alpha)(\sigma\eta - 1) = 0$

Equation (32) - Coefficient of marginal cost

$\pi_{H,t} = \beta\mathbb{E}_t\{\pi_{H,t+1}\} + \lambda\widehat{mc}_t$ where $\lambda = \frac{(1-\theta)(1-\beta\theta)}{\theta}$

Log-linearizing the equation above around the steady-state with the Taylor expansion gives

$$P_{H,t} \approx [\theta(P_H)^{1-\varepsilon} + (1-\theta)(P_H)^{1-\varepsilon}]^{\frac{1}{1-\varepsilon}} + \frac{1}{1-\varepsilon}\left[(P_H)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}-1}\theta(1-\varepsilon)P_H^{-\varepsilon}(P_{H,t-1} - P_H)$$

$$+\frac{1}{1-\varepsilon}\left[(P_H)^{1-\varepsilon}\right]^{\frac{\varepsilon}{1-\varepsilon}}(1-\theta)(1-\varepsilon)P_H^{-\varepsilon}(\overline{P}_{H,t} - P_H)$$

$P_{H,t} \approx P_H + \theta(P_H)^{\varepsilon}(P_H)^{-\varepsilon}(P_{H,t-1} - P_H) + (1-\theta)(P_H)^{\varepsilon}(P_H)^{-\varepsilon}(\overline{P}_{H,t} - P_H) = P_H + \theta P_{H,t-1} - \theta P_H + \overline{P}_{H,t} - P_H - \theta\overline{P}_{H,t} + \theta P_H$

$P_{H,t} = \theta P_{H,t-1} + (1-\theta)\overline{P}_{H,t} \quad \Rightarrow \quad \dfrac{P_{H,t} - P_{H,t-1}}{P_H} = \dfrac{\theta P_{H,t-1} - P_{H,t-1} + (1-\theta)\overline{P}_{H,t}}{P_H} \quad \Rightarrow \quad \pi_{H,t} = (1-\theta)(\overline{p}_{H,t} - p_{H,t-1})$

Substituting $\overline{p}_{H,t} - p_{H,t-1}$ we have

$\pi_{H,t} = (1-\theta)(\beta\theta\mathbb{E}_t\{\overline{p}_{H,t+1} - p_{H,t} + \pi_{H,t} + (1-\beta\theta)\widehat{mc}_t\}) = (1-\theta)\left(\beta\theta\mathbb{E}_t\left\{\dfrac{\pi_{H,t+1}}{1-\theta} + \pi_{H,t} + (1-\beta\theta)\widehat{mc}_t\right\}\right)$

$\theta\pi_{H,t} = \beta\theta\mathbb{E}_t\{\pi_{H,t+1}\} + (1-\theta)(1-\beta\theta)\widehat{mc}_t \quad \Rightarrow \quad \pi_{H,t} = \beta\mathbb{E}_t\{\pi_{H,t+1}\} + \dfrac{(1-\theta)(1-\beta\theta)}{\theta}\widehat{mc}_t$

which gives $\pi_{H,t} = \beta\mathbb{E}_t\{\pi_{H,t+1}\} + \lambda\widehat{mc}_t$ where $\lambda = \dfrac{(1-\theta)(1-\beta\theta)}{\theta}$

5 - Inflation dynamics

Equation (33) - Log-linearized marginal cost

$mc_t = -\nu + \sigma y_t^* + \varphi y_t + s_t - (1+\varphi)a_t$

The two relations below were derived in the previous sections

$\pi_{H,t} = \beta\mathbb{E}_t\{\pi_{H,t+1}\} + \lambda\widehat{mc}_t$ where $\lambda \equiv \dfrac{(1-\beta\theta)(1-\theta)}{\theta}$

The log-linearized equation of marginal cost is $mc_t = -\nu + w_t - p_{H,t} - a_t$ where $\nu \equiv -\ln(1-\tau)$ and τ is the subsidy. Thus, $mc_t = -\nu + (w_t - p_t) - (p_{H,t} - p_t) - a_t$

From the log-linearized FOC, we have $w_t - p_t = \sigma c_t + \varphi n_t$

Thus, $mc_t = -\nu + \sigma c_t + \varphi n_t + \alpha s_t - a_t$

Using $c_t = c_t^* + \left(\dfrac{1-\alpha}{\sigma}\right)s_t$ and $y_t = a_t + n_t$, we have

$mc_t = -\nu + \sigma\left[c_t^* + \left(\dfrac{1-\alpha}{\sigma}\right)s_t\right] + \varphi(y_t - a_t) + \alpha s_t - a_t = -\nu + \sigma c_t^* + (1-\alpha)s_t + \varphi(y_t - a_t) + \alpha s_t - a_t$

As the world consumption is equal to its production (international market clearing),

$mc_t = -\nu + \sigma y_t^* + \varphi y_t + s_t - (1+\varphi)a_t$

Equation (34) - Alternative log-linearized marginal cost

$mc_t = -\nu + (\sigma_\alpha + \varphi)y_t + (\sigma - \sigma_\alpha)y_t^* - (1+\varphi)a_t$

Using $y_t = y_t^* + \dfrac{1}{\sigma_\alpha}s_t$ we can substitute for s_t in the expression above

$mc_t = -\nu + \sigma y_t^* + \varphi y_t + \sigma_\alpha(y_t - y_t^*) - (1+\varphi)a_t \quad \Rightarrow \quad mc_t = -\nu + (\sigma_\alpha + \varphi)y_t + (\sigma - \sigma_\alpha)y_t^* - (1+\varphi)a_t$

Equation (35) - domestic natural output

$\overline{y_t} = \Omega + \Gamma a_t + \alpha\Psi y_t^*$

The output gap is defined as $x_t \equiv y_t - \overline{y}_t$

To find The domestic natural level of output we impose $mc_t = -\mu$ and solving for domestic output as $y_t = \overline{y}_t$

$-\mu = -\nu + (\sigma_\alpha + \varphi)\overline{y_t} + (\sigma - \sigma_\alpha)y_t^* - (1+\varphi)a_t \quad \Rightarrow \quad (\sigma_\alpha + \varphi)\overline{y_t} = \nu - \mu + (1+\varphi)a_t - (\sigma - \sigma_\alpha)y_t^*$

$\overline{y_t} = \dfrac{\nu - \mu}{\sigma_\alpha + \varphi} + \dfrac{1+\varphi}{\sigma_\alpha + \varphi}a_t - \dfrac{\sigma - \sigma_\alpha}{\sigma_\alpha + \varphi}y_t^*$

As $\Theta = (\sigma\gamma - 1) + (\sigma\eta - 1)(1-\alpha)$, $\omega = \sigma\gamma + (\sigma\eta - 1)(1-\alpha) = \Theta + 1$ $\sigma_\alpha = \dfrac{\sigma}{(1-\alpha)+\alpha\omega}$, we have

$\sigma - \sigma_\alpha = \sigma - \dfrac{\sigma}{(1-\alpha)+\alpha\omega} = \dfrac{-\alpha\sigma + \alpha\omega\sigma}{(1-\alpha)+\alpha\omega}$

Let's verify that $\sigma - \sigma_\alpha = -\alpha\Theta\sigma_\alpha$

$\alpha\Theta\sigma_\alpha = \alpha(\omega - 1)\dfrac{\sigma}{(1-\alpha)+\alpha\omega} = \dfrac{\alpha\sigma\omega - \alpha\sigma}{(1-\alpha)+\alpha\omega} = \sigma - \sigma_\alpha$

Thus, $\overline{y_t} = \dfrac{\nu - \mu}{\sigma_\alpha + \varphi} + \dfrac{1+\varphi}{\sigma_\alpha + \varphi}a_t - \dfrac{\alpha\Theta\sigma_\alpha}{\sigma_\alpha + \varphi}y_t^* \quad \Rightarrow \quad \overline{y_t} = \Omega + \Gamma a_t + \alpha\Psi y_t^*$ where

$\Omega = \dfrac{\nu - \mu}{\sigma_\alpha + \varphi}$, $\Gamma = \dfrac{1+\varphi}{\sigma_\alpha + \varphi} > 0$ as $\sigma_\alpha > 0$ and $\Psi = -\dfrac{\Theta\sigma_\alpha}{\sigma_\alpha + \varphi}$

Equation (36) - New Keynesian Phillips Curve

$\pi_{H,t} = \beta\mathbb{E}_t\{\pi_{H,t+1}\} + \kappa_\alpha x_t$

Substituting y_t in $mc_t = -\nu + (\sigma_\alpha + \varphi)y_t + (\sigma - \sigma_\alpha)y_t^* - (1+\varphi)a_t$,

we have $mc_t = -\nu + (\sigma_\alpha + \varphi)(x_t + \overline{y_t}) + (\sigma - \sigma_\alpha)y_t^* - (1+\varphi)a_t$

