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An exact algorithm for the multicriteria ordered clustering problem

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ABSTRACT

In the context of multicriteria decision aid, we address the problem of regrouping alternatives into completely ordered categories based on valued preference degrees. We assume that the number of groups is fixed *a priori*. This will be referred to as the multicriteria ordered clustering problem. The model is based on the definition of an inconsistency matrix and only uses the ordinal properties of the pairwise preference relations. An exact algorithm is proposed to find the ordered partition and is applied as illustration to the Human Development Index.

1. Introduction

The problem of grouping 'similar' objects into homogeneous groups has been extensively studied in the literature [1] and is commonly encountered in different fields [2,3] such as finance, health care, agriculture, marketing, and image processing.

Generally, two distinct problems can be considered. In the first case, the groups are unknown *a priori*. Thus, the aim is to elicit them and the information contained in them (this is called *data abstraction*). In this case, the problem is referred to as *clustering problem* [3,4] and the groups are called 'clusters'. In the second case, once the groups are defined *a priori*, the problem of assigning an object to one of them is referred to as *classification problem* and these groups are called 'classes'.

In the context of multicriteria decision aid (MCDA), many authors have been interested in assigning objects to predefined groups. These groups can, for instance, be either completely ordered or such that no relation exists between them. In the former case, we speak about *sorting problems* as in the latter about nominal classification problems. A number of approaches have been proposed to tackle this problem. Among others, let us mention Electre TRI [5–8], FlowSort [9], UTADIS [10], SMAA-TRI [11], PROAFTN [12] or interactive sorting methods [13,14]. A distinctive feature related to these techniques is to characterize the groups by 'limiting' or 'central' objects (sometimes called 'profiles' or 'prototypes').

From our point of view, not enough attention has been paid to the problem of defining these categories. Therefore, the aim of this paper is to propose a method that helps the decision maker to obtain what we call 'ordered clusters' (i.e., ordered groups of objects). This will be referred to as the *multicriteria ordered clustering* problem.

The identification of ordered clusters may help the decision maker to identify profiles or prototypes that will later be used in sorting problems. But *multicriteria ordered clustering* techniques might also offer a new perspective in a world where more and more rankings are developed. One might think, for instance, about the human development index, the academic ranking of world universities, the environmental performance index, the world happiness rankings and so on. A lot of criticisms have been raised against these rankings. One could ask if the identification of ordered clusters would not be more accurate than trying to obtain a complete ranking of all the alternatives. For example, in the famous Shangai ranking, only the first hundred universities are ranked. The others are then presented by groups of 50 or 100.

To our knowledge, the first who have addressed the problem of identifying ordered clusters in a multicriteria context are De Smet and Gilbart [15,16] for the country risk evaluation problem [15,17–20]. At first, they have proposed an extension of the PROMETHEE method [21,22] in order to rank groups of alternatives. This one was mainly based on the definition of a preference index between the groups (which was computed as being the average of preference values between the alternatives). Then, they have presented an optimization model to find the ordered partition that was the most compatible with the obtained

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ranking. This model was solved by means of a heuristic approach and illustrated on the problem of country risk assessment. Despite the originality of this first contribution, a few limitations have been pointed out. Among them we can cite the lack of model's justifications or the fact that the heuristic was not performing well for large data sets.

In this contribution, our aim was to pursue and to improve this initial work. Therefore, our goal was to develop an optimization model that only uses the ordinal properties of a preference matrix and to provide an exact algorithm to solve it.

An obvious attempt to obtain ordered clusters could be to perform at first, a ranking of the objects, and then, to group them with respect to their rank. Several MCDA methods ([7,21,23–25], etc.) lead to rankings while considering multiple criteria. Ordered groups can then be obtained by *ordinal sorting* for instance (e.g. in class 1: the first 10 objects, in class 2: the next 10 objects, etc.). Statistical approaches such as multiple or logistic regression, discriminant analysis [26], regression analysis, regression trees [27] and cluster analysis [28] have also been widely used to obtain rankings. Nevertheless, the statistical methods present some drawbacks, like for instance the definition of dependent variables, and the exclusion of important factors [29].

In this paper, we present a new approach for solving the *multicriteria ordered clustering* problem where actions are compared by means of a multicriteria outranking degree. As a consequence, it can be seen as a new way of exploiting an outranking relation for clustering problems. The main idea of the proposed method is to characterize any given ordered partition by an inconsistency matrix. The latter allows to quantify the incompatibility between a valued preference matrix (given as input) and the preferential information underlying the ordered partition. The preference matrix can be obtained by different outranking methods such as Promethee, ELECTRE III, etc. ([7,22], …). This will lead us to formalize the aforementioned problem as an optimization one in Section 2 (without defining a preference structure between the clusters). An exact algorithm is then proposed to solve the optimization problem (see Section 3). In order to demonstrate our approach, we will apply the proposed algorithm to the measure of the Human Development Index (see Section 4).

2. Formalization

Let $A=\{a_1,a_2,\ldots,a_n\}$ be the set of alternatives (or objects) of interest and let $F=\{f_1,\ldots,f_q\}$ be the set of q criteria. We assume that the model takes as input both the number of clusters, denoted as K, and a valued preference matrix denoted as π (π_{ij} has to be interpreted as the strict preference value of alternative a_i over alternative a_j). We assume that the absence of any preference between a_i over a_j (i.e. a_i and a_j are thus indifferent) corresponds to $\pi_{ij}=0$ and that $\forall a_i,a_j \in A : \pi_{ij} \geq 0$. In what follows, we will only use the ordinal information contained in π.

An *ordered partition* of A into K clusters, noted $P_K(A)=[C_1,\ldots,C_K]$, is defined as follows:

- $A=\bigcup_{i=1,\ldots,K} C_i$,
- $\forall i \neq j : C_i \cap C_j = \emptyset$,
- $C_1 \succ C_2 \succ \cdots \succ C_K$,

where the symbol $\succ$ in $C_i \succ C_j$ denotes that cluster C_i has a lower rank than cluster C_j.[1] Obviously $S(n,K).K!$ ordered partitions of K clusters can be considered for a set of n elements (where $S(n,K)$ denotes the Stirling number of the second kind).

