Logarithm and Exponential

MATH 312 Final Project

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Exponentiation

- $x^n = (x \cdot x \cdots x)_{n \text{ times}}$.
- $x^{n/m} = \sqrt[m]{x^n} = \sup\{y \in \mathbb{R} : y^m < x^n\}$.
- How to define x^r, $r \in \mathbb{R}$?
- How to differentiate x^r, r^x, x^x?

The Logarithm

Proposition

There exists a unique function $L : (0, \infty) \to \mathbb{R}$ such that

1. $L(1) = 0$;
2. L is differentiable and $L'(x) = \frac{1}{x}$;
3. $L(xy) = L(x) + L(y)$ for all $x, y \in (0, \infty)$;
4. If $q \in \mathbb{Q}$ and $x > 0$, $L(x^q) = qL(x)$;
5. L is strictly increasing, bijective, and

$$\lim_{x \to 0} L(x) = -\infty, \ \lim_{x \to \infty} L(x) = \infty.$$

Proof

Let

$$L(x) = \int_1^x \frac{1}{t} dt.$$

1. $L(1) = \int_1^1 \frac{1}{t} dt = 0.$
2. By fundamental thm., $L'(x) = \frac{1}{x}$.

For uniqueness, suppose L and M both satisfy 1 and 2.
$L'(x) - M'(x) = \frac{1}{x} - \frac{1}{x} = 0$. $(L - M)(x)$ is a constant function.

$(L - M)(x) = (L - M)(1) = 0$. $L = M$.

Arithmetic Properties

$$
\begin{aligned}
L(xy) &= \int_1^{xy} \frac{1}{t} dt \\
&= \int_1^{x} \frac{1}{t} dt + \int_x^{xy} \frac{1}{t} dt \\
\text{Let } t &= xu, dt = xdu. \\
&= \int_1^{x} \frac{1}{t} dt + \int_1^{y} \frac{1}{u} du \\
&= L(x) + L(y).
\end{aligned}
$$

Arithmetic Properties

For $m, n \in \mathbb{N}$,

- $L(x^n) = L(x \cdot x \cdot x ...)_{n \text{ times}} = nL(x)$;
- $L(x^{1/m}) = \frac{1}{m} mL(x^{1/m}) = \frac{1}{m} L(x)$;
- $L(x^{-1}) = L(xx^{-1}) - L(x) = L(1) - L(x) = -L(x)$.

For all $m, n \in \mathbb{Z}$, $L(x^{n/m}) = \frac{n}{m} L(x)$.

Bijection

The Expoential

Proposition

There exists a unique function $E : \mathbb{R} \to (0, \infty)$ such that

1. $E(0) = 1$;
2. E is differentiable and $E'(x) = E(x)$;
3. $E(x + y) = E(x)E(y)$ for all $x, y \in \mathbb{R}$;
4. If $q \in \mathbb{Q}$, then $E(qx) = E(x)^q$;
5. E is strictly increasing, bijective, and

$$\lim_{x \to -\infty} E(x) = 0, \lim_{x \to \infty} E(x) = \infty.$$

Proof

Let

$$E(x) = L^{-1}(x).$$

- $E(0) = 1$;
- Bijection;
- Strictly increasing.

Inverse Function Theorem

Proposition

If $f : I \to J$ is strictly monotone, onto, differentiable at $x_0 \in I$, and $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$.

$$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))} = \frac{1}{f'(x_0)}.$$

Proof

Let $y = f(x)$. If $x \neq x_0$ and $y \neq y_0$,

$$\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{f^{-1}(f(x)) - f^{-1}(f(x_0))}{f(x) - f(x_0)} = \frac{x - x_0}{f(x) - f(x_0)}.$$

$\square$

Inverse Function Theorem

Let $Q(x) = \begin{cases} \frac{x - x_0}{f(x) - f(x_0)} & \text{if } x \neq x_0, \\ \frac{1}{f'(x_0)} & \text{if } x = x_0. \end{cases}$

$$\lim_{x \to x_0} Q(x) = \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0)} = Q(x_0).$$

Q is continuous at x_0. f^{-1} is continuous, so $Q(f^{-1}(y))$ is continuous at y_0.

$$\begin{aligned} (f^{-1})'(y_0) &= \lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} \\ &= \lim_{y \to y_0} Q(f^{-1}(y)) \\ &= Q(f^{-1}(y_0)) = \frac{1}{f'(f^{-1}(y_0))}. \end{aligned}$$

Properties of the Exponential

Let $x = L(a)$ and $y = L(b)$.

$$\begin{aligned} E(x+y) &= E(L(a)+L(b)) \\ &= E(L(ab)) \\ &= ab \\ &= E(x)E(y); \end{aligned}$$

$$\begin{aligned} E(qx) &= E(qL(a)) \\ &= E(L(a^q)) \\ &= a^q \\ &= (E(x))^q. \end{aligned}$$

Uniqueness

Let E and F be two functions satisfying $E(0) = F(0) = 1$ and $E'(x) = E(x)$, $F'(x) = F(x)$.

$$\begin{aligned} \frac{d}{dx}(E(x)F(-x)) &= E'(x)F(-x) + E(x)F'(-x) \\ &= E(x)F(-x) - E(x)F(-x) \\ &= 0. \end{aligned}$$

$E(x)F(-x)$ is a constant function. For all x,

$$\begin{aligned} E(x)F(-x) &= E(0)F(0) = 1 \\ E(x)F(-x) - F(x)F(-x) &= 1 - 1 \\ (E(x) - F(x))F(-x) &= 0. \end{aligned}$$

$F(-x) > 0$ for all x, so $E(x) - F(x) = 0$ and $E(x) = F(x)$.

Exponentiation

Definition

- $\ln(x) = L(x)$;
- $\exp(x) = E(x)$;
- $e = \exp(1)$.

For $q \in \mathbb{Q}$, $x^q = \exp(\ln(x^q)) = \exp(q\ln(x))$. Now we can define x^y for all $y \in \mathbb{R}$.

Definition

If $x > 0$ and $y \in \mathbb{R}$, let

$$x^y = \exp(y\ln(x)).$$

Power Rule

Now, we can derive the power rule for any real power:

$$
\begin{aligned}
\frac{d}{dx}(x^r) &= \frac{d}{dx}(\exp(r\ln(x))) \\
&= \exp(r\ln x)(r \cdot {}^1\!/\!_x) \\
&= x^r \cdot {}^r\!/\!_x \\
&= rx^{r-1}.
\end{aligned}
$$

Derivatives of the Exponential

$$\begin{aligned}\frac{d}{dx}(b^x) &= \frac{d}{dx}(\exp(x\ln(b))) \\ &= \exp(x\ln(b))\ln(b) \\ &= \ln(b)b^x.\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}(x^x) &= \frac{d}{dx}(\exp(x\ln(x))) \\ &= \exp(x\ln(x))(\ln(x) + \frac{x}{x}) \\ &= x^x(\ln(x) + 1).\end{aligned}$$

Works Cited I

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The Bright Side of Mathematics.
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