$mc_t = -\nu + (\sigma_\alpha + \varphi)\left(x_t + \dfrac{\nu - \mu}{\sigma_\alpha + \varphi} + \dfrac{1+\varphi}{\sigma_\alpha + \varphi}a_t - \dfrac{\alpha\Theta\sigma_\alpha}{\sigma_\alpha + \varphi}y_t^*\right) + (\sigma - \sigma_\alpha)y_t^* - (1+\varphi)a_t$

$mc_t = -\nu + (\sigma_\alpha + \varphi)x_t + \nu - \mu - \alpha\Theta\sigma_\alpha y_t^* + (\sigma - \sigma_\alpha)y_t^*$

$mc_t - (-\mu) = (\sigma_\alpha + \varphi)x_t - \mu - \alpha\left[\omega - 1\right]\dfrac{\sigma}{(1-\alpha)+\alpha\omega}y_t^* + \dfrac{-\alpha\sigma + \alpha\omega\sigma}{(1-\alpha)+\alpha\omega}y_t^* + \mu$

$\widehat{mc_t} = (\sigma_\alpha + \varphi)x_t + \dfrac{-\alpha\omega\sigma + \alpha\sigma}{(1-\alpha)+\alpha\omega}y_t^* + \dfrac{-\alpha\sigma + \alpha\omega\sigma}{(1-\alpha)+\alpha\omega}y_t^* \quad \Rightarrow \quad \widehat{mc_t} = (\sigma_\alpha + \varphi)x_t$

Substituting the equation above into $\pi_{H,t} = \beta\mathbb{E}_t\{\pi_{H,t+1} + \lambda\widehat{mc_t}\}$, we have the a version of the New Keynesian Phillips Curve (NKPC)

$\pi_{H,t} = \beta\mathbb{E}_t\{\pi_{H,t+1}\} + \lambda(\sigma_\alpha + \varphi)x_t = \beta\mathbb{E}_t\{\pi_{H,t+1}\} + \kappa_\alpha x_t$, where $\kappa_\alpha \equiv \lambda(\sigma_\alpha + \varphi)$

Equation (37) - output gap

$x_t = \mathbb{E}_t\{x_{t+1}\} - \dfrac{1}{\sigma_\alpha}(r_t - \mathbb{E}_t\{\pi_{H,t+1}\} - \overline{rr}_t)$

Substituting y_t and $\mathbb{E}_t\{y_{t+1}\}$ in $y_t = \mathbb{E}_t\{y_{t+1}\} - \dfrac{1}{\sigma_\alpha}(r_t - \mathbb{E}_t\{\pi_{H,t+1}\} - \rho) + \alpha\Theta\mathbb{E}_t\{\Delta y_{t+1}^*\}$, we have

$x_t + \Omega + \Gamma a_t + \alpha\Psi y_t^* = \mathbb{E}_t\{x_{t+1} + \Omega + \Gamma a_{t+1} + \alpha\Psi y_{t+1}^*\} - \frac{1}{\sigma_\alpha}(r_t - \mathbb{E}_t\{\pi_{H,t+1}\} - \rho) + \alpha\Theta\mathbb{E}_t\{\Delta y_{t+1}^*\}$

$x_t = \mathbb{E}_t\{x_{t+1}\} + \Gamma(\rho_a a_t - a_t) + \alpha\Psi\mathbb{E}_t\{\Delta y_{t+1}^*\} - \dfrac{1}{\sigma_\alpha}(r_t - \mathbb{E}_t\{\pi_{H,t+1}\} - \rho) + \alpha\Theta\mathbb{E}_t\{\Delta y_{t+1}^*\}$

$x_t = \mathbb{E}_t\{x_{t+1}\} - \dfrac{1}{\sigma_\alpha}(r_t - \mathbb{E}_t\{\pi_{H,t+1}\} - [\rho - \sigma_\alpha\Gamma(1-\rho_a)a_t + \alpha\sigma_\alpha(\Theta + \Psi)\mathbb{E}_t\{\Delta y_{t+1}^*\}])$

$x_t = \mathbb{E}_t\{x_{t+1}\} - \dfrac{1}{\sigma_\alpha}(r_t - \mathbb{E}_t\{\pi_{H,t+1}\} - \overline{rr}_t)$

where $\overline{rr}_t \equiv \rho - \sigma_\alpha\Gamma(1-\rho_a)a_t + \alpha\sigma_\alpha(\Theta + \Psi)\mathbb{E}_t\{\Delta y_{t+1}^*\}$ is the small open economy's natural rate of interest.

Optimal Monetary Policy: a special case

The problem of the central planner is:

$\max E_0 \sum_{i=0}^{\infty} \beta^t U(C_t, N_t)$ subject to $Y_t = A_t N_t$, $C_t = C_t^i \mathcal{Q}_{i,t}$ and $Y_t = \dfrac{P_t}{P_{H,t}}C_t$, as $\eta = \sigma = \gamma = 1$.

In this case, $Y_t = \frac{(P_{H,t})^{1-\alpha}(P_{F,t})^{\alpha}}{P_{H,t}} C_t = \frac{(P_{F,t})^{\alpha}}{(P_{H,t})^{\alpha}} C_t = \mathcal{S}_t^{\alpha} C_t \;\; \Rightarrow \;\; Y_t = \mathcal{S}_t^{\alpha} C_t \;\; \Rightarrow \;\; \mathcal{S}_t = \left(\frac{Y_t}{C_t}\right)^{\frac{1}{\alpha}}$.

By equation 29, we have $y_t = y_t^* + s_t$ or $Y_t = Y_t^* \mathcal{S}_t = Y_t^* \left(\frac{Y_t}{C_t}\right)^{\frac{1}{\alpha}} \;\; \Rightarrow \;\; C_t Y_t^{\alpha} = (Y_t^*)^{\alpha} Y_t \;\; \Rightarrow \;\; C_t = Y_t^{1-\alpha}(Y_t^*)^{\alpha}$.

The Central planner problem then becomes

$E_0 \max \sum_{i=0}^{\infty} \beta^t U(C_t, N_t)$ subject to $Y_t = A_t N_t$ and $C_t = Y_t^{1-\alpha}(Y_t^*)^{\alpha}$.

The Lagrangean can be written as

$$\mathcal{L} = E_0 \sum_{i=0}^{\infty} \beta^t \left\{ U(C_t, N_t) + \Lambda_t (A_t N_t - Y_t) + \Phi_t \left(Y_t^{1-\alpha}(Y_t^*)^{\alpha} - C_t \right) \right\}$$

which yields the FOCS:

(C_t) $U_c(C_t, N_t) = \Phi_t$

(N_t) $U_N(C_t, N_t) = -\Lambda_t A_t$

(Y_t) $\Phi_t (1-\alpha) Y_t^{-\alpha}(Y_t^*)^{\alpha} = \Lambda_t$

$U_N(C_t, N_t) = -A_t \Phi_t (1-\alpha) Y_t^{-\alpha}(Y_t^*)^{\alpha} = -A_t U_c(C_t, N_t)(1-\alpha) Y_t^{-\alpha}(Y_t^*)^{\alpha}$

$-\frac{U_N(C_t, N_t)}{U_c(C_t, N_t)} = \frac{Y_t}{N_t}(1-\alpha) Y_t^{-\alpha}(Y_t^*)^{\alpha} = (1-\alpha)\frac{C_t}{N_t}$

Using the utility function $U(C_t, N_t) = \log(C_t) - \frac{N_t^{1+\varphi}}{1+\varphi}$ as $\sigma = 1$.

We have $U_C(C_t, N_t) = \frac{1}{C_t}$ and $U_N(C_t, N_t) = -N_t^{\varphi}$.

Substituting in the FOCs relation, we have

$N_t^{\varphi} C_t = (1-\alpha)\frac{C_t}{N_t} \;\; \Rightarrow \;\; N_t^{1+\varphi} = (1-\alpha) \;\; \Rightarrow \;\; N_t = N = (1-\alpha)^{\frac{1}{1+\varphi}}$,

which is a constant employment.

We know that, under flexible prices, $\frac{\overline{MC}_t^n}{\overline{P}_{H,t}} = \overline{MC}_t = \frac{\varepsilon - 1}{\varepsilon}$ (see section 9 for the derivation).

From the representative consumer FOCs, we have the standard relation: $-\frac{U_C(\overline{C}_t, \overline{N}_t)}{U_N(\overline{C}_t, \overline{N}_t)} = \frac{\overline{W}_t}{\overline{P}_t}$.

The subsidy τ is chosen to achieve the optimal level of production if the prices were fully flexible.

Thus, we have $\overline{MC}_t = \frac{\varepsilon - 1}{\varepsilon} = \frac{\overline{W}_t(1-\tau)}{\overline{P}_{H,t} A_t} = -\frac{U_C(\overline{C}_t, \overline{N}_t)}{U_N(\overline{C}_t, \overline{N}_t)} \frac{\overline{P}_t(1-\tau)}{\overline{P}_{H,t} A_t}$.

