

1 Theory

1.1 Relation between momentum and acceleration

In classical mechanics, Newton's second law defines momentum and states

$$\vec{\mathbf{F}} = \frac{\partial \vec{\mathbf{p}}}{\partial t} = m \frac{\partial \vec{\mathbf{v}}}{\partial t} = m \vec{\mathbf{a}} \quad (1)$$

In special relativity however, relativistic momentum is defined as $\vec{p} = \gamma m_0 \vec{v}$ where gamma is the Lorentz Factor;

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - \beta^2}}$$

Considering these two statements, we can find the relation between momentum and acceleration as follows

$$\begin{aligned} \frac{\partial \vec{p}}{\partial t} &= m_0 \frac{\partial(\gamma \vec{v})}{\partial t} = m_0 \left\{ \frac{\partial \gamma}{\partial t} \vec{v} + \gamma \frac{\partial \vec{v}}{\partial t} \right\} \\ \frac{\partial \gamma}{\partial t} &= \gamma^3 \vec{\beta} \cdot \frac{\partial \vec{\beta}}{\partial t} = \frac{\gamma^3}{c} \vec{\beta} \cdot \vec{a} \\ \frac{\partial \vec{p}}{\partial t} &= m_0 \left\{ \frac{\gamma^3}{c} (\vec{\beta} \cdot \vec{a}) \vec{v} + \gamma \frac{\partial \vec{v}}{\partial t} \right\} \\ \vec{\mathbf{F}} &= \gamma m_0 \{ \vec{a} + \gamma^2 (\vec{\beta} \cdot \vec{a}) \vec{\beta} \} \end{aligned} \quad (2)$$

It is clear that acceleration is not necessarily parallel to the force. To start separating the parallel and perpendicular components relative to $\vec{\beta}$, we can find $\vec{a}_{||}$ and $\vec{\mathbf{F}}_{||}$;

$$\vec{a}_{||} = \frac{(\vec{a} \cdot \vec{\beta})}{\beta^2} \vec{\beta} \quad \vec{\mathbf{F}}_{||} = \frac{(\vec{\mathbf{F}} \cdot \vec{\beta})}{\beta^2} \vec{\beta} \quad (3)$$

$$\begin{aligned} \vec{\mathbf{F}} \cdot \vec{\beta} &= \gamma m_0 \{ \vec{a} \cdot \vec{\beta} + \gamma^2 (\vec{\beta} \cdot \vec{a}) \beta^2 \} \\ &= \gamma m_0 (\vec{a} \cdot \vec{\beta}) \{ \gamma^2 \beta^2 + 1 \} \end{aligned}$$

Using $\gamma^2 \beta^2 + 1 = \gamma^2$ we have,

$$\vec{\mathbf{F}} \cdot \vec{\beta} = m_0 \gamma^3 (\vec{a} \cdot \vec{\beta})$$

Inserting this

$$\begin{aligned} \vec{\mathbf{F}}_{||} &= \frac{(\vec{\mathbf{F}} \cdot \vec{\beta})}{\beta^2} \vec{\beta} \\ &= m_0 \gamma^3 \frac{(\vec{a} \cdot \vec{\beta})}{\beta^2} \vec{\beta} \\ &= m_0 \gamma^3 \vec{a}_{||} \end{aligned} \quad (4)$$

Therefore from *equations 2 and 4*

$$\begin{aligned}
\vec{\mathbf{F}} &= m_0 \gamma^3 \vec{\mathbf{a}}_{||} \beta^2 + m_0 \gamma \vec{\mathbf{a}} \\
&= m_0 \gamma^3 \vec{\mathbf{a}}_{||} \beta^2 + m_0 \gamma \{ \vec{\mathbf{a}}_{||} + \vec{\mathbf{a}}_{\perp} \} \\
&= m_0 \vec{\mathbf{a}}_{||} \gamma \{ \gamma^2 \beta^2 + 1 \} + m_0 \gamma \vec{\mathbf{a}}_{\perp} \\
&= m_0 \vec{\mathbf{a}}_{||} \gamma^3 + m_0 \gamma \vec{\mathbf{a}}_{\perp} \\
&= \vec{\mathbf{F}}_{||} + m_0 \gamma \vec{\mathbf{a}}_{\perp}
\end{aligned}$$

$$\vec{\mathbf{F}}_{||} = \gamma^3 m_0 \vec{\mathbf{a}}_{||} \quad \vec{\mathbf{F}}_{\perp} = \gamma m_0 \vec{\mathbf{a}}_{\perp} \quad (5)$$

