Exponential and Logarithm

Ming Gong

May 2024

1 Introduction

In real analysis, we briefly touched upon the concept of exponentiation, where conveniently express

$$(x \cdot x \cdots x)_{n \text{ times}}$$

as x^n. The roots can be characterized using the supremum, where $x^{n/m} = \sqrt[m]{x^n} = \sup\{y \in \mathbb{R} : y^m < x^n\}$. However, challenges arise when the exponent is irrational. How can we define x^r for $r \in \mathbb{R}$ in general? This project will formally define the exponential and logarithm functions and explore their properties. With these foundations, we will extend the definition of exponentiation to all real numbers and use this to derive various theorems in calculus.

2 The Logarithm

There are many definitions of the logarithm. Here, I will propose a function with the following properties and call this the logarithm.

Proposition. There exists a unique function $L : (0, \infty) \to \mathbb{R}$ such that

1. $L(1) = 0$;
2. L is differentiable and $L'(x) = \frac{1}{x}$;
3. $L(xy) = L(x) + L(y)$ for all $x, y \in (0, \infty)$;
4. If q is a rational number and $x > 0$, then $L(x^q) = qL(x)$;
5. L is strictly increasing, bijective, and

$$\lim_{x \to 0} L(x) = -\infty \text{ and } \lim_{x \to \infty} L(x) = \infty.$$

Proof. I will define a candidate, prove it satisfies all the properties, and show that it is unique. Let

$$L(x) = \int_1^x \frac{1}{t} dt.$$

1. $L(1) = \int_1^1 \frac{1}{t} dt = 0$.
2. The function $\frac{1}{t}$ is continuous on $(0, \infty)$. By the fundamental theorem of calculus, L is differentiable for all $x \in (0, \infty)$, and $L'(x) = \frac{1}{x}$.
3. **Lemma** (integration by substitution) Let $\phi : [a, b] \to \mathbb{R}$ be a continuously differentiable function, and $f : [c, d] \to \mathbb{R}$ be continuous. Suppose $\phi([a, b]) \subset [c, d]$. Then,

$$\int_a^b f(\phi(x))\phi'(x) dx = \int_{\phi(a)}^{\phi(b)} f(y) dy.$$

Proof. The functions f, ϕ, ϕ' are all continuous, so $f(g(x))g'(x)$ is Riemann integrable.

Define $F : [c,d] \to \mathbb{R}$ by $F(y) = \int_{\phi(a)}^{y} f(s)ds$.

By the second fundamental theorem of calculus, $F'(y) = f(y)$. From the chain rule,

$$(F \circ \phi)'(x) = F'(\phi(x))\phi'(x) = f(\phi(x))\phi'(x).$$

$$\int_a^b f(\phi(x))\phi'(x)dx = \int_a^b (F \circ \phi)'(x)dx = F(\phi(b)) - F(\phi(a)) = \int_{\phi(a)}^{\phi(b)} f(y)dy.$$

□

With substitution, we can prove the additive property of logarithm by splitting the integral into two portions. Let $x, y \in (0, \infty)$.

$$\begin{aligned}
L(xy) &= \int_1^{xy} \frac{1}{t}dt \\
&= \int_1^{x} \frac{1}{t}dt + \int_x^{xy} \frac{1}{t}dt \\
\text{Let } t &= xu, dt = xdu. \\
&= \int_1^{x} \frac{1}{t}dt + \int_1^{y} \frac{1}{u}du \\
&= L(x) + L(y).
\end{aligned}$$

4. Let's first look at the case where $q = n$, where $n \in \mathbb{N}$. We can write multiplication as a sequence of additions and use the additive property.

$$L(x^n) = L(x \cdot x \cdot x ...)_{n \text{ times}} = nL(x).$$

Using this property backwards, if $q = {}^{n}/_{m}$, where $n, m \in \mathbb{N}$,

$$L(x^{n/m}) = \frac{n}{m} mL(x^{1/m}) = \frac{n}{m} L(x).$$

Finally, if q is negative, we once again use the additive property.

$$L(x^{-q}) = L(x^q x^{-q}) - L(x^q) = L(1) - qL(x) = -qL(x).$$

Therefore, for all $m, n \in \mathbb{Z}$, $L(x^{n/m}) = \frac{n}{m} L(x)$.

5. The derivative $L'(x) = \frac{1}{x} > 0$ for all $x \in (0, \infty)$. The function L is strictly increasing and therefore injective.

   To prove surjection, let $y \in \mathbb{R}$ be given.