When considering a given K-ordered partition of A, information is provided regarding the assignment of the alternatives in the ordered clusters. Intuitively:

- Two alternatives belonging to the same cluster should be considered as being **indifferent or similar**. Therefore, their mutual preference values should be as low as possible.
- If an alternative a_i is assigned to a cluster of a lower rank than another alternative a_j, a_i is considered to be better than a_j. Therefore π_{ji} should be as low as possible.

Among all the possible K-ordered partitions of A, we are looking for the one that will be the most compatible with the preference information contained in π. In other words, we will identify the K-ordered partition that minimizes the inconsistencies with respect to the two previous conditions. Two kinds of inconsistencies might exist:

- $a_i \in C_l$ and $a_j \in C_m$ with $l<m$ while $\pi_{ji}>0$. Indeed, in an ideal case, we should have $\pi_{ij}>0$ and $\pi_{ji}=0$.
- $a_i,a_j \in C_l$ while $\pi_{ij}>0$ or $\pi_{ji}>0$. Indeed, in an ideal case, we should have $\pi_{ij}=\pi_{ji}=0$.

Of course the severity of these inconsistencies will depend on the values of π_{ij}. As we will see, the inconsistencies tackled by the algorithm depend on the preference values.

In order to characterize the quality of a K-ordered partition, let us define the inconsistency matrix between π and $P_K(A)$ as follows:

$$I(\pi,P_K(A))_{ij}=\begin{cases}0 & \text{if } a_i \in C_l,\ a_j \in C_m,\ l<m \\ \pi_{ij} & \text{otherwise.}\end{cases}$$

The $I(\pi,P_K(A))$ matrix is equal to π, except where the preference values between alternatives are compatible with the grouping and ordering of $P_K(A)$. For those elements, the value is equal to zero. The $I(\pi,P_K(A))$ matrix allows us to identify all the preference values that are not compatible with the grouping of the elements and the order between the categories. In an ideal case, the inconsistency matrix should be equal to the null matrix.

Our goal is to find the K-ordered partition characterized by the best inconsistency matrix. Therefore, we are going to compare all the K ordered partitions on the basis of their inconsistency matrix: $P_K(A)$ will be better than $\tilde{P}_K(A)$ if and only $I(\pi,P_K(A)) \succ_{\mathcal{L}} I(\pi,\tilde{P}_K(A))$ where $\succ_{\mathcal{L}}$ denotes the lexicographical order defined on the elements of the inconsistency matrices.

Intuitively, working with the lexicographic order ensures that we first try to minimize the highest preference value that is not compatible with the information provided by the K-ordered partition, then, in case of equality, we try to minimize the second highest preference value, and so on. Additionally, it is worth reminding that, by doing so, we only use the ordinal information contained in π.

In order to illustrate the previous developments let us consider the following idealized example (see Fig. 1). Four alternatives are evaluated according to two criteria that have to be minimized. The aim is to find a partition of three ordered clusters. In such contexts, we will necessarily have a cluster constituted by two alternatives and two clusters reduced to a single alternative. At first, let us remark that the application of a traditional clustering procedure such as a k-means algorithm based on the Euclidean distance would lead to $\{(a),(b),(c,d)\}$, $\{(a),(c),(b,d)\}$,$\{(a,b),(c),(d)\}$… but never to $\{(a),(c,b),(d)\}$ which is the most meaningful ordered partition from a multicriteria point of view (a being the

[1] In this context, we assume that the smaller the rank, the better the cluster. As a consequence, C_1 is considered to be the best cluster.

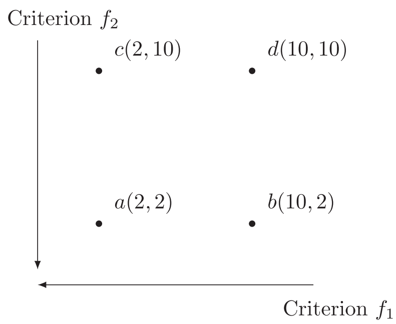

Fig. 1. Example of a bi-criteria problem.

Table 1
Preference matrix for the alternatives *a*,*b*,*c*,*d*.

	a	*b*	*c*	*d*
a	0	1	1	1
b	0	0	0.5	1
c	0	0.5	0	1
d	0	0	0	0

Table 2
Inconsistency matrix for the partition $[\{a\}\{b,c\}\{d\}]$.

	a	*b*	*c*	*d*
a	0	0	0	0
b	0	0	0.5	0
c	0	0.5	0	0
d	0	0	0	0

best alternative as smaller values are better, d being dominated by all the other alternatives and (c,b) being intermediate solutions).

For simplicity reasons we have built the preference matrix as follows:

$$\pi_{ij} = \sum_{k|f_k(a_i) \leq f_k(a_j)} w_k,$$

where f_k corresponds to criterion k, w_k to the weight of f_k and such that $w_1 = w_2 = \frac{1}{2}$. The resulting preference matrix is listed in Table 1.

A direct consequence of these values is that $\{a\}\{b,c\}\{d\}$ is the best three-ordered partition. Indeed, if we want to avoid an inconsistency value equal to 1 we have the following constraint: a should be assigned in a better category than $\{b,c,d\}$, b and c should be assigned to a better category than d. Since, we have requested a partition of three clusters the only remaining solution is $[\{a\}\{b,c\}\{d\}]$. This partition leads to a maximal inconsistency value equal to 0.5 which is the lowest achievable value. In this case, the inconsistency matrix is given in Table 2.

