

# COMSW 4731 Computer Vision, Fall 2017

## Homework 2

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### Problem 1

Show that the extreme values of moment of inertia( $E$ ) for a 2D binary object are given by the eigenvalues of the  $2 \times 2$  matrix:

$$\begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$$

1. where  $a, b, c$ , and  $E$  are as defined in the lecture notes. (3 points)

**Proof:**

The characteristic equation of the matrix above is

$$|\mathbf{A} - \lambda \cdot \mathbf{I}| = \left| \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} a - \lambda & b/2 \\ b/2 & c - \lambda \end{vmatrix} = (a - \lambda)(c - \lambda) - (b/2)^2 = 0$$

$$\lambda^2 - (a + c)\lambda + ac - \frac{b^2}{4} = 0$$

and the two eigenvalues are

$$\begin{aligned} \lambda_{1,2} &= \frac{a + c \pm \sqrt{(a + c)^2 - 4ac + b^2}}{2} \\ &= \frac{a + c \pm \sqrt{(a - c)^2 + b^2}}{2} \end{aligned}$$

By the definition of  $a, b, c$ , and  $E$  in the lecture notes.

$$E = a \sin^2 \theta - b \sin \theta \cos \theta + c \cos^2 \theta$$

$$E = \frac{1}{2}(a + c) - \frac{1}{2}(a - c) \cos 2\theta - \frac{1}{2} \sin 2\theta$$

When  $\tan 2\theta = \frac{b}{a - c}$ , that is to say  $\sin 2\theta = \pm \frac{b}{\sqrt{b^2 + (a - c)^2}}$  and  $\cos 2\theta = \pm \frac{a - c}{\sqrt{b^2 + (a - c)^2}}$ ,  $E$  has extreme values.

When  $\sin 2\theta = \frac{b}{\sqrt{b^2 + (a-c)^2}}$  and  $\cos 2\theta = \frac{a-c}{\sqrt{b^2 + (a-c)^2}}$ ,  $E$  has the minimum value:

$$E = \frac{1}{2}(a+c) - \frac{(a-c)^2}{2\sqrt{b^2 + (a-c)^2}} - \frac{b^2}{2\sqrt{b^2 + (a-c)^2}}$$

$$E = \frac{1}{2}(a+c) - \frac{\sqrt{b^2 + (a-c)^2}}{2}$$

When  $\sin 2\theta = -\frac{b}{\sqrt{b^2 + (a-c)^2}}$  and  $\cos 2\theta = -\frac{a-c}{\sqrt{b^2 + (a-c)^2}}$ ,  $E$  has the maximum value:

$$E = \frac{1}{2}(a+c) + \frac{(a-c)^2}{2\sqrt{b^2 + (a-c)^2}} + \frac{b^2}{2\sqrt{b^2 + (a-c)^2}}$$

$$E = \frac{1}{2}(a+c) + \frac{\sqrt{b^2 + (a-c)^2}}{2}$$

As we can see above, the extreme values of moment of inertia( $E$ ) for a 2D binary object are given by the eigenvalues of the matrix.

**2.** Argue that  $E$  is real and non-negative, and hence prove that  $4ac \geq b^2$ . **(2 points)**

**Proof:**

By the definition of moment of inertia  $E$  for a 2D binary object

$$E = \int \int_I r^2 b(x, y) dx dy$$

Where  $r$  is the perpendicular distance from point  $(x, y)$  to a line. Obviously,  $r^2$  is real and nonnegative.

Since the area of a object is

$$A = \int \int_I b(x, y) dx dy$$

which is real and nonnegative. So  $E$  is real and nonnegative.

And the minimum value of  $E$  is nonnegative

$$\begin{aligned} \frac{1}{2}(a+c) - \frac{\sqrt{b^2 + (a-c)^2}}{2} &\geq 0 \\ (a+c)^2 &\geq b^2 + (a-c)^2 \\ 4ac &\geq b^2 \end{aligned}$$

**3.** What kind of object gives  $E$  equal to zero ? **(1 point)**

**Solution:**

When the binary object is a straight line,  $E$  is zero.

Because  $E$  is the integer of the square of the distance from the point  $(x, y)$  to a line, if  $E$  is zero, then every distance is zero, which means every point is on the line, and the binary object is a straight line.