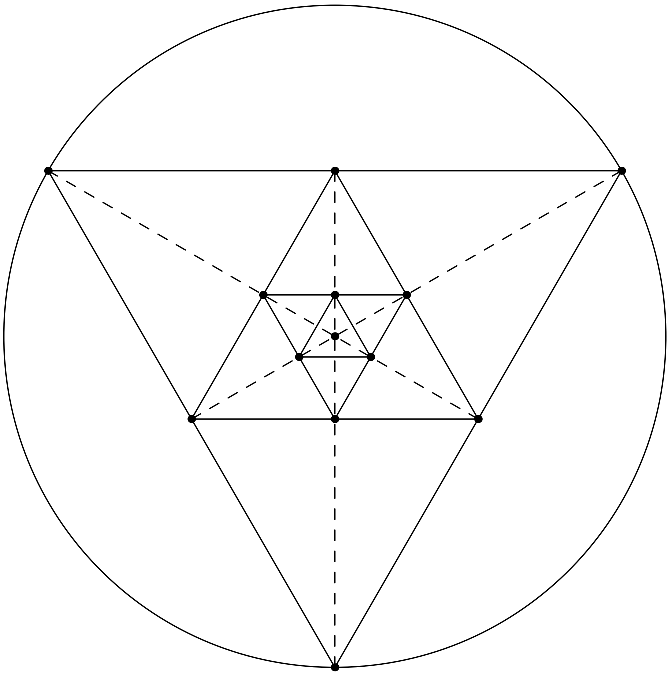


A Geometric Approach To Matrices

Peter Herreshoff
Henry M. Gunn High School
Analysis Honors

Contents

1 Trigonometry Review

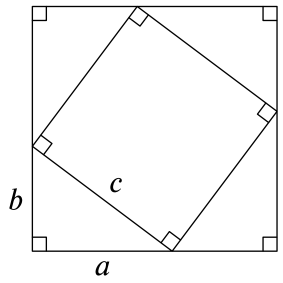


Figure 1: Scenario in Problem 1.

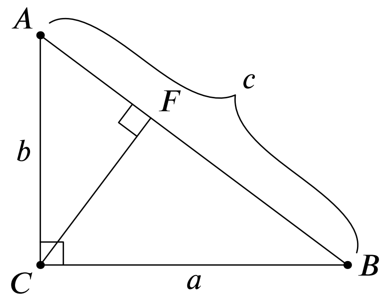


Figure 2: Scenario in Problem 2.

1. Prove the Pythagorean theorem using "conservation of area." Start with Figure 1.

In Figure 1, the larger square has side length $a+b$. The smaller, nested square has side length c. Four copies of the right triangle with side lengths a, b, c are placed around the square. We have

$$\begin{aligned}
A_{\text{triangles}} + A_{\text{small sq.}} &= A_{\text{big sq.}} && \text{[Conservation of area]}\\
4A_{\text{triangle}} + A_{\text{small sq.}} &= A_{\text{big sq.}} \\
4\left(\frac{1}{2}ab\right) + c^2 &= (a+b)^2 && \text{[Areas of triangle, square]}\\
2ab + c^2 &= a^2 + 2ab + b^2 && \text{[Expanding]}\\
c^2 &= a^2 + b^2. && \text{Q.E.D.}
\end{aligned}$$

2. Prove the Pythagorean theorem using a right triangle with an altitude drawn to its hypotenuse, as shown in Figure 2, making use of similar right triangles.

Let $h = CF$, the length of the altitude to the hypotenuse. $\triangle ACF \sim \triangle ABC$ by AA Similarity because they share an angle and both have a right angle. Therefore, $\frac{AF}{AC} = \frac{AC}{AB}$. Substituting named variables for these lengths, we get

$$\frac{AF}{b} = \frac{b}{c} \Longrightarrow AF = \frac{b^2}{c}.$$

Applying the same logic to $\triangle CFB$, we get $\triangle CFB \sim \triangle ABC$, so $\frac{BF}{BC} = \frac{BC}{AB}$. Substituting, we get

$$\frac{BF}{a} = \frac{a}{c} \Longrightarrow BF = \frac{a^2}{c}.$$

Since F is between A and B, we have $AB = AF + FB$; substituting our found values for AF and FB, we get

$$\begin{aligned}
c = AB &= AF + FB \\
c &= \frac{b^2}{c} + \frac{a^2}{c} \\
c^2 &= b^2 + a^2. && \text{Q.E.D.}
\end{aligned}$$

3. We now prove the trigonometric identities.

(a) Draw and label a right triangle and a unit circle, then write trig definitions for $\cos$, $\sin$, $\tan$, and $\sec$ in terms of your drawing.

The scenario is depicted in Figure 3. By the definition of sine and cosine, we have $\sin\theta = AP$ and $\cos\theta = OA$. Since $\triangle OAP \sim \triangle OPT$ by AA Similarity, we have $\frac{TP}{OP} = \frac{AP}{OA}$. Substituting known values, we get

$$\frac{TP}{1} = \frac{\sin\theta}{\cos\theta} \Longrightarrow TP = \tan\theta.$$

Also, $\triangle OAP \sim \triangle OKS$ by AA, so $\frac{OS}{OK} = \frac{1}{\cos\theta}$. Similarly, we have

$$\frac{OS}{1} = \frac{1}{\cos\theta} \implies OS = \sec\theta.$$

Finally, as an alternate interpretation of $\tan$, we have $\frac{KS}{OK} = \frac{AP}{OA}$, so

$$\frac{KS}{1} = \frac{\sin\theta}{\cos\theta} \implies KS = \tan\theta.$$

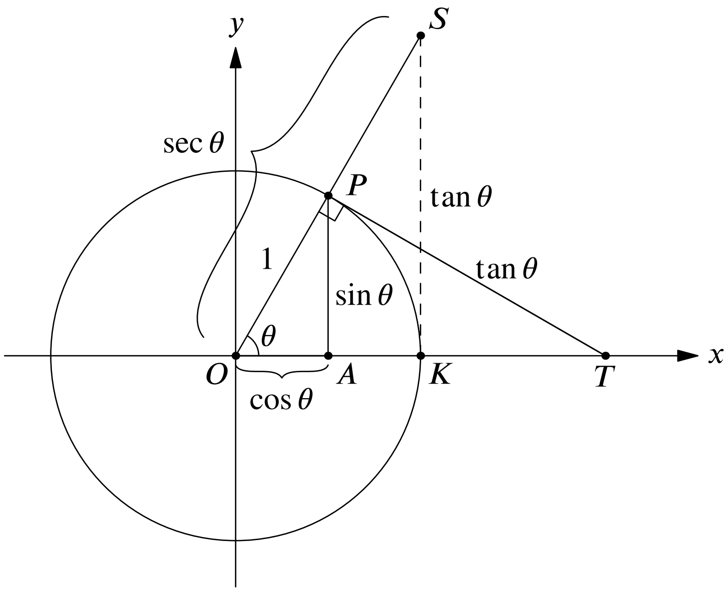


Figure 3: The right triangle and unit circle.

(b) Use a right triangle and the definitions of $\sin$ and $\cos$ to find and prove a value for $\sin^2\theta + \cos^2\theta$.

Referring back to Figure 3, focus on $\triangle OAP$. It is a right triangle with side lengths $a = \cos\theta$, $b = \sin\theta$, and $c = 1$. By the Pythagorean theorem, we have

$$\begin{aligned} OA^2 + AP^2 &= OP^2 && \text{[Pythagorean theorem]} \\ \cos^2\theta + \sin^2\theta &= 1^2 && \text{[Substitution]} \\ \sin^2\theta + \cos^2\theta &= 1. && \text{[Rearrange]} \end{aligned}$$

(c) Use the picture of the unit circle in Figure 4 to find and prove a value for $\cos(A - B)$. Note that D_1 and D_2 are the same length because they subtend the same size arc of the circle. Set them equal and work through the algebra, using the distance formula and part (b) of this problem.

We have $D_1 = D_2$, so

$$\begin{aligned} D_1^2 &= D_2^2 \\ (\cos A - \cos B)^2 + (\sin A - \sin B)^2 &= (\cos(A-B) - 1)^2 + \sin^2(A-B) \\ \cos^2 A - 2\cos A\cos B + \cos^2 B + \sin^2 A - 2\sin A\sin B + \sin^2 B &= \cos^2(A-B) - 2\cos(A-B) + \\ &\quad\; 1 + \sin^2(A-B) \\ (\cos^2 A + \sin^2 A) + (\cos^2 B + \sin^2 A) - 2\sin A\sin B &= (\cos^2(A-B) + \sin^2(A-B)) + \\ &\quad\; 1 - 2\cos(A-B) \\ \cancelto{0}{1+1} - 2\sin A\sin B - 2\cos A\cos B &= \cancelto{0}{1+1} - 2\cos(A-B) \\ 2\sin A\sin B + 2\cos A\cos B &= 2\cos(A-B) \\ \sin A\sin B + \cos A\cos B &= \cos(A-B). \end{aligned}$$

Q.E.D.

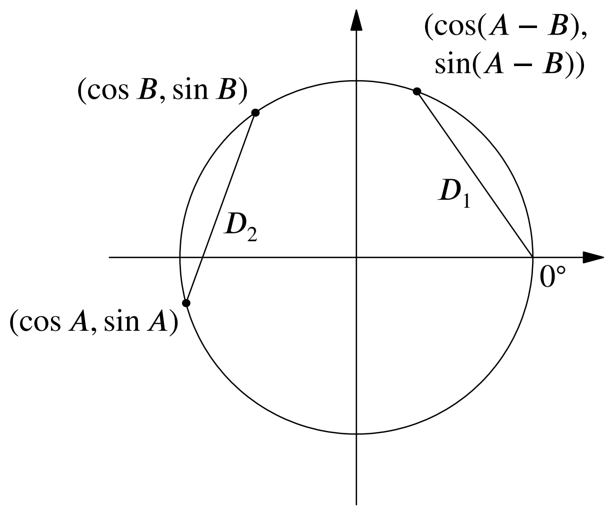


Figure 4: Scenario in Problem 3.

4. Write down as many trig identities as you can—no need to prove these.

$$\begin{array}{rrr}
\sin(A+B) = & \sin(A-B) = & \cos(A+B) = \\
\tan(A+B) = & \tan(A-B) = & \sin(2A) = \\
\cos(2A) = & \tan(2A) = & \sin\left(\frac{A}{2}\right) = \\
\cos\left(\frac{A}{2}\right) = & \tan\left(\frac{A}{2}\right) = &
\end{array}$$

You should probably memorize these for convenience.

$$\begin{aligned}
\sin(A+B) &= \sin A \cos B + \cos A \sin A \\
\sin(A-B) &= \sin A \cos B - \cos A \sin B \\
\cos(A+B) &= \cos A \cos B - \sin A \sin B \\
\tan(A+B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \\
\tan(A-B) &= \frac{\tan A - \tan B}{1 + \tan A \tan B} \\
\sin(2A) &= 2 \sin A \cos A \\
\cos(2A) &= 2\cos^2 A - 1 = 1 - 2\sin^2 A = \cos^2 A - \sin^2 A \\
\tan(2A) &= \frac{2 \tan A}{1 - \tan^2 A} \\
\sin\left(\frac{A}{2}\right) &= \pm\sqrt{\frac{1 - \cos A}{2}} \\
\cos\left(\frac{A}{2}\right) &= \pm\sqrt{\frac{1 + \cos A}{2}} \\
\tan\left(\frac{A}{2}\right) &= \frac{\sin A}{1 + \cos A} = \frac{1 - \cos A}{\sin A}
\end{aligned}$$

5. Let's review complex numbers and DeMoivre's theorem.

(a) Recall that you can write a complex number both in Cartesian and polar forms. Let

$$a + bi = (a, b) = (r\cos\theta, r\sin\theta) = r\cos\theta + ir\sin\theta.$$

What is r in terms of a and b?

r is just the distance to the origin from $a + bi$. Draw a right triangle as shown in Figure 5. By the pythagorean theorem, $r = \sqrt{a^2 + b^2}$.

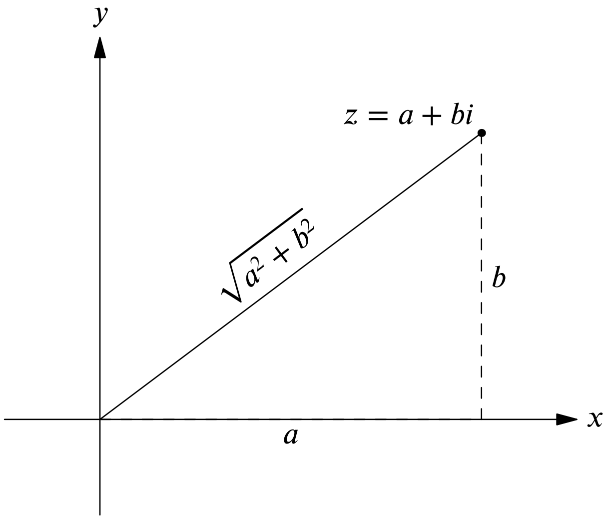


Figure 5: $a + bi$ in the complex plane.

(b) Expand $(a + bi)(c + di)$ the usual way.

$$\begin{aligned}(a + bi)(c + di) &= ac + adi + bci + (bi)(di)\\ &= ac + (ad + bc)i - bd\\ &= ac - bd + (ad + bc)i.\end{aligned}$$

(c) Let $a + bi = r_1(\cos\theta + i\sin\theta)$ and $c + di = r_2(\cos\phi + i\sin\phi)$. Multiply them, and use your results from Problems 3c and 3d to show that multiplying two complex numbers involves multiplying their lengths and adding their angles. This is DeMoivre's theorem!

$$\begin{aligned}r_1(\cos\theta + i\sin\theta)r_2(\cos\phi + i\sin\phi) &= r_1r_2(\cos\theta\cos\phi - \sin\theta\sin\phi + i(\sin\theta\cos\phi + \cos\theta\sin\phi))\\ &= r_1r_2(\cos(\theta+\phi) + i\sin(\theta+\phi)).\end{aligned}$$

(d) Use part (c) to simplify $(\sqrt{3} + i)^{18}$.

We have $\sqrt{3} + i = r(\cos\theta + i\sin\theta) = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$.

$$\begin{aligned}(2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right))^{18} &= 2^{18}\cdot\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^{18}\\ &= 2^{18}\cdot\underbrace{\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\cdots\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)}_{\text{18 copies}}\\ &= 2^{18}\cdot\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\underbrace{\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\cdots\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)}_{\text{16 copies}}\\ &= \vdots\\ &= 2^{18}\cdot(\cos 3\pi + i\sin 3\pi)\\ &= 2^{18}\cdot -1\\ &= -2^{18}.\end{aligned}$$

6. Here is a review of 2D rotation.

(a) Recall that we can graph complex numbers as ordered pairs in the complex plane. Now, consider the complex number $z = \cos\theta + i\sin\theta$, where θ is fixed. What is the magnitude of z?

We have

$$|z| = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1.$$

(b) Multiplying $z \cdot (x + yi)$ yields a rotation of the point (x, y) counterclockwise around the origin by the angle θ. Notice that rotating the graph counterclockwise around the origin has the same effect as rotating the coordinate axes clockwise around the origin by the same angle θ. What if we wanted to rotate clockwise by θ instead?

We can multiply by the conjugate of z, since

$$\overline{z} = \cos\theta - i\sin\theta = \cos -\theta + i\sin -\theta.$$

Thus, the operation is $\overline{z} \cdot (x + yi)$ to rotate clockwise by θ.

7. Rotate the following conics by (i) $30°$, (ii) $45°$, and (iii) θ:

(a) $x^2 - y^2 = 1$

i. $30°$

We make the substitution $x' = x\cos 30° - y\sin 30° = \frac{\sqrt{3}}{2}x - \frac{y}{2}$ and $y' = x\sin 30° + y\cos 30° = \frac{x}{2} + \frac{\sqrt{3}}{2}y$:

$$\begin{aligned} x'^2 - y'^2 &= 1 \\ \left(\frac{\sqrt{3}}{2}x - \frac{y}{2}\right)^2 - \left(\frac{x}{2} + \frac{\sqrt{3}}{2}y\right)^2 &= 1 \\ x^2/2 - \sqrt{3}xy - y^2/2 &= 1. \end{aligned}$$

ii. $45°$

We make the substitution $x' = x\cos 45° - y\sin 45° = \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y$ and $y' = x\sin 45° + y\cos 45° = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y$:

$$\begin{aligned} x'^2 - y'^2 &= 1 \\ \left(\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y\right)^2 - \left(\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right)^2 &= 1 \\ -2xy &= 1. \end{aligned}$$

iii. θ

We make the substitution $x' = x\cos\theta - y\sin\theta$ and $y' = x\sin\theta + y\cos\theta$:

$$\begin{aligned} x'^2 - y'^2 &= 1 \\ (x\cos\theta - y\sin\theta)^2 - (x\sin\theta + y\cos\theta)^2 &= 1. \end{aligned}$$

(b) $\frac{x^2}{16} - \frac{y^2}{9} = 1$

i. $30°$

We make the substitution $x' = x\cos 30° - y\sin 30° = \frac{\sqrt{3}}{2}x - \frac{y}{2}$ and $y' = x\sin 30° + y\cos 30° = \frac{x}{2} + \frac{\sqrt{3}}{2}y$:

$$\begin{aligned} \frac{x'^2}{16} - \frac{y'^2}{9} &= 1 \\ \frac{\left(\frac{\sqrt{3}}{2}x - \frac{y}{2}\right)^2}{16} - \frac{\left(\frac{x}{2} + \frac{\sqrt{3}}{2}y\right)^2}{9} &= 1 \\ \frac{1}{576}(11x^2 - 50\sqrt{3}xy - 39y^2) &= 1. \end{aligned}$$

ii. 45°

We make the substitution $x' = x\cos 45° - y\sin 45° = \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y$ and $y' = x\sin 45° + y\cos 45° = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y$:

$$\begin{aligned}\frac{x'^2}{16} - \frac{y'^2}{9} &= 1\\ \frac{\left(\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y\right)^2}{16} - \frac{\left(\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right)^2}{9} &= 1\\ \frac{1}{288}(-x-7y)(7x+y) &= 1.\end{aligned}$$

iii. θ

We make the substitution $x' = x\cos\theta - y\sin\theta$ and $y' = x\sin\theta + y\cos\theta$:

$$\begin{aligned}\frac{x'^2}{16} - \frac{y'^2}{9} &= 1\\ \frac{(x\cos\theta - y\sin\theta)^2}{16} - \frac{(x\sin\theta + y\cos\theta)^2}{9} &= 1.\end{aligned}$$

(c) $y^2 = 4Cx$

i. 30°

We make the substitution $x' = x\cos 30° - y\sin 30° = \frac{\sqrt{3}}{2}x - \frac{y}{2}$ and $y' = x\sin 30° + y\cos 30° = \frac{x}{2} + \frac{\sqrt{3}}{2}y$:

$$\begin{aligned}y'^2 &= 4Cx'\\ \left(\frac{x}{2} + \frac{\sqrt{3}}{2}y\right)^2 &= 4C\left(\frac{\sqrt{3}}{2}x - \frac{y}{2}\right).\end{aligned}$$

ii. 45°

We make the substitution $x' = x\cos 45° - y\sin 45° = \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y$ and $y' = x\sin 45° + y\cos 45° = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y$:

$$\begin{aligned}y'^2 &= 4Cx'\\ \left(\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right)^2 &= 4C\left(\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y\right)\\ \frac{1}{2}(x+y)^2 &= 2C\sqrt{2}(x-y).\end{aligned}$$

iii. θ

We make the substitution $x' = x\cos\theta - y\sin\theta$ and $y' = x\sin\theta + y\cos\theta$:

$$\begin{aligned}y'^2 &= 4Cx'\\ (x\cos\theta - y\sin\theta)^2 &= 4C\,(x\sin\theta + y\cos\theta).\end{aligned}$$

2 It's a Snap

$\bullet$	I	A	B	C	D	E
I						
A			E			
B						
C						
D						
E						

Figure 1: Unfilled 3-post snap group table.

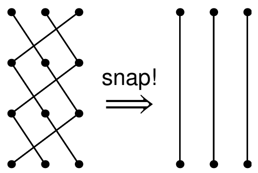


Figure 2: $E \bullet E \bullet E = I$; E has period 3.

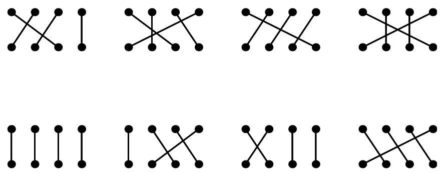


Figure 3: Some 4-post group elements.

1. **Fill out a 6×6 table like the one in Figure 1, showing the results of each of the 36 possible snaps, where $X \bullet Y$ is in X's row and Y's column. $A \bullet B = E$ is done for you.**

$\bullet$	I	A	B	C	D	E
I	I	A	B	C	D	E
A	A	I	E	D	C	B
B	B	D	I	E	A	C
C	C	E	D	I	B	A
D	D	B	C	A	E	I
E	E	C	A	B	I	D

2. **Which of the elements is the identity element K, such that $X \bullet K = K \bullet X = X$ for all X?**

The identity element is I, since $I \bullet A = A \bullet I = A$, $I \bullet B = B \bullet I = B$, and so forth.

3. **Does every element have an inverse? In other words, can you get to the identity element from every element using only one snap?**

Yes you can. The inverses are shown below.

$I \leftrightarrow I$ $\quad$ $A \leftrightarrow A$ $\quad$ $B \leftrightarrow B$ $\quad$ $C \leftrightarrow C$ $\quad$ $D \leftrightarrow E$

Note that the inverse of an element X is denoted X^{-1}.

4. **(a) Is the snap operation commutative (does $X \bullet Y = Y \bullet X$ for all X, Y)?**

No, the snap operation is not commutative. For example, $A \bullet B = E$, but $B \bullet A = D$.

(b) Is the snap operation associative (does $(X \bullet Y) \bullet Z = X \bullet (Y \bullet Z)$ for all X, Y, Z)?

Yes, the snap operation is associative. You can rationalize this as the fact that a 4×3 grid of posts is snapped to a single configuration, regardless of which middle row you remove first. This is shown in Figure 4.

5. **(a) For any elements X, Y, is there always an element Z so that $X \bullet Z = Y$?**

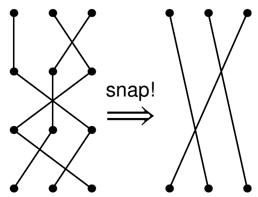


Figure 4: A 4×3 grid of posts has a unique result after the snap operation.

Yes, there is always a way to get from one element to another in one snap. You can prove this by construction. If element X connects n_1 to n_1', n_2 to n_2', and n_3 to n_3', and element Y connects m_1 to m_1', m_2 to m_2', and m_3 to m_3', then the solution Z to $X \bullet Z = Y$ connects m_1 to $n_{m_1'}$, m_2 to $n_{m_2'}$, and m_3 to $n_{m_3'}$.

That's probably a bit hard to understand, but a more clever solution uses inverses. We multiply X by X^{-1}, then by Y:

$$X \bullet X^{-1} \bullet Y = Y.$$

But since every element has an inverse, and the snap operation is associative, we have

$$\begin{aligned} X \bullet (X^{-1} \bullet Y) &= Y \\ \implies Z &= X^{-1} \bullet Y. \end{aligned}$$

In this way, we have constructed the element Z.

(b) For (a), is Z always unique?

Yes. To show this, we use a proof by contradiction. Suppose we have two solutions Z_1 and Z_2 so that $Z_1 \neq Z_2$ and

$$\begin{aligned} X \bullet Z_1 &= Y \\ X \bullet Z_2 &= Y. \end{aligned}$$

We multiply to the left by Y^{-1}. Note that since the snap operation is not commutative, we need to multiply both sides on a specific side:

$$\begin{aligned} Y^{-1} \bullet X \bullet Z_1 &= Y^{-1} \bullet Y = I \\ Y^{-1} \bullet X \bullet Z_2 &= I. \end{aligned}$$

So Z_1, Z_2 are the inverses of $Y^{-1} \bullet X$. But the inverse of an element is unique; we've showed this by listing them all out! Thus, $Z_1 = Z_2$, contradicting our assumption and proving that Z is unique in $X \bullet Z = Y$.

6. **If you constructed a 5×5 table using only five of the snap elements, the table would not describe a group, because there would be entries in the table outside of those 5. Therefore, a group must be closed under its operation. *Some* subsets of our six elements, however, do happen to be closed. Write valid group tables using exactly one, two, and three elements from the snap group. These are known as subgroups.**

Here are tables with 1, 2, and 3 elements:

$\bullet$	I
I	I

$\bullet$	I	A
I	I	A
A	A	I

$\bullet$	I	D	E
I	I	D	E
D	D	E	I
E	E	I	D

7. **What do you guess is a good definition of a mathematical group? (Hint: consider your answers to Problems 2–6.)**

(Answers may vary.)

Definition of **group**: A group G is a set of elements together with a **binary operation** that meets the following criteria:

(a) Identity: There is an element $I \in G$ such that for all $X \in G$, $X \bullet I = I \bullet X = X$.

(b) Closure: If X, Y are elements of the group, then $X \bullet Y$ is also an element of the group.

(c) Invertibility: Each element X has an inverse X^{-1} such that $X \bullet X^{-1} = X^{-1} \bullet X = I$.

(d) Associativity: For all elements X, Y, and Z, $X \bullet (Y \bullet Z) = (X \bullet Y) \bullet Z$.

8. Notice that $E \bullet E \bullet E = I$ (See Figure 2). This means that E has a period of 3 when acting upon itself. Which elements have a period of

(a) 1**?**

I is the only element with a period of 1, since $I = I$.

(b) 2**?**

A, B, and C have periods of 2, since for each $X \in A, B, C$ we have $X \bullet X = I$.

(c) 3**?**

D and E have periods of 3, since for each $Y \in D, E$ we have $Y \bullet Y \neq I$, but $Y \bullet Y \bullet Y = I$.

9. Answer the following with the one, two, and four-post snap groups S_1, S_2 and S_4. These are just the analogous groups for connections between one, two, and four posts.

(a) How many elements would there be?

S_1 has $1! = 1$ elements. S_2 has $2! = 2$ elements. S_4 has $4! = 24$ elements.

(b) Systematically draw and name them.

Figure 5: Elements of S_1.

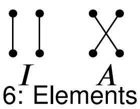


Figure 6: Elements of S_2.

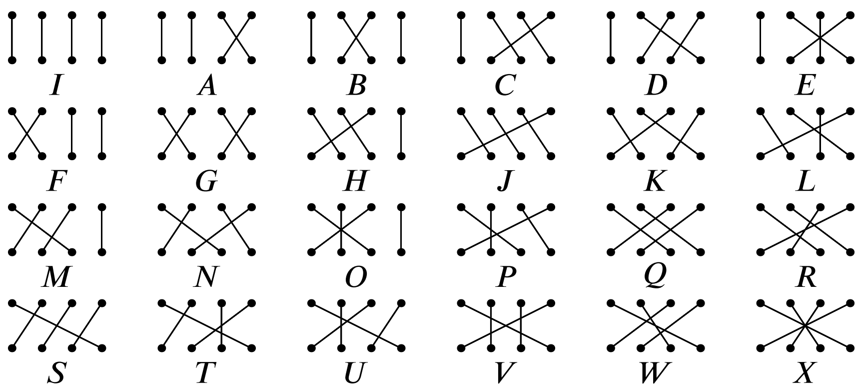


Figure 7: Elements of S_4.

(c) Make a group table of these elements. For four posts, instead of creating the massive table, give the number of elements that the table would have.

Here are group tables for S_1 and S_2.

The table for S_4 is given at the end of the section in Figure 12 for the curious.

•	I
I	I

Figure 8: Group table for S_1.

•	I	A
I	I	A
A	A	I

Figure 9: Group table for S_2.

(d) What is the relationship of your original table to this new table?

Both S_1's and S_2's tables are subgroups of the original table for S_3. In turn, S_3 is a subgroup of S_4.

10. Can you think of a shortcut to generate a snap group table without drawing every possible configuration?

(Answers may vary.)

One way to do it is to treat each element as a list of indices. For example, I is the ordered triple $(1, 2, 3)$ because it takes column 1 to 1, 2 to 2, and 3 to 3. A is $(1, 3, 2)$, because it takes 1 to 1, 2 to 3, and 3 to 2.

This makes it a bit easier to calculate, because you can simply substitute indices for each configuration rather than make a drawing. It also makes it easy to write a program to calculate; this is actually how all the tables in this answer key were generated.

11. (a) How many elements would there be in the five-post snap group?

There would be $5! = 120$ elements in S_5.

(b) How many entries would its table have?

There would be $5!^2 = 14400$ entries in S_5's table.

(c) What possible periods would its elements have? Make sure you include a period of six!

This is a more difficult question. We must ask what characteristics of an element determine its period.

If we observe the periodicity of an element with a pretty large period, say one from S_5 with a period of 6, you can see how a large period can arise. This is shown in Figure 11.

We can split up this element into two components: a component with period 3 and one with period 2. Let's call these components C_3 and C_2. After 2 steps, the C_3 has not completed one period, even though C_2. After 3 steps, C_3 has completed one period, but C_2 has gone through $\frac{3}{2}$. It takes $\mathrm{lcm}(2, 3) = 6$ steps before both components "line up!"

All elements can be split up into some number of these cyclic components, even if it doesn't look like it at first glance. For example, the element from S_8 shown in Figure 10 is actually two size 3 and size 2 components. It therefore has a period of $\mathrm{lcm}(2, 3, 3) = 6$. Note that it does *not* have a period of $2 \cdot 3 \cdot 3 = 18$.

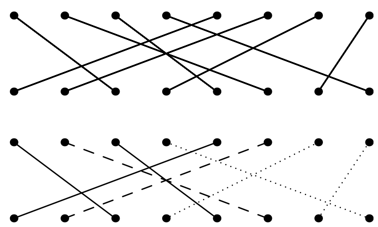


Figure 10: This element from S_8 has components of size $2, 3, 3$.

For S_5, we can split it up into components of size $1, 1, 1, 1, 1$, giving period 1; components of size $1, 1, 1, 2$, giving period 2; components of size $1, 1, 3$, giving period 3; components of size $1, 4$, giving period 4; a component of size 5, giving period 5; and component of size $1, 2, 3$, giving period 6. Thus, periods $1, 2, 3, 4, 5, 6$ are achievable.

(d) Extend your answers for Problems a through c to M posts per row.

This is rather straightforward. There are (a) $M!$ elements in the M-post snap group, and thus (b) $M!^2$ elements in the corresponding group table. The possible periods are harder to calculate, but they can be generated like so:

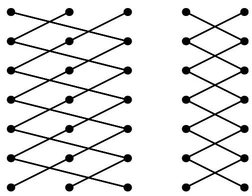


Figure 11: This element from S_5 has a period of 6.

Let integers $x_i > 0$ and $\sum_i x_i = M$. In other words, the sum of all x_i is M. Then $\operatorname{lcm}(x_1, x_2, \cdots, x_n)$ is a valid period; the least common multiple of all x_i is a possible period.

For fun: in set builder notation, we have the set of possible periods P_n for the n-post snap group as

$$P_n = \left\{ \operatorname{lcm}(x_1, x_2, \cdots, x_n) \;\middle|\; x_i \in \mathbb{Z}^+ \wedge \sum_i x_i = n \right\}.$$

The maximum such period (i.e. $\max P_n$) is actually known as Landau's function, $g(n)$.

12. A permutation of a set of things is an order in which they can be arranged. What is the relationship between the set of permutations of m things and the m-post snap group?

We can make a pretty simple correspondence between a permutation of m things and an element of the m-post snap group. If we think back to the idea of treating each element of the group as a list of indices, the correspondence is obvious. For example, I is the ordered triple $(1, 2, 3)$ because it takes column 1 to 1, 2 to 2, and 3 to 3. A is $(1, 3, 2)$, because it takes 1 to 1, 2 to 3, and 3 to 2. But each ordered triple is a permutation of $1, 2, 3$! This extends to any m.

$\bullet$	I	A	B	C	D	E	F	G	H	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X
I	I	A	B	C	D	E	F	G	H	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X
A	A	I	D	E	B	C	G	F	K	L	H	J	S	T	U	V	W	X	M	N	O	P	Q	R
B	B	C	I	A	E	D	M	N	O	P	Q	R	F	G	H	J	K	L	T	S	W	X	U	V
C	C	B	E	D	I	A	N	M	Q	R	O	P	T	S	W	X	U	V	F	G	H	J	K	L
D	D	E	A	I	C	B	S	T	U	V	W	X	G	F	K	L	H	J	N	M	Q	R	O	P
E	E	D	C	B	A	I	T	S	W	X	U	V	N	M	Q	R	O	P	G	F	K	L	H	J
F	F	G	H	J	K	L	I	A	B	C	D	E	O	P	M	N	R	Q	U	V	S	T	X	W
G	G	F	K	L	H	J	A	I	D	E	B	C	U	V	S	T	X	W	O	P	M	N	R	Q
H	H	J	F	G	L	K	O	P	M	N	R	Q	I	A	B	C	D	E	V	U	X	W	S	T
J	J	H	L	K	F	G	P	O	R	Q	M	N	V	U	X	W	S	T	I	A	B	C	D	E
K	K	L	G	F	J	H	U	V	S	T	X	W	A	I	D	E	B	C	P	O	R	Q	M	N
L	L	K	J	H	G	F	V	U	X	W	S	T	P	O	R	Q	M	N	A	I	D	E	B	C
M	M	N	O	P	Q	R	B	C	I	A	E	D	H	J	F	G	L	K	W	X	T	S	V	U
N	N	M	Q	R	O	P	C	B	E	D	I	A	W	X	T	S	V	U	H	J	F	G	L	K
O	O	P	M	N	R	Q	H	J	F	G	L	K	B	C	I	A	E	D	X	W	V	U	T	S
P	P	O	R	Q	M	N	J	H	L	K	F	G	X	W	V	U	T	S	B	C	I	A	E	D
Q	Q	R	N	M	P	O	W	X	T	S	V	U	C	B	E	D	I	A	J	H	L	K	F	G
R	R	Q	P	O	N	M	X	W	V	U	T	S	J	H	L	K	F	G	C	B	E	D	I	A
S	S	T	U	V	W	X	D	E	A	I	C	B	K	L	G	F	J	H	Q	R	N	M	P	O
T	T	S	W	X	U	V	E	D	C	B	A	I	Q	R	N	M	P	O	K	L	G	F	J	H
U	U	V	S	T	X	W	K	L	G	F	J	H	D	E	A	I	C	B	R	Q	P	O	N	M
V	V	U	X	W	S	T	L	K	J	H	G	F	R	Q	P	O	N	M	D	E	A	I	C	B
W	W	X	T	S	V	U	Q	R	N	M	P	O	E	D	C	B	A	I	L	K	J	H	G	F
X	X	W	V	U	T	S	R	Q	P	O	N	M	L	K	J	H	G	F	E	D	C	B	A	I

Figure 12: Group table for S_4.

3 From Snaps to Flips

1. **Is the list of six operations complete? (Are there any other isometries of the equilateral triangle that preserve its shape and location?)**

There are no other isometries for this triangle; our list of operations is complete. To see why, note that the vertices must exchange places. At most there is $3! = 6$ ways to do this, so we have already achieved the maximum possible number of isometries.

$\cdot$	I	A	B	C	D	E
I						
A					B	
B						
C						
D						
E						

Figure 1: Unfilled D_3 group table.

2. **As with the snap group, we can make a group table for the dihedral group. Fill out a table like the one in Figure 1 in your notebook. Like the snap group table, the top row indicates what operation is done first and the left column indicates what's done second. In other words, XY is in the X's' row and Y's column. $AD = B$ is done for you.**

The completed table is shown in Figure 2.

$\cdot$	I	A	B	C	D	E
I	I	A	B	C	D	E
A	A	I	D	E	B	C
B	B	E	I	D	C	A
C	C	D	E	I	A	B
D	D	C	A	B	E	I
E	E	B	C	A	I	D

Figure 2: Completed D_3 group table.

$\bullet$	I	A	B	C	D	E
I	I	A	B	C	D	E
A	A	I	E	D	C	B
B	B	D	I	E	A	C
C	C	E	D	I	B	A
D	D	B	C	A	E	I
E	E	C	A	B	I	D

Figure 3: Completed S_3 group table from the last chapter.

3. **What is the relationship between the tables for the snap group S_3 and the dihedral group D_3?**

D_3's table is S_3's table flipped over the top-left–bottom-right diagonal, and vice versa. Contrast D_3 from Figure 2 to S_3 in Figure 3. If these were matrices, one would be the transpose of the other: we'll get to that later.

4. **Check your understanding by defining isomorphic in your own words.**

(Answers may vary.)

Isomorphic means that two groups have the same structure. Isomorphic means that there is a correspondence between the elements of two groups so that the correspondence preserves the order. In the language of abstract algebra, an isomorphism between groups A and B exists if there is a homomorphism from A to B and from B to A.

$\cdot$	I	D	E
I	I	D	E
D	D	E	I
E	E	I	D

5. (a) **Make a table for only the rotations of D_3, a subgroup of D_3.**

The table is shown above. Note that the identity element I is a rotation of 0.

Interestingly, this subgroup is a commutative group, also known as an abelian group.

(b) Which subgroup of the snap group S_3 is isomorphic to the subgroup in (a)?

The same elements (nominally) make the same subgroup:

$\cdot$	I	D	E
I	I	D	E
D	D	E	I
E	E	I	D

6. What shape's dihedral group is isomorphic to

(a) the two post snap group S_2?

The dihedral group of a line segment is isomorphic to S_2. After all, you can only reflect it over its midpoint, which is the other element of S_2 besides the identity. We can also think of this as permuting the two endpoints or vertices of a line segment.

(b) the one post snap group S_1?

The dihedral group of a point is isomorphic to S_1, because the only element is the identity element. This is permuting the one vertex of a point.

(c) the four post snap group S_4?

For this question we need to think 3 dimensions. There are four vertices to permute, but we can't do that on a square since diagonal points will remain on diagonals, as shown in Figure 4.

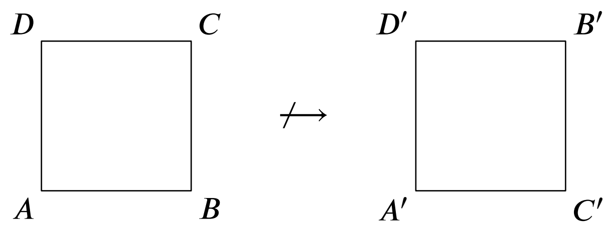


Figure 4: At right is a valid permutation of the vertices, but not a valid isometry of the square.

Instead, we choose the regular tetrahedron, so that there are no "diagonals"; every permutation is achievable. Note that rotations and reflections are now in 3 dimensional space, which is a bit difficult to visualize. A sample rotation is depicted in Figure 5.

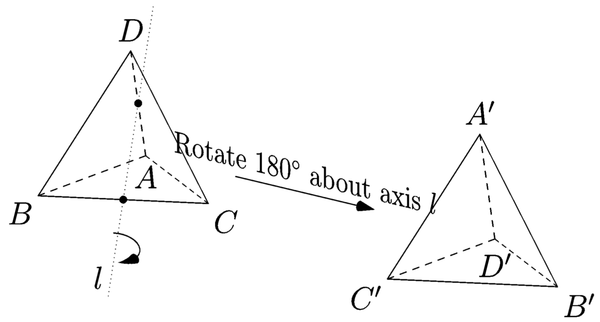


Figure 5: A rotation of the tetrahedron (orthographic view).

(d) the five post snap group S_5?

This is isomorphic to the dihedral group of the 4-dimensional equivalent of the tetrahedron, also known as the regular 4-simplex. A projection is shown in Figure 6, but it cannot be faithfully represented on this paper.

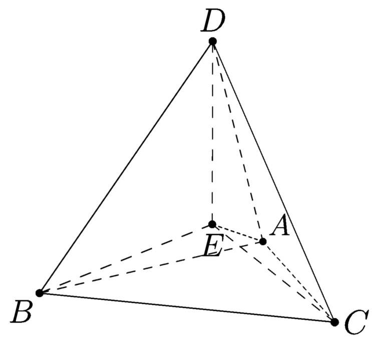


Figure 6: A 3D projection of the regular 4-simplex. In a true realization, every line segment here would be the same length.

7. **Find a combination of A and D that yields C.**

8. **We call A and D generators of the group because every element of the group is expressible as some combination of As and Ds. For convenience, let's call A "f" since it's a flip, and call D "r" meaning a $120\,\text{deg}$ rotation counterclockwise. Then, for example, fr^2 is a rotation of $2 \cdot 120\,\text{deg} = 240\,\text{deg}$, followed by a flip across the A axis, equivalent to our original C (see Figure ??). Make a new table using I, r, r^2, f, fr, and fr^2 as elements, like the one in Figure 7. *Note that the element order is different!***

$\cdot$	I	r	r^2	f	fr	fr^2
I						
r				fr^2		
r^2						
f						
fr						
fr^2						

Figure 7: Unfilled alternate D_3 table.

The filled table is shown in Figure 8 below.

$\cdot$	I	r	r^2	f	fr	fr^2
I	I	r	r^2	f	fr	fr^2
r	r	r^2	I	fr^2	f	fr
r^2	r^2	I	r	fr	fr^2	f
f	f	fr	fr^2	I	r	r^2
fr	fr	fr^2	f	r^2	I	r
fr^2	fr^2	f	fr	r	r^2	I

Figure 8: Completed alternate D_3 table.

Note that $I = I$, $A = f$, $B = fr$, $C = fr^2$, $D = r$, and $E = r^2$.

9. **What other pairs of elements could you have used to generate the table?**

You could also use any of the following pairs: $\{A, E\}$, $\{B, D\}$, $\{B, E\}$, $\{C, D\}$, $\{C, E\}$, $\{A, B\}$, $\{B, C\}$, $\{A, C\}$. In essence, you can generate it with any rotation element and any reflection element, or with any two reflection elements.

10. **Notice the 3×3 table of a group you've already described in the top-left corner of your table. What is it, and what are the two possible generators of this three-element group?**

This is the cyclic group of order 3, C_3, also known as the rotation group of the equilateral triangle. The two possible generators are r and r^2.

11. Explain why each element of the dihedral group D_3 has the period it has.

I has a period of 1 because it is the identity. A, B, C have periods of 2 because they are reflections, so they are their own inverse transformation. D and E are rotations of a multiple of $1/3$ of a turn. Since 3 is a prime, they take 3 iterations to resolve, and thus have period 3.

12. Some pairs of elements of the dihedral group are two-element subgroups. Which pairs are they?

These would be the pairs I, A, I, B, and I, C, since $A \cdot A = B \cdot B = C \cdot C = I$ so the subgroup is closed. These are shown in Figure 9.

$\cdot$	I	A
I	I	A
A	A	I

$\cdot$	I	B
I	I	B
B	B	I

$\cdot$	I	C
I	I	C
C	C	I

Figure 9: The three two-element subgroups.

13. One of the elements forms a one-element subgroup. Which is it?

The element I forms the so-called trivial group, or the only group of order 1; this is shown in Figure 10. It is not very interesting.

$\cdot$	I
I	I

Figure 10: The trivial group.

14. The addition of two numbers is a binary operation, while the addition of three numbers is not. In logic, $\wedge$ (and) and $\vee$ (or) are binary operations, but $\neg$ (not) is not. Define binary operation in your own words, and name some other binary operations.

(Answers may vary.)
A binary operation is an operation with two arguments.
Some binary operations:

1. multiplication
2. exponentiation
3. addition
4. subtraction
5. division
6. modulo operator
7. bitwise OR
8. bitwise AND
9. snap operation ($\bullet$)
10. function convolution

15. In your original dihedral group table, what is

(a) the identity element?

The identity element is I.

(b) the inverse of A?

The inverse of A is also A, since it is a reflection.

(c) the inverse of E?

The inverse of E is D, since $240° + 120° \equiv 0°$.

4 Rotation and Reflection Groups

1. **Notice that the original dihedral group had twice as many elements as the rotation group. Why?**

(Answers may vary.)

There are a couple ways to think about this, but an intuitive way is to consider a "mirror world" of reflection and the "normal world" where the orientation in normal. Here, orientation is not absolute orientation, but the difference between clockwise and counterclockwise. For chemistry nerds, it is like chirality. Rotation preserves orientation, but reflection does not. Instead, it takes us between these two "worlds." Thus, it allows twice the number of elements.

2. **Make and justify a conjecture extending this observation to the dihedral groups of other shapes like rectangles, squares, and hexagons, as well as the symmetry group of the cube.**

(Answers may vary.)

Conjecture: The dihedral groups of a shape has twice the order of its rotation group.

Informal Justification: A shape can be flipped or not, and it can have whatever rotational isometries applied to it whether it's flipped or not. Thus, the dihedral group allows for twice the number of elements as the rotation group.

3. **Let r be a $180°$ rotation, x be a reflection over the x-axis, and y be a reflection over the y-axis. Write a table for the dihedral group of the rectangle, recalling that the allowed isometries are reflections and rotations. How does this table differ from the dihedral group of the equilateral triangle?**

$\cdot$	I	r	x	y
I	I	r	x	y
r	r	I	y	x
x	x	y	I	r
y	y	x	r	I

The table is shown above. The four elements are shown acting on a rectangle with "P" painted on it in Figure 1 to show the transformation a bit better.

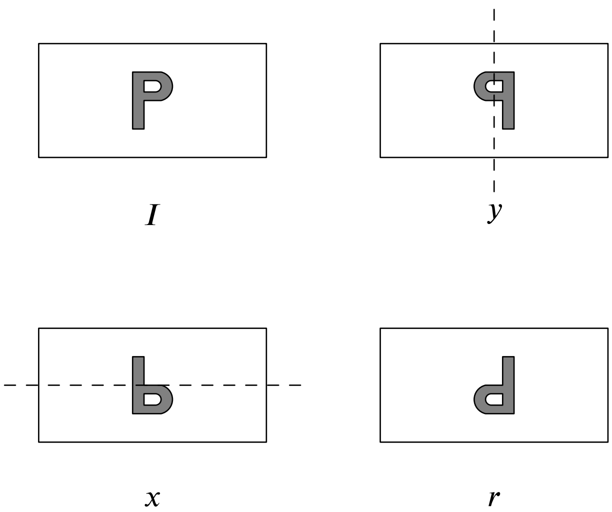


Figure 1: A rectangle AMBULATES and FLIPS around.

This differs from the dihedral group of the equilateral triangle, D_3, in several ways. The most obvious is that there are only 4 elements. Also, all elements besides I in this group have a period of 2, while D_3 has two elements with a period of 3.

4. **Write a table for the rotation group of the square, with 4 elements and 16 entries. Compare this table to Problem 3.**

$\cdot$	I	r	r^2	r^3
I	I	r	r^2	r^3
r	r	r^2	r^3	I
r^2	r^2	r^3	I	r
r^3	r^3	I	r	r^2

The elements are $I = r_0$, $r = r_{90}$, $r^2 = r_{180}$, and $r^3 = r_{270}$. The table is shown above.

While this has the same order as the rectangle's dihedral group, it has a different structure. There are two elements with period 4 (r, r^3) and one element with period 2 (r^2).

For each of the following problems, find the following:

(a) The number of elements; this is known as the **order**. More formally known as **cardinality**

(b) If order < 10, name the set of elements; otherwise, explain how you know the order

(c) A smallest possible **generating set**; in other words, a list of elements which generate a group[1]

(d) Whether the group is **commutative**; in other words, whether its operation $\cdot$ satisfies $X \cdot Y = Y \cdot X$ for all X, Y

5. Rectangle under rotation

(a) Number of elements

This group has two elements, the identity and the rotation of $180°$.

(b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

As stated, they are the identity I and the rotation r of $180°$, as shown in Figure 2.

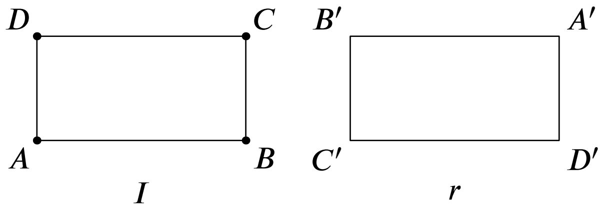


Figure 2: Rectangle under rotation.

(c) A smallest possible generating set

The smallest possible generating set is the singleton $\{r\}$.

(d) Whether the group is commutative

The group is commutative, since it's only comprised of rotations, which commute.

6. Rectangle under reflection

We already considered this in Problem 3.

(a) Number of elements

There are 4 elements in this group.

(b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

The elements are the identity I, rotation r by $180°$, reflection x over the x axis, and reflection y over the y axis.

(c) A smallest possible generating set

[1] There may be multiple generating sets of the same size.

(Answers may vary.)

$\{r, x\}$, $\{r, y\}$, and $\{x, y\}$ all generate the group. No single element, however, can generate the group.

(d) Whether the group is commutative

This group is commutative.

7. Square under rotation

Again, we have considered this group before.

(a) Number of elements

There are 4 elements.

(b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

The elements are rotations $I = r_0$, $r = r_{90}$, $r^2 = r_{180}$, and $r^3 = r_{270}$.

(c) A smallest possible generating set

(Answers may vary.)

Both $\{r\}$ and $\{r^3\}$ generate the group, because $1, 3$ are coprime to 4.

(d) Whether the group is commutative

The group is commutative, since it consists of all rotations.

8. Square under reflection

(a) Number of elements

There are 8 elements in this group. We can quickly see this by noting that it is the dihedral group of the square, which has twice the order of the rotation group of the square. We just found that had 4 elements, and $2 \cdot 4 = 8$.

(b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

The elements are as follows:

Rotations $I = r_0$, $r = r_{90}$, $r^2 = r_{180}$, and $r^3 = r_{270}$; reflections $f =$ flip over the x-axis, $fr = r$ then f, fr^2 and fr^3.

Recall that rotations can be generated by a sequence of two reflections.

Each of these elements is shown in Figure 3.

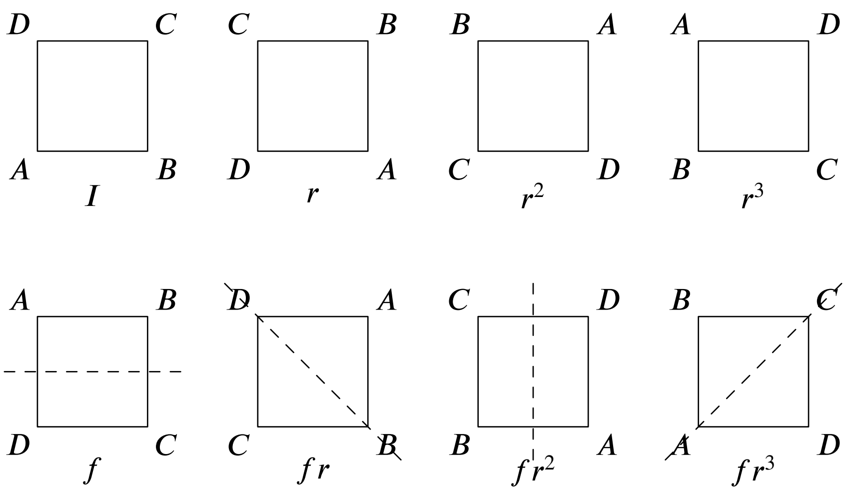


Figure 3: Reflections of a square.

(c) A smallest possible generating set

(Answers may vary.)

Any pair of a rotation and flip will generate the set, except for $\{r^2, fr^2\}$ and $\{r^2, f\}$; these will produce the rectangle group instead. Any pair of two flips, except for $\{f, fr^2\}$, will also work. As an example of both of these categories, both $\{r^2, fr^3\}$ and $\{f, fr\}$ will generate the group.

(d) Whether the group is commutative

This group is not commutative. For example, $fr = fr$, but $rf = fr^3$.

