Algebra

$\sin n\pi = 0$
$1 - \cos n\pi = 2$ for odd n
$\arctan(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$, $\arctan(\sqrt{3}) = \frac{\pi}{3}$

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$
$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$
$\cos x \cos y = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$
$\sin x \cos y = \frac{1}{2}[\sin(x-y) + \sin(x+y)]$

$\sin a \pm \sin b = 2 \sin \frac{a+b}{2} \cos \frac{a \mp b}{2}$
$\cos a + \cos b = 2 \cos \frac{a+b}{2} \cos \frac{a-b}{2}$
$\cos a - \cos b = -2 \sin \frac{a+b}{2} \sin \frac{a-b}{2}$

$C\mathrm{c}\,(\omega_0 t + \theta) = C\mathrm{c}\,(\theta)\mathrm{c}\,(\omega_0 t) - C\mathrm{s}\,(\theta)\mathrm{s}\,(\omega_0 t)$
$C\mathrm{s}\,(\omega_0 t + \theta) = C\mathrm{s}\,(\theta)\mathrm{c}\,(\omega_0 t) + C\mathrm{c}\,(\theta)\mathrm{s}\,(\omega_0 t)$

$\theta = \tan^{-1}(-\frac{b}{a})$, $\pm\pi$ when $a < 0$
$\sin t = \cos(t - \frac{\pi}{2})$
$-\cos t = \sin(t - \frac{\pi}{2})$

$\cos x = \frac{1}{2}\left[e^{jx} + e^{-jx}\right]$
$\sin x = \frac{1}{2j}\left[e^{jx} - e^{-jx}\right]$
$e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)$

$z^* = a - jb = re^{-j\theta}$
$u^* v^* = (uv)^*$

$\angle z = \tan^{-1}(\frac{b}{a})$, $\pm\pi$ in Q2 and Q3

$z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{j\frac{\theta + 2\pi m}{n}}$

Integrals

$\int \cos^2 at\, dt = \frac{t}{2} + \frac{\sin 2at}{4a}$

$\int t \cos at\, dt = \frac{1}{a^2}(\cos at + at \sin at)$
$\int t \sin at\, dt = \frac{1}{a^2}(\sin at - at \cos at)$
$\int t^2 \mathrm{c}\, at\, dt = \frac{1}{a^3}(2at\mathrm{c}\, at - 2\mathrm{s}\, at + a^2 t^2 \mathrm{s}\, at)$
$\int t^2 \mathrm{s}\, at\, dt = \frac{1}{a^3}(2at\mathrm{s}\, at + 2\mathrm{c}\, at - a^2 t^2 \mathrm{c}\, at)$

$\int te^{at}\, dt = \frac{1}{a^2} e^{at}(at - 1)$
$\int t^2 e^{at}\, dt = \frac{1}{a^3} e^{at}(a^2 t^2 - 2at + 2)$

$\int e^{at} \cos bt\, dt = \frac{1}{a^2 + b^2} e^{at}(a \cos bt + b \sin bt)$
$\int e^{at} \sin bt\, dt = \frac{1}{a^2 + b^2} e^{at}(a \sin bt - b \cos bt)$

$\int \frac{1}{x^2 + a^2}\, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$

Signals

$\mathcal{E}_f = \int_{-\infty}^{\infty} |f(t)|^2 dt$ (complex);
$P_f = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 dt$;
rms power $= \sqrt{P_f}$

Cont; analog; periodic (extension); (non/anti)causal; energy/power (both); deterministic/stochastic (info)

$\int f(t) \cdot \delta(t - t_0) dt = f(t_0)$ (f cont at t_0)

$f(2x - 6)$: shift by 6, scale by 2;
$f(2(x - 6))$: scale by 2, shift by 6

$f_e(t) = \frac{1}{2}[f(t) + f(-t)]$
$f_o(t) = \frac{1}{2}[f(t) - f(-t)]$

Square $\frac{4}{\pi}(\cos t - \frac{1}{3} \cos 3t + \frac{1}{5} \cos 5t - \ldots)$
Triangle $\frac{8}{\pi^2}(\sin t - \frac{1}{9} \sin 3t + \frac{1}{25} \sin 5t - \ldots)$
Sawtooth $\frac{2}{\pi}(\sin t - \frac{1}{2} \sin 2t + \frac{1}{3} \sin 3t - \ldots)$
δ train $\delta_{T_0}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$

