Algorithmic Framework for Quasi-Static Crack Growth in Viscoelastic Transversely Isotropic Media

Mikhailo Selivanov

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Abstract

This document presents a baseline formulation of a numerical framework for modeling delayed fracture and quasi-static crack growth in viscoelastic transversely isotropic solids using a cohesive zone approach and incremental constitutive equations.

The constitutive modeling, fracture criteria, and finite element formulation are described in a consistent manner. However, the algorithm for determining time increments during crack incubation and propagation is acknowledged to be provisional. In particular, for certain load and crack configurations, the time increment could not be reliably determined, which may affect quantitative conclusions drawn from earlier numerical results.

The present document therefore serves as a reference formulation for further algorithmic refinement and code development rather than as a source of validated numerical predictions.

1 Scope and limitations of the present formulation

The purpose of this document is to fix the mechanical model, notation, and overall algorithmic intent before revising the numerical procedure for time integration. While the viscoelastic constitutive relations and the cohesive-zone fracture framework are considered reliable, the time-stepping strategy used in earlier computations requires reassessment.

Consequently, numerical results reported previously should be interpreted as illustrative only and not as quantitatively validated predictions.

2 Problem statement

We consider a solid body containing a pre-existing crack subjected to quasi-static loading. The material outside the fracture process zone is modeled as a linear viscoelastic transversely isotropic medium. Crack initiation and propagation are described using a cohesive zone model with a prescribed traction–separation law.

The analysis is restricted to small deformations and isothermal conditions. Body forces are neglected.

3 Viscoelastic constitutive formulation

The stress–strain relationship is written in the Boltzmann–Volterra form

$$\sigma_{ij}(t) = \int_{-\infty}^{t} C_{ijkl}(t-\tau)\,\frac{\partial \varepsilon_{kl}(\tau)}{\partial \tau}\,\mathrm{d}\tau, \tag{1}$$

where $C_{ijkl}(t)$ are the relaxation functions of a non-aging viscoelastic transversely isotropic material.

For numerical implementation, the constitutive equations are rewritten in incremental form using internal variables, allowing efficient evaluation within a finite element framework.

3.1 Incremental quasi-static formulation

The evolution of the crack and the associated viscoelastic response is modeled as a sequence of quasi-static increments. Each increment connects two equilibrium states and is characterized by an unknown time duration $\Delta t > 0$, which is determined as part of the solution rather than prescribed *a priori*.

At the beginning of an increment, the current state of the system is assumed known, including the displacement field, the internal viscoelastic history variables, and the position of the active cohesive zone. Within the increment, the response is governed by a viscoelastic constitutive law and a nonlinear cohesive traction–separation relation.

Incremental equilibrium problem. For a given trial value of Δt, the viscoelastic constitutive update yields an effective tangent stiffness operator $\mathbf{K}(\Delta t)$ and a history-dependent residual force vector $\mathbf{F}^{t\sigma}(\Delta t)$. The incremental equilibrium problem is then formulated as a coupled system for the displacement increment $\Delta\mathbf{u}$ and a scalar load parameter Σ:

$$\mathbf{K}(\Delta t)\,\Delta\mathbf{u} - \big(\Sigma\,\mathbf{r}_{\text{ext}} + \mathbf{r}_{\text{c}}(\Delta\mathbf{u}) - \mathbf{F}^{p\sigma} + \mathbf{F}^{t\sigma}(\Delta t)\big) = \mathbf{0}, \tag{2}$$

where $\mathbf{r}_{\text{ext}}$ denotes the external load vector, $\mathbf{r}_{\text{c}}$ is the cohesive force vector, and $\mathbf{F}^{p\sigma}$ represents the contribution of stresses accumulated in previous increments.

To ensure a unique solution of the incremental problem, an additional kinematic constraint is imposed. In the present work, the constraint prescribes the opening displacement at a selected cohesive index c_i,

$$u_{c_i} + \Delta u_{c_i} = \frac{k_D D_{\text{ym}}}{2}, \tag{3}$$

where k_D is a dimensionless control parameter and D_{ym} is a characteristic opening scale.