9. Square prism under rotation

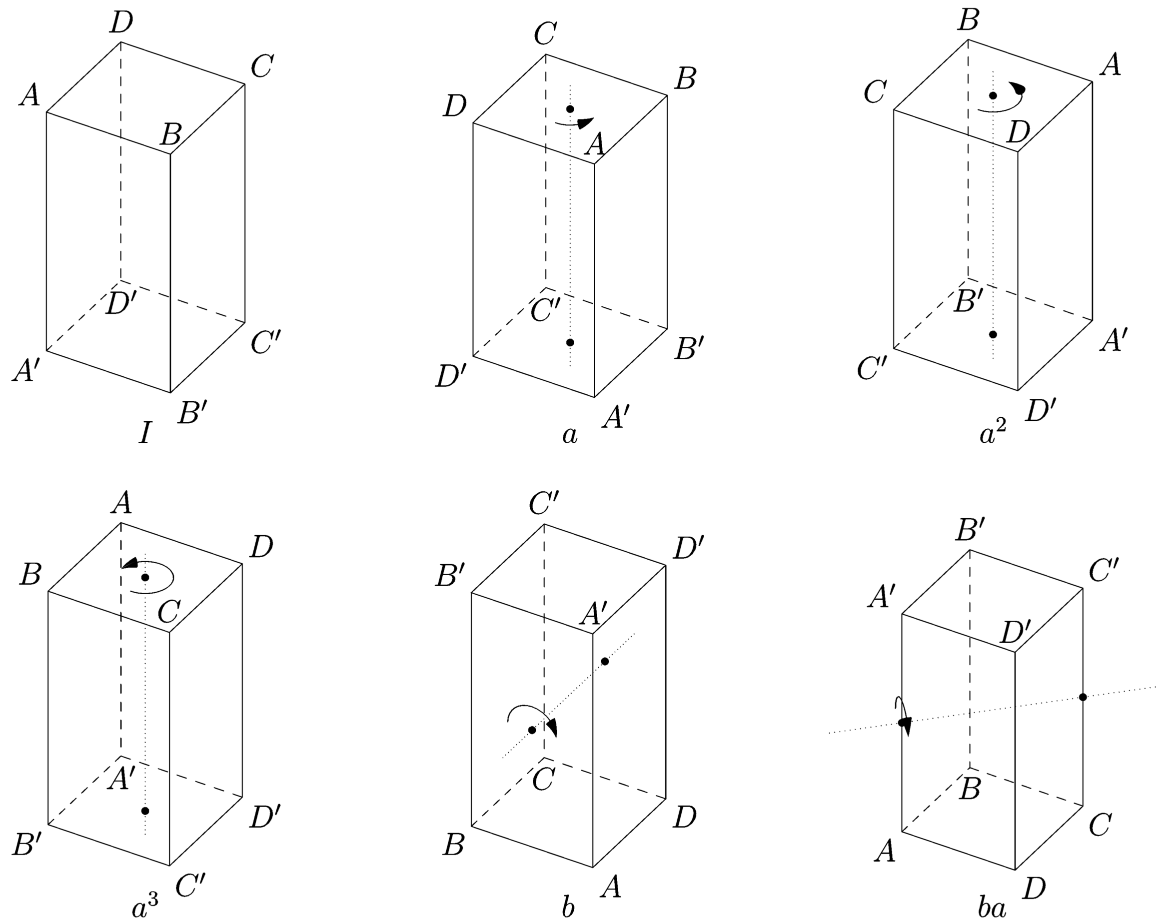
This group is isomorphic to the dihedral group of the square in Problem 8.

(a) Number of elements

This is a bit more difficult than the previous questions, because we need to understand what elements are possible. We can rotate the prism about its central axis, which an action analogous to just rotating a square: 4 elements. But we can also rotate the prism $180°$ on a axis through the middle (pictures are shown in the next subpart). This switches the top square face with the bottom face, giving another 4 elements. In total, we have 8 elements.

(b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

The set of elements are shown in Figure 4 below. Let a be a rotation of $90°$ counterclockwise—as viewed from the top—around the central axis, going through the centers of both square faces; let b be a rotation of $180°$ around an axis going through the centers of faces $\square ABB'A'$ and $\square DCC'D'$.



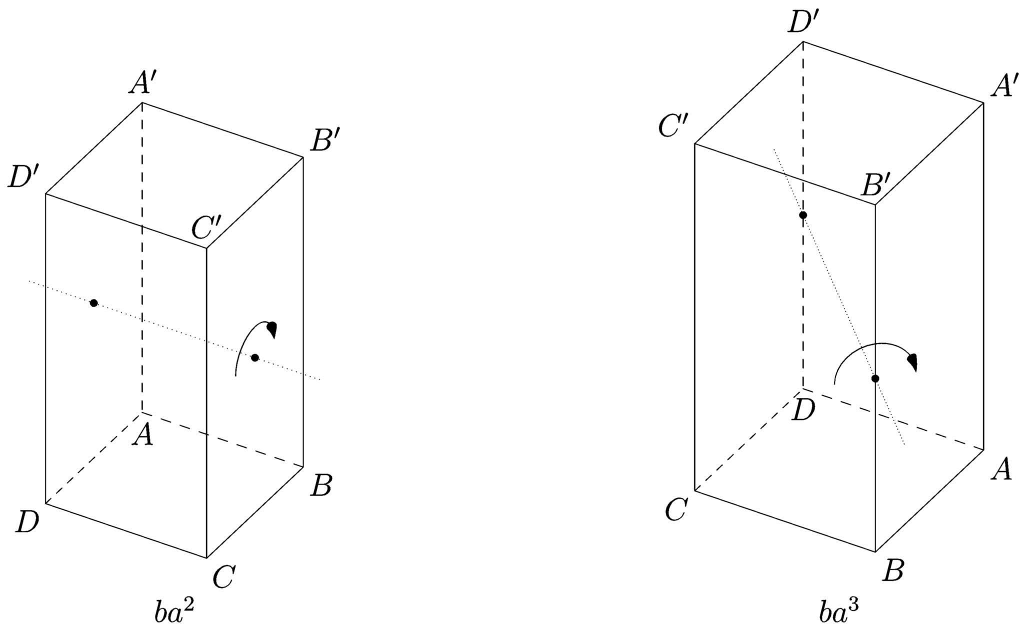


Figure 4: The elements of the rotation group of the rectangular prism.

(c) A smallest possible generating set

(Answers may vary.) The elements with b in their name are equivalent to the reflections in the dihedral group of the square. Thus, we need a "reflection" ba^n and a rotation a^m, or two separate reflections. All such pairs work except for $\{a^2, ba^2\}$, $\{a^2, b\}$ and $\{b, ba^2\}$. An example from each category: $\{a, b\}$, $\{b, ba\}$.

(d) Whether the group is commutative

This group is not commutative. For example, $ba = ba$, but $ab = ba^3$.

10. Square prism under reflection

(a) Number of elements

If the previous group—the rotation group of the square prism—had 8 elements, then this group should have 16 elements.

(b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

We know the order because the previous group has 8 elements, and dihedral groups have twice the number of elements of the rotation group, this group has 16 elements.

(c) A smallest possible generating set

(Answers may vary significantly.)

Since we could generate the previous group with (most) pairs of $\{ba^n, a^m\}$, or (most) pairs of $\{a^n, a^m\}$, we could just add another element c which is a true geometric reflection about, say, the midplane P between $\square DCD'C'$ and $\square ABB'A$ as shown in Figure 5.

Thus, $\{a, b, c\}$ can generate the group. You can prove that two generators are impossible, but the proof either requires making the group table or some more sophisticated abstract algebra. I will give the latter for those who are well-versed in group theory already, but it will probably be inaccessible to most.

The rotation group generated by $\{a, b\}$ is D_4. Define a new element d which is the reflection through the midplane P between the two square faces.[2] This is crudely shown in Figure 6; I couldn't be bothered to make a nicer figure. Then the reflection group generated by $\{d\}$ is Z_2. Furthermore, the operation sets $\{a, b\}$ and $\{d\}$ are separable, in that $a^x b^y d = d a^x b^y$ [3]. Thus, the group G described in this problem is (isomorphic to) the direct product:

$$G \cong D_4 \times Z_2.$$

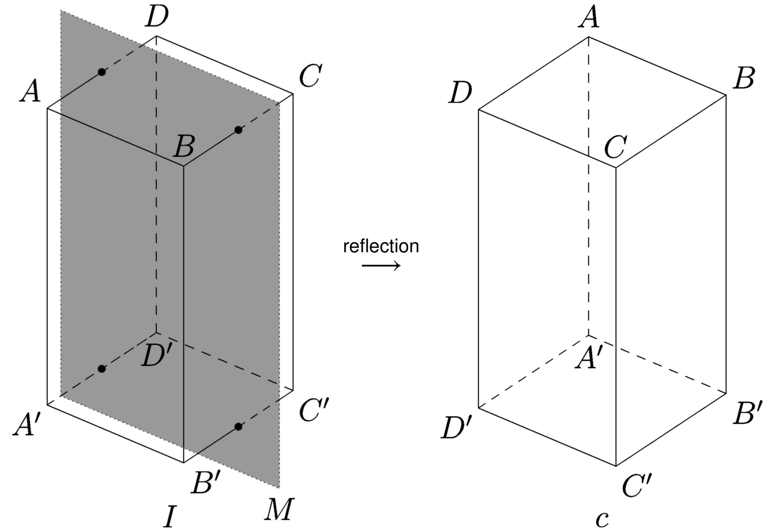


Figure 5: 3D reflection over the midplane M is c.

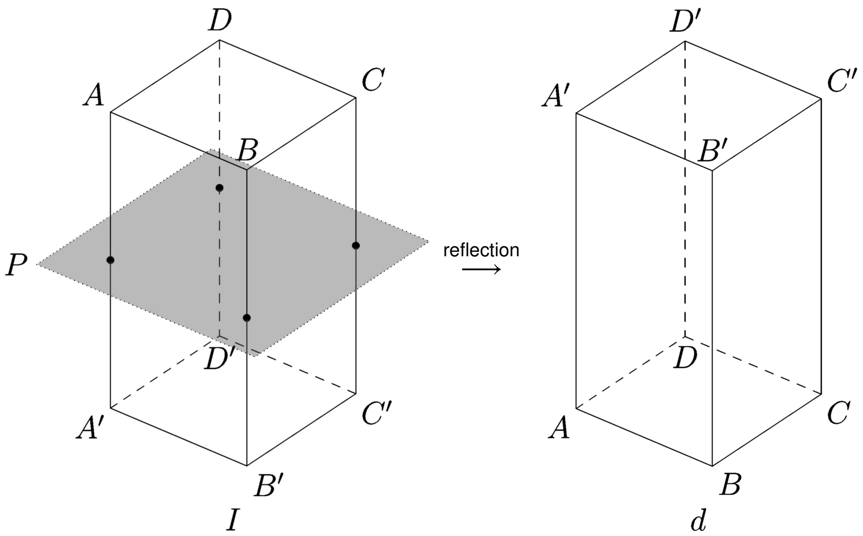


Figure 6: d is the reflection through midplane P.

We wish to show that $Z_2 \times Z_2 \times Z_2$ is a quotient of this group. That is, we wish to find a normal subgroup N such that

$$G/N = Z_2 \times Z_2 \times Z_2.$$

If this is true, then the minimal generating set of G has at least cardinality 3. All that remains is to find N and G/N.

It suffices to show that $Z_2 \times Z_2 \triangleleft D_4$, since then $Z_2 \times Z_2 \times Z_2 \triangleleft D_4 \times Z_2$. We have $|Z_2 \times Z_2| = 2^2 = 4$, so we want $|D_4/N| = 4$. We know $|D_4| = 8$, so by Lagrange's theorem, $|N| = 2$.
A normal subgroup of D_4 is $N = \{1, a^2\}$. It is normal because for $x \in \{0, 1, 2, 3\}$ and $y \in \{0, 1\}$:

[2] For the curious, $d = cba^2$.
[3] This can be shown concretely by simply showing geometrically that $ad = da$ and $bd = db$.

$$
\begin{aligned}
(b^x a^y)a^2(b^x a^y)^{-1} &= (b^x a^y)a^2(a^{-y}b^{-x}) \\
&= b^x a^{2+y-y} b^{-x} \\
&= b^x a^2 b^{-x} \\
&= b^x b^{-x} a^2 \\
&= a^2 \in \{1, a^2\}.
\end{aligned}
$$

The corresponding quotient group is

$$D_4/N = \{\{1, a^2\}, \{a, a^3\}, \{b, ba^2\}, \{ba, ba^3\}\}.$$

We have the isomorphism $\{b^x a^y, b^x a^{y+2}\} \leftrightarrow (x, y)$ under the operation of element-wise addition modulo 2. After all,

$$\{b^{x_1} a^{y_1}, b^{x_1} a^{y_1+2}\} \cdot \{b^{x_2} a^{y_2}, b^{x_1} a^{y_2+2}\} = \{b^{x_1+x_2} a^{y_1+y_2}, b^{x_1+x_2} a^{y_1+y_2+2}\}.$$

Therefore,

$$D_4/N \cong Z_2 \times Z_2,$$

so

$$Z_2 \times Z_2 \times Z_2 \lhd D_4 \times Z_2 = G.$$

Since the minimal generating set of $Z_2 \times Z_2 \times Z_2$ is 3, G's generating set is at least 3. But we've already found the set $\{a, b, c\}$ which generates G![4] Thus, it is minimal.

(d) Whether the group is commutative

As we found in the previous problem, the rotation group of the square prism is not commutative, and since that's a subgroup of this group, this group certainly isn't commutative.

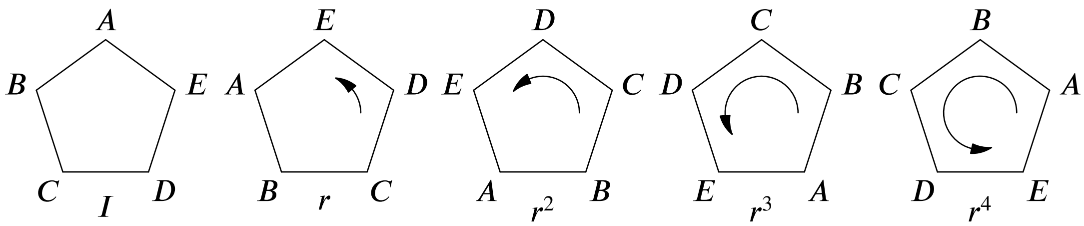
11. Regular pentagon under rotation

(a) Number of elements

This is just the cyclic group of order 5, so there are 5 elements.

(b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

The elements are rotations of $I = r_0$, $r = r_{72}$, $r^2 = r_{144}$, $r^3 = r_{216}$, $r^4 = r_{288}$. They are shown below.



Pentagons should always wear helmets, lest they want to damage their vertices.

(c) A smallest possible generating set

Any rotation by itself $\{r^n\}$ works, since 5 is a prime.

(d) Whether the group is commutative

The group is indeed commutative, since all operations are rotations.

12. Regular pentagon under reflection

[4] $\{a, b, d\}$ also generates G.

(a) Number of elements

This is the dihedral group of the pentagon, which has $2 \cdots 5 = 10$ elements.

(b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

We know the order because it should have twice the number of elements as the rotation group, which has 5 elements, giving 10 elements total.

(c) A smallest possible generating set

We can either do a rotation and a reflection or two reflections. Since 5 is prime, all pairs work (unlike for the square). Let f is a flip over the vertical axis. Examples of each are $\{r, f\}$ and $\{f, fr\}$.

(d) Whether the group is commutative

The group is not commutative. For example, $fr = fr$, but $rf = fr^4$.

13. Regular pentagonal prism under rotation

This is isomorphic to the dihedral group of the pentagon, which is Problem 12. The reason is the same as for Problem 9's dependence on 8, thus I will not explain it.

14. Regular pentagonal prism under reflection

This is akin to Problem 10.

(a) Number of elements

$2 \cdot 10 = 20$.

(b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

We know the order because the rotation group of the pentagonal prism has 10 elements, so its dihedral group has 20 elements.

(c) A smallest possible generating set

If a is a rotation of $72°$ about the central axis, b is a rotation of $180°$ about a horizontal axis, and d is a reflection across the midplane between the two pentagonal faces, then $\{a, b, d\}$ generates the set, since $\{a, b\}$ generates all rotations and d turns them into their mirror images. But this isn't the right answer.

Are there any smaller generating sets? The previous trick asserting no using more advanced abstract algebra doesn't actually work.[5] We have $ad = da$ and $bd = db$ (you can verify this geometrically). So to have a two element subgroup we likely need something like $a^n d$ and ba^m for some integers n, m, so that we can potentially generate a, b and d.

Let's try ad and b. Taking successive powers of ad, we get

$$\begin{aligned}
ad &= ad \\
(ad)^2 &= a^2 \\
(ad)^3 &= a^3 d \\
(ad)^4 &= a^4 \\
(ad)^5 &= a^5 d = d \\
(ad)^6 &= a
\end{aligned}$$

We've just generated d and a from ad alone! Since we have b already, we have created $\{a, b, d\}$ from $\{ad, b\}$. Thus, the smallest generating set has size 2. (We can't have size 1 because then the group would be cyclic and thus commutative, which this group certainly isn't.)

This is a hard problem. Don't worry if you didn't get it.

[5] If you understand it, it's because $Z_2 \times Z_2 \times Z_2$ isn't a quotient of this group, since this group $D_5 \times Z_2$ has order 20 which is not divisible by 8.

(d) Whether the group is commutative

The dihedral group of the pentagon is a subgroup of this group, and is not commutative, so this group is not commutative.

15. Regular pentagonal pyramid under rotation

This is just isomorphic to the rotation group of the pentagon, or Problem 11.

16. Regular pentagonal pyramid under reflection

This is just isomorphic to the reflection group of the pentagon, or Problem 12.

17. Regular tetrahedron (triangular pyramid) under rotation

This is isomorphic to a subgroup of S_4, the snap group of order 24.

(a) Number of elements

The snap group includes reflections, but this does not: thus, this group has $\frac{4!}{2} = 12$ elements.

(b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

We know the order because this is the rotation group of a tetrahedron, and the reflection group of a tetrahedron has 24 elements, so this must have half that.

(c) A smallest possible generating set

Another difficult problem!

Let's figure out where the rotation axes actually are. There are 4 axes going through a vertex—let's call these *vertex* axes v_i. There are also 3 axes going through the midpoints of opposite edges: let's call these *edge* axes e_i. These axes are enumerated and shown in Figure 7.

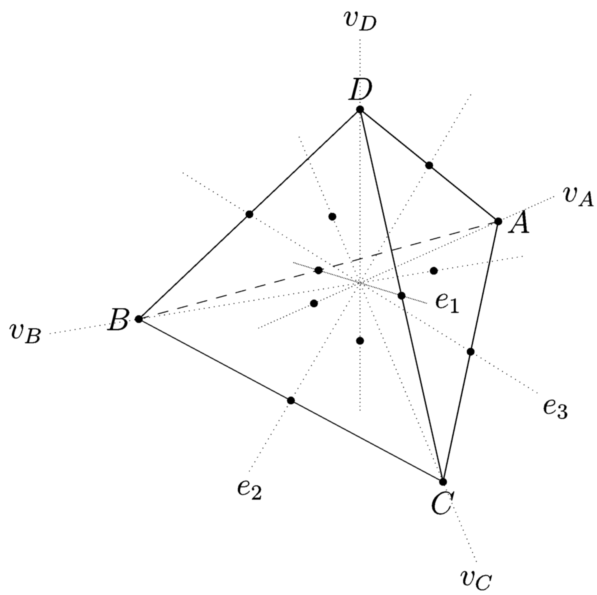


Figure 7: Regular tetrahedron's succulent rotation axes.

We can rotate by $120°$ or $240°$ (counterclockwise as viewed from the vertex) about any v_i, but only by $180°$ about any e_i. Along with the identity, this gives all $2 \cdot 4 + 3 + 1 = 12$ elements.

To make manipulating these elements easier, treat them as moving elements in a list. We name this list with indices as shown in Figure 8. Thus, the identity element I is (A, B, C, D). A rotation of $240°$ around v_A swaps vertices in positions $(3 \quad 4)$ then $(2 \quad 3)$, so $v_A = (A, D, B, C)$ as shown in Figure 9.

If we take a look at an edge rotation, say e_1, you will see it also swaps two vertices: in this case, $(3 \quad 4)$ and $(1 \quad 2)$. In general, any edge rotation or vertex rotation will swap two vertices—you can see this by plain symmetry or if you want, working it out for each rotation.

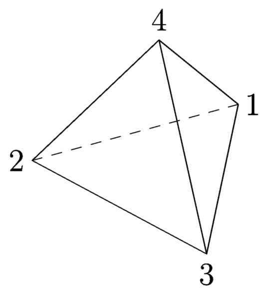


Figure 8: Regular tetrahedron's indices.

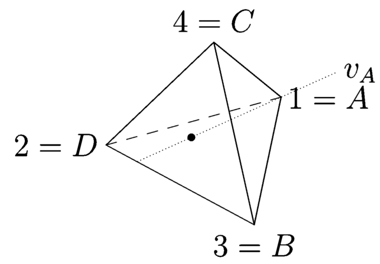


Figure 9: $v_A = (A, D, B, C)$.

We now have a more abstract representation of this group: namely, it is the group of *even permutations* of (A, B, C, D). Even permutations are permutations made by swapping two pairs at a time. For example, (B, A, D, C) is even, but (A, B, D, C) is not. The group operation is composing two permutations by chaining them together. Note that the identity, (A, B, C, D) is considered even, just as 0 is considered even.

One element is clearly not enough, because this group is not cyclic. Can we do it in two elements though?

Consider two vertex rotations, which cycle (without loss of generality) the first three vertices and the last three vertices. That is, $a = (3, 1, 2, 4)$ and $b = (1, 4, 2, 3)$. Can we get every even permutation with combinations of a and b? Let's try list them out:

$$\begin{aligned}
a &= (3,1,2,4), & a^2 &= (2,3,1,4), & a^3 = I &= (1,2,3,4), & b &= (1,4,2,3),\\
b^2 &= (1,3,4,2), & \cancel{b^3} &= I = a^3, & ab &= (2,1,4,3), & ab^2 &= (4,1,3,2),\\
a^2b &= (4,2,1,3), & a^2b^2 &= (3,4,1,2), & \cancel{ba} &= a^2b^2, & b^2a &= (3,2,4,1),\\
ba^2 &= (2,4,3,1), & \cancel{b^2a^2} &= ab, & \cancel{bab} &= a^2, & \cancel{b^2ab} &= ba^2,\\
\cancel{bab^2} &= a^2b, & b^2ab^2 &= (4,3,2,1) = ba^2b
\end{aligned}$$

We have successfully generated all 12 elements with the set $\{a, b\}$. Thus, a two element generating set is sufficient! Interestingly, this means you can turn a tetrahedron however you want by holding it at two corners and twisting it with each.

For the curious, this group is known as the alternating group A_4.

(d) Whether the group is commutative

The group is clearly not commutative, since $ab \neq ba$.

18. Regular tetrahedron under reflection

This is just the snap group of order 4, S_4.

(a) Number of elements

As we found in the first problem, S_4 has $4! = 24$ elements.

(b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

As we found in the first problem, S_4 has $4! = 24$ elements.

(c) A smallest possible generating set

This is tricky.

The obvious thing to do is keep $\{a, b\}$ from the previous problem and add some reflection c. Then $\{a, b, c\}$ has all 24 elements, since $\{a, b\}$ makes 12 elements and c makes a copy of each "in the mirror world." This is not, however, the right answer.

A generating of 2 elements is actually possible! There are several ways to see this, but I find a permutation argument easiest to follow.

S_4 is not just the reflection group of the tetrahedron, but also the group of all permutations of $(1, 2, 3, 4)$. Consider the permutation $j = (4, 1, 2, 3)$, which cycles all the elements, and the permutation $k = (2, 1, 3, 4)$, which swaps the first elements. Then

$$j = (4,1,2,3), \quad j^2 = (3,4,1,2), \quad j^3 = (2,3,4,1), \quad j^4 = I = (1,2,3,4).$$

We can flip any two adjacent elements (as well as the first and last elements) by doing the following:

1. Cycle using powers of j until the two elements in question are the first two elements.
2. Swap them with an application of k.
3. Cycle back to the starting position with powers of j.

In more mathematical terms, we can swap indices i and $i+1$, where $1 \le i \le 3$, with the following element.

$$j^{i-1} k j^{5-i}.$$

Intuitively, if you can swap any two adjacent elements, you can make any permutations. The proof of this is pretty standard and outside the scope of this answer key.

For fun, let's see what the elements j and k actually are, operating on the tetrahedron.

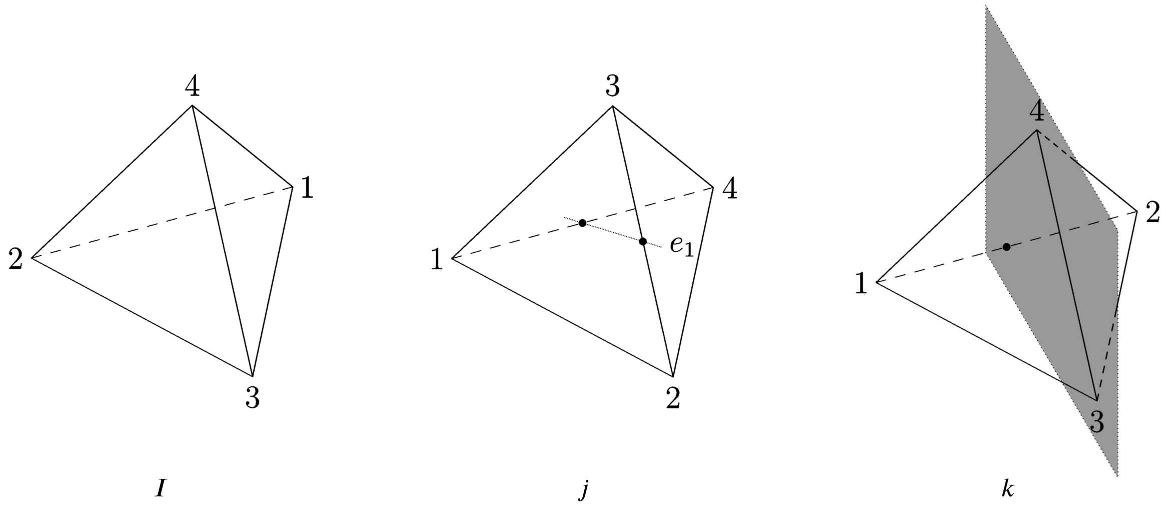


Figure 10: The two elements j and k generate the full symmetry group of the tetrahedron.

Thus, the true minimal generating set is $\{j, k\}$ as described.

(d) Whether the group is commutative

This group is certainly not commutative, since the previous group from Problem 17 was not commutative and is a subgroup of this group.

19. Cube under rotation

There are a couple ways to analyze this. My favorite one is to choose a face to make the top face, which can be done in 6 ways, then choose which rotation that face should be in, which can be done in 4 ways.

(a) Number of elements

Since we choose a front face in 6 ways, and its rotations in 4 ways, we have $6 \cdot 4 = 24$ total rotations.

(b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

The order is found above.

(c) A smallest possible generating set

This is tough until you make a key observation. If we label space-diagonally opposite vertices (that is, vertices which don't share a face) with the same number, as shown in Figure 11, then we can easily enumerate valid rotations.

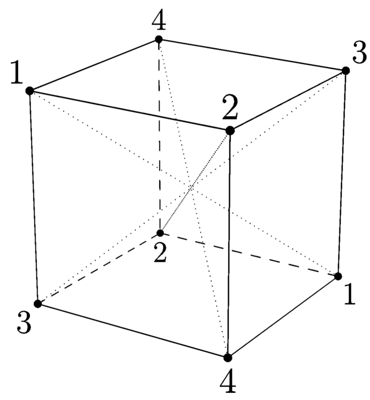


Figure 11: Marking opposite pairs of vertices.

The front face starts off saying "1,2,3,4." I claim that the $4!$ permutations of these four labels on the front face yields every rotation, and only rotations. This can be manually verified, but the higher-level argument isn't too bad.

First, note that you will always see the numbers "1,2,3,4" in *some order* on the front face; you cannot see two of one number because all numbers are placed on diagonals of each other, and never share a side.

Second, note that the list of four numbers on the front face uniquely determines the other labels, since each has exactly one pair on the back face. For example, if there is a 3 in the closest corner to the camera, then there *must* be a 3 in the furthest corner of the camera.

Third, we demonstrate that the permutation of labels can always be represented as a rotation. There are six fundamentally different types of label squares under rotation *rotation*, as shown in Figure 12. But all appear somewhere as a face on the cube, as shown in Figure 13.

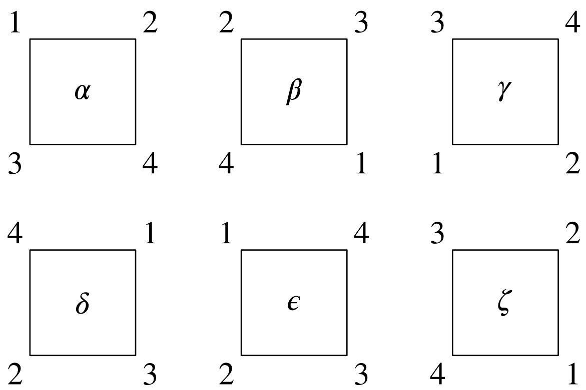


Figure 12: The six different labelings of a square.

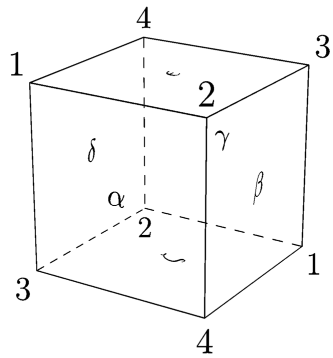


Figure 13: The six different labelings indeed appear on the cube!

We have demonstrated that every permutation of the front face labels 1. creates a unique orientation of the cube and 2. that orientation is a rotations. Since there are 24 unique permutations and 24 unique rotations, every rotation has exactly one corresponding permutation and vice versa. We can now construct an isomorphism! The set of label permutations under the operation of composing permutations (as we did with the tetrahedron) and the set of rotations under the operation of composing rotations are *isomorphic*. In symbols, $S_4 \cong G$, our group.

So the group of rotations of a cube is actually S_4, the permutation group of 4 elements. I find this incredible.

Back to the main question: what is the minimal generating set? In the previous question, we found that the permutations $(4, 1, 2, 3)$—cycling all elements forward—and $(2, 1, 3, 4)$—swapping the first two elements—generate S_4. For the cube, those are two rotations a and b as shown in Figure 14.

(d) Whether the group is commutative

This group is not commutative, since S_4 is not commutative.

20. Cube under reflection

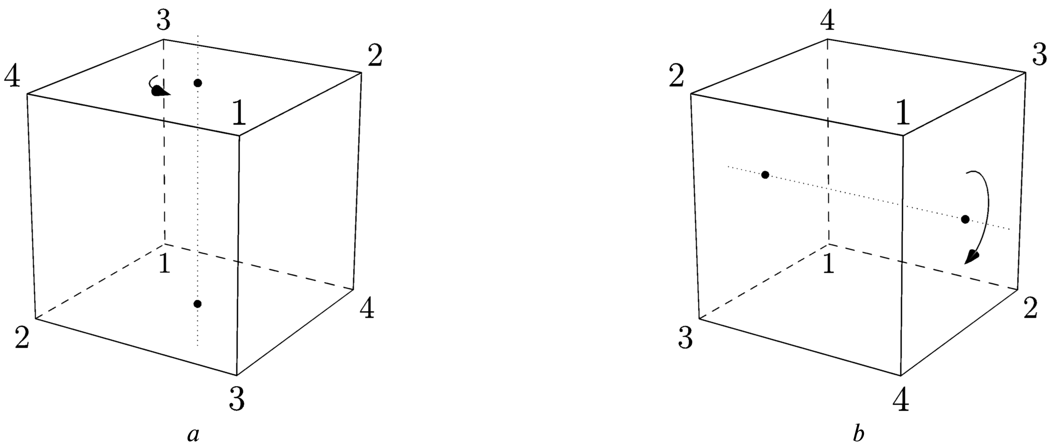


Figure 14: The two rotations a and b.

(a) Number of elements

There are 24 elements in the rotation group of the cube, so naturally there are 48 elements in the reflection group.

(b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

For each of the 24 rotations of the cube, there is also a reflected version over some plane. This gives $2 \cdot 24 = 48$ total elements in this group.

(c) A smallest possible generating set

If c is a reflection about, say, the origin of the cube, then $\{a, b, c\}$ (where a, b are the rotations from before) would generate the whole group, since $\{a, b\}$ generates all rotations and $\{c\}$ generates their respective reflections. But can we do it in two?

As usual it seems, the answer is yes! The proof is not mine, because I couldn't figure it out, but due to math.SE user **verret**. It does require some more advanced concepts, so it is probably inaccessible to most.

The group we've been analyzing is $S_4 \times Z_2$. Let S_4 be permuting elements $\{1, 2, 3, 4\}$ and Z_2 be permuting elements $\{5, 6\}$ (note that $Z_2 = S_2$). Then given two elements $g = (4, 1, 2, 3, 5, 6)$ in our notation, meaning that indices $(1, 2, 3, 4)$ are cycled, and $h = (3, 1, 2, 4, 6, 5)$, meaning that indices $(1, 2, 3)$ and $(5, 6)$ are cycled, we can construct the group.

Note that $h^2 = (2, 3, 1, 5, 6)$ is in S_4, since it does not permute indices $5, 6$. It has a period of 3, and thus generates a subgroup of order 3. Furthermore, h^3 only permutes $(5, 6)$. Furthermore, g is an element in S_4 and has a period of 4. Thus, since $\gcd(3, 4) = 1$, by Lagrange's theorem we know that $\{h^2, g\}$ generates a subgroup of S_4 of at least order $3 \cdot 4 = 12$.

The only such subgroup, besides S_4 itself, is the alternating group A_4. But g is outside of A_4, since it is an odd permutation:

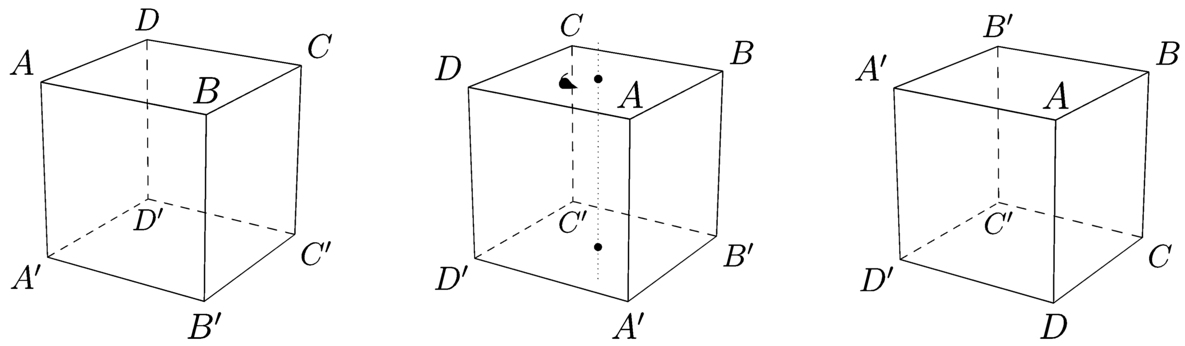
$$(1, 2, \boxed{3, 4}) \to (1, \boxed{2, 4}, 3) \to (\boxed{1, 4}, 2, 3) \to (4, 1, 2, 3).$$

Thus, $\{h^2, g\}$ does not generate A_4, and must generate S_4. Adding h^3, the generator for Z_2, to this set gives the full S_4, Z_2. The minimal generating set is therefore $\{g, h\}$ as defined.

For the curious, using our vertex "labeling" convention as before, the elements g and h are shown in Figure 15.

(d) Whether the group is commutative

The subgroup of rotations of the cube, S_4, is not commutative, so this group is definitely not commutative.



I g h

Figure 15: Elements g and h. Note that h is not solely a reflection about a mirror plane, but actually a combination of a rotation and reflection: a so-called rotoreflection!

5 Infinite Groups

Note: an **injection** f is a function taking A into B such that for all $a \in A$, $f(a) \in B$ and $f(a)$ is unique. In other words, there are no two $a_1, a_2 \in A$, $a_1 \neq a_2$ such that $f(a_1) = f(a_2)$.

1. **Where have you come across the roots *iso-* and *-morphic* before?**

(Answers may vary.)

Iso- is a root meaning equal. You might have seen it in isometry, isometric (paper), isomer, isosceles, isotonic, isotropy, and isotope. *Morph* means "form" or "shape." You might have seen it in metamorphosis, amorphous, anthropomorphism, or morpheme.

2. **Can two groups be isomorphic if they have different orders?**

No. Suppose we have groups A and B such that $|A| > |B|$ (A is bigger than B). Then we can't have a one-to-one correspondence between the elements of A and B, because there will always be elements in A without a "partner" in B. Thus, they cannot be isomorphic.

3. **The rotation group of the regular hexagon, also known as the cyclic group of order 6, C_6, has six elements: the identity, and rotations of $\frac{\pi}{3}$, $\frac{2\pi}{3}$, π, $\frac{4\pi}{3}$, $\frac{5\pi}{3}$ radians. A rotation of $\frac{\pi}{3}$ generates the group.**

   (a) Which other rotation can generate the group?

The other rotation which generates the group is $\frac{5\pi}{3}$, because 5 is coprime with 6. This is necessary because otherwise a subgroup of the full C_6 is formed. For example, $\frac{2\pi}{3}$ generates

$$\left\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\right\},$$

which is merely C_3. Lame!

   (b) What is the period of each element?

$$\begin{aligned} 0 \text{ or } I &: 1 \\ \frac{\pi}{3} &: 6 \\ \frac{2\pi}{3} &: 3 \\ \pi &: 2 \\ \frac{4\pi}{3} &: 3 \\ \frac{5\pi}{3} &: 6 \end{aligned}$$

4. **C_6 has the same number of elements as the dihedral group D_3.**

   (a) Are the two groups isomorphic? How do you know?

No, the two groups are not isomorphic, although they are the same size. An easy way to see this is that D_3 has three reflections, which have period 2, but H only has one element of period 2.

   (b) What is the period of each element of D_3?

$$\begin{aligned} I &: 1 \\ r &: 3 \\ r^2 &: 3 \\ f &: 2 \\ fr &: 2 \\ fr^2 &: 2 \end{aligned}$$

(c) What can you say if the sets of the periods of the elements of each group are not the same?

If the periods of each group can't be paired up, then the elements cannot be paired up either; after all, isomorphism is a structure-preserving operation. Thus, the two groups are not isomorphic.

(d) Which subgroups of the cyclic group C_6 and D_3 are isomorphic?

One is C_2, which is $\{0, \pi\}$ in C_6 and $\{I, \text{any reflection}\}$ in D_3. The other non-trivial one is C_3, which is $\left\{0, \frac{(3\pm1)\pi}{3}\right\}$ in C_6 and $\{I, \text{any rotation}\}$ in D_3. Both also have the trivial subgroup $\{I\}$ of just the identity element.

5. Could an infinite group and a finite group be isomorphic?

No, because their sizes are not the same; a one-to-one correspondence cannot be constructed.

6. Do you think all infinite groups are isomorphic to each other? Find a counterexample if you can.

Not all infinite groups are isomorphic. For example, the set of rotations about the origin has only one element of period 2, namely r_{180°. But the set of reflections about the origin has infinitely many elements of period 2. Both, however, are infinite in size.

7. Make guesses to the relative sizes of the following pairs of sets. You may use shorthand like $|a| < |b|$, $|a| > |b|$, $|a| = |b|$. After you have made your guesses, we will analyze some of the cases and you can find out how good your intuition was.

(Answers may vary, but the "correct" answers are shown.)

(a) natural numbers, $\mathbb{N}$ vs. positive even numbers, $2\mathbb{N}$

$$|\mathbb{N}| = |2\mathbb{N}|$$

(b) natural numbers, $\mathbb{N}$ vs. positive rational numbers, $\mathbb{Q}^+$

$$|\mathbb{N}| = \left|\mathbb{Q}^+\right|$$

(c) natural numbers, $\mathbb{N}$ vs. real numbers between zero and one, $[0, 1)$

$$|\mathbb{N}| < |[0, 1)|$$

(d) real numbers, $\mathbb{R}$ vs. complex numbers, $\mathbb{C}$

$$|\mathbb{R}| = |\mathbb{C}|$$

(e) real numbers, $\mathbb{R}$ vs. points on a line

$$|\mathbb{R}| = |\text{points on a line}|$$

(f) points on a line vs. points on a line segment

$$|\text{points on a line}| = |\text{points on a line segment}|$$

(g) points on a line vs. points on a plane

$$|\text{points on a line}| = |\text{points on a plane}|$$

(h) rational numbers, $\mathbb{Q}$ vs. Cantor set (look this up or ask your teacher)

$$|Q| < |C|$$

8. Now, please return to Problem 7 and revise your answers. Justify each answer by producing a one-to-one correspondence, or showing the impossibility of doing so. Part (h) is an optional challenge.

(a) natural numbers, $\mathbb{N}$ vs. positive even numbers, $2\mathbb{N}$

This one is pretty straightforward. We have the following injection from $\mathbb{N}$ to $2\mathbb{N}$:

$$s \in \mathbb{N} \to 2s \in 2\mathbb{N}.$$

We have the following injection from $2\mathbb{N}$ to $\mathbb{N}$:

$$s \in \mathbb{N} \to \frac{s}{2} \in \mathbb{N}.$$

Since we can go both ways, we have $|\mathbb{N}| = |2\mathbb{N}|$, even though $\mathbb{N} \subset 2\mathbb{N}$ ($\mathbb{N}$ is a subset of $2\mathbb{N}$).[6]

(b) natural numbers, $\mathbb{N}$ vs. positive rational numbers, $\mathbb{Q}^+$

Surprisingly, we can make a one-to-one correspondence. If we list out the positive rationals in reduced form ($\frac{p}{q}$ with p, q coprime), ordered by increasing denominator, we can create the correspondence:

$$\begin{array}{ccccccccccccc} \mathbb{Q}^+ & \frac{0}{1} & \frac{1}{1} & \frac{1}{2} & \frac{2}{1} & \frac{1}{3} & \frac{3}{1} & \frac{1}{4} & \frac{2}{3} & \frac{3}{2} & \frac{4}{1} & \cdots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \cdots \\ \mathbb{N} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \end{array}$$

More details of this construction are given later in the textbook chapter. In any case, $|\mathbb{Q}^+| = |\mathbb{N}|$.

(c) natural numbers, $\mathbb{N}$ vs. real numbers between zero and one, $[0, 1)$

A one-to-one correspondence cannot exist between these two sets, so $|\mathbb{N}| < |[0, 1)|$. The classic proof of this is Cantor's diagonal argument, which is given in the textbook.

(d) real numbers, $\mathbb{R}$ vs. complex numbers, $\mathbb{C}$

This is a pretty tough problem to do in a logically sound way. The key is to represent complex numbers $a + bi$ as the ordered pair (a, b) where $a, b \in \mathbb{R}$. The set of all (a, b) is denoted $\mathbb{R}^2$.

Here is the route we will take:

1. Construct a one-to-one correspondence between the interval $[0, 1)$ and $\mathbb{R}$.
2. Use (1) to construct a similar correspondence between $[0, 1)^2$ and $\mathbb{R}^2$. That is, we will construct a correspondence between ordered pairs of reals in $[0, 1)$ and ordered pairs of any reals.
3. We find an injection from $[0, 1)$ into $[0, 1)^2$.
4. We find an injection from $[0, 1)^2$ into $[0, 1)$. This shows there is a one-to-one correspondence between $[0, 1)$ and $[0, 1)^2$.
5. We "chain" the correspondences together:

$$\mathbb{R} \leftrightarrow [0, 1) \leftrightarrow [0, 1)^2 \leftrightarrow \mathbb{R}^2.$$

Step 1: The most straight forward way to do this is to show there is an injection from $[0, 1)$ into $\mathbb{R}$, and vice versa.[7] We have $f(x) = x$ as a straightforward injection from $[0, 1)$ into $\mathbb{R}$, and

$$g(x) = \frac{1}{1 + e^{-x}}$$

as an injection $g : \mathbb{R} \to [0, 1)$. Thus, there exists a one-to-one correspondence H between $\mathbb{R}$ and $[0, 1)$.

Step 2: If H is the function from Step 1, we have

$$J(a, b) = (H(a), H(b))$$

as a one-to-one correspondence between $\mathbb{R}^2$ and $[0, 1)^2$.

[6] We didn't explicitly state it because it's pretty intuitive, but this is using the Cantor-Schröder-Bernstein theorem (CSB).
[7] Again, this uses the Cantor-Schröder-Bernstein theorem.

Step 3: An injection from $[0,1)$ into $[0,1)^2$ is straightforward:

$$k_1(x) = (x, 0).$$

Step 4: An injection from $[0,1)^2$ into $[0,1)$ is the more challenging portion. The basic idea is to interleave digits like so:

$$(0.123456789..., 0.314159265) \overset{k_2}{\rightarrow} 0.132134415569728695...$$

The main issue with this construction is that $0.5 = 0.4999...$ gives two different outputs, so this mapping isn't even a function:

$$(0.5, 0.0) \rightarrow 0.50$$

$$(0.499..., 0.0) \rightarrow 0.409090... \neq 0.50.$$

The easiest thing to do here is arbitrarily choose one of these mappings. In particular, we represent a number with an infinite sequence of trailing zeroes $0.a_1a_2 \cdots a_n00000...$ with the numerically equivalent

$$0.a_1a_2 \cdots (a_n - 1)9999....$$

Now, our function k_2 is a true injection, since $k(a,b) \in [0,1)$ for all $(a,b) \in [0,1)^2$ and $k(a_1,b_1) \neq k(a_2,b_2)$ for $(a_1,b_1) \neq (a_2,b_2)$.

Step 5: We have constructed an injection k_1 from $[0,1) \rightarrow [0,1)^2$ and an injection k_2 from $[0,1)^2 \rightarrow [0,1)$. Thus, there exists a one-to-one correspondence K between $[0,1)$ and $[0,1)^2$.

We chain the correspondences, finally proving that there exists a one-to-one correspondence between $\mathbb{R}$ and $\mathbb{R}^2$:

$$\mathbb{R} \overset{H}{\leftrightarrow} [0,1) \overset{K}{\leftrightarrow} [0,1)^2 \overset{J}{\leftrightarrow} \mathbb{R}^2.$$

Thus, $|\mathbb{R}| = \left|\mathbb{R}^2\right|$.

(e) real numbers, $\mathbb{R}$ vs. points on a line

This is pretty straightforward if you think of points on a line as points on a number line. We arbitrarily choose a point on the line for 0 and a point for 1. In this regime, each point on the line corresponds with a unique real number. Thus, $|\mathbb{R}| = |\text{points on a line}|$.

(f) points on a line vs. points on a line segment

The simplest way to do this is, once again, to show there is an injection going both ways. We can go from segment → line by observing that a segment is just a subset of a line. We can go from line to segment by representing each point as a real number $\mathbb{R}$ as we already did, then taking the function

$$f(x) = \frac{1}{1 + e^{-x}}$$

which turns that point into a real number in the interval $(0,1)$. This can be mapped onto the line segment by simply choosing one endpoint to be 0 and the other to be 1.

|points on a line segment| = |points on a line|

(g) points on a line vs. points on a plane

We can represent points on a line, as usual, with $\mathbb{R}$. We can represent points on a plane by arbitrarily choosing non-collinear points for $(0,0)$, $(1,0)$ and $(0,1)$ and letting this be a coordinate space where points (a,b) are expressed as

$$a < 1, 0 > +b < 0, 1 > .$$

Note that the two vectors don't have to be perpendicular. This shows that we can represent points on a plane by $\mathbb{R}^2$. But we've already proved $\left|\mathbb{R}^2\right| = |\mathbb{R}|$! Thus,

$$|\text{points on a plane}| = |\text{points on a line}|.$$

(h) rational numbers, $\mathbb{Q}$ vs. Cantor set (look this up or ask your teacher)

The Cantor set C is formed by iteratively deleting the open middle third of segments, starting with the unit segment. We start with the interval $[0, 1]$, then split it into two intervals: $\left[0, \frac{1}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$. This set, let's call it C_1, has total length $\frac{2}{3}$. We split up each of C_1's intervals again, forming $C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \left[\frac{8}{9}, 1\right]$. Note what happened here: the first interval in C_1 had its middle third deleted, splitting it into our first two intervals; and the same with the second interval. C_2 has total length $\frac{4}{9}$. We repeat this process to infinity, so that (informally speaking) $C_\infty = C$, the Cantor set. The set has total length 0, but it's not empty! 0, $\frac{1}{3}$, and $\frac{7}{9}$ are all members of C, for example.
For a visual, the construction is shown in Figure 1.

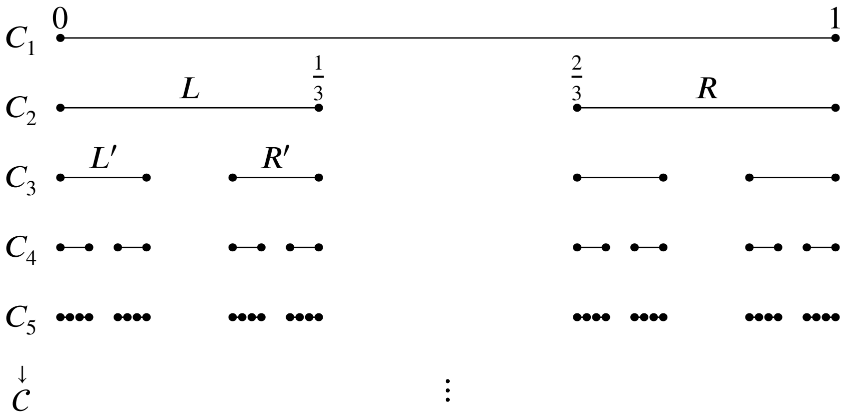


Figure 1: The construction of the Cantor set C.

How do we attack this problem? It's not immediately clear how to tell whether a number x is in C. It's not something as simple as $\frac{p}{3^n}$ for some integers p, n, because $\frac{4}{9}$ is in the deleted interval $\left[\frac{1}{3}, \frac{2}{3}\right]$, and is not in C. We should represent the set as something we know how to deal with, keeping in mind that answering our question of cardinality only involves making a one-to-one correspondence, not a complete description of the set.

Consider how we would choose a random point in C. Starting at C_1, we can either go to the left segment or the right segment (marked as L and R in Figure 1). If we choose L, which is $\left[0, \frac{1}{3}\right]$, then we once again have 2 choices: to go to L', which is $\left[0, \frac{1}{9}\right]$, or R', which is $\left[\frac{2}{9}, \frac{1}{3}\right]$. This continues forever, and every element of C can be uniquely obtained this way. Thus, we can correspond each element $x \in C$ with a binary number in the interval $[0, 1]$.

As an example, suppose we choose segments in the sequence $LRRLLLRLLLLLLLL$... with infinite trailing L's. Then the corresponding binary number is

$$0.0110001\bar{0}_2 = \frac{39}{128} \in [0, 1].$$

Thus, we have a one-to-one correspondence between the elements of C and $[0, 1]$.[8] The question is asking the relative sizes of $\mathbb{Q}$ and C. We already know (by Cantor's diagonal argument or otherwise) that $|\mathbb{Q}| < |[0, 1]|$. Therefore, we have

$$|\mathbb{Q}| < |[0, 1]| = |C| \implies |\mathbb{Q}| < |C|.$$

This might be counterintuitive, that a set with "length" 0 is still bigger than all the rational numbers on the great big number line. Frankly, without some formal language (particularly from topology), it's hard to describe this set with any rigor. The Wikipedia article on the Cantor set may guide you further!

9. Here's a list of infinite sets, each with an operation. For each pair, answer: (i) Does it form a group? (ii) Which previous group(s) is it isomorphic to?

(a) natural numbers, addition

[8]Note it is inclusive because $0.\bar{1}_2 = 1$ and $0.\bar{0}_2 = 0$.

i. **Does it form a group?**

Nope! It cannot satisfy the identity, since for $x + I = I + x = I$ to be true for all x we need $I = 0$. If you're a fan of the standard ISO 80000, and include $0 \in \mathbb{N}$, then it still doesn't form a group, since it can't satisfy the invertibility property. For example, the inverse of 1 should be -1 so that $1 + (-1) = 0$, but $-1 \notin \mathbb{N}$.

ii. **Which previous group(s) is it isomorphic to?**

Not a group, oof.

(b) **integers, addition**

i. **Does it form a group?**

It does form a group. The identity element is 0, and it satisfies all necessary properties:
Identity: $x + 0 = 0 + x = x$.
Closure: If $x, y \in \mathbb{Z}$, then $x + y \in \mathbb{Z}$.
Associativity: We have $x + (y + z) = (x + y) + z$ for all $x, y, z \in \mathbb{Z}$.
Inverse: The inverse of x is $-x$, since $x + (-x) = (-x) + x = 0$.

ii. **Which previous group(s) is it isomorphic to?**

None, I wonder why.

(c) **even integers, addition**

i. **Does it form a group?**

It does form a group. The identity element is 0, and it satisfies all necessary properties:
Identity: $x + 0 = 0 + x = x$.
Closure: If $x, y \in 2\mathbb{Z}$, then $x + y = 2s + 2t = 2(s + t) \in 2\mathbb{Z}$.
Associativity: We have $x + (y + z) = (x + y) + z$ for all $x, y, z \in 2\mathbb{Z}$.
Inverse: The inverse of x is $-x$, since $x + (-x) = (-x) + x = 0$.

ii. **Which previous group(s) is it isomorphic to?**

It is isomorphic to integers under addition, because we can simply correspond $2n \in 2\mathbb{Z}$ with $n \in \mathbb{Z}$. All the group structure is preserved, since $2m + 2n \in 2\mathbb{Z}$ corresponds with $m + n \in \mathbb{Z}$.

(d) **odd integers, addition**

i. **Does it form a group?**

This does not form a group, since it cannot satisfy the identity property. There is no odd integer I such that $x + I = I + x = x$.

ii. **Which previous group(s) is it isomorphic to?**

Not a group, oof.

(e) **rational numbers, addition**

i. **Does it form a group?**

Yes, this forms a group with identity element 0. It satisfies all necessary properties:
Identity: $\frac{p}{q} + 0 = 0 + \frac{p}{q} = \frac{p}{q}$.
Closure: If $\frac{p_1}{q_1}, \frac{p_2}{q_2} \in \mathbb{Q}$, then

$$\frac{p_1}{q_1} + \frac{p_2}{q_2} = \frac{p_1 q_2 + p_2 q_1}{q_1 q_2} \in \mathbb{Q}.$$

Associativity: We have $x + (y + z) = (x + y) + z$ for all $x, y, z \in 2\mathbb{Z}$.
Inverse: The inverse of $\frac{p}{q}$ is $-\frac{p}{q}$, since

$$\frac{p}{q} + \left(-\frac{p}{q}\right) = \left(-\frac{p}{q}\right) + \frac{p}{q} = 0.$$

ii. Which previous group(s) is it isomorphic to?