1.2 Lorentz Force

Force acting on a charged particle moving in electromagnetic fields is called Lorentz Force and is given by the formula

$$\frac{\partial \vec{p}}{\partial t} = \vec{F}_L = q(\vec{E} + \vec{v} \times \vec{B}) \quad (6)$$

where the q is the charge and \vec{v} is the velocity of the particle. Replacing

1.3 Relativistic Lorentz Force

Similar to non-relativistic version, relativistic Lorentz Force is given by the following 4-vector equality

$$\frac{\partial p^\mu}{\partial \tau} = q F^{\mu\nu} u_\nu \quad (7)$$

Where $\partial \tau = \partial t / \gamma$,

$$p^\mu = \begin{bmatrix} W/c \\ p_x \\ p_y \\ p_z \end{bmatrix} \quad F^{\mu\nu} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix} \quad u_\nu = \gamma \begin{bmatrix} c \\ -v_x \\ -v_y \\ -v_z \end{bmatrix} \quad (8)$$

Where W is the energy of the particle and $\gamma = 1/\sqrt{1 - v^2/c^2}$ is the lorentz factor.

For $\mu = 0$, we have the time component of the equation;

$$\frac{\gamma}{c} \frac{\partial W}{\partial t} = \frac{q \gamma \vec{E} \cdot \vec{v}}{c} = \frac{q \gamma}{c} \frac{\vec{E} \cdot \partial \vec{r}}{\partial t} \quad (9)$$

$$\frac{\partial W}{\partial t} = q \frac{\vec{E} \cdot \partial \vec{r}}{\partial t} \quad (10)$$

This is the definition of work done by an electric field. For $\mu = 1, 2, 3$, we have the spacial components;

$$\frac{\partial \vec{p}}{\partial \tau} = \gamma \frac{\partial \vec{p}}{\partial t} = q\gamma(\vec{E} + \vec{v} \times \vec{B})$$

Which simplifies to non-relativistic Lorentz Force in *equation 6*.

1.4 Acceleration caused by lorentz force

Due to the nature of the cross product, lorentz force caused by a magnetic field is always perpendicular to the velocity of the particle. Therefore the acceleration of the magnetic field is straightforward

$$\vec{F}_B = \vec{F}_\perp = \gamma m_0 \vec{a}_\perp = \gamma m_0 \vec{a}_B$$

The same thing cannot be said about electric field however. It can create force in any direction with respect to velocity. Therefore, we have the following equality;

$$\vec{a}_\parallel = \frac{q}{\gamma^3 m_0} \vec{E}_\parallel \quad \vec{a}_\perp = \frac{q}{\gamma m_0} \{\vec{E}_\perp + \vec{v} \times \vec{B}\} \quad (11)$$

Acceleration due to electric field can be simplified as;

$$\begin{aligned} \vec{a}_{\{B=0\}} = \vec{a}_E &= \vec{a}_\parallel + \vec{a}_{\perp\{B=0\}} \\ &= \frac{q}{m_0 \gamma} \left\{ \frac{\vec{E}_\parallel}{\gamma^2} + \vec{E}_\perp \right\} \\ &= \frac{q}{m_0 \gamma} \{ \{1 - \beta^2\} \vec{E}_\parallel + \vec{E}_\perp \} \\ &= \frac{q}{m_0 \gamma} \{ \vec{E}_\parallel + \vec{E}_\perp - \beta^2 \vec{E}_\parallel \} \\ &= \frac{q}{m_0 \gamma} \{ \vec{E} - \beta^2 \vec{E}_\parallel \} \end{aligned}$$

Using the fact that $\vec{E}_\parallel = \vec{\beta}(\vec{E} \cdot \vec{\beta})/\beta^2$, we finally have

$$\vec{a}_E = \frac{q}{m_0 \gamma} \left\{ \vec{E} - \vec{v} \frac{(\vec{E} \cdot \vec{v})}{c^2} \right\} \quad \vec{a}_B = \frac{q}{\gamma m_0} (\vec{v} \times \vec{B}) \quad (12)$$