COMSW 4731 Computer Vision, Fall 2017
Homework 2

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Problem 1

Show that the extreme values of moment of inertia(E) for a 2D binary object are given by the eigenvalues of the 2 x 2 matrix:

$$\begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$$

1. where a, b, c, and E are as defined in the lecture notes. **(3 points)**
Proof:
The characteristic equation of the matrix above is

$$\left|\mathbf{A} - \lambda \cdot \mathbf{I}\right| = \left|\begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right| = 0$$

$$\begin{vmatrix} a-\lambda & b/2 \\ b/2 & c-\lambda \end{vmatrix} = (a-\lambda)(c-\lambda) - (b/2)^2 = 0$$

$$\lambda^2 - (a+c)\lambda + ac - \frac{b^2}{4} = 0$$

and the two eigenvalues are

$$\begin{aligned} \lambda_{1,2} &= \frac{a+c \pm \sqrt{(a+c)^2 - 4ac + b^2}}{2} \\ &= \frac{a+c \pm \sqrt{(a-c)^2 + b^2}}{2} \end{aligned}$$

By the definition of a, b, c, and E in the lecture notes.

$$\begin{aligned} E &= asin^2\theta - bsin\theta cos\theta + ccos^2\theta \\ E &= \frac{1}{2}(a+c) - \frac{1}{2}(a-c)cos2\theta - \frac{1}{2}sin2\theta \end{aligned}$$

When $tan2\theta = \dfrac{b}{a-c}$, that is to say $sin2\theta = \pm\dfrac{b}{\sqrt{b^2+(a-c)^2}}$ and $cos2\theta = \pm\dfrac{a-c}{\sqrt{b^2+(a-c)^2}}$, E has extreme values.

When $sin2\theta = \frac{b}{\sqrt{b^2+(a-c)^2}}$ and $cos2\theta = \frac{a-c}{\sqrt{b^2+(a-c)^2}}$, E has the minimum value:

$$E = \frac{1}{2}(a+c) - \frac{(a-c)^2}{2\sqrt{b^2+(a-c)^2}} - \frac{b^2}{2\sqrt{b^2+(a-c)^2}}$$
$$E = \frac{1}{2}(a+c) - \frac{\sqrt{b^2+(a-c)^2}}{2}$$

When $sin2\theta = -\frac{b}{\sqrt{b^2+(a-c)^2}}$ and $cos2\theta = -\frac{a-c}{\sqrt{b^2+(a-c)^2}}$, E has the maximum value:

$$E = \frac{1}{2}(a+c) + \frac{(a-c)^2}{2\sqrt{b^2+(a-c)^2}} + \frac{b^2}{2\sqrt{b^2+(a-c)^2}}$$
$$E = \frac{1}{2}(a+c) + \frac{\sqrt{b^2+(a-c)^2}}{2}$$

As we can see above, the extreme values of moment of inertia(E) for a 2D binary object are given by the eigenvalues of the matrix.

2. Argue that E is real and non-negative, and hence prove that $4ac \geq b^2$. **(2 points)**
Proof:
By the definition of moment of inertia E for a 2D binary object

$$E = \int\int_I r^2 b(x,y)dxdy$$

Where r is the perpendicular distance from point (x,y) to a line. Obviously, r^2 is real and nonnegative.
Since the area of a object is

$$A = \int\int_I b(x,y)dxdy$$

which is real and nonnegative. So E is real and nonnegative.
And the minimum value of E is nonnegative

$$\frac{1}{2}(a+c) - \frac{\sqrt{b^2+(a-c)^2}}{2} \geq 0$$
$$(a+c)^2 \geq b^2 + (a-c)^2$$
$$4ac \geq b^2$$

3. What kind of object gives E equal to zero ? **(1 point)**
Solution:
When the binary object is a straight line, E is zero.
Because E is the integer of the square of the distance from the point (x,y) to a line, if E is zero, then every distance is zero, which means every point is on the line, and the binary object is a straight line.