None. It's not isomorphic to integers under addition because for each element $x \in \mathbb{Q}$, there exists an element $y = \frac{x}{2} \in \mathbb{Q}$ such that

$$y + y = x.$$

This is impossible for any odd integers (analogously, for the group of even integers under addition, impossible for any elements not divisible by 4).

(f) real numbers, addition

i. Does it form a group?

Yes, this forms a group with identity element 0. It satisfies all necessary properties:
Identity: $x + 0 = 0 + x = x$.
Closure: If $x, y \in \mathbb{R}$, then $x + y \in \mathbb{R}$.
Associativity: We have $x + (y + z) = (x + y) + z$ for all $x, y, z \in \mathbb{R}$.
Inverse: The inverse of x is $-x$, since $x + (-x) = (-x) + x = 0$.

ii. Which previous group(s) is it isomorphic to?

None. After all, $\mathbb{R}$ is uncountable, while the groups we've seen so far are countable.

(g) complex numbers, addition

i. Does it form a group?

Yes, this forms a group with identity element $0 = 0 + 0i$. It satisfies all necessary properties:
Identity: $x + 0 = 0 + x = x$.
Closure: If $x, y \in \mathbb{C}$, then $x + y \in \mathbb{C}$.
Associativity: We have $x + (y + z) = (x + y) + z$ for all $x, y, z \in \mathbb{C}$.
Inverse: The inverse of x is $-x$, since $x + (-x) = (-x) + x = 0$.

ii. Which previous group(s) is it isomorphic to?

Assuming the axiom of choice[9], it is actually isomorphic to $\mathbb{R}$ under addition. Since $\mathbb{C}$ is uncountable, this is the only candidate.

Proving they are isomorphic is tough[10] without the introduction of vector spaces (specifically, $\mathbb{Q}$-vector spaces). I was originally going to put it in, but I couldn't get it under a satisfactory length. If you're really curious, check out https://math.stackexchange.com/a/1511685/677124 for a mildly accessible view of the subject... if you already understand the basics of vector spaces. In summary, both $\mathbb{R}$ and $\mathbb{R}^2$ are vector spaces over the rational numbers $\mathbb{Q}$, and since $|\mathbb{R}| = |\mathbb{R}^2|$ they are isomorphic as vector spaces. This also implies that they are isomorphic as additive groups.

(h) integers, multiplication

i. Does it form a group?

No, this does not form a group. The identity element would be 1, so that $1 \cdot x = x \cdot 1 = x$, but $1 \cdot 0 = 0 \neq 1$, so it cannot satisfy invertibility. Even if we removed 0, for any $p \neq \pm 1$ there is no integer q such that $pq = 1$.

ii. Which previous group(s) is it isomorphic to?

Not a group, oof.

(i) integer powers of 2, multiplication

i. Does it form a group?

[9] The axiom of choice states that for every indexed family of sets $(S_i)_{i \in I}$, where $S_i \neq \emptyset$, there exists an indexed family of elements $(x_i)_{i \in I}$ such that $x_i \in S_i$ for all $i \in I$. Intuitively, this means that given a list of non-empty sets, you can select exactly one item from each set.

[10] In fact, it is impossible to construct an "explicit" isomorphism because $\mathbb{R} \ncong \mathbb{C}$ is consistent with the axiom of choice.

Yes! Let the group be called $\mathcal{W} = \{2^x : x \in \mathbb{Z}\}$ for fun. The identity element is $2^0 = 1$. The group properties are satisfied:

Identity: $2^x \cdot 1 = 1 \cdot 2^x = 2^x$.

Closure: $2^x \cdot 2^y = 2^{x+y} \in \mathcal{W}$.

Associativity: We have $2^x(2^y \cdot 2^z) = (2^x \cdot 2^y)2^z = 2^{x+y+z}$ for all $x, y, z \in \mathbb{Z}$.

Inverse: The inverse of 2^x is 2^{-x}, since $2^x 2^{-x} = 2^{-x} 2^x = 2^0 = 1$.

ii. Which previous group(s) is it isomorphic to?

It is isomorphic to integers under addition. $2^n \in \mathcal{W}$ corresponds with $n \in \mathbb{Z}$, since we have

$$2^m \cdot 2^n = 2^{m+n} \leftrightarrow m + n.$$

(j) rational numbers, multiplication

i. Does it form a group?

The rational numbers under multiplication do not form a group; the identity element must be 1, but then 0 would not have an inverse since nothing times 0 is 1.

ii. Which previous group(s) is it isomorphic to?

Not a group, oof.

(k) rational numbers excluding 0, multiplication

i. Does it form a group?

Yes! The rational numbers excluding 0, written $\mathbb{Q} \setminus 0$, form a group under multiplication with identity element 1. The group properties are satisfied:

Identity: $x \cdot 1 = 1 \cdot x = x$.

Closure: If $x, y \in \mathbb{Q} \setminus 0$, then $x \cdot y \in \mathbb{Q} \setminus 0$; the product of two nonzero rational numbers is rational and nonzero.

Associativity: We have $x(yz) = (xy)z$ for all $x, y, z \in \mathbb{Q} \setminus 0$.

Inverse: The inverse of $x \in \mathbb{Q} \setminus 0$ is $\frac{1}{x}$, since $x\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)x = 1$ and $x \neq 0$.

ii. Which previous group(s) is it isomorphic to?

It is not isomorphic to any. Let this group be $\mathcal{M}$. The only candidates are other groups with countably infinite order, aka addition of rational numbers ($\mathcal{R}$) and addition of integers ($\mathcal{J}$). $\mathcal{M}$ can't be isomorphic to $\mathcal{R}$, because all elements of $\mathcal{R}$ have a "half," while the elements of $\mathcal{M}$ don't all have an analogous square root. To be more explicit, all elements $k' = \frac{p}{q}$ in $\mathcal{R}$ have a corresponding element j' such that $j' + j' = k'$. But not all elements k in $\mathcal{R}$ have a corresponding element j such that $j \cdot j = k$, since, for example, no element q of $\mathcal{M}$ satisfies $q \cdot q = -\frac{1}{2}$.

Thinking about $j' + j' = k'$ and $j \cdot j = k$ also helps us prove that $\mathcal{M}$ can't be isomorphic to $\mathcal{J}$. Take the element $k = 4$ in $\mathcal{M}$, for example. Then $j = 2$ and $j = -2$ both square to k. In contrast, no element k' of $\mathcal{J}$ has the property that there are *two different values of j'* which satisfy $j' + j' = k'$, since either is no solution or the unique solution $j' = k'/2$.

(l) real numbers excluding 0, multiplication

i. Does it form a group?

Yes! The real numbers excluding 0, written $\mathbb{R} \setminus 0$, form a group under multiplication with identity element 1. The group properties are satisfied:

Identity: $x \cdot 1 = 1 \cdot x = x$.

Closure: If $x, y \in \mathbb{R} \setminus 0$, then $x + y \in \mathbb{R} \setminus 0$; the product of two nonzero rational numbers is rational and nonzero.

Associativity: We have $x(yz) = (xy)z$ for all $x, y, z \in \mathbb{R} \setminus 0$.

Inverse: The inverse of $x \in \mathbb{R} \setminus 0$ is $\frac{1}{x}$, since $x\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)x = 1$ and $x \neq 0$.

ii. Which previous group(s) is it isomorphic to?

It is not isomorphic to any of the previous groups.

(m) complex numbers, multiplication

i. Does it form a group?

No, because 0 prevents the group from satisfying invertibility.

ii. Which previous group(s) is it isomorphic to?

Not a group, oof.

(n) rotation by a rational number of degrees

i. Does it form a group?

Yes! The identity is 0 and it can simply by thought of adding rationals modulo 360. The group properties are satisfied:

Identity: $x \cdot 1 = 1 \cdot x = x$.

Closure: If $x, y \in \mathbb{Q}$ then $x + y \in \mathbb{Q}$ and it

Associativity: We have $x(yz) = (xy)z$ for all $x, y, z \in \mathbb{Q} \setminus 0$.

Inverse: The inverse of $x \in \mathbb{Q} \setminus 0$ is $\frac{1}{x}$, since $x\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)x = 1$ and $x \neq 0$.

ii. Which previous group(s) is it isomorphic to?

None. The easiest way to see this is that the element $\frac{360}{n}^\circ$, where n is an integer, has period n, so we can construct elements of arbitrary periods. No previous groups have elements of arbitrary periods.

(o) rotation by a rational number of radians

i. Does it form a group?

Yes! The identity element is 0 and it is totally equivalent to the rational numbers under addition. That's because for any rational radian rotations r_{p_1/q_1} and r_{p_2/q_2}, where $p_1, q_1, p_2, q_2 \in \mathbb{Z}$, $q_1, q_2 \neq 0$ and $p_1/q_1 \neq p_2/q_2$, we have $r_{p_1/q_1} \cong r_{p_2/q_2}$. The easiest way to understand this is that for two such rotations to be equal, there must be some integer k such that

$$\frac{p_1}{q_1} = \frac{p_2}{q_2} + 2\pi k.$$

There are two cases to consider: $k = 0$ and $k \neq 0$. If $k = 0$, then $\frac{p_1}{q_1} = \frac{p_2}{q_2}$, which violates our assumption. If $k \neq 0$, then since π is irrational, the RHS is irrational while the LHS is rational. Since an irrational and rational cannot be equal, such an integer k cannot exist and $r_{p_1/q_1} \ncong r_{p_2/q_2}$.

The group properties are straightforward and identical to the rationals under addition.

ii. Which previous group(s) is it isomorphic to?

As explained, it is isomorphic to the rational numbers under addition.

(p) rotation by an integer number of radians

i. Does it form a group?

Yes. The identity element is 0 and it is equivalent to the integers under addition. We proceed in a similar method to the previous problem: consider two integer radian rotations r_a and r_b where $a, b \in \mathbb{Z}$ and $a \neq b$.

Suppose $r_a \cong r_b$. Then there is some integer k such that

$$a = b + 2\pi k.$$

If $k = 0$, then $a = b$, contradicting our assumption that $a \neq b$. If $k \neq 0$, then the RHS is irrational while the LHS is rational. This is impossible to satisfy, so k does not exist and $r_a \ncong r_b$.

The group properties are straightforward and identical to the integers under addition.

ii. Which previous group(s) is it isomorphic to?

As explained, it is isomorphic to the integers under addition.

10. Can an irrational number taken to an irrational power ever be rational? Consider the potential example $a = \sqrt{2}^{\sqrt{2}}$. To help you answer this question, let $b = a^{\sqrt{2}}$. Simplify b, and explain why we don't need to know whether a is rational or irrational.

We have

$$b = a^{\sqrt{2}} = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^{2} = 2.$$

Let's write out the facts we know.

1. We know that $\sqrt{2}$ is irrational.
2. By (1), we know that a is the result of an irrational number to an irrational power.
3. We know that 2 is the result of a to an irrational power.

Suppose a is rational. Then $a = \sqrt{2}^{\sqrt{2}}$ is our desired example, by (2)! So, suppose a is irrational. Then by (3), 2 is the result of an irrational number (namely, a) to an irrational power (namely, $\sqrt{2}$). So then $2 = a^{\sqrt{2}}$ is our desired example! a has to be irrational or rational; it can't be something else. But in either case, we can produce a number which satisfies the requirements. Thus, we can answer in the affirmative, but we can't give an explicit example!

In fact, a is the irrational one by the Gelfond-Schneider theorem. Proving this theorem requires some pretty advanced analysis, yet we are able to derive related results with simple logic! Exciting.

6 Geometry of Complex Numbers

1. Explain why iz is perpendicular to z, without using DeMoivre's theorem.

Let $z = a + bi$. Then $iz = i(a + bi) = -b + ai$, which is the transformation $(a, b) \to (-b, a)$. Drawing this out on the 2D plane makes clear that the angle between the two points and the origin is $90°$, simply by subtracting angles: $(90° + \theta) - \theta = 90°$. This is shown in Figure 1.

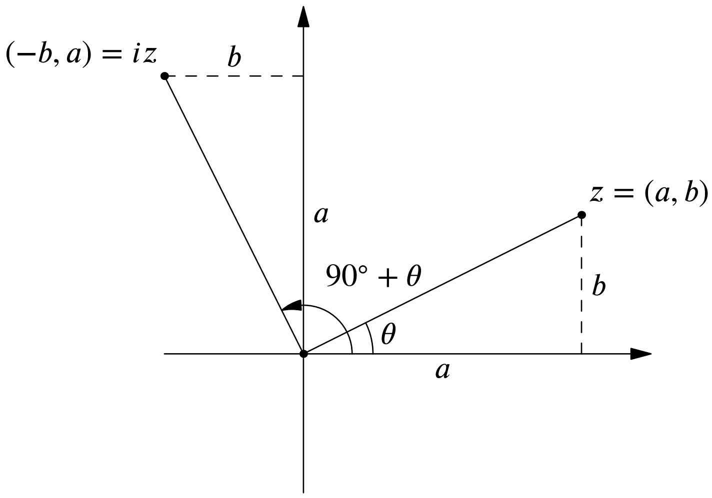


Figure 1: iz is perpendicular to z as long as $z \neq 0$.

You can also rationalize it by the fact that the lines through z/iz and the origin have slopes of $\frac{b}{a}$ and $-\frac{a}{b}$, respectively, so they must be perpendicular. Also, $< a, b > \bullet < -b, a >= 0$.

2. How does $\operatorname{Arg} \overline{z}$ relate to $\operatorname{Arg} z$? (Hint: symmetry!)

Again, let $z = a + bi$. $\overline{z} = a - bi$ is flipped over the x-axis, since the imaginary part is negated. Thus, $\operatorname{Arg} \overline{z} = -\operatorname{Arg} z$ due to congruent triangles formed by z and $\overline{z}$.[11] The geometric interpretation is shown in Figure 2.

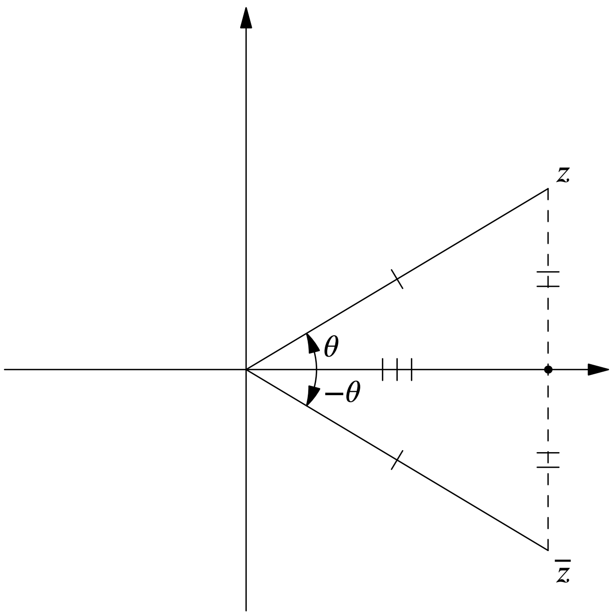


Figure 2: z and $\overline{z}$ form congruent triangles, showing that $\operatorname{Arg} z = -\operatorname{Arg} \overline{z}$.

3. Compute $z\overline{z}$ and relate it to the cis form of z.

[11] If we are to be pedantic, we'd either use the multivalued function $\arg$ or say this is modulo 2π.

Once more, let $z = a + bi$. Then

$$z\overline{z} = (a+bi)(a-bi) = a^2 - (bi)^2 = a^2 + b^2.$$

If $z = r\operatorname{cis}\theta$, then $z\overline{z} = r^2$. In other words, it is the square of the distance from z to the origin.

4. Explain, using a picture, why $\tan(\operatorname{Arg} z) = \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}$.

This is basically just an application of soh-cah-toa to a triangle in the complex plane. The details are shown in Figure 3.

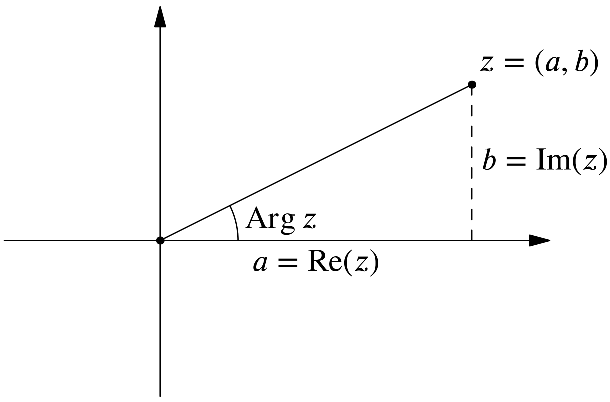


Figure 3: $\tan(\operatorname{Arg} z) = \frac{b}{a} = \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}$.

5. Divide $\frac{a+bi}{c+di}$ by rationalizing the denominator.

$$\begin{aligned}\frac{a+bi}{c+di}\cdot\frac{c-di}{c-di} &= \frac{(a+bi)(c-di)}{c^2+d^2}\\ &= \frac{ac+bd+(bc-ad)i}{c^2+d^2}.\end{aligned}$$

6. Divide $\frac{r_1\operatorname{cis}\theta}{r_2\operatorname{cis}\phi}$ using DeMoivre's theorem.

We don't have a rule yet for applying DeMoivre's theorem for division, but we can quickly derive it. We have

$$\begin{aligned}\frac{r_1\operatorname{cis}\theta}{r_2\operatorname{cis}\phi}\cdot\frac{\overline{\operatorname{cis}\phi}}{\overline{\operatorname{cis}\phi}} &= \frac{r_1\operatorname{cis}\theta\,\overline{\operatorname{cis}\phi}}{r_2\underbrace{\operatorname{cis}\phi\,\overline{\operatorname{cis}\phi}}_{=1}} && \text{Multiplying by conjugate}\\ &= \frac{r_1\operatorname{cis}\theta\operatorname{cis}(-\phi)}{r_2} && \text{Using } \operatorname{Arg} z = -\operatorname{Arg}\overline{z}\\ &= \frac{r_1}{r_2}\operatorname{cis}(\theta-\phi). && \text{Use DeMoivre's theorem}\end{aligned}$$

7. Compare and contrast the methods of division in Problems 5 and 6. Which is more convenient? Or does it depend on the circumstance?

Opinions may vary, but 6 is definitely faster to do if the dividend and divisor are already in cis form. 5 is likely convenient than converting from rectangular to cis, then back to rectangular.

8.

(a) If $z = r\operatorname{cis}\theta$, what is $\frac{1}{z}$?

As we hinted at in the previous problem, $\frac{1}{z} = \frac{1}{r}\operatorname{cis}(-\theta)$:

$$\begin{aligned}
\frac{1}{r\operatorname{cis}\theta} \cdot \frac{\overline{\operatorname{cis}\theta}}{\overline{\operatorname{cis}\theta}} &= \frac{\overline{\operatorname{cis}\theta}}{r\operatorname{cis}\theta\,\overline{\operatorname{cis}\theta}} \\
&= \frac{\operatorname{cis}(-\theta)}{r|\operatorname{cis}\theta|^2} \\
&= \frac{1}{r}\operatorname{cis}(-\theta).
\end{aligned}$$

(b) Explain how this shows $\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}$, without having to rationalize the denominator. (Hint: use problems 3, 4, and 7.)

Let $a + bi = r\operatorname{cis}\theta$. We have

$$\begin{aligned}
\frac{1}{r\operatorname{cis}\theta} &= \frac{1}{r}\operatorname{cis}(-\theta) \\
&= \frac{r\operatorname{cis}(-\theta)}{r^2} \\
&= \frac{a-bi}{a^2+b^2}.
\end{aligned}$$

9. Compute $(1+i)^{13}$; pencil, paper, and brains only. No calculators!

We have $1 + i = \sqrt{2}\operatorname{cis}\frac{\pi}{4}$, since it forms a $45°$ angle with the x-axis. Applying DeMoivre's theorem,

$$\begin{aligned}
(1+i)^{13} &= \left(\sqrt{2}\operatorname{cis}\frac{\pi}{4}\right)^{13} \\
&= (\sqrt{2})^{13}\operatorname{cis}\frac{13\pi}{4} \\
&= 64\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) \\
&= 64\sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) \\
&= 64(-1-i) \\
&= -64-64i.
\end{aligned}$$

10. Compute $\frac{(1+i\sqrt{3})^3}{(1-i)^2}$ without a calculator.

We convert to cis form and apply DeMoivre's theorem.

$$\begin{aligned}
\frac{(1+i\sqrt{3})^3}{(1-i)^2} &= \frac{\left(2\operatorname{cis}(\frac{\pi}{3})\right)^3}{\left(\sqrt{2}\operatorname{cis}(-\frac{\pi}{4})\right)^2} \\
&= \frac{8\operatorname{cis}(\pi)}{2\operatorname{cis}(-\frac{\pi}{2})} \\
&= \frac{8\cdot -1}{2\cdot -i} \\
&= \frac{4}{i}\cdot\frac{-i}{-i} \\
&= -4i.
\end{aligned}$$

11. Draw $\operatorname{cis}\left(\frac{\pi}{4}\right) + \operatorname{cis}\left(\frac{\pi}{2}\right)$. Use your picture to prove an expression for $\tan\left(\frac{3\pi}{8}\right)$. (Hint: add them as vectors.)

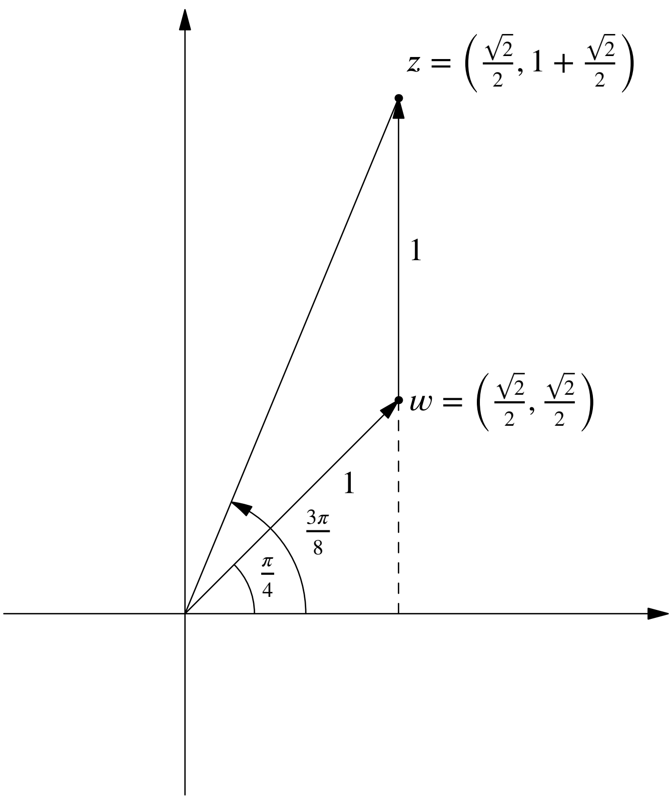


Figure 4: Addition of $\operatorname{cis}\left(\frac{\pi}{4}\right) + \operatorname{cis}\left(\frac{\pi}{2}\right)$ as vectors.

The drawing is shown in Figure 4. The first vector, starting at the origin, is $\operatorname{cis}\frac{\pi}{4}$. The second vector, starting at the endpoint of the first vector, is $\operatorname{cis}\frac{\pi}{2}$. The origin, along with points w and z, form an isosceles triangle. Furthermore, the apex of this triangle, at w, has a measure of $\pi - \frac{\pi}{4} = \frac{3\pi}{4}$ radians. Thus, the base angles of the isosceles triangle are

$$\frac{\pi - \frac{3\pi}{4}}{2} = \frac{\pi}{8}.$$

Adding this with $\frac{\pi}{4}$ shows that the angle z forms with the x-axis is

$$\frac{\pi}{8} + \frac{\pi}{4} = \frac{3\pi}{8},$$

our desired angle to analyze. We wish to find the tangent of this angle, which is just $\tan(\operatorname{Arg} z)$. But we know how to compute that!

$$\begin{aligned}
\tan(\operatorname{Arg} z) = \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)} &= \frac{1 + \frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \\
&= \frac{\sqrt{2}+1}{1} \\
\tan\left(\frac{3\pi}{8}\right) &= \sqrt{2}+1.
\end{aligned}$$

12. Solve $z^3 = 1$, and show that its solutions under the operation of multiplication form a group, isomorphic to the rotation group of the equilateral triangle. Write a group table!

There's numerous ways to solve this, but let's use cis form as usual. Let $z = r\operatorname{cis}\theta$. Then

$$
\begin{aligned}
z^3 = r^3 \operatorname{cis} 3\theta &= 1 \\
\implies r &= 1 \\
\operatorname{cis} 3\theta &= 1 \\
\cos 3\theta &= 1 \\
3\theta &= 2\pi k && \text{For } k \in \mathbb{Z} \\
\theta &= \frac{2\pi k}{3} \implies \theta \in \left\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\right\} \\
\implies z \in \left\{\operatorname{cis} 0, \operatorname{cis} \frac{2\pi}{3}, \operatorname{cis} \frac{4\pi}{3}\right\}.
\end{aligned}
$$

Under multiplication, these three values of z indeed form a group isomorphic to the rotation group of the equilateral triangle, C_3. In particular, $\operatorname{cis} 0$ is the identity, $\operatorname{cis} \frac{2\pi}{3}$ is a rotation by $120°$ counterclockwise, and $\operatorname{cis} \frac{4\pi}{3}$ is a rotation by $240°$ counterclockwise. Let $I = \operatorname{cis} 0$, $r = \operatorname{cis} \frac{2\pi}{3}$, and $r^2 = \operatorname{cis} \frac{4\pi}{3}$. Then, we have the following group table:

$\cdot$	I	r	r^2
I	I	r	r^2
r	r	r^2	I
r^2	r^2	I	r

13.

(a) Find multiplication groups of complex numbers which are isomorphic to the rotation groups for

i. a non-square rectangle

Since this rotation group is just the identity and a rotation of $180°$, we can just choose the group $\{-1, 1\}$ under multiplication. 1 is the identity, and $-1 = \operatorname{cis} 180°$ is the rotation.

ii. a regular hexagon.

We following in the footsteps of the equivalent problem for the equilateral triangle. We have elements

$$\left\{\operatorname{cis} 0, \operatorname{cis} 60°, \operatorname{cis} 120°, \operatorname{cis} 180°, \operatorname{cis} 240°, \circ 300°\right\}.$$

As should be obvious, these are rotations of 0, $60°$, $120°$, $180°$, $240°$, and $300°$ respectively. Under multiplication, this is isomorphic to the rotation group of the hexagon, C_6.

(b) Make a table for each group.

i. a non-square rectangle

Let I be the identity and r be the rotation of $180°$.

$\cdot$	I	r
I	I	r
r	r	I

ii. a regular hexagon.

Let I be the identity and r be the rotation of $60°$. r^n is defined in the natural way, by raising $\operatorname{cis} 60°$ to the power n.

$\cdot$	I	r	r^2	r^3	r^4	r^5
I	I	r	r^2	r^3	r^4	r^5
r	r	r^2	r^3	r^4	r^5	I
r^2	r^2	r^3	r^4	r^5	I	r
r^3	r^3	r^4	r^5	I	r	r^2
r^4	r^4	r^5	I	r	r^2	r^3
r^5	r^5	I	r	r^2	r^3	r^4

(c) Compare the regular hexagon's group to the dihedral group of the equilateral triangle, D_3. Consider: how are they the same? How are they different? Is the difference fundamental?

The two groups are not isomorphic, although they are the same size; the difference is fundamental. The hexagon's rotation group, C_6, has elements of periods $\{1, 2, 3, 3, 6, 6\}$, while D_3 has elements of periods $\{1, 2, 2, 2, 3, 3\}$. They do share some subgroups however: the trivial subgroup of just the identity, and the subgroups generated by r^2 and by r^3 in C_6, which are C_3 and C_2 respectively.

14. Which of the following sets is a group under (i) addition and (ii) multiplication?

(a) $\{0\}$

This is a group under (i) addition, since it has an identity 0, is closed, has 0 as 0's inverse, and $0+(0+0) = (0+0)+0$. It also is a group under (ii) multiplication, for the same reasons.

(b) $\{1\}$

This is not a group under (i) addition, since $1 + 1 = 2 \notin \{1\}$. It is a group under multiplication, though, since $1 \cdot 1 = 1$ and all other properties are satisfied.

(c) $\{0, 1\}$

This is not a group under (i) addition, since $1+1 = 2 \notin \{0, 1\}$. It is also not a group under (ii) multiplication. 0 can't be the identity, since $1 \cdot 0 = 0 \neq 1$. 1 also can't be the identity, since then 0 has no inverse K such that $0 \cdot K = 1$.

(d) $\{-1, 1\}$

This is not a group under (i) addition, since $1 + (-1) = 0 \notin \{\pm 1\}$. It is a group under (ii) multiplication, since it satisfies the group properties:

1. Identity: 1 is the identity
2. Associativity: Multiplication is associative
3. Invertibility: Each element is its own inverse
4. Closure: $(\pm 1)(\pm 1) \in \{\pm 1\}$

(e) $\{1, -1, i, -i\}$

This is not a group under (i) addition, since the sum of any two of the elements takes you out of the set. It is a group under (ii) multiplication, however. One easy way to see this is that $1 = \operatorname{cis} 0$, $-1 = \operatorname{cis} \pi$, $i = \operatorname{cis} \frac{\pi}{2}$, and $-i = \frac{3\pi}{2}$, which are all rotations of multiples of $90°$. In particular, it is isomorphic to C_4, the rotation group of the square.

(f) $\{$**naturals**$\}$

This is not a group under (i) addition, because it cannot satisfy invertibility. There is no element $X \in \mathbb{N}$ such that $1 + X = I = 0$, for example. This is also not a group under (ii) multiplication for the same reason.

(g) $\{$**integers**$\}$

This is a group under (i) addition, because all the group properties are satisfied. The inverse of an element n is just $-n$, addition is associative, the identity is 0, and the sum of two integers is another integer. It is a not a group under (ii) multiplication, because no numbers except ± 1 have integer multiplicative inverses.

(h) $\{$**rationals**$\}, \mathbb{Q}$

This is a group under (i) addition with identity element 0. The inverse of an element $\frac{p}{q}$ is $-\frac{p}{q}$, addition is associative, and the sum of two rational numbers is another rational number. It is not a group under (ii) multiplication, because no number is the multiplicative inverse of 0.

(i) $\{\mathbb{Q}$ **without zero**$\}$

This is no longer a group under (i) addition, since the identity element needs to be 0. It is now, however, group under (ii) multiplication, because all numbers have their inverses. Multiplication is associative, the inverse of $\frac{p}{q}$ is $\frac{q}{p}$, and the product of two rationals is another rational.

(j) $\{$**complex numbers**$\}, \mathbb{C}$

This is a group under (i) addition with identity element 0. The inverse of an element z is $-z$, addition is associative, and the sum of two complex numbers is another complex number. This is not a group under (ii) multiplication, because again, no number is the multiplicative inverse of 0.

(k) $\{\mathbb{C}$ **without zero**$\}$

This is no longer a group under (i) addition, since the identity element needs to be 0. It is now, however, a group under (ii) multiplication, because all numbers have their inverses. Multiplication is associative, the inverse of z is $\frac{1}{z}$, and the product of two complex numbers is another complex number.

15. Prove that $(r_1 \operatorname{cis}\theta)(r_2 \operatorname{cis}\phi) = r_1 r_2 \operatorname{cis}(\theta+\phi)$ using brute force and the angle-sum trig identities for $\cos$ and $\sin$. Do you prefer this method or the one on the previous page? Which method gives you a better understanding of why DeMoivre's works?

$$\begin{aligned}
(r_1 \operatorname{cis}\theta)(r_2 \operatorname{cis}\phi) &= r_1 r_2(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\
&= r_1 r_2(\cos\theta\cos\phi + i\cos\theta\sin\phi + i\sin\theta\cos\phi - \sin\theta\sin\phi) \\
&= r_1 r_2((\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \sin\theta\cos\phi)) \\
&= r_1 r_2(\cos(\theta+\phi) + i\sin(\theta+\phi)) \\
&= r_1 r_2 \operatorname{cis}(\theta+\phi)
\end{aligned}$$

(Opinions may vary.) I actually prefer this because it's kind of satisfying, but the previous way likely gives a better understanding of the underlying mechanics.

16. Find an identity for $\sin 3\theta$ as we have done for $\cos$. Most of the work is already done for you!

We already know that

$$\cos 3\theta + i\sin 3\theta = \operatorname{cis} 3\theta = (c^3 - 3c^2) + i(3c^2 s - s^3),$$

where $c = \cos\theta$ and $s = \sin\theta$. Equating imaginary parts, we have

$$\begin{aligned}
\sin 3\theta &= 3c^2 s - s^3 \\
&= 3\cos^2\theta\sin\theta - \sin^3\theta.
\end{aligned}$$

17. Your friend's textbook says $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, different from our identity. Who's right?

Both are right. Our identity is

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta.$$

Remembering that $\sin^2\theta = 1 - \cos^2\theta$, we can pretty easily change the form:

$$\begin{aligned}
\cos^3\theta - 3\cos\theta\sin^2\theta &= \cos^3\theta - 3\cos\theta(1-\cos^2\theta) \\
&= \cos^3\theta + 3\cos^3\theta - 3\cos\theta \\
&= 4\cos^3\theta - 3\cos\theta.
\end{aligned}$$

18. Now you can finish the rest of the proof.

If you need context for this answer, check out the relevant textbook section.

(a) Draw a, b, c, d, m, n approximately for the quadrilateral on the previous page.

The quadrilateral is shown in Figure 5.

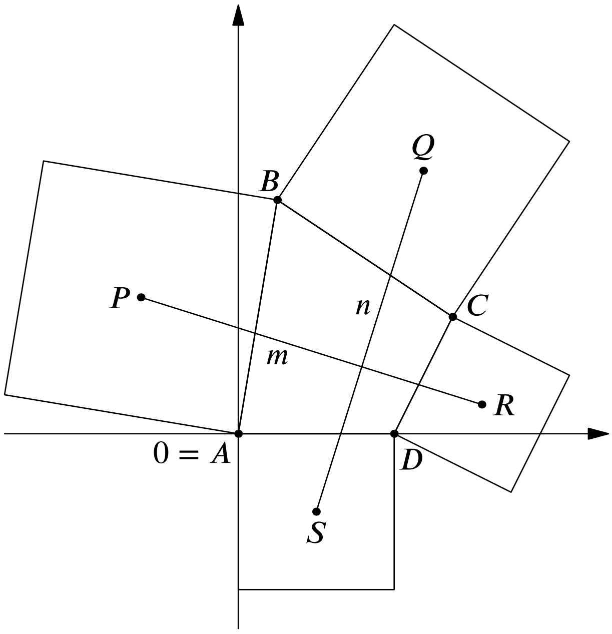


Figure 5: The quadrilateral to analyze.

The relative magnitudes and directions are shown in Figure 6 below. We find a, b, c, d from halving the sides of the quadrilateral. m and n are just the vectors from P to R and Q to S, respectively.

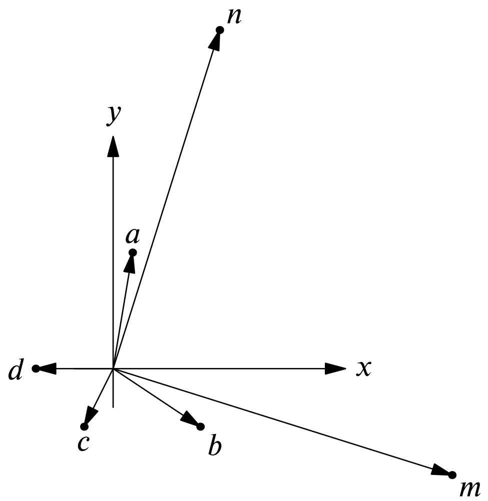


Figure 6: The relative magnitudes and directions of a, b, c, d, m, n.

(b) Why does showing $n = \pm im$ prove the segments are (i) perpendicular and (ii) the same length?

They are (i) perpendicular because iz is perpendicular to z for all $z \neq 0$, and (ii) are the same length because $|n| = |\pm im| = |im| = |m|$.

(c) Explain why $Q = 2a + b + ib$.

The justification is geometric. We know that $B = 2a$, and we can get to the midpoint of $\overline{BC}$ by adding b. Then, we go up to the center of the square on $\overline{BC}$ by adding ib. This process is shown in Figure 7.

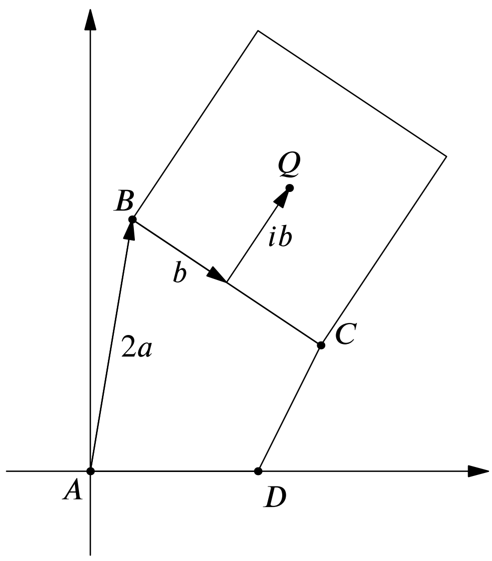


Figure 7: $Q = 2a + b + ib$.

(d) Find formulae for R and S in terms of c and d.

In a similar fashion, we have $S = -d + id$ (note that it is $-d$ because we are going counterclockwise now) and $R = -2d - c + ic$. The interpretations of these are shown in Figure 8 below.

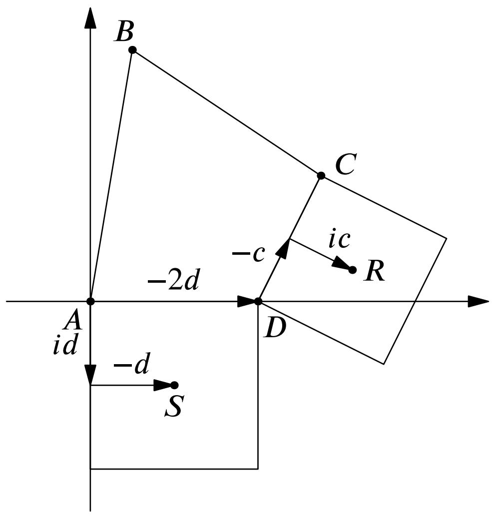


Figure 8: $S = -d + id$ and $R = -2d - c + ic$.

(e) Find m and n in terms of a, b, c, and d.

We have $m = R - P = (-2d - c + ic) - (a + ia)$ and $n = Q - S = (2a + b + ib) - (-d + id)$.

(f) Check that $n - im = 0$, using the fact that $a + b + c + d = 0$.

We evaluate straightforwardly:

$$
\begin{aligned}
(2a + b + ib + d - id) - i(-2d - c + ic - a - ia) &= 2a + b + ib + d - id + 2id + ic + c + ia - a \\
&= a + b + c + d + ia + ib + ic + id \\
&= (a + b + c + d)(1 + i) \\
&= 0.
\end{aligned}
$$

19. In the previous problem, we drew squares outside a quadrilateral and connected their centers. Conjecture what happens if we draw equilateral triangles outside a triangle and connect their centers. Prove your conjecture using complex numbers.

We conjecture that following this construction leads to connecting together another equilateral triangle. An example, with the variables we'll use labeled, is shown in Figure 9.

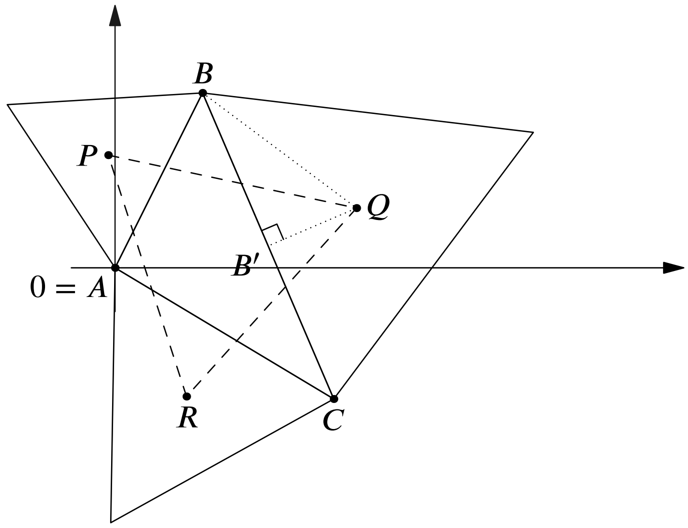


Figure 9: Equilateral triangles around a central, arbitrary triangle $\triangle ABC$ with A at the origin.

Similar to the last problem, let A, B, and C be numbers in the complex plane. Without loss of generality, let $A = 0$ be the origin. Also, define $a = \frac{B-A}{2}$, $b = \frac{C-B}{2}$, and $c = \frac{A-C}{2}$ to be the vectors going halfway along each of $\overrightarrow{AB}$, $\overrightarrow{BC}$, and $\overrightarrow{CA}$. Finally, let P, Q, and R be the centers of the triangles on sides AB, BC, and CA respectively.

Consider Q in the figure. It is on the $60°$ vertex of a 30-60-90 triangle $\triangle BB'Q$, outlined in dotted line. We know that $\overrightarrow{BB'} = b$. Thus, since $BB' : B'Q = \sqrt{3} : 1$,

$$B'Q = \frac{|b|}{\sqrt{3}}.$$

Furthermore, since $\overrightarrow{B'Q} \perp \overrightarrow{BB'}$, we know that it is $s \cdot ib$ for some real s. Combining these facts,

$$B'Q = \frac{ib}{|ib|}\frac{|b|}{\sqrt{3}} = \frac{ib}{\sqrt{3}}.$$

Since $Q = \overrightarrow{AB} + \overrightarrow{BB'} + \overrightarrow{B'Q}$, we have

$$Q = 2a + b + \frac{ib}{\sqrt{3}}.$$

With similar logic, we know that

$$P = a + \frac{ia}{\sqrt{3}}$$
$$R = -c + \frac{ic}{\sqrt{3}}$$

Like with the quadrilateral, we know $a + b + c = 0$, since $2(a + b + c) = 0$. To prove the dashed triangle is indeed equilateral, we can just show that $P - R = (Q - P)\operatorname{cis} 120°$. After all, if the vectors $\overrightarrow{RP}$ and $\overrightarrow{PQ}$ have an angle of $120°$ between them and they have the same magnitude, $\triangle PQR$ is equilateral by SAS Congruence as shown in Figure 10. Substituting in our found values for P, Q, R in terms of a, b, c, we get

$$
\begin{aligned}
P - R &= (Q - P)\operatorname{cis} 120^\circ \\
a + \frac{ia}{\sqrt{3}} - \left(-c + \frac{ic}{\sqrt{3}}\right) &= \left(2a + b + \frac{ib}{\sqrt{3}} - \left(a + \frac{ia}{\sqrt{3}}\right)\right)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\
(a+c) + \frac{ia - ic}{\sqrt{3}} &= \left(a + b + \frac{ib - ia}{\sqrt{3}}\right)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\
&= -\frac{1}{2}a - \frac{1}{2}b - \frac{ib - ia}{2\sqrt{3}} + \frac{\sqrt{3}}{2}ia + \frac{\sqrt{3}}{2}ib + \frac{ib - ia}{2} \cdot i \\
&= \left(-\frac{1}{2}a - \frac{1}{2}b + \frac{ib - ia}{2} \cdot i\right) + \left(-\frac{ib - ia}{2\sqrt{3}} + \frac{\sqrt{3}}{2}ia + \frac{\sqrt{3}}{2}ib\right) \\
&= \left(-\frac{1}{2}a - \frac{1}{2}b - \frac{1}{2}b + \frac{1}{2}a\right) + \left(\frac{ia - ib + 3ia + 3ib}{2\sqrt{3}}\right) \\
&= (-b) + \left(\frac{4ia + 2ib}{2\sqrt{3}}\right) \\
&= (-b + a + b + c) + \left(\frac{i(2a+b)}{\sqrt{3}}\right) \\
&= (a+c) + \frac{i(2a + b - (a + b + c))}{\sqrt{3}} \\
(a+c) + \frac{ia - ic}{\sqrt{3}} &= (a+c) + \frac{ia - ic}{\sqrt{3}}
\end{aligned}
$$

Tedious, but it worked.

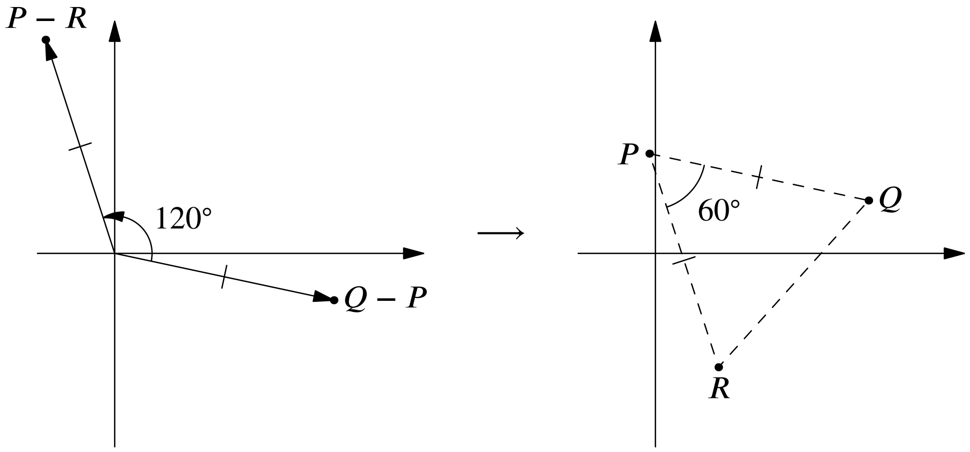


Figure 10: SAS Congruence lets us say that $P - R = (Q - P)\operatorname{cis} 120^\circ$ is sufficient to prove the triangle $\triangle PQR$ is equilateral.

20. The hard way to find an identity for $\tan 3\theta$ is to divide the identity for $\sin$ and $\cos$ that we already found. Try this. Make sure your answer is in terms of $\tan$ only!

We have found that $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ and $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$. We set $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and evaluate:

$$\begin{aligned}
\tan\theta = \frac{\sin\theta}{\cos\theta} &= \frac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta\sin^2\theta} \\
&= \frac{\sin\theta}{\cos\theta}\cdot\frac{3\cos^2\theta - \sin^2\theta}{\cos^2\theta - 3\sin^2\theta}\cdot\frac{\frac{1}{\cos^2\theta}}{\frac{1}{\cos^2\theta}} \\
&= \tan\theta\cdot\frac{3-\frac{\sin^2\theta}{\cos^2\theta}}{1-\frac{3\sin^2\theta}{\cos^2\theta}} \\
&= \tan\theta\cdot\frac{3-\tan^2\theta}{1-3\tan^2\theta} \\
&= \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}.
\end{aligned}$$

21. The easier way to get an identity for $\tan 3\theta$ starts with setting $z = 1 + i\tan\theta$.

(a) Why is $\operatorname{Arg} z = \theta$?

You can see this pretty quickly with a diagram, like in Figure 11. More algebraically, we have

$$\begin{aligned}
\tan(\operatorname{Arg} z) = \operatorname{Im}(z) = \tan(\theta) \\
\implies \operatorname{Arg} z = \theta.
\end{aligned}$$

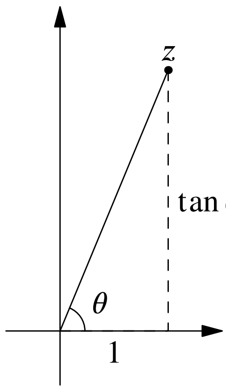


Figure 11: $\operatorname{Arg} z = \tan^{-1}\left(\frac{\tan\theta}{1}\right) = \theta$.

(b) Why is $\tan 3\theta = \frac{\operatorname{Im}(z^3)}{\operatorname{Re}(z^3)}$?

We have

$$\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta} = \frac{\operatorname{Im}(\operatorname{cis} 3\theta)}{\operatorname{Re}(\operatorname{cis} 3\theta)}.$$

But since z makes an angle of θ with the x-axis, we can express it as $r\operatorname{cis}\theta$ for some real r. Thus,

$$\frac{\operatorname{Im}(z^3)}{\operatorname{Re}(z^3)} = \frac{\operatorname{Im}(r^3\operatorname{cis} 3\theta)}{\operatorname{Re}(r^3\operatorname{cis} 3\theta)} = \frac{\operatorname{Im}(\operatorname{cis} 3\theta)}{\operatorname{Re}(\operatorname{cis} 3\theta)},$$

which matches the expression for $\tan 3\theta$.

(c) Use (b) to find an identity for $\tan 3\theta$.

We expand out z^3 and factor into real and imaginary parts:

$$\begin{aligned}
z^3 = (1 + i\tan\theta)^3 &= 1^3 + 3i\tan\theta - 3\tan^2\theta - i\tan^3\theta \\
&= (1 - 3\tan^2\theta) + i(3\tan\theta - \tan^3\theta).
\end{aligned}$$

Then we use our expression for $\tan 3\theta$ in terms of z^3:

$$\begin{aligned}\tan 3\theta &= \frac{\text{Im}(z^3)}{\text{Re}(z^3)}\\ &= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.\end{aligned}$$

22. Find multiplication groups of complex numbers isomorphic to rotation groups for the

(a) regular octagon.

We choose complex numbers corresponding to rotations of $0, 45°, \cdots, 315°$:

$$z = \left\{\text{cis}\,0, \text{cis}\,\frac{\pi}{4}, \text{cis}\,\frac{\pi}{2}, \text{cis}\,\frac{3\pi}{4}, \text{cis}\,\pi, \text{cis}\,\frac{5\pi}{4}, \text{cis}\,\frac{3\pi}{2}, \text{cis}\,\frac{7\pi}{4}\right\}.$$

(b) regular pentagon.

We simply choose complex numbers corresponding to rotations of $0, 72°, \cdots, 288°$:

$$z = \left\{\text{cis}\,0, \text{cis}\,\frac{2\pi}{5}, \text{cis}\,\frac{4\pi}{5}, \text{cis}\,\frac{6\pi}{5}, \text{cis}\,\frac{8\pi}{5}\right\}.$$

23. Make tables for

(a) the rotation group of the regular octagon.

There are 8 elements. If r is a rotation by $45°$, then the elements are $I, r, r^2, ..., r^7$. The group table is shown below.

$\cdot$	I	r	r^2	r^3	r^4	r^5	r^6	r^7
I	I	r	r^2	r^3	r^4	r^5	r^6	r^7
r	r	r^2	r^3	r^4	r^5	r^6	r^7	I
r^2	r^2	r^3	r^4	r^5	r^6	r^7	I	r
r^3	r^3	r^4	r^5	r^6	r^7	I	r	r^2
r^4	r^4	r^5	r^6	r^7	I	r	r^2	r^3
r^5	r^5	r^6	r^7	I	r	r^2	r^3	r^4
r^6	r^6	r^7	I	r	r^2	r^3	r^4	r^5
r^7	r^7	I	r	r^2	r^3	r^4	r^5	r^6

(b) the dihedral group of the square.

There are, once again, $4 \cdot 2 = 8$ elements. Let r be a rotation by $90°$, and f be a flip about say, the x-axis. The group table is shown below.

$\cdot$	I	r	r^2	r^3	f	fr	fr^2	fr^3
I	I	r	r^2	r^3	f	fr	fr^2	fr^3
r	r	r^2	r^3	I	fr^3	f	fr	fr^2
r^2	r^2	r^3	I	r	fr^2	fr^3	f	fr
r^3	r^3	I	r	r^2	fr	fr^2	fr^3	f
f	f	fr	fr^2	fr^3	I	r	r^2	r^3
fr	fr	fr^2	fr^3	f	r^3	I	r	r^2
fr^2	fr^2	fr^3	f	fr	r^2	r^3	I	r
fr^3	fr^3	f	fr	fr^2	r	r^2	r^3	I

(c) Is the difference between them fundamental?

Yes, the difference is fundamental, even though they have the same order. The easiest way to see this is that the latter group has 4 elements of order 2, but the former group has only 1 such element.

24. Which of the following tables defines a group? Why or why not?

(d)

\$	I	A	B	C	D
I	I	A	B	C	D
A	A	C	D	B	I
B	B	I	C	D	A
C	C	D	A	I	B
D	D	B	I	A	C

This table does not define a group, because it does not follow associativity. For example, $(D\$A)\$A = B\$A = I$, but $D\$(A\$A) = D\$C = A$.

(e)

#	I	A	B	C	D
I	I	A	B	C	D
A	A	B	C	D	I
B	B	C	D	I	A
C	C	D	I	A	B
D	D	I	A	B	C

This table is a group; in fact, it is a commutative group. The quickest way to see this is noting that it is (up to isomorphism) the cyclic group of order 5, where $A = r$, $B = r^2$, $C = r^3$, and $D = r^4$.

25. Name some subsets of the complex numbers that are groups under multiplication. I can name an infinite number of both finite and infinite groups with this property, so after you list a few of each type, try to generalize.

Some simple examples: $\{1\}$, $\{\pm 1\}$, $\{\pm 1, \pm i\}$.

In general, we choose the nth roots of unity: the numbers of the form $\operatorname{cis} \frac{2\pi k}{n}$ for $k \in \mathbb{Z}$. Each rotation is a symmetry of the n-gon, and thus this set under multiplication is isomorphic to the cyclic group of order n.