Equations (2) and (3) form a coupled nonlinear system for $(\Delta\mathbf{u}, \Sigma)$, which is solved quasi-statically for each trial value of Δt.

Continuation condition for the time increment. For a fixed internal state and control parameter k_D, the solution of the incremental problem yields a scalar quantity $\Sigma(\Delta t)$, representing the external load required to satisfy equilibrium and the imposed opening constraint over an increment of duration Δt.

The admissible time increment is defined by the scalar continuation condition

$$\Sigma(\Delta t) = \Sigma_{\text{ext}}, \tag{4}$$

where Σ_{ext} is the prescribed external load level. The existence of a quasi-static increment therefore reduces to the existence of a root of Eq. (4) for $\Delta t > 0$.

Physical interpretation. In the adopted viscoelastic formulation, increasing Δt enhances stress relaxation and reduces the effective stiffness of the structure. As a result, the function $\Sigma(\Delta t)$ is expected to decrease with increasing Δt along a continuously tracked equilibrium branch. The limiting case $\Delta t \to 0^+$ corresponds to an instantaneous elastic response, while large values of Δt approach the fully relaxed regime.

If Eq. (4) admits no solution for $\Delta t > 0$, no quasi-static increment exists that satisfies the prescribed loading and constraint. This criterion provides a physically meaningful distinction between the non-existence of quasi-static solutions and purely numerical convergence issues.

4 Cohesive zone model

Crack growth is modeled by a cohesive zone located ahead of the crack tip. The normal cohesive traction is assumed to depend on the local crack opening displacement Δ via a prescribed traction–separation law

$$T = T(\Delta), \tag{5}$$

which is assumed to be non-negative and vanishing at a critical opening $\Delta_{\max}$.

Crack propagation is governed by the condition

$$\Delta(\lambda, t) = \Delta_{\max}, \tag{6}$$

where λ denotes the current crack length.

5 Algorithmic structure

The computational procedure conceptually consists of the following stages:

1. instantaneous elastic response at load application;
2. crack incubation under viscoelastic deformation at fixed crack length;
3. quasi-static crack growth under sustained loading.

At each stage, the displacement field is obtained by enforcing equilibrium and the cohesive traction–separation law. Time increments are determined by solving a nonlinear auxiliary problem that links the applied load, crack geometry, and crack opening.

6 Current status of the time-increment procedure

The determination of time increments represents the most sensitive part of the algorithm. In its current form, the procedure may fail to identify a valid time increment for certain steps of crack incubation or propagation. This deficiency directly affects the reliability of time-dependent results, such as predicted incubation times and crack growth histories.

For this reason, the time-integration strategy must be revised and validated before any quantitative conclusions can be drawn.

6.1 Solvability of the time-increment equation

The key computational step of the proposed framework is the determination of the time increment Δt_n from the scalar nonlinear equation

$$\Sigma(\Delta t_n, \Delta_n, \lambda_n) = \sigma^{(\mathrm{ext})}, \tag{7}$$

where $\Sigma(\Delta t, \Delta, \lambda)$ denotes the external load level required to achieve (after the viscoelastic increment of duration Δt) the prescribed crack-tip opening Δ for a prescribed crack geometry characterized by λ. Equation (7) is the natural coupling condition between (i) the incremental viscoelastic update and (ii) the fracture constraint $\Delta(\lambda_n, t_n) = \Delta_n$ imposed at the crack tip while satisfying the cohesive traction–separation law along the cohesive segment.

Expected qualitative properties. For fixed (Δ_n, λ_n), the function $\Sigma(\Delta t, \Delta_n, \lambda_n)$ is expected to be non-increasing in Δt under bounded creep: increasing the time available for relaxation and creep reduces the load required to attain a given opening. In addition,

$$\lim_{\Delta t \to 0^+} \Sigma(\Delta t, \Delta_n, \lambda_n) = \Sigma_{\text{inst}}(\Delta_n, \lambda_n), \tag{8}$$

corresponding to the instantaneous (elastic) response.

