1 Theory

1.1 Relation between momentum and acceleration

In classical mechanics, Newton's second law defines momentum and states

$$\vec{\mathbf{F}} = \frac{\partial \vec{\mathbf{p}}}{\partial t} = m\frac{\partial \vec{\mathbf{v}}}{\partial t} = m\vec{\mathbf{a}} \tag{1}$$

In special relativity however, relativistic momentum is defined as $\vec{p} = \gamma m_0 \vec{v}$ where gamma is the Lorentz Factor;

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1-\beta^2}}$$

Considering these two statements, we can find the relation between momentum and acceleration as follows

$$\begin{aligned}
\frac{\partial \vec{p}}{\partial t} &= m_0\frac{\partial(\gamma\vec{v})}{\partial t} = m_0\{\frac{\partial\gamma}{\partial t}\vec{v} + \gamma\frac{\partial\vec{v}}{\partial t}\} \\
\frac{\partial\gamma}{\partial t} &= \gamma^3\vec{\beta}\cdot\frac{\partial\vec{\beta}}{\partial t} = \frac{\gamma^3}{c}\vec{\beta}\cdot\vec{a} \\
\frac{\partial \vec{p}}{\partial t} &= m_0\{\frac{\gamma^3}{c}(\vec{\beta}\cdot\vec{a})\vec{v} + \gamma\frac{\partial\vec{v}}{\partial t}\}
\end{aligned}$$

$$\vec{\mathbf{F}} = \gamma m_0\{\vec{\mathbf{a}} + \gamma^2(\vec{\beta}\cdot\vec{\mathbf{a}})\vec{\beta}\} \tag{2}$$

It is clear that acceleration is not necessarily parallel to the force. To start seperating the parallel and perpendicular components relative to $\vec{\beta}$, we can find $\vec{\mathbf{a}}_{||}$ and $\vec{\mathbf{F}}_{||}$;

$$\vec{\mathbf{a}}_{||} = \frac{(\vec{a}\cdot\vec{\beta})}{\beta^2}\beta \qquad\qquad \vec{\mathbf{F}}_{||} = \frac{(\vec{F}\cdot\vec{\beta})}{\beta^2}\beta \tag{3}$$

$$\begin{aligned}
\vec{\mathbf{F}}\cdot\vec{\beta} &= \gamma m_0\{\vec{\mathbf{a}}\cdot\vec{\beta} + \gamma^2(\vec{\beta}\cdot\vec{\mathbf{a}})\beta^2\} \\
&= \gamma m_0(\vec{\mathbf{a}}\cdot\vec{\beta})\{\gamma^2\beta^2 + 1\}
\end{aligned}$$

Using $\gamma^2\beta^2 + 1 = \gamma^2$ we have,

$$\vec{\mathbf{F}}\cdot\vec{\beta} = m_0\gamma^3(\vec{\mathbf{a}}\cdot\vec{\beta})$$

Inserting this

$$\begin{aligned}
\vec{\mathbf{F}}_{||} &= \frac{(\vec{\mathbf{F}}\cdot\vec{\beta})}{\beta^2}\vec{\beta} \\
&= m_0\gamma^3\frac{(\vec{\mathbf{a}}\cdot\vec{\beta})}{\beta^2}\vec{\beta} \\
&= m_0\gamma^3\vec{\mathbf{a}}_{||} \qquad (4)
\end{aligned}$$

Therefore from *equations 2 and 4*

$$
\begin{aligned}
\vec{\mathbf{F}} &= m_0\gamma^3\vec{\mathbf{a}}_{||}\beta^2 + m_0\gamma\vec{\mathbf{a}} \\
&= m_0\gamma^3\vec{\mathbf{a}}_{||}\beta^2 + m_0\gamma\{\vec{\mathbf{a}}_{||} + \vec{\mathbf{a}}_{\perp}\} \\
&= m_0\vec{\mathbf{a}}_{||}\gamma\{\gamma^2\beta^2 + 1\} + m_0\gamma\vec{\mathbf{a}}_{\perp} \\
&= m_0\vec{\mathbf{a}}_{||}\gamma^3 + m_0\gamma\vec{\mathbf{a}}_{\perp} \\
&= \vec{\mathbf{F}}_{||} + m_0\gamma\vec{\mathbf{a}}_{\perp}
\end{aligned}
$$

$$
\vec{\mathbf{F}}_{||} = \gamma^3 m_0\vec{\mathbf{a}}_{||} \qquad\qquad \vec{\mathbf{F}}_{\perp} = \gamma m_0\vec{\mathbf{a}}_{\perp} \tag{5}
$$

1.2 Lorentz Force

Force acting on a charged particle moving in electromagnetic fields is called Lorentz Force and is given by the formula

$$
\frac{\partial\vec{p}}{\partial t} = \vec{F}_L = q(\vec{E} + \vec{v}\times\vec{B}) \tag{6}
$$

where the q is the charge and $\vec{v}$ is the velocity of the particle. Replacing

1.3 Relativistic Lorentz Force

Similar to non-relativistic version, relativistic Lorentz Force is given by the following 4-vector equality

$$
\frac{\partial p^\mu}{\partial\tau} = qF^{\mu\nu}u_\nu \tag{7}
$$

Where $\partial\tau = \partial t/\gamma$,

$$
p^\mu = \begin{bmatrix} W/c \\ p_x \\ p_y \\ p_z \end{bmatrix} \qquad F^{\mu\nu} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix} \qquad u_\nu = \gamma\begin{bmatrix} c \\ -v_x \\ -v_y \\ -v_z \end{bmatrix} \tag{8}
$$

Where W is the energy of the particle and $\gamma = 1/\sqrt{1 - v^2/c^2}$ is the lorentz factor.