To conclude, let us remark that the computation of preference degrees will lead to an asymmetric matrix π (since in most cases $\pi_{ij} \neq \pi_{ji}$). However, most of traditional clustering techniques are based on a distance measure that is, by definition, symmetric (and thus on a distance matrix, denoted as D, such that $d_{ij} = d_{ji}$). This constitutes a major difference since the proposed algorithm handles non-symmetric measures. Being able to differentiate the intensity of preferences π_{ij} and π_{ji} allows us to consider a complete order on the clusters (indeed if we have $\pi_{ij} \gg \pi_{ji}$ it seems natural to put a_i in a 'better' cluster than a_j). We refer the reader to [30] for a detailed discussion about these issues. Another distinctive feature of our model is the fact that we are not looking for an optimal partition based on a single scalar fitness value. The comparison of K-ordered partitions is based on a lexicographic order.

3. Proposed algorithm

In this section, we propose an exact algorithm to identify the best K-ordered partition based on a lexicographic comparison of the inconsistency matrices. We assume that all the values π_{ij} are different (except if they are null). If it is not the case, we add, without loss of generality, a constant ϵ that is lower than the smallest difference between the π_{ij} of π.

We start by giving an intuitive description of the algorithm. Let M be a $n \times n$ binary matrix. At the beginning, M is a null matrix. $M_{ij}=1$ means that alternative a_i is put in a better cluster than alternative a_j. All the elements of the π matrix are successively considered in descending order (only n^2-n elements have to be tested since the elements of the diagonal are equal to zeros). Every time that a new element π_{ij} is considered, we test if putting $M_{ij}=1$ (i.e. putting a_i in a better cluster than a_j) creates a cycle or a path longer than $K-1$ in the graph induced by the new M matrix. If neither of these conditions are satisfied, the value π_{ij} will be compatible with a K-ordered partition (as a consequence, at the end of the procedure, the value $I(\pi,P_K(A))_{ij}$ will be set to zero). If at least one of the two previous conditions is satisfied, we cannot put a_i in a better cluster than a_j without creating a cycle or path longer than $K-1$. Therefore the preference value will remain in the inconsistency matrix. Furthermore, we force $M_{ij}=0$ so that we are sure that the M matrix is acyclic and only contains paths smaller than K at each step of the algorithm. Finally, the K-ordered partition is obtained by computing the ranks of the graph induced by M.

Algorithm 1. Determine the K-ordered partition $P_K^*(A)$.

Inputs: A,π,K
Output: $M,I(\pi,P_K(A))$
$M \leftarrow 0_{n\times n}$
$I \leftarrow 0_{n\times n}$
while $\max_{k,l}\{\pi_{kl}\} > 0$ **do**
  $M^* = M$
  Determine $(i,j)|\pi_{ij} = \max_{k,l}\{\pi_{kl}\}$
  $M^*_{ij} = 1$
  if has.no.cyle $(M^*) \wedge$ max.rank $(M^*) \leq K-1$ **then**
    $M_{ij} = 1$
  else
    $I_{ij} = \pi_{ij}$
  end if
  $\pi_{ij} = 0$
end while
The K-ordered partition is given by the determination of the ranks of the graph induced by M.

The algorithm performs at most n^2-n main iterations. Within each iteration the ranks of the graph are computed in order to detect a cycle or a path longer than $K-1$. The complexity of this test is $\mathcal{O}(K \times n)$. Therefore, the complexity of the first algorithm is $\mathcal{O}(n^3 \times K)$. It is worth noting that in the case of multiple optima the algorithm will return a solution that depends on the order the alternatives are analyzed. The assumption that all the positive π_{ij} values are different allows to avoid this situation. Nevertheless, in the case of multiple optima for the initial problem, we suggest to accept ties and to present all the solutions to the decision maker (this can be done by analyzing the sensitivity of the results with respect to ϵ). Finally, let us mention that the lexicographic strategy used to limit the search in the solutions space is similar to the work of Dias and Lamboray [31] for ranking problems.

In order to further explain the application of the algorithm, we will come back to the illustrative example shown in Section 2. At first, we have to make sure that all the preference values are different in order to respect our assumptions. Without loss of generality, we can slightly modify the preference values (this step will just induce a particular exploration sequence of the π matrix, see Table 3).

Table 4 illustrates the different steps of the algorithm and Fig. 2 gives a representation of the corresponding matrix M.

At iteration 7 the algorithm stops since there are only null values remaining in the π matrix. We have

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Obviously, the determination of rank in the graph induced by M graph leads to $[\{a\}\{b,c\}\{d\}]$.

It is easy to understand that the application of the algorithm leads to the best K-ordered partition (in the sense of the lexicographic order applied on the elements of the inconsistency matrices). Let $P^*_K(A)$ denote the K-ordered partition obtained at the end of the algorithm. Let us assume that another K-ordered partition, denoted $\tilde{P}_K(A)$ is such that $I(\pi,\tilde{P}_K(A)) \succ_{\mathcal{L}} I(\pi,P^*_K(A))$. Without loss of generality, it means that $I(\pi,\tilde{P}_K(A))$ and $I(\pi,P^*_K(A))$ have their m highest elements being equal ($m \geq 0$). Let π_{kl} (respectively π_{hg}) denote the $m+1$ highest element of $I(\pi,P^*_K(A))$ (respectively $I(\pi,\tilde{P}^*_K(A))$). Since $\pi_{kl} > \pi_{hg}$, a_k has been placed in a better cluster than a_l in $\tilde{P}_K(A)$. According to the building of $P^*_K(A)$ by the algorithm, $\tilde{P}_K(A)$ either contains a path of length greater or equal to K, or, it contains a cycle. As a consequence $\tilde{P}_K(A)$ is not a K-ordered partition.

Table 3
Preference matrix for the alternatives a,b,c,d.

	a	b	c	d
a	0	1.05	1.04	1.03
b	0	0	0.51	1.02
c	0	0.5	0	1.01
d	0	0	0	0

Table 4
Execution of the algorithm on an illustrative example.

