DFT and Fourier Transform

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1 Relationship between DFT and Fourier Transform

DFT:

$$X_d[k] = \sum_{n=0}^{N-1} x_d[n] \exp\left(-i\frac{2\pi}{N}nk\right) \quad (1)$$

$$x_d[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_d[k] \exp\left(i\frac{2\pi}{N}nk\right). \quad (2)$$

Fourier transform:

$$X(f) = \int_{t=-\infty}^{+\infty} x(t) \exp\left(-i2\pi ft\right) \mathrm{d}t \quad (3)$$

$$x(t) = \int_{f=-\infty}^{+\infty} X(f) \exp\left(i2\pi ft\right) \mathrm{d}f. \quad (4)$$

Suppose that $x(t) = 0$ for $t < 0$ and $t > T$. Suppose that you want to approximate Eq. 3 by an N-point integration rule with equally-spaced abscissas ($= \delta = T/N$) and constant weights ($= w = T/N = \delta$). Let the approximate answer be denoted by $X_a(f)$. Let $x_d[n] = x(n\delta)$. Then

$$X_a(f) = \sum_{n=0}^{N-1} x(n\delta) \exp\left(-i2\pi fn\delta\right) \delta \quad (5)$$

$$= \delta \sum_{n=0}^{N-1} x_d[n] \exp\left(-i2\pi fn\frac{T}{N}\right) \quad (6)$$

$$= \delta \sum_{n=0}^{N-1} x_d[n] \exp\left(-i\frac{2\pi}{N}(fT)n\right). \quad (7)$$

Suppose you focus on f such that $fT = k \in \{0, \ldots, N-1\}$. Then

$$X_a(f = k/T) = \delta \sum_{n=0}^{N-1} x_d[n] \exp\left(-i\frac{2\pi}{N}kn\right) \quad (8)$$

$$= \quad \delta X_d[k]. \tag{9}$$

Since $T/N = \delta$, I can express T in the form $T = N\delta$ and then

$$X_a\left(f = \frac{k}{N\delta}\right) = \delta X_d[k] \quad (k \in \{0, \dots, N-1\}). \tag{10}$$

2 Sinusoidal signals

In continuous time,

$$x(t) = \exp(i2\pi f_0 t) \quad \leftrightarrow \quad X(f) = \delta(f - f_0). \tag{11}$$

In discrete time, for $k_0 \in \{0, \dots, N-1\}$ let

$$x_d[n] = \exp\left(i\frac{2\pi}{N}nk_0\right). \tag{12}$$

Then,

$$\begin{aligned} X_d[k] &= \sum_{n=0}^{N-1} x_d[n] \exp\left(-i\frac{2\pi}{N}nk\right) && (13)\\ &= \sum_{n=0}^{N-1} \exp\left(i\frac{2\pi}{N}nk_0\right) \exp\left(-i\frac{2\pi}{N}nk\right) && (14)\\ &= \sum_{n=0}^{N-1} \exp\left(-i\frac{2\pi}{N}n(k-k_0)\right). && (15) \end{aligned}$$

Suppose $k = k_0$. Then

$$\begin{aligned} X_d[k] &= \sum_{n=0}^{N-1} \exp\left(-i\frac{2\pi}{N}n0\right) && (16)\\ &= \sum_{n=0}^{N-1} 1 && (17)\\ &= N. && (18) \end{aligned}$$

Now suppose $k \neq k_0$. The geometric sum is

$$\sum_{n=0}^{N-1} \rho^n = \frac{1-\rho^N}{1-\rho}. \tag{19}$$

Eq. 15 is a geometric sum with

$$\rho = \exp\left(-i\frac{2\pi}{N}(k-k_0)\right) \tag{20}$$

so that

$$
\begin{aligned}
X_d[k] &= \frac{1-\exp\left(-i\frac{2\pi}{N}(k-k_0)\right)^N}{1-\exp\left(-i\frac{2\pi}{N}(k-k_0)\right)} && (21)\\
&= \frac{1-\exp\left(-i2\pi(k-k_0)\right)}{1-\exp\left(-i\frac{2\pi}{N}(k-k_0)\right)} && (22)\\
&= \frac{1-1}{1-\exp\left(-i\frac{2\pi}{N}(k-k_0)\right)} && (23)\\
&= 0. && (24)
\end{aligned}
$$

Therefore, for $k_0 \in \{0, \ldots, N-1\}$,

$$
x_d[n] = \exp\left(i\frac{2\pi}{N}nk_0\right) \quad \leftrightarrow \quad X_d[k] = N\delta_{k,k_0}. \tag{25}
$$

Suppose you want to think of a continuous-time sinusoidal signal of the form shown in Eq. 11 that is sampled to give $x_d[n]$ of the form shown in Eq. 12, that is,

$$
\begin{aligned}
x_d[n] &= x(n\delta) && (26)\\
&= \exp(i2\pi f_0 n\delta). && (27)
\end{aligned}
$$

Therefore we need

$$
f_0\delta = \frac{k_0}{N}. \tag{28}
$$

So the only continuous time signals that will work are those with

$$
f_0 = \frac{k_0}{N\delta} \tag{29}
$$

where $k_0 \in \{0, \ldots, N-1\}$.