Systems

$\mathcal{T}$: $\sum_{k=0} a_k D^k\, y(t) = \sum_{l=0} b_l D^l\, f(t)$

Linear $\mathcal{T}[kf_1(t) + f_2(t)] = ky_1(t) + y_2(t)$.
Lin if a_k, b_l are not func of $y(t), f(t)$
E. $\sin \dot{y}(t) + t^2 y(t) = (t+3) f(t)$

Time-inv $\mathcal{T}[f(t - \tau)] = y(t - \tau)$.
a_k, b_l indep of t (const coeff)
Let $g(t) \equiv f(t - \tau)$, find $z(t) = \mathcal{T}[g(t)]$, cmp $y(t - \tau)$

Causal $y(t)$ dep only on $f(\tau)$, $\tau \le t$.
Compare t and τ.

Ins/dyn y only dep f at present (no $\int$, no memory)

Invertible given $y(t)$, we can know $f(t)$

Conv prop

$c(t) \equiv \int_{-\infty}^{\infty} f(\tau) g(t - \tau)\, d\tau$
$c[n] \equiv \sum_{m=-\infty}^{\infty} f[m] g[n - m]$

$f * g = g * f$
$f * (g + h) = f * h + g * h$
$f * (g * h) = (f * g) * h$
pf: $f * (g * h) = f * (h * g)$
$= \int f(\tau_1) \int h(\tau_2)\, g(t - \tau_2 - \tau_1)\, d\tau_2\, d\tau_1$
$= \int h(\tau_2) \int f(\tau_1)\, g(t - \tau_1 - \tau_2)\, d\tau_1\, d\tau_2$

$f(t - T_1) * g(t - T_2) = c(t - T_1 - T_2)$

$f(at) * g(at) = |\frac{1}{a}|\, c(at)$ (even/odd)

$f^{(m)}(t) * g^{(n)}(t) = c^{(m+n)}(t)$
pf: $\dot{f}(\tau) = \lim_{T \to 0} f(\tau) - f(\tau - T)$

Graph: shift left by $+t$, and reflect;
Every τ replaced by $t - \tau$; Reverted

Conv table

$f(t) * \delta(t - T) = f(t - T)$
$u(t) * u(t) = t\, u(t)$

$e^{at}\, u(t) *\, u(t) = \frac{1 - e^{at}}{-a}\, u(t)$
$e^{at}\, u(t) * e^{bt}\, u(t) = \frac{e^{at} - e^{bt}}{a - b}\, u(t)$
$a = b$, $te^{at}\, u(t)$
$e^{at}\, u(t) * e^{bt}\, u(-t) = \frac{e^{at}\, u(t) + e^{bt}\, u(-t)}{b - a}$

$te^{at}\, u(t) * e^{at}\, u(t) = \frac{1}{2}\, t^2 e^{at}\, u(t)$
$t^m\, u(t) * t^n\, u(t) = \frac{m!\, n!}{(m+n+1)!}\, t^{m+n+1}\, u(t)$

Don't forget $[u(t + T_1) - u(t - T_2)]$ term

LTI response

$Q(D)y(t) = P(D)f(t)$, typically integrating f
Assume causal input $f(t)u(t)$

$y_{zs}(t) = f(t) * h(t)$ from input
$y_{zs}(0^-) = 0$, $y_{zs}(0^+) \neq 0$

Let $h(t) = \mathcal{T}[\delta(t)]$ (impulse response)
$y_{zs}(t) = \mathcal{T}[f(t)] = \mathcal{T}[f(t) * \delta(t)]$
$= \mathcal{T}[\lim \sum f(n\Delta\tau)\delta(t - n\Delta\tau)\Delta\tau]$

$= \lim \sum f(n\Delta\tau) h(t - n\Delta\tau)\Delta\tau = f * h$

$y_{zi}(t)$ from ini, $f(t) = 0$, $Qy_{zi}(t) = 0$
$y_{zi}(0^-) = y_{zi}(0^+)$, $\dot{y_{zi}}(0^-) = \dot{y_{zi}}(0^+)$

Ortho set

$\mathcal{E}_e = \int_{t_1}^{t_2} [e(t)]^2 dt = \int_{t_1}^{t_2} f^2(t) dt$
$-2 \sum c_i \int f(t) x_i(t) dt + \int (\sum c_i x_i(t))^2 dt$
$= \mathcal{E}_f - 2 \sum \langle f, x_i \rangle + (\sum c_i^2 \int x_i(t)^2 dt +$
$\sum_{i \neq j} c_i c_j \int_{t1}^{t2} x_i(t) x_j(t) dt)$