Iteration	(i,j)	M	I
1	(1,2)	$M(1,2)=1$	
2	(1,3)	$M(1,3)=1$	
3	(1,4)	$M(1,4)=1$	
4	(2,4)	$M(2,4)=1$	
5	(3,4)	$M(3,4)=1$	
6	(2,3)		$I(2,3)=\pi_{23}$
7	(3,2)		$I(3,2)=\pi_{32}$

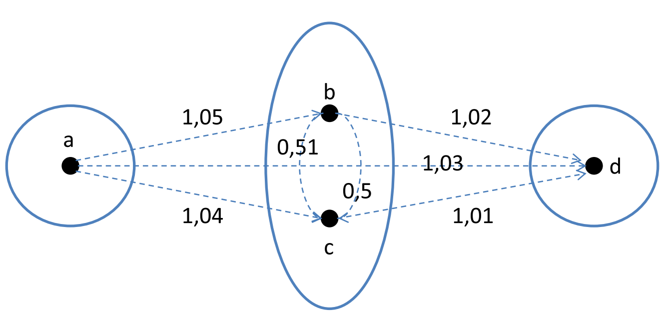

Fig. 2. Representation of matrix M. Null preference values are not represented.

4. Case study: the human development index (2008)

In order to illustrate our approach, let us consider the Human Development Index Problem (HDI problem). The United Nations Development Program (UNDP) has proposed the so-called HDI ranking where 179 United Nations countries are evaluated on the basis of three criteria: the life expectancy, the education and the income index. These three criteria, considered as equally weighted, are then aggregated to a final score (the Human Development Index, HDI) by adding their values. This leads to the HDI complete ranking. The interested reader may find the data as well as the adopted methodology at the following website: http://hdr.undp.org/en/statistics/.

Several authors (e.g. [32]) have criticized the methodology adopted by the UNDP, but in this paper we will not concentrate on this aspect. In this section, our aim is to regroup the countries into homogeneous and ordered clusters.

For that purpose, let us denote by a_i the country ranked at the ith place in the HDI ranking. So for instance, a_1 corresponds to Iceland, a_2 to Norway, …, a_{17} to Belgium, etc. The complete list of the countries, as well as their performances on the three criteria are given in the Appendix for the year 2008. Moreover, Fig. 3 gives the score of each country (the x-axis represents the label of a country i.e., its rank, whereas the y-axis, its score). Thus, we may see that it is not easy *a priori* to regroup the countries on the basis of their score given the shape of the curve. No clear threshold exists to determine the groups.

At first, we compute preference degrees between every pair of countries. We used the PROMETHEE method [21,22] although another outranking method may be used. For each criterion we considered the linear preference function and defined the thresholds as follows: the indifference threshold q is fixed at 0 and the preference threshold p is equal to the highest difference between the countries' performances on the particular criterion. All criteria have an equal weight noted as w_i (see Table 5). Of course, these choices are motivated for simplicity reasons. The focus of this section being the illustration of the proposed algorithm and not the detailed (and probably questionable) modeling of the problem.

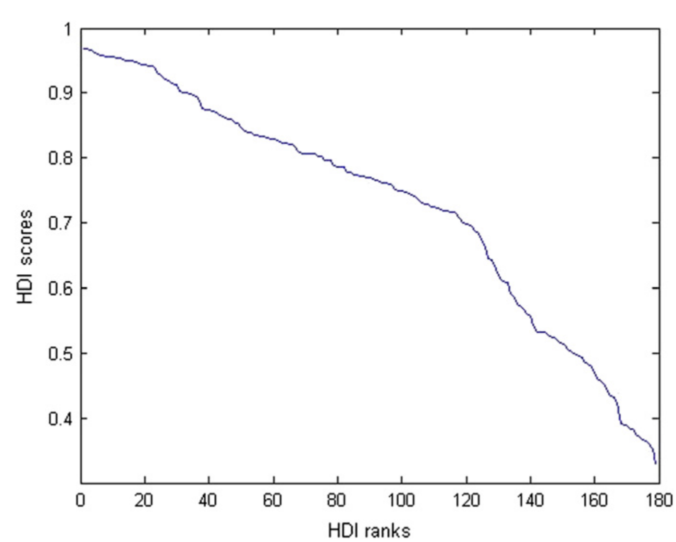

Fig. 3. Representation of the HDI-index (score ranging between 0.971 and 0.34) for the different countries with respect to their rank.

Table 5
Parameters for the computation of the preference degrees.

Parameters	Life expectancy (g_1)	Adult literacy index (g_2)	GDP (g_3)
p_i	0.704	0.719	0.828
q_i	0	0	0
w_i	0.333	0.333	0.333

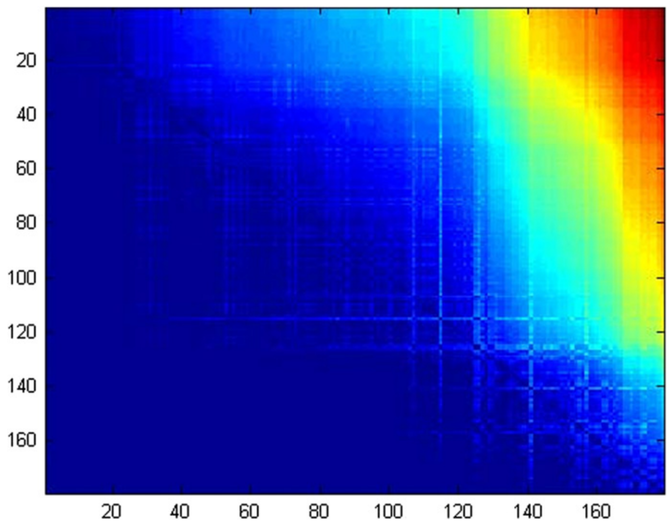

Fig. 4. Representation of the pair-wise preference degrees between the 179 countries; the color 'red' indicates a high preference degree whereas the blue color indicates a small preference degree. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Fig. 4 represents the preference matrix when applying the PROMETHEE method with the parameters given in Table 2. The lines and columns of this matrix have been sorted according to their label (i.e., the rank induced by the HDI index).

At the intersection of the line i and column j we find the value of the preference degree π_{ij} (given by the color) between alternative i and j. A blue zone represents low preference degrees while a red zone represents a high preference degrees. Thus, we may at first conclude that the preference degrees are coherent with the HDI ranking. No alternative with a high rank is strongly preferred to an alternative of a lower rank and alternatives of similar rank have low preference degrees.