26. Prove with a diagram that if $|z| = 1$, then $\operatorname{Im}\left(\frac{z}{(z+1)^2}\right) = 0$.

To draw a diagram, we need to interpret these expressions as points on the complex plane. $|z| = 1$ implies that z is 1 away from the origin. $z + 1$ is z, translated right by 1 unit in the x-axis. Let $z + 1 = r \operatorname{cis} \theta$. Then $(z+1)^2 = r^2 \operatorname{cis} 2\theta$, so it forms an angle of 2θ with the origin.

If the quotient $\frac{z}{(z+1)^2}$ has no imaginary part, then $(z+1)^2$ is a real scalar times z. In other words, the two numbers have the same complex argument. Thus, we wish to prove that $\operatorname{Arg} z$ and $\operatorname{Arg}(z+1)^2$ are equal.

The scenario is shown in Figure 12.

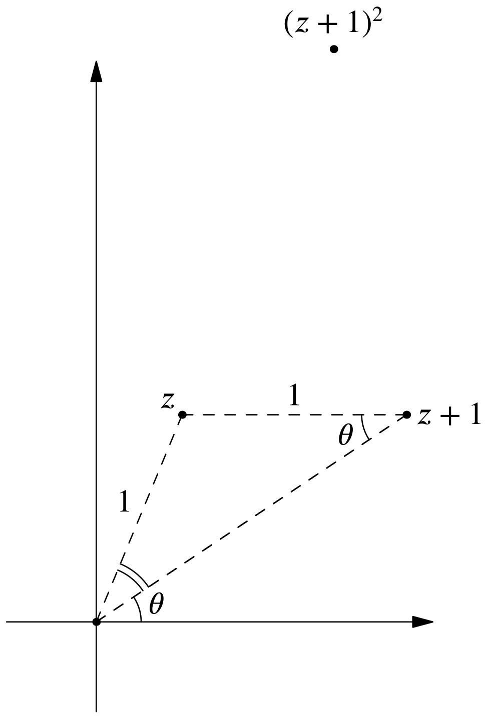


Figure 12: A graph of z, $z + 1$, and $(z+1)^2$.

As shown in the diagram, $\operatorname{Arg}(z+1) = \theta$. The triangle formed by O, z and $z+1$ is isosceles, since it has two sides of length 1. Furthermore, it has a base angle of θ by the Parallel Postulate. Thus, the angle marked with a double line is also θ, and $\operatorname{Arg} z = 2\theta$. But $\operatorname{Arg}(z+1)^2 = 2\theta$! Thus, we have

$$\begin{aligned} \operatorname{Im}\left(\frac{z}{(z+1)^2}\right) &= \operatorname{Im}\left(\frac{r_1 \operatorname{cis} 2\theta}{r_2 \operatorname{cis} 2\theta}\right) \\ &= \operatorname{Im}\left(\frac{r_1}{r_2} + 0i\right) \\ &= 0. \end{aligned}$$

This is not truly complete, because we have only considered z in the first quadrant. In this case, extending it to other locations of z is pretty trivial. Nonetheless, I provide an algebraic solution for fun.

We wish to show that $(z+1)^2 = kz$ for some real k. Express z as $\operatorname{cis}\theta$. Then

$$(\operatorname{cis}\theta + 1)^2 = \operatorname{cis}^2\theta + 2\operatorname{cis}\theta + 1.$$

Our supposed k is

$$\begin{aligned} k = \frac{(z+1)^2}{z} &= \frac{\operatorname{cis}^2\theta + 2\operatorname{cis}\theta + 1}{\operatorname{cis}\theta} \\ &= \operatorname{cis}\theta + 2 + \operatorname{cis}(-\theta) \\ &= \operatorname{cis}\theta + 2 + \overline{\operatorname{cis}\theta} \\ &= 2\operatorname{Re}(\operatorname{cis}\theta) + 2, \end{aligned}$$

which is indeed real. It's interesting what the scale factor actually is. Furthermore, since $\operatorname{Re}(\operatorname{cis}\theta) = \cos\theta$, we have a polar equation

$$|kz| = r = 2\cos\theta + 2,$$

hinting that the path traced out by $(z+1)^2$ is in fact a cardioid! The cardioid produced is shown in Figure 13 below.

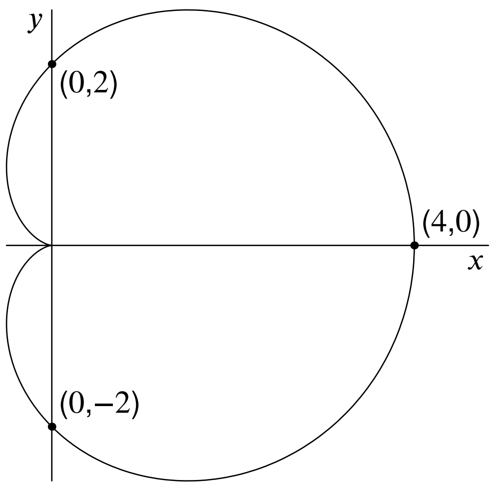


Figure 13: The cardioid produced by $(z+1)^2$ for $|z| = 1$.

It's all connected, guys.

27. Prove geometrically that if $|z| = 1$, then $|1 - z| = \left|2\sin\left(\frac{\operatorname{Arg} z}{2}\right)\right|$.

To prove this geometrically, we must again consider what the various expressions in the desired equation mean. $|z| = 1$ means that z is distance 1 from the origin. $1 - z$ is the reflection of z across the origin, then translated by 1 units right.

To geometrically interpret $A = 2\sin\left(\frac{\operatorname{Arg} z}{2}\right)$, we halve the angle $\theta = \operatorname{Arg} z$ and draw a vector of length 2; the imaginary component, or y height, of this new point is the desired quantity.

The diagram of all this is shown in Figure 14.

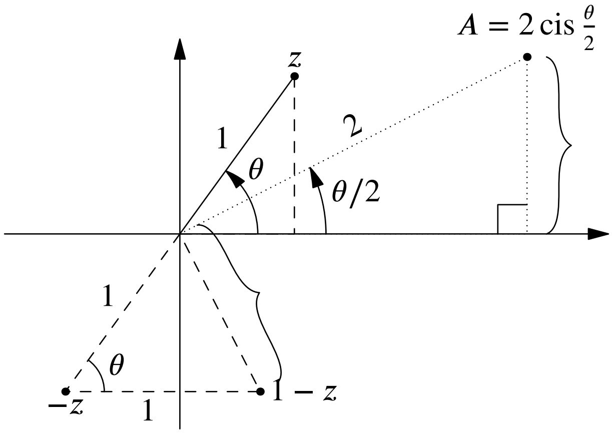


Figure 14: Graph of z, $1-z$, and $\sin\left(\frac{\theta}{2}\right)$.

We wish to show that the two lengths indicated in braces are equal. There's a couple of ways to do this; perhaps the most natural is to find a triangle congruent to the one formed by the origin, $-z$, and $1-z$. This is the other dashed triangle shown in Figure 15.

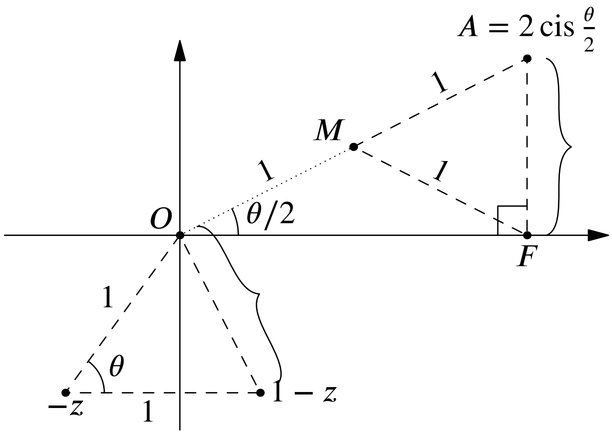


Figure 15: The succulent triangle,

Because the angles of a triangle sum to π, we know the angle $\angle MAF$ is $\pi - \frac{\theta}{2} - \frac{\pi}{2} = \frac{\pi-\theta}{2}$. Furthermore, $\triangle AMF$ is isosceles with an apex at M, since the midpoint of the hypotenuse of a right triangle is equidistant from all vertices. Thus, $\angle MAF = \angle MFA$, and we have

$$\angle AMF = \pi - \angle MAF - \angle MFA = \pi - \frac{\pi-\theta}{2}\cdot 2 = \theta.$$

Thus, by SAS Congruence, the two dashed triangles are congruent. Finally, pairing up the previously indicated sides, we have

$$|1-z| = \overline{AF} = \left|\operatorname{Im}\left(2\operatorname{cis}\frac{\theta}{2}\right)\right|,$$

as desired.

Technically, this proof is slightly incomplete, because some of these triangles do not exist as described for $\theta \geq 90°$. You can extend it to these cases with no problem, but I'd also like to give a algebraic proof to show its perks.

By the half-angle identity,

$$\left|2\sin\left(\frac{\operatorname{Arg} z}{2}\right)\right| = \left|\pm 2\sqrt{\frac{1-\cos\operatorname{Arg} z}{2}}\right| = \sqrt{2(1-\cos\operatorname{Arg} z)}.$$

Let $z = a+bi = \operatorname{cis}\theta$; note that $r = 1$ since $|z| = 1$. We know that $\cos\operatorname{Arg} z = a$. Then

$$\begin{aligned}
|1-(a+bi)| &= |(1-a)-bi| \\
&= \sqrt{b^2+(1-a)^2} \\
&= \sqrt{1-a^2+(1-2a+a^2)} \\
&= \sqrt{2-2a} \\
&= \sqrt{2(1-a)} \\
&= \sqrt{2(1-\cos \operatorname{Arg} z)}.
\end{aligned}$$

This matches our expression using half-angle for $\left|2\sin\left(\frac{\operatorname{Arg} z}{2}\right)\right|$.

Personally, I strongly prefer the algebraic solution because it is quick, easy to understand, and truly complete. Nonetheless, the geometric solution gives a better idea of *why* the equation is true.

28.

(a) Prove that if $(z-1)^{10} = z^{10}$, then $\operatorname{Re}(z) = \frac{1}{2}$. (Hint: if two numbers are equal, they have the same magnitude.)

We do as the hint suggests. We know that $|(z-1)^{10}| = |z^{10}|$. Expanding this out would be rough, but we can take the exponents out of the inside of the magnitude symbols[12].

So $|z-1|^{10} = |z|^{10}$. Since $|n| \geq 0$, we have $|z-1| = |z|$.

Let $z = a+bi$. Then $|a+bi-1| = \sqrt{(a-1)^2+b^2}$ and $|a+bi| = \sqrt{a^2+b^2}$. We set these equal and solve:

$$\begin{aligned}
\sqrt{(a-1)^2+b^2} &= \sqrt{a^2+b^2} \\
(a-1)^2+b^2 &= a^2+b^2 \\
(a-1)^2 &= a^2 \\
a-1 &= \pm a.
\end{aligned}$$

If $a-1=a$, then $-1=0$, which is dumb. Thus, $a-1=-a$, so $a=\frac{1}{2}$ and indeed, $\operatorname{Re}(z) = \frac{1}{2}$ as desired.

(b) How many solutions does this equation have?

We have $(z-1)^{10} = z^{10}$, so $(z-1)^{10} - z^{10} = P(z) = 0$, where P is a polynomial of degree 9. Thus, by the Fundamental Theorem of Algebra, there are 9 solutions... if there aren't any repeated roots. So this is only truly complete if we know there are no roots which appear in the factorization twice or more. Unfortunately, I can't think of a way to do this without calculus[13].

Let's start over. We should use the fact that $\operatorname{Re}(z) = \frac{1}{2}$. A simple diagram reveals that z and $z-1$ are symmetric about the y-axis, since $\operatorname{Re}(z) = \operatorname{Re}\left(\frac{1}{2}+bi\right) = -\operatorname{Re}\left(\frac{1}{2}+bi-1\right)$. The diagram is shown in Figure 16.

[12] This is true because $|(r\operatorname{cis}\theta)^n| = |r^n \operatorname{cis} n\theta| = |r^n|$.

[13] For that route, we simply check that $P''(z) = 0$ for all solutions, which isn't pleasant until a clever rearrangement and substitution. Try it if you know how!

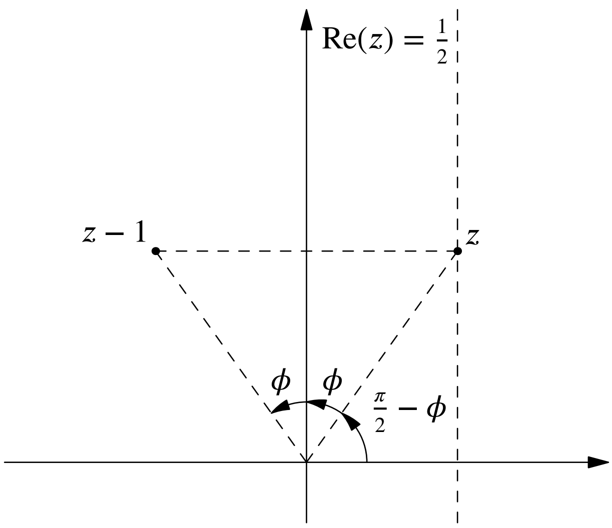


Figure 16: z and $1-z$, residents of the complex plane.

Let z in the first quadrant make a angle ϕ to the $\pm y$-axis. Note that we're using the y-axis, not the x-axis, for mathematical convenience. In general, for z in the first and fourth quadrants[14], we have $\operatorname{Arg} z = \frac{\pi}{2} - \phi$ and $\operatorname{Arg}(z-1) = \frac{\pi}{2} + \phi$. Since $|z| = |z-1| = r$, we have

$$z = r \operatorname{cis}\left(\frac{\pi}{2} - \phi\right); \; z - 1 = r \operatorname{cis}\left(\frac{\pi}{2} + \phi\right).$$

Since $(z-1)^{10} = z^{10}$, we have

$$\begin{aligned}
\left(r \operatorname{cis}\left(\frac{\pi}{2} - \phi\right)\right)^{10} &= \left(r \operatorname{cis}\left(\frac{\pi}{2} + \phi\right)\right)^{10} \\
r^{10} \operatorname{cis}(5\pi - 10\phi) &= r^{10} \operatorname{cis}(5\pi + 10\phi) \\
\operatorname{cis}(5\pi - 10\phi) &= \operatorname{cis}(5\pi + 10\phi) \\
5\pi - 10\phi + 2\pi k &= 5\pi + 10\phi && \text{For some } k \in \mathbb{Z} \\
20\phi &= 2\pi k \\
\phi &= \frac{2\pi k}{20}.
\end{aligned}$$

To find all unique solutions, we restrict k to the range $0 \le k \le 19$... wait, isn't that 20 solutions?

The issue is that z must be in the first or fourth quadrant, so that our premise $|z| = |z-1|$ is true. That means $0 < \phi < \pi$, a strict relation because $\phi = 0$ or $\phi = \pi$ only gives values along the y-axis, which does not intersect with $\mathrm{Re}(z) = 0$. Solving this gives

$$\begin{gathered}
0 < \frac{2\pi k}{20} < \pi \\
0 < \pi k < 10\pi \\
0 < k < 10 \\
k \in \{1, 2, \cdots, 8, 9\},
\end{gathered}$$

which is 9 solutions, in agreement with our polynomial argument.

29. I claim that $e^{i\theta} = \cos\theta + i\sin\theta = \operatorname{cis}\theta$, for θ in radians.

(a) Find e^{-it}.

$$e^{-it} = \cos(-t) + i\sin(-t) = \cos t - i \sin t.$$

(b) Find $\frac{e^{i\theta}+e^{-i\theta}}{2}$.

[14] If z is in the fourth quadrant, then you'd define ϕ as π + angle to negative $y$axis, where the angle is taken clockwise so it's positive.

$$\frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{\cos\theta + i\sin\theta + \cos\theta - i\sin\theta}{2}$$
$$= \cos\theta.$$

(c) Find $\frac{e^{i\theta}-e^{-i\theta}}{2i}$.

$$\frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{\cos\theta + i\sin\theta - (\cos\theta - i\sin\theta)}{2i}$$
$$= \sin\theta.$$

30. Use your new, complex definitions for $\cos$ and $\sin$ to find:

(a) $\cos^2\theta + \sin^2\theta$

$$\begin{aligned}\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^2 + \left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^2 &= \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^2 - \left(\frac{e^{i\theta}-e^{-i\theta}}{2}\right)^2\\ &= \left(\frac{e^{i\theta}+e^{-i\theta}}{2} + \frac{e^{i\theta}-e^{-i\theta}}{2}\right)\left(\frac{e^{i\theta}+e^{-i\theta}}{2} - \frac{e^{i\theta}-e^{-i\theta}}{2}\right)\\ &= \left(e^{-i\theta}\right)\left(e^{i\theta}\right)\\ &= e^{-i\theta+i\theta}\\ &= e^0\\ &= 1.\end{aligned}$$

That was expected.

(b) $\tan\theta$

$$\begin{aligned}\tan\theta = \frac{\sin\theta}{\cos\theta} &= \frac{\frac{e^{i\theta}-e^{-i\theta}}{2i}}{\frac{e^{i\theta}+e^{-i\theta}}{2}}\\ &= \frac{e^{i\theta}-e^{-i\theta}}{i(e^{i\theta}+e^{-i\theta})}.\end{aligned}$$

(c) $\cos 2\theta$

$$\cos 2\theta = \frac{e^{2i\theta}+e^{-2i\theta}}{2}.$$

(d) $\sin 2\theta$

$$\sin 2\theta = \frac{e^{2i\theta}-e^{-2i\theta}}{2i}.$$

(e) What kind of group is generated by $\left\{e^{i\theta}, e^{-i\theta}\right\}$ under the operation of multiplication if θ is an integer? A rational multiple of π?

If $\theta = 0$, then the group is the trivial group of order 1. If θ is any other integer, then a group isomorphic to the additive group of the integers is generated. We correspond $e^{ik\theta}$ with the integer k, so that

$$e^{ik_1\theta}\cdot e^{ik_2\theta} = e^{i(k_1+k_2)\theta} \leftrightarrow k_1 + k_2.$$

If θ is a rational multiple of π, say $\frac{p}{q}\cdot 2\pi$ where $\gcd(p,q) = 1$, then we get (up to isomorphism) cyclic group of order q.

31. You've used the quadratic equation throughout high school, but there's also a cubic equation that finds the roots of any cubic. Let's derive it, starting with the cubic $x^3 + bx^2 + cx + d = 0$.

(a) Make the substitution $x = y - \frac{b}{3}$. Combine like terms to create an equation of the form $y^3 - 3py - 2q = 0$, with p, q in terms of b, c, and d.

$$\begin{aligned}
\left(y - \frac{b}{3}\right)^3 + b\left(y - \frac{b}{3}\right)^2 + c\left(y - \frac{b}{3}\right) + d &= 0 \\
\left(y^3 - 3 \cdot \frac{by^2}{3} + 3 \cdot \frac{b^2 y}{9} - \frac{b^3}{27}\right) + \left(by^2 - \frac{2b^2 y}{3} + \frac{b^3}{9}\right) + \left(cy - \frac{bc}{3}\right) + d &= 0 \\
y^3 + (-b + b)\, y^2 + \left(\frac{b^2 y}{3} - \frac{2b^2 y}{3} + c\right) y + \left(-\frac{b^3}{27} + \frac{b^3}{9} - \frac{bc}{3} + d\right) &= 0 \\
y^3 + \left(c - \frac{b^2}{3}\right) y + \left(d - \frac{bc}{3} + \frac{2b^3}{27}\right) &= 0 \\
y^3 - 3\underbrace{\left(\frac{b^2}{9} - \frac{c}{3}\right)}_{p} y - 2\underbrace{\left(\frac{bc}{6} - \frac{b^3}{27} - \frac{d}{2}\right)}_{q} &= 0.
\end{aligned}$$

Thus, $p = \frac{b^2}{9} - \frac{c}{3}$ and $q = \frac{bc}{6} - \frac{b^3}{27} - \frac{d}{2}$.

(b) Rearrange this equation as $y^3 = 3py + 2q$.

$$y^3 = 3\left(\frac{b^2}{9} - \frac{c}{3}\right) y + 2\left(\frac{bc}{6} - \frac{b^3}{27} - \frac{d}{2}\right).$$

(c) Make the substitution $y = s + t$ into (b), and prove that y is a solution of the cubic in part (a) if $st = p$ and $s^3 + t^3 = 2q$.

We substitute $y = s + t$ and use the fact that $st = p$ and $s^3 + t^3 = 2q$ to simplify.

$$\begin{aligned}
(s + t)^3 &= 3p(s + t) + 2q \\
s^3 + 3s^2 t + 3st^2 + t^3 &= 3ps + 3pt + 2q \\
3s(st) + 3t(st) + s^3 + t^3 &= 3ps + 3pt + 2q \\
3sp + 3tp + 2q &= 3ps + 3pt + 2q \\
0 &= 0.
\end{aligned}$$

This checks out.

(d) Eliminate t between these two equations to get a quadratic in s^3.

We have $t^3 = 2q - s^3$. Also, $(st)^3 = p^3$, so

$$\begin{aligned}
(st)^3 = s^3 t^3 = s^3(2q - s^3) &= p^3 \\
-(s^3)^2 + 2qs^3 - p^3 &= 0 \\
(s^3)^2 - 2qs^3 + p^3 &= 0.
\end{aligned}$$

(e) Solve this quadratic to find s^3. By symmetry, what is t^3?

Let $w = s^3$. Then the above quadratic is $w^2 - 2qw + p^3 = 0$. The solutions are

$$s^3 = w = \frac{2q \pm \sqrt{4q^2 - 4p^3}}{2} = \frac{2q \pm 2\sqrt{q^2 - p^3}}{2} = q \pm \sqrt{q^2 - p^3}.$$

We have $t^3 = 2q - s^3 = 2q - (q \pm \sqrt{q^2 - p^3}) = q \mp \sqrt{q^2 - p^3}$. This inverted $\pm$ sign, $\mp$, means that when s^3's $\pm$ sign is positive, the $\mp$ sign is negative, and vice versa.

(f) Find a formula for y in terms of p and q. What about a formula for x?

Taking cube roots of both sides of our expressions for t^3 and s^3, we find that

$$\begin{aligned} s &= \sqrt[3]{q \pm \sqrt{q^2 - p^3}}, \\ t &= \sqrt[3]{q \mp \sqrt{q^2 - p^3}}. \end{aligned}$$

We must keep in mind, however, that over the complex numbers, cube rooting has 3 possible values. Thus, the three solutions for s and t are

$$\begin{aligned} s &= \sqrt[3]{q \pm \sqrt{q^2 - p^3}} \cdot \operatorname{cis} \frac{2\pi k}{3}, \\ t &= \sqrt[3]{q \mp \sqrt{q^2 - p^3}} \cdot \operatorname{cis} \left(2\pi - \frac{2\pi k}{3}\right), \end{aligned}$$

where $k \in \{0, 1, 2\}$ and the cube root is taking its principal value. We multiply them by cis with these angles to preserve the fact that $st = q$, since otherwise it would produce another result:

$$\begin{aligned} \sqrt[3]{q \pm \sqrt{q^2 - p^3}} \cdot \operatorname{cis} \frac{2\pi k}{3} \cdot \sqrt[3]{q \mp \sqrt{q^2 - p^3}} \cdot \operatorname{cis} \left(2\pi - \frac{2\pi k}{3}\right) &= \sqrt[3]{\left(q \pm \sqrt{q^2 - p^3}\right)\left(q \mp \sqrt{q^2 - p^3}\right)} \cdot \operatorname{cis} 2\pi \\ &= \sqrt[3]{q^2 - (q^2 + p^3)} \\ &= p. \end{aligned}$$

Thus, we have

$$y = s + t = \sqrt[3]{q \pm \sqrt{q^2 - p^3}} \cdot \operatorname{cis} \frac{2\pi k}{3} + \sqrt[3]{q \mp \sqrt{q^2 - p^3}} \cdot \operatorname{cis} \left(2\pi - \frac{2\pi k}{3}\right).$$

To get x, we substitute $x = y - \frac{b}{3}$ to get

$$x = \sqrt[3]{q \pm \sqrt{q^2 - p^3}} \cdot \operatorname{cis} \frac{2\pi k}{3} + \sqrt[3]{q \mp \sqrt{q^2 - p^3}} \cdot \operatorname{cis} \left(2\pi - \frac{2\pi k}{3}\right) - \frac{b}{3}.$$

You could substitute our values of p, q in terms of b, c, d to get a monstrous equation for x in terms of only b, c, d... but no thanks.

(g) What if we started with $ax^3 + bx^2 + cx + d = 0$, with a coefficient in front of the x^3 term as well? Can you come up with a formula for x?

Sure! We divide through the equation by a:

$$\begin{aligned} \frac{ax^3 + bx^2 + cx + d}{a} &= 0 \\ \implies x^3 + \frac{b}{a} + \frac{c}{a} + \frac{d}{a} &= 0. \end{aligned}$$

We can then attack this as we already did, setting $b' = \frac{b}{a}$, $c' = \frac{c}{a}$ and $d' = \frac{d}{a}$, then applying the formula.

32. Starting with the same cubic as in problem 31b.

(a) Let $c = \cos\theta$. Remember that $\cos 3\theta = 4c^3 - 3c$, as we proved. Substitute $y = 2c\sqrt{p}$ into $y^3 = 3py + 2q$ to obtain $4c^3 - 3c = \frac{q}{p^{3/2}}$.

We substitute and proceed:

$$\begin{aligned} y^3 &= 3py + 2q \\ (2c\sqrt{p})^3 &= 3p(2c\sqrt{p}) + 2q \\ 8c^3p^{3/2} &= 6cp^{3/2} + 2q \\ 8c^3p^{3/2} - 6cp^{3/2} &= 2q \\ 4c^3 - 3c &= \frac{2q}{2p^{3/2}} = \frac{q}{p^{3/2}}. \end{aligned}$$

(b) Provided that $q^2 \leq p^3$, show that $y = 2\sqrt{p}\cos\left(\frac{1}{3}(\theta + 2\pi n)\right)$, where n is an integer. Why does this yield all three solutions?

This isn't actually hard. We know that $\cos 3\theta = 4c^3 - 3c = \frac{q}{p^{3/2}}$, so there are three possible values for $\cos\theta = c$. Namely, if $\theta_0 = \frac{1}{3}\cos^{-1}\frac{q}{p^{3/2}}$ is the principal value, then we also have unique solutions

$$\theta_1 = \frac{2\pi}{3} + \theta_0,\ \theta_2 = \frac{4\pi}{3} + \theta_0,$$

because multiplying these by 3 to get 3θ just adds a multiple of 2π. Indeed, we have

$$c = \cos\frac{1}{3}(\theta + 2\pi n)$$

as a solution for any integer n. Substituting into the expression for y, we get

$$y = 2c\sqrt{p} = 2\cos\left(\frac{1}{3}(\theta + 2\pi n)\right)\sqrt{p},$$

as desired. This yields all three solutions because, as we observed, the only unique values this makes are for $n \in \{0, 1, 2\}$.

Note that if $q^2 > p^3$, then this strategy actually still works, but you have to define $\cos$ (and $\cos^{-1}$) over a larger, complex domain. This is certainly possible though!

(c) Explain how you would find θ from p and q, and how we would use what we have found to solve an arbitrary cubic $ax^3 + bx^2 + cx + d = 0$.

We have $\cos 3\theta = \frac{q}{p^{3/2}}$, so

$$\theta = \cos^{-1}\frac{q}{p^{3/2}}.$$

The steps to solving an arbitrary cubic are the following:

1. Divide through by a to get a new cubic $x^3 + b'x^2 + c'x + d' = 0$.
2. Compute $p = \frac{b'^2}{9} - \frac{c'}{3}$ and $q = \frac{b'c'}{6} - \frac{b'^3}{27} - \frac{d'}{2}$.
3. Compute $\theta = \cos^{-1}\frac{q}{p^{3/2}}$.
4. Substitute this value of θ into $x = y - \frac{b}{3} = 2\sqrt{p}\cos\left(\frac{1}{3}(\theta + 2\pi n)\right) - \frac{b}{3}$, where $n \in \{0, 1, 2\}$.

These two problems were a doozy!

7 Your Daily Dose of Vitamin i

1. **We will use complex numbers to find identities for** $\cot$**. Use Pascal's triangle to expand the following:**

   (a) $(a+b)^3$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

   (b) $(a+b)^4$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

   (c) $(a+b)^5$

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

1. **(cont.) Then substitute** $b = i = \sqrt{-1}$ **and expand:**

   (d) $(a+i)^3$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + 3a^2i - 3a - i.$$

   (e) $(a+i)^4$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 = a^4 + 4a^3i - 6a^2 - 4ai + 1.$$

   (f) $(a+i)^5$

$$(a+i)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 = a^5 + 5a^4i - 10a^3 - 10a^2i + 5a + i.$$

1. **(cont.) Finally, substitute** $a = \cot\theta$ **and expand:**

   (g) $(\cot\theta + i)^3$

$$(\cot\theta + i)^3 = a^3 + 3a^2i - 3a - i = (\cot^3\theta - 3\cot\theta) + i(3\cot^2\theta - 1).$$

   (h) $(\cot\theta + i)^4$

$$(\cot\theta + i)^4 = a^4 + 4a^3i - 6a^2 - 4ai + 1 = (\cot^4\theta - 6\cot^2\theta + 1) + (4\cot^3\theta - 4\cot\theta).$$

   (i) $(\cot\theta + i)^5$

$$(\cot\theta + i)^5 = a^5 + 5a^4i - 10a^3 - 10a^2i + 5a + i = (\cot^5\theta - 10\cot^3\theta + 5\cot\theta) + i(5\cot^4\theta - 10\cot^2\theta + 1).$$

1. **(cont.) Consider** $z = i + \cot\theta$**.**

   (j) Use the above results to find identities for (i) $\cot 3\theta$**, (ii)** $\cot 4\theta$**, and (iii)** $\cot 5\theta$**.**

      i. $\cot 3\theta$

Given the right triangle formed by $z = i + \cot\theta$ in Figure 7, we have $\tan(\operatorname{Arg} z) = \frac{1}{\cot\theta} = \tan\theta$, so $\operatorname{Arg} z = \theta$ and $z = r\operatorname{cis}\theta$.

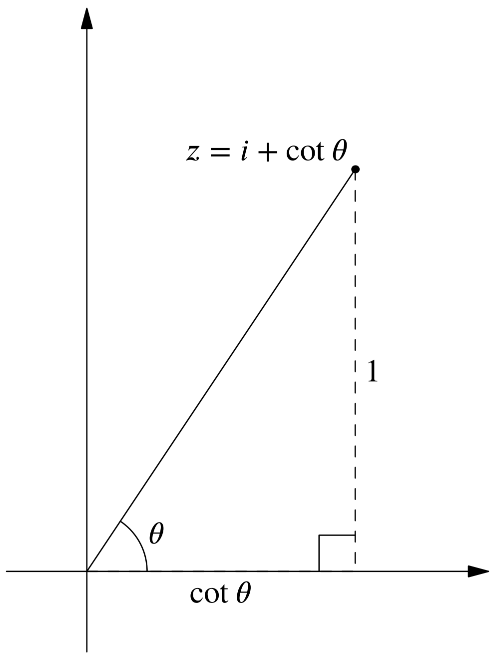


Figure 1: $\mathrm{Arg}(i + \cot\theta) = \theta$.

Thus, we have

$$\begin{aligned}
\cot 3\theta &= \frac{\cos 3\theta}{\sin 3\theta} \\
&= \frac{\mathrm{Re}(\mathrm{cis}\, 3\theta)}{\mathrm{Im}(\mathrm{cis}\, 3\theta)} \\
&= \frac{\mathrm{Re}(r^3 \,\mathrm{cis}\, 3\theta)}{\mathrm{Im}(r^3 \,\mathrm{cis}\, 3\theta)} \\
&= \frac{\mathrm{Re}(z^3)}{\mathrm{Im}(z^3)}.
\end{aligned}$$

We substitute in our expression for z^3, $(\cot^3\theta - 3\cot\theta) + i(3\cot^2\theta - 1)$:

$$\cot 3\theta = \frac{\cot^3\theta - 3\cot\theta}{3\cot^2\theta - 1}.$$

i. $\cot 4\theta$

We proceed in the same way as the last subproblem.

$$\begin{aligned}
\cot 4\theta &= \frac{\cos 4\theta}{\sin 4\theta} \\
&= \frac{\mathrm{Re}(\mathrm{cis}\, 4\theta)}{\mathrm{Im}(\mathrm{cis}\, 4\theta)} \\
&= \frac{\mathrm{Re}(r^4 \,\mathrm{cis}\, 4\theta)}{\mathrm{Im}(r^4 \,\mathrm{cis}\, 4\theta)} \\
&= \frac{\mathrm{Re}(z^4)}{\mathrm{Im}(z^4)} \\
\cot 4\theta &= \frac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta}.
\end{aligned}$$

i. $\cot 5\theta$

We proceed in the same way as the last subproblem.

$$\begin{aligned}
\cot 5\theta &= \frac{\cos 5\theta}{\sin 5\theta} \\
&= \frac{\mathrm{Re}(\mathrm{cis}\, 5\theta)}{\mathrm{Im}(\mathrm{cis}\, 5\theta)} \\
&= \frac{\mathrm{Re}(r^5 \,\mathrm{cis}\, 5\theta)}{\mathrm{Im}(r^5 \,\mathrm{cis}\, 5\theta)} \\
&= \frac{\mathrm{Re}(z^5)}{\mathrm{Im}(z^5)} \\
&= \frac{\cot^5\theta - 10\cot^3\theta + 5\cot\theta}{5\cot^4\theta - 10\cot^2\theta + 1}.
\end{aligned}$$

(k) Graph z, z^2, z^3, z^4, and z^5, with $\theta \approx 75°$. What is your solution method?

To graph these, I first calculated the approximate magnitude of z, which is how many times each subsequent power will be scaled by. We have $|1 + \cot 75°| \approx 1.268$, so we only need to scale by about $\frac{5}{4}$ each time. Of course, we rotate by about $75°$ each time.

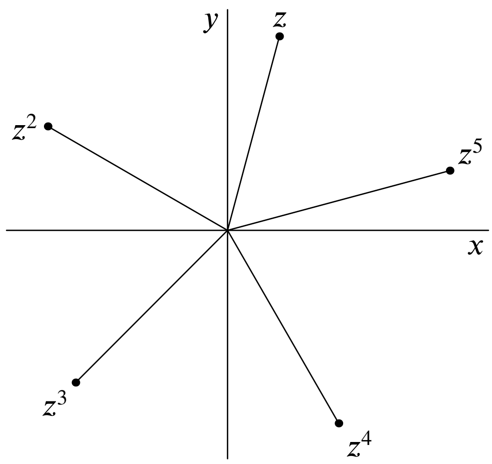


Figure 2: Graphs of z, z^2, z^3, z^4, and z^5.

2. Compute $(1+i)^n$ for $n = 3, 4, 5, \ldots$. Can you find a general pattern?

We have

$$\begin{aligned}
(1+i)^3 &= 1^3 + 3i - 3 - i &&= -2 - 2i \\
(1+i)^4 &= 1^4 + 4i - 6 - 6i + 1 &&= -4 - 2i \\
(1+i)^5 &= 1^5 + 5i - 10 - 10i + 5 + i &&= -4 - 4i.
\end{aligned}$$

We can find the pattern by representing $1 + i = \sqrt{2}\operatorname{cis} 45°$. This shows that it has period 8 and let's us find an expression for $(1+1)^n$:

$$(1+i)^n = \left(\sqrt{2}\operatorname{cis} 45°\right)^n = 2^{n/2}\operatorname{cis}\left(\frac{n\pi}{4}\right).$$

3. Expand and graph $\operatorname{cis}^n \theta$ for $n = 2, 3, 4, \ldots$.

Let $\cos\theta = c$ and $\sin\theta = s$. We have

$$\begin{aligned}
(c+is)^2 &= c^2 + 2csi - s^2 = (c^2 - s^2) + i(2cs) \\
(c+is)^3 &= c^3 + 3c^2si - 3cs^2 - s^3i = (c^3 - 3cs^2) + i(3c^2s - s^3) \\
(c+is)^4 &= c^4 + 4c^3si - 6c^2s^2 - 4cs^3i + s^4 = (c^4 - 6c^2s^2 + s^4) + i(4c^3s - 4cs^3) \\
(c+is)^5 &= c^5 + 5c^4si - 10c^3s^2 - 10c^2s^3i + 5cs^4 + s^5i = (c^5 - 10c^3s^2 + 5cs^4) + i(5c^4s - 10c^2s^3 + s^5).
\end{aligned}$$

The graphs of $\operatorname{cis}^n \theta$ for $\theta \approx 50°$ are shown in Figure 3 below.

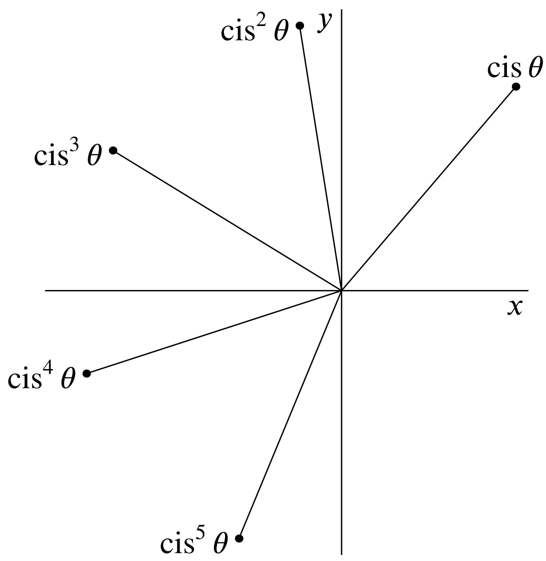


Figure 3: Graphs of $\mathrm{cis}^n\,\theta$ for $\theta \approx 50°$.

(a) Why is the real part $\cos n\theta$ and the imaginary part $\sin n\theta$?

By DeMoivre's theorem, $\mathrm{cis}^n\,\theta = \mathrm{cis}\,n\theta$, which by definition has $\mathrm{Im}(\mathrm{cis}\,n\theta) = \cos n\theta$ and $\mathrm{Re}(\mathrm{cis}\,n\theta) = \sin n\theta$.

(b) Use your results to write identities for $\cos n\theta$ and $\sin n\theta$ for $n = 2, 3, 4, 5$.

Here they are. Again, let $\cos\theta = c$ and $\sin\theta = s$:

$$\begin{aligned}
\cos 2\theta &= \mathrm{Re}(\mathrm{cis}\,2\theta) = c^2 - s^2\\
\cos 3\theta &= \mathrm{Re}(\mathrm{cis}\,3\theta) = c^3 - 3cs^2\\
\cos 4\theta &= \mathrm{Re}(\mathrm{cis}\,4\theta) = c^4 - 6c^2s^2 + s^4\\
\cos 5\theta &= \mathrm{Re}(\mathrm{cis}\,5\theta) = c^5 - 10c^3s^2 + 5cs^4\\
\sin 2\theta &= \mathrm{Im}(\mathrm{cis}\,2\theta) = 2cs\\
\sin 3\theta &= \mathrm{Im}(\mathrm{cis}\,3\theta) = 3c^2s - s^3\\
\sin 4\theta &= \mathrm{Im}(\mathrm{cis}\,4\theta) = 4c^3s - 4cs^3\\
\sin 5\theta &= \mathrm{Im}(\mathrm{cis}\,5\theta) = 5c^4s - 10c^2s^3 + s^5.
\end{aligned}$$

4. Compute $\cos 7° + \cos 79° + \cos 151° + \cos 223° + \cos 295°$ without a calculator. (Hint: what does this have to do with complex numbers?)

These numbers look random, but a closer inspection reveals they are in arithmetic progression, with starting term 7 and increasing 72° each time. That's the rotation of a pentagon!

We rewrite this as the real component of a sum of cises, then manipulate and evaluate:

$$\begin{aligned}
\cos 7° + \cos 79° + \cos 151° + \cos 223° + \cos 295° &= \mathrm{Re}(\mathrm{cis}\,7° + \mathrm{cis}\,79° + \mathrm{cis}\,151° + \mathrm{cis}\,223° + \mathrm{cis}\,295°)\\
&= \mathrm{Re}((\mathrm{cis}\,7°)(\mathrm{cis}\,0° + \mathrm{cis}\,72\circ + \mathrm{cis}\,144° + \mathrm{cis}\,216° + \mathrm{cis}\,288°))\\
&= \mathrm{Re}((\mathrm{cis}\,7°)(0))\\
&= \mathrm{Re}(0)\\
&= 0.
\end{aligned}$$

5. Factor the following:

(a) $x^6 - 1$ as a difference of squares

We substitute $y = x^3$, giving $y^2 - 1 = (y+1)(y-1)$. Substituting back in, we get

$$(x^3+1)(x^3-1).$$

(b) $x^6 - 1$ **as a difference of cubes**

We now substitute $y = x^2$, giving $y^3 - 1 = (y-1)(y^2+y+1)$. Substituting back in, we get

$$(x^2-1)(x^4+x^2+1)$$

.

(c) $x^4 + x^2 + 1$ **over the real numbers**

This one isn't as obvious. We substitute $y = x^2$ to get $y^2 + y + 1$ and find the quadratic's zeroes: $y = \frac{-1\pm\sqrt{1-4}}{2} = \frac{-1\pm i\sqrt{3}}{2}$.
So it is irreducible over the reals.

(d) $x^6 - 1$ **completely**

We already broke it down into (x^3+1) and (x^3-1). Going further, we have $x^3 + 1 = (x+1)(x^2-x+1)$ and $x^3 - 1 = (x-1)(x^2+x+1)$. To break apart the last two quadratics, we find their zeros:

$$x^2 - x + 1 = 0 \Longrightarrow x = \frac{1 \pm i\sqrt{3}}{2} \Longrightarrow \left(x - \frac{1-i\sqrt{3}}{2}\right)\left(x - \frac{1+i\sqrt{3}}{2}\right).$$

$$x^2 + x + 1 = 0 \Longrightarrow x = \frac{-1 \pm i\sqrt{3}}{2} \Longrightarrow \left(x + \frac{1-i\sqrt{3}}{2}\right)\left(x + \frac{1+i\sqrt{3}}{2}\right).$$

Combining all these, we get the complete factorization over the complex numbers:

$$x^6 - 1 = (x+1)\left(x - \frac{1-i\sqrt{3}}{2}\right)\left(x - \frac{1+i\sqrt{3}}{2}\right)(x-1)\left(x + \frac{1-i\sqrt{3}}{2}\right)\left(x + \frac{1+i\sqrt{3}}{2}\right).$$

(e) $x^4 + x^2 + 1$ **completely**

We could do a lot of work again, or we could observe that $x^4 + x^2 + 1 = \frac{x^6-1}{x^2-1} = \frac{x^6-1}{(x+1)(x-1)}$. Removing the denominator's terms from our factorization of $x^6 - 1$ we found in the last subproblem, we get

$$x^4 + x^2 + 1 = \left(x - \frac{1-i\sqrt{3}}{2}\right)\left(x - \frac{1+i\sqrt{3}}{2}\right)\left(x + \frac{1-i\sqrt{3}}{2}\right)\left(x + \frac{1+i\sqrt{3}}{2}\right).$$

6. Let $f(z) = \frac{z+1}{z-1}$**.**

(a) Without a calculator, compute $f^{2014}(z)$**.**

This seems terrifying. Let's try computing $f^2(z)$ and perhaps $f^3(z)$.

$$f^2(z) = \frac{f(z)+1}{f(z)-1} = \frac{\frac{z+1}{z-1}+1}{\frac{z+1}{z-1}-1} = \frac{\frac{2z}{z-1}}{\frac{2}{z-1}} = z.$$

Oh.
Since 2014 is even, we have $f^{2014}(z) = (f^2)^{1007}(z) = z$.

(b) What if you replace 2014 **with the current year?**

Let y be the current year. As I write this, it is 1492.
If y is even, then $f^y(z) = (f^2)^{y/2}(z) = z$. If y is odd, then $f^y(z) = f((f^2)^{(y-1)/2}(z)) = f(z) = \frac{z+1}{z-1}$.

7. Find $\operatorname{Im}\left((\operatorname{cis} 12^\circ + \operatorname{cis} 48^\circ)^6\right)$**.**

These are some weird looking angles. Thinking back to some older problems, however, the resultant angle of the addition may be tractable. We draw a diagram, shown in Figure 4.

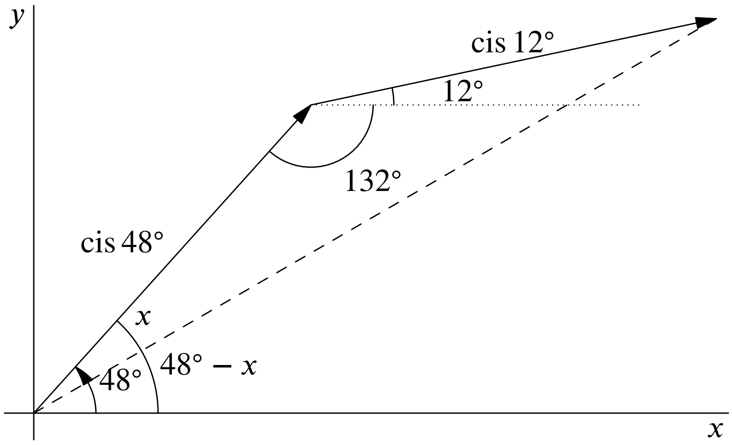


Figure 4: Adding $\operatorname{cis} 12° + \operatorname{cis} 48°$.

Consider the isosceles triangle. The apex has angle measure $132° + 12° = 144°$, so the base angles are each $x = \frac{180°-144°}{2} = 18°$. But $\operatorname{Arg}(\operatorname{cis} 12° + \operatorname{cis} 48°) = 48° - x = 30°$!

That's a familiar angle. Indeed, we have $z = \operatorname{cis} 12° + \operatorname{cis} 48° = r \operatorname{cis} 30°$ for some r. It doesn't really matter which r, because

$$\operatorname{Im}((r \operatorname{cis} 30°)^6) = \operatorname{Im}(r^6 \operatorname{cis} 180°) = \operatorname{Im}(-r^6) = 0.$$

8. Let x satisfy the equation $x + \frac{1}{x} = 2\cos\theta$.

(a) Compute $x^2 + \frac{1}{x^2}$ in terms of θ.

Squaring the left hand side will get us some terms that look close to what we want.

$$\left(x + \frac{1}{x}\right)^2 = x^2 + 2 + \frac{1}{x^2}.$$

So $x^2 + \frac{1}{x^2} = (2\cos\theta)^2 - 2 = 4\cos^2\theta - 2 = 2(2\cos^2\theta - 1) = 2\cos 2\theta$. Huh.

(b) Compute $x^n + \frac{1}{x^n}$ in terms of n and θ.

We conjecture that this is equal to $2\cos n\theta$. To do this, we let $x = \operatorname{cis}\frac{\theta}{n}$, so $x^n = \operatorname{cis}\theta$, and compute. That should give us some similar looking terms:

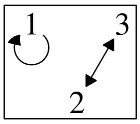
$$\begin{aligned}
x^n + \frac{1}{x^n} &= \operatorname{cis}\theta + \frac{1}{\operatorname{cis}\theta} \\
&= \operatorname{cis}\theta + \operatorname{cis}(-\theta) \\
&= \operatorname{cis}\theta + \overline{\operatorname{cis}\theta} \\
&= 2\operatorname{Re}(\operatorname{cis}\theta) \\
&= 2\cos\theta.
\end{aligned}$$

This proves the relationship.

8 Matrix Multiplication

1. The three-post snap group can be represented by a set of graphs, each with three towns. The posts are the towns and the elastic bands are the roads. For example,

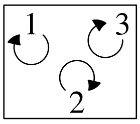
$$A = \begin{array}{c c} & \textbf{to} \\ & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \textbf{from} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array} \longleftrightarrow$$



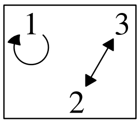
(a) Draw the graphs and transportation matrices for this group.

Here they are!

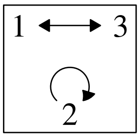
$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longleftrightarrow$$



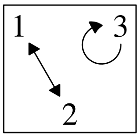
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \longleftrightarrow$$



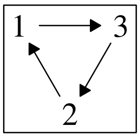
$$B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \longleftrightarrow$$



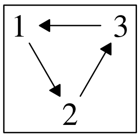
$$C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longleftrightarrow$$



$$D = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \longleftrightarrow$$



$$E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \longleftrightarrow$$



(b) Try a few multiplications and notice the isomorphism to the snap group.

Before, we found that $A \bullet B = E$. But does this work with the matrices? We have

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \langle 1,0,0\rangle \cdot \langle 0,0,1\rangle & \langle 1,0,0\rangle \cdot \langle 0,1,0\rangle & \langle 1,0,0\rangle \cdot \langle 1,0,0\rangle \\ \langle 0,0,1\rangle \cdot \langle 0,0,1\rangle & \langle 0,0,1\rangle \cdot \langle 0,1,0\rangle & \langle 0,0,1\rangle \cdot \langle 1,0,0\rangle \\ \langle 0,1,0\rangle \cdot \langle 0,0,1\rangle & \langle 0,1,0\rangle \cdot \langle 0,1,0\rangle & \langle 0,1,0\rangle \cdot \langle 1,0,0\rangle \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = D. \end{aligned}$$

Huh?

The issue is simple. Matrix multiplication, just like the snap operation, is not commutative, and we need to flip the order of the matrices so it represents taking B first, then A. After all, that's what we defined $A \bullet B$ to be.

$$
\begin{aligned}
BA &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \langle 0,0,1\rangle \cdot \langle 1,0,0\rangle & \langle 0,0,1\rangle \cdot \langle 0,0,1\rangle & \langle 0,0,1\rangle \cdot \langle 0,1,0\rangle \\ \langle 0,1,0\rangle \cdot \langle 1,0,0\rangle & \langle 0,1,0\rangle \cdot \langle 0,0,1\rangle & \langle 0,1,0\rangle \cdot \langle 0,1,0\rangle \\ \langle 1,0,0\rangle \cdot \langle 1,0,0\rangle & \langle 1,0,0\rangle \cdot \langle 0,0,1\rangle & \langle 1,0,0\rangle \cdot \langle 0,1,0\rangle \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = E.
\end{aligned}
$$

This works for any of the matrices.