As $\eta = 1$, $\overline{P}_t = (\overline{P}_{H,t})^{1-\alpha}(\overline{P}_{F,t})^{\alpha}$. Substituting,

$$\overline{MC}_t = -\frac{U_C(\overline{C}_t, \overline{N}_t)}{U_N(\overline{C}_t, \overline{N}_t)} \frac{(\overline{P}_{H,t})^{1-\alpha}(\overline{P}_{F,t})^{\alpha}(1-\tau)}{\overline{P}_{H,t}\overline{A}_t} = -\frac{U_C(\overline{C}_t, \overline{N}_t)}{U_N(\overline{C}_t, \overline{N}_t)} \left(\frac{\overline{P}_{F,t}}{\overline{P}_{H,t}}\right)^{\alpha} \frac{1-\tau}{\overline{A}_t} = -\frac{(1-\tau)}{\overline{A}_t} (\overline{\mathcal{S}}_i)^{\alpha} \frac{U_C(\overline{C}_t, \overline{N}_t)}{U_N(\overline{C}_t, \overline{N}_t)}$$

As $\sigma = 1$, $-\frac{U_C(\overline{C}_t, \overline{N}_t)}{U_N(\overline{C}_t, \overline{N}_t)} = \overline{C}_t \overline{N}_t^{\varphi} \;\; \Rightarrow \;\; \overline{MC}_t = \frac{(1-\tau)}{\overline{A}_t} (\overline{\mathcal{S}}_i)^{\alpha} \overline{C}_t \overline{N}_t^{\varphi}$

From equation 26 (equilibrium part), we know that $\overline{Y}_t = \overline{C}_t \overline{\mathcal{S}}_t^{\alpha} \;\; \Rightarrow \;\; \overline{\mathcal{S}}_t^{\alpha} = \frac{\overline{Y}_t}{\overline{C}_t}$. Thus, $\overline{MC}_t = \frac{(1-\tau)}{\overline{A}_t} \frac{\overline{Y}_t}{\overline{C}_t} \overline{C}_t \overline{N}_t^{\varphi}$

Substituting the technological constraint

$$1 - \frac{1}{\varepsilon} = \overline{MC}_t = \frac{(1-\tau)}{\overline{A}_t} \overline{A}_t \overline{N}_t \overline{N}_t^{\varphi} = (1-\tau)\overline{N}_t^{1+\varphi} = (1-\tau)\left((1-\alpha)^{1+\varphi}\right)^{\frac{1}{1+\varphi}} = (1-\tau)(1-\alpha).$$

Hence, if τ is set to satisfy $(1-\tau)(1-\alpha) = 1 - \frac{1}{\varepsilon}$, we have also the log-linear form $\nu = \mu + \log(1-\alpha)$, where μ (defined earlier) is $\log(1-\tau)$ and the flexible price allocation is guaranteed.

Equation (38) - Central bank commitment rule

Solving forward equation (36), we have

$$\pi_{H,t} = \beta \mathbb{E}_t\{\pi_{H,t+1}\} + \kappa_\alpha x_t = \beta \mathbb{E}_t\{\beta \mathbb{E}_t\{\pi_{H,t+2}\} + \kappa_\alpha x_{t+1}\} + \kappa_\alpha x_t = \beta^T \mathbb{E}_t\{\pi_{H,T}\} + \sum_{j=t}^{T} \beta^{j-t} \kappa_\alpha x_j$$

To stabilize inflation, $\pi_{H,t} = \lim_{T\to\infty} \beta^T \mathbb{E}_t\{\pi_{H,T}\} + \sum_{j=t}^{T} \beta^{j-t} \kappa_\alpha x_j$ which will stabilized ($\pi_{H,t} = 0$) only if the output gap is zero for every period.

As $x_t \equiv y_t - \overline{y}_t \quad \Rightarrow \quad y_t = \overline{y}_t$.

In equation (37) we have $x_t = \mathbb{E}_t\{x_{t+1}\} - \frac{1}{\sigma_\alpha}(r_t - \mathbb{E}_t\{\pi_{H,t+1}\} - \overline{rr}_t)$.

In this case, we have $\mathbb{E}_t\{x_{t+1}\} = 0$, $\mathbb{E}_t\{\pi_{H,t+1}\} = 0$, and $\sigma_\alpha = 1$, as

$$\sigma_\alpha \equiv \frac{\sigma}{(1-\alpha) + \alpha\omega} = \frac{1}{(1-\alpha) + \alpha(\sigma\gamma + (\sigma\eta - 1)(1-\alpha))} = \frac{1}{(1-\alpha) + \alpha(1 + (1-1)(1-\alpha))} = 1.$$

Thus, $r_t = \overline{rr}_t$.

Equation (38) comes from a hypothesis that the Central Bank commits itself with the rule $r_t = \overline{rr}_t + \phi_\pi \pi_{H,t} + \phi_x x_t$.

Equation (C.1) - Difference equation system

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \mathbf{A_O} \begin{bmatrix} \mathbb{E}_t\{x_{t+1}\} \\ \mathbb{E}_t\{\pi_{t+1}\} \end{bmatrix}$$

After setting $r_t = \overline{rr}_t$ in a closed economy we have

$$x_t = \mathbb{E}_t\{x_{t+1}\} - \frac{1}{\sigma_\alpha}(r_t - \mathbb{E}_t\{\pi_{H,t+1}\} - \overline{rr}_t) = \mathbb{E}_t\{x_{t+1}\} - \frac{1}{\sigma}(\overline{rr}_t - \mathbb{E}_t\{\pi_{H,t+1}\} - \overline{rr}_t) = \mathbb{E}_t\{x_{t+1}\} - \frac{1}{\sigma}\mathbb{E}_t\{\pi_{H,t+1}\}$$

We have now a system with 2 equations

$$x_t = \mathbb{E}_t\{x_{t+1}\} + \frac{1}{\sigma}\mathbb{E}_t\{\pi_{H,t+1}\}$$

$$\pi_{H,t} = \beta \mathbb{E}_t\{\pi_{H,t+1}\} + \kappa_\alpha x_t,$$

which can be summarized as $\begin{bmatrix} 1 & 0 \\ -\kappa & 1 \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} 1 & \sigma^{-1} \\ 0 & \beta \end{bmatrix} \begin{bmatrix} \mathbb{E}_t\{x_{t+1}\} \\ \mathbb{E}_t\{\pi_{t+1}\} \end{bmatrix}$

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \kappa & 1 \end{bmatrix} \begin{bmatrix} 1 & \sigma^{-1} \\ 0 & \beta \end{bmatrix} \begin{bmatrix} \mathbb{E}_t\{x_{t+1}\} \\ \mathbb{E}_t\{\pi_{t+1}\} \end{bmatrix} = \begin{bmatrix} 1 & \sigma^{-1} \\ \kappa & \kappa\sigma^{-1} + \beta \end{bmatrix} \begin{bmatrix} \mathbb{E}_t\{x_{t+1}\} \\ \mathbb{E}_t\{\pi_{t+1}\} \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} 1 & \sigma^{-1} \\ \kappa & \beta + \kappa\sigma^{-1} \end{bmatrix} \begin{bmatrix} \mathbb{E}_t\{x_{t+1}\} \\ \mathbb{E}_t\{\pi_{t+1}\} \end{bmatrix}.$$

Defining $\mathbf{A_O} = \begin{bmatrix} 1 & \sigma^{-1} \\ \kappa & \beta + \kappa\sigma^{-1} \end{bmatrix}$, we get the expression

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \mathbf{A_O} \begin{bmatrix} \mathbb{E}_t\{x_{t+1}\} \\ \mathbb{E}_t\{\pi_{t+1}\} \end{bmatrix}$$

Equations (C.2) and (C.3) - Non-explosive solution

$$\kappa(\phi_\pi - 1) + \phi_y(1-\beta) > 0$$

From the system above, we can calculate its eigenvalues:

$$\begin{vmatrix} 1-\xi & \sigma^{-1} \\ \kappa & \beta + \kappa\sigma^{-1} - \xi \end{vmatrix} = (1-\xi)(\beta + \kappa\sigma^{-1} - \xi) - \kappa\sigma^{-1} = \beta + \kappa\sigma^{-1} - \xi - \beta\xi - \kappa\sigma^{-1}\xi + \xi^2 - \kappa\sigma^{-1}$$

Now we have a quadratic function whose roots are the eigenvalues of the system $f(\xi) = \xi^2 - (1 + \beta + \kappa\sigma^{-1})\xi + \beta$, with the product of roots being $\beta < 1$ and the sum of the roots greater than 1 $1 + \beta + \kappa\sigma^{-1}$.