On the basis of these pair-wise comparisons, the countries will be regrouped. A distinguishing feature of our approach is that the decision maker may choose the number of clusters. So, let us first fix the numbers of clusters to 4. This may correspond to the following clusters: very high human developed countries, high human developed countries, medium human developed countries and low human developed countries.

The obtained partition is represented in Fig. 5. The x-axis corresponds to the labels of the alternative (from 1 to 179) and the y-axis to the number of cluster (from 1 to 4) of each country. For instance, alternative 40 belongs to cluster C_1 whereas alternative 140 to cluster C_3. Since the label of the alternatives corresponds to their rank in the HDI-ranking, we can easily observe that there is a very high consistency with the ordered grouping and the HDI-ranking. Very few inconsistencies are present such as for alternatives 85, 132 and 159. We may furthermore notice that the groups are unevenly distributed. Groups C_1 and C_2 contain most of the alternatives. This is coherent with the HDI ranking since they have very similar HDI-indexes. For instance, all the scores of the countries in the two first clusters are between 0.968 and 0.705 (with an overall minimum score of

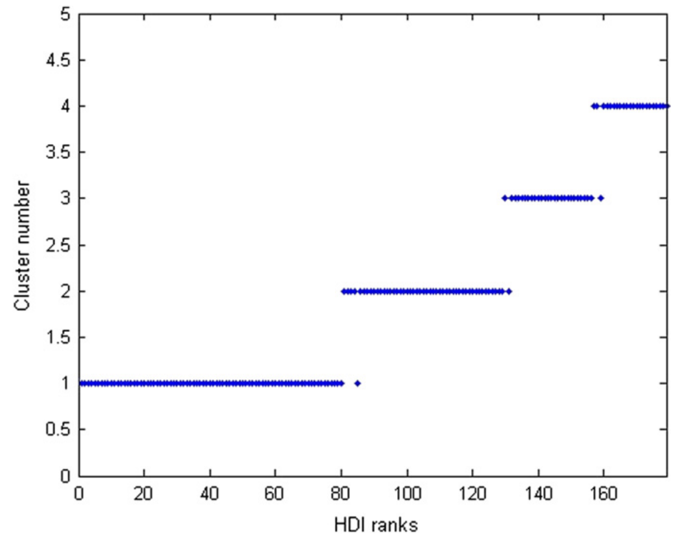

Fig. 5. Resulting ordered partition for four clusters. The x-axis representing the countries by their label and the y-axis their group to which they belong.

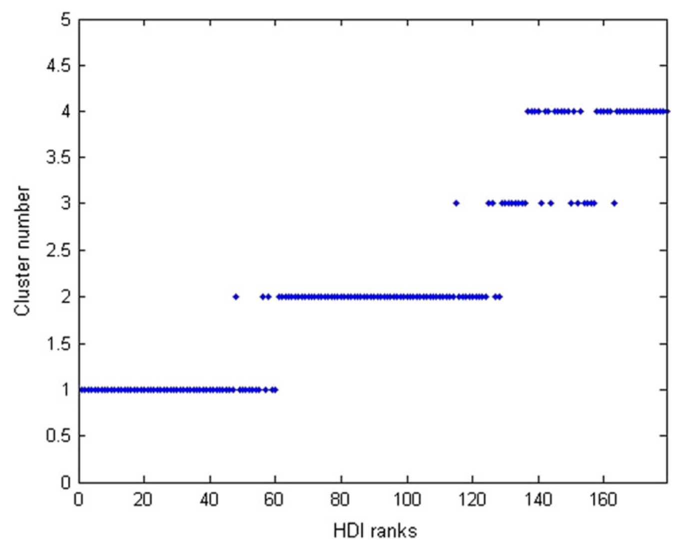

Fig. 6. Partition obtained when using the K-means method (four clusters).

Table 6
Confusion matrix between the partition obtained by the K-means and by the proposed algorithm.

K-means / our approach	1	2	3	4
1	57	0	0	0
2	24	44	0	0
3	0	5	13	2
4	0	0	14	20

0.329). This was predictable considering the small preference degrees, represented in Fig. 4, between alternatives 1 and 120. Let us remark that the partition does indeed not depend on the label given to the alternatives.

When applying the classical K-means clustering method on the data set constituted by the evaluations of the different countries (for which the resulting partition is given in Fig. 6) we may notice that there exist many more rank inconsistencies. Table 6 presents the confusion matrix denoted by CM. The value CM(i,j) indicates the number of objects that simultaneously belong to cluster

i (obtained when the K-means algorithm is applied) and to cluster j (when the proposed algorithm is run). The comparison between the two partitions leads to a total number of 134 alternatives that belong to the same groups i.e., that are simultaneously assigned to the same cluster i when using the two different approaches. Assignment differences encompass 45 countries. The confusion matrix evaluates how different the two partitions are while using two different algorithms. We can remark that the difference between assignments is maximum 1: actions are assigned to the same cluster or to a neighboring cluster. This means that is not any high inconsistency between the two approaches and thus, that the ordered partition preserves the idea of regrouping similar actions together whilst introducing an order on the clusters.

Moreover, the clusters obtained with the K-means are distributed more equably. This may be explained by the fact that the K-means uses a symmetrical distance to regroup the countries together. Finally, let us remind the reader that the application of a K-means algorithm leads to a partition without any relation between the clusters. On the contrary, our method provides not only clusters but also a total order on these clusters.

However, let us consider partitions with different number of clusters (see Figs. 7–9). Let us at first remark that there are no rank inconsistencies when $K=2$ but the number of rank inconsistencies seems to increase with the number of clusters. This may be explained by the fact that the HDI ranking is based on a final score whereas the clustering method is based on preference degrees. In the former score, the ‘preference’ are indeed transitive. In the latter cases, there might be some ties since the preference degrees are not necessarily transitive. Moreover, preference degrees may pinpoint the presence of incomparable alternatives when $\pi_{ij}=\pi_{ji}=0.5$. As one may notice, increasing the number of clusters by one does not necessarily imply that one cluster will further be subdivided in the new partition. This is indicated by the resulting partitions for $K=3$ and $K=4$.