2. Using 3×3 matrices A and B from this section, compute

For reference, the matrices are

$$
A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}.
$$

(a) $AA = A^2$

We use the column vector/row vector approach.

$$
\begin{aligned}
AA &= \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \langle 1,1,2,2\rangle \cdot \langle 1,1,2,2\rangle & \langle 1,1,2,2\rangle \cdot \langle 1,1,1,0\rangle & \langle 1,1,2,2\rangle \cdot \langle 2,1,1,1\rangle & \langle 1,1,2,2\rangle \cdot \langle 2,0,1,1\rangle \\ \langle 1,1,1,0\rangle \cdot \langle 1,1,2,2\rangle & \langle 1,1,1,0\rangle \cdot \langle 1,1,1,0\rangle & \langle 1,1,1,0\rangle \cdot \langle 2,1,1,1\rangle & \langle 1,1,1,0\rangle \cdot \langle 2,0,1,1\rangle \\ \langle 2,1,1,1\rangle \cdot \langle 1,1,2,2\rangle & \langle 2,1,1,1\rangle \cdot \langle 1,1,1,0\rangle & \langle 2,1,1,1\rangle \cdot \langle 2,1,1,1\rangle & \langle 2,1,1,1\rangle \cdot \langle 2,0,1,1\rangle \\ \langle 2,0,1,1\rangle \cdot \langle 1,1,2,2\rangle & \langle 2,0,1,1\rangle \cdot \langle 1,1,1,0\rangle & \langle 2,0,1,1\rangle \cdot \langle 2,1,1,1\rangle & \langle 2,0,1,1\rangle \cdot \langle 2,0,1,1\rangle \end{bmatrix} \\
&= \begin{bmatrix} 10 & 4 & 7 & 6 \\ 4 & 3 & 4 & 3 \\ 7 & 4 & 7 & 6 \\ 6 & 3 & 6 & 6 \end{bmatrix}.
\end{aligned}
$$

(b) AB

$$
\begin{aligned}
AB &= \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \langle 1,1,2,2\rangle \cdot \langle 1,1,0,0\rangle & \langle 1,1,2,2\rangle \cdot \langle 1,1,0,0\rangle & \langle 1,1,2,2\rangle \cdot \langle 1,1,0,0\rangle & \langle 1,1,2,2\rangle \cdot \langle 1,1,0,0\rangle \\ \langle 1,1,1,0\rangle \cdot \langle 0,1,1,0\rangle & \langle 1,1,1,0\rangle \cdot \langle 0,1,1,0\rangle & \langle 1,1,1,0\rangle \cdot \langle 0,1,1,0\rangle & \langle 1,1,1,0\rangle \cdot \langle 0,1,1,0\rangle \\ \langle 2,1,1,1\rangle \cdot \langle 0,0,1,1\rangle & \langle 2,1,1,1\rangle \cdot \langle 0,0,1,1\rangle & \langle 2,1,1,1\rangle \cdot \langle 0,0,1,1\rangle & \langle 2,1,1,1\rangle \cdot \langle 0,0,1,1\rangle \\ \langle 2,0,1,1\rangle \cdot \langle 1,0,0,1\rangle & \langle 2,0,1,1\rangle \cdot \langle 1,0,0,1\rangle & \langle 2,0,1,1\rangle \cdot \langle 1,0,0,1\rangle & \langle 2,0,1,1\rangle \cdot \langle 1,0,0,1\rangle \end{bmatrix} \\
&= \begin{bmatrix} 3 & 2 & 3 & 4 \\ 1 & 2 & 2 & 1 \\ 3 & 3 & 2 & 2 \\ 3 & 2 & 1 & 2 \end{bmatrix}.
\end{aligned}
$$

(c) BA

$$BA = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \langle 1,1,0,0\rangle \cdot \langle 1,1,2,2\rangle & \langle 1,1,0,0\rangle \cdot \langle 1,1,1,0\rangle & \langle 1,1,0,0\rangle \cdot \langle 2,1,1,1\rangle & \langle 1,1,0,0\rangle \cdot \langle 2,0,1,1\rangle \\ \langle 0,1,1,0\rangle \cdot \langle 1,1,2,2\rangle & \langle 0,1,1,0\rangle \cdot \langle 1,1,1,0\rangle & \langle 0,1,1,0\rangle \cdot \langle 2,1,1,1\rangle & \langle 0,1,1,0\rangle \cdot \langle 2,0,1,1\rangle \\ \langle 0,0,1,1\rangle \cdot \langle 1,1,2,2\rangle & \langle 0,0,1,1\rangle \cdot \langle 1,1,1,0\rangle & \langle 0,0,1,1\rangle \cdot \langle 2,1,1,1\rangle & \langle 0,0,1,1\rangle \cdot \langle 2,0,1,1\rangle \\ \langle 1,0,0,1\rangle \cdot \langle 1,1,2,2\rangle & \langle 1,0,0,1\rangle \cdot \langle 1,1,1,0\rangle & \langle 1,0,0,1\rangle \cdot \langle 2,1,1,1\rangle & \langle 1,0,0,1\rangle \cdot \langle 2,0,1,1\rangle \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 3 & 2 \\ 3 & 2 & 2 & 1 \\ 4 & 1 & 2 & 2 \\ 3 & 1 & 3 & 3 \end{bmatrix}.$$

(d) B^2

$$BA = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \langle 1,1,0,0\rangle \cdot \langle 1,0,0,1\rangle & \langle 1,1,0,0\rangle \cdot \langle 1,1,0,0\rangle & \langle 1,1,0,0\rangle \cdot \langle 0,1,1,0\rangle & \langle 1,1,0,0\rangle \cdot \langle 0,0,1,1\rangle \\ \langle 0,1,1,0\rangle \cdot \langle 1,0,0,1\rangle & \langle 0,1,1,0\rangle \cdot \langle 1,1,0,0\rangle & \langle 0,1,1,0\rangle \cdot \langle 0,1,1,0\rangle & \langle 0,1,1,0\rangle \cdot \langle 0,0,1,1\rangle \\ \langle 0,0,1,1\rangle \cdot \langle 1,0,0,1\rangle & \langle 0,0,1,1\rangle \cdot \langle 1,1,0,0\rangle & \langle 0,0,1,1\rangle \cdot \langle 0,1,1,0\rangle & \langle 0,0,1,1\rangle \cdot \langle 0,0,1,1\rangle \\ \langle 1,0,0,1\rangle \cdot \langle 1,0,0,1\rangle & \langle 1,0,0,1\rangle \cdot \langle 1,1,0,0\rangle & \langle 1,0,0,1\rangle \cdot \langle 0,1,1,0\rangle & \langle 1,0,0,1\rangle \cdot \langle 0,0,1,1\rangle \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \end{bmatrix}.$$

(e) Which one (AB and BA) represents taking a step by walking, then by bus?

Since A is walking and B is bus, we know that AB is a step by walking, then by bus. Unlike most of the operations we've been doing, it is not right-to-left!

One way to conceptualize this is to draw an arrow from "from" to "to" for each matrix, then join the arrows. For example, for the product AB, it should look like this:

$$AB = \text{from}\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\!\!\begin{array}{c} \begin{array}{cccc} & & \text{to} & \\ A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 \end{bmatrix} \end{array} \quad \text{from}\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\!\!\begin{array}{c} \begin{array}{cccc} & & \text{to} & \\ A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

This arrow shows the order in which paths are taken.

(f) Use your calculator to check your computations of A^2, AB, BA, and B^2.

Here's some instructions on multiplying matrices on various TI calculators:

TI-83/TI-84: Press "2nd" and "x^{-1}," or if your calculator has it, the "MATRIX" button, to enter the matrix editing page. Navigate to the "EDIT" menu, then navigate to the desired name for the first matrix. Press enter to select that matrix, then type in the size and values of the matrix. Repeat this for the second matrix. When you're ready to multiply them, press "2nd" and "x^{-1}" again, but stay in the "NAMES" menu. Navigate to the first matrix to multiply, and press enter. Repeat this for the second matrix. Finally, pressing enter to calculate will give us the result of the multiplication (or an error if the dimensions are incorrect).

TI Nspire: Press the button to the bottom left of "$\overset{\text{del}}{\leftarrow}$" that looks like "$|\blacksquare|\ \begin{cases}\blacksquare\\ \blacksquare\end{cases}$". Navigate to the button that looks like a blank 3×3 matrix and press enter. Enter in the size of the first matrix, then the values. Repeat this process for the second matrix, then multiply the two matrices by pressing enter.

Assuming I didn't make a large oopsie, those answers are all correct. :P

3. Write a 3×3 matrix T that shows the following scenario: you can go from town B to C, C to D, and D to B by train, in exactly one way each, and not backwards.

$$T = \begin{array}{cc} & \text{to} \\ & \begin{array}{ccc} B & C & D \end{array} \\ \text{from}\begin{array}{c} B \\ C \\ D \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \end{array}.$$

(a) Why can't you add this matrix to matrices A or B?

They have different dimensions!

(b) Rewrite matrix T so that it *can* be meaningfully added to matrices A and B. What did you do to its dimensions?

We need T to be 4×4, and we want the entries A, B, C, D to line up properly. Thus, we insert 0s in the A column and A row, as shown:

$$T = \begin{array}{cc} & \text{to} \\ & \begin{array}{cccc} A & B & C & D \end{array} \\ \text{from}\begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array}.$$

4. Evaluate the following:

(a) $\sum_{k=1}^{4} k$

$$\sum_{k=1}^{4} k = 1 + 2 + 3 + 4 = 10.$$

(b) $\sum_{k=0}^{5} k^2$

$$\sum_{k=0}^{5} k^2 = 0^2 + 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 0 + 1 + 4 + 9 + 16 + 25 = 55.$$

(c) $\sum_{k=1}^{10} 3$

$$\sum_{k=1}^{10} 3 = \underbrace{3 + 3 + \cdots + 3}_{10} = 10 \cdot 3 = 30.$$

(d) $\sum_{k=1}^{n} k$

$$\begin{aligned}\sum_{k=1}^{n} k &= \underbrace{1+2+\cdots+n}_{n} \\ &= \underbrace{(n+1)+(n-1+2)+\cdots n/2} \\ &= \frac{n(n+1)}{2}.\end{aligned}$$

(e) $\sum_{k=1}^{n} n$

$$\sum_{k=1}^{n} n = \underbrace{n+n+\cdots+n}_{n} = n^2.$$

(f) $\sum_{k=1}^{n} 1$

$$\sum_{k=1}^{n} 1 = \underbrace{1+1+\cdots+1}_{n} = n.$$

5. **The matrix C^T whose rows are the same as the respective columns of matrix C is called the transpose of C. For example,**

$$C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \; C^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}.$$

   (a) **Let the elements of C be c_{ij} and the elements of C^T be c'_{ij}. Write a formula for C^T in terms of these elements. That is, c'_{ij} =?**

We simply have $c'_{ij} = c_{ji}$; the indices are swapped.

   (b) **Write $\begin{bmatrix} 2 & 1 & 5 \\ 4 & -2 & 0 \end{bmatrix}^T$.**

We flip it over the main diagonal; since the matrix is not square, we get a matrix with different dimensions!

$$\begin{bmatrix} 2 & 1 & 5 \\ 4 & -2 & 0 \end{bmatrix}^T = \begin{bmatrix} 2 & 4 \\ 1 & -2 \\ 5 & 0 \end{bmatrix}$$

6. **Fill in the blanks: Multiplying an $m \times n$ matrix by a(n) ___ $\times k$ matrix gives a(n) ___ $\times$ ___ matrix.**

Multiplying an $m \times n$ matrix by a(n) $\underline{n} \times k$ matrix gives a(n) $\underline{m} \times \underline{n}$ matrix.

7. **Dogs can eat cats, rats, or mice; cats can eat rats or mice; rats can eat mice.**

   (a) **Make a matrix E showing what can eat what.**

Let dogs be D, cats be C, rats be R, and mice be M. Then the matrix is straightforward. Note that we put the prey on the left and predator on top because then following the matrices is going through each step in the food chain.

$$T = \text{prey}\begin{array}{c} \\ D \\ C \\ R \\ M \end{array}\!\!\begin{array}{c} \text{predator} \\ \begin{matrix} D & C & R & M \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}.$$

Note that the diagonal is all 0s because no animal eats their own species.

(b) Draw a directed graph.

The graph is shown in Figure 1.

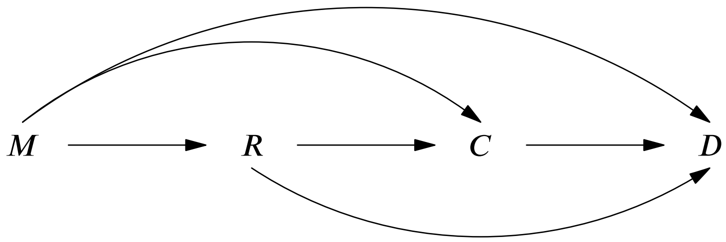


Figure 1: The directed graph of E.

(c) Calculate and interpret E^2, E^3, E^4.

We have

$$
\begin{aligned}
E^2 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

This means that there are two ways for a mouse's nutrients to find its way to a dog in two steps (namely, through a rat and through a cat). Also, there is only one way for a rat to get to a dog in two steps, and only one way for a mouse to get to a cat in two steps.

We have

$$
\begin{aligned}
E^3 = EE^2 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

This means a mouse can get to a dog in three steps in only one way.

We have

$$
\begin{aligned}
E^4 = EE^3 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

A matrix of all zeroes! That means there are no paths which take four steps, which makes sense.

8.

Note that as given in the problem, we have

$$S = \begin{bmatrix} 5 & 1 & 0 & \frac{4}{3} & 1 & 1 & 0 & 0 \\ 5 & 1 & 1 & \frac{5}{4} & \frac{3}{4} & 0 & \frac{1}{3} & 2 \end{bmatrix};$$
$$C = \begin{bmatrix} 5 & 20 & 10 & 0 & 1 & 2 & 5 & 12 \end{bmatrix}.$$

(a) Unfortunately, if you try to multiply S and C as given, it won't work. Why not?

The dimensions aren't right! S is a 2×8 matrix, while C is a 1×8 matrix.

(b) What do you need to do to C so they can be multiplied? Explain the dimensions of each matrix.

We need to transpose C, since C^T is a 8×1 matrix. This lets it be multiplied by S.

(c) Once you've fixed matrix C, do the multiplication. What are the dimensions of your answer?

We do the multiplication:

$$\begin{aligned} SC^T &= \begin{bmatrix} 5 & 1 & 0 & \frac{4}{3} & 1 & 1 & 0 & 0 \\ 5 & 1 & 1 & \frac{5}{4} & \frac{3}{4} & 0 & \frac{1}{3} & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 20 \\ 10 \\ 0 \\ 1 \\ 2 \\ 5 \\ 12 \end{bmatrix} \\ &= \begin{bmatrix} 48 \\ \frac{977}{12} \end{bmatrix}. \end{aligned}$$

We end up with a 2×1 matrix. Indeed, $M_{2\times 8} M_{8\times 1} = M_{2\times 1}$; in some sense the inner dimensions VANISH, ANNIHILATE, whatever you like, to leave the outer dimensions behind.

9. Matrix multiplication is not necessarily commutative, even when the dimensions of the matrices suggest it might be. How do we know? Be specific.

One way to know is to just multiply two random matrices together (chosen for computational convenience) and check for commutativity:

$$C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}; \quad D = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

We compute CD and DC:

$$\begin{aligned} CD &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} . DC & &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Longrightarrow CD \neq DC. \end{aligned}$$

Oof! Matrix multiplication is not commutative.

10. Matrix multiplication is associative, though. Prove that $(PX)T = P(XT)$ for

$$P = \begin{bmatrix} m & n \\ p & q \end{bmatrix}, \; X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}, \; Y = \begin{bmatrix} r & s \\ t & u \end{bmatrix}.$$

With great dread, we compute $(PX)T$ and $P(XT)$.

$$\begin{aligned}
(PX)T &= \left(\begin{bmatrix} m & n \\ p & q \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right)\begin{bmatrix} r & s \\ t & u \end{bmatrix} \\
&= \begin{bmatrix} mx+nz & my+nw \\ px+qz & py+qw \end{bmatrix}\begin{bmatrix} r & s \\ t & u \end{bmatrix} \\
&= \begin{bmatrix} r(mx+nz)+t(my+nw) & s(mx+nz)+u(my+nw) \\ r(px+qz)+t(py+qw) & s(px+qz)+u(py+qw) \end{bmatrix} \\
&= \begin{bmatrix} mrx+mty+nrz+ntw & msx+muy+nsz+nuw \\ prx+pty+qrz+qtw & psx+puy+qsz+quw \end{bmatrix}. \\
P(XT) &= \begin{bmatrix} m & n \\ p & q \end{bmatrix}\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} r & s \\ t & u \end{bmatrix}\right) \\
&= \begin{bmatrix} m & n \\ p & q \end{bmatrix}\begin{bmatrix} xr+yt & xs+yu \\ zr+wt & zs+wu \end{bmatrix} \\
&= \begin{bmatrix} m(xr+yt)+n(zr+wt) & m(xs+yu)+n(zs+wu) \\ p(xr+yt)+q(zr+wt) & p(xs+yu)+q(zs+wu) \end{bmatrix} \\
&= \begin{bmatrix} mrx+mty+nrz+ntw & msx+muy+nsz+nuw \\ prx+pty+qrz+qtw & psx+puy+qsz+quw \end{bmatrix}.
\end{aligned}$$

They are equal!

Note that this doesn't prove it's associative for all qualifying[15] matrices because we've only shown it for 2×2 matrices.

11. Prove that matrix multiplication is distributive: $P(X+T) = PX + PT$.

This isn't as bad.

$$\begin{aligned}
P(X+T) &= \begin{bmatrix} m & n \\ p & q \end{bmatrix}\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix} + \begin{bmatrix} r & s \\ t & u \end{bmatrix}\right) \\
&= \begin{bmatrix} m & n \\ p & q \end{bmatrix}\begin{bmatrix} x+r & y+s \\ z+t & w+u \end{bmatrix} \\
&= \begin{bmatrix} m(x+r)+n(z+t) & m(y+s)+n(w+u) \\ p(x+r)+n(z+t) & p(y+s)+q(w+u) \end{bmatrix} \\
&= \begin{bmatrix} (mx+nz)+(mr+nt) & (my+nw)+(ms+nu) \\ (px+nz)+(pr+nt) & (py+qw)+(ps+qu) \end{bmatrix} \\
&= \begin{bmatrix} m & n \\ p & q \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} + \begin{bmatrix} m & n \\ p & q \end{bmatrix}\begin{bmatrix} r & s \\ t & u \end{bmatrix} \\
&= PX + PT.
\end{aligned}$$

Indeed, they are equal!

12. When does $PX = XP$? Don't worry if you get some messy equations in your answer.

Let's try it.

$$\begin{aligned}
PX &= \begin{bmatrix} m & n \\ p & q \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} \\
&= \begin{bmatrix} mx+nz & my+nw \\ px+qz & py+qw \end{bmatrix} \\
XP &= \begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} m & n \\ p & q \end{bmatrix} \\
&= \begin{bmatrix} xm+yp & xn+yq \\ zm+wp & zn+wq \end{bmatrix}
\end{aligned}$$

[15] In terms of dimension.

Equating terms, we get

$$\begin{cases} mx + nz &= xm + yp \\ my + nw &= xn + yq \\ px + qz &= zm + wp \\ py + qw &= zn + wq \end{cases}$$

$$\implies \begin{cases} nz &= yp \\ my + nw &= xn + yq \\ px + qz &= zm + wp \\ py &= zn \end{cases}.$$

13. Cook's Seafood Restaurant in Menlo Park sells fish and chips. The Captain's order is two pieces of fish and one order of chips, while the Regular order is one piece of fish and one order of chips.

(a) Write a matrix representing these facts, with clear labels on your rows and columns.

Let Captain = C and Regular = R. Then the matrix is:

$$M = \begin{array}{c c} & \begin{array}{cc} \text{fish} & \text{chips} \end{array} \\ \begin{array}{c} C \\ R \end{array} & \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \end{array}.$$

I'm writing it this way rather than the transpose so that the order of the matrices in the next problem is the same as the problems appear. Otherwise, you'd have to flip the order of multiplication (remember, it's not commutative!).

(b) The restaurant management estimates their cost at 0.75 for each piece of fish and 0.50 for each order of chips. Represent this as a matrix, then use matrix multiplication to calculate the cost of the two possible orders.

$$N = \begin{array}{c c} & \text{cost (\$)} \\ \begin{array}{c} \text{fish} \\ \text{chips} \end{array} & \begin{bmatrix} 0.75 \\ 0.50 \end{bmatrix} \end{array}.$$

Now we just multiply the matrices:

$$\begin{aligned} MN &= \begin{array}{c c} & \begin{array}{cc} \text{fish} & \text{chips} \end{array} \\ \begin{array}{c} C \\ R \end{array} & \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \end{array} \begin{array}{c c} & \text{cost (\$)} \\ \begin{array}{c} \text{fish} \\ \text{chips} \end{array} & \begin{bmatrix} 0.75 \\ 0.50 \end{bmatrix} \end{array} \\ &= \begin{array}{c c} & \text{cost (\$)} \\ \begin{array}{c} C \\ R \end{array} & \begin{bmatrix} 2 \\ 1.25 \end{bmatrix} \end{array}. \end{aligned}$$

Thus, the cost of a Captain's order is \$2 and the cost of a Regular order is \$1.25 (for the restaurant).

(c) For a party, Cook's provides 10 Captain's orders and 5 Regular orders. Write this as a matrix and use matrix multiplication to find how many pieces of fish and orders of chips are provided.

We want to multiply this matrix by M and get a 2×1 or 1×2 matrix of fish and chips. Thus, we choose the 1×2 matrix

$$P = \begin{array}{cc} C & R \\ \left[10\right. & \left.5\right] \end{array},$$

so that the product is simply

$$PM = \begin{array}{cc} C & R \\ \left[10\right. & \left.5\right] \end{array} \begin{array}{c} \\ C \\ R \end{array} \begin{array}{cc} \text{fish} & \text{chips} \\ \multicolumn{2}{c}{\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}} \end{array} = \begin{array}{cc} \text{fish} & \text{chips} \\ \left[25\right. & \left.15\right] \end{array}.$$

Thus, 25 pieces of fish and 15 orders of chips are provided.

(d) Now use matrix multiplication to find out the cost of the party.

We need to multiply PM by N to get a 1×1 matrix:

$$PMN = \begin{array}{cc} \text{fish} & \text{chips} \\ \left[25\right. & \left.15\right] \end{array} \begin{array}{c} \text{fish} \\ \text{chips} \end{array} \begin{array}{c} \text{cost (\$)} \\ \begin{bmatrix} 0.75 \\ 0.50 \end{bmatrix} \end{array} = \left[26.25\right].$$

Thus, the party costs \$26.25 for the restaurant.

14. We will find coefficient matrices to be particularly useful for solving systems of linear equations. For instance,

$$\begin{cases} 3x + 4y &= 5 \\ 6x + 4y &= 8 \end{cases} \longleftrightarrow \begin{bmatrix} 3 & 4 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}.$$

Rewrite

$$\begin{cases} 2x + 3y + 4z &= 5 \\ 5x - 4y + 2z &= 2 \\ x + 2y &= 7 \end{cases}$$

as a matrix equation in this way.

We rewrite the last equation as $x + 2y + 0z = 7$ and proceed to tabulate the coefficients:

$$\underbrace{\begin{bmatrix} 2 & 3 & 4 \\ 5 & -4 & 2 \\ 1 & 2 & 0 \end{bmatrix}}_{M} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 7 \end{bmatrix}.$$

15.

(a) What is the transpose of the 3×3 matrix M from the previous problem?

M is shown in the previous problem above. We flip it, obtaining

$$M^T = \begin{bmatrix} 2 & 5 & 1 \\ 3 & -4 & 2 \\ 4 & 2 & 0 \end{bmatrix}.$$

(b) Use M^T to rewrite the system in the previous problem.

At first, one might try flipping the order of the column matrix to say z, y, x (from top to bottom). But this doesn't work.

Thinking in terms of what columns and rows mean, we can label them in the original equation from the previous problem:

$$\begin{array}{c} \text{Eq. 1} \\ \text{Eq. 2} \\ \text{Eq. 3} \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 2 & 3 & 4 \\ 5 & -4 & 2 \\ 1 & 2 & 0 \end{bmatrix} \end{array} \begin{array}{c} x \\ y \\ z \end{array} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{array}{c} \text{Eq. 1} \\ \text{Eq. 2} \\ \text{Eq. 3} \end{array} \begin{bmatrix} 5 \\ 2 \\ 7 \end{bmatrix}.$$

Doing the same for M^T, we get

$$M^T = \begin{array}{c} \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{ccc} \text{Eq. 1} & \text{Eq. 2} & \text{Eq. 3} \end{array} \\ \begin{bmatrix} 2 & 5 & 1 \\ 3 & -4 & 2 \\ 4 & 2 & 0 \end{bmatrix} \end{array}.$$

We realize that to get a matrix containing the values of Equations 1-3, we need to left-multiply M^T by a 1×3 matrix $\begin{bmatrix} x & y & z \end{bmatrix}$:

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 2 & 5 & 1 \\ 3 & -4 & 2 \\ 4 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 7 \end{bmatrix}.$$

This looks quite similar to the previous problem! In symbols, if $A = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 2 \\ 7 \end{bmatrix}$, we have

$$\underbrace{MA = B}_{\text{prev. prob.}} \longleftrightarrow A^T M^T = B^T.$$

(c) What is the transpose of the transpose matrix, $(M^T)^T$?

Since we're just reflecting over the diagonal twice, we have $(M^T)^T = M$.

This brings up an interesting fact, thinking back to the last subproblem. In general for any matrices P, Q, R, we have

$$PQ = R \longleftrightarrow Q^T P^T = R^T.$$

Taking the transpose of both sides of the right equation, we get

$$(Q^T P^T)^T = (R^T)^T = R.$$

Equating this with the left equation, we get

$$PQ = (Q^T P^T)^T.$$

Succulent!

9 Mapping the Plane with Matrices

$$\overset{M}{\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}} \overset{P}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} = \overset{P'}{\begin{bmatrix} 7 \\ -1 \end{bmatrix}}$$

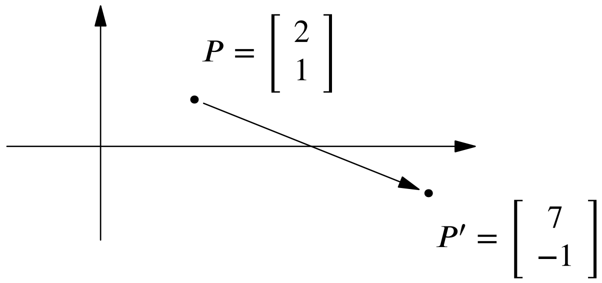


Figure 1: Matrix multiplication is a transformation.

1.

(a) Use the 2×2 matrix from Figure 1 to operate on the points $(0, 0)$, $(1, 0)$, and $(0, 1)$. What are their images? Graph them.

The matrix is $M = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$. We multiply this by the column vectors $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to get

$$\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Thus, the right hand sides are the images of the left hand side. I graphed the transformation in Figure 2.

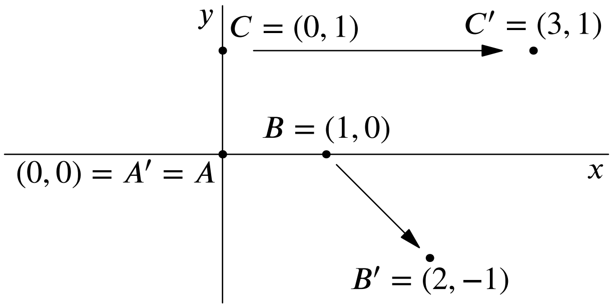


Figure 2: The matrix operating on points $(0, 0)$, $(1, 0)$, and $(0, 1)$.

(b) The preimage includes two perpendicular unit vectors, $(0, 1)$ and $(1, 0)$. What is the (i) ratio of the lengths of their images and (ii) angle between the images?

The (i) ratio of the lengths of their images is

$$\frac{|C'|}{|B'|} = \frac{\sqrt{3^2 + 1^2}}{\sqrt{2^2 + (-1)^2}} = \sqrt{\frac{10}{5}} = \sqrt{2}.$$

The (ii) angle between their images is a bit less straightforward, but we can compute it by summing the angle from C' to the x-axis with the (positive) angle from B' to the x-axis:

$$\tan^{-1} \frac{1}{3} + \left| \tan^{-1} \frac{-1}{2} \right| = \frac{\pi}{4} = 45°.$$

Oh. Well another way to find the angle is to draw the $45 - 45 - 90$ triangle between A, B' and C', which shows that $\angle C'AB' = 45°$. We know it's $45 - 45 - 90$ because the side lengths are $\sqrt{5}$, $\sqrt{5}$ and $\sqrt{10}$ as determined by the Pythagorean Theorem. This triangle is shown in Figure 3.

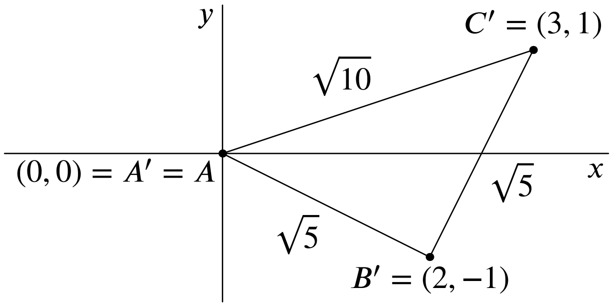


Figure 3: A helpful $45-45-90$ triangle $\triangle AB'C'$.

(c) You can conclude that multiplication by matrices does not, in general, preserve which two quantities between the image and preimage?

It does not preserve the ratio of lengths or the angle between two vectors. After all, the angle was $90°$ and is now $45°$. Also, the ratio used to be 1, but is now $\sqrt{2}$ (or $\sqrt{2}2$).

2.

(a) Now, use the 2×2 **matrix from Figure 1 to operate on each of these points:** $(2, 1)$**,** $(1, 0)$**,** $(0, -1)$ **and** $(-1, -2)$**. Do this by consolidating all the points into one matrix, with each point as a column vector, then performing a multiplication:**

$$\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & -1 \\ 1 & 0 & -1 & -2 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \end{bmatrix}.$$

We perform the multiplication:

$$\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & -1 \\ 1 & 0 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 7 & 2 & -3 & -8 \\ -1 & -1 & -1 & -1 \end{bmatrix}.$$

(b) Graph and label the preimage and the image of each point onto the same set of axes.

The graph is shown in Figure 4.

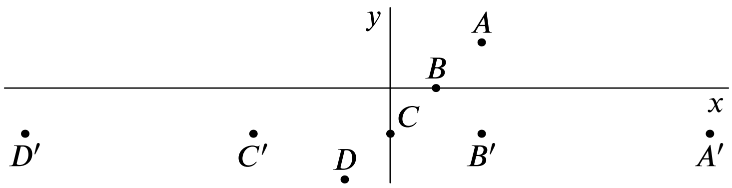


Figure 4: The preimage and image.

(c) The points in the preimage are discontinuous, but they belong to a particular, infinite set of points. Write the equation of that set. (Hint: What is y in terms of x?)

They belong to a line! More precisely, they are all on the line $y = x - 1$.

(d) Write an equation for the image of that set.

It appears the equation of the image is simply $y = -1$. We can verify this by multiplying M by $\begin{bmatrix} t+1 \\ t \end{bmatrix}$:

$$\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} t+1 \\ t \end{bmatrix} = \begin{bmatrix} 2(t+1)+3t \\ -(t+1)+t \end{bmatrix} = \begin{bmatrix} 5t+2 \\ -1 \end{bmatrix}.$$

Indeed, the y coordinate of the image is -1, and the x coordinate can be any real number.

(e) What other characteristic of the preimage points also applies to the image?

The points in the preimage are equidistant, taken as consecutive pairs. This is also true of the image.

(f) Name two things that seem to be conserved when mapping points with a matrix.

It seems (i) collinearity and (ii) equally spaced points have their characteristics preserved. Note that not all equidistant points will have this property conserved. Think back to the first problem, for example, where the pairs of points $((0,0),(1,0))$ and $((0,0),(0,1))$ started off equidistant, but ended up not equidistant. Indeed, they have to be collinear for this to be preserved.

3.

(a) Choose a different 2×2 matrix and a different set of three collinear, equally spaced unique points. Perform the appropriate matrix multiplication.

I'm gonna choose the transformation matrix $M = \begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix}$, and the points $\begin{bmatrix} 2 & 4 & 6 \\ -1 & 0 & 1 \end{bmatrix}$. The multiplication is straightforward:

$$\begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 4 & 6 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 12 & 16 \\ 1 & 8 & 15 \end{bmatrix}.$$

(b) Graph and label the preimage points and the image points.

The picture is shown in Figure 5.

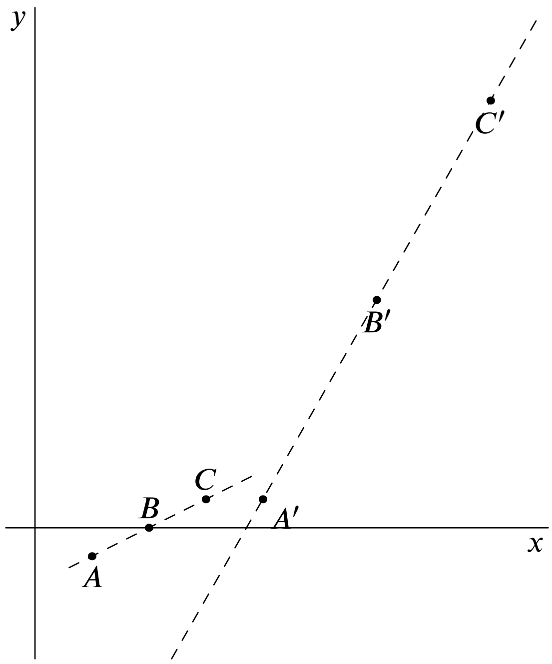


Figure 5: The preimage and image.

(c) Have the collinearity and equal spacing been preserved?

Indeed! The vector from $(8,1)$ to $(12,8)$ is $\langle 4,7\rangle$ and the vector from $(12,8)$ to $(16,15)$ is also $\langle 4,7\rangle$.

(d) Make a conjecture about when a matrix will preserve collinearity and when a matrix will preserve equal spacing.

Since two random matrices have both done it, we conjecture that all matrices do so.

4. Now, we will check your conjecture.

(a) Start with a general 2×2 matrix and three equally spaced points on a line, and multiply the two matrices:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x-h & x & x+h \\ m(x-h)+k & mx+k & m(x+h)+k \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}.$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x-h & x & x+h \\ m(x-h)+k & mx+k & m(x+h)+k \end{bmatrix}$$

$$= \begin{bmatrix} a(x-h)+b(m(x-h)+k) & ax+b(mx+k) & a(x+h)+b(m(x+h)+k) \\ c(x-h)+d(m(x-h)+k) & cx+d(mx+k) & c(x+h)+d(m(x+h)+k) \end{bmatrix}.$$

(b) How do you know that the second matrix indeed represents collinear and equally spaced points?

There's a couple of ways to rationalize this, but my favorite is to simply consider the vector between consecutive points.

Our second matrix is $\begin{bmatrix} x-h & x & x+h \\ m(x-h)+k & mx+k & m(x+h)+k \end{bmatrix}$. The vector from the first point to the second is $\langle h, mh \rangle$; similarly, the vector from the second point to the third is $\langle h, mh \rangle$. This means the points are collinear, because the vectors between each pair have the same direction, and equidistant, because each consecutive pair has the same distance.

(c) Are there any sets of collinear points that aren't representable by the 2×3 matrix?

Yes there are. We cannot represent collinear points that go directly vertically, because (informally) that would mean $h = 0$ and $m = \infty$. More precisely, we must have $h = 0$, but then there is no real m such that $m(x-h)+k \neq mx+k \neq m(x+h)+k$. As an example, we cannot represent the points

$$\begin{bmatrix} 1 & 1 & 1 \\ -50 & 0 & 50 \end{bmatrix}.$$

(d) Are the points in the image collinear? Show why or why not.

Yes they are. Again, we think about the vector between consecutive points: for the first pair it's

$$\begin{aligned} V_1 &= \langle (ax+b(mx+k)) - (a(x-h)+b(m(x-h)+k)), (cx+d(mx+k)) - (c(x-h)+d(m(x-h)+k)) \rangle \\ &= \langle ah+bmh, ch+dmh \rangle; \end{aligned}$$

for the second pair it's

$$\begin{aligned} V_2 &= \langle (a(x+h)+b(m(x+h)+k)) - (ax+b(mx+k)), (c(x+h)+d(m(x+h)+k)) - (cx+d(mx+k)) \rangle \\ &= \langle ah+bmh, ch+dmh \rangle = V_1. \end{aligned}$$

Since $V_2 = V_1$, the points are collinear (and equidistant).

(e) Can you find values for a, b, c, and d so that the image does not lie on a unique line? (Hint: all of the points in the image must lie on no line, or on multiple lines.)

At first this seems like it's contradicting our conjecture, but the devil's in the details. If all the points are coincident—that is, they're all equal—then there are infinitely many lines passing through it.

We set $a = b = c = d = 0$ so that $M = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, which maps every point to $(0,0)$. Then there are infinitely many lines going through the image.

(f) Use the distance formula—or some other justification—to answer whether the points in the image are equally spaced.

We showed that they're equidistant two subproblems ago already, with vectors. But I'll also show it with the distance formula since that what the question suggests.

The distance between the first two points is

$$\sqrt{((ax+b(mx+k)) - (a(x-h)+b(m(x-h)+k)))^2 + ((cx+d(mx+k)) - (c(x-h)+d(m(x-h)+k)))^2}$$

$$= \sqrt{(ah+bmh)^2+(ch+dmh)^2}.$$

Looks familiar.... The distance between the second points is

$$\sqrt{((a(x+h)+b(m(x+h)+k))-(ax+b(mx+k)))^2+((c(x+h)+d(m(x+h)+k))-(cx+d(mx+k)))^2}$$

$$= \sqrt{(ah+bmh)^2+(ch+dmh)^2}.$$

The distances are equal; they are equidistant.

5. There is a point which remains fixed—its image is the same as its preimage—when multiplied by the matrix $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$. That is, $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$.

(a) Solve the above matrix equation for x and y to find the point.

We multiply out the right side:

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+3y \\ 4x+5y \end{bmatrix}.$$

Equating corresponding entries, we get the system of equations

$$\begin{cases} x = 2x+3y \\ y = 4x+5y \end{cases}.$$

Solving this system gives $(x, y) = (0, 0)$. How mundane....

(b) There is a point $Q = \left[\begin{smallmatrix} e \\ f \end{smallmatrix}\right]$ that remains fixed no matter what matrix you multiply it by. Can you guess what point that is?

Looks like the point is $Q = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, given the answer to the previous subproblem.

(c) Prove your conjecture by plugging your point Q into $\begin{bmatrix} a & b \\ c & d \end{bmatrix} Q = Q$.

We do the multiplication:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = Q.$$

Indeed, Q always remains fixed.

6. Begin with a triangle with vertices $(5, 0)$, $(10, 0)$, and $(5, 10)$.

(a) Map the vertices with the following matrices.

i. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

We multiply them:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 10 & 5 \\ 0 & 0 & 10 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 5 \\ 0 & 0 & 10 \end{bmatrix}.$$

ii. $\begin{bmatrix} .6 & -.8 \\ .8 & .6 \end{bmatrix}$

$$\begin{bmatrix} .6 & -.8 \\ .8 & .6 \end{bmatrix} \begin{bmatrix} 5 & 10 & 5 \\ 0 & 0 & 10 \end{bmatrix} = \begin{bmatrix} 3 & 6 & -5 \\ 4 & 8 & 10 \end{bmatrix}.$$

iii. $\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}$

$$\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix} \begin{bmatrix} 5 & 10 & 5 \\ 0 & 0 & 10 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 11 \\ 4 & 8 & -2 \end{bmatrix}.$$

(b) Why will the new triangle defined by these vertices be the image of the starting triangle?

We only transformed the vertices, but because matrix multiplication is a linear transformation, lines are mapped to lines. Thus, the new triangle defined by these vertices is the image of the old triangle, in the sense that the sides of the original are mapped onto this new triangle as well.

(c) Accurately graph the preimage, then the image for each matrix on three separate sets of axes.

(i) is shown in Figure 6; (ii) is shown in Figure 7; (iii) is shown in Figure 8.

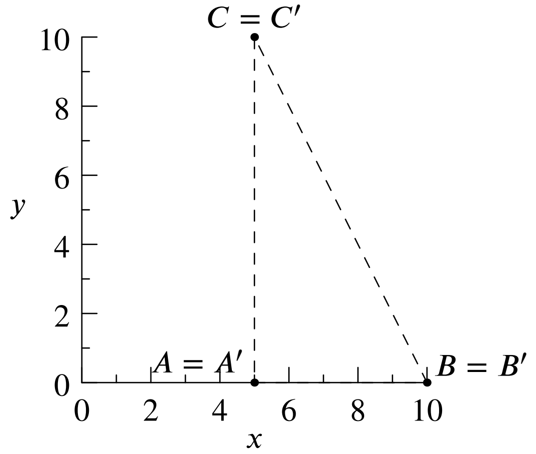


Figure 6: Preimage and image for (i).

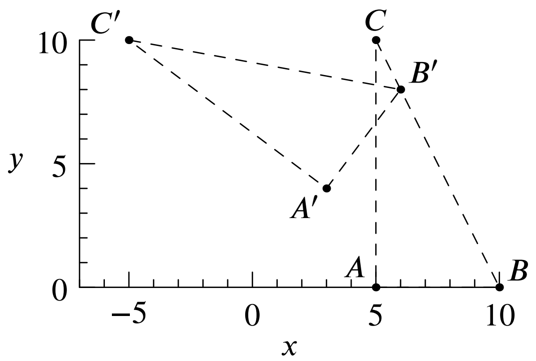


Figure 7: Preimage and image for (ii).

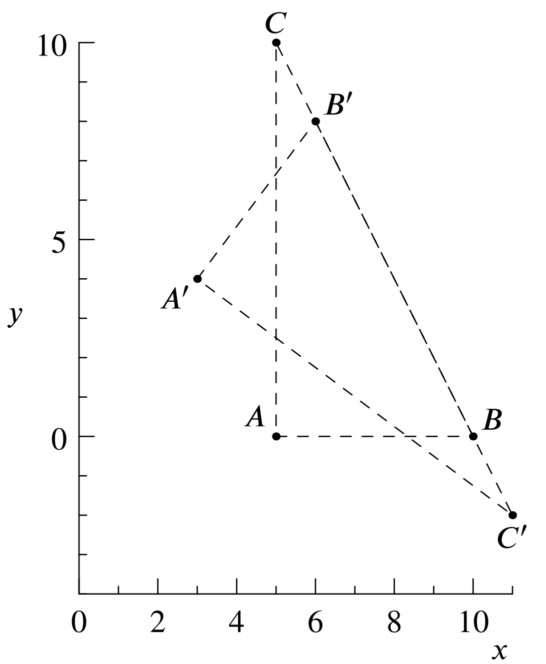


Figure 8: Preimage and image for (iii).

(d) For each, describe the transformation as fully as you can. Try to classify them on the transformations we mentioned earlier, and quantify them if necessary (e.g. to describe the line of reflection or angle of rotation).

(i) is the identity transformation; it does absolutely nothing. Indeed, the matrix $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called the identity matrix of size 2.

(ii) is a rotation about the origin by $\tan^{-1}\frac{4}{3} \approx 53.13°$. This is found by taking $\tan^{-1}\frac{y}{x}$ for the point $(x, y) = A' = (3, 4)$.

(iii) is a reflection about the line $y = \frac{x}{2}$. The easiest way to see this is that because the origin is fixed, the reflection line must pass through $(0, 0)$, so it is $y = mx$ for some real number m. But it also must pass through the midpoint of $\overline{AA'}$ (and $\overline{BB'}$, $\overline{CC'}$), which is $(4, 2)$. Thus, $m = \frac{2}{4} = \frac{1}{2}$.

7. Soon, we will map the unit square: it has vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$. We could actually get the entire image from the image of the unit vectors $(1, 0)$ and $(0, 1)$, which will be useful later.

(a) How can we obtain the image of $(1, 1)$ from the images of $(1, 0)$ and $(0, 1)$?

We can use the associative property of matrix multiplication to help us! If the image of a point P is i_P, then we have

$$i_{(1,1)} = M\begin{bmatrix}1\\1\end{bmatrix} = M\left(\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\right) = M\begin{bmatrix}1\\0\end{bmatrix} + M\begin{bmatrix}0\\1\end{bmatrix} = i_{(1,0)} + i_{(0,1)}.$$

In words, we sum the images of $(1, 0)$ and $(0, 1)$ to get the image of $(1, 1)$. Neat!

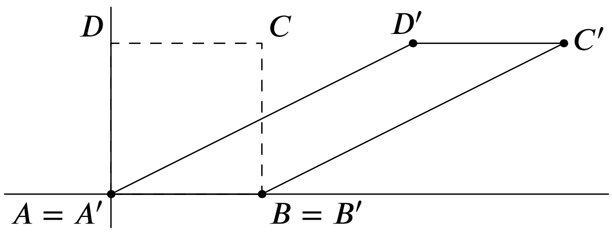
(b) Of $(0, 0)$?

This is sort of a trick question. We have $i_{(0,0)} = M\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$, so the image is always $(0, 0)$.

8.

(a) Take the matrix $\left[\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right]$ and see what it does to the unit square. Please graph this, being careful to label each point and its image. The multiplication is done for you below.

$$\begin{bmatrix}1&2\\0&1\end{bmatrix}\begin{array}{cccc}A&B&C&D\\ \left[\begin{matrix}0&1&1&0\\0&0&1&1\end{matrix}\right.&&&\left.\right]\end{array} = \begin{array}{cccc}A'&B'&C'&D'\\ \left[\begin{matrix}0&1&3&2\\0&0&1&1\end{matrix}\right.&&&\left.\right]\end{array}.$$



(b) What happens to the area of the image versus the preimage?

They are equal. After all, both are parallelograms with base 1 and height 1.

(c) We have $AB = BC$, but is $A'B'$ equal to $B'C'$? Should it?

No, $A'B' \neq B'C'$. It doesn't have to, because though the points are equidistant, they are not collinear, and don't have to be equidistant in the image.

9.

(a) When is the ratio of distances between points in the image the same as in the preimage?

This is true when the image is a dilation, reflection, or rotation: or combination of the three.

(b) What is the image of the origin under any matrix mapping?

The image of $(0, 0)$ is always $(0, 0)$.

(c) What are the images of the points $(1, 0)$ and $(0, 1)$ under the mapping $\begin{bmatrix}a&b\\c&d\end{bmatrix}$?

The image of $(1,0)$ is just (a,c). The image of $(0,1)$ is (b,d).

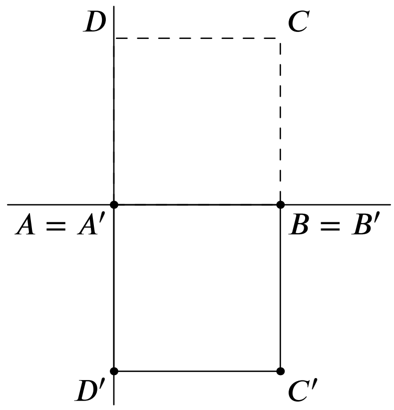
(d) Knowing the images of $(1,0)$ and $(0,1)$, how do we find the image of $(1,1)$ algebraically and geometrically?

The image of $(1,1)$, as we found earlier, is the sum of the images of $(1,0)$ and $(0,1)$. Thus, it is $(a+b, c+d)$. Geometrically, it forms a parallelogram with the images of $(1,0)$ and $(0,1)$.

10. How do these matrices map the plane? For each mapping, write a matrix for the images of the four corners of the unit square, then graph the preimage and image. Describe the mapping using words from geometry such as congruent, similar, rotate, reflect, shear, stretch, magnitude, and direction.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

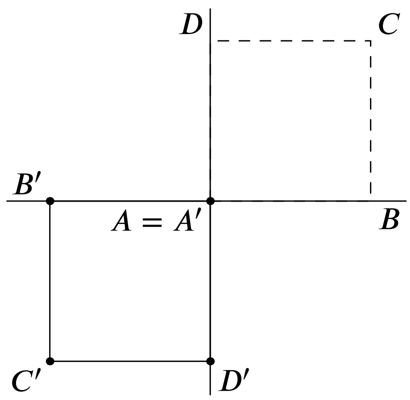
Image matrix: $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}$.



This is a reflection over the x-axis.

(b) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

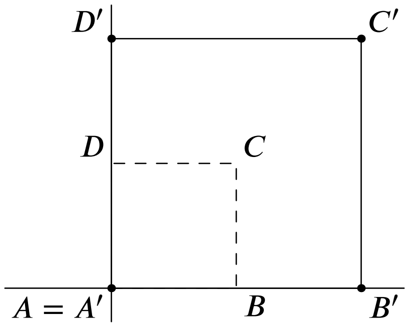
Image matrix: $\begin{bmatrix} 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}$.



This is a reflection about the origin $(0,0)$.

(c) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

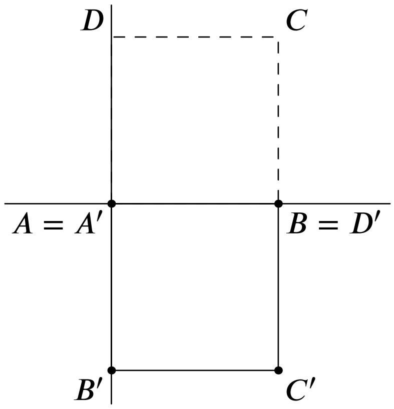
Image matrix: $\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$.



This is a dilation about the origin $(0,0)$ by a factor of 2.

(d) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

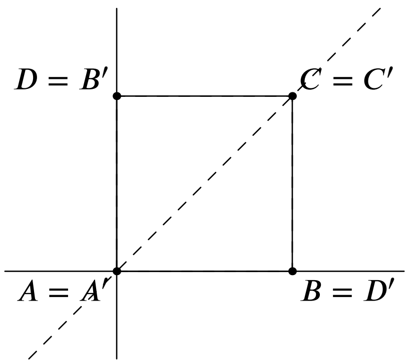
Image matrix: $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix}$.



This is a rotation by $90° = \frac{\pi}{2}$ clockwise, or $270° = \frac{3\pi}{2}$ counterclockwise.

(e) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

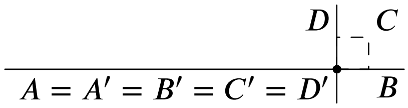
Image matrix: $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$.



This is a reflection about the line $y = x$, which is marked in the diagram.

(f) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

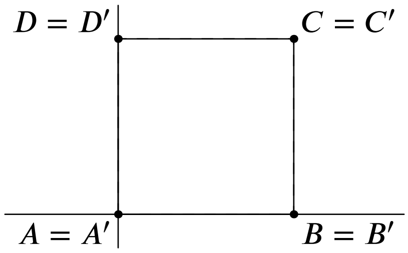
Image matrix: $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.



This is a projection onto the point $(0, 0)$. Every point goes to $(0, 0)$!

(g) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

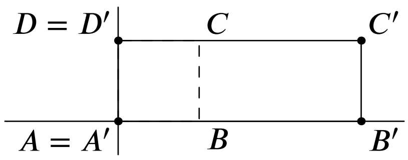
Image matrix: $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.



This is the identity transformation: it does nothing. Every point is mapped to itself.

(h) $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

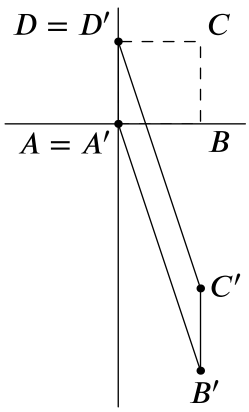
Image matrix: $\begin{bmatrix} 0 & 3 & 3 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.



This is a stretch by a factor of 3 along the x-axis.

(i) $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

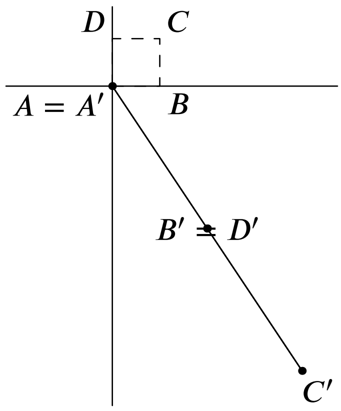
Image matrix: $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & -3 & -2 & 1 \end{bmatrix}$.



This is a shear by a factor of 3 along the negative y-axis.

(j) $\begin{bmatrix} 2 & 2 \\ -3 & -3 \end{bmatrix}$

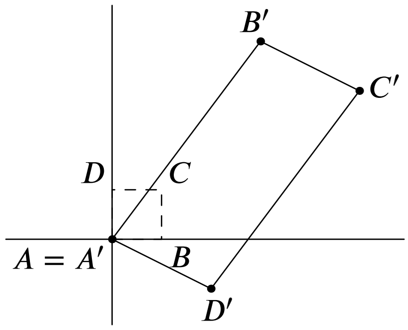
Image matrix: $\begin{bmatrix} 0 & 2 & 4 & 2 \\ 0 & -3 & -6 & -3 \end{bmatrix}$.



This is a mapping onto the line $-3x = 2y \implies 3x + 2y = 0$. Note that this isn't a *projection*, since that has extra constraints.

(k) $\begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}$

Image matrix: $\begin{bmatrix} 0 & 3 & 5 & 2 \\ 0 & 4 & 3 & -1 \end{bmatrix}$.

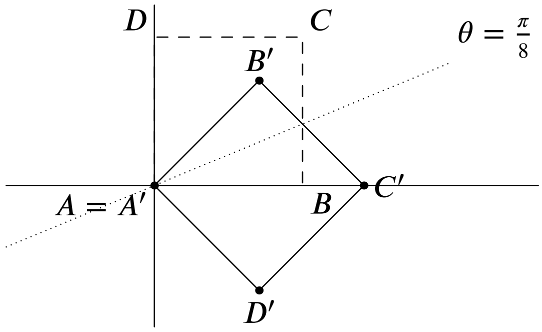


This is a rather complex transformation. We'll learn later how to tackle this properly. If you're curious, this matrix is the following operations in order:

- Shear by $\frac{2}{3}$ along the (positive) x-axis
- Stretch by $-\frac{11}{3}$ along the y-axis
- Shear by 4 along the y-axis
- Stretch by 3 along the x-axis.

(l) $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$

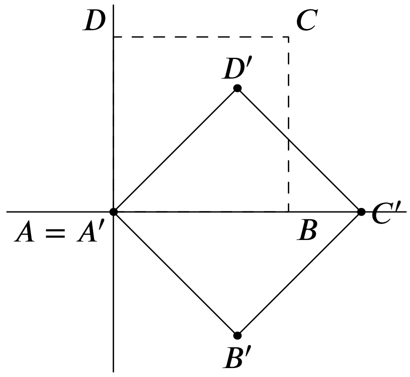
Image matrix: $\begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & \sqrt{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \end{bmatrix}$.



This a reflection over the line $\theta = 22.5° = \frac{\pi}{8}$, or $y = (\sqrt{2} - 1)x$.

(m) $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

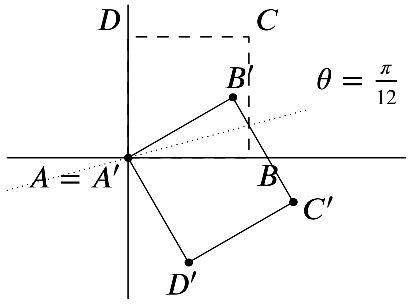
Image matrix: $\begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & \sqrt{2} & \frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$.



This is a rotation by $45° = \frac{\pi}{4}$ clockwise about the origin, or a rotation of $315° = \frac{7\pi}{4}$ counterclockwise about the origin.

(n) $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$

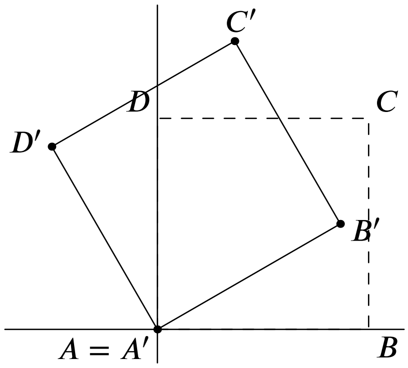
Image matrix: $\begin{bmatrix} 0 & \frac{\sqrt{3}}{2} & \frac{1+\sqrt{3}}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1-\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$.