We can see that if $f(\xi) > 0$ if $\xi \to \infty$, $f(0) = \beta > 0$ and $f(1) = 1^2 - (1 + \beta + \kappa\sigma^{-1})1 + \beta = -\kappa\sigma^{-1} < 0$, as $\kappa = \lambda(\sigma+\varphi) = \dfrac{(1-\beta\theta)(1-\theta)}{\theta}(\sigma+\varphi) > 0$. Thus, we have one root between 0 and 1 and another one greater than 1 (from the intermediate value theorem). As there's one eigenvalue outside the unit root circle, there are infinite solution for this system, as both variables are forward looking.

Now if the Central bank has commits to the rule $r_t = \overline{rr}_t + \phi_\pi \pi_t + \phi_x x_t$, which is the equation equation (C.2), we have

$$x_t = \mathbb{E}_t\{x_{t+1}\} - \frac{1}{\sigma}(\overline{rr}_t + \phi_\pi\pi_t + \phi_x x_t - \mathbb{E}_t\{\pi_{t+1}\} - \overline{rr}_t) = -\phi_\pi\sigma^{-1} - \pi_t + \phi_x\sigma^{-1}x_t + \mathbb{E}_t\{x_{t+1}\} + \sigma^{-1}\mathbb{E}_t\{\pi_{t+1}\}$$

and the system becomes

$$(1+\phi_x\sigma^{-1})x_t + \phi_\pi\sigma^{-1}\pi_t = \mathbb{E}_t\{x_{t+1}\} + \sigma^{-1}\mathbb{E}_t\{\pi_{t+1}\}$$

$$-\kappa x_t + \pi_{H,t} = \beta\mathbb{E}_t\{\pi_{H,t+1}\},$$

$$\begin{bmatrix} 1+\phi_x\sigma^{-1} & \phi_\pi\sigma^{-1} \\ -\kappa & 1 \end{bmatrix}\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} 1 & \sigma^{-1} \\ 0 & \beta \end{bmatrix}\begin{bmatrix} \mathbb{E}_t\{x_{t+1}\} \\ \mathbb{E}_t\{\pi_{t+1}\} \end{bmatrix}$$

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \frac{1}{1+\phi_x\sigma^{-1}+\phi_\pi\sigma^{-1}\kappa}\begin{bmatrix} 1 & -\phi_\pi\sigma^{-1} \\ \kappa & 1+\phi_x\sigma^{-1} \end{bmatrix}\begin{bmatrix} 1 & \sigma^{-1} \\ 0 & \beta \end{bmatrix}\begin{bmatrix} \mathbb{E}_t\{x_{t+1}\} \\ \mathbb{E}_t\{\pi_{t+1}\} \end{bmatrix} = \frac{\sigma}{\sigma+\phi_x+\phi_\pi\kappa}\begin{bmatrix} 1 & \sigma^{-1}(1-\beta\phi_\pi) \\ \kappa & \sigma^{-1}(\kappa+\beta\sigma+\beta\phi_x) \end{bmatrix}\begin{bmatrix} \mathbb{E}_t\{x_{t+1}\} \\ \mathbb{E}_t\{\pi_{t+1}\} \end{bmatrix}$$

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \frac{1}{\sigma+\phi_x+\phi_\pi\kappa}\begin{bmatrix} \sigma & 1-\beta\phi_\pi \\ \kappa\sigma & \kappa+\beta(\sigma+\phi_x) \end{bmatrix}\begin{bmatrix} \mathbb{E}_t\{x_{t+1}\} \\ \mathbb{E}_t\{\pi_{t+1}\} \end{bmatrix} = \mathbf{A}_T\begin{bmatrix} \mathbb{E}_t\{x_{t+1}\} \\ \mathbb{E}_t\{\pi_{t+1}\} \end{bmatrix}, \text{ where}$$

$$\mathbf{A}_T \equiv \Omega\begin{bmatrix} \sigma & 1-\beta\phi_\pi \\ \kappa\sigma & \kappa+\beta(\sigma+\phi_x) \end{bmatrix} \text{ and } \Omega \equiv \frac{1}{\sigma+\phi_x+\phi_\pi\kappa}$$

If we restrict $\phi_\pi > 0$ and $\phi_x > 0$, $\Omega > 0$.

To satisfy the Blanchard and Khan conditions, we need that both eigenvalues are inside the unit circle.

$$|\mathbf{A}_T - \lambda\mathbf{I}| = 0$$

$$\left|\frac{1}{\sigma+\phi_x+\phi_\pi\kappa}\begin{bmatrix} \sigma & 1-\beta\phi_\pi \\ \kappa\sigma & \kappa+\beta(\sigma+\phi_x) \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| = 0$$

$$\left|\begin{bmatrix} \dfrac{\sigma}{\sigma+\phi_x+\phi_\pi\kappa} - \lambda & \dfrac{1-\beta\phi_\pi}{\sigma+\phi_x+\phi_\pi\kappa} \\ \dfrac{\kappa\sigma}{\sigma+\phi_x+\phi_\pi\kappa} & \dfrac{\kappa+\beta(\sigma+\phi_x)}{\sigma+\phi_x+\phi_\pi\kappa} - \lambda \end{bmatrix}\right| = 0$$

$$\frac{\sigma\kappa+\beta\sigma(\sigma+\phi_y)}{(\sigma+\phi_x+\phi_\pi\kappa)^2} - \frac{\sigma+\kappa+\beta(\sigma+\phi_y)}{(\sigma+\phi_x+\phi_\pi\kappa)}\lambda + \lambda^2 - \frac{\sigma\kappa-\beta\phi_\pi\sigma\kappa}{(\sigma+\phi_x+\phi_\pi\kappa)^2} = \lambda^2 - \frac{\sigma+\kappa+\beta(\sigma+\phi_y)}{(\sigma+\phi_x+\phi_\pi\kappa)}\lambda + \frac{\sigma\beta(\sigma+\phi_y+\phi_\pi\kappa)}{(\sigma+\phi_x+\phi_\pi\kappa)^2} = 0$$

$$\lambda^2 - \frac{\sigma+\kappa+\beta(\sigma+\phi_y)}{\sigma+\phi_x+\phi_\pi\kappa}\lambda + \frac{\sigma\beta}{\sigma+\phi_x+\phi_\pi\kappa} = 0$$

LaSalle (1986) showed that both roots of the equation $x^2+bx+c=0$ are less than 1 if and only if $|c|<1$ and $|b|<1+c$. This comment was taken from Drago Bergholt notes.

We have that $\left|\dfrac{\sigma\beta}{\sigma+\phi_y+\phi_\pi\kappa}\right| < 1$

As $\sigma>0$, $\beta>0$, $\kappa>0$, $\phi_\pi>0$ and $\phi_y>0$

$\dfrac{\sigma\beta}{\sigma+\phi_y+\phi_\pi\kappa} < 1 \quad\Rightarrow\quad \sigma\beta < \sigma+\phi_y+\phi_\pi\kappa \quad\Rightarrow\quad \sigma(\beta-1) < \phi_y+\phi_\pi\kappa$ This condition is always satisfied, as $\beta<1$.

The second condition is $\left|\dfrac{\sigma+\kappa+\beta(\sigma+\phi_y)}{\sigma+\phi_y+\phi_\pi\kappa}\right| < 1 + \dfrac{\sigma\beta}{\sigma+\phi_y+\phi_\pi\kappa}$

$$\frac{\sigma+\kappa+\beta(\sigma+\phi_y)}{\sigma+\phi_y+\phi_\pi\kappa} < 1 + \frac{\sigma\beta}{\sigma+\phi_y+\phi_\pi\kappa}$$

$$\sigma+\kappa+\beta(\sigma+\phi_y) < \sigma+\phi_y+\phi_\pi\kappa+\sigma\beta \quad\Rightarrow\quad \kappa+\beta\phi_y < \phi_y+\phi_\pi\kappa \quad\Rightarrow\quad \kappa(\phi_\pi-1)+\phi_y(1-\beta) > 0$$

Equation (39) - Equation (C.3) for the open economy

$\kappa_\alpha(\phi_\pi - 1) + \phi_y(1-\beta) > 0$

The system is analogous to the matrix equation (C.1). The only difference is that the parameter κ is now κ_α, as defined in equation (36). In our particular case $\sigma = \eta = \gamma = 1$, $\sigma_\alpha = \sigma$ and $\kappa_\alpha = \kappa$, the standard parameter in the New Keynesian Phillips Curve.

Macroeconomic implications

By equation (35) $\overline{y}_t = \Omega + \Gamma a_t - \alpha\Psi y_t^* = \dfrac{\nu-\mu}{\sigma_\alpha+\varphi} + \dfrac{1+\varphi}{\sigma_\alpha+\varphi}a_t - \alpha\dfrac{\Theta\sigma_\alpha}{\sigma_\alpha+\varphi}y_t^*$, we can see that a technological shock always increase the output level, as $\sigma_\alpha > 0$ and $\varphi > 0$.