Since the number of clusters has to be fixed, the decision maker may need some help in order to determine this number. For this purpose we can determine the best lexicographic vector for different values of K (see Fig. 10—this vector is obtained by listing the elements of the best different inconsistency matrices by descending order). From this graph, we can see the gain in

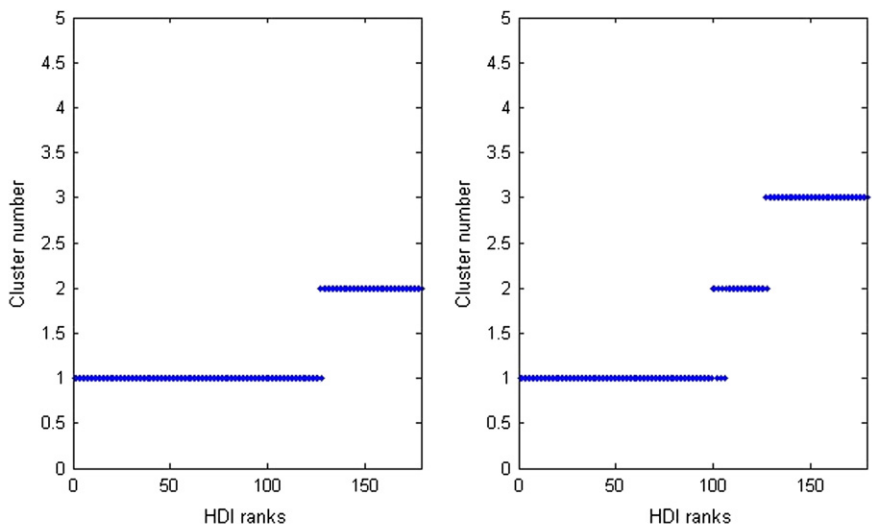

Fig. 7. Resulting ordered partition for two and three clusters.

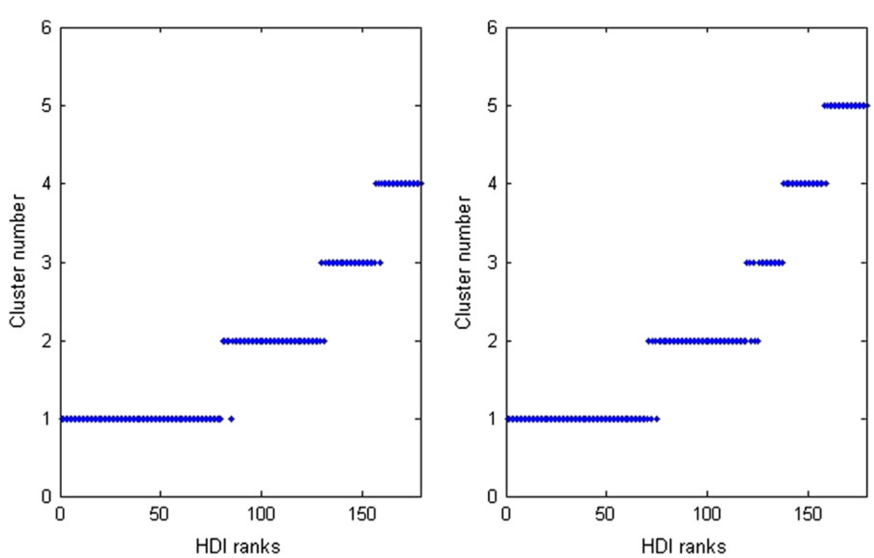

Fig. 8. Resulting ordered partition for four and five clusters.

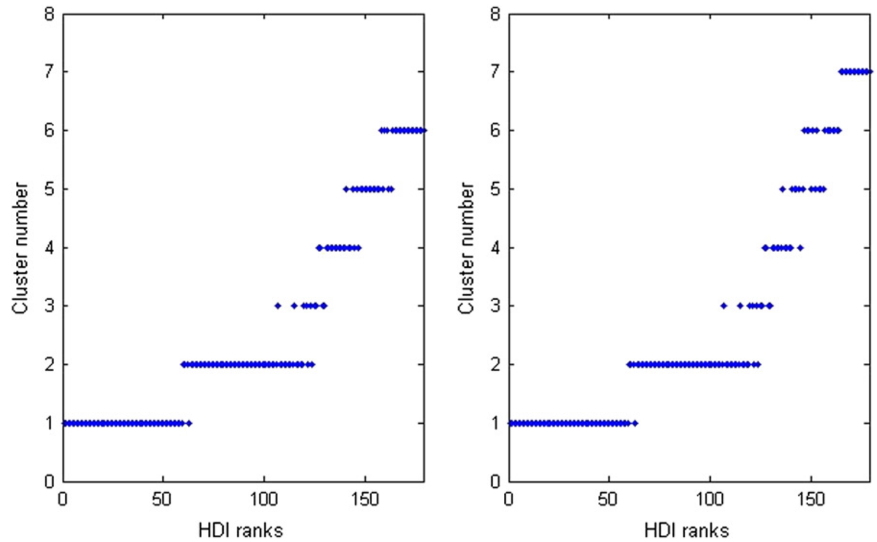

Fig. 9. Resulting ordered partition for six and seven clusters.

