Numerical Methods

Set 6: Numerical Integration

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Problem 1

Combining Simpson's 1/3 rule with the Romberg method.

Proving Simpson's 1/3 rule:

If we have equidistant points from x_0 to x_n, with step $h=\frac{x_n-x_0}{n}$ and values of a function y_i at each point $x_i=i\cdot h+x_0$, then we interpolate each interval of 3 consecutive points (in non-overlapping intervals) with a forward-differences parabola, so with a Newton polynomial of 2nd order:

$$\int_{x_0}^{x_n} f(x)\,\mathrm{d}x=\sum_{i=0}^{n/2-1}\int_{x_{2i}}^{x_{2i+2}} P_2(x)\,\mathrm{d}x$$

- In order not to have problems at the boundaries or with overlapping intervals, it's best to have an even number of points $n=2m$ (with convention of starting at x_0), unless of course we have a really small step h or equivalently a really large number of points and we can ignore some non-irregular interval of 1 step.

Let's say we have the interval $[x_{2i}, x_{2i+2}]$, then we find the polynomial $P_2(s)$ as:

x	**y**	**Δy**	**Δ²y**
x_{2i}	y_{2i}	$y_{2i+1}-y_{2i}$	$y_{2i+1}+y_{2i}-2y_{2i+1}$
x_{2i+1}	y_{2i+1}	$y_{2i+2}-y_{2i+1}$	
x_{2i+2}	y_{2i+2}		

$$P_2(x_s)=\sum_{k=0}^{2}\binom{s}{k}\Delta^k f_{2i}=f_{2i}+s(f_{2i+1}-f_{2i})+\frac{1}{2}s(s-1)(y_{2i+2}+y_{2i}-2y_{2i+1})$$

Then, to integrate , we can simply substitute x with s, dx = h ds and calculating the integrals:

$$\int_0^2 s\,\mathrm{d}s=\left[\frac{s^2}{2}\right]_0^2=2$$

$$\int_0^2 \frac{s(s-1)}{2}\,\mathrm{d}s=\left[\frac{s^3}{6}-\frac{s^2}{4}\right]_0^2=\frac{8}{6}-1=\frac{1}{3}$$

$$\Rightarrow\int_{x2i}^{x_{2i+2}} P_2(x)\,\mathrm{d}x=h\int_0^2 P_2 s\,\mathrm{d}s=¿$$

$$h\left(2y_{2i}+2(y_{2i+1}-y_{2i})+\frac{1}{3}(y_{2i+2}+y_{2i}-2y_{2i+1})\right)=\frac{1}{3}h(y_{2i+2}+y_{2i}+4y_{2i+1})$$

and due to the “coincidence”, the error is known to be:

$$E_{gl}=\frac{-\Delta x}{180}h^4 f^{(4)}(\xi)$$

for the whole interval Δx = b- a, after the summation, or just for a single interval $[x_{2i},x_{2i+2}]$:

$$E=\frac{-1}{90}h^5 f^{(4)}(\xi_i)$$

Applying Romberg’s method:

For a half-step optimization we need in total 5 points, x_0,x_1,x_2,x_3,x_4

For a full step $h=x_2-x_0$ we have the numerical result:

$$I_1=\frac{h}{3}(y_4+4y_2+y_0)$$

And for a half step $h/2=x_1-x_0$:

$$I_2=\frac{h}{6}(y_4+4y_3+2y_2+4y_1+y_0)$$

$$\Rightarrow I_2-I_1=\frac{h}{6}(4y_3+4y_1-y_4-y_0-6y_2)$$

Since the global error in Simpson's 1/3 method is h^4, we apply Romberg's method with n=4:

$$A = I_2 + \frac{I_2 - I_1}{2^n - 1} = I_2 + \frac{I_2 - I_1}{15}$$

$$\Rightarrow A = \frac{h}{90}\left(4y_3 + 4y_1 - y\,y_4 - y_0 - 6y_2 + 15y_4 + 60y_3 + 30y_2 + 60y_1 + 15y_0\right) = \frac{h}{90}\left(14y_0 + 64y_1 + 24y_2 + 64y_3 +\right.$$

Problem 2

For the Gauss-Legendre we can do the following rescaling for random limits of integration:

$$\int_a^b f(x)\,dx = \lambda \int_{-1}^{1} f(\mu + \lambda z)\,dz$$

Where we want $a = \mu - \lambda$ and $b = \mu + \lambda$:

$$\mu = \frac{a+b}{2}$$

$$\lambda = \frac{b-a}{2}$$

Problem 3

We use a nested method to compute the integral and we go from out to the inside. The code has been slightly modified to account for 2-variable functions. See .py file.

The third integral is:

$$\int_0^2 \int_0^{x/2} x y^2\,dy\,dx = \int_0^2 x \left[\frac{y^3}{3}\right]_0^{x/2} dx = \frac{1}{24}\int_0^2 x^4\,dx = \frac{1}{24}\left[\frac{x^5}{5}\right]_0^2 = \frac{8*4}{8*15} = \frac{4}{15}$$