This is a reflection about the line $\theta = 15° = \frac{\pi}{12}$, or the line $y = (2 - \sqrt{3})x$.

(o) $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

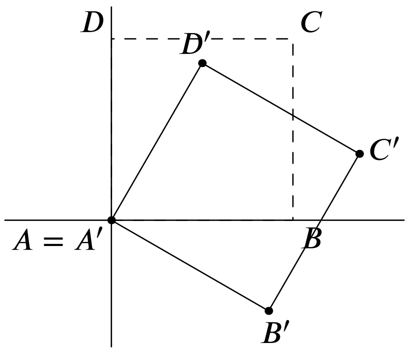
Image matrix: $\begin{bmatrix} 0 & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}-1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1+\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$.



This is a rotation of $30° = \frac{\pi}{6}$ counterclockwise about the origin.

(p) $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

Image matrix: $\begin{bmatrix} 0 & \frac{\sqrt{3}}{2} & \frac{1+\sqrt{3}}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}-1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$.



This is rotation of $30° = \frac{\pi}{6}$ *clockwise* about the origin, or $330° = \frac{11\pi}{6}$ counterclockwise about the origin.

11. Carry out the following multiplications and convince yourself they are equivalent mappings of the x and y coordinates.

(a) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$

This is simple:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} au + bv \\ cu + dv \end{bmatrix}.$$

(b) $\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} au + bv + 0 \cdot 1 \\ cu + dv + 0 \cdot 1 \\ 0u + 0v + 0 \cdot 1 \end{bmatrix} \begin{bmatrix} au + bv \\ cu + dv \\ 1 \end{bmatrix}.$$

They are equivalent because the x, y components of each are the same, and z is unchanged in the second equation.

12.

(a) Multiply these matrices: $\begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$.

$$\begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} u+\alpha \\ v+\beta \\ 1 \end{bmatrix}.$$

(b) Fill in the blanks: The result of the above multiplication is that the point $(u, v, 1)$ has been translated by ____ in the x direction, ____ in the y direction, and is still anchored to the plane $z =$ ___.

The result of the above multiplication is that the point $(u, v, 1)$ has been translated by $\underline{\alpha}$ in the x direction, $\underline{\beta}$ in the y direction, and is still anchored to the plane $z = \underline{1}$.

13.

(a) Write a matrix which translates a point $(x, y, 1)$ 4 units in the x direction and 7 units in the y direction, leaving z fixed at 1.

We have $\alpha = 4$ and $\beta = 7$:

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+4 \\ y+7 \\ 1 \end{bmatrix}.$$

(b) Check your work by applying your matrix to the point $(3, 5, 1)$.

The expected result is $(3 + 4, 5 + 7, 1) = (7, 12, 1)$. We multiply:

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 12 \\ 1 \end{bmatrix}.$$

Looks good.

14. Do these two multiplications. What does each represent?

(a) $\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix}$

We multiply:

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b & a\alpha + b\beta \\ c & d & c\alpha + d\beta \\ 0 & 0 & 1 \end{bmatrix}$$

Because we have chosen the column vector to represent points, the transformations take place from right to left. Thus, the right matrix is the first transformation, and the left matrix is the second transformation. Therefore, this multiplication represents a translation by $\langle \alpha, \beta, 0 \rangle$, then a 2D matrix transformation by $\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}$.

(b) $\begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$

We multiply:

$$\begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b & \alpha \\ c & d & \beta \\ 0 & 0 & 1 \end{bmatrix}.$$

Well that was easy. In the same logic as the previous subproblem, this multiplication represents a 2D matrix transformation by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, then a translation by $\langle \alpha, \beta, 0 \rangle$.

15. What does each of these matrices represent?

(a) $\begin{bmatrix} a & b & \alpha \\ c & d & \beta \\ 0 & 0 & 1 \end{bmatrix}$

This is the result to Problem 14b, so it's a 2D matrix transformation by $\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}$, then a translation by $\langle \alpha, \beta, 0 \rangle$.

(b) $\begin{bmatrix} \cos\theta & -\sin\theta & \alpha \\ \sin\theta & \cos\theta & \beta \\ 0 & 0 & 1 \end{bmatrix}$

We recognize the 2D matrix transformation here as a rotation by θ radians counterclockwise. Thus, this is a rotation by θ radians counterclockwise, followed by a translation by $\langle \alpha, \beta, 0 \rangle$.

16.

(a) Rewrite your translation matrix and your preimage vector from Problem a so that you do not restrict your translations to the plane $z = 1$, but can translate in the x, y, and z directions. (Hint: think four dimensions!)

We do as the problem hints, and construct a 4×4 matrix which translates by $\langle \alpha, \beta, \gamma, 0 \rangle$ in four dimensions.

$$\begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & \gamma \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + \alpha \\ y + \beta \\ z + \gamma \\ 1 \end{bmatrix}.$$

(b) Write a matrix product that translates the point $(2, 3, -5)$ by the vector $(4, -1, 2)$.

We have $(x, y, z) = (2, 3, -5)$ and $(\alpha, \beta, \gamma) = (4, -1, 2)$:

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ -3 \\ 1 \end{bmatrix}.$$

10 Rotations of the Plane

1.

(a) Which matrix changes nothing, so that the image is the same as the preimage?

We already found this; this is the identity matrix I. The 2×2 identity matrix is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(b) Which complex number changes nothing?

1, since $1 \cdot x = x \cdot 1 = x$. Although it might not seem complex at first sight, real numbers are complex too.

2.

(a) Which matrix doubles the length of every vector but leaves angles unchanged?

This is scaling up by a factor of 2 in all directions, which is

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

(b) Which complex number corresponds to the same transformation?

This is just 2, since multiplying 2 produces the desired effect of scaling.

3. Based on your answers to the previous problems, which matrix corresponds to the real number r? Let's call this $\mathrm{M}(r)$ for short.

It looks like

$$\mathrm{M}(r) = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}.$$

4. Explain why $\mathrm{M}(u) + \mathrm{M}(v) = \mathrm{M}(u + v)$.

From the perspective of complex numbers, the relationship $(u)+(v) = u+v$ holds. It also holds for matrices:

$$\mathrm{M}(u) + \mathrm{M}(v) = \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} + \begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix} = \begin{bmatrix} u+v & 0 \\ 0 & u+v \end{bmatrix} = \mathrm{M}(u+v).$$

5. Under a $90°$ counterclockwise rotation, what is the image of (a) $(1, 0)$ and (b) $(0, 1)$?

The image of $(1, 0)$ is $(0, 1)$, and the image of $(0, 1)$ is $(-1, 0)$. You can verify this by drawing it out if you want.

6.

(a) Which matrix corresponds to a $90°$ rotation?

This is just the rotation matrix $\left[\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right]$, with $\theta = 90° = \frac{\pi}{2}$:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

(b) Which complex number corresponds to the same rotation?

Multiplying by i produces the same rotation.

7. Based on your answers to Problems 1 to 6, what matrix corresponds to the complex number $x + yi$? Let's extend our function M and call this $\mathrm{M}(x + yi)$ for short.

It looks like

$$\mathrm{M}(x+yi) = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}.$$

8. **Check that $\mathrm{M}(a+bi)+\mathrm{M}(c+di) = \mathrm{M}((a+bi)+(c+di))$. That is, prove that M has the same addition rules as complex numbers.**

We have

$$\mathrm{M}(a+bi)+\mathrm{M}(c+di) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} a+c & -b-d \\ b+d & a+c \end{bmatrix} = \mathrm{M}((a+c)+(b+d)i) = \mathrm{M}((a+bi)+(c+di)).$$

9. **Check that $\mathrm{M}(a+bi)\,\mathrm{M}(c+di) = \mathrm{M}((a+bi)(c+di))$. That is, prove that M has the same multiplication rules as complex numbers.**

We have

$$\begin{aligned}
\mathrm{M}(a+bi)\,\mathrm{M}(c+di) &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \\
&= \begin{bmatrix} ac-bd & -ad-bc \\ ad+bc & ac-bd \end{bmatrix} \\
&= \mathrm{M}((ac-bd)+(ad+bc)i) \\
&= \mathrm{M}((a+bi)(c+di)).
\end{aligned}$$

10. **Recall that multiplying by $\operatorname{cis}\theta$ rotates a complex number by θ radians.**

   (a) **Find $\mathrm{M}(\operatorname{cis}\theta)$.**

Since $\operatorname{cis}\theta = \cos\theta + i\sin\theta$, we have

$$\mathrm{M}(\operatorname{cis}\theta) = \mathrm{M}(\cos\theta + i\sin\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

   (b) **To prove that this matrix really does rotate by θ:**

      i. **Check that the image and preimage have the same length;**

Let the preimage by $\begin{bmatrix} x \\ y \end{bmatrix}$, which has length $\sqrt{x^2+y^2}$. Then the image is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix},$$

which has length

$$\begin{aligned}
\sqrt{(x\cos\theta - y\sin\theta)^2 + (x\sin\theta + y\cos\theta)^2} &= \sqrt{x^2\cos^2\theta - 2xy\cos\theta\sin\theta + y^2\sin^2\theta + x^2\sin^2\theta + 2xy\cos\theta\sin\theta + y^2\cos^2\theta} \\
&= \sqrt{x^2(\cos^2\theta + \sin^2\theta) + y^2(\sin^2\theta + \cos^2\theta)} \\
&= \sqrt{x^2+y^2}.
\end{aligned}$$

Indeed, this matches up with the length of the preimage.

      ii. **Check that the angle of the image with the x-axis is θ more than the preimage.**

This is a bit unpleasant. The angle of the image with the x-axis is $\tan^{-1}\frac{y}{x}$... we can make this more pleasant by representing our preimage as

$$\begin{bmatrix} r\cos\phi \\ r\sin\phi \end{bmatrix},$$

which is a point r away from the origin and making an angle of ϕ with the x-axis.
Then the image is

$$\begin{aligned}\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}r\cos\phi\\ r\sin\phi\end{bmatrix} &= \begin{bmatrix}r\cos\phi\cos\theta - r\sin\phi\sin\theta\\ r\sin\phi\cos\theta + r\cos\phi\sin\theta\end{bmatrix}\\ &= \begin{bmatrix}r(\cos\phi\cos\theta - \sin\phi\sin\theta)\\ r(\sin\phi\cos\theta + \cos\phi\sin\theta)\end{bmatrix}\\ &= \begin{bmatrix}r\cos(\phi+\theta)\\ r\sin(\phi+\theta)\end{bmatrix}.\end{aligned}$$

This is a point r away from the origin and making an angle of $\theta + \phi$ with the x-axis, an angle θ more than the original ϕ as desired.

11.

(a) Find $\mathrm{M}(r \operatorname{cis}\theta)$.

$$\mathrm{M}(r\operatorname{cis}\theta) = \mathrm{M}(r\cos\theta + ir\sin\theta) = \begin{bmatrix}r\cos\theta & -r\sin\theta\\ r\sin\theta & r\cos\theta\end{bmatrix}.$$

(b) To prove that this matrix really does rotate by θ and stretch by r:

i. Check that the length of the image is r times the length of the preimage;

Let the preimage be $\begin{bmatrix}x\\ y\end{bmatrix}$, which has length $\sqrt{x^2+y^2}$. Then the image is

$$\begin{bmatrix}r\cos\theta & -r\sin\theta\\ r\sin\theta & r\cos\theta\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = r\underbrace{\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}}_{\mathrm{M}(\operatorname{cis}\theta)\cdot\left[\begin{smallmatrix}x\\ y\end{smallmatrix}\right]}.$$

This has length $r\sqrt{x^2+y^2}$ by the previous problem, as desired.

ii. Check that the angle of the image with the x-axis is θ more than the preimage. (Hint: You may want to use the previous problem, or the tangent addition formulas.)

Let our preimage be

$$\begin{bmatrix}\rho\cos\phi\\ \rho\sin\phi\end{bmatrix},$$

which makes an angle of ϕ with the x-axis. Then the image is

$$\begin{bmatrix}r\cos\theta & -r\sin\theta\\ r\sin\theta & r\cos\theta\end{bmatrix}\begin{bmatrix}\rho\cos\phi\\ \rho\sin\phi\end{bmatrix} = r\underbrace{\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\rho\cos\phi\\ \rho\sin\phi\end{bmatrix}}_{\mathrm{M}(\operatorname{cis}\theta)\cdot\left[\begin{smallmatrix}\rho\cos\phi\\ \rho\sin\phi\end{smallmatrix}\right]},$$

which makes an angle of $\phi + \theta$ with the x-axis via the last problem, as desired.

12.

(a) What matrix reflects over the x-axis, taking $(x, y) \to (x, -y)$?

This is the matrix $\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$, since this flips the y coordinate.

(b) What is the complex number operation equivalent to this transformation?

The equivalent operation is complex conjugation, denoted $\overline{a+bi} = a - bi$.

(c) Is there a complex number multiplication equivalent to this transformation? Justify your answer.

There is not. Suppose there was a complex number $r \operatorname{cis} \theta$ which satisfied

$$(a+bi)r \operatorname{cis} \theta = a - bi.$$

Then since $|a+bi| = |a-bi|$, $r = 1$. So $(a+bi)\operatorname{cis} \theta = a - bi$. But the transformation described is a reflection, while this is a rotation! Thus, no such complex number exists.

13.

(a) What matrix reflects through the origin, taking $(x, y) \to (-x, -y)$?

This is the matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, since this flips both the x and y coordinates.

(b) What is the complex number operation equivalent to this transformation?

This is negating the complex number: $f(z) = -z$.

(c) Is there a complex number multiplication equivalent to this transformation? Justify your answer.

Yes there is! $f(z) = -1 \cdot z$ is equivalent; we have $-1 \cdot (a+bi) = -a - bi$ as desired.

14.

(a) Which of the 16 matrices on page 87, for Problem 10, have corresponding complex numbers?

i. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

No, since $1 \neq -1$, and the entries on the top left–bottom right diagonal need to be the same.

ii. $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Yes; the complex number is -1.

iii. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Yes; the complex number is 2.

iv. $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Yes; the complex number is $-i$.

v. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

No, since $1 \neq -(1)$; the entries on the bottom left–top right diagonal need to be the opposite of each other.

vi. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Yes; the complex number is 0.

vii. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Yes; the complex number is 1.

viii. $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

No, since $3 \neq 1$.

ix. $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

No, since $-(-3) \neq 0$.

x. $\begin{bmatrix} 2 & 2 \\ -3 & -3 \end{bmatrix}$

No, since $2 \neq -3$.

xi. $\begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}$

No, since $3 \neq -1$.

xii. $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$

No, since $\frac{\sqrt{2}}{2} \neq -\frac{\sqrt{2}}{2}$ (comparing TR and BL corners).

xiii. $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

Yes; the complex number is $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$.

xiv. $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$

No, since $\frac{1}{2} \neq -\frac{1}{2}$.

xv. $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

Yes; the complex number is $\frac{\sqrt{3}}{2} + \frac{1}{2}i$.

xvi. $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

Yes; the complex number is $\frac{\sqrt{3}}{2} - \frac{1}{2}i$.

(b) How can you tell algebraically?

The matrix must be of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ for real (but not necessarily positive) a, b.

(c) How can you tell geometrically?

If the matrix is purely a rotation and dilation, then it has an associated complex number. Note that the zero matrix is a dilation of 0.

15. Make multiplication tables with the set of matrices which correspond to the elements of the rotation group for the square (a 4×4 table) and the equilateral triangle (a 3×3 table).

Square: Define $r = \mathrm{M}(i) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, which is a rotation $90°$ counterclockwise. Then we have

$$r^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}; \qquad r^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}; \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The table is shown below.

$\cdot$	I	r	r^2	r^3
I	I	r	r^2	r^3
r	r	r^2	r^3	I
r^2	r^2	r^3	I	r
r^3	r^3	I	r	r^2

Equilateral triangle: Define $r = \mathrm{M}(\operatorname{cis} 120°) = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$. Then we have

$$r^2 = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}; \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The table is shown below.

$\cdot$	I	r	r^2
I	I	r	r^2
r	r	r^2	I
r^2	r^2	I	r

16.

(a) Write a matrix for a rotation of θ around the origin followed by a translation by (a, b).

Recall that for translations, we need 3×3 matrices. The matrix is

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Write a matrix for a translation by (a, b) followed by a rotation of θ around the origin.

The matrix is:

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & a\cos\theta - b\sin\theta \\ \sin\theta & \cos\theta & a\cos\theta + b\sin\theta \\ 0 & 0 & 1 \end{bmatrix}.$$

17. Use matrix multiplication to find the image (x', y') of a point (x, y) rotated by θ.

We represent (x, y) as $\begin{bmatrix} x \\ y \end{bmatrix}$. With matrix multiplication, we get

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}.$$

Thus, $(x', y') = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$.

18.

(a) Given the parabola $x = t, y = t^2$, use matrix multiplication to rotate it by $45°$.

Let the new axes be x' and y'. Then we have

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} t \\ t^2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2}t - \frac{\sqrt{2}}{2}t^2 \\ \frac{\sqrt{2}}{2}t + \frac{\sqrt{2}}{2}t^2 \end{bmatrix}.$$

Thus, $x' = \frac{\sqrt{2}}{2}t - \frac{\sqrt{2}}{2}t^2$ and $y' = \frac{\sqrt{2}}{2}t + \frac{\sqrt{2}}{2}t^2$.

(b) Graph the new parametric equations on your calculator.

Here you go! We let $x = x'$ and $y = y'$ for graphing purposes, which rotates it.

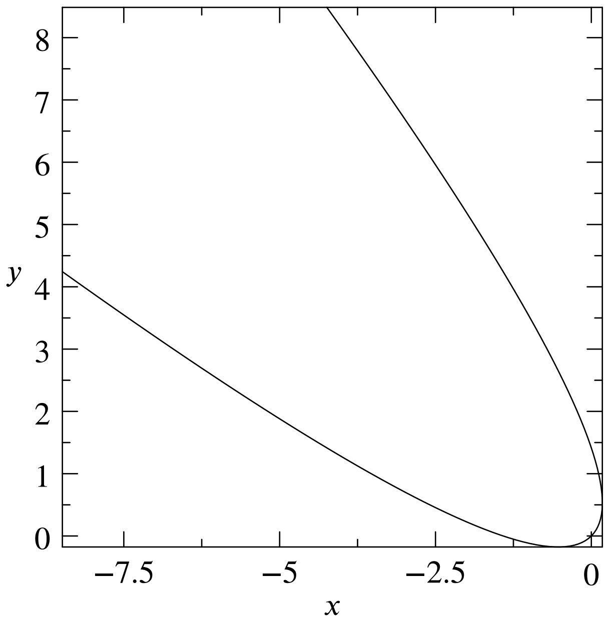


Figure 1: Rotated parabola.

Challenge from Tim: find the maximum x value of this! Requires either some ingenuity or some calculus.[16]

(c) Does it look like a rotation clockwise or counterclockwise? Why?

Looks like a rotation by 45° counterclockwise, since that's what the matrix $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$ does!

[16]Answer: Many non-calculus ways to do this challenge. One way is to ask what t maximizes $x(t)$.

11 Matrices Generate Groups

1. **Analyze this group with the following elements, following the form of Example 1. What makes this group fundamentally different from the example?**

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

i. Specify the elements of the matrix group, unless they are all given.

They are all given.

ii. Describe what each matrix does to the plane.

I does nothing. A rotates by $90° = \frac{\pi}{2}$ counterclockwise. B reflects over the origin, or rotates $180° = \pi$ counterclockwise. C rotates by $90° = \frac{\pi}{2}$ clockwise, or $270° = \frac{3\pi}{2}$ counterclockwise.

iii. Construct a group table; you can use a calculator.

This is pretty simple. Everything is a rotation by a factor of $90° = \frac{\pi}{2}$.

$\cdot$	I	A	B	C
I	I	A	B	C
A	A	B	C	I
B	B	C	I	A
C	C	I	A	B

iv. Decide which symmetry group your matrix is isomorphic to.

This is the cyclic group C_4, which is (up to isomorphism) the rotation group of the square.

2. **The matrix $\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$ generates a group of order 3. Enumerate the elements of this group and analyze per the example.**

i. Specify the elements of the matrix group, unless they are all given.

Let the given matrix be $M = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$. Then

$$M^2 = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix};$$

$$M^3 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

ii. Describe what each matrix does to the plane.

M rotates by $120° = \frac{2\pi}{3}$ counterclockwise. M^2 rotates by $240° = \frac{4\pi}{3}$ counterclockwise, or $120° = \frac{2\pi}{3}$ clockwise. I does nothing.

iii. Construct a group table; you can use a calculator.

These are all rotations by a factor of $120° = \frac{2\pi}{3}$.

$\cdot$	I	M	M^2
I	I	M	M^2
M	M	M^2	I
M^2	M^2	I	M

iv. Decide which symmetry group your matrix is isomorphic to.

This is the cyclic group C_3, which is (up to isomorphism) the rotation group of the triangle.

3. The matrices $\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$ **and** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **generate a group of order** 6**, of which the group in problem 2 is a subgroup. Enumerate the elements of the group and analyze per the example.**

i. Specify the elements of the matrix group, unless they are all given.

We know that the first matrix is a rotation by $120° = \frac{2\pi}{3}$, and the second matrix is a reflection about the x-axis, since it flips the y coordinate. Thus, let the first matrix be r and the second matrix be f. Note how understanding transformations helps us find the other matrices without much work.

The six elements are shown below.

$$r = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$
$$f = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$r^2 = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$
$$fr = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$
$$fr^2 = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$
$$I = f^2 = r^3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

ii. Describe what each matrix does to the plane.

r is a rotation by $120° = \frac{2\pi}{3}$ counterclockwise. f is a reflection about the x-axis. r^2 is a rotation by $240° = \frac{4\pi}{3}$ counterclockwise, or $120° = \frac{2\pi}{3}$ clockwise. fr is a reflection about the line $\theta = 120° = \frac{2\pi}{3}$. fr^2 is a reflection about the line $\theta = 240° = \frac{4\pi}{3}$. I does nothing.

iii. Construct a group table; you can use a calculator.

Here you go.

$\cdot$	I	r	r^2	f	fr	fr^2
I	I	r	r^2	f	fr	fr^2
r	r	r^2	I	fr^2	f	fr
r^2	r^2	I	r	fr	fr^2	f
f	f	fr	fr^2	I	r	r^2
fr	fr	fr^2	f	r^2	I	r
fr^2	fr^2	f	fr	r	r^2	I

iv. Decide which symmetry group your matrix is isomorphic to.

This is the dihedral group of order 6 (D_3), or the symmetry group of the triangle (S_3).

v. What other sets of matrices could have generated this group?

The sets $\{r, fr\}$, $\{r, fr^2\}$, $\{r^2, f\}$, $\{r^2, fr\}$, $\{r^2, fr^2\}$, $\{f, fr\}$, $\{f, fr^2\}$, and $\{fr, fr^2\}$. In fact, any two non-identity elements can together generate the group, except for $\{r, r^2\}$.

4. The matrix $\begin{bmatrix} \frac{\sqrt{5}-1}{4} & -\frac{\sqrt{10+2\sqrt{5}}}{4} \\ \frac{\sqrt{10+2\sqrt{5}}}{4} & \frac{\sqrt{5}-1}{4} \end{bmatrix}$ **generates a group of order** 5**. Enumerate the elements of the group and analyze per the example; you can use a calculator.**

i. Specify the elements of the matrix group, unless they are all given.

We'd expect this to be a rotation matrix of some multiple of 72°. Thus, let's call it r for now. It isn't immediately clear, however, how to compute $\cos 72°$. All available sum and difference expressions seem useless. We'll defer this computation to part (b). Here are the elements of the matrix group:

$$r = \begin{bmatrix} \frac{\sqrt{5}-1}{4} & -\frac{\sqrt{10+2\sqrt{5}}}{4} \\ \frac{\sqrt{10+2\sqrt{5}}}{4} & \frac{\sqrt{5}-1}{4} \end{bmatrix}$$
$$r^2 = \begin{bmatrix} -\frac{1+\sqrt{5}}{4} & -\frac{\sqrt{10-2\sqrt{5}}}{4} \\ \frac{\sqrt{10-2\sqrt{5}}}{4} & -\frac{1+\sqrt{5}}{4} \end{bmatrix}$$
$$r^3 = \begin{bmatrix} -\frac{1+\sqrt{5}}{4} & \frac{\sqrt{10-2\sqrt{5}}}{4} \\ -\frac{\sqrt{10-2\sqrt{5}}}{4} & -\frac{1+\sqrt{5}}{4} \end{bmatrix}$$
$$r^4 = \begin{bmatrix} \frac{\sqrt{5}-1}{4} & \frac{\sqrt{10+2\sqrt{5}}}{4} \\ -\frac{\sqrt{10+2\sqrt{5}}}{4} & \frac{\sqrt{5}-1}{4} \end{bmatrix}$$
$$I = r^5 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Note that there are many equivalent ways to write the entries of these matrices. For example,

$$-\frac{\sqrt{10-2\sqrt{5}}}{4} = -\sqrt{2\left(5+\sqrt{5}\right)} + \sqrt{10\left(5+\sqrt{5}\right)}.$$

The latter is what WolframAlpha gives; I prefer the former form.

ii. Describe what each matrix does to the plane.

We'd guess that r is a rotation of $\frac{360°}{5} = 72°$, but to prove this we need to find $\cos 72°$ and $\sin 72°$.

Consider $\cos 72°$. Because of symmetry around $2.5 \cdot 72° = 180°$, we have $\cos(2 \cdot 72°) = \cos(3 \cdot 72°)$. By the double-angle and triple-angle (which we found in the complex numbers section) formulae,

$$\cos 2x = 2\cos^2 x - 1;$$
$$\cos 3x = 4\cos^3 x - 3\cos x.$$

Let $c = \cos 72°$. Then we have

$$2c^2 - 1 = 4c^3 - 3c$$
$$4c^3 - 2c^2 - 3c + 1 = 0$$
$$(4c^2 + 2c - 1)(c - 1) = 0,$$

We know $\cos 72° \neq \cos 0° = 1$, so

$$\begin{aligned}4c^2 + 2c - 1 &= 0\\ c &= \frac{-1 \pm \sqrt{5}}{4}.\end{aligned}$$

Since $0 < 72° < 90°$, we have $c > 0$, so $c = \frac{\sqrt{5}-1}{4}$. To find $\sin 72°$ we use the Pythagorean identity and choose the positive root:

$$\begin{aligned}\sin 72° = \sqrt{1-c^2} &= \sqrt{1 - \frac{5 - 2\sqrt{5} + 1}{16}}\\ &= \sqrt{\frac{16 - 6 + 2\sqrt{5}}{16}}\\ &= \frac{\sqrt{10 + 2\sqrt{5}}}{4}.\end{aligned}$$

Indeed, we have

$$\begin{bmatrix}\cos 72° & -\sin 72° \\ \sin 72° & \cos 72°\end{bmatrix} = \begin{bmatrix} c & -s \\ s & c\end{bmatrix} = \begin{bmatrix}\frac{\sqrt{5}-1}{4} & -\frac{\sqrt{10+2\sqrt{5}}}{4} \\ \frac{\sqrt{10+2\sqrt{5}}}{4} & \frac{\sqrt{5}-1}{4}\end{bmatrix} = r.$$

iii. Construct a group table; you can use a calculator.

Here it is:

$\cdot$	I	r	r^2	r^3	r^4
I	I	r	r^2	r^3	r^4
r	r	r^2	r^3	r^4	I
r^2	r^2	r^3	r^4	I	r
r^3	r^3	r^4	I	r	r^2
r^4	r^4	I	r	r^2	r^3

iv. Decide which symmetry group your matrix is isomorphic to.

This the cyclic group of order 5, or the rotation group of the regular pentagon.

v. What other sets of matrices could have generated this group?

Any matrix in this group, except the identity matrix, would generate the whole group, since 5 is a prime number.

5. Let $A = \begin{bmatrix}\cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} \\ \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n}\end{bmatrix}$, $B = \begin{bmatrix}\cos\frac{2\pi}{n} & \sin\frac{2\pi}{n} \\ \sin\frac{2\pi}{n} & -\cos\frac{2\pi}{n}\end{bmatrix}$, $C = \begin{bmatrix}1 & 0 \\ 0 & -1\end{bmatrix}$, and n be an integer. What group is generated by the following sets of generators? Describe them geometrically.

(a) $\{A\}$

A is a rotation matrix rotating by $\frac{2\pi}{n}$ radians, which is the angle subtended by one of the sides of an n-gon:

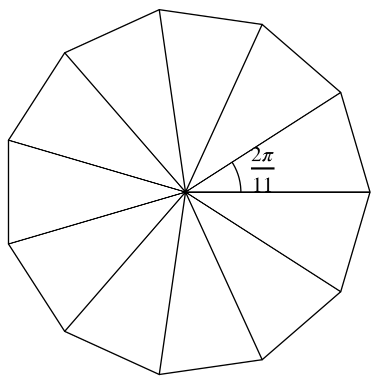


Figure 1: A rotation of $\frac{2\pi}{n}$ radians is a symmetry of the n-gon. Here, $n = 11$.

(b) $\{B\}$

$B = \begin{bmatrix} \cos\frac{2\pi}{n} & \sin\frac{2\pi}{n} \\ \sin\frac{2\pi}{n} & -\cos\frac{2\pi}{n} \end{bmatrix}$ initially appears to be a rotation matrix, but the right column is negated. What could this mean?!

Well, notice that $BC = A$. Since C is just a reflection over the x-axis, $C^2 = I$, so we have $BCC = AC$ and thus $B = AC$. In geometric terms, B is a reflection over the x-axis, followed by a rotation of $2\pi/n$ radians counterclockwise; recall that our matrices transform right-to-left. But a non-zero rotation followed by a reflection is just a reflection about a different axis! So $\{B\}$ generates the cyclic group of order 2, which is the rotation group of the rectangle or the symmetry group of the line segment.

To confirm this, we can show that $B^2 = I$:

$$B^2 = \begin{bmatrix} \cos\frac{2\pi}{n} & \sin\frac{2\pi}{n} \\ \sin\frac{2\pi}{n} & -\cos\frac{2\pi}{n} \end{bmatrix} \begin{bmatrix} \cos\frac{2\pi}{n} & \sin\frac{2\pi}{n} \\ \sin\frac{2\pi}{n} & -\cos\frac{2\pi}{n} \end{bmatrix} = \begin{bmatrix} \cos^2\frac{2\pi}{n} + \sin^2\frac{2\pi}{n} & \cos\frac{2\pi}{n}\sin\frac{2\pi}{n} - \sin\frac{2\pi}{n}\cos\frac{2\pi}{n} \\ \sin\frac{2\pi}{n}\cos\frac{2\pi}{n} - \cos\frac{2\pi}{n}\sin\frac{2\pi}{n} & \sin^2\frac{2\pi}{n} + \cos^2\frac{2\pi}{n} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

(c) $\{A, B\}$

Thinking of these as transformations, we see that B is a reflection across the line $\theta = \pi/n$ (not $2\pi/n$!), which is a symmetry of the n-gon[17]. Combined with the rotation of $2\pi/n$, this generates the dihedral group of order $2n$: the symmetry group of the n-gon.

(d) $\{B, C\}$

Since $A = BC$ and $A^{n-1}B = C$, this problem's set can generate A and the previous problem's set can generate C. Thus, they are the same; $\{B, C\}$ generates the dihedral group of order $2n$: the symmetry group of the n-gon.

6. Given $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, what is the order of the group generated by the following sets of generators?

(a) $\{C\}$

This has order 2, since C is just a reflection over the x-axis. In algebraic terms, $C^2 = I$.

(b) $\{D\}$

Interestingly, $D^2 = I$, so this again has order 2. Truly succulent!

(c) $\{C, D\}$

[17]To be pedantic, the n-gon centered on the origin and with a vertex on the x-axis.

What new do we get from a set of two matrices? Well, consider $CD = J$ (how fun) and $DC = K$ (less fun) which are just $J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $K = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, respectively. Let's analyze products of J and K.

Well, $JK = KJ = I$. More interesting stuff happens when we multiply them repeatedly.

$$J^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$J^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$K^2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$K^3 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

$$J^3K^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = J$$

$$K^3J^2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = K.$$

Interesting! It seems a product of J's and K's leads to a matrix of the form $\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$, where n is an integer. We can also multiply this by C, which gives us the matrices $\begin{bmatrix} 1 & m \\ 0 & -1 \end{bmatrix}$, where m is an integer. The order of this group is countably infinite, since we can enumerate all of the elements in a list.

7. **What matrix could generate a group isomorphic to the cyclic group of order n, C_n?**

The matrix $\begin{bmatrix} \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} \\ \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{bmatrix}$ could do so.

8. **What set of two matrices could generate a group isomorphic to the dihedral group of order $2n$, D_n?**

9. **Look at Problem 1 on page 69. The adjacency matrices map to a subgroup of the full cube symmetry group. What rotations/reflections do they map to?**

10. **Given $P = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, and $R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, try understanding the groups generated by:**

Fair warning: these problems are challenging. With more advanced tools of abstract algebra, however, they are much easier. Nonetheless, I will present "elementary" solutions.

(a) $\{P\}$

We have $P^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P^3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$. Thus, $\{P\}$ generates the cyclic group of order 4 (C_4): the rotation group of the square.

(b) $\{Q\}$

We have $Q^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $Q^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$. Thus, $\{Q\}$ generates the cyclic group of order 3 (C_3): the rotation group of the equilateral triangle.

(c) $\{R\}$

Clearly, $R^2 = I$, so R generates the cyclic group of order 2 (C_2): the rotation group of the rectangle.

(d) $\{P, Q\}$

This is where the complexity begins.

Approach 1: Purely in Matrices

This way is kind of silly, so I would suggest you read Approach 2.

To wrap our heads around this group, we consider what left-multiplying by P and Q does to a matrix's entries. Multiplying P by some matrix M, we get

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} -d & -e & -f \\ a & b & c \\ g & h & i \end{bmatrix}.$$

Thus, P swaps the top two rows and negates the topmost row, in that order. Multiplying Q by M, we get

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} d & e & f \\ g & h & i \\ a & b & c \end{bmatrix}.$$

Thus, Q cycles the rows "upward," where row 1 goes to row 3, row 2 goes to row 1, and row 3 goes to row 2. So P and Q are just operations on rows; columns don't matter.

Consider the sequence of multiplications

$$\underbrace{PQQQPQP \cdots PPP}_{\text{some sequence}} Q^3 = \underbrace{PQQQPQP \cdots PPP}_{\text{some sequence}} I.$$

On one hand, every matrix that can be generated by P and Q can be written in this form. On the other hand, we can treat the sequence of P and Q as (left-to-right) sequential row operations on the identity matrix I.

Think of the identity matrix as the ordered triple (R_1, R_2, R_3), where R_i is row vector i. Then P is the function $f(r_1, r_2, r_3) = (-r_2, r_1, r_3)$ and Q is the function $f(r_1, r_2, r_3) = (r_2, r_3, r_1)$.

The order of this group is clearly finite; a quick upper bound is the number of permutations of (R_1, R_2, R_3), along with any combination of negations of elements. This is

$$3! \cdot 2^3 = 48.$$

We can reduce this upper bound by constructing an invariant: something that neither of these operations change. This invariant is rather simple:

$$K = \begin{cases} 1 & \text{cyclic order preserved} \\ 0 & \text{cyclic order not preserved} \end{cases} + \begin{cases} -1 & \text{even number of negated rows} \\ 0 & \text{odd number of negated rows} \end{cases}.$$

"Cyclic order" is what I'm calling the property that you can start at R_1 and continue reading right, looping back if necessary, to read R_1, R_2, R_3. Cyclic order is not preserved if you read R_1, R_3, R_2.

The invariant for the identity is $K = 0$. For all attainable row triples, the invariant is $K = 0$. That's because Q just cycles the elements, changing nothing about the invariant, while P changes the parity of negated rows *and* either restores or removes "cyclic order." This will produce canceling effects in the two terms of K.[18]

Half of row triples have $K = 0$. A simple way to see this is that cyclic orderedness and negation of rows can be chosen independently (among all possible triples), and since there are two possibilities for each, there is a $1/4$ chance of any particular state. Since 2 states are zero, we have $2/4 = 1/2$ of row triples that have $K = 0$ and can be attained. That's $1/2 \cdot 48 = 24$ total row triples for a new upper bound.

Let's see if we can construct all 24 of these triples using P and Q. There are two cases: cyclic order preserved and even number of negated rows, and non-cyclic order preserved and odd number of negated rows.

If we can construct $\pm R_1, \pm R_2, \pm R_3$ (where the number of -s is even, totaling 4 cases) and $\pm R_1, \pm R_3, \pm R_2$ (where the number of -s is odd, totaling 4 cases), then applying Q iteratively will give us all $(4 + 4) \cdot 3 = 24$ possible cases. Let's see if this is possible.

[18] Verify this yourself if you don't believe me, I'm getting tired.

Case 1: $\pm R_1, \pm R_2, \pm R_3$, even number of negations.

Subcase a: 0 negations. This is just the identity, or Q^3. Subcase b: 2 negations.

Ssubcase i: $-R_1, -R_2, R_3$. We can get this by applying P twice: this is just P^2. Ssubcase ii: $-R_1, R_2, -R_3$. We cycle the elements until R_3 and R_1 are first, then apply P^2, then cycle back to the original order. In this case, it is QP^2Q^2:

$$(R_1, R_2, R_3) \to^{Q^2} (R_3, R_1, R_2) \to^{P^2} (-R_3, -R_1, R_2) \to^{Q} (-R_1, R_2, -R_3).$$

Ssubcase iii: $R_1, -R_2, -R_3$. We apply the same concept as ssubcase ii. In this case, it is Q^2P^2Q.

$$(R_1, R_2, R_3) \to^{Q} (R_2, R_3, R_1) \to^{P^2} (-R_2, -R_3, R_1) \to^{Q^2} (R_1, -R_2, -R_3).$$

Case 2: $\pm R_1, \pm R_3, \pm R_2$, odd number of negations.

Subcase a: 1 negation. Ssubcase i: $-R_1, R_3, R_2$. We cycle to R_3, R_1, R_2, then apply P. This is just PQ^2. Ssubcase ii: $R_1, -R_3, R_2$. We cycle to R_2, R_3, R_1, then apply P, then cycle to $R_1, -R_3, R_2$. This is just Q^2PQ. Ssubcase iii: $R_1, R_3, -R_2$. We apply P, then cycle to $R_1, R_3, -R_2$. This is just QP.

Subcase b: 3 negations. We can get this by taking ssubcase iii, then left-multiplying by P^2, which negates the first two rows. This is just P^2QP.

In matrix and row form, these results are summarized like so:

$$I = Q^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftrightarrow (R_1, R_2, R_3)$$

$$P^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftrightarrow (-R_1, -R_2, R_3)$$

$$QP^2Q^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftrightarrow (-R_1, R_2, -R_3)$$

$$Q^2P^2Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \leftrightarrow (R_1, -R_2, -R_3)$$

$$PQ^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \leftrightarrow (-R_1, R_3, R_2)$$

$$Q^2PQ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \leftrightarrow (R_1, -R_3, R_2)$$

$$QP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \leftrightarrow (R_1, R_3, -R_2)$$

$$P^2QP = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \leftrightarrow (-R_1, -R_3, -R_2).$$

Left-multiplying each of these by I (nothing), Q, and Q^2 yield all 24 elements of our group. Since this is a lower bound, and we found 24 to also be an upper bound, this group has order 24.

The underlying structure is a bit unclear still, and honestly this solution method is very tedious and prone to error. I would't be surprised if I made a silly error somewhere. This next approach will make it a bit clearer though.

Approach 2: Geometric Visualization

We observe matrix P and notice it looks rather like a rotation matrix in 2D, but taken to 3D in the manner we discussed in Mapping the Plane with Matrices. In particular, it is a rotation counterclockwise by $\sin^{-1} 1 = 90° = \frac{\pi}{2}$ about the z-axis. (Note that rotating about the origin in 2D is rotating about the z-axis in 3D.)

What transformation is matrix Q then? Well just as we could apply a matrix to the points $(1,0)$ and $(0,1)$ to understand its behavior in 2D, we can apply a matrix to the points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. This tells us that $(1,0,0)$ is mapped to $(0,0,1)$, $(0,1,0)$ is mapped to $(1,0,0)$, and $(0,0,1)$ is mapped to $(1,0,0)$ (we get this by reading off the column vectors). Drawing out this motion makes it clear that it is a rotation, as shown in Figure 11.

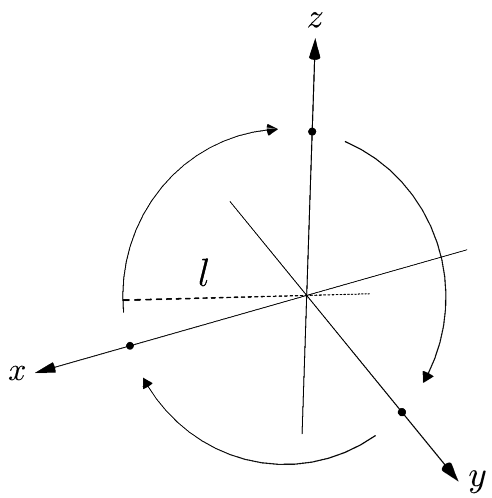


Figure 2: Q is a rotation about an interesting axis l, the one going between $(0,0,0)$ and $(1,1,1)$.

Let's try to find an object which has these two rotations as symmetries. An obvious one is the cube, centered at $(0,0,0)$. Arbitrarily, let's let the vertices be $(\pm1,\pm1,\pm1)$. Then these two matrix rotations look like so:

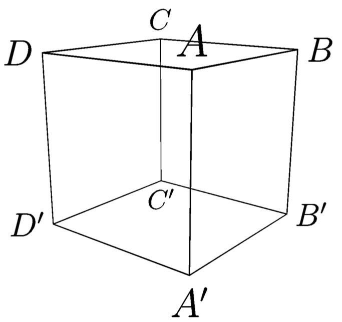


Figure 3: Rotation I.

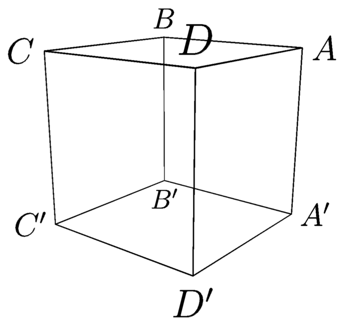


Figure 4: Rotation P.

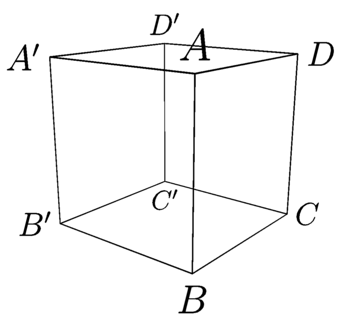


Figure 5: Q.

So since both generators are valid rotations of the cube, we at least know that this group is a subgroup of the rotation group of the cube, which we previously found to be S_4, the permutation group on 4 elements. To figure out which subgroup, we recall the methodology we used to prove that the rotation group of the cube was isomorphic to S_4. We labeled opposite vertices with the same number, going 1 through 4. Then each valid rotation of the cube corresponds with a permutation of $(1,2,3,4)$ on a chosen face. In this case, we will use the top face, and consider the vertices counterclockwise starting from the corner facing the camera.

We find which permutations P and Q correspond to:

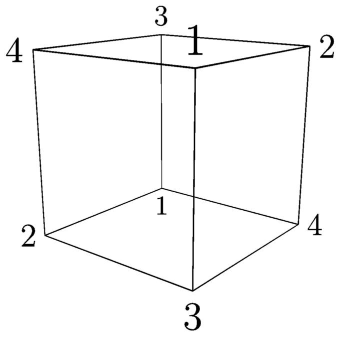


Figure 6: $I \leftrightarrow (1,2,3,4)$, according to the top face.

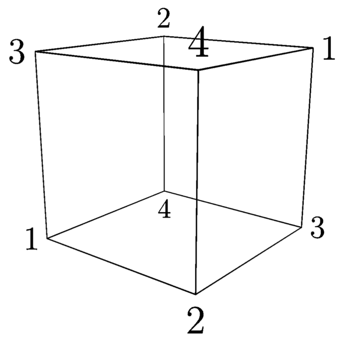


Figure 7: $P \leftrightarrow (4,1,2,3)$.

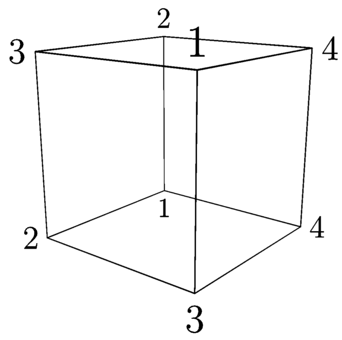


Figure 8: $Q \leftrightarrow (1,4,2,3)$.

So P is a cyclic permutation of all four elements (which makes sense, since the top face is just rotating 90°), and Q is a cyclic permutation of the last three elements. This makes sense; P has period 4 and Q has period 3. What permutations can we generate with these two operations though?

We show that we can swap any two adjacent elements. If this is possible, then any permutation can definitely be reached.

First, we show we can swap the first two elements. This is done by the sequence P^3Q^2:

$$(1,2,3,4) \to^{Q^2} (1,3,4,2) \to^{P^3} (2,1,3,4).$$

Then, to swap elements in positions i and $i+1$, where $1 \le i \le 3$, we 1. cycle the elements so that the elements once in positions i and $i+1$ are now in positions 1 and 2; 2. perform the aforementioned swap; 3. cycle the elements back to their original positions.

For example, suppose we want to swap the second and third elements in $(1,2,3,4)$. Then we start with P^3, which makes the sequence $(2,3,4,1)$. Continuing with the swap, P^3Q^2, takes us to $(3,2,4,1)$. Then P takes us to $(1,3,2,4)$. Overall, this transformation is $PP^3Q^2P^3$, which actually simplifies to Q^2P^3.

A bit more formally, the swap between positions i and $i+1$ is

$$P^{i-1}P^3Q^2P^{5-i}.$$

(Note that $P^0 = I$.) In our example, $i = 2$.

So we can swap any two elements. This means we can get any permutation[19].

So $\{P, Q\}$ generates the rotation group of the cube, which is the permutation group on 4 elements S_4.

(e) $\{P, R\}$

This isn't too hard to do with matrices, but a geometric way is more fun. Observe the matrices' resemblance to 2D transformation matrices (we've already observed this for P):

$$P = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

P is thus a rotation counterclockwise by 90°, and R is a reflection over the y-axis, since the x coordinate is being negated. This makes clear that it is the dihedral group of the square, D_4, which has order 8.

(f) $\{Q, R\}$

This is a bit trickier, because Q is not reducible to some plain 2D transformation matrix. Luckily, left-multiplying by R *is* a row operation, so we recall the idea of naming the rows of the identity matrix R_1, R_2, and R_3:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We represent I as the ordered triple (R_1, R_2, R_3). Then left-multiplying by Q is the function $f(r_1, r_2, r_3) = (r_3, r_1, r_2)$, and left-multiplying by R is the function $f(r_1, r_2, r_3) = (-r_1, r_2, r_3)$.

[19]This is intuitive but we haven't actually proved it. Perhaps I'll add the proof later? Nah.

It's pretty clear what's going on now. Since we can cycle the rows as we please, and negate any one of them, we have 3 orders[20] and 2^3 possible negation patterns, we have $3 \cdot 2^3 = 24$ total elements.

But what is the structure of this group? Is it again, the rotation group of the cube? Let's keep in mind the fact that this group has order 24 and enter the geometric realm.

We know that R is a reflection of the x-coordinate, and thus through the xy-plane. Q and R are indeed symmetries of the cube, but since the full symmetry group of the cube (including reflections) has 48 elements, we need something a bit more restrictive.

This may seem like a leap in logic, but bear with me for a moment. Have you ever looked carefully at a standard volleyball, like the one in Figure 11?[21] It has six sides, but the sides have an extra feature on them: the seams, which are directional.

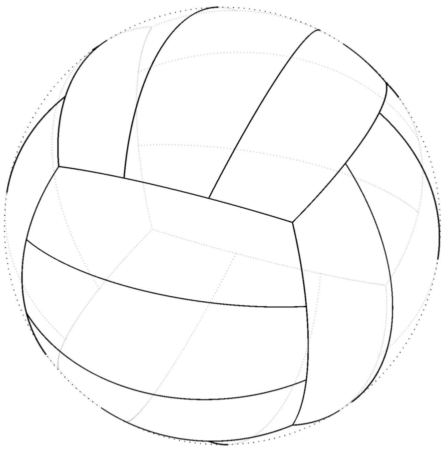


Figure 9: A standard volleyball, with its interesting striations.

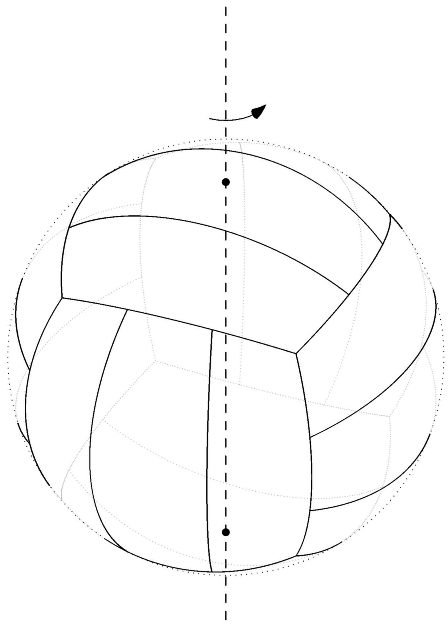


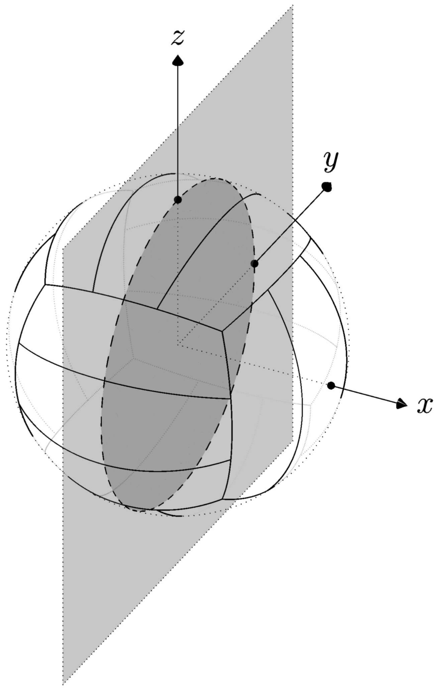
Figure 10: This rotation of $90°$ is a symmetry of the cube, but *not* of the volleyball.

This is similar to a cube, in that it has six sides, but it has restricted symmetries. For example, we can't rotate it $90°$ about an axis going straight through two opposite faces, since then the seams would not line up. This is shown in Figure 11.

The motivation for analyzing this shape is that it *does* accept the matrices R and Q as symmetries. R, or flipping the volleyball over a midplane between two opposite sides, is a symmetry, as shown in Figure 11. Q, or rotating the volleyball $120°$ around a vertex-vertex axis, is also a symmetry, as shown in Figure 11;

[20] Not 6 orders, since the "cyclic order" is preserved. That is, something like (R_1, R_3, R_2) is not achievable.

[21] Not to brag, but a funny memory I had in 9th grade P.E. was looking at a volleyball and explaining to someone—I forget who—how it's similar to a cube. Little did I know how useful that memory would be a year and a half later!



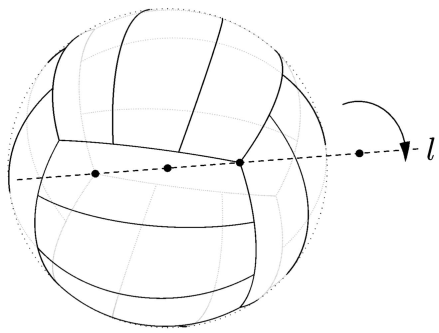


Figure 12: Q, a rotation around the axis l, is a symmetry of the volleyball.

Figure 11: R, a reflection through the yz-plane, is a symmetry of the volleyball.

Indeed, since there are two possible directions for each set of seams, it seems like this should have half the symmetries of the cube. To make this concrete, we observe that there are six faces. Each face can be oriented to face the top, and there are two choices—not four—for its orientation, since it only has bilateral symmetry and not four-fold symmetry like a square. This gives $6 \cdot 2$ choices. But we can also mirror the volleyball into the "mirror world," which multiplies the number of choices by 2. Thus, there are $6 \cdot 2 \cdot 2 = 24$ symmetries of the volleyball.

We know that Q and R are symmetries of the volleyball, and, via our matrix logic, generate a group of order 24. But if they generated a group other than the symmetry group of the volleyball—let's call it V for short—then V isn't closed, since some elements $Q, R \in V$ generate a different group of the same order.

So $\{Q, R\}$ generates the symmetries of a volleyball! I think this is wonderful. For the curious, this is known as **pyritohedral** symmetry. Spicy. As an abstract group, this is $A_4 \times C_2$, where A_4 is the alternating group on 4 elements.

(g) $\{P, Q, R\}$.