   - If $y \geq 0$, by the Archimedean property, there exists $n \in \mathbb{N}$ such that $nL(2) > y$.

     $$0 = L(1) \leq y < nL(2) = L(2^n).$$

     Because L is continuous, by the intermediate value theorem, there exists $c \in [1, 2^n)$ with $L(c) = y$.

     $$\lim_{x \to \infty} L(x) = \infty.$$

   - If $y < 0$, there exists n such that $nL(2) > -y$. Similarly,

     $$L(2^{-n}) = -nL(2) < y < 0 = L(1).$$

     By the IVT, there exists $c \in (2^{-n}, 1)$ with $L(c) = y$. The function L is onto.

$$\lim_{x\to 0} L(x) = \lim_{x\to 0} -L(x^{-1}) = \lim_{x\to\infty} -L(x) = -\infty.$$

For uniqueness, suppose L and M both satisfy $L(1) = M(1) = 0$ and $L'(x) = M'(x) = \frac{1}{x}$. Then, $L'(x) - M'(x) = \frac{1}{x} - \frac{1}{x} = 0$. $(L - M)(x)$ is a constant function.

Therefore, for all $x \in (0, \infty)$, $(L - M)(x) = (L - M)(1) = L(1) - M(1) = 0$. Therefore, $L = M$. The logarithm function is unique.

□

3 The Exponential

Proposition. There exists a unique function $E : \mathbb{R} \to (0, \infty)$ such that

1. $E(0) = 1$;
2. E is differentiable and $E'(x) = E(x)$;
3. $E(x + y) = E(x)E(y)$ for all $x, y \in \mathbb{R}$;
4. If $q \in \mathbb{Q}$, then $E(qx) = E(x)^q$;
5. E is strictly increasing, bijective, and

$$\lim_{x\to -\infty} E(x) = 0 \text{ and } \lim_{x\to\infty} E(x) = \infty.$$

Again, I will name a candidate and show that it is unique and that it satisfies all the properties.

Proof. Let
$$E(x) = L^{-1}(x).$$

1. $E(0) = L^{-1}(0) = 1$.
2. To show differentiability, we would need the inverse function theorem.

   Proposition. If $f : [a, b] \to \mathbb{R}$ is strictly monotone, then the inverse $f^{-1} : f([a, b]) \to [a, b]$ is continuous.

   Proof. Let $\epsilon > 0$ be given, and let $\epsilon' = \min\{\epsilon, \frac{b-x}{2}, \frac{x-a}{2}\}$. Because f is monotone, $f(x + \epsilon') = y + \delta_1$, $f(x - \epsilon') = y - \delta_2$ for some $\delta_1, \delta_2 > 0$. Let $\delta = \min\{\delta_1, \delta_2\}$. If $|y - c| < \delta$, because f is monotone,
   $$|f^{-1}(y) - f^{-1}(c)| < |f^{-1}(y) - f^{-1}(\delta)| \le \epsilon' \le \epsilon.$$

   □

   Lemma. If $f : I \to J$ is strictly monotone, onto, differentiable at $x_0 \in I$, and $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and
   $$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))} = \frac{1}{f'(x_0)}.$$

   Proof. The function f is continuous and strictly monotone, so it has a continuous inverse $f^{-1} : J \to I$. Let $y = f(x)$. If $x \neq x_0$ and $y \neq y_0$,
   $$\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{f^{-1}(f(x)) - f^{-1}(f(x_0))}{f(x) - f(x_0)} = \frac{x - x_0}{f(x) - f(x_0)}.$$

Let $Q(x) = \begin{cases} \frac{x-x_0}{f(x)-f(x_0)} & \text{if } x \neq x_0, \\ \frac{1}{f'(x_0)} & \text{if } x = x_0. \end{cases}$ The function Q can be seen as the inverse of the secant line slope. At the limit when $x \to x_0$, Q becomes the inverse of the tangent line slope.

$$\lim_{x\to x_0} Q(x) = \lim_{x\to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0)} = Q(x_0).$$

The function Q is continuous at x_0. Since $f^{-1}(y)$ is continuous, the composition $Q(f^{-1}(y))$ is continuous at $f(x_0) = y_0$. Now we can directly evaluate the limit.

$$(f^{-1})'(y_0) = \lim_{y\to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \lim_{y\to y_0} Q(f^{-1}(y)) = Q(f^{-1}(y_0)) = \frac{1}{f'(f^{-1}(y_0))}.$$