If $\Sigma(\Delta t, \Delta_n, \lambda_n)$ is continuous in Δt and admits the limit

$$\Sigma_\infty(\Delta_n, \lambda_n) = \lim_{\Delta t \to \infty} \Sigma(\Delta t, \Delta_n, \lambda_n),$$

then a sufficient condition for existence of a solution of Eq. (7) is

$$\Sigma_\infty(\Delta_n, \lambda_n) \le \sigma^{(\text{ext})} \le \Sigma_{\text{inst}}(\Delta_n, \lambda_n). \tag{9}$$

When the inequality is strict and Σ is strictly monotone, the solution Δt_n is unique.

Interpretation of numerical difficulties. In the present context, Eq. (7) is retained as the correct governing relation for determining Δt_n. The main issue observed in earlier computations is that the numerical procedure used to solve Eq. (7) may fail to identify Δt_n at some steps, even though a root may exist. This motivates the development of a more robust root-finding strategy (e.g., bracketing plus safeguarded iterations) and improved diagnostics for verifying the existence condition (9) at each step.

6.2 Properties of the continuation function $\Sigma(\Delta t)$

In the incremental quasi-static formulation adopted in this work, the time increment Δt is not prescribed *a priori*, but is determined from a scalar continuation condition. For a fixed internal state at the beginning of an increment and a prescribed kinematic control parameter, the equilibrium problem yields a scalar quantity $\Sigma(\Delta t)$, which represents the remote load (or stress) required to satisfy equilibrium and the imposed constraint over the increment of duration Δt.

The admissible time increment is therefore defined implicitly by the scalar equation

$$\Sigma(\Delta t) = \Sigma_{\text{ext}}, \tag{10}$$

where Σ_{ext} denotes the prescribed external load level.

Dependence on Δt. In the present viscoelastic formulation, the function $\Sigma(\Delta t)$ depends on Δt through two mechanisms. First, the effective tangent modulus entering the incremental stiffness operator decreases with increasing Δt, reflecting the transition from instantaneous to relaxed material response. Second, the contribution of the viscoelastic history term increases with Δt, providing an additional internal driving force that reduces the external load required to reach the prescribed kinematic state. Consequently, along a continuously tracked equilibrium branch, $\Sigma(\Delta t)$ is expected to be a decreasing function of Δt.

Limiting behavior. In the limit $\Delta t \to 0^+$, the viscoelastic relaxation is negligible and the incremental response approaches the instantaneous elastic behavior. In this limit, $\Sigma(\Delta t)$ attains its maximum value. Conversely, for large Δt, the response approaches the fully relaxed regime and $\Sigma(\Delta t)$ attains its minimum value. This behavior provides a natural basis for determining Δt from Eq. (10).

Existence of quasi-static increments. Equation (10) admits a quasi-static solution if and only if a root of $\Sigma(\Delta t) - \Sigma_{\text{ext}} = 0$ exists for $\Delta t > 0$. The absence of such a root indicates that no quasi-static increment can satisfy the prescribed constraint at the given load level. This criterion provides a physically meaningful distinction between the existence of quasi-static solutions and purely numerical convergence issues.

Remarks on monotonicity. While the above arguments suggest a monotone decrease of $\Sigma(\Delta t)$ with Δt, strict monotonicity is not guaranteed in the presence of material and geometric nonlinearities, such as cohesive softening or changes in the active constraint set. For this reason, the determination of Δt is performed using a bracketing strategy based on the sign change of $\Sigma(\Delta t) - \Sigma_{\text{ext}}$, rather than relying solely on the convergence behavior of nonlinear solvers.

7 Outlook

Future work will focus on:

- reformulating the time-increment determination problem;
- improving robustness and existence guarantees of the auxiliary solver;
- reassessing numerical results after algorithmic correction.

The present document will serve as a stable reference for these developments.