While this may look terrifying at first, we can reuse lots of information from the previous problems. We know that P, Q, R are all symmetries of the cube (including reflections). Thus, $\{P, Q, R\}$ generates a subgroup of the symmetry group of the cube, which we previously found has order 48 (and abstract structure $S_4 \times C_2$)

We also know that $\{P, Q\}$ generates the rotational group of the cube, which has order 24. Thus, since R is a plain old reflection, it takes every element of the rotational group to the "mirror world," for a total of $24 \cdot 2 = 48$ elements. The only subgroup with order 48 of a group of order 48 is the group itself. Therefore, $\{P, Q, R\}$ generates the full symmetry group of the cube, which has order 48.

11. The matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ produces a shear. What is its inverse—what undoes the shear?

The matrix $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ undoes the shear, since the product of these two matrices is the identity matrix.

12. The complex numbers, excluding zero, form a group under multiplication. What set of matrices is isomorphic to the same group under multiplication?

As we found previously, the set of matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ where $a, b \in \mathbb{R}$ under multiplication is isomorphic, since we have the one-to-one correspondence $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \leftrightarrow a + bi$.

13. Does the set of all 2×2 matrices form a group under multiplication? Why or why not?

No, because there is no matrix M such that $M \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; it cannot satisfy invertibility.

12 Composite Mappings of the Plane

1. **For Problems a through e, fill in the blank.**

   (a) **We start by finding the images of our points under the $-90°$ rotation.**

      i. **Find the matrix R which results in a $-90°$ rotation.**

That's just $R = \begin{bmatrix} \cos -90° & -\sin -90° \\ \sin -90° & \cos -90° \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

      ii. **Multiply R by our unit vectors and point (u, v):**

$$\begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} 1 & 0 & u \\ 0 & 1 & v \end{bmatrix} = \begin{bmatrix} & & \\ & & \end{bmatrix}.$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & u \\ 0 & 1 & v \end{bmatrix} = \begin{bmatrix} 0 & 1 & v \\ -1 & 0 & -u \end{bmatrix}.$$

   (b) **Next, we reflect those intermediate image points over the line $y = 0$.**

      i. **Find the matrix S which does this.**

We want to flip the y coordinate, so $S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

      ii. **Multiply S by the result of Problem ii.**

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & v \\ -1 & 0 & -u \end{bmatrix} = \begin{bmatrix} 0 & 1 & v \\ 1 & 0 & u \end{bmatrix}.$$

   (c) **You should notice that the net result of the two transformations taken together is a reflection over the line $y = x$. Which matrix represents this transformation?**

The matrix that represents a reflection over $y = x$ is $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

   (d) **Notice that what we did to achieve this mapping was**

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & u \\ 0 & 1 & v \end{bmatrix} = \begin{bmatrix} & & \\ & & \end{bmatrix},$$

   where we multiplied the two rightmost matrices first but didn't use the associative property to multiply the two leftmost matrices first. See what happens when you multiply the two left hand matrices together:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}.$$

   Look familiar?

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

This is our matrix M from part (c)!

   (e) **See what happens when you reverse the order of multiplication:**

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

(f)

i. What transformation does this new matrix result in?

This is a reflection about the line $y = -x$. After all, we have the mappings $(1, 0) \to (0, -1)$ and $(0, 1) \to (-1, 0)$:

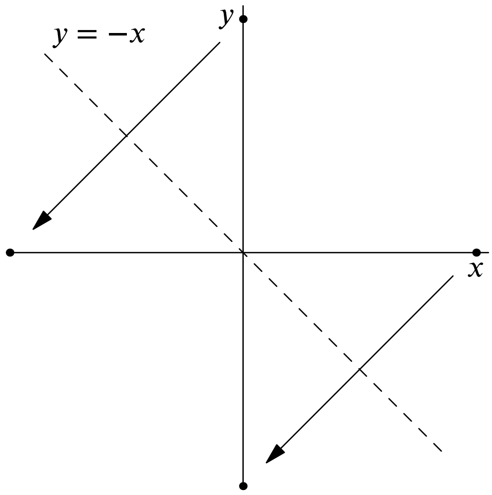


Figure 1: The mapping is a reflection over $y = -x$.

ii. How is a reflection followed by a rotation different from a rotation followed by a reflection? Visualize this by following what happens to a point under both sets of transformations.

A reflection followed by a rotation and a rotation followed by a reflection are both (at least in 2D) reflections overall. In our example, the first ordering of the matrices is the rotation followed by the reflection (recall we're working from right to left), and the second ordering is the reflection followed by the rotation.

In a reflection followed by a rotation clockwise (a.k.a. our rotation of $-90°$), the line of reflection is moved clockwise by half the (positive) angle of rotation. In a rotation clockwise followed by a reflection, the line of reflection is moved *counterclockwise* by half the angle of rotation.

(g) Notice that we apply the transformations from right to left. If you wanted to read from left to right, what would you have to change about the way you wrote the mapping matrices, the vectors representing points, and the order of the matrices?

We would have to reverse the order of the matrices, write the vectors representing points as row vectors, and transpose the matrices. For example, here is our original notation:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ -a \end{bmatrix},$$

and here is it in a left to right format:

$$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -b & -a \end{bmatrix}.$$

(h) How does our convention for ordering transformation matrices compare...

i. ... to the convention for writing composite functions, like $f(g(x))$?

It is similar in order to composing functions, because these both are evaluated sequentially from the right to the left.

ii. ... to the "followed by" convention we used for "From Snaps to Flips?"

We evaluated the flip elements from right to left, so it is like our transformation matrices.

iii. ... to the "from _ to _" convention for transportation matrices?

We wrote transportation matrices with the destinations between on top (as columns), and the origins on the left side (as rows). Thus, they are evaluated from left to right, unlike our transformation matrices.

2. **There are two, infinite classes of matrices which comprise all isometries of the plane which keep the origin fixed. These are the rotation matrix and reflection matrix. Let's look first at the rotation matrix and make sure that it really always works the way it should.**

   (a) **What is the result of a rotation by an angle θ followed by one of ϕ?**

It is a rotation by $\theta + \phi$.

   (b) **Multiply their rotation matrices:** $\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}.$

$$\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\phi\cos\theta - \sin\phi\sin\theta & -\cos\phi\sin\theta - \sin\phi\cos\theta \\ \sin\phi\cos\theta + \cos\phi\sin\theta & -\sin\phi\sin\theta + \cos\phi\cos\theta \end{bmatrix}.$$

   (c) **Use the angle addition formulae to simplify your answer.**

$$\begin{bmatrix} \cos\phi\cos\theta - \sin\phi\sin\theta & -\cos\phi\sin\theta - \sin\phi\cos\theta \\ \sin\phi\cos\theta + \cos\phi\sin\theta & -\sin\phi\sin\theta + \cos\phi\cos\theta \end{bmatrix} = \begin{bmatrix} \cos(\phi+\theta) & -\sin(\phi+\theta) \\ \sin(\phi+\theta) & \cos(\phi+\theta) \end{bmatrix}.$$

   (d) **Should the result be the same if you reverse the order of rotation?**

Yes, since rotation is (unlike most other planar transformations) commutative.

   (e) **What happens to the points $(1,0)$, $(0,1)$, and (x,y) when you operate on them with the rotation matrix?**

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \end{bmatrix}.$$

We multiply as directed:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & x\cos\theta - y\sin\theta \\ \sin\theta & \cos\theta & x\sin\theta + y\cos\theta \end{bmatrix}.$$

3. **Now let's check for the generalized reflection matrix.**

   (a) **Take the matrix which results in a reflection over the line $y = x\tan\frac{\theta}{2}$ and reflect over that line twice:**

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \cos\theta\sin\theta \\ \sin\theta\cos\theta - \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}.$$

   (b) **Simplify your answer and explain the result.**

$$\begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \cos\theta\sin\theta \\ \sin\theta\cos\theta - \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This happens because a reflection is its own inverse. How boring!

   (c) **Let's do a reflection over the line $y = \tan\frac{\theta}{2}$ followed by a reflection over the line $y = \tan\frac{\phi}{2}$:**

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{bmatrix}.$$

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\phi + \sin\theta\sin\phi & \cos\theta\sin\phi - \sin\theta\cos\phi \\ \sin\theta\cos\phi - \cos\theta\sin\phi & \sin\theta\sin\phi + \cos\theta\cos\phi \end{bmatrix}.$$

(d) Simplify your answer using the angle addition formulae, and interpret.

$$\begin{bmatrix} \cos\theta\cos\phi + \sin\theta\sin\phi & \cos\theta\sin\phi - \sin\theta\cos\phi \\ \sin\theta\cos\phi - \cos\theta\sin\phi & \sin\theta\sin\phi + \cos\theta\cos\phi \end{bmatrix} = \begin{bmatrix} \cos(\theta-\phi) & \sin(\phi-\theta) \\ \sin(\theta-\phi) & \cos(\theta-\phi) \end{bmatrix} = \begin{bmatrix} \cos(\theta-\phi) & -\sin(\theta-\phi) \\ \sin(\theta-\phi) & \cos(\theta-\phi) \end{bmatrix}.$$

This is rotation by $\theta - \phi$ counterclockwise.

(e) Does it make a difference which reflection comes first? Do the matrix multiplication to confirm your answer.

In general, reflections aren't commutative, so it probably will make a difference.

$$\begin{bmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} = \begin{bmatrix} \cos\phi\cos\theta + \sin\phi\sin\theta & \cos\phi\sin\theta - \sin\phi\cos\theta \\ \sin\phi\cos\theta - \cos\phi\sin\theta & \sin\phi\sin\theta + \cos\phi\cos\theta \end{bmatrix} = \begin{bmatrix} \cos(\phi-\theta) & -\sin(\phi-\theta) \\ \sin(\phi-\theta) & \cos(\phi-\theta) \end{bmatrix}.$$

This is different from the original. In fact, it is the opposite rotation (and thus the matrix's inverse).

We didn't really have to do the matrix multiplication. We could have substituted the rather confusing $\theta = \phi'$ and $\phi = \theta'$ into the original expression and gotten the same result. Oh well.

4. We've found specific matrices which map the plane in the following ways:

- **identity;**
- **rotation about the origin by θ;**
- **reflection over a line $y = x\tan\frac{\theta}{2}$;**
- **size change by some factor centered at the origin;**
- **stretching along a specific line through the origin by some factor;**
- **shearing perpendicular to a specific line through the origin by some factor.**

We want to generalize those ideas. What does each of the following matrices do? Be quantitative by specifying angle, equation of line, and/or factor:

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

This is the identity matrix; it does nothing.

(b) $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$

This matrix scales (or dilates) through the origin by a factor of a.

(c) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

This matrix rotates counterclockwise by θ.

(d) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

This is initially rather mysterious, but if we recall that $\cos\theta = \cos -\theta$ and $\sin -\theta = -\sin\theta$, we realize that this matrix can be rewritten:

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos -\theta & -\sin -\theta \\ \sin -\theta & \cos -\theta \end{bmatrix}.$$

This makes it clear that it is a rotation clockwise by θ.

(e) $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$

This is a stretch (or squish) along the x-axis by a factor of a.

(f) $\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$

This is a stretch (or squish) along the y-axis by a factor of a.

(g) $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$

This is a shear along the x-axis by a factor of a.

(h) $\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$

This is a shear along the y-axis by a factor of a.

(i) $\begin{bmatrix} a & b \\ ca & cb \end{bmatrix}$

Again, this seems a bit foreign. Multiplying it by a point (u, v), though, we see its true meaning:

$$\begin{bmatrix} a & b \\ ca & cb \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} au + bv \\ c(au + bv) \end{bmatrix}.$$

Letting $t = au + bv$, we see that this parametrizes the line $t \begin{bmatrix} 1 \\ c \end{bmatrix}$. In standard form, this is $cx - y = 0$; in sane person's form, this is $y = cx$.

(j) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

This is a negation of the x coordinate, or a reflection about the y-axis.

(k) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

As we found earlier, this is a reflection about the line $y = -x$.

(l) $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$

Another rather foreign one. We notice that it is similar to the rotation matrix by 2θ, but the right column is negated. As a matrix multiplication, we have

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Thus, this matrix is a reflection about the x-axis, followed by a rotation of 2θ counterclockwise. We know a reflection followed by a rotation is a reflection, so what axis is this reflection about?

Well, an easy way to find out is to note which points remain *fixed* after the matrix transformation, as these will be precisely the points on the line of reflection. Let the point be $(c, 1)$, since then (tc, t) gives every point on the line of reflection. We wish to solve the system of equations

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} c \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ 1 \end{bmatrix}.$$

Multiplying out the left side, we get

$$\begin{bmatrix} c\cos 2\theta + \sin 2\theta \\ c\sin 2\theta - \cos 2\theta \end{bmatrix} = \begin{bmatrix} c \\ 1 \end{bmatrix}.$$

Equating corresponding parts, we get the system of equations

$$\begin{cases} c\cos 2\theta + \sin 2\theta = c \\ c\sin 2\theta - \cos 2\theta = 1 \end{cases}.$$

The second equation gives us $c = \frac{1+\cos 2\theta}{\sin 2\theta}$. By the double angle formula, this is

$$c = \frac{1 + 2\cos^2\theta - 1}{2\cos\theta\sin\theta} = \frac{2\cos^2\theta}{2\cos\theta\sin\theta} = \frac{\cos\theta}{\sin\theta} = \cot\theta.$$

Thus, the line can be parameterized as $(t\cot\theta, t)$. In standard form, this is $x - y\cot\theta = 0$; in sane person's form, this is $y = x\tan\theta$. Thus, this matrix is a reflection over the line $y = x\tan\theta$, which is the line $\theta = \theta$ in polar coordinates. That's some unfortunate notation, but I hope you get what I mean.

5. **What matrix/transformation undoes each of a through l? For instance, matrix x is a rotation of θ. It is undone by a rotation of $-\theta$.**

   (a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The inverse of the identity matrix is itself: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

   (b) $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$

The inverse is a scaling by $\frac{1}{a}$: $\begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{a} \end{bmatrix}$.

   (c) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

This matrix rotates counterclockwise by θ. Thus, the inverse is a rotation by $-\theta$—or a rotation clockwise by θ—which is $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ (the subject of the next problem).

   (d) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

This is a rotation clockwise by θ, so the inverse is a matrix rotating counterclockwise by θ: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, the subject of the previous problem.

   (e) $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$

This is a stretch (or squish) along the x-axis by a factor of a. Thus, the inverse is a stretch along the x-axis by a factor of $\frac{1}{a}$: $\begin{bmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{bmatrix}$.

   (f) $\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$

This is a stretch (or squish) along the y-axis by a factor of a. Thus, the inverse is a stretch along the y-axis by a factor of $\frac{1}{a}$: $\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{a} \end{bmatrix}$.

   (g) $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$

This is a shear along the x-axis by a factor of a. Thus, the inverse is a shear along the x-axis by a factor of $-a$: $\begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix}$.

(h) $\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$

This is a shear along the y-axis by a factor of a. Thus, the inverse is a shear along the y-axis by a factor of $-a$: $\begin{bmatrix} 1 & 0 \\ -a & 1 \end{bmatrix}$.

(i) $\begin{bmatrix} a & b \\ ca & cb \end{bmatrix}$

This matrix doesn't have an inverse, because multiple points can be mapped to the same point. For example, if $a = b = c = 1$, then the matrix is $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then for example, both $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1 \end{bmatrix}$ are mapped to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(j) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

This is a negation of the x coordinate, or a reflection about the y-axis. Since it's a reflection, the inverse is itself: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

(k) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

As we found earlier, this is a reflection about the line $y = -x$. Since it's a reflection, the inverse is itself: $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

(l) $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$

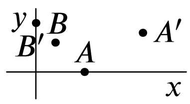
We found that this matrix is a reflection over the line $y = x \tan\theta$. Since it's a reflection, it is its own inverse: $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$.

6. **In this problem, you will observe the effects of multiplying two or more matrices. Do the following matrix multiplications, graph the preimage $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and image, then identify the transformations and their order. Note the effect of order on the outcome!**

(a) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} .6 & -.8 \\ .8 & .6 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} .6 & -.8 \\ .8 & .6 \end{bmatrix} = \begin{bmatrix} 2.2 & 0.4 \\ 0.8 & 0.6 \end{bmatrix}.$$

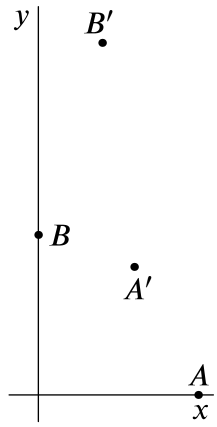
This is a rotation of $\tan^{-1}\frac{4}{3} \approx 53.13°$, followed by a shear along the x-axis by a factor of 2.



(b) $\begin{bmatrix} .6 & -.8 \\ .8 & .6 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} .6 & -.8 \\ .8 & .6 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.4 \\ 0.8 & 2.2 \end{bmatrix}.$$

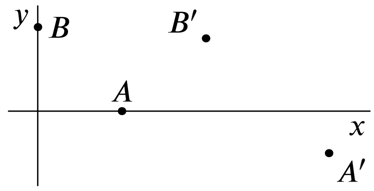
This is a shear along the x-axis by a factor of 2, followed by a rotation of $\tan^{-1}\frac{4}{3} \approx 53.13°$. The order does change the outcome as compared with the previous problem.



(c) $\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

$$\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 2\sqrt{3} & 2 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

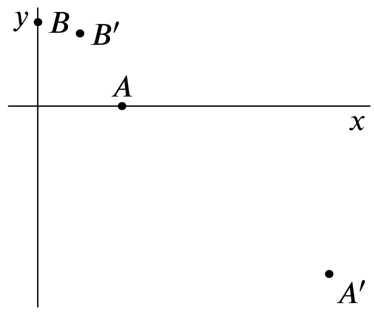
This is a rotation of $60°$, followed by a stretch along the x-axis by a factor of 4.



(d) $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2\sqrt{3} & \frac{1}{2} \\ -2 & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

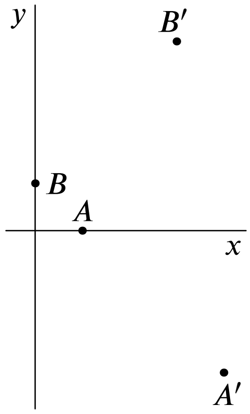
This is a stretch along the x-axis by a factor of 4, followed by a rotation of $60°$.



(e) $\begin{bmatrix} .8 & .6 \\ -.6 & .8 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

$$\begin{bmatrix} .8 & .6 \\ -.6 & .8 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}.$$

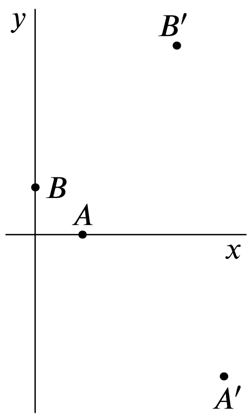
This is a dilation by a factor of 5, followed by a rotation of $\tan^{-1} -\frac{3}{4} \approx -36.87°$.



(f) $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} .8 & .6 \\ -.6 & .8 \end{bmatrix}$

$$\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} .8 & .6 \\ -.6 & .8 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}.$$

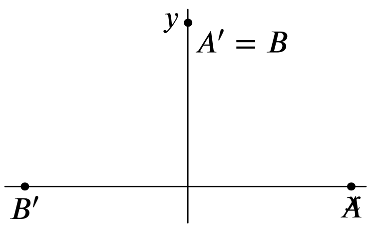
This is a rotation of $\tan^{-1} -\frac{3}{4} \approx -36.87°$, followed by a dilation by a factor of 5. In this case, order doesn't matter.



(g) $\begin{bmatrix} .6 & -.8 \\ .8 & .6 \end{bmatrix} \begin{bmatrix} .8 & -.6 \\ .6 & .8 \end{bmatrix}$

$$\begin{bmatrix} .6 & -.8 \\ .8 & .6 \end{bmatrix} \begin{bmatrix} .8 & -.6 \\ .6 & .8 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

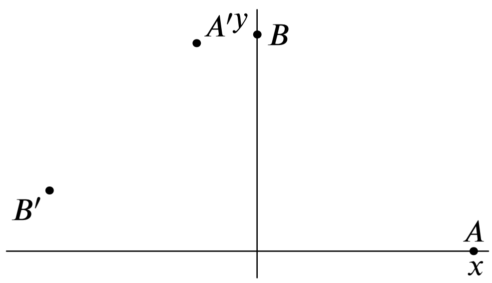
This is a rotation of $\tan^{-1} \frac{3}{4} \approx 36.87°$, followed by a rotation by $\tan^{-1} \frac{4}{3} \approx 53.13°$.



(h) $\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix} \begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}$

$$\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix} \begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

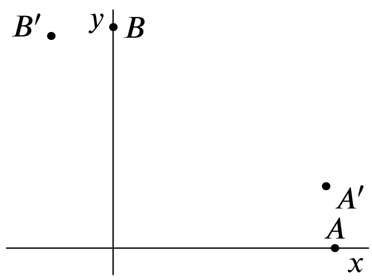
This is a reflection about the line $\theta = \frac{1}{2} \tan^{-1} \frac{4}{3} \approx 26.57°$, followed by the same reflection, which yields the identity.



(i) $\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix} \begin{bmatrix} .8 & .6 \\ .6 & -.8 \end{bmatrix}$

$$\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix} \begin{bmatrix} .8 & .6 \\ .6 & -.8 \end{bmatrix} = \begin{bmatrix} 0.96 & -0.28 \\ 0.28 & 0.96 \end{bmatrix}.$$

This is a reflection about the line $\theta = \frac{1}{2}\tan^{-1}\frac{3}{4} \approx 18.43°$, followed by a reflection about the line $\theta = \frac{1}{2}\tan^{-1}\frac{4}{3} \approx 26.57°$.



7. **A linear mapping f is one in which all lines are mapped to lines and the origin remains a fixed point. Algebraically, $f\left(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\right) = xf\left(\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\right) + yf\left(\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\right)$. I claim that we can build any linear mapping of the plane by multiplying together some combination of the matrices from Problem 4. Only two classes of matrix, however, are necessary; all other matrices are products or examples of these. Which two classes of matrix do you think comprise the minimum set from which the others can be composed? Be able to justify your choice.**

(Answers may vary.) Stretches and shears seem like the most general classes of transformations. After all, they can be done in two directions, and unlike reflections and rotations, don't include an "angle"; they have an infinite range of possibilities. Of course, I have the benefit of foresight.

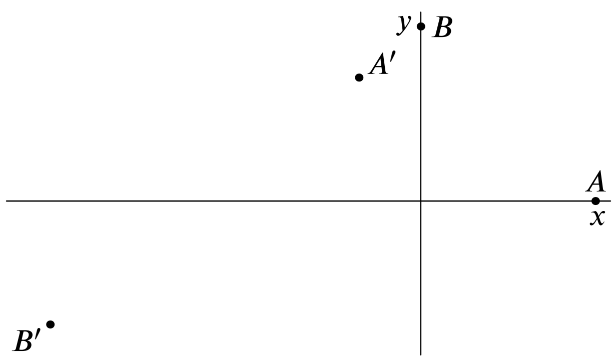
8. **Write matrix products that perform the following mappings. Do the indicated multiplication and graph the preimage and image when applied to $(1, 0)$ and $(0, 1)$.**

   (a) Rotation by $135°$ followed by a shear by a factor of $\frac{1}{2}$ *perpendicular* to the y-axis

The matrix multiplication is as follows:

$$\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 135° & -\sin 135° \\ \sin 135° & \cos 135° \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{4} & -\frac{3\sqrt{2}}{4} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}.$$

Note that a shear *perpendicular* to the y-axis is *parallel* with the x-axis. The graph is below.

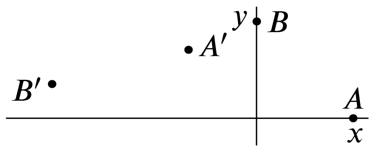


(b) Same transformations as in (a), but reversed

The matrix multiplication is as follows:

$$\begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{3\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{4} \end{bmatrix}.$$

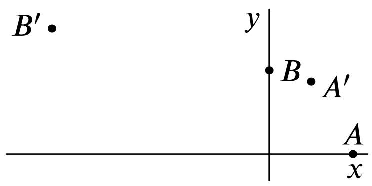
The graph is below.



(c) Stretch in the y direction by a factor of 3 followed by a rotation of $60°$

The matrix multiplication is as follows:

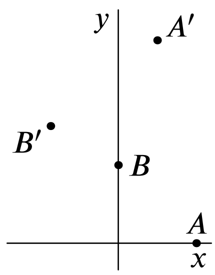
$$\begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{3\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{3}{2} \end{bmatrix}.$$



(d) Same transformations as in (c), but reversed

The matrix multiplication is as follows:

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{3\sqrt{3}}{2} & \frac{3}{2} \end{bmatrix}.$$



(e) Projection onto the line $y = 5x$

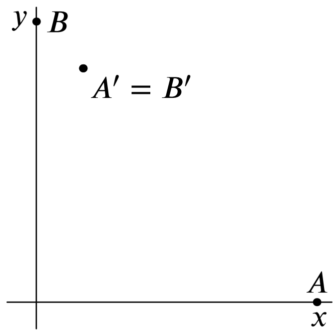
We might think that any old matrix like $\begin{bmatrix} 1 & 1 \\ 5 & 5 \end{bmatrix}$, which takes every point to a point on the line $y = 5x$, may work. This matrix, however, is not a *projection*. In a projection to a line, the image of a point is the foot of the altitude from the point to the line.

To map every point onto $y = 5x$, the matrix must be of the form $c \begin{bmatrix} 1 & 1 \\ 5 & 5 \end{bmatrix}$ for some real constant c. To make it a true projection, we choose a point that's already on $y = 5x$ and note that it must map to itself. In this case, we'll choose $(1, 5)$. then

$$c \begin{bmatrix} 1 & 1 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$c \begin{bmatrix} 6 \\ 30 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$
$$\Longrightarrow c = \frac{1}{6}.$$

Thus, the transformation matrix is $\begin{bmatrix} \frac{1}{6} & \frac{1}{6} \\ \frac{5}{6} & \frac{5}{6} \end{bmatrix}$.



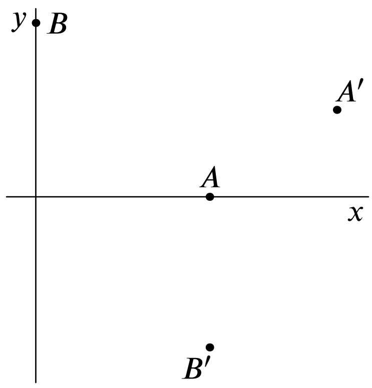
(f) Reflection over $\theta = \frac{\pi}{12}$ followed by a stretch in the x direction by a factor of 2

We recall that a reflection over the line $\theta = \phi$ is the matrix $\begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix}$. Substituting $\phi = \frac{\pi}{12}$ yields the matrix

$$\begin{bmatrix} \cos \frac{\pi}{6} & \sin \frac{\pi}{6} \\ \sin \frac{\pi}{6} & -\cos \frac{\pi}{6} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}.$$

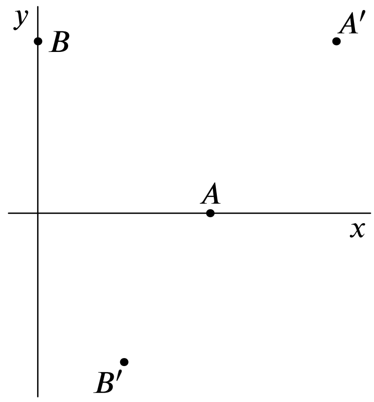
Stretching in the x direction by a factor of 2 is just $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. Thus, the total transformation matrix is

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 1 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}.$$



(g) Same transformations as in (f), but reversed

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3} & \frac{1}{2} \\ 1 & -\frac{\sqrt{3}}{2} \end{bmatrix}$$



9. Write a set of matrices which undoes Problems a to g. You will find one of them impossible to undo; explain why.

(a) Rotation by $135°$ followed by a shear by a factor of $\frac{1}{2}$ *perpendicular* to the y-axis

We shear by $-\frac{1}{2}$ along the x-axis, then rotate $-135°$.

$$\begin{bmatrix} \cos -135° & -\sin -135° \\ \sin -135° & \cos -135° \end{bmatrix}\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{3\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{4} \end{bmatrix}$$

(b) Same transformations as in (a), but reversed

We rotate $-135°$, then shear by $-\frac{1}{2}$ along the x-axis.

$$\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{3\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{4} \end{bmatrix}.$$

(c) Stretch in the y direction by a factor of 3 followed by a rotation of $60°$

We rotate by $-60°$, then stretch by $\frac{1}{3}$ in the y direction.

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} \cos -60° & -\sin -60° \\ \sin -60° & \cos -60° \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{6} & \frac{1}{6} \end{bmatrix}.$$

(d) Same transformations as in (c), but reversed

We stretch by $1/3$ in the y direction, then rotate by $-60°$.

$$\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{6} \\ -\frac{\sqrt{3}}{2} & \frac{1}{6} \end{bmatrix}.$$

(e) Projection onto the line $y = 5x$

This matrix doesn't have an inverse, because multiple points map to the same point. For example, both $(0, 0)$ and $(5, -1)$ project to $(0, 0)$

(f) Reflection over $\theta = \frac{\pi}{12}$ followed by a stretch in the x direction by a factor of 2

We stretch by a factor of $\frac{1}{2}$ in the x direction, then apply our old reflection matrix:

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

(g) Same transformations as in (f), but reversed

We first apply our old reflection matrix, then stretch by a factor of $\frac{1}{2}$ in the x direction:

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

10.

(a) Find the height of the parallelogram in Figure 2 in terms of b and a trig function in terms of φ.

We see that $\sin\varphi = h/b$, so $h = b\sin\varphi$.

(b) Find the area of the parallelogram in terms of a, b, and φ.

Let A be the area of the parallelogram. We have $A = ah$, and using information from the previous problem, we know that $A = ab\sin\varphi$.

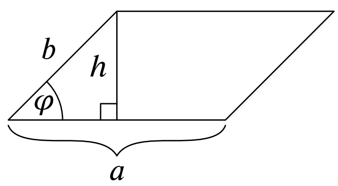


Figure 2: A parallelogram.

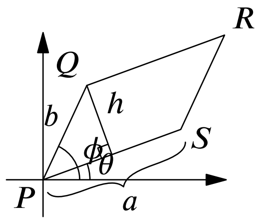


Figure 3: The parallelogram in the xy plane.

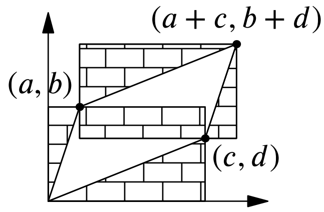


Figure 4: Scenario for Problem 12.

11. In Figure 3, we have put our parallelogram onto the xy plane so that a makes an angle of θ with the x axis and b makes an angle of ϕ with the x axis. Thus, $\varphi = \phi - \theta$.

(a) Rewrite the equation for the area of the parallelogram in terms of θ and ϕ.

We substitute $\varphi = \phi - \theta$ to find that $A = ab\sin(\phi - \theta)$.

(b) Find the x and y coordinates of P, Q, R, S in terms of a, b, ϕ, θ.

The vector $\overrightarrow{PS}$ is just $\langle a\cos\theta, a\sin\theta\rangle$, and the vector $\overrightarrow{PQ}$ is just $\langle b\cos\phi, b\sin\phi\rangle$. Thus, we have the following coordinates for P, Q, R, S:

$$P = (0,0); \quad Q = P+\overrightarrow{PQ} = (b\cos\phi, b\sin\phi); \quad R = P+\overrightarrow{PQ}+\overrightarrow{PS} = (a\cos\theta+b\cos\phi, a\sin\theta+b\sin\phi); \quad S = P+\overrightarrow{PS} = (a\cos\theta, a$$

(c) Write a matrix so that the first column contains the coordinates of Q and the second column contains the coordinates of S. This matrix maps the plane.

The matrix is $M = \begin{bmatrix} b\cos\phi & a\cos\theta \\ b\sin\phi & a\sin\theta \end{bmatrix}$.

(d) Your matrix has two diagonals. One rises from left to right and the other descends from left to right. Subtract the product of the entries of the ascending diagonal from the product of those of the descending diagonal.

This is $ab\cos\phi\sin\theta - ab\cos\theta\sin\phi$.

(e) Use angle addition formulas to simplify your answer.

$$ab\cos\phi\sin\theta - ab\cos\theta\sin\phi = ab(\cos\phi\sin\theta - \cos\theta\sin\phi) = ab\sin(\theta-\phi).$$

(f) You should find some relationship between your answers to problems 11a and 11d. What is it?

The answers have the same magnitude, but opposite sign. Indeed, $-ab\sin(\theta-\phi) = ab\sin(\phi-\theta)$.

(g) The difference of the products of the two diagonals of a 2×2 matrix is called the determinant of the matrix, written $\det\left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right] = ad - bc$. What does it measure?

It measures the area (in terms of magnitude) of the parallelogram formed by the two vectors $\langle a, b\rangle$ and $\langle c, d\rangle$.

(h) Find a matrix which produces a rotation. What is its determinant?

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is such a matrix. It has a determinant of $(0)(0) - (-1)(1) = 1$.

(i) Find a matrix that produces a reflection.

$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ is such a matrix. It has a determinant of $(-1)(1) - (0)(0) = -1$.

(j) What is the absolute value of its determinant?

The absolute value of its determinant is $|-1| = 1$.

(k) How does its determinant differ from that of a rotation matrix?

It is negative, while a rotation matrix's determinant is positive.

(l) What property is not conserved under reflection?

Orientation (or "handedness," "chirality," whatever you want to call it) is not preserved.

(m) What does the size of a determinant indicate?

This size of a determinant indicates the amount by which areas will be scaled by. If it is negative, than the orientation is changed. If it is zero, then the transformation is degenerate; it's a mapping to a line or the origin.

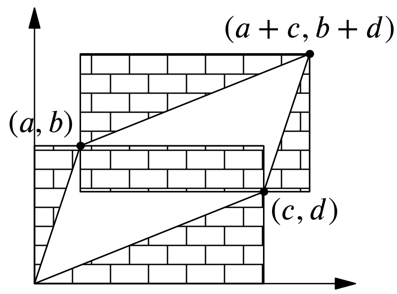


Figure 5: Scenario for Problem 12.

12. Here is another way to think about the area of the image of the unit square under a linear transformation. First, we use the matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ to transform the unit square into a parallelogram. Then, we graph the image.

(n) There are three rectangles and four triangles in Figure 5. Find the dimensions and the area of each one. You can use this information to figure out the area of the parallelogram in terms of a, b, c, and d. Write a sentence or equation explaining how you can use the seven areas to find the area of the parallelogram.

Possible sentence: The area of the parallelogram is the sum of the areas of the two big rectangles, minus the sum of all the areas shaded in bricks. Possible equation: $A_{\text{parallelogram}} = A_{\text{large rectangles}} - A_{\text{triangles}} - A_{\text{small rectangle}}$.

(o) Carry out the algebra to find the area.

The large rectangles are each c units wide and b units tall, so they have a total area of $2bc$. There are two sets of two congruent triangles here. We can combine each set together to create two rectangles with an area of ab and cd, respectively. Finally, the small rectangle has area $(c-a)(b-d)$. The area of the parallelogram is

$$2bc - (ab + cd) - (c - a)(b - d) = 2bc - (ab + cd) - (bc + ad - ab - cd) = bc - ad.$$

(p) Calculate the determinant of the matrix.

The determinant is $\det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc$.

(q) What is the relationship between the determinant of the matrix and the area of its associated parallelogram?

The determinant of the matrix is the area of its associated parallelogram negated. In mathematical terms, $A_{parallelogram} = bc - ad = -(ad - bc) = -\det \begin{bmatrix} a & c \\ b & d \end{bmatrix}$.

(r) Consider what happens if (a, b) and (c, d) switch places in the graph.

i. How would the area you calculated be different?

The new area would be $ad - bc$, and thus the negative of what it was before!

ii. What property would now be preserved by the transformation?

Orientation (or chirality or handedness whatever) would now be preserved.

iii. What isometry would have been included in any composition of simple transformations yielding the mapping?

Any such composition would require a reflection, because all the other simple transformations do not flip.

iv. What would be true of the determinant?

The determinant would be positive, since it is now equal to the (necessarily positive) area of the parallelogram.

(s)

i. What does a reversal of the orientation of figure in its image say about the determinant of the transformation matrix?

It says that the determinant is negative.

ii. What does that same property of the determinant imply that a transformational matrix does?

It implies that a transformation matrix reverses the orientation of a figure.

iii. What isometry reverses orientation?

Reflections reverse orientation; rotations don't.

(t)

i. What would have happened to the parallelogram if we replaced c, d in the matrix with kc, kd for some $k > 0$, so that the transformation matrix is $\left[\begin{smallmatrix} a & kc \\ b & kd \end{smallmatrix}\right]$?

The parallelogram would become scaled in the direction of (c, d), by a factor of k. See the diagram:

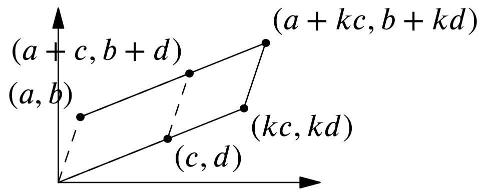


Figure 6: The parallelogram gets scaled by a factor of k. In this case, $k = 1.7$.

ii. What would its area be?

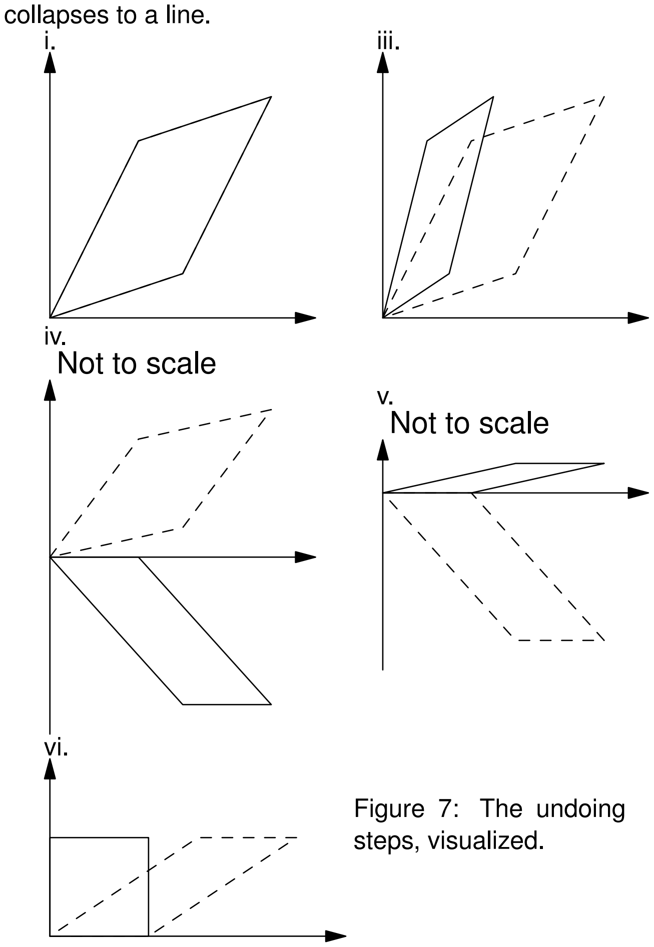
Its area would be scaled by a factor of k; it would be $k(bc - ad)$.

iii. What would the determinant of the matrix be?

The determinant of the matrix would be $k(ad - bc)$.

iv. What if $[\,b\ d\,] = r[\,a\ c\,]$? That is, what if the second row of the matrix was a linear multiple of the first row?

In this case, the the two vectors constructing the parallelogram are collinear. Thus, the transformation collapses to a line.



i. The determinant is not 0, so we continue.

ii. $a \neq 0$, so we continue.

iii. $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2} \\ 4 & 1 \end{bmatrix}$.

iv. $\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{3}{2} \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & -5 \end{bmatrix}$.

v. $\begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix}$.

vi. $\begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Figure 7: The undoing steps, visualized.

13. Look at Figure 7 and describe the transformation in each step.

iii. We stretch by a factor of $\frac{1}{2}$ in the x direction. iv. We shear by a factor of -4 in the y direction (perpendicular to the x direction). v. We stretch by a factor of $-\frac{1}{5}$ in the y direction. vi. We shear by a factor of $-\frac{1}{2}$ in the x direction.

14.

(a) How do you undo a shear in the x direction? $\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(b) How do you undo a stretch along the x-axis? $\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} \frac{1}{x} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(c) How do you undo a shear in the y direction? $\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ -s & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(d) How do you undo a stretch along the y-axis? $\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{y} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

15. Now let's put this all together. Undo each of the operations in turn, until only matrix $\left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right]$ remains on the left side. Remember that what you do on the left side of the expression must also be done to the right side, so on the right side you will see the basic operations from which $\left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right]$ is composed. Order is important!

undoes
undoes
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$$\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{bmatrix} 1 & -\frac{c}{a} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{a}{ad-bc} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

These are the matrices all filled in:

$$\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{ad-bc}{a} \end{bmatrix}\begin{bmatrix} 1 & \frac{c}{a} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

16. Each step in the decomposition of $\begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix}$ is explained below.

(i) Stretch along the x-axis by factor of $\frac{1}{3}$.

$$\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & \frac{4}{3} \\ 2 & -5 \end{bmatrix}$$

(ii) Shear perpendicular to the x-axis by -2

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{4}{3} \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & \frac{4}{3} \\ 0 & -\frac{23}{3} \end{bmatrix}$$

(iii) Stretch along y-axis by $-\frac{3}{23}$

$$\begin{bmatrix} 1 & 0 \\ 0 & -\frac{3}{23} \end{bmatrix}\begin{bmatrix} 1 & \frac{4}{3} \\ 0 & -\frac{23}{3} \end{bmatrix} = \begin{bmatrix} 1 & \frac{4}{3} \\ 0 & 1 \end{bmatrix}$$

(iv) Shear perpendicular to the y-axis by $-\frac{4}{3}$

$$\begin{bmatrix} 1 & -\frac{4}{3} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{4}{3} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Taken all together, the decomposition is:

$$\begin{bmatrix}1 & -\frac{4}{3}\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -\frac{3}{23}\end{bmatrix}\begin{bmatrix}1 & 0\\ -2 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{3} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}3 & 4\\ 2 & -5\end{bmatrix}.$$

Therefore:

$$\begin{bmatrix}3 & 4\\ 2 & -5\end{bmatrix} = \begin{bmatrix}3 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -\frac{23}{3}\end{bmatrix}\begin{bmatrix}1 & \frac{4}{3}\\ 0 & 1\end{bmatrix}.$$

What does each matrix do?

Going from right to left, the first matrix shears by $\frac{4}{3}$ in the x direction (perpendicular to the y-axis). The second matrix stretches by $-\frac{23}{3}$ in the y direction. The third matrix shears by 2 in the y direction (perpendicular to the x-axis). The fourth and final matrix stretches by 3 in the x direction.

17. Here is another way that you could have decomposed the above matrix.

$$\overset{\textbf{i}}{\begin{bmatrix}1 & 0\\ 0 & \frac{13}{23}\end{bmatrix}}\overset{\textbf{ii}}{\begin{bmatrix}1 & -\frac{2}{23}\\ 0 & 1\end{bmatrix}}\overset{\textbf{iii}}{\begin{bmatrix}\frac{1}{\sqrt{13}} & 0\\ 0 & \frac{1}{\sqrt{13}}\end{bmatrix}}\overset{\textbf{iv}}{\begin{bmatrix}\frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{13}}\\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}}\end{bmatrix}}\overset{\textbf{v}}{\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}}\begin{bmatrix}3 & 4\\ 2 & -5\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

(a) Identify what matrices i through v each do.

i scales by $\frac{13}{23}$ along the y-axis. ii shears by $-\frac{2}{23}$. iii scales by $\frac{1}{\sqrt{13}}$. iv rotates by $\tan^{-1}\frac{2}{3}$. v reflects over the x-axis (since the y coordinate is being flipped).

17. (cont.) Next, we undo this sequence of operations by working backwards.

$$\overset{\text{i}}{\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}}\overset{\text{ii}}{\begin{bmatrix}\frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}}\\ -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}}\end{bmatrix}}\overset{\text{iii}}{\begin{bmatrix}\sqrt{13} & 0\\ 0 & \sqrt{13}\end{bmatrix}}\overset{\text{iv}}{\begin{bmatrix}1 & \frac{2}{23}\\ 0 & 1\end{bmatrix}}\overset{\text{v}}{\begin{bmatrix}1 & 0\\ 0 & \frac{23}{13}\end{bmatrix}}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}3 & 4\\ 2 & -5\end{bmatrix}.$$

(b) Explain what happens at each matrix, i through v.

i reflects over the x-axis, since the y coordinate is being flipped. ii rotates by $-\tan^{-1}\frac{2}{3}$. iii scales by $\sqrt{13}$. iv shears by $\frac{2}{23}$ along the x-axis. Finally, v scales by $\frac{23}{13}$.

18. Find a set of basic transformations which is equivalent to each of the following matrices.

(a) $\begin{bmatrix}12 & 8\\ 5 & 15\end{bmatrix}$

(Answers may vary.)
Using the method we described, we first stretch along the x-axis by $\frac{1}{12}$:

$$\begin{bmatrix}\frac{1}{12} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}12 & 8\\ 5 & 15\end{bmatrix} = \begin{bmatrix}1 & \frac{2}{3}\\ 5 & 15\end{bmatrix}.$$

We then shear along the y-axis by -5:

$$\begin{bmatrix}1 & 0\\ -5 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{12} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}12 & 8\\ 5 & 15\end{bmatrix} = \begin{bmatrix}1 & \frac{2}{3}\\ 0 & \frac{35}{3}\end{bmatrix}.$$

We then stretch along the y-axis by $\frac{3}{35}$:

$$\begin{bmatrix}1 & 0\\ 0 & \frac{3}{35}\end{bmatrix}\begin{bmatrix}1 & 0\\ -5 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{12} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}12 & 8\\ 5 & 15\end{bmatrix} = \begin{bmatrix}1 & \frac{2}{3}\\ 0 & 1\end{bmatrix}.$$

Finally, we shear by $-\frac{2}{3}$ along the x-axis:

$$\begin{bmatrix} 1 & -\frac{2}{3} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{35} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{12} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 12 & 8 \\ 5 & 15 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Now we undo the operations in turn:

$$\underbrace{\begin{bmatrix} 12 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{35}{3} \end{bmatrix}\begin{bmatrix} 1 & \frac{2}{3} \\ 0 & 1 \end{bmatrix}}\begin{bmatrix} 1 & -\frac{2}{3} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{35} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{12} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 12 & 8 \\ 5 & 15 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\Longrightarrow \begin{bmatrix} 12 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{35}{3} \end{bmatrix}\begin{bmatrix} 1 & \frac{2}{3} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 8 \\ 5 & 15 \end{bmatrix}.$$

(b) $\begin{bmatrix} 3 & 24 \\ 4 & 7 \end{bmatrix}$

(Answers may vary.)
Using the method we described, we first stretch along the x-axis by $\frac{1}{3}$:

$$\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 24 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 4 & 7 \end{bmatrix}.$$

We then shear along the y-axis by -4:

$$\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 24 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 0 & -25 \end{bmatrix}.$$

We then stretch along the y-axis by $-\frac{1}{25}$:

$$\begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{25} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 24 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix}.$$

Finally, we shear by -8 along the x-axis:

$$\begin{bmatrix} 1 & -8 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{25} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 24 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We undo the operations in turn:

$$\underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -25 \end{bmatrix}\begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix}}\begin{bmatrix} 1 & -8 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{25} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 24 \\ 4 & 7 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -25 \end{bmatrix}\begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 24 \\ 4 & 7 \end{bmatrix}.$$

(c) $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

This matrix can't be decomposed with our usual method! That's because it lacks an inverse transformation, since it's projecting to a line (namely, the line $y = 2x$).

Let's think a bit more laterally here. What of our operations take the whole plane to a line? Shears don't, rotations and reflections don't, but stretches with a factor of 0 can. For example, we can take everything to the x-axis with the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, we should be able to decompose our matrix as

$$A\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}B = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$$

for some matrices A, B. If we want to take the x-axis to the line $y = 2x$, which is the job of matrix A (remember, right to left!), we'll need A to be a shear by a factor of 2 along the y-axis. (See Figure 8.) Thus,

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

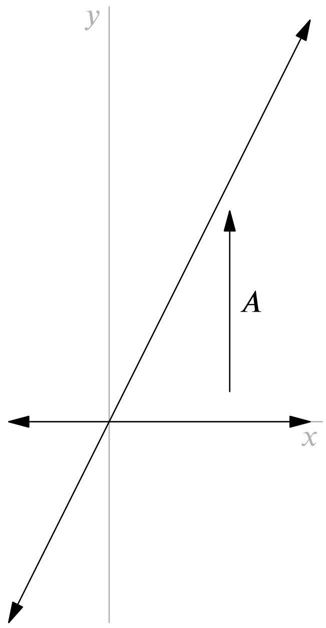

Figure 8: The shearing action of A on the x-axis.

To find B, we solve the matrix equation. We have

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} B = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} B = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}.$$

Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}.$$

$$\begin{bmatrix} a & b \\ 2a & 2b \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}.$$

Interesting! c, d can be any real numbers, because their effect is nullified by the mapping to a line. But $a = 2$ and $b = 3$. Let's just choose $c = 0$ and $d = 1$ for simplicity of decomposition. We have

$$\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = B$$

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We undo the new matrices:

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\implies \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix} = B.$$

Thus, our full decomposition is

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}}_{A} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix}}_{B} = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}.$$

In words, this is a shear of $\frac{3}{2}$ along the x-axis, followed by a stretch of a factor of 2 along the x-axis, a stretch of a factor of 0 along the y-axis, and a shear of 2 along the y-axis.

As an aside, here's what happens if you try using the usual method. We first stretch along the x-axis by $\frac{1}{2}$, then shear along the y-axis by -2:

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2} \\ 2 & 3 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 0 \end{bmatrix}.$$

Here, our problem arises. We cannot make the bottom-right corner 1.

19. One of the matrices in Problem 18 is a projection onto a line.

(a) Which matrix is it?

Problem (c): $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$ is the matrix.

(b) What line does it project onto?

It projects onto the line $y = 2x$.

(c) If you try to decompose this matrix to the identity matrix, what happens? Why?

You get stuck, because it lacks an inverse! The details are given as an aside at the end of that problem.

20. Onto what line does $\left[\begin{smallmatrix} a & b \\ 2a & 2b \end{smallmatrix}\right]$ map the plane? Solve for a and b such that the matrix projects perpendicular onto the line. You can do this because you know that a point on the line should not move under the projection and a point on a line perpendicular to the line has its image on the origin. Using this information you can set up two equations with two unknowns.

This matrix projects points onto the line $y = 2x$. To project perpendicular on the line, we do as the problem suggests. Consider the point $(1, 2)$, which is on the line. If the transformation is a true projection, then the point should not move under the transformation. Thus,

$$\begin{bmatrix} a & b \\ 2a & 2b \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

The left side expands out to $\begin{bmatrix} a+2b \\ 2a+4b \end{bmatrix}$. Thus, as long as $a + 2b = 1$, the matrix will be a projection onto $y = 2x$. A simple example is $a = b = \frac{1}{3}$, so a possible matrix is

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix}.$$

21. Use Problem 20 to decompose $\left[\begin{smallmatrix} 2 & 3 \\ 4 & 6 \end{smallmatrix}\right]$ into a projection to a line followed by a size change.

We know $\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix}$ projects to the line $y = 2x$. The desired matrix is just this matrix scaled by 6. Thus, the decomposition is

$$\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix}.$$

22. Decompose $\left[\begin{smallmatrix}1 & 2\\ 3 & 6\end{smallmatrix}\right]$ into a projection perpendicular to a line followed by a size change.

This matrix maps onto the line $y = 3x$, since it's of the form $\begin{bmatrix} a & b \\ 3a & 3b \end{bmatrix}$. We try a similar method as the last problem to find the matrix of a true projection, choosing the point $(1, 3)$:

$$\begin{bmatrix} a & b \\ 3a & 3b \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$
$$\begin{bmatrix} a + 3b \\ 3a + 9b \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

So $a + 3b = 1$. We just choose $a = b = \frac{1}{4}$ for simplicity, giving the projection matrix

$$\begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{3}{4} \end{bmatrix}.$$

Scaling this by 4 gives the desired matrix, so the decomposition is

$$\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{3}{4} \end{bmatrix}.$$

23. Write matrices which project onto the following lines:

(a) $y = x$

The matrix will be of the form $\begin{bmatrix} a & b \\ a & b \end{bmatrix}$. We choose the fixed point $(1, 1)$:

$$\begin{bmatrix} a & b \\ a & b \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} a + b \\ a + b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

So $a + b = 1$. For simplicity, we choose $a = b = \frac{1}{2}$, giving the matrix

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

(b) $y = 5x$

The matrix will be of the form $\begin{bmatrix} a & b \\ 5a & 5b \end{bmatrix}$. We choose the fixed point $(1, 5)$:

$$\begin{bmatrix} a & b \\ 5a & 5b \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$
$$\begin{bmatrix} a + 5b \\ 5a + 25b \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

So $a + 5b = 1$. For simplicity, we choose $a = b = \frac{1}{6}$, giving the matrix

$$\begin{bmatrix} \frac{1}{6} & \frac{1}{6} \\ \frac{5}{6} & \frac{5}{6} \end{bmatrix}.$$

(c) $y = mx$

This is pretty easy to generalize. The matrix will be of the form $\begin{bmatrix} a & b \\ ma & mb \end{bmatrix}$. We choose the fixed point $(1, m)$:

$$\begin{bmatrix} a & b \\ ma & mb \end{bmatrix} \begin{bmatrix} 1 \\ m \end{bmatrix} = \begin{bmatrix} 1 \\ m \end{bmatrix}$$

$$\begin{bmatrix} a + mb \\ ma + m^2 b \end{bmatrix} = \begin{bmatrix} 1 \\ m \end{bmatrix}.$$

So $a + mb = 1$. For simplicity, we choose $a = b = \frac{1}{m+1}$, giving the matrix

$$\begin{bmatrix} \frac{1}{m+1} & \frac{1}{m+1} \\ \frac{m}{m+1} & \frac{m}{m+1} \end{bmatrix}.$$

13 Inverses

1.