Computing the natual level of the terms of trade, we have

$$\overline{s}_t = \sigma_\alpha(\overline{y}_t - y_t^*) = \sigma_\alpha\left(\frac{\nu-\mu}{\sigma_\alpha+\varphi} + \frac{1+\varphi}{\sigma_\alpha+\varphi}a_t - \alpha\frac{\Theta\sigma_\alpha}{\sigma_\alpha+\varphi}y_t^* - y_t^*\right) = \sigma_\alpha\left(\frac{\nu-\mu}{\sigma_\alpha+\varphi} + \frac{1+\varphi}{\sigma_\alpha+\varphi}a_t - \frac{\alpha\Theta\sigma_\alpha+\sigma_\alpha+\varphi}{\sigma_\alpha+\varphi}y_t^*\right)$$

$$\overline{s}_t = \sigma_\alpha\left(\frac{\nu-\mu}{\sigma_\alpha+\varphi} + \frac{1+\varphi}{\sigma_\alpha+\varphi}a_t - \frac{\alpha(\omega-1)\sigma_\alpha+\sigma_\alpha+\varphi}{\sigma_\alpha+\varphi}y_t^*\right) = \sigma_\alpha\left(\frac{\nu-\mu}{\sigma_\alpha+\varphi} + \frac{1+\varphi}{\sigma_\alpha+\varphi}a_t - \frac{(\alpha\omega-\alpha+1)\dfrac{\sigma}{1-\alpha+\alpha\omega}+\varphi}{\sigma_\alpha+\varphi}y_t^*\right)$$

$$\overline{s}_t = \sigma_\alpha\left(\frac{\nu-\mu}{\sigma_\alpha+\varphi} + \frac{1+\varphi}{\sigma_\alpha+\varphi}a_t - \frac{\sigma+\varphi}{\sigma_\alpha+\varphi}y_t^*\right) = \sigma_\alpha\Omega + \sigma_\alpha\Gamma a_t - \sigma_\alpha\Phi y_t^*, \text{ where } \Phi = \frac{\sigma+\varphi}{\sigma_\alpha+\varphi} > 0$$

As $\overline{p}_t = (1-\alpha)\overline{p}_{H,t} + \alpha\overline{p}_{F,t} = (1-\alpha)\overline{p}_{H,t} + \alpha(\overline{e}_t + p_t^*)$. As domestic prices are fully stabilized, $(1-\alpha)\overline{p}_{H,t}$ is a constant, so $\overline{p}_t$ is proportional to $\alpha(\overline{e}_t + p_t^*) = \alpha\overline{s}_t$

6 - The welfare costs of deviations from the optimal

Equation (40) - Function with welfare losses

$$\mathbb{W} = -\frac{(1-\alpha)}{2}\sum_{i=0}^{\infty}\beta^t\left[\frac{\varepsilon}{\lambda}\pi_{H,t}^2 + (1+\varphi)x_t^2\right] + t.i.p + o(||a||^3)$$

The calculations for this expressions are summarized in appendix D. By the Taylor's rule, the second order approximation is:

$f(x_t) = f(a) + f'(a)(x_t - a) + \dfrac{1}{2}f''(a)(x_t-a)^2 + \dfrac{1}{6}f'''(\widetilde{a})(x_t-a)^3$. We assume that the third term is small, as the deviations from the steady-state are assumed to be small (by the intermediate value theorem, $\widetilde{a}$ is between x_t and a.

We can take

$\dfrac{Y_t}{Y} = e^{\ln\frac{Y_t}{Y}} = 1 + \ln\dfrac{Y_t}{Y} + \dfrac{1}{2}\left(\ln\dfrac{Y_t}{Y}\right)^2 + \dfrac{1}{3!}\left(\ln\dfrac{Y_t}{Y}\right)^3 + ... = 1 + y_t + \dfrac{1}{2}(y_t)^2 + o(||a||^n)$, there a is the bound for the high order terms.

$\dfrac{Y_t - Y}{Y} = y_t + \dfrac{y_t^2}{2} + o(||a||^n)$

Combining equations (18): $c_t = c_t^* + \left(\dfrac{1-\alpha}{\sigma}\right)s_t$ and (29): $y_t = y_t^* + \dfrac{1}{\sigma_\alpha}s_t = y_t^* + \dfrac{1}{\sigma}s_t$ as $\omega = 1$, we have:

$c_t = c_t^* + \left(\dfrac{1-\alpha}{\sigma}\right)\sigma(y_t - y_t^*)$. As $c_t^* = y_t^*$ because of global market clearing, $c_t = y_t^* + y_t - y_t^* - \alpha y_t + \alpha y_t^* \quad\Rightarrow\quad c_t = (1-\alpha)y_t + \alpha y_t^*$.

As $x_t \equiv y_t - \overline{y}_t$, in the stabilized economy, $x_t = 0$ and $y_t = \overline{y}_t$. Thus, $\overline{c}_t = (1-\alpha)(0 - \overline{y}_t) + \alpha y_t^* = \alpha y_t^* - (1-\alpha)\overline{y}_t$

Substituting, we get: $c_t = (1-\alpha)(\overline{y}_t + x_t) + \alpha y_t^* = (1-\alpha)\overline{y}_t + \alpha y_t^* + (1-\alpha)x_t \quad\Rightarrow\quad c_t = \overline{c}_t + (1-\alpha)x_t$.

Expanding the log-deviation of the disutility of work, we have:

$$\left(\frac{N_t}{\overline{N}}\right)^{1+\varphi} = \exp[(1+\varphi)\widetilde{n}] = 1 + (1+\varphi)\widetilde{n}_t + \frac{1}{2}\widetilde{n}_t^2 + o(||a||^3) \quad \Rightarrow \quad N_t^{1+\varphi} = \overline{N}^{1+\varphi}\left(1 + (1+\varphi)\widetilde{n}_t + \frac{1}{2}\widetilde{n}_t^2 + o(||b||^3)\right)$$

$$\frac{N_t^{1+\varphi}}{1+\varphi} = \frac{\overline{N}^{1+\varphi}}{1+\varphi} + \overline{N}^{1+\varphi}\left[\widetilde{n}_t + \frac{1}{2}(1+\varphi)\widetilde{n}_t^2\right] + o(||a||^3)$$

Using the fact that $N_t = \left(\frac{Y_t}{A_t}\right)\int_0^1 \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} di$, $\int_0^1 \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} di = \frac{N_t A_t}{Y_t}$

$$\Rightarrow \quad \log \int_0^1 \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} di = \log\left(\frac{N_t A_t}{Y_t}\right) = n_t + a_t - y_t$$

If we define $z_t \equiv \log \int_0^1 \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} di$, then $z_t = n_t + a_t - y_t = n_t + a_t - (\overline{y}_t + x_t)$.

When prices are stabilized, $P_{H,t}(i) = P_{H,t}$ and $\overline{z}_t = 0$. Also, there are no productivity shocks, so $a_t = \overline{y}_t - \overline{n}_t$. Thus,

$$z_t = \overline{n}_t + \widetilde{n}_t + \overline{y}_t - \overline{n}_t - (\overline{y}_t + x_t) \quad \Rightarrow \quad \widetilde{n}_t = z_t + x_t$$

Lemma 1 (appendix D): The proof is in the paper. There's just one passage that it took time to figure out what happened.

From $\mathbb{E}_i\{\widehat{p}_{H,t}(i)\} = \frac{(\varepsilon-1)}{2}\mathbb{E}_i\{\widehat{p}_{H,t}(i)^2\}$ and $\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} = 1 - \varepsilon\widehat{p}_{H,t}(i) + \frac{\varepsilon^2}{2}\widehat{p}_{H,t}(i)^2 + o(||a||^3)$

$$\mathbb{E}_i\left[\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon}\right] = \mathbb{E}_i\left[1 - \varepsilon\widehat{p}_{H,t}(i) + \frac{\varepsilon^2}{2}\widehat{p}_{H,t}(i)^2 + o(||a||^3)\right]$$

$$\int_0^1 \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} di = 1 - \varepsilon\mathbb{E}_i[\widehat{p}_{H,t}(i)] + \frac{\varepsilon^2}{2}\mathbb{E}_i[\widehat{p}_{H,t}(i)^2] = 1 - \varepsilon\frac{(\varepsilon-1)}{2}\mathbb{E}_i\{\widehat{p}_{H,t}(i)^2\} + \frac{\varepsilon^2}{2}\mathbb{E}_i[\widehat{p}_{H,t}(i)^2] = 1 + \frac{\varepsilon}{2}\mathbb{E}_i\{\widehat{p}_{H,t}(i)^2\}$$

$$\Rightarrow \quad \int_0^1 \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} di = 1 + \frac{\varepsilon}{2}\mathbb{V}_i\{p_{H,t}(i)\}$$

$$z_t = \log \int_0^1 \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} di = \log\left(1 + \frac{\varepsilon}{2}\mathbb{V}_i\{p_{H,t}(i)\}\right) = \frac{\varepsilon}{2}\mathbb{V}_i\{p_{H,t}(i)\} + o(||a||^3)$$