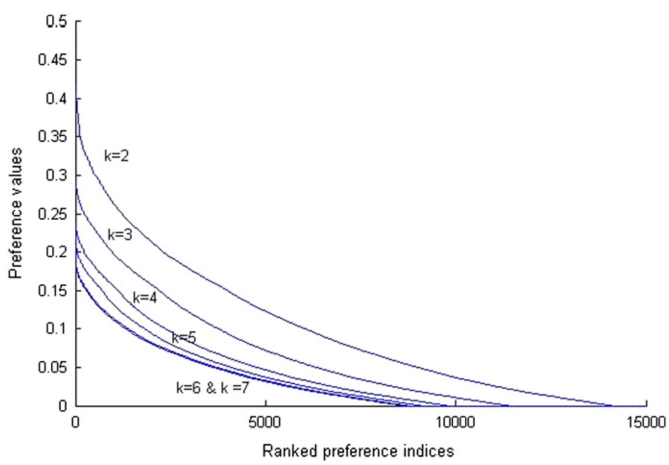

Fig. 10. Best lexicographic vectors related to ordered partition (2–7 clusters).

homogeneity and order coherency that is achieved by increasing the number of clusters from $K=2$ to $K=5$. Further increasing the number of clusters ($K=6$ or $K=7$) does not lead to a substantial gain. Thus, this kind of graph may clearly help the decision maker if he has no constraint or idea about the number of clusters of his partition.

5. Conclusion and future research

In this paper we have proposed a new formalization for the *multicriteria ordered clustering problem*, i.e., the elicitation of ordered groups of alternatives in a multicriteria context. In comparison with traditional clustering techniques, the problem addressed here is different. At first, the input is a preference matrix that is most of time asymmetric (contrary to distance or similarity measure). Then, the output is not only limited to a grouping of the elements into K clusters, it also provides an ordering of these groups. An exact method has been proposed in order to obtain the best partition defined on the basis of an inconsistency matrix. Moreover, the algorithm only uses the ordinal information of the preference matrix.

Our approach has been illustrated on real data concerning the Human Development Index. In this example we have shown the consistency between the obtained ordered partition and the HDI

Table A1
Evaluations of the first 53 countries.
Source: 2008, http://hdr.undp.org/en/statistics/.

Rank	Country name	Life expectancy	Adult literacy index	GDP
1	Iceland	0.944	0.98	0.982
2	Norway	0.916	0.989	1
3	Canada	0.924	0.991	0.986
4	Australia	0.934	0.993	0.968
5	Ireland	0.894	0.985	1
6	Netherlands	0.907	0.985	0.983
7	Sweden	0.928	0.974	0.973
8	Japan	0.957	0.949	0.962
9	Luxembourg	0.893	0.975	1
10	Switzerland	0.941	0.936	0.989
11	France	0.923	0.978	0.963
12	Finland	0.901	0.993	0.967
13	Denmark	0.884	0.993	0.978
14	Austria	0.91	0.962	0.98
15	United States	0.884	0.968	1
16	Spain	0.928	0.971	0.948
17	Belgium	0.901	0.974	0.969
18	Greece	0.901	0.98	0.959
19	Italy	0.923	0.965	0.945
20	New Zealand	0.916	0.993	0.923
21	United Kingdom	0.903	0.957	0.966
22	Hong Kong, China (SAR)	0.951	0.879	0.996
23	Germany	0.904	0.954	0.962
24	Israel	0.925	0.947	0.918
25	Korea (Republic of)	0.887	0.988	0.908
26	Slovenia	0.878	0.969	0.922
27	Brunei Darussalam	0.865	0.892	1
28	Singapore	0.911	0.843	1
29	Kuwait	0.873	0.864	1
30	Cyprus	0.901	0.909	0.927
31	United Arab Emirates	0.891	0.818	1
32	Bahrain	0.84	0.89	0.975
33	Portugal	0.882	0.927	0.891
34	Qatar	0.838	0.857	1
35	Czech Republic	0.853	0.938	0.9
36	Malta	0.904	0.88	0.898
37	Barbados	0.865	0.94	0.862
38	Hungary	0.802	0.96	0.868
39	Poland	0.839	0.952	0.833
40	Chile	0.891	0.918	0.812
41	Slovakia	0.824	0.928	0.865
42	Estonia	0.771	0.964	0.877
43	Lithuania	0.795	0.968	0.844
44	Latvia	0.788	0.961	0.841
45	Croatia	0.842	0.915	0.828
46	Argentina	0.834	0.946	0.799
47	Uruguay	0.851	0.955	0.772

Table A1 (*continued*)

Rank	Country name	Life expectancy	Adult literacy index	GDP
48	Cuba	0.882	0.976	0.706
49	Bahamas	0.797	0.878	0.886
50	Costa Rica	0.893	0.882	0.767
51	Mexico	0.847	0.879	0.801
52	Libyan Arab Jamahiriya	0.81	0.894	0.817
53	Oman	0.838	0.787	0.892

Table A2
Evaluations of the countries 54–108.
Source: 2008, http://hdr.undp.org/en/statistics/.

Rank	Country name	Life expectancy	Adult literacy index	GDP
54	Seychelles	0.783	0.886	0.837
55	Saudi Arabia	0.791	0.815	0.901
56	Bulgaria	0.798	0.93	0.773
57	Trinidad and Tobago	0.74	0.861	0.898
58	Panama	0.838	0.887	0.771
59	Antigua and Barbuda	0.795	0.832	0.863
60	Saint Kitts and Nevis	0.77	0.896	0.824
61	Venezuela (Bolivarian Republic of)	0.807	0.886	0.786
62	Romania	0.786	0.914	0.776
63	Malaysia	0.815	0.848	0.806
64	Montenegro	0.82	0.891	0.756
65	Serbia	0.813	0.891	0.76
66	Saint Lucia	0.806	0.896	0.761
67	Belarus	0.73	0.958	0.764
68	Macedonia (TFYR)	0.816	0.879	0.73
69	Albania	0.856	0.886	0.68
70	Brazil	0.783	0.888	0.75
71	Kazakhstan	0.689	0.966	0.766
72	Ecuador	0.83	0.877	0.713
73	Russian Federation	0.669	0.933	0.815
74	Mauritius	0.793	0.836	0.778
75	Bosnia and Herzegovina	0.827	0.874	0.704
76	Turkey	0.776	0.824	0.792
77	Dominica	0.818	0.848	0.725
78	Lebanon	0.778	0.845	0.765
79	Peru	0.766	0.885	0.711
80	Colombia	0.792	0.875	0.694
81	Thailand	0.75	0.886	0.723
82	Ukraine	0.712	0.956	0.689
83	Armenia	0.78	0.903	0.649
84	Iran (Islamic Republic of)	0.759	0.804	0.769
85	Tonga	0.8	0.92	0.602
86	Grenada	0.724	0.884	0.714
87	Jamaica	0.789	0.83	0.694
88	Belize	0.851	0.762	0.701
89	Suriname	0.747	0.848	0.715
90	Jordan	0.786	0.88	0.641
91	Dominican Republic	0.78	0.837	0.686
92	Saint Vincent and the Grenadines	0.772	0.817	0.71
93	Georgia	0.763	0.909	0.616
94	China	0.795	0.849	0.642
95	Tunisia	0.811	0.766	0.708
96	Samoa	0.768	0.905	0.608
97	Azerbaijan	0.704	0.881	0.688
98	Paraguay	0.775	0.864	0.617
99	Maldives	0.71	0.884	0.653
100	Algeria	0.783	0.743	0.719
101	El Salvador	0.776	0.798	0.668
102	Philippines	0.772	0.887	0.576
103	Fiji	0.725	0.868	0.637
104	Sri Lanka	0.781	0.834	0.611
105	Syrian Arab Republic	0.814	0.769	0.625
106	Occupied Palestinian Territories	0.802	0.884	0.506
107	Gabon	0.522	0.838	0.827
108	Turkmenistan	0.63	0.907	0.647

