STAT 309: MATHEMATICAL COMPUTATIONS I
FALL 2015
LECTURE 4

1. EIGENVALUE DECOMPOSITION

- recall our two fundamental problems:
$$A\mathbf{x} = \mathbf{b} \quad \text{and} \quad A\mathbf{x} = \lambda\mathbf{x}$$
- even if we are just interested to solve $A\mathbf{x} = \mathbf{b}$ and its variants, we will need to understand eigenvalues and eigenvectors
- we will use properties of eigenvalues and eigenvectors but will only briefly describe its computation (towards the last few lectures)
- recall: $A \in \mathbb{C}^{n\times n}$, if there exists $\lambda \in \mathbb{C}$ and $\mathbf{0} \neq \mathbf{x} \in \mathbb{C}^n$ such that
$$A\mathbf{x} = \lambda\mathbf{x},$$
we call λ an *eigenvalue* of A and $\mathbf{x}$ an *eigenvector* of A corresponding to λ or λ-eigenvector
- some basic properties
  - eigenvector is a scale invariant notion, if $\mathbf{x}$ is a λ-eigenvector, than so is $c\mathbf{x}$ for any $c \in \mathbb{C}^\times$
  - we usually, but not always, require that $\mathbf{x}$ be a unit vector, i.e., $\|\mathbf{x}\|_2 = 1$
  - note that if $\mathbf{x}_1$ and $\mathbf{x}_2$ are λ-eigenvectors, then so is $\mathbf{x}_1 + \mathbf{x}_2$
  - for an eigenvalue λ, the subspace $V_\lambda = \{\mathbf{x} \in \mathbb{C}^n : A\mathbf{x} = \lambda\mathbf{x}\}$ is called the λ-*eigenspace* of A and is the set of all λ-eigenvectors of A together with $\mathbf{0}$
  - the set of all eigenvalues of A is called its *spectrum* and often denoted $\lambda(A)$
  - an $n \times n$ matrix always have n eigenvalues in $\mathbb{C}$ counted with multiplicty
  - however an $n \times n$ matrix may not have n linear independent eigenvectors
  - an example is
$$J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{1.1}$$
which has eigenvalue 0 with multiplicity 2 but only one eigenvector (up to scaling) $\mathbf{x} = [1, 0]^{\mathsf{T}}$
- an $n \times n$ matrix A that has n linear independent eigenvectors $\mathbf{x}_1, \dots, \mathbf{x}_n$ is called a *diagonalizable matrix* since if we write these as columns of a matrix $X = [\mathbf{x}_1, \dots, \mathbf{x}_n]$, then X is necessarily nonsingular and
$$AX = [A\mathbf{x}_1, \dots, A\mathbf{x}_n] = [\lambda_1\mathbf{x}_1, \dots, \lambda_n\mathbf{x}_n] = X \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} =: X\Lambda \tag{1.2}$$
and so
$$A = X\Lambda X^{-1} \tag{1.3}$$
where $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ is a diagonal matrix of eigenvalues
- the decomposition (1.3) is called the *eigenvalue decomposition* (EVD) of A
- not every matrix has an EVD, an example is the J in (1.1)

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- summary: a matrix has an EVD iff it has n linearly independent eigenvectors iff it is diagonalizable
- normally we will sort the eigenvalues in descending order of magnitude

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$$

- λ_1, also denoted $\lambda_{\max}$, is called the *principle eigenvalue* of A and a $\lambda_{\max}$-eigenvector is called a *principal eigenvector*
- since $\mathbf{x}_1, \dots, \mathbf{x}_n$ form a basis for the domain of A, we call this an *eigenbasis*
- note that the matrix of eigenvectors X in (1.3) is only required to be non-singular (a.k.a. invertible)

2. SPECTRAL THEOREMS

- in general it is difficult to check whether a matrix is diagonalizable
- however there is a special class of matrices for which we check diagonalizability easily, namely, the normal matrices
- a normal matrix is one that commutes with its adjoint, i.e. $A^*A = AA^*$
- recall that $A^* = \bar{A}^{\mathsf{T}}$ is the *adjoint* or Hermitian conjugate of A
- the matrix J above is *not* normal

Theorem 1 (Spectral Theorem for Normal Matrices)**.** *Let $A \in \mathbb{C}^{n\times n}$. Then A is unitarily diagonalizable iff A has an orthonormal eigenbasis iff A is a normal matrix, i.e.*

$$A^*A = AA^*,$$

iff A has an EVD *of the form*

$$A = V\Lambda V^* \tag{2.1}$$

where $V \in \mathbb{C}^{n\times n}$ is unitary and $\Lambda \in \mathbb{C}^{n\times n}$ is diagonal.