Rewriting the second-order approximation disutility of labor:

$$\frac{N_t^{1+\varphi}}{1+\varphi} = \frac{\overline{N}^{1+\varphi}}{1+\varphi} + \overline{N}^{1+\varphi}\left[x_t + z_t + \frac{1}{2}(1+\varphi)(x_t+z_t)^2\right] + o(||a||^3)$$

$$\frac{N_t^{1+\varphi}}{1+\varphi} = \frac{\overline{N}^{1+\varphi}}{1+\varphi} + \overline{N}^{1+\varphi}\left[x_t + z_t + \frac{1}{2}(1+\varphi)(x_t^2 + 2x_t z_t + z_t^2)\right] + o(||a||^3)$$

As z_t is a variance (second-order term), z_t^2 and $x_t z_t$ are terms of greater order, so they can be included in the remanescent terms $o(||a||^3)$. Thus, we get

$$\frac{N_t^{1+\varphi}}{1+\varphi} = \frac{\overline{N}^{1+\varphi}}{1+\varphi} + \overline{N}^{1+\varphi}\left[x_t + z_t + \frac{1}{2}(1+\varphi)x_t^2\right] + o(||a||^3).$$

Under the optimal subsidy assumption, from the consumer's FOC, we have that $\overline{N}_t^{1+\varphi} = (1-\alpha)$ (constant employment). Thus,

$$U(C_t, N_t) \equiv \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{\overline{N}^{1+\varphi}}{1+\varphi} - \overline{N}^{1+\varphi}\left[x_t + z_t + \frac{1}{2}(1+\varphi)x_t^2\right] + o(||a||^3)$$

$$U(C_t, N_t) = -\frac{(1-\alpha)}{1+\varphi} - (1-\alpha)\left[x_t + z_t + \frac{1}{2}(1+\varphi)x_t^2\right] + \frac{C_t^{1-\sigma}}{1-\sigma} + o(||a||^3)$$

$U(C_t, N_t) = -(1-\alpha)\left[x_t + z_t + \frac{1}{2}(1+\varphi)x_t^2\right] + \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{(1-\alpha)}{1+\varphi}o(||a||^3) = -(1-\alpha)\left[z_t + \frac{1}{2}(1+\varphi)x_t^2\right] + t.i.p + o(||a||^3)$, there t.i.p. denotes terms independent of policy and under the optimal policy, $x_t = 0$.

$$\mathbb{W} = \sum_{i=0}^{\infty}\beta^t U(C_t, N_t) = \sum_{i=0}^{\infty}\beta^t\left[-(1-\alpha)\left(z_t + \frac{1}{2}(1+\varphi)x_t^2\right) + t.i.p + o(||a||^3)\right]$$

$$\mathbb{W} = \sum_{i=0}^{\infty} \beta^t \left[-(1-\alpha) \left(\frac{\varepsilon}{2} \mathbb{V}_i[p_{H,t}(i)] + o(||a||^3) + \frac{1}{2}(1+\varphi)x_t^2 \right) \right] + t.i.p + o(||a||^3)$$

Taking out the constant ($t.i.p$), which are the terms independent of monetary policy and $o(||a||^3)$, which is a third order term, we achieve the desired welfare loss equation:

$$\mathbb{W} = \sum_{i=0}^{\infty} \beta^t \left[-(1-\alpha) \left(\frac{\varepsilon}{2} \mathbb{V}_i[p_{H,t}(i)] + \frac{1}{2}(1+\varphi)x_t^2 \right) \right].$$

Lemma 2 from Woodford(2003): $\sum_{t=0}^{\infty} \beta^t \mathbb{V}_i[p_{H,t}(i)] = \frac{1}{\lambda} \sum_{t=0}^{\infty} \beta^t \pi_{H,t}^2$, where $\lambda \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta}$

Using Lemma 2, we have:

$$\mathbb{W} = -\frac{(1-\alpha)}{2} \sum_{i=0}^{\infty} \beta^t \left[\frac{\varepsilon}{\lambda} \pi_{H,t}^2 + (1+\varphi)x_t^2 \right] + t.i.p + o(||a||^3)$$

To arrive at Lemma 2, we can calculate the price variance:

$$\mathbb{V}_i = \mathbb{E}_i[(p_{H,t}(i) - \mathbb{E}_i[p_{H,t-1}(i)])^2] - (\mathbb{E}_i[p_{H,t}(i)] - \mathbb{E}_i[p_{H,t-1}(i)])^2$$

Because only an exogenous draw of $1-\theta$ firms are able to reset their price:

$$\mathbb{E}_i[(p_{H,t}(i) - \mathbb{E}_i[p_{H,t-1}(i)])^2] = \theta \mathbb{E}_i[(p_{H,t-1}(i) - \mathbb{E}_i[p_{H,t-1}(i)])^2] + (1-\theta)\mathbb{E}_i[(p_{H,t}^* - \mathbb{E}_i[p_{H,t-1}(i)])^2]$$

The expected price of good i produced at home country is

$$\mathbb{E}_i[(p_{H,t}(i)] = (1-\theta)p_{H,t}^* + \theta \mathbb{E}_i[(p_{H,t-1}(i)].$$

Solving for p_t^*, we have

$$(1-\theta)p_{H,t}^* = \mathbb{E}_i[(p_{H,t}(i)] - \theta \mathbb{E}_i[(p_{H,t-1}(i)] \quad \Rightarrow \quad p_{H,t}^* = \frac{1}{1-\theta} \mathbb{E}_i[(p_{H,t}(i)] - \frac{\theta}{1-\theta} \mathbb{E}_i[(p_{H,t-1}(i)]$$

Substituting $p_{H,t}^*$ in the above equation, we have

$$\mathbb{E}_i[(p_{H,t}(i) - \mathbb{E}_i[p_{H,t-1}(i)])^2] =$$

$$= \theta \mathbb{E}_i[(p_{H,t-1}(i) - \mathbb{E}_i[p_{H,t-1}(i)])^2] + (1-\theta)\mathbb{E}_i \left[\left(\left[\frac{1}{1-\theta} \mathbb{E}_i[(p_{H,t}(i)] - \frac{\theta}{1-\theta} \mathbb{E}_i[(p_{H,t-1}(i)] \right] - \mathbb{E}_i[p_{H,t-1}(i)] \right)^2 \right]$$

$$= \theta \mathbb{E}_i[(p_{H,t-1}(i) - \mathbb{E}_i[p_{H,t-1}(i)])^2] + (1-\theta)\mathbb{E}_i \left[\left(\frac{1}{1-\theta} \mathbb{E}_i[(p_{H,t}(i)] - \frac{1}{1-\theta} \mathbb{E}_i[(p_{H,t-1}(i)] \right)^2 \right]$$

$$\mathbb{E}_i[(p_{H,t}(i) - \mathbb{E}_i[p_{H,t-1}(i)])^2] = \theta \mathbb{E}_i[(p_{H,t-1}(i) - \mathbb{E}_i[p_{H,t-1}(i)])^2] + \frac{1}{1-\theta} \left(\mathbb{E}_i[(p_{H,t}(i)] - \mathbb{E}_i[(p_{H,t-1}(i)] \right)^2$$

$$\mathbb{V}_i[p_{H,t}(i)] = \mathbb{E}_i[(p_{H,t}(i) - \mathbb{E}_i[p_{H,t-1}(i)])^2] - (\mathbb{E}_i[p_{H,t}(i)] - \mathbb{E}_i[p_{H,t-1}(i)])^2$$

Substituting the expression above into the variance, we get

$$\mathbb{V}_i[p_{H,t}(i)] = \theta \mathbb{E}_i[(p_{H,t-1}(i) - \mathbb{E}_i[p_{H,t-1}(i)])^2] + \frac{1}{1-\theta} \left(\mathbb{E}_i[(p_{H,t}(i)] - \mathbb{E}_i[(p_{H,t-1}(i)] \right)^2 - (\mathbb{E}_i[p_{H,t}(i)] - \mathbb{E}_i[p_{H,t-1}(i)])^2$$

$$\mathbb{V}_i = \theta \mathbb{E}_i[(p_{H,t-1}(i) - \mathbb{E}_i[p_{H,t-1}(i)])^2] + \frac{\theta}{1-\theta} \left(\mathbb{E}_i[(p_{H,t}(i)] - \mathbb{E}_i[(p_{H,t-1}(i)] \right)^2 \approx \theta \mathbb{V}_i[p_{H,t-1}] + \frac{\theta}{1-\theta} \pi_{H,t}^2$$

$$\mathbb{V}_i[p_{H,t}(i)] \approx \theta \mathbb{V}_i[p_{H,t-1}] + \frac{\theta}{1-\theta} \pi_{H,t}^2$$