$\frac{\partial \mathcal{E}_e}{\partial c_i} = 0 = -2\langle f(t), x_i(t) \rangle + 2\mathcal{E}_i c_i$
$\mathcal{E}_e^{\min} = \mathcal{E}_f - \sum_{i=1}^{N} c_i^2 \mathcal{E}_i$
$c_i = \frac{1}{\mathcal{E}_i} \langle f, x_i \rangle = \frac{\int f(t) x(t) dt}{\int x_i^2(t) dt}$

For ortho, $E_z = E_x + E_y$
$|u + v|^2 = |u|^2 + |v|^2 + u^* v + v^* u$
$\langle x(t), y(t) \rangle = \int_{t_1}^{t_2} x(t) y(t)^*\, dt$

FS

$a_0 = \frac{1}{T_0} \int_{T_0} f(t)\, dt$
$a_n = \frac{2}{T_0} \int_{T_0} f(t) \cos(n\omega_0 t)\, dt$
$b_n = \frac{2}{T_0} \int_{T_0} f(t) \sin(n\omega_0 t)\, dt$
Energy: T_0 for $n = 0$; $T_0/2$ else

Half wave sym $f(t - \frac{T_0}{2}) = -f(t)$
$a_{n_{\text{odd}}} = \frac{4}{T_0} \int_0^{T_0/2} f(t) \cos(n\omega_0 t)\, dt$

$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$
$F_n = \frac{1}{T_0} \int_{T_0} f(t) e^{-jn\omega_0 t}\, dt$

$C_n \cos(n\omega_0 t + \theta_n) =$
$\frac{C_n}{2} \left(e^{j(n\omega_0 t + \theta_n)} + e^{-j(n\omega_0 t + \theta_n)}\right)$
$= (\frac{C_n}{2} e^{j\theta_n}) e^{jn\omega_0 t} + (\frac{C_n}{2} e^{-j\theta_n}) e^{-jn\omega_0 t}$

$F_n = \frac{C_n}{2}\, e^{j\theta_n} = \frac{1}{2}(a_n - jb_n) = |F_n| e^{j\angle F_n}$
$F_{-n} = \frac{C_n}{2}\, e^{-j\theta_n} = \frac{1}{2}(a_n + jb_n)$

Weak: finite $\int$, fin bounds a, b, fin power
Strong: fin m/m/disc over T_0, Converge

FS prop

Time shift $f(t - t_0) \to F_n e^{-jn(\omega_0 t_0)}$
$|F_n|$ same; $\angle F_n$ shifted by $-(\omega_0 t_0) n$

Reversal $f(-t) \to F_{-n}$
Scaling $T = \frac{T_0}{a}$, $\omega = a\omega_0$

Multiplication (same T_0):
$f(t)g(t) \to F_n * G_n$
$\frac{1}{T_0} \int_{T_0} f(t) g(t) e^{jn\omega_0 t}\, dt$
$=$
$\frac{1}{T_0} \int (\sum F_m e^{jm\omega_0 t})(\sum G_k e^{jk\omega_0 t}) e^{-jn\omega_0 t}\, dt$
$= \sum_m \sum_k F_m G_k \frac{1}{T_0} \int_{T_0} e^{j(m+k-n)\omega_0 t}\, dt$
$= \sum_m \sum_k F_m G_k \langle e^{j(m+k)\omega_0 t}, e^{jn\omega_0 t} \rangle$
$= \sum_{k=-\infty}^{\infty} G_k F_{n-k}$

Conjugation $f(t)^* = F_{-n}^*$

Parseval (power sig):
$P_f = \frac{1}{T_0} \int_{T_0} f(t) f(t)^* dt$
$= \frac{1}{T_0} \int_{T_0} (\sum_n F_n e^{jn\omega_0 t})(\sum_m F_m e^{jm\omega_0 t})^* dt$
$= \sum_n \sum_m F_n F_m^* \frac{1}{T_0} \int_{T_0} e^{j(n-m)\omega_0 t} dt$
$= \sum_n |F_n|^2 \cdot 1$

f real $\to |F|$ even, $\angle F$ odd
f real, even $\to F$ re, e; $F_{-n} = F_n = F_n^*$
f re, od $\to F$ im, o; $-F_{-n} = F_n = -F_n^*$

$f_e(t) \to \Re\{F_n\}$
$f_o(t) \to j\,\Im\{F_n\}$