- as in (1.2), $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ consists of the eigenvalues of A and the columns of $V = [\mathbf{v}_1, \dots, \mathbf{v}_n]$ are the eigenvectors of A and are mutually orthonormal, i.e.

$$\mathbf{v}_i^*\mathbf{v}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

- note that saying the column vectors of $V = [\mathbf{v}_1, \dots, \mathbf{v}_n]$ are mutually orthonormal is the same as saying $V^*V = I = VV^*$ and is the same as saying that V is unitary
- a special class of normal matrices are the ones that are equal to their adjoint, i.e. $A^* = A$,and these are called *Hermitian* or self-adjoint matrices
- for Hermitian matrices, we can say more — the diagonal matrix Λ in (2.1) is real

Theorem 2 (Spectral Theorem for Hermitian Matrices)**.** *Let $A \in \mathbb{C}^{n\times n}$. Then A is unitarily diagonalizable with a real diagonal matrix iff A has an orthonormal eigenbasis and all eigenvalues real iff A is a Hermitian matrix, i.e.*

$$A^* = A,$$

iff iff A has an EVD *of the form*

$$A = V\Lambda V^*$$

where $V \in \mathbb{C}^{n\times n}$ is unitary and $\Lambda \in \mathbb{R}^{n\times n}$ is diagonal.

- if we had start from a real matrix $A \in \mathbb{R}^{n\times n}$, then Theorem 2 holds true with 'Hermitian' replaced by *symmetric* (i.e., $A^{\mathsf{T}} = A$) and 'unitary' replaced by *orthogonal* (i.e., $V^{\mathsf{T}}V = I = VV^{\mathsf{T}}$)
- we have strict inclusions

$$\{\text{symmetric}\} \subsetneq \{\text{Hermitian}\} \subsetneq \{\text{normal}\} \subsetneq \{\text{diagonalizable}\} \subsetneq \mathbb{C}^{n\times n}$$

3. Jordan canonical form

- if A is not diagonalizable and we want something like a diagonalization, then the best we could do is a *Jordan canonical form* or Jordan normal form where we get
$$A = XJX^{-1} \tag{3.1}$$
  - the matrix J has the following characteristics
    * it is not diagonal but it is the next best thing to diagonal, namely, *bidiagonal*, i.e. only the entries a_{ii} and $a_{i,i+1}$ can be non-zero, every other entry in J is 0
    * the diagonal entries of J are precisely the eigenvalues of A, counted with multiplicity
    * the superdiagonal entries $a_{i,i+1}$ are as simple as they can be — they can take one of two possible values $a_{i,i+1} = 0$ or 1
    * if $a_{i,i+1} = 0$ for all i, then J is in fact diagonal and (3.1) reduces to the eigenvalue decomposition
  - the matrix J is more commonly viewed as a block diagonal matrix
$$J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{bmatrix}$$
    * each block J_r, for $r = 1, \dots, k$, has the form
$$J_r = \begin{bmatrix} \lambda_r & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_r \end{bmatrix}$$
      where J_r is $n_r \times n_r$
    * clearly $\sum_{r=1}^{k} n_r = n$
  - the set of column vectors of X are called a *Jordan basis* of A
- in general the Jordan basis X include all eigenvectors of A but also additional vectors that are not eigenvectors of A
- the Jordan canonical form provides valuable information about the eigenvalues of A
- the values λ_j, for $j = 1, \dots, k$, are the eigenvalues of A
- for each distinct eigenvalue λ, the number of Jordan blocks having λ as a diagonal element is equal to the number of linearly independent eigenvectors associated with λ, this number is called the *geometric multiplicity* of the eigenvalue λ
- the sum of the sizes of all of these blocks is called the *algebraic multiplicity* of λ
- we now consider J_r's eigenvalues,
$$\lambda(J_r) = \lambda_r, \dots, \lambda_r$$
  where λ_r is repeated n_r times, but because
$$J_r - \lambda_r I = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$$
  is a matrix of rank $n_r - 1$, it follows that the homogeneous system $(J_r - \lambda_r I)\mathbf{x} = \mathbf{0}$ has only one vector (up to a scalar multiple) for a solution, and therefore there is only one eigenvector associated with this Jordan block
- the unique unit vector that solves $(J_r - \lambda_r I)\mathbf{x} = \mathbf{0}$ is the vector $\mathbf{e}_1 = [1, 0, \dots, 0]^{\mathsf{T}}$