ranking. These results are encouraging and allow us to hope to tackle and resolve new problems of this type in other fields such as environmental management, finance, health care, etc.

Further research is needed to determine the changing sensibility of partitions with respect to varying the number of alternatives. Another possible line of research is in extending this method to be partially ordered clusters.

Finally, let us mention that recent works also investigate other potential synergies between multicriteria decision aid and the field of classification. For example, Peng et al. [33] apply multicriteria methods to assess and to select classification algorithms based on different criteria (such as accuracy, computation time, …). To our point, this shows that the complementary nature of

Table A3
Evaluations of the countries 108–162.
Source: 2008, http://hdr.undp.org/en/statistics/.

Rank	Country name	Life expectancy	Adult literacy index	GDP
109	Indonesia	0.752	0.834	0.591
110	Guyana	0.68	0.939	0.555
111	Bolivia	0.668	0.885	0.615
112	Mongolia	0.688	0.913	0.561
113	Moldova	0.727	0.9	0.53
114	Viet Nam	0.816	0.81	0.528
115	Equatorial Guinea	0.43	0.787	0.935
116	Egypt	0.766	0.731	0.651
117	Honduras	0.745	0.8	0.596
118	Cape Verde	0.771	0.787	0.558
119	Uzbekistan	0.698	0.89	0.515
120	Nicaragua	0.789	0.774	0.533
121	Guatemala	0.75	0.709	0.628
122	Kyrgyzstan	0.678	0.919	0.484
123	Vanuatu	0.743	0.723	0.592
124	Tajikistan	0.691	0.896	0.464
125	South Africa	0.418	0.84	0.753
126	Botswana	0.399	0.783	0.809
127	Morocco	0.762	0.563	0.612
128	Sao Tome and Principe	0.669	0.805	0.456
129	Namibia	0.448	0.808	0.647
130	Congo	0.492	0.769	0.596
131	Bhutan	0.669	0.553	0.616
132	India	0.652	0.638	0.537
133	Lao People's Democratic Republic	0.645	0.682	0.498
134	Solomon Islands	0.637	0.676	0.461
135	Myanmar	0.604	0.787	0.363
136	Cambodia	0.561	0.7	0.465
137	Comoros	0.659	0.649	0.408
138	Yemen	0.616	0.563	0.521
139	Pakistan	0.665	0.492	0.528
140	Mauritania	0.643	0.537	0.491
141	Swaziland	0.253	0.731	0.643
142	Ghana	0.574	0.605	0.421
143	Madagascar	0.564	0.671	0.363
144	Kenya	0.462	0.69	0.445
145	Nepal	0.634	0.571	0.384
146	Sudan	0.547	0.539	0.49
147	Bangladesh	0.641	0.524	0.408
148	Haiti	0.584	0.578	0.402
149	Papua New Guinea	0.534	0.518	0.496
150	Cameroon	0.416	0.622	0.504
151	Djibouti	0.487	0.554	0.497
152	Tanzania (United Republic of)	0.443	0.661	0.404
153	Senegal	0.627	0.417	0.462
154	Nigeria	0.36	0.648	0.487
155	Lesotho	0.289	0.753	0.445
156	Uganda	0.424	0.692	0.365
157	Angola	0.285	0.535	0.633
158	Timor-Leste	0.586	0.545	0.317
159	Togo	0.55	0.543	0.345
160	Gambia	0.567	0.439	0.408
161	Benin	0.514	0.44	0.423
162	Malawi	0.366	0.679	0.325

Table A4
Evaluations of the countries 162–179.
Source: 2008, http://hdr.undp.org/en/statistics/.

Rank	Country name	Life expectancy	Adult literacy index	GDP
163	Zambia	0.27	0.664	0.425
164	Eritrea	0.536	0.514	0.275
165	Rwanda	0.346	0.607	0.351
166	Côte d'Ivoire	0.378	0.45	0.466
167	Guinea	0.505	0.361	0.403
168	Mali	0.478	0.3	0.394
169	Ethiopia	0.454	0.39	0.325
170	Chad	0.424	0.293	0.449
171	Guinea-Bissau	0.351	0.541	0.257
172	Burundi	0.399	0.546	0.201
173	Burkina Faso	0.445	0.274	0.398
174	Niger	0.521	0.286	0.302
175	Mozambique	0.291	0.474	0.334
176	Liberia	0.335	0.555	0.202
177	Congo (Democratic Republic of the)	0.351	0.559	0.172
178	Central African Republic	0.317	0.419	0.32
179	Sierra Leone	0.285	0.396	0.307

the two domains will certainly lead to rapid developments in the near future.

Appendix

Evaluations of the countries 1–179 are given in Tables A1–A4

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