Using the approximation above and Iterating backwards, we have:

$$\mathbb{V}_i[p_{H,t}(i)] = \frac{\theta}{1-\theta} \pi_t^2 + \theta \left\{ \frac{\theta}{1-\theta} \pi_{t-1}^2 + \theta \left[\frac{\theta}{1-\theta} \pi_{t-2}^2 + \theta \left(\frac{\theta}{1-\theta} \pi_{t-3}^2 + \theta \mathbb{V}_i[p_{H,t-4}(i)] \right) \right] \right\}$$

$$\mathbb{V}_i[p_{H,t}(i)] = \theta^{t+1} \mathbb{V}_i[p_{H,-1}(i)] + \sum_{s=0}^{t} \theta^s \frac{\theta}{1-\theta} \pi_{H,t-s}^2$$

Taking the discounted value of $\mathbb{V}_i[p_{H,t}(i)]$ over all periods, as $t \to \infty$, the first term of equation above converges to zero almost surely as $\theta < 1$. Then we have,

$$\sum_{t=0}^{\infty}\beta^t\mathbb{V}_i[p_{H,t}(i)] = \sum_{t=0}^{\infty}\beta^t\sum_{t=0}^{\infty}\theta^t\frac{\theta}{1-\theta}\pi_{H,t}^2 = \frac{\theta}{1-\theta}\sum_{t=0}^{\infty}\sum_{t=0}^{\infty}\beta^t(\beta\theta)^t\pi_{H,t}^2 = \frac{\theta}{(1-\theta)(1-\beta\theta)}\sum_{t=0}^{\infty}\beta^t\pi_{H,t}^2 = \frac{1}{\lambda}\sum_{t=0}^{\infty}\beta^t\pi_{H,t}^2$$

Equation (41) - Variances of inflation and output gap

$$\mathbb{V} = -\frac{(1-\alpha)}{2}\sum_{i=0}^{\infty}\beta^t\left[\frac{\varepsilon}{\lambda}\mathbb{V}(\pi_{H,t}) + (1+\varphi)\mathbb{V}(x_t)\right]$$

Now $\mathbb{E}_t\left[-\frac{(1-\alpha)}{2}\sum_{i=0}^{\infty}\beta^t\left[\frac{\varepsilon}{\lambda}\pi_{H,t}^2 + (1+\varphi)x_t^2\right] + t.i.p + o(||a||^3)\right] = -\frac{(1-\alpha)}{2}\sum_{i=0}^{\infty}\beta^t\left[\frac{\varepsilon}{\lambda}\mathbb{V}(\pi_{H,t}) + (1+\varphi)\mathbb{V}(x_t)\right]$

as $E[x_t] = E[\pi_t] = 0$

If $\beta \to 1$, any policy deviation in a period can be calculated as

$$\mathbb{V} = -\frac{(1-\alpha)}{2}\sum_{i=0}^{\infty}\beta^t\left[\frac{\varepsilon}{\lambda}\mathbb{V}(\pi_{H,t}) + (1+\varphi)\mathbb{V}(x_t)\right]$$

Appendix E - Optimal policy for the modification

$$\mathbb{W} = -\frac{(1-\alpha)}{2}\sum_{i=0}^{\infty}\beta^t\left[\frac{\varepsilon}{\lambda}\pi_{H,t}^2 + \frac{\zeta}{\Lambda}\pi_{w,t}^2 + (1+\varphi)x_t^2\right] + t.i.p + o(||a||^3)$$

Here we'll derive the loss function with wage rigidity, adapted from Rhee. Then, considering the limit case, we get the original paper formula. By the Taylor's rule, the second order approximation is:

$Y_t = Y + y_t + \frac{1}{2}(y_t)^2 + o(||a||^n) \quad\Rightarrow\quad \frac{Y_t - Y}{Y} = y_t + \frac{y_t^2}{2} + o(||a||^n) \approx y_t + \frac{y_t^2}{2}$, there a is the bound for the high order terms.

Let's consider and additive welfare function, defined as the sum of the utility of all households:

$$\mathbb{W} = \int_0^1 U_t(\ell)d\ell = \int_0^1 E_0\sum_{t=0}^{\infty}\beta^t U(C_t(\ell),N_t(\ell))d\ell = E_0\sum_{t=0}^{\infty}\beta^t\int_0^1 U(C_t(\ell),N_t(\ell))d\ell = E_0\sum_{t=0}^{\infty}\beta^t\int_0^1\left(\log C_t(\ell) - \frac{N_t(\ell)}{1+\varphi}\right)d\ell$$

In the extended model, as the demanded work for each type of labor is different (as they are inside a continuum of measure 1), the consumption will also be different and will depend on the total income. To make the problem tractable, we will suppose that the government makes a lump-sum transfer for each household to compensate the income differences so everyone consume the same basket and the only source of heterogeneity is the disutility of labor. We also consider the special parametrization $\sigma = \gamma = \eta = 1$

With flexible salary prices, there's no monopolistic competition in the labor market. Thus, we have $U_C(C_t,N_t) = \frac{1}{C_t}$ and $U_N(C_t,N_t) = -N_t^{\varphi}$. Substituting in the FOCs relation, with flexible prices we have

$\overline{N}_t^{\varphi}\overline{C}_t = (1-\alpha)\frac{\overline{C}_t}{\overline{N}_t} \quad\Rightarrow\quad \overline{N}_t^{\varphi+1} = \overline{N}^{1+\varphi}(1-\alpha) \quad\Rightarrow\quad \overline{N} = (1-\alpha)^{\frac{1}{1+\varphi}}$ which is a constant employment.

From log-linearized equations in the special case, we have

$$c_t = c_t^* + (1-\alpha)s_t \quad\Rightarrow\quad s_t = \frac{c_t - c_t^*}{1-\alpha}$$

$$y_t = c_t + \alpha s_t = c_t + \alpha\frac{c_t - c_t^*}{1-\alpha} \quad\Rightarrow\quad (1-\alpha)y_t = (1-\alpha)c_t + \alpha c_t - \alpha c_t^* \quad\Rightarrow\quad c_t = (1-\alpha)y_t + \alpha y_t^*$$

Back to our welfare function

$$\mathbb{W} \approx E_0\sum_{t=0}^{\infty}\beta^t\left[(1-\alpha)y_t + \alpha y_t^* - \int_0^1\left(\frac{(N_t(\ell)-\overline{N}_t)^{\varphi+1}}{\varphi+1}\right)d\ell\right] \approx E_0\sum_{t=0}^{\infty}\beta^t\left[(1-\alpha)y_t + \alpha y_t^* - \int_0^1\left(\frac{\overline{N}_t^{\varphi+1} + (1+\varphi)N_t(\ell)\overline{N}_t^{\varphi}}{1+\varphi}\right)d\ell\right]$$

$$\mathbb{W} = E_0\sum_{t=0}^{\infty}\beta^t\left[(1-\alpha)y_t + \alpha y_t^* - \frac{\overline{N}_t^{\varphi+1}}{1+\varphi}\int_0^1\left(1 + (1+\varphi)\frac{N_t(\ell)}{\overline{N}_t}\right)d\ell\right]$$

$$\mathbb{W} \approx E_0\sum_{t=0}^{\infty}\beta^t\left[(1-\alpha)y_t + \alpha y_t^* - \frac{1-\alpha}{1+\varphi} - (1-\alpha)\int_0^1\left(\widehat{n}_t(\ell) + \frac{1+\varphi}{2}\widehat{n}_t(\ell)^2\right)d\ell\right]$$

$$\mathbb{W} \approx E_0 \sum_{t=0}^{\infty} \beta^t \left[(1-\alpha) y_t - (1-\alpha)\widehat{n}_t - \frac{(1-\alpha)(1+\varphi)}{2} \int_0^1 \widehat{n}_t(\ell)^2 d\ell + t.i.p \right],$$

where $t.i.p$ contains the terms independent of policy, which we remove from the function, as they are constant terms from the point of view of the policy maker.