- now consider the matrix

$$(J_r - \lambda_r I)^2 = \begin{bmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & \ddots & 0 \\ & & & & 0 \end{bmatrix}$$

- it is easy to see that $(J_r - \lambda_r I)^2 \mathbf{e}_2 = 0$
- continuing in this fashion, we can conclude that

$$(J_r - \lambda_r I)^k \mathbf{e}_k = \mathbf{0}, \quad k = 1, \ldots, n_r - 1$$

- the Jordan form can be used to easily compute powers of a matrix
- for example,

$$A^2 = XJX^{-1}XJX^{-1} = XJ^2X^{-1}$$

  and, in general,

$$A^k = XJ^kX^{-1}$$

- due to its structure, it is easy to compute powers of a Jordan block J_r:

$$\begin{aligned} J_r^k &= \begin{bmatrix} \lambda_r & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_r \end{bmatrix}^k \\ &= (\lambda_r I + K)^k, \quad K = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \\ &= \sum_{j=0}^{k} \binom{k}{j} \lambda_r^{k-j} K^j \end{aligned}$$

  which yields, for $k > n_r$,

$$J_r^k = \begin{bmatrix} \lambda_r^k & \binom{k}{1}\lambda_r^{k-1} & \binom{k}{2}\lambda_r^{k-2} & \cdots & \binom{k}{n_r-1}\lambda_r^{k-(n_r-1)} \\ & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & \vdots \\ & & & & \lambda_r^k \end{bmatrix} \tag{3.2}$$

- for example,

$$\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}^3 = \begin{bmatrix} \lambda^3 & 3\lambda^2 & 3\lambda \\ 0 & \lambda^3 & 3\lambda^2 \\ 0 & 0 & \lambda^3 \end{bmatrix}$$

- we now consider an application of the Jordan canonical form
  - consider the system of differential equations

$$\mathbf{y}'(t) = A\mathbf{y}(t), \quad \mathbf{y}(t_0) = \mathbf{y}_0$$

– using the Jordan form, we can rewrite this system as

$$\mathbf{y}'(t) = XJX^{-1}\mathbf{y}(t)$$

– ultiplying through by X^{-1} yields

$$X^{-1}\mathbf{y}'(t) = JX^{-1}\mathbf{y}(t)$$

which can be rewritten as

$$\mathbf{z}'(t) = J\mathbf{z}(t)$$

where $\mathbf{z} = Q^{-1}\mathbf{y}(t)$

– this new system has the initial condition

$$\mathbf{z}(t_0) = \mathbf{z}_0 = Q^{-1}\mathbf{y}_0$$

– if we assume that J is a diagonal matrix (which is true in the case where A has a full set of linearly independent eigenvectors), then the system decouples into scalar equations of the form

$$z_i'(t) = \lambda_i z_i(t),$$

where λ_i is an eigenvalue of A

– this equation has the solution

$$z_i(t) = e^{\lambda_i(t-t_0)}z_i(0),$$

and therefore the solution to the original system is

$$\mathbf{y}(t) = X \begin{bmatrix} e^{\lambda_1(t-t_0)} & & \\ & \ddots & \\ & & e^{\lambda_n(t-t_0)} \end{bmatrix} X^{-1}\mathbf{y}_0$$

- Jordan canonical form suffers however from one major defect that makes them useless in practice: they cannot be computed in finite precision or in the presence of rounding errors in general, a result of Golub and Wilkinson
- that is why you won't find a Matlab function for Jordan canonical form

