Numerical Methods

Marios Gavriil Petrakis

§3. Interpolation

Prob. I-B Padé approximations of $f(x)=e^x$

◇ $P[2,3] = \dfrac{a_0 + a_1 x + a_2 x^2}{1 + b_1 x + b_2 x^2 + b_3 x^3}$, order $N = 2+3=5$

- To find the coefficients we equate with the McLaurin ~~series~~ expansion of the same order x^n of the same order & match the coefficients:

$$P[5,0] - P[2,3] = 0 \Rightarrow \underbrace{\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}\right)\left(1 + b_1 x + b_2 x^2 + b_3 x^3\right)}_{\cdot\ \text{denom.}} = a_0 + a_1 x + a_2 x^2$$

$\Downarrow$

$$\begin{aligned}
n=0 &: \boxed{1 = a_0} \\
1 &: b_1 + 1 = a_1 \\
2 &: b_2 + b_1 + \tfrac{1}{2} = a_2 \\
3 &: b_3 + b_2 + \tfrac{b_1}{2} + \tfrac{1}{6} = 0 \\
4 &: b_3 + \tfrac{b_2}{2} + \tfrac{b_1}{6} + \tfrac{1}{24} = 0 \\
5 &: \tfrac{b_3}{2} + \tfrac{b_2}{6} + \tfrac{b_1}{24} + \tfrac{1}{120} = 0
\end{aligned}
\quad \Bigg\} \quad
\left\{\begin{aligned}
& b_1 + 1 = a_1 \\
& 2(b_2 + b_1) + 1 = 2a_2 \\
& 6(b_3 + b_2) + 3b_1 + 1 = 0 \\
& 24b_3 + 12b_2 + 4b_1 + 1 = 0 \\
& 60b_3 + 20b_2 + 5b_1 + 1 = 0
\end{aligned}\right.$$

- From Mathematica: $a_1 = \frac{2}{5}$ $\quad b_1 = -\frac{3}{5}$ $\quad b_3 = -\frac{1}{60}$

$a_2 = \frac{1}{20}$ $\quad b_2 = \frac{3}{20}$

$$\Rightarrow P[2,3] = \frac{1 + \frac{2}{5}x + \frac{1}{20}x^2}{1 - \frac{3}{5}x + \frac{3}{20}x^2 - \frac{1}{60}x^3} = \frac{60 + 24x + 3x^2}{60 - 36x + 9x^2 - x^3}$$

$$\Rightarrow \quad P[4,1] = \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4}{1 + b_1 x}$$

likewise $\quad P[5,0] - P[4,1] = 0 \Rightarrow$

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}\right)(1 + b_1 x)$$

$\{$

$n=0 \cdot \boxed{a_0 = 1}$

$n=1 \cdot \ a_1 = b_1 + 1$

$n=2 \cdot \ a_2 = b_1 + \frac{1}{2}$

$n=3 \cdot \ a_3 = \frac{1}{2} b_1 + \frac{1}{6}$

$n=4 \cdot \ a_4 = \frac{b_1}{6} + \frac{1}{24}$

$n=5 \cdot \ \emptyset = \frac{b_1}{24} + \frac{1}{5 \cdot 24} \Rightarrow \boxed{b_1 = -\frac{1}{5}}$

$\Rightarrow a_1 = 1 - \frac{1}{5} = 4/5$

$\Rightarrow a_2 = \frac{1}{2} - \frac{1}{5} = \frac{3}{10}$

$\Rightarrow a_3 = \frac{1}{6} - \frac{1}{10} = \frac{4}{60} = \frac{1}{15}$

$\Rightarrow a_4 = \frac{1}{24} - \frac{1}{30} = \frac{1}{120}$

$$\Rightarrow \quad P[4,1] = \frac{1 + \frac{4}{5}x + \frac{3}{10}x^2 + \frac{1}{15}x^3 + \frac{1}{120}x^4}{1 - \frac{1}{5}x}$$