Using equation proposed in the modification $N_t(j,\ell) = \left(\frac{W_t(\ell)}{W_t}\right)^{-\zeta} N_t(j)$, where $W_t \equiv \left(\int_0^1 N_t(j,\ell)^{\frac{\zeta-1}{\zeta}} d\ell\right)^{\frac{\zeta}{1-\eta}}$

and summing the labor for all j firms, we have $N_t(j,\ell) = \left(\frac{W_t(\ell)}{W_t}\right)^{\zeta} N_t(j) \quad \Rightarrow \quad \int_0^1 N_t(j,\ell) dj = \int_0^1 \left(\frac{W_t(\ell)}{W_t}\right)^{\zeta} N_t(j) dj \quad \Rightarrow$ $N_t(\ell) = \left(\frac{W_t(\ell)}{W_t}\right)^{\zeta} N_t$

Taking the log and subtracting the same equation in the steady-state, we have

$n_t(\ell) = \zeta(w_t(\ell) - w_t) + n_t \quad \Rightarrow \quad \widehat{n}_t(\ell) = \zeta(\widehat{w}_t(\ell) - \widehat{w}_t) + n_t \quad \Rightarrow \quad \widehat{n}_t(\ell) - n_t = \zeta(\widehat{w}_t(\ell) - \widehat{w}_t)$

$$\int_0^1 \widehat{n}_t(j)^2 dj = \int_0^1 (\zeta(\widehat{w}_t(\ell) - \widehat{w}_t) + \widehat{n}_t)^2 dj = \zeta^2 \int_0^1 (\widehat{w}_t(\ell) - \widehat{w}_t)^2 d\ell + 2\zeta \int_0^1 (\widehat{w}_t(\ell) - \widehat{w}_t) d\ell + \int_0^1 \widehat{n}_t^2 d\ell = \widehat{n}_t^2 + \zeta^2 \mathbb{V}[w_t(\ell)]$$

using the fact that $\int_0^1 (\widehat{w}_t(\ell) - \widehat{w}_t) d\ell = 0$ and $\int_0^1 (\widehat{w}_t(\ell) - \widehat{w}_t)^2 d\ell = \mathbb{V}[w_t(\ell)]$

Now we need to write the employment in terms of output and prices:

$$N_t = \int_0^1 \int_0^1 N_t(j,\ell) d\ell dj = \int_0^1 N_t(j) \int_0^1 \frac{N_t(j,\ell)}{N_t(\ell)} d\ell dj = \Delta_{w,t} \int_0^1 N_t(j) dj$$

where $\Delta_{w,t} \equiv \int_0^1 \frac{N_t(j,\ell)}{N_t(\ell)} d\ell = \int_0^1 \left(\frac{W_t(\ell)}{W_t}\right)^{-\zeta} d\ell$.

Now we'll derive the demand curve for $Y_t(j)$. Good market clearing in the representative small open economy ("home") requires that the total produced inside the small economy is consumed either by the households of this economy or imported (from the small economy, subscript H) from households of any other country (superscript i).

$$Y_t(j) = C_{H,t}(j) + \int_0^1 C^i_{H,t}(j) di = \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\varepsilon} C_{H,t} + \int_0^1 \left(\frac{P^i_{H,t}(j)}{P^i_{H,t}}\right)^{-\varepsilon} C^i_{H,t} di$$

$$Y_t(j) = \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\varepsilon} (1-\alpha) \left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t + \int_0^1 \left(\frac{P^i_{H,t}(j)}{P^i_{H,t}}\right)^{-\varepsilon} \left(\frac{P^i_{H,t}}{P^i_{F,t}}\right)^{-\gamma} C^i_{F,t} di$$

$$Y_t(j) = \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\varepsilon} (1-\alpha) \left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t + \int_0^1 \left(\frac{P^i_{H,t}(j)}{P^i_{H,t}}\right)^{-\varepsilon} \left(\frac{P_{H,t}}{\mathcal{E}_{i,t} P^i_{F,t}}\right)^{-\gamma} \alpha \left(\frac{P^i_{F,t}}{P^i_t}\right)^{-\eta} C^i_t di$$

Assuming symmetric preferences across countries, the home country will sell the variety of good j for the same price, independently of which country is buying, which implies $\frac{P^i_{H,t}(j)}{P^i_{H,t}} = \frac{P_{H,t}(j)}{P_{H,t}}$

Thus, $Y_t(j) = \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\varepsilon} \left[(1-\alpha) \left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t + \alpha \int_0^1 \left(\frac{P_{H,t}}{\mathcal{E}_{i,t} P^i_{F,t}}\right)^{-\gamma} \left(\frac{P^i_{F,t}}{P^i_t}\right)^{-\eta} C^i_t di \right] = \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\varepsilon} Y_t$

Back to the employment expression, using the above expression and using $Y_t(j) = A_t N_t(j)$ and

$$N_t \equiv \left(\int_0^1 N_t(j,\ell)^{\frac{\zeta-1}{\zeta}} d\ell\right)^{\frac{\zeta}{\zeta-1}}$$

in the modification proposed, we have

$$N_t = \Delta_{w,t} \int_0^1 \frac{Y_t(j)}{A_t} dj = \Delta_{w,t} \frac{Y_t}{A_t} \int_0^1 \frac{Y_t(j)}{Y_t} dj = \Delta_{w,t} \frac{Y_t}{A_t} \int_0^1 \frac{Y_t(j)}{Y_t} dj = \Delta_{w,t} \Delta_{P_H,t} \frac{Y_t}{A_t}$$

where $\Delta_{P_H,t} = \int_0^1 \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\varepsilon} dj$

Log-linearizing around the steady-state (the potential, or natural product), where there are no differences between firms and households we have:

$\widehat{n}_t = y_t - \overline{y}_t + d_{w,t} + d_{P_H,t}$, where $d_{w,t} = \log \Delta_{w,t}$ and $d_{P_H,t} = \log \Delta_{P_H,t}$

Lemma 1: $d_{P_H,t} = \frac{\varepsilon}{2}\mathbb{V}_j[P_{H,t}(j)]$. The proof is in the appendix D of the original paper.

Lemma 2: $d_{p_H,t} = \frac{\zeta}{2}\mathbb{V}_\ell[w_t(\ell)]$. Proof is Erced et al.

Lemma 3: (proof in Woodford, chapter 6)

$$\sum_{i=0}^{\infty} \beta^t \mathbb{V}_j[p_{H,t}(j)] = \frac{\theta}{(1-\beta\theta)(1-\theta)} \sum_{i=0}^{\infty} \beta^t \pi_{H,t}^2$$

$$\sum_{i=0}^{\infty} \beta^t \mathbb{V}_\ell[w_t(\ell)] = \frac{\varsigma}{(1-\beta\varsigma)(1-\varsigma)} \sum_{i=0}^{\infty} \beta^t \pi_{w,t}^2$$

Back to the welfare function, we have

$$\mathbb{W} = E_0 \sum_{t=0}^{\infty} \beta^t \left[(1-\alpha)y_t - (1-\alpha)(y_t - \overline{y}_t + d_{w,t} + d_{P_H,t}) - \frac{(1-\alpha)(1+\varphi)}{2}(\widehat{n}_t^2 + \zeta^2 \mathbb{V}_\ell[w_t(\ell)]) + t.i.p\right]$$

$$\mathbb{W} = E_0 \sum_{t=0}^{\infty} \beta^t \left[-(1-\alpha)\left(\frac{\varepsilon}{2}\mathbb{V}_j[p_{H,t}(j)] + \frac{\zeta}{2}\mathbb{V}_\ell[w_t(\ell)]\right) - \frac{(1-\alpha)(1+\varphi)}{2}(\widehat{n}_t^2 + \zeta^2 \mathbb{V}_\ell[w_t(\ell)]) + t.i.p\right]$$

$$\mathbb{W} = -\frac{1-\alpha}{2} E_0 \sum_{t=0}^{\infty} \beta^t \left[\varepsilon \mathbb{V}_j[p_{H,t}(j)] + \zeta \mathbb{V}_\ell[w_t(\ell)] + (1+\varphi)(x_t^2 + \zeta^2 \mathbb{V}_\ell[w_t(\ell)])\right] + t.i.p$$

$$\mathbb{W} = -\frac{1-\alpha}{2} E_0 \sum_{t=0}^{\infty} \beta^t \left\{(1+\varphi)x_t^2 + \varepsilon \mathbb{V}_j[p_{H,t}(j)] + [\zeta(1+\varphi\zeta)]\mathbb{V}_\ell[w_t(\ell)]\right\} + t.i.p,$$

where $\zeta^2 \mathbb{V}_\ell[w_t(\ell)]$ is a term independent of policy.

Using both equations from lemma 3, we have

$$\mathbb{W} = -\frac{1-\alpha}{2} E_0 \sum_{t=0}^{\infty} \beta^t \left\{(1+\varphi)x_t^2 + \varepsilon \frac{\theta}{(1-\beta\theta)(1-\theta)} \pi_{H,t}^2 + \zeta(1+\varphi\zeta) \frac{\varsigma}{(1-\beta\varsigma)(1-\varsigma)} \pi_{w,t}^2\right\} + t.i.p,$$

$$\mathbb{W} = -\frac{1-\alpha}{2} E_0 \sum_{t=0}^{\infty} \beta^t \left\{(1+\varphi)x_t^2 + \frac{\varepsilon}{\lambda}\pi_{H,t}^2 + \frac{\zeta}{\Lambda}\pi_{w,t}^2\right\} + t.i.p,$$

where $\lambda = \frac{\theta}{(1-\beta\theta)(1-\theta)}$ and $\Lambda = \frac{\varsigma(1-\beta\varsigma)}{(1-\beta\varsigma)(1-\varsigma)}$

In the limit case, when $\Lambda \to \infty$, the third term of the equation above vanishes and we have the result for the original paper, with flexible prices in the labor market.