4. spectral radius

- matrix 2-norm is also known as the *spectral norm*
- name is connected to the fact that the norm is given by the square root of the largest eigenvalue of $A^{\mathsf{T}}A$, i.e., largest singular value of A (more on this later)
- in general, the *spectral radius* $\rho(A)$ of a matrix $A \in \mathbb{C}^{n\times n}$ is defined in terms of its largest eigenvalue

$$\rho(A) = \max\{|\lambda_i| : A\mathbf{x}_i = \lambda_i\mathbf{x}_i,\ \mathbf{x}_i \neq \mathbf{0}\}$$

- note that the spectral radius does not define a norm on $\mathbb{C}^{n\times n}$
- for example the non-zero matrix

$$J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

has $\rho(J) = 0$ since both its eigenvalues are 0
- there are some relationships between the norm of a matrix and its spectral radius
- the easiest one is that

$$\rho(A) \leq \|A\|$$

for any matrix norm that satisfies the inequality $\|A\mathbf{x}\| \leq \|A\|\|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{C}^n$

– here's a proof:

$$A\mathbf{x}_i = \lambda_i \mathbf{x}_i$$

taking norms,

$$\|A\mathbf{x}_i\| = \|\lambda_i \mathbf{x}_i\| = |\lambda_i|\|\mathbf{x}_i\|$$

therefore

$$|\lambda_i| = \frac{\|A\mathbf{x}_i\|}{\|\mathbf{x}_i\|} \le \|A\|$$

since this holds for any eigenvalue of A, it follows that

$$\max_i |\lambda_i| = \rho(A) \le \|A\|$$

– in particular this is true for any operator norm

– this is in general not true for norms that do not satisfy the inequality $\|A\mathbf{x}\| \le \|A\|\|\mathbf{x}\|$ (thanks to Likai Chen for pointing out); for example the matrix

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

is orthogonal and therefore $\rho(A) = 1$ but $\|A\|_{H,\infty} = 1/\sqrt{2}$ and so $\rho(A) > \|A\|_{H,\infty}$

– exercise: show that any eigenvalue of a unitary or an orthogonal matrix must have absolute value 1

- on the other hand, the following characterization is true for *any* matrix norm, even the inconsistent ones

$$\rho(A) = \lim_{m\to\infty} \|A^m\|^{1/m}$$

- we can also get an upper bound for any particular matrix (but not for all matrices)

Theorem 3. *For every $\varepsilon > 0$, there exists an operator norm $\|\cdot\|_\alpha$ such that*

$$\|A\|_\alpha \le \rho(A) + \epsilon.$$

The norm is dependent on A and ε.

- this result suggests that the largest eigenvalue of a matrix can be easily approximated
- here is an example, let

$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & 1 \\ & & & -1 & 2 \end{bmatrix}$$

– the eigenvalues of this matrix, which arises frequently in numerical methods for solving differential equations, are known to be

$$\lambda_j = 2 + 2\cos\frac{j\pi}{n+1}, \quad j = 1, 2, \dots, n$$

the largest eigenvalue is

$$|\lambda_1| = 2 + 2\cos\frac{\pi}{n+1} \le 4$$

and $\|A\|_\infty = 4$, so in this case, the ∞-norm provides an excellent approximation

– on the other hand, suppose

$$A = \begin{bmatrix} 1 & 10^6 \\ 0 & 1 \end{bmatrix}$$

we have $\|A\|_\infty = 10^6 + 1$, but $\rho(A) = 1$, so in this case the norm yields a poor approximation
however, suppose

$$D = \begin{bmatrix} \varepsilon & 0 \\ 0 & 1 \end{bmatrix}$$

then

$$DAD^{-1} = \begin{bmatrix} 1 & 10^6\varepsilon \\ 0 & 1 \end{bmatrix}$$

and $\|DAD^{-1}\|_\infty = 1 + 10^{-6}\epsilon,$, which for sufficiently small ε, yields a much better approximation to $\rho(DAD^{-1}) = \rho(A)$.

- if $\|A\| < 1$ for some submultiplicative norm, then $\|A^m\| \leq \|A\|^m \to 0$ as $m \to \infty$
- since $\|A\|$ is a continuous function of the elements of A, it follows that $A^m \to O$, i.e., every entry of A^m goes to 0
- however, if $\|A\| > 1$, it does not follow that $\|A^m\| \to \infty$
- for example, suppose

$$A = \begin{bmatrix} 0.99 & 10^6 \\ 0 & 0.99 \end{bmatrix}$$

in this case, $\|A\|_\infty > 1$, we claim that because $\rho(A) < 1$, $A^m \to O$ and so $\|A^m\| \to 0$
- let us prove this more generally, in fact we claim the following

Lemma 1. $\lim_{m\to\infty} A^m = O$ *if and only if* $\rho(A) < 1$.