$$= \frac{5 + 4x + (3/2)x^2 + (1/3)x^3 + (1/24)x^4}{5 - x}$$

Values	e^x	$P[5,0]$	$P[2,3]$	$P[4,1]$	
at $x=0.5$	1.64872	1.6487	1.64873	1.64873	all in agreement
at $x=1$	2.71828	2.71667	2.71875	2.71875	still good agreement
at $x=2$	7.38906	7.26667	7.5	7.444	small deviations begun
at $x=5$	148.413	91.4167	-12.75	$\pm\infty$	all approx. fail

Taylor shows larger deviations as we get further away

$\varepsilon \propto \delta x^6$

$P[2,3]$ has a singularity before $x=5$ & changes sign $\to$ invalid

$x=5$ its singularity $\downarrow$ invalid close to 5

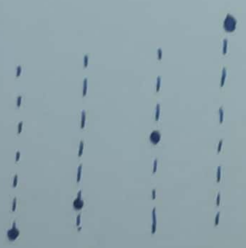
but away from the poles the Padé approx show better convergence

$\Rightarrow$ good comparison with logplot $\to$ Mathematica notebook

---

Prob. II

const. step $h=2$



k	x_k	$f(x_k)$	$\Delta f(x_k)$	$\Delta^2 f(x_k)$	$\Delta^3 f(x_k)$
0	4	1	2	3	4
1	6	3	5	7	
2	8	8	12		
3	10	20			

$k=0$

(forward) difference matrix

(of 3rd degree)

◇ So, the forward Newton polynomial is:

$$P_3(x_k) = \sum_{i=0}^{3} \binom{k}{i} \Delta^i f_0 = f_0 + k\Delta f_0 + \frac{k!}{2!(k-2)!}\Delta^2 f_0 + \frac{k!}{3!(k-3)!}\Delta^3 f_0$$

$$\Rightarrow P(x^k) = 1 + 2k + \frac{3k(k-1)}{2} + \frac{4k(k-1)(k-2)}{6}$$

$$= 1 + \underbrace{2k} + \frac{3}{2}k^2 \underbrace{- \frac{3}{2}k} + \frac{2}{3}k^3 - 2k^2 + \frac{4k}{3}$$

$$= 1 + \frac{11}{6}k - \frac{1}{2}k^2 + \frac{2}{3}k^3$$

but $x_k = x_0 + kh = 4 + 2k \Rightarrow k = \frac{x_k - 4}{2}$

- $k(k-1) = \frac{x_k - 4}{2}\,\frac{x_k - 6}{2} = \frac{x_k^2 - 10x_k + 24}{4}$

- $k(k-1)(k-2) = \frac{x_k^2 - 10x_k + 24}{4}\,\frac{x_k - 8}{2} = \frac{x_k^3 - 18x_k^2 + 104x_k - 192}{8}$

$$\Rightarrow P(x_k) = 1 + x_k - 4 + 3\,\frac{x_k^2 - 10x_k + 24}{8} + \frac{x_k^3 - 18x_k^2 + 104x_k - 192}{12}$$

$$= \underbrace{\Big(\overbrace{-3 + 3^2}^{+6} - \frac{\cancel{192}^{16}}{\cancel{12}}\Big)}_{6-16=-10} + \underbrace{\left(1 - \frac{15}{4} + \frac{26}{3}\right)}_{\frac{24}{3} - \frac{15}{4} = \frac{116-45}{12} = \frac{71}{12}}x_k + \left(\frac{3}{8} - \frac{3}{2}\right)x_k^2 + \frac{x_k^3}{12}$$

$$= \frac{1}{12}x_k^3 - \frac{9}{8}x_k^2 + \frac{71}{12}x_k - 10 =$$

$$= \frac{1}{24}\left(2x_k^3 - 27x_k^2 + 142x_k - 240\right)$$