For $\mu = 0$, we have the time component of the equation;

$$
\frac{\gamma}{c}\frac{\partial W}{\partial t} = \frac{q\gamma\vec{E}\cdot\vec{v}}{c} = \frac{q\gamma}{c}\frac{\vec{E}\cdot\partial\vec{r}}{\partial t} \tag{9}
$$

$$
\frac{\partial W}{\partial t} = q\frac{\vec{E}\cdot\partial\vec{r}}{\partial t} \tag{10}
$$

This is the definition of work done by an electric field. For $\mu = 1, 2, 3$, we have the spacial components;

$$\frac{\partial \vec{p}}{\partial \tau} = \gamma \frac{\partial \vec{p}}{\partial t} = q\gamma(\vec{E} + \vec{v} \times \vec{B})$$

Which simplifies to non-relativistic Lorentz Force in *equation 6*.

1.4 Acceleration caused by lorentz force

Due to the nature of the cross product, lorentz force caused by a magnetic field is always perpendicular to the velocity of the particle. Therefore the acceleration of the magnetic field is straightforward

$$\vec{\mathbf{F}}_B = \vec{\mathbf{F}}_\perp = \gamma m_0 \vec{\mathbf{a}}_\perp = \gamma m_0 \vec{\mathbf{a}}_B$$

The same thing cannot be said about electric field however. It can create force in any direction with respect to velocity. Therefore, we have the following equality;

$$\vec{\mathbf{a}}_{||} = \frac{q}{\gamma^3 m_0}\vec{\mathbf{E}}_{||} \qquad \vec{\mathbf{a}}_\perp = \frac{q}{\gamma m_0}\{\vec{\mathbf{E}}_\perp + \vec{\mathbf{v}} \times \vec{\mathbf{B}}\} \tag{11}$$

Acceleration due to electric field can be simplified as;

$$\begin{aligned}
\vec{\mathbf{a}}_{\{B=0\}} = \vec{\mathbf{a}}_E &= \vec{\mathbf{a}}_{||} + \vec{\mathbf{a}}_{\perp\{B=0\}} \\
&= \frac{q}{m_0\gamma}\{\frac{\vec{\mathbf{E}}_{||}}{\gamma^2} + \vec{\mathbf{E}}_\perp\} \\
&= \frac{q}{m_0\gamma}\{\{1-\beta^2\}\vec{\mathbf{E}}_{||} + \vec{\mathbf{E}}_\perp\} \\
&= \frac{q}{m_0\gamma}\{\vec{\mathbf{E}}_{||} + \vec{\mathbf{E}}_\perp - \beta^2\vec{\mathbf{E}}_{||}\} \\
&= \frac{q}{m_0\gamma}\{\vec{\mathbf{E}} - \beta^2\vec{\mathbf{E}}_{||}\}
\end{aligned}$$

Using the fact that $\vec{\mathbf{E}}_{||} = \vec{\beta}(\vec{\mathbf{E}} \cdot \vec{\beta})/\beta^2$, we finally have

$$\vec{\mathbf{a}}_E = \frac{q}{m_0\gamma}\{\vec{\mathbf{E}} - \vec{\mathbf{v}}\frac{(\vec{\mathbf{E}} \cdot \vec{\mathbf{v}})}{c^2}\} \qquad \vec{\mathbf{a}}_B = \frac{q}{\gamma m_0}(\vec{\mathbf{v}} \times \vec{\mathbf{B}}) \tag{12}$$

1.5 Leap Frog

Leapfrog is a method that is used to numarically intergenerate that are in the form of

$$\frac{\partial^2 x}{\partial t^2} = f(x)$$

It is also known as the Störmer-Verlet method, commonly used to numerically calculate the trejectory of particles. The name comes from the fact that calculation of updated $\mathbf{x}$ and $\mathbf{v}$ are done in some order and calculated for different time slices. The energy is approximately conserved during the calculation. It is stable in oscillatory motion as long as the time-step Δt is constant, and satisfies $\Delta t \leq 2/\omega$ [1]. The idea is straight forward; in the time interval Δt,

$$\begin{aligned}
a(t_0) &= f(x_0) \\
x(t_0 + \Delta t) &= x(t_0) + v(t_0)\Delta t + a(t_0)\frac{\Delta t^2}{2} \qquad (13) \\
v(t_0 + \Delta t) &= v(t_0) + \{a(t_0) + a(t_0 + \Delta t)\}\frac{\Delta t}{2} \qquad (14)
\end{aligned}$$

For more stability, this version can be rearranged to what is called 'kick-drift-kick' form,

$$\begin{aligned}
v(t_0 + \Delta t/2) &= v(t_0) + a(t_0)\frac{\Delta t}{2} \\
x(t_0 + \Delta t) &= x(t_0) + v(t_0 + \Delta t/2)\Delta t \qquad (15) \\
v(t_0 + \Delta t) &= v(t_0 + \Delta t/2) + a(t_0 + \Delta t)\frac{\Delta t}{2}
\end{aligned}$$

This version provides more time resolution to our calculation; however, it increases the number of calculations needed by about 50%.

1.6 Runge Kutta

Runge Kutta is another

References

[1] C. K. Birdsall and A. B. Langdon, *Plasma Physics via Computer Simulations*, McGraw-Hill Book Company, 1985, p. 56.