Proof. ($\Rightarrow$) Let $A\mathbf{x} = \lambda\mathbf{x}$ with $\mathbf{x} \neq \mathbf{0}$. Then $A^m\mathbf{x} = \lambda^m\mathbf{x}$. Taking limits

$$\left(\lim_{m\to\infty} \lambda^m\right)\mathbf{x} = \lim_{m\to\infty} \lambda^m\mathbf{x} = \lim_{m\to\infty} A^m\mathbf{x} = \left(\lim_{m\to\infty} A^m\right)\mathbf{x} = O\mathbf{x} = \mathbf{0}.$$

Since $\mathbf{x} \neq \mathbf{0}$, we must have $\lim_{m\to\infty} \lambda^m = 0$ and thus $|\lambda| < 1$. Hence $\rho(A) < 1$.

($\Leftarrow$) Since $\rho(A) < 1$, there exists some operator norm $\|\cdot\|_\alpha$ such that $\|A\|_\alpha < 1$ by Theorem 3. So $\|A^m\|_\alpha \leq \|A\|_\alpha^m \to 0$ and so $A^m \to O$. Alternatively this part may also be proved directly via the Jordan form of A and (3.2) (without using Theorem 3). ☐

- in Homework 1 we will see that if for some operator norm, $\|A\| < 1$, then $I - A$ is nonsingular

5. Gerschgorin's theorem

- $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, for $i = 1, \dots, n$, we define the *Gerschgorin's discs*

$$G_i := \{z \in \mathbb{C} : |z - a_{ii}| \leq r_i\}$$

where

$$r_i := \sum_{j\neq i} |a_{ij}|$$

- Gerschgorin's theoerm says that the n eigenvalues of A are all contained in the union of G_i's
- before we prove this, we need a result that is by itself useful
- a matrix $A \in \mathbb{C}^{n\times n}$ is called *strictly diagonally dominant* if

$$|a_{ii}| > \sum_{j\neq i} |a_{ij}|$$

- it is called *weakly diagonally dominant* if the '$>$' is replaced by '$\geq$'

Lemma 2. *A strictly diagonally dominant matrix is nonsingular.*

Proof. Let A be strictly diagonally dominant. Suppose $A\mathbf{x} = \mathbf{0}$ for some $\mathbf{x} \neq \mathbf{0}$. Let $k \in \{1, \dots, n\}$ be such that $|x_k| = \max_{i=1,\dots,n}|x_i|$. Since $\mathbf{x} \neq \mathbf{0}$, we must have $|x_k| > 0$. Now observe that

$$a_{kk}x_k = -\sum_{j\neq k} a_{kj}x_j$$

and so by the triangle inequality,

$$|a_{kk}||x_k| = \left|\sum_{j\neq k} a_{kj}x_j\right| \leq \sum_{j\neq i}|a_{ij}||x_j| \leq \left(\sum_{j\neq k}|a_{kj}|\right)|x_k| < |a_{kk}||x_k|$$

where the last inequality is by strict diagonal dominance. This is a contradiction. In other words, there are no non-zero vector with $A\mathbf{x} = \mathbf{0}$. So $\ker(A) = \{\mathbf{0}\}$ and so A is nonsingular. $\square$

- we are going to use this to prove the first part of Gerschgorin theorem
- the second part requires a bit of topology

Theorem 4 (Gerschgorin)**.** *The spectrum of A is contained in the union of its Gerschgorin's discs, i.e.*

$$\lambda(A) \subseteq \bigcup\nolimits_{i=1}^{n} G_i.$$

Furthermore, the number of eigenvalues (counted with multiplicity) in each connected component of $\bigcup_{i=1}^{n} G_i$ is equal to the number of Gerschgorin discs that constitute that component.