□

From the lemma,

$$\begin{aligned} (L^{-1})'(x) &= \frac{1}{L'(L^{-1}(x))} \\ E'(x) &= \frac{1}{L'(E(x))} = \frac{1}{1/E(x)} = E(x). \end{aligned}$$

3. The arithmetic properties of E can be derived from the properties of L, using the fact that $E = L^{-1}$. Let $x, y \in \mathbb{R}$ and $q \in \mathbb{Q}$. Let $x = L(a)$ and $y = L(b)$.

$$\begin{aligned} E(x+y) &= E(L(a) + L(b)) \\ &= E(L(ab)) \\ &= ab \\ &= E(x)E(y). \end{aligned}$$

4.

$$\begin{aligned} E(qx) &= E(qL(a)) \\ &= E(L(a^q)) \\ &= a^q \\ &= (E(x))^q. \end{aligned}$$

5. The function L is strictly increasing and bijective, so does its inverse E.

   The function E is subjective over $(0, \infty)$, so for every given $M > 0$, there exists $c \in \mathbb{R}$ such that $E(c) = M$ and $E(c) \geq M$ for all $x \geq x_0$. For every given $\epsilon > 0$, there also exists $d \in \mathbb{R}$ such that $E(d) = \epsilon$ and $E(x) < \epsilon$ for all $x < x_0$. As a result,

$$\lim_{x\to-\infty} E(x) = 0 \text{ and } \lim_{x\to\infty} E(x) = \infty.$$

For uniqueness, let E and F be two functions satisfying properties $E(0) = F(0) = 1$ and $E'(x) = E(x)$, $F'(x) = F(x)$. We take the derivative of $F(x)E(-x)$. Using the chain rule and the derivative of E,

$$\begin{aligned} \frac{d}{dx}(E(x)F(-x)) &= E'(x)F(-x) + E(x)F'(-x) \\ &= E(x)F(-x) - E(x)F(-x) \\ &= 0. \end{aligned}$$

The function $E(x)F(-x)$ is a constant function. For all $x \in \mathbb{R}$,

$$\begin{aligned} E(x)F(-x) &= E(0)F(0) = 1 \\ E(x)F(-x) - F(x)F(-x) &= 1 - 1 \\ (E(x) - F(x))F(-x) &= 0. \end{aligned}$$

Since $F(-x) > 0$ for all x, the other term $F(x) - E(x) = 0$. $F(x) = E(x)$. The exponential function is unique. $\square$

4 Exponentiation

Now we can formally define the logarithm and the exponential function:

Definition. Define $\ln(x) = L(x)$, and $\exp(x) = E(x)$, where $L(x)$, $E(x)$ are specified previously.

There are also alternative definitions of the expoential function, and logarithm will be defined as its inverse.

- Using infinite series,
$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + ...$$
- Using the limit definition,
$$\exp(x) = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n.$$
- exp can also be defined by the unique solution to the differential equation $y' = y$ with $y(0) = 1$.

Definition. Let $e = \exp(1)$. e is also called Euler's number or the base of the natural log.

For all $q \in \mathbb{Q}$, $x^q = \exp(\ln(x^q)) = \exp(q \ln(x))$. We want to generalize exponentiation from $\mathbb{Q}$ to $\mathbb{R}$.

Definition. For all $x > 0$ and $y \in \mathbb{R}$, define

$$x^y = \exp(y \ln(x)).$$

5 Power rule

Now, we can derive the power rule for any real-numbered power. For all $r \in \mathbb{R}$,

$$\begin{aligned} \frac{d}{dx}(x^r) &= \frac{d}{dx}(\exp(r \ln(x)) \\ &= \exp(r \ln x)(r \cdot {}^1\!/\!_x) \\ &= x^r \cdot {}^r\!/\!_x \\ &= rx^{r-1}. \end{aligned}$$

We can also compute the derivative for exponential functions with an arbitrary base. For all $b > 0$,

$$\begin{aligned} \frac{d}{dx}(b^x) &= \frac{d}{dx}(\exp(x \ln(b))) \\ &= \exp(x \ln(b)) \ln(b) \\ &= \ln(b) b^x. \end{aligned}$$

$$\begin{aligned} \frac{d}{dx}(x^x) &= \frac{d}{dx}(\exp(x \ln(x))) \\ &= \exp(x \ln(x))(\ln(x) + {}^x\!/\!_x) \\ &= x^x(\ln(x) + 1). \end{aligned}$$

References

[1] J. Lebl. *Basic Analysis: Introduction to Real Analysis*. CreateSpace Independent Publishing Platform, 2014.