Proof. Suppose $z \notin \bigcup_{i=1}^{n} G_i$. Then $A - zI$ is a strictly diagonal dominant matrix (check!) and therefore nonsingular by the above lemma. Hence $\det(A - zI) \neq 0$ and so z is not an eigenvalue of A. This proves the first part. For the second part, consider the matrix

$$A(t) := \begin{bmatrix} a_{11} & ta_{12} & \cdots & ta_{1n} \\ ta_{21} & a_{22} & \cdots & ta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ta_{n1} & ta_{n2} & \cdots & a_{nn} \end{bmatrix}$$

for $t \in [0, 1]$. Note that $A(0) = \operatorname{diag}(a_{11}, \dots, a_{nn})$ and $A(1) = A$. We will let $G_i(t)$ be the ith Gerschgorin disc of $A(t)$. So

$$G_i(t) = \{z \in \mathbb{C} : |z - a_{ii}| \leq tr_i\}.$$

Clearly $K_i(t) \subseteq K_i$ for any $t \in [0, 1]$. By the first part, all eigenvalues of all the matrices $A(t)$ are contained in $\bigcup_{i=1}^{n} G_i$. Since the set of eigenvalues of the matrices $A(t)$ depends continuously on the parameter t, $A(0)$ must have the same number of eigenvalues as $A(1)$ in each connected component of $\bigcup_{i=1}^{n} G_i$. Now just observe that the eigenvalues of $A(0)$ are simply the centers a_{kk} of each discs in a connected component. $\square$

6. SCHUR DECOMPOSITION

- suppose we want a decomposition for arbitrary matrices $A \in \mathbb{C}^{n\times n}$ like the EVD for normal matrices (2.1), i.e., diagonalizing with unitary matrices
- the way to obtain such a decomposition is to relax the requirement of having a diagonal matrix Λ in (2.1) but instead allow it to be upper-triangular
- this gives the *Schur decomposition*:

$$A = QRQ^* \tag{6.1}$$

- as in (2.1), Q is a unitary matrix but

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & r_{nn} \end{bmatrix}$$

- note that we the eigenvalues of A are precisely the diagonal entries of R: $r_{11}, \dots, r_{nn}$
- unlike the Jordan canonical form, the Schur decomposition is readily computable in finite-precision via the QR algorithm
- in its most basic form, QR algorithm does the following:

INPUT:	$A_0 = A$;	
STEP k:	$A_k = Q_k R_k$;	perform QR decomposition
STEP $k+1$:	$A_{k+1} = R_k Q_k$;	multiply QR factors in reverse order

- under suitable conditions, one may show that $Q_k \to Q$ and $R_k \to R$ where Q and R are as the requisite factors in (6.1)
- in most undergraduate linear algebra classes, one is taught to find eigenvalues by solving for the roots of the characteristic polynomial

$$p_A(x) = \det(xI - A) = 0$$

- this is almost never the case in practice
- for one, there is no finite algorithms for finding the roots of a polynomial when the degree exceeds four — by the famous impossibility result of Abel–Galois
- in fact what happens is the opposite — the roots of a univariate polynomial (divide by the coefficient of the highest degree term first so that it becomes a monic polynomial)

$$p(x) = c_0 + c_1 x + \cdots + c_{n-1} x^{n-1} + x^n$$

  are usually obtained as the eigenvalues of its companion matrix

$$C_p = \begin{bmatrix} 0 & 0 & \dots & 0 & -c_0 \\ 1 & 0 & \dots & 0 & -c_1 \\ 0 & 1 & \dots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -c_{n-1} \end{bmatrix}$$

  using the QR algorithm
- exercise: show that $\det(xI - C_p) = p(x)$

7. SINGULAR VALUE DECOMPOSITION

- let $A \in \mathbb{C}^{m \times n}$, we can always write

$$A = U \Sigma V^* \tag{7.1}$$

  – $U \in \mathbb{C}^{m \times m}$ and $V \in \mathbb{C}^{n \times n}$ are both unitary matrices

$$U^* U = I_m = U U^*, \quad V^* V = I_n = V V^*$$

  – $\Sigma \in \mathbb{C}^{m \times n}$ is a diagonal matrix in the sense that $\sigma_{ij} = 0$ if $i \neq j$

- if $m > n$, then Σ looks like

$$\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

- if $m < n$, then Σ looks like

$$\Sigma = \begin{bmatrix} \sigma_1 & & & 0 & \cdots & 0 \\ & \ddots & & \vdots & & \vdots \\ & & \sigma_m & 0 & \cdots & 0 \end{bmatrix}$$

- if $m = n$, then Σ looks like

$$\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_m \end{bmatrix}$$

- the diagonal elements of Σ, denoted σ_i, $i = 1, \ldots, n$, are all nonnegative, and can be ordered such that

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0, \quad \sigma_{r+1} = \cdots = \sigma_{\min(m,n)} = 0$$

- r is the rank of A

• this decomposition of A is called the *singular value decomposition*, or SVD
  - the values σ_i, for $i = 1, 2, \ldots, n$, are the *singular values* of A
  - the columns of U are the *left singular vectors*
  - the columns of V are the *right singular vectors*

• an alternative decomposition of A omits the singular values that are equal to zero:

$$A = \tilde{U}\tilde{\Sigma}\tilde{V}^*$$

  - $\tilde{U} \in \mathbb{C}^{m\times r}$ is a matrix with orthonormal columns, i.e. satisfying $\tilde{U}^*\tilde{U} = I_r$ (but not $\tilde{U}\tilde{U}^* = I_m$!)
  - $\tilde{V} \in \mathbb{C}^{n\times r}$ is also a matrix with orthonormal columns, i.e. satisfying $\tilde{V}^*\tilde{V} = I_r$ (but again not $\tilde{V}\tilde{V}^* = I_n$!)
  - $\tilde{\Sigma}$ is an $r \times r$ diagonal matrix with diagonal elements $\sigma_1, \ldots, \sigma_r$
  - again $r = \operatorname{rank}(A)$
  - the columns of $\tilde{U}$ are the left singular vectors corresponding to the nonzero singular values of A, and form an orthonormal basis for the range of A
  - the columns of $\tilde{V}$ are the right singular vectors corresponding to the nonzero singular values of A, and form an orthonormal basis for the cokernel of A
  - this is called the *condensed* or *compact* or *reduced* SVD
  - note that in this case, $\tilde{\Sigma}$ is a square matrix

• the form in (7.1) is sometimes called the *full* SVD

• we may also write the reduced SVD as a sum of rank-1 matrices

$$A = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^* + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^* + \cdots + \sigma_r \mathbf{u}_r \mathbf{v}_r^*$$

  - $\tilde{U} = [\mathbf{u}_1, \ldots, \mathbf{u}_r]$, i.e. $\mathbf{u}_1, \ldots, \mathbf{u}_r \in \mathbb{C}^m$ are the left singular vectors of A
  - $\tilde{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_r]$, i.e. $\mathbf{v}_1, \ldots, \mathbf{v}_r \in \mathbb{C}^n$ are the right singular vectors of A

- note that for $\mathbf{x} = [x_1, \dots, x_m]^\mathsf{T} \in \mathbb{C}^m$ and $\mathbf{y} = [y_1, \dots, y_n]^\mathsf{T} \in \mathbb{C}^n$,

$$\mathbf{x}\mathbf{y}^\mathsf{T} = \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_n \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_n \\ \vdots & \vdots & & \vdots \\ x_m y_1 & x_m y_2 & \cdots & x_m y_n \end{bmatrix}, \quad \mathbf{x}\mathbf{y}^* = \begin{bmatrix} x_1 \bar{y}_1 & x_1 \bar{y}_2 & \cdots & x_1 \bar{y}_n \\ x_2 \bar{y}_1 & x_2 \bar{y}_2 & \cdots & x_2 \bar{y}_n \\ \vdots & \vdots & & \vdots \\ x_m \bar{y}_1 & x_m \bar{y}_2 & \cdots & x_m \bar{y}_n \end{bmatrix}$$

- if neither $\mathbf{x}$ nor $\mathbf{y}$ is the zero vector, then

$$\operatorname{rank}(\mathbf{x}\mathbf{y}^\mathsf{T}) = \operatorname{rank}(\mathbf{x}\mathbf{y}^*) = 1$$

- furthermore if $\operatorname{rank}(A) = 1$, then there exists $\mathbf{x} \in \mathbb{C}^m$ and $\mathbf{y} \in \mathbb{C}^n$ so that $A = \mathbf{x}\mathbf{y}^*$