(a) With real numbers, one of the important purposes of division is that it lets you solve equations like $ax = b$ for x. Solve this by division (difficult!).

We divide both sides by a, to get $x = \frac{b}{a}$.

(b) If division didn't exist, you could still solve this equation by multiplication. The number you'd multiply by is called the "multiplicative inverse" of a. What is the property that defines this special number?

The property defining the number is that it is the unique number which, when multiplying by a, yields 1. That is, if the number is c, then $ac = 1$.

(c) The multiplicative inverse of a is often written a^{-1}. Why does this notation make sense?

Since $a = a^1$, we have $a^{-1}a^1 = a^{1-1} = a^0 = 1$.

2.

(a) For fixed a, b, you might think that the equation $ax = b$ has only one solution, but sometimes it can have zero or infinitely many. Give an example of both cases.

There are zero solutions if $a = 0$ and $b \neq 0$, for example $(a, b) = (0, 1)$. There are infinite solutions if $a = 0$ and $b = 0$.

(b) How does the existence of a unique solution relate to the idea of multiplicative invertibility?

The unique solution can be found by multiplying both sides by a^{-1}... if it exists. If it doesn't exist, then there isn't a unique solution. In this case, 0 does not have a multiplicative inverse[22], so it causes either no solution or multiple solutions to exist.

(c) Are there any other possible numbers of solutions?

In this case, no. The easiest way is to simply construct all solutions for various values of a, b. We have $x = \frac{b}{a}$ for $a \neq 0$, which is only one solution, and the other two cases we've already described; they have either an infinite number of solutions or no solutions.

3.

(a) Define "one-to-one" function.

A one-to-one function, also known as a bijection, is a function between two sets which maps each element of each set to exactly one element of the other set, and vice versa.

(b) Is $f(x) = ax$ a one-to-one function for all real a? (Hint: Look for the silly exception(s)!)

It is one-to-one for $a \neq 0$, but $f(x) = 0x = 0$ maps every real number to 0, which is certainly not a bijection; the mapping is not unique.

4. Would your answers to the previous numbers change if you were talking about complex numbers instead of just real numbers? Why or why not?

They would not change. $f(x) = cx$ is still one-to-one for complex c, x, except for $c = 0$. After all, the inverse function is $f^{-1}(x) = \frac{1}{c} \cdot x$, and $\frac{1}{c}$ is defined for all $c \neq 0$.

5. In the following problems, x can be any integer from 0 to 11.

(a) Find all solutions of $5x \equiv 7$ in clock arithmetic.

[22] Unless you're Brandon and trying to be annoying in Bio

We see that we must solve $5x = 7 + 12a$ for integer solutions. While we could check a values manually, a simpler method is to notice that we just need to find a so that $x = \frac{7+12a}{5}$ is an integer. We have

$$\frac{7+12a}{5} = 1 + \frac{2+(10+2)a}{5} = 1 + 2a + \frac{2+2a}{5}.$$

This is only an integer when $2 + 2a$ is divisible by 5, which first happens when $a = 4$. This yields the solution 11. Thus, $x = 11$ is the only solution.

(b) Find all solutions of $2x \equiv 6$ in clock arithmetic.

We want to solve $2x = 6 + 12a$ for integer solutions. Since x is an integer for all integers a, we just need $x = 3 + 6a$. This gives $x = 3, 9$.

(c) Find all solutions of $6x \equiv 6$ in clock arithmetic.

We want to solve $6x = 6 + 12a$ for integer solutions. Again, x is an integer for all integers a, so we just need $x = 1 + 2a$. This yields $x = 1, 3, 5, 7, 9, 11$.

(d) Find all solutions of $2x \equiv 7$ in clock arithmetic.

This has no solutions, since $7 + 12a$ is never even for $a \in \mathbb{Z}$.

(e) For integers a, b, what are all possible numbers of solutions that $ax \equiv b$ can have in clock arithmetic?

We could try every pair (a, b), but that's 144 combinations; no thanks.

We have $ax = b + 12k$ for some integer k. Solving for k in terms of a, b, x, we find that $k = \frac{ax-b}{12}$. Thus, for x taking on values 0 through 11, we must find how many such values have $ax - b$ divisible by 12.

We know that 0 values are possible; $(a, b) = (2, 7)$ is an example. Suppose, however, that we have a solution, say x_1, so that $ax_1 - b = 12k_1$ for some value of k_1. Let the other solutions be $x_1 + m$ for some integer m, so $a(x_1 + m) - b = 12k_2$. But the left side is $ax_1 - b + am$, so am must be divisible by 12. The values of m which make this true depend on $\gcd(a, 12)$; we have

$$m = \frac{12}{\gcd(a, 12)} n$$

for integers n. Thus, the solutions are

$$x = x_1 + \frac{12}{\gcd(a, 12)} n.$$

Note that n can be any integer, not just among the positive integers. Since $0 \le x \le 11$, the number of solutions is bounded. Substituting x and subtracting x_1 from all sides, we find that

$$-x_1 \le \frac{12}{\gcd(a, 12)} n \le 11 - x_1$$

In the twelve consecutive integers between $-x_1$ and $11 - x_1$ inclusive, there are always going to be $\gcd(a, 12)$ integers divisible by $\frac{12}{\gcd(a,12)}$. The possible values of $\gcd(a, 12)$ are $1, 2, 3, 4, 6$, and 12; the divisors of 12.

To recap: we know that 0 values are possible. If a solution for some pair (a, b) exists, then there are $\gcd(a, 12)$ solutions for that pair in total. The possible numbers of solutions for x are $0, 1, 2, 3, 4, 6$, and 12.

6. How does the number of solutions to $ax \equiv b$ relate to the idea of multiplicative inverse? (Hint: You can try solving for $a = 5, 7, 11$ and $b = 1$. What numbers would be 5^{-1}, 7^{-1}, 11^{-1} in clock arithmetic?)

If we multiply both sides by the "inverse" of a, then we get $x \equiv ba^-1$. This would give us a quick way to solve for x. The question is whether such an "inverse" exists.

Well, this equation $x \equiv ba^-1$ is only valid if there is a single solution for x; otherwise, since $0 \le x \le 11$, this wouldn't encapsulate all the possible values of x. Thus, $\gcd(a, 12) = 1$. This is true for all the a values in the hint.

We can find 5^{-1}, 7^{-1}, and 11^{-1} by just trying numbers.[23] For example, 5^{-1} is just the solution to the equation $5a \equiv 1$, which is clearly 5, since $25 = 2 \cdot 12 + 1$. 7^{-1} is 7, since $7 \cdot 7 \equiv 1$. Finally, 11^{-1} is 11, since $11 \cdot 11 \equiv 1$. Interesting!

[23] There is a better way to do this, called the Extended Euclidean Algorithm. Check it out if you're bored!

7. How does this all relate to groups?

(a) The clock numbers are a group under clock addition. Name that group!

The group is the cyclic group of order 12.

(b) They are not a group under clock multiplication. Why?

1 can't be the identity element, because there's no element x such that $0 \cdot x = 1$. Without an identity element, it can't be a group.

(c) A subset of four of the clock numbers form a group under the operation of clock multiplication. Find them, and write a group table.

It's not immediately obvious how we'd find this subset besides trying various pairs of elements and seeing what they generate. If we remember the invertibility property, however, we realize that all elements of this group must be coprime (i.e. not share any factors besides 1 with) 12. The only four elements which satisfy this requirement are $1, 5, 7$, and 11; notice how we've already seen three of these elements.

We can now write a group table to expose the structure of the group:

$\cdot$	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

(d) Describe this group. What is the inverse of each element?

Every element's inverse is itself. The group is commutative (abelian); however it is not the cyclic group of order 4.

(e) What symmetry group is it isomorphic to?

It is isomorphic to the symmetry group of the rectangle: the dihedral group of order 4, or D_2.[24]

8. If the numbers on an advanced Mars clock went from 0 to 4,

(a) They would form a group under addition. Make a group table!

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

(b) What group is this isomorphic to?

This is isomorphic to the cyclic group of order 5.

(c) A subset of four of these numbers forms a group under multiplication. Find them and write a group table.

Analogous to the original clock, which was modulo 12, the numbers which form the group under multiplication must be coprime to 5. But since 5 is a prime, this is just $\{1, 2, 3, 4\}$.

$\cdot$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

[24]This group is also sometimes known as the Klein four-group, denoted V or K_4.

(d) Describe this multiplication group.

The group has order 4. It is commutative, and cyclic, with each element generated by a single element.

(e) What symmetry group is it isomorphic to?

Based on the above properties, this is isomorphic to the rotation group of the square: the cyclic group of order 4, C_4.

9.

(a) Find all solutions (x, y) of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$, by multiplying out the left side and rewriting this as a system of equations.

The left side multiplied out is $\begin{bmatrix} x+2y \\ 3x+4y \end{bmatrix}$. Thus, we have the system of equations

$$\begin{cases} x+2y=5 \\ 3x+4y=6 \end{cases}.$$

There are a couple ways to solve this. Perhaps the easiest way is to double the first equation and subtract it from the second equation:

$$\begin{array}{r} 3x+4y=6 \\ \hline -2\cdot(x+2y=5) \\ x \qquad =-4 \end{array}$$

If $x=-4$, then $-4+2y=5$, so $y=\frac{9}{2}$. We can verify our solution by multiplying out the original matrix form with the substitution $(x, y) = \left(4, \frac{9}{2}\right)$.

(b) Find all solutions (x, y) of $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

Multiplying out the left side and comparing corresponding parts, we get the following system of equations:

$$\begin{cases} x+2y=5 \\ 2x+4y=6 \end{cases}.$$

Multiplying the top equation by 2 and subtracting it from the bottom equation, we get the following system of equations:

$$\begin{array}{r} 2x+4y=6 \\ \hline -2\cdot(x+2y=5) \\ 0\ =-4 \end{array}$$

This is a contradiction, so there are no solutions (x, y) to this equation.

(c) Find all solutions (x, y) of $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$

Multiplying out the left side and comparing corresponding parts, we get the following system of equations:

$$\begin{cases} x+2y=5 \\ 2x+4y=10 \end{cases}.$$

Multiplying the top equation by 2 and subtracting it from the bottom equation, we get the following system of equations:

$$\begin{array}{r} 2x+4y=10 \\ \hline -2\cdot(x+2y=5) \\ 0\ =0 \end{array}$$

This is always true. This seems to imply that the original equation is true for all (x, y), but this is not the case. Solving for x in terms of y for each of the equations, we get $x = 5 - 2y$ for each. Also, if $x = 5 - 2y$, then both equations are true. Thus, $x = 5 - 2y$ is both a necessary and sufficient condition for the equation to be true.

In any case, this yields an infinite number of solutions.

(d) What are all possible numbers of solutions that $AX = B$ can have, where A, B are 2×2 and 2×1 matrices respectively and $X = \begin{bmatrix} x \\ y \end{bmatrix}$? Use your knowledge of the properties of systems of equations.

Letting $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} f \\ g \end{bmatrix}$, we get the system of equations

$$\begin{cases} ax + by = f \\ cx + dy = g \end{cases}.$$

With our knowledge of system of equations, we know that this can have 0, 1, or infinite solutions, depending on a, b, c, d, f, g.

10. Now, let's relate the two 2×2 matrices from the previous problem to the transformations we know.

(a) Contrast the mapping properties of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ projects to a line, and is thus not invertible, while $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ does not project to a line and is thus invertible.

(b) Find the determinants of these matrices. What do you notice?

We have $\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = -1$ and $\det \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = 0$. The matrix which maps to a line has 0 determinant, which is hardly a coincidence; since the (absolute value of the) determinant is the area of the unit square after the mapping, this makes sense. The unit square is mapped to a line segment, which has 0 area.

(c) When is $f(X) = AX$ a one-to-one function? That is, in mapping the plane, when does each point in the image have exactly one preimage?

$f(X) = AX$ is a one-to-one function when A has nonzero determinant. We'll see how to prove this shortly, but it makes sense that a matrix with zero determinant, being a mapping to a line, is not one-to-one.

(d) Compare how you find the number of solutions of the real number equation $ax = b$ with how you find the number of solutions of the matrix equation $AX = B$.

For $ax = b$, we have three cases:

$$\text{\# solutions} = \begin{cases} \infty & (a, b) = (0, 0) \\ 0 & a = 0, b \neq 0 \\ 1 & \text{otherwise} \end{cases}.$$

For $AX = B$, we have three cases[25]:

$$\text{\# solutions} = \begin{cases} \infty & \det A = 0, B \text{ is on the line } A \text{ maps to} \\ 1 & \det A = 0, B \text{ is not on the line } A \text{ maps to} \\ 1 & \text{otherwise} \end{cases}.$$

Thus, the condition that $\det A = 0$ is analogous to $a = 0$, and the condition that B is on the line A projects to is analogous to $b = 0$.

[25] There is technically a "fourth case," when A is the matrix of all 0s, where there are infinite solutions if and only if $B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, but I have kept it out of the main solution for simplicity. This is a "fourth case" because A no longer projects to a line.

11. Let $K = \begin{bmatrix} 5 & 7 \\ 8 & -3 \end{bmatrix}$.

(a) Find all solutions to $K \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$.

Expanding out the left side and comparing corresponding entries, we get the system of equations

$$\begin{cases} 5x + 7y = 10 \\ 8x - 3y = 2 \end{cases}.$$

This is a bit unpleasant to solve, but we can multiply the first equation by $\frac{8}{5}$ and subtract the second equation:

$$\begin{array}{r} \frac{8}{5} \cdot (5x + \ 7y \ = 10) \\ \underline{\quad 8x - \ 3y \ = 2} \\ \frac{71}{5} y = 14 \end{array}$$

Thus, $y = \frac{5}{71} \cdot 14 = \frac{70}{71}$. We can get x by substituting y back into either equation. Choosing the second equation, we get

$$8x - 3\left(\frac{70}{71}\right) = 10$$
$$x = \frac{10 + \frac{210}{71}}{8} = \frac{44}{71}.$$

Thus, $(x, y) = \left(\frac{44}{71}, \frac{70}{71}\right)$.

(b) If we knew a matrix which was the inverse of K, written K^{-1}, we could write the following equation:

$$K^{-1} K \begin{bmatrix} x \\ y \end{bmatrix} = K^{-1} \begin{bmatrix} 5 \\ 10 \end{bmatrix}.$$

What would the left side reduce to?

The left side would reduce to $I \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$, since by definition $K^{-1}K = I$.

12. Consider the following matrix inverses:

$$\begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{5}{23} & -\frac{4}{23} \\ \frac{2}{23} & \frac{3}{23} \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}^{-1} = \frac{1}{10} \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}$$

(a) Look for a pattern in these inverses.

It appears that they are some fraction of a matrix with the top-left and bottom-right entries swapped, and the bottom-left and top-right entries negated. With some closer inspection, the fraction appears to be $\frac{1}{\det M}$.

(b) Describe the inverse of an arbitrary matrix: $\begin{bmatrix} a & c \\ b & d \end{bmatrix}^{-1} = \frac{1}{\quad} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$. Use the word determinant in your answer.

The inverse of an arbitrary matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ is

$$\frac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}.$$

In words, we swap the top-left and bottom-right entries, then negate the other two entries, and divide by the determinant of the matrix.

(c) We've been writing the inverse of matrix A as A^{-1}. Why does this notation make sense?

This makes sense because $AA^{-1} = I$, and I is analogous to 1 in $aa^{-1} = 1$ in that multiplying by it does nothing.

13. Now, see what happens when you multiply the following matrices:

(a) $-\frac{1}{2}\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix}$

$$-\frac{1}{2}\begin{bmatrix} 2\cdot5+3\cdot-4 & 2\cdot-3+3\cdot2 \\ 4\cdot5+5\cdot-4 & 4\cdot-3+5\cdot2 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(b) $\frac{1}{71}\begin{bmatrix} 5 & 7 \\ 8 & -3 \end{bmatrix}\begin{bmatrix} 3 & 7 \\ 8 & -5 \end{bmatrix}$

$$\frac{1}{71}\begin{bmatrix} 5\cdot3+7\cdot8 & 5\cdot7+7\cdot-5 \\ 8\cdot3-3\cdot8 & 8\cdot7+-3\cdot-5 \end{bmatrix} = \frac{1}{71}\begin{bmatrix} 71 & 0 \\ 0 & 71 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(c) $\begin{bmatrix} a & c \\ b & d \end{bmatrix}\frac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

$$\frac{1}{ad-bc}\begin{bmatrix} ad-cb & -ac+ca \\ bd-db & -cb+ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(d) $\frac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$$\frac{1}{ad-bc}\begin{bmatrix} da-cb & dc-cd \\ -ba+ab & -bc+ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

14. For another approach to finding the inverse of a matrix, solve the following for w, x, y, z in terms of a, b, c, d by converting the matrix equations into a set of four linear equations:

$$\begin{bmatrix} w & y \\ x & z \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The left side is $\begin{bmatrix} wa+yb & wc+yd \\ xa+zb & xc+zd \end{bmatrix}$. This yields the system of equations

$$\begin{cases} wa+yb = 1 \\ wc+yd = 0 \\ xa+zb = 0 \\ xc+zd = 1 \end{cases}.$$

This does not look pleasant. No factors are shared at all.
From the second equation, we see that $w = -\frac{yd}{c}$. Substituting into the first equation, we get

$$\begin{aligned}\left(-\frac{yd}{c}\right)a + yb &= 1\\ y\left(\frac{-ad+bc}{c}\right) &= 1\\ y &= \frac{c}{-ad+bc}\\ y &= -\frac{c}{ad-bc}.\end{aligned}$$

Progress! We substitute this into our expression for w:

$$w = -\frac{yd}{c} = -\frac{-\frac{c}{ad-bc}\cdot d}{c} = \frac{d}{ad-bc}.$$

We can apply the same logic to the latter two equations. From the third equation, we see that $x = -\frac{zb}{a}$. Substituting this into the fourth equation, we get

$$\begin{aligned}\left(-\frac{zb}{a}\right)c + zd &= 1\\ z\left(\frac{ad-bc}{a}\right) &= 1\\ z &= \frac{a}{ad-bc}.\end{aligned}$$

We substituting this back into our expression for x:

$$x = -\frac{\frac{a}{ad-bc}\cdot b}{a} = -\frac{b}{ad-bc}.$$

Overall we get the following answer for the inverse matrix:

$$\begin{bmatrix} w & y \\ x & z \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & -\frac{c}{ad-bc} \\ -\frac{b}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}.$$

This agrees with our previous observations.

15. Rewrite each system of equations in matrix form. Use your calculator to calculate a matrix inverse, solve the system, and finally, check your answer. Remember to make clear in your work when you have used a calculator.

(a) $\begin{cases} 2x+3y &= 5 \\ 4x+5y &= 7 \end{cases}$

In matrix form:

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}.$$

We left-multiply both sides by the inverse of $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, which WolframAlpha informs us is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}^{-1}\begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

Thus, $(x, y) = (-2, 3)$.

(b) $\begin{cases} 37x+12y &= 65 \\ 93x+40y &= 156 \end{cases}$

$$\begin{bmatrix} 37 & 12 \\ 93 & 40 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 65 \\ 156 \end{bmatrix}.$$

We left-multiply both sides by the inverse of $\begin{bmatrix} 37 & 12 \\ 93 & 40 \end{bmatrix}$, which WolframAlpha informs us is

$$\begin{bmatrix} 37 & 12 \\ 93 & 40 \end{bmatrix}^{-1} \begin{bmatrix} 65 \\ 156 \end{bmatrix} = \begin{bmatrix} 2 \\ -\frac{3}{4} \end{bmatrix}.$$

Thus, $(x, y) = \left(2, -\frac{3}{4}\right)$.

(c) $\begin{cases} 2x + 5y + 3z & = 5 \\ 3x + 2y + 4z & = 7 \\ 13x + 16y + 18z & = 4 \end{cases}$

$$\begin{bmatrix} 2 & 5 & 3 \\ 3 & 2 & 4 \\ 13 & 16 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 4 \end{bmatrix}.$$

We left-multiply both sides by the inverse of the 3×3 matrix, which WolframAlpha informs us... doesn't exist.

But are there infinite solutions or no solutions for (x, y, z)? Well, we can try to derive a contradiction with the equations we have, though that's kind of ugly. Subtracting 8 times the second equation from the third equation, we get

$$\begin{aligned} &13x + 16y + 18z && = 4 \\ &-8(3x + 2y + 4z && = 7) \\ &-11x + 14z && = -52. \quad (1) \end{aligned}$$

We can do this with several combinations:

$$\begin{aligned} &2(3x + 2y + 4z && = 7) \\ &-3(2x + 5y + 3z && = 5) \\ &-11y - z && = -1; \quad (2) \end{aligned}$$

$$\begin{aligned} &6(2x + 5y + 3z && = 5) \\ &-(13x + 16y + 18z && = 4) \\ &x + 14y && = 26. \quad (3) \end{aligned}$$

There seem to be some shared numbers cropping up. We multiply Equation (2) by 14 and add it to 11 times Equation (3):

$$\begin{aligned} &14(-11y - z && = -1) \\ &11(x + 14y && = 26) \\ &-14z + 11x && = 272 \\ &-11x + 14z && = -272. \end{aligned}$$

Combining this with our first equation, we get $-52 = -272$, a contradiction. Thus, there are no solutions to this system of equations.

As an aside, there's a significantly nicer way to check whether there's no or infinite solutions using *reduced row-echelon form*, which is a function called *rref* on your calculator. Basically, apply *rref* to the matrix

$$M = \begin{bmatrix} 2 & 5 & 3 & 5 \\ 3 & 2 & 4 & 7 \\ 13 & 16 & 18 & 4 \end{bmatrix},$$

and if the last row is $[0 \quad 0 \quad 0 \quad x]$ where $x \neq 0$, there is no solution. Otherwise, there are infinite solutions. In this case,

$$\operatorname{rref}(M) = \begin{bmatrix} 1 & 0 & \frac{14}{11} & 0 \\ 0 & 1 & \frac{1}{11} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

so $x = 1$ and there are no solutions. As for what *rref* really is... Wikipedia is your friend!

(d) $\begin{cases} w + 2x + 3y + 4z & = 7 \\ 3w - x - 2y - 5z & = 5 \\ 5w + 3x - y - 4z & = 3 \\ 7w + 9x + 5y - 2z & = 2 \end{cases}$

In matrix form:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & -1 & -2 & -5 \\ 5 & 3 & -1 & -4 \\ 7 & 9 & 5 & -2 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 3 \\ 2 \end{bmatrix}.$$

Left-multiplying by the inverse of the 4×4 matrix gives, according to WolframAlpha:

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & -1 & -2 & -5 \\ 5 & 3 & -1 & -4 \\ 7 & 9 & 5 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ 5 \\ 3 \\ 2 \end{bmatrix} = \frac{1}{226} \begin{bmatrix} 871 \\ -696 \\ -257 \\ 333 \end{bmatrix}.$$

Disgusting! So the solution is $(w, x, y, z) = \left(\frac{871}{226}, \frac{-696}{226}, \frac{-257}{226}, \frac{333}{226}\right)$.

(e) $\begin{cases} 2x + 5y + 2z & = 1 \\ 3x + 2y + 4z & = 1 \\ 13x + 16y + 18z & = 5 \end{cases}$

This looks rather similar to two problems ago, but the coefficients are slightly different. Indeed, such a small change can permit an inverse.

In matrix form:

$$\begin{bmatrix} 2 & 5 & 2 \\ 3 & 2 & 4 \\ 13 & 16 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}.$$

Left-multiplying by the inverse of the 3×3 matrix with WolframAlpha gives

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 5 & 2 \\ 3 & 2 & 4 \\ 13 & 16 & 18 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{3}{11} \\ \frac{1}{11} \\ 0 \end{bmatrix}.$$

Thus, $(x, y, z) = \left(\frac{3}{11}, \frac{1}{11}, 0\right)$.

(f) When can you use matrix inverses to solve a system of equations?

You can use matrix inverses when the system is a system of n *linear* equations in n variables, as such a system always be rearranged to what we have dealt with in the past few problems. Furthermore, they will find the solution if it exists, but if there are zero or infinite solutions, it cannot differentiate between the two cases.

16. You can fit a polynomial to any set of points in the plane, so long as the points pass the Vertical Line Test.

(a) What is the least degree polynomial through

i. One point?

A polynomial of degree 0 can pass through a point. After all, if the point is (a, b), then the polynomial $y = b$ passes through the point.

ii. Two points?

A polynomial of degree 1 (i.e. a line) can pass through two points.

iii. Three points?

A polynomial of degree 2 (i.e. a quadratic) can pass through three points.

iv. n points?

A polynomial of degree $n-1$ can pass through n points. The easiest way to see this is to suppose we have points $(x_1, y_1), \cdots, (x_n, y_n)$ and a general degree $n-1$ polynomial

$$P(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0.$$

Then the polynomial passing through the points (x_i, y_i) is equivalent to the system of equations

$$\begin{cases} a_{n-1}x_1^{n-1} + a_{n-2}x_1^{n-2} + \cdots + a_1 x_1 + a_0 = y_1 \\ a_{n-1}x_2^{n-1} + a_{n-2}x_2^{n-2} + \cdots + a_1 x_2 + a_0 = y_2 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \vdots \quad \vdots \\ a_{n-1}x_n^{n-1} + a_{n-2}x_n^{n-2} + \cdots + a_1 x_n + a_0 = y_n \end{cases}.$$

We can express this as the matrix equation

$$\begin{bmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\ x_2^{n-1} & x_2^{n-2} & \cdots & x_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_n^{n-1} & x_n^{n-2} & \cdots & x_n & 1 \end{bmatrix} \begin{bmatrix} a_{n-1} \\ a_{n-2} \\ \vdots \\ a_0 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

Note that the matrices here are $n \times n$ and $n \times 1$. We can solve this as usual by left-multiplying by the inverse of the big square matrix, which provides a solution for $(a_{n-1}, a_{n-2}, \cdots, a_0)$, and thus a polynomial satisfying the requirements.

For the interested: we actually have to prove that the big square matrix M is invertible. Thus, we need to show that $\det M \neq 0$. This is the kind of thing that more advanced linear algebra is useful for, but it turns out that the determinant of M is

$$\det M = \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} (x_j - x_i).$$

This may look terrifying, but this means we take the product of $(x_j - x_i)$ for indices $1 \leq i < j \leq n$. That is, we take the product of that expression for all (i, j) pairs where j is strictly greater than i. The proof of this is beyond the scope of this book.

Recalling the zero product property, that expression for $\det M$ is 0 if and only if one of the products is 0. Thus, it's 0 if and only if $x_j = x_i$ for some $i < j$. This makes sense! If there are two (or more) points with the same x coordinate, then the determinant is 0, and the matrix is not invertible. If all x coordinates are unique, however, then the determinant is nonzero, and the inverse and solution exist.

M is a special type of matrix known as a *Vandermonde matrix*.

(b) Find a polynomial of least degree that passes through $(0,3)$, $(1,5)$, $(2,-3)$, $(3,4)$, and $(4,7)$.

We apply our findings from the previous problem, setting $n = 5$, $(x_1, x_2, x_3, x_4, x_5) = (0, 1, 2, 3, 4)$ and $(y_1, y_2, y_3, y_4, y_5) = (3, 5, -3, 4, 7)$. The matrix equation is

$$\begin{bmatrix} 0^4 & 0^3 & 0^2 & 0 & 1 \\ 1^4 & 1^3 & 1^2 & 1 & 1 \\ 2^4 & 2^3 & 2^2 & 2 & 1 \\ 3^4 & 3^3 & 3^2 & 3 & 1 \\ 4^4 & 4^3 & 4^2 & 4 & 1 \end{bmatrix} \begin{bmatrix} a_4 \\ a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ -3 \\ 4 \\ 7 \end{bmatrix}$$

$$\implies \begin{bmatrix} a_4 \\ a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 16 & 8 & 4 & 2 & 1 \\ 81 & 27 & 9 & 3 & 1 \\ 256 & 64 & 16 & 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 5 \\ -3 \\ 4 \\ 7 \end{bmatrix} \overset{\text{WolframAlpha}}{=} = \frac{1}{6} \begin{bmatrix} -11 \\ 91 \\ -226 \\ 158 \\ 18 \end{bmatrix}.$$

Terrifeunt! Thus, the polynomial is

$$y = -\frac{11}{6}x^4 + \frac{91}{6}x^3 - \frac{226}{6}x^2 + \frac{158}{6}x + \frac{18}{6}$$
$$y = -\frac{11}{6}x^4 + \frac{91}{6}x^3 - \frac{113}{3}x^2 + \frac{79}{3}x + 3.$$

14 Multiplication Modulo m Meets Groups

1. **Clearly some of these numbers cannot be elements of a group. For instance, in both cases, 0 cannot be used, since it prevents the existence of an inverse. In the case of mod 4, 2 cannot be used either. Why not?**

2 can't be used because it lacks an inverse. After all, there is no integer x such that $2x \equiv 1 (\mathrm{mod}\, 4)$.

2. **How could we have known that these numbers would not work in advance?**

They share a common factor with the modulus m; we must have $\gcd(x, m) = 1$ where x is the element of the group.

3. **Euler's totient function $\varphi(m)$ tells us how many numbers are relatively prime to a given number m. That is, $\varphi(m)$ is the count of numbers n such that $\gcd(m, n) = 1$. What does the maximum size of a group under multiplication mod m have to do with this function?**

The size of the largest group under multiplication modulo m is $\varphi(m)$, since that's the group of numbers which have inverses modulo m. The other group properties are satisfied; associativity because multiplication is associative, closure because the product of two numbers coprime to m is also coprime to m, and identity, since 1 is always (considered) coprime to m.

4. **We will write tables for the largest possible groups under multiplication mod 5 and mod 8.**

   (a) **Make a prediction as to how many elements will be in each group.**

(Answers may vary.)
Since 5 is prime, there should be 4 elements in its group since $\varphi(p)$ with p prime is $p - 1$. For 8, the elements should be $1, 3, 5, 7$: the odd numbers, or 4 elements.

   (b) **Which numbers can you eliminate from consideration?**

We can eliminate numbers that are coprime to the modulus. For 5, this is $\{0\}$. For 8, this is $\{0, 2, 4, 6\}$.

   (c) **Do you think that the groups will be isomorphic to those of multiplication mod 3 and mod 4, or to each other?**

(Answers may vary.)
The group mod 5, since 5 is prime, should just be the cyclic group of order 4. 8 is harder to predict; it turns out to be a different group of order 4. Since multiplication groups mod $3, 4$ have order 2, it can't be isomorphic to these.

   (d) **Find the period of each element in the groups and write their orbits: the list of its powers until it reaches the identity.**

mod 5:

Element	Orbit	Period
1	$1, (1)$	1
2	$2, 4, 3, 1, (2)$	4
3	$3, 4, 2, 1, (3)$	4
4	$4, 1, (4)$	2

mod 8:

Element	Orbit	Period
1	$1, (1)$	1
3	$3, 1, (3)$	3
5	$5, 1, (5)$	5
7	$7, 1, (7)$	7

   (e) **Make the tables, and analyze them to confirm/correct your predictions.**

The multiplicative group of integers mod 5 is indeed the cyclic group of order 4, since there are elements of periods $1, 2$, and 4. If we want to write a correspondence, let the cyclic group of order 4 be $\{I, r, r^2, r^3\}$ under multiplication. Then $1 \leftrightarrow I$, $2 \leftrightarrow r$, $3 \leftrightarrow r^3$, $4 \leftrightarrow r^2$.
The multiplicative group of integers mod 8 is the dihedral group of order 4, or D_2.

   (f) **Are there any subgroups?**

Yes; $\{1, 4\}$ is a subgroup of the mod 5 multiplicative group, and $\{1, 3\}$, $\{1, 5\}$, $\{1, 7\}$ are subgroups of the mod 8 multiplicative group. $\{1\}$ is a subgroup of both groups.

5. Now use a program to find the largest possible group under multiplication mod 14.

Here's my Python code for this, compressed to fit:

```
#!/usr/bin/env python
from fractions import gcd

# run in a terminal as python -i mod_m_find.py,
# so you can interact with it as a REPL

modulus = 10
group_elements = []
for i in xrange(1, modulus):
    if gcd(i, modulus) == 1:
        group_elements.append(i)
def pretty_print_elements():
    print(pp_list(group_elements))
def pp_list(arr):
    return ", ".join(map(lambda s: "$%s$" % s, arr))
def make_orbit_table(sort_by_orbit=True):
    print("\\begin{tabular}{c|l|c}\nElement & Orbit & Period \\\\ \\hline")
    orbits = []
    for elem in group_elements:
        x = elem
        orbit = []
        while True:
            orbit.append(str(x))
            x *= elem
            x %= modulus

            if x == elem:
                orbit.append("(%s)" % elem)
                break
        orbits.append(orbit)
    if sort_by_orbit:
        orbits.sort(key=len)
    for orbiit in orbits:
        print("$%s$ & %s & $%s$ \\\\" % (orbiit[0], pp_list(orbiit), len(orbiit) - 1))
    print("\\end{tabular}")
def create_group_table(sort_by=None):
    if not sort_by:
        group_elements.sort()
    else:
        group_elements.sort(key=sort_by)
    hline = "\\hline"
    print("\\begin{array}{c|%s}" % ("c|" * len(group_elements)))
    print("\\cdot & " + " & ".join(map(str, group_elements)) + " \\\\ " + hline)
    for i in group_elements:
        row = "%s" % i
        for j in group_elements:
            row += " & %s" % ((i * j) % modulus)
        row += " \\\\ " + hline
        print(row)
    print("\\end{array}")
```

You can find this code at *https://github.com/anematode/gatm/blob/master/accessories/scripts/mod_m_find.py*.

(a) What are its elements?

```
Timothys-Pro:gatm timoothy$ python -i accessories/scripts/mod_m_find.py
```

```
>>> pretty_print_elements()
$1$, $3$, $7$, $9$
```

My program is customized for LaTeX of course, since that's what I'm writing this text in. But the elements are $\{1,3,7,9\}$.

(b) Make a table of the group's orbits.

```
>>> make_orbit_table(False)
\begin{tabular}{c|l|c}
Element & Orbit & Period \\
$1$ & $1$, $(1)$ & $1$ \\
$3$ & $3$, $9$, $7$, $1$, $(3)$ & $4$ \\
$7$ & $7$, $9$, $3$, $1$, $(7)$ & $4$ \\
$9$ & $9$, $1$, $(9)$ & $2$ \\
\end{tabular}
```

This renders to the following table of orbits.

Element	Orbit	Period
1	1, (1)	1
3	3, 9, 7, 1, (3)	4
7	7, 9, 3, 1, (7)	4
9	9, 1, (9)	2

(c) Make a group table.

```
>>> create_group_table()
\begin{array}{c|c|c|c|c|}
\cdot & 1 & 3 & 7 & 9 \\ \hline
1 & 1 & 3 & 7 & 9 \\ \hline
3 & 3 & 9 & 1 & 7 \\ \hline
7 & 7 & 1 & 9 & 3 \\ \hline
9 & 9 & 7 & 3 & 1 \\ \hline
\end{array}
```

This gives the following table:

$\cdot$	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

(d) It might be good to order the numbers at the top of the table so that they start with a 1 and go by successive powers of 3.

The successive powers of 3 are $1,3,9,7$.

```
>>> create_group_table([1,3,9,7].index)
\begin{array}{c|c|c|c|c|}
\cdot & 1 & 3 & 9 & 7 \\ \hline
1 & 1 & 3 & 9 & 7 \\ \hline
3 & 3 & 9 & 7 & 1 \\ \hline
9 & 9 & 7 & 1 & 3 \\ \hline
7 & 7 & 1 & 3 & 9 \\ \hline
\end{array}
```

This gives the following table:

$\cdot$	1	3	9	7
1	1	3	9	7
3	3	9	7	1
9	9	7	1	3
7	7	1	3	9

(e) What group is it isomorphic to?

It is now clear that this is isomorphic to the cyclic group of order 4, C_4.

(f) Does it have any subgroups; if so, what are they?

It does have subgroups: the trivial subgroup $\{1\}$ and the cyclic group of order 2, $\{1, 9\}$.

6. Now, a surprise: find the powers of 10, mod 14.

Here's my one-liner in Python to do this:

```
>>> print(";\\quad ".join("10^%s \\equiv %s (\\operatorname{mod} 14)" %
  (n, 10**n % 14) for n in range(1,8)))
```

$$10^1 \equiv 10(\operatorname{mod} 14); \quad 10^2 \equiv 2(\operatorname{mod} 14); \quad 10^3 \equiv 6(\operatorname{mod} 14); \quad 10^4 \equiv 4(\operatorname{mod} 14);$$

$$10^5 \equiv 12(\operatorname{mod} 14); \quad 10^6 \equiv 8(\operatorname{mod} 14); \quad 10^7 \equiv 10(\operatorname{mod} 14).$$

It cycles!

(a) How long is the period of this orbit?

The orbit appears to be 6 elements long, since $7 - 1 = 6$.

(b) What number appears to be the identity element?

$10^6 = 8$ appears to be the identity element, since multiplying it by 10 yields 10.

(c) Make a table in which the identity element comes first.

```
>>> elements = [8,10,2,6,4,12];print("\\cdot & " + " & ".join(
  map(str,elements)) + " \\\\ \\hline");print("\n".join(
    " & ".join([str(i)] + map(lambda x: str((x * i) % 14), elements))
\cdot & 8 & 10 & 2 & 6 & 4 & 12 \\ \hline
8 & 8 & 10 & 2 & 6 & 4 & 12 \\ \hline
10 & 10 & 2 & 6 & 4 & 12 & 8 \\ \hline
2 & 2 & 6 & 4 & 12 & 8 & 10 \\ \hline
6 & 6 & 4 & 12 & 8 & 10 & 2 \\ \hline
4 & 4 & 12 & 8 & 10 & 2 & 6 \\ \hline
12 & 12 & 8 & 10 & 2 & 6 & 4 \\ \hline
```

Here's how it looks:

$\cdot$	8	10	2	6	4	12
8	8	10	2	6	4	12
10	10	2	6	4	12	8
2	2	6	4	12	8	10
6	6	4	12	8	10	2
4	4	12	8	10	2	6
12	12	8	10	2	6	4

(d) Find a number besides 10 whose group of powers mod 14 is isomorphic to this group.

This is clearly the cyclic group of order 6, so we can choose any generator of this group to get the same group. We might choose 12 for example.

```
>>> print(";\\quad ".join("12^%s \\equiv %s (\\operatorname{mod} 14)" %
  (n, 12**n % 14) for n in range(1,8)))
```

$$12^1 \equiv 12 (\bmod 14); \quad 12^2 \equiv 4 (\bmod 14); \quad 12^3 \equiv 6 (\bmod 14); \quad 12^4 \equiv 2 (\bmod 14);$$
$$12^5 \equiv 10 (\bmod 14); \quad 12^6 \equiv 8 (\bmod 14); \quad 12^7 \equiv 12 (\bmod 14)$$

Indeed, we get the same numbers, but in a different order.

(e) Are these groups isomorphic to a multiplication group of a smaller modulus?

Yes, this group is isomorphic to the multiplicative group mod 7, since 7 is a prime and would produce the cyclic group of order 6.

7. To really tell if two groups are isomorphic, you can write their tables in such an order that they would be identical if you substituted them in place. Why is it helpful to first note the periods and orbits of each element?

This is helpful because then you know which elements could possibly correspond, and which elements definitely don't correspond (i.e. the ones with different periods). Also, once you have chosen one correspondence, all the elements in that orbit are determined.

For the suggested problems, you can write your own program (or use my program) to investigate!

15 Eigenvectors and Eigenvalues

1. **Consider the matrix equation $\begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 6x+y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$. We wish to find an eigenvector $\begin{bmatrix} x \\ y \end{bmatrix}$.**

   (a) **On graph paper, draw what the matrix $\begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}$ does to the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.**

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ goes to $\begin{bmatrix} 0 \\ 6 \end{bmatrix}$ (dashed lines) and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ goes to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (solid lines):

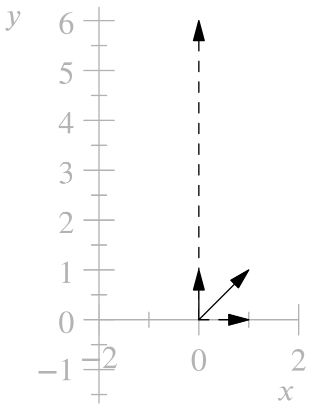

Figure 1: The mapping of the matrix.

   (b) **In your picture, draw a rough line through the origin where you think a family of eigenvectors may be.**

This is the line where vectors should change direction. Thus, it should be roughly where the diagram is "flipped," though this definitely is not a pure reflection.

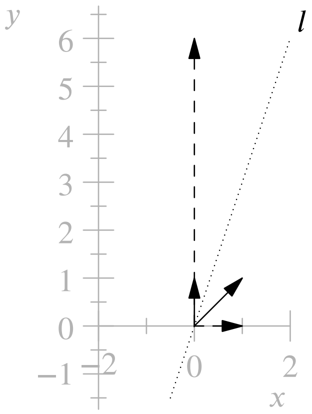

Figure 2: The mapping of the matrix, with the suspected eigenvectors indicated by l.

   (c) **Try some lattice points, say $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 1 \\ 4 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 1 \\ 5 \end{smallmatrix}\right]$. What does the matrix transform each vector into?**

These points are transformed as follows:

$$\begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix};$$

$$\begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix};$$

$$\begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix};$$
$$\begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \end{bmatrix};$$
$$\begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}.$$

(d) Which of these is an eigenvector?

$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is an eigenvector, since the image is $\begin{bmatrix} 3 \\ 9 \end{bmatrix}$, which is just the original vector times 3.

(e) Does it lie near the line you drew earlier?

Well, it lies *on* the line I drew because I'm using the computer, but it should lie close. Here it is superimposed on the previous graph:

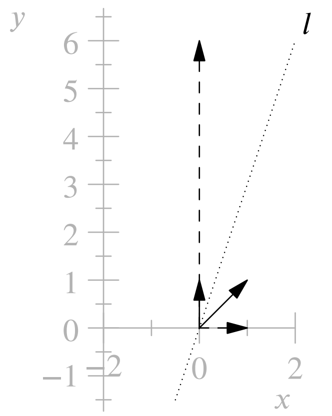

Figure 3: The mapping of the matrix, with the suspected eigenvectors indicated by l.

2. **This guess-and-check process for finding eigenvectors is terrible, so let's develop a procedure to find the eigenvalues and eigenvectors for any 2×2 matrix. We will use the same example.**

$$\begin{aligned} \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \lambda \begin{bmatrix} x \\ y \end{bmatrix} && \textbf{Definition of eigenvector} \\ &= \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ \implies \left(\begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} && \textbf{Subtraction and factoring} \\ \implies \begin{bmatrix} -\lambda & 1 \\ 6 & 1-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

(a) If $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \neq \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$, then

$$\det \begin{bmatrix} -\lambda & 1 \\ 6 & 1-\lambda \end{bmatrix} = 0.$$

Why? Think inverses.

If we left-multiply both sides of the last equation above by the inverse of the 2×2 matrix, we'll get

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ 6 & 1-\lambda \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

But we assumed $\begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so the inverse must not exist—that's the only other case. Thus,

$$\det \begin{bmatrix} -\lambda & 1 \\ 6 & 1-\lambda \end{bmatrix} = 0.$$

(b) Find the above determinant in terms of λ and solve for the eigenvalues.

We have
$$\det \begin{bmatrix} -\lambda & 1 \\ 6 & 1-\lambda \end{bmatrix} = -(1-\lambda)\lambda - 1(6) = \lambda^2 - \lambda - 6.$$
This is just a quadratic in λ, which factors as
$$(\lambda+2)(\lambda-3) = 0.$$
Thus, $\lambda = -2, 3$.

(c) One eigenvalue is $\lambda = 3$. We solve for the associated eigenvector like so:

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} -\lambda & 1 \\ 6 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} -3 & 1 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ \implies \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} -3x+y \\ 6x-2y \end{bmatrix} \\ &\implies y = 3x \to \begin{bmatrix} x \\ y \end{bmatrix} = s \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \textbf{(for some } s\textbf{)} \end{aligned}$$

Solve for the other eigenvector using the other eigenvalue from part (b).

The other eigenvalue is -2.

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} -\lambda & 1 \\ 6 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ \implies \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 2x+y \\ 6x+3y \end{bmatrix} \\ &\implies y = -2x \to \begin{bmatrix} x \\ y \end{bmatrix} = s \begin{bmatrix} 1 \\ -2 \end{bmatrix}. \end{aligned}$$

(d) Check your work by multiplying the original matrix by the eigenvector!

$$\begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} s \begin{bmatrix} 1 \\ -2 \end{bmatrix} = s \begin{bmatrix} 0\cdot 1 - 2\cdot 1 \\ 6\cdot 1 - 2\cdot 1 \end{bmatrix} = s \begin{bmatrix} -2 \\ 4 \end{bmatrix} = -2 \cdot \left(s \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right).$$
Indeed, the image of $s \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is the pre-image scaled by -2.

3. Solve for the eigenvectors and eigenvalues of the following matrices:

(a) $\begin{bmatrix} 3 & 24 \\ 4 & 7 \end{bmatrix}$

$$\begin{aligned}
\begin{bmatrix} 3 & 24 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \lambda\begin{bmatrix} x \\ y \end{bmatrix} && \text{Definition of eigenvector} \\
&= \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\
\Longrightarrow \left(\begin{bmatrix} 3 & 24 \\ 4 & 7 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} && \text{Subtraction and factoring} \\
\Longrightarrow \begin{bmatrix} 3-\lambda & 24 \\ 4 & 7-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\det\begin{bmatrix} 3-\lambda & 24 \\ 4 & 7-\lambda \end{bmatrix} &= 0 \\
(3-\lambda)(7-\lambda) - 24(4) &= 0 \\
\lambda^2 - 10\lambda - 75 &= 0 \\
(\lambda+5)(\lambda-15) &= 0 \\
\lambda &= -5, 15.
\end{aligned}$$

Thus, the eigenvalues are -5 and 15. We now find the corresponding eigenvectors.
-5:

$$\begin{aligned}
\begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 3-\lambda & 24 \\ 4 & 7-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\
&= \begin{bmatrix} 8 & 24 \\ 4 & 12 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\
\Longrightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 8x + 24y \\ 4x + 12y \end{bmatrix} \\
\Longrightarrow y = -3x &\to \begin{bmatrix} x \\ y \end{bmatrix} = s\begin{bmatrix} 1 \\ -3 \end{bmatrix}.
\end{aligned}$$

The first eigenvalue-eigenvector pair is $\left\{-5, \begin{bmatrix} 1 \\ -3 \end{bmatrix}\right\}$.
15:

$$\begin{aligned}
\begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 3-\lambda & 24 \\ 4 & 7-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\
&= \begin{bmatrix} -12 & 24 \\ 4 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\
\Longrightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} -12x + 24y \\ 4x - 8y \end{bmatrix} \\
\Longrightarrow y = 2x &\to \begin{bmatrix} x \\ y \end{bmatrix} = s\begin{bmatrix} 1 \\ 2 \end{bmatrix}.
\end{aligned}$$

The second eigenvalue-eigenvector pair is $\left\{15, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$.

(b) $\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$

$$
\begin{aligned}
\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \lambda \begin{bmatrix} x \\ y \end{bmatrix} && \text{Definition of eigenvector} \\
&= \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\
\implies \left(\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} && \text{Subtraction and factoring} \\
\implies \begin{bmatrix} 3-\lambda & 1 \\ 2 & 4-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\det \begin{bmatrix} 3-\lambda & 1 \\ 2 & 4-\lambda \end{bmatrix} &= 0 \\
(3-\lambda)(4-\lambda) - 1(2) &= 0 \\
(\lambda-2)(\lambda-5) &= 0 \\
\lambda &= 2, 5.
\end{aligned}
$$

Thus, the eigenvalues are 2 and 5. We now find the corresponding eigenvectors.
2:

$$
\begin{aligned}
\begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 3-\lambda & 1 \\ 2 & 4-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\
\implies \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} x+y \\ 2x+2y \end{bmatrix} \\
&\implies y = -x \to \begin{bmatrix} x \\ y \end{bmatrix} = s\begin{bmatrix} 1 \\ -1 \end{bmatrix}.
\end{aligned}
$$

The first eigenvalue-eigenvector pair is $\left\{2, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$.
5:

$$
\begin{aligned}
\begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 3-\lambda & 1 \\ 2 & 4-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\
&= \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\
\implies \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} -2x+y \\ 2x-y \end{bmatrix} \\
&\implies y = 2x \to \begin{bmatrix} x \\ y \end{bmatrix} = s\begin{bmatrix} 1 \\ 2 \end{bmatrix}.
\end{aligned}
$$

The second eigenvalue-eigenvector pair is $\left\{5, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$.

(c) $\begin{bmatrix} 1 & -1 \\ 4 & 6 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{Definition of eigenvector}$$
$$= \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$
$$\implies \left(\begin{bmatrix} 1 & -1 \\ 4 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{Subtraction and factoring}$$
$$\implies \begin{bmatrix} 1-\lambda & -1 \\ 4 & 6-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\det \begin{bmatrix} 1-\lambda & -1 \\ 4 & 6-\lambda \end{bmatrix} = 0$$
$$(1-\lambda)(6-\lambda) - (-1)(4) = 0$$
$$(\lambda - 2)(\lambda - 5) = 0$$
$$\lambda = 2, 5.$$

Thus, the eigenvalues are 2 and 5. Interestingly, these are the same eigenvalues as the previous problem. We now find the corresponding eigenvectors.

2:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1-\lambda & -1 \\ 4 & 6-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$
$$= \begin{bmatrix} -1 & -1 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$
$$\implies \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x+y \\ 4x+4y \end{bmatrix}$$
$$\implies y = -x \to \begin{bmatrix} x \\ y \end{bmatrix} = s\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The first eigenvalue-eigenvector pair is $\left\{2, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$.

5:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1-\lambda & -1 \\ 4 & 6-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$
$$= \begin{bmatrix} -4 & -1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$
$$\implies \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -4x-y \\ 4x+y \end{bmatrix}$$
$$\implies y = -4x \to \begin{bmatrix} x \\ y \end{bmatrix} = s\begin{bmatrix} 1 \\ -4 \end{bmatrix}.$$

The second eigenvalue-eigenvector pair is $\left\{5, \begin{bmatrix} 1 \\ -4 \end{bmatrix}\right\}$.

4. The image of an eigenvector will have the same ________ when acted on by the transformation ______ for which it is an eigenvector. The image of the eigenvector is simply the eigenvector itself multiplied by its corresponding __________.

The image of an eigenvector will have the same direction when acted on by the transformation matrix for which it is an eigenvector. The image of the eigenvector is simply the eigenvector itself multiplied by its corresponding eigenvalue.

5.

(a) If the transformation matrix were a reflection over a line $y = x\tan\theta$, in what directions would the two eigenvectors point? Think geometrically.

Geometrically, the eigenvectors would be 1. along the line $y = x\tan\theta$ and 2. perpendicular to the line. Observe the figure below if you're confused.

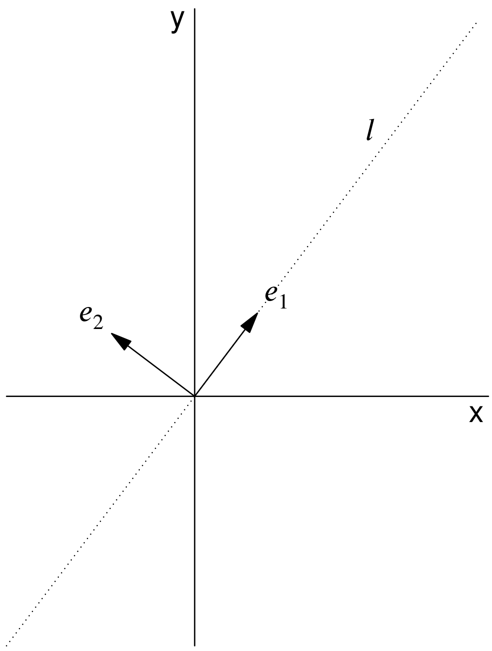

Figure 4: The eigenvectors e_1, e_2 of reflection over the line l.

(b) What would the angle between them be?

The angle between them is $90°$, since one is along a line and the other is perpendicular to that line.

(c) What would their eigenvalues be?

Referring to the above figure, e_1 would have an eigenvalue of 1, since its magnitude and direction is completely preserved, while e_2 would have an eigenvalue of -1, since it is multiplied by -1 to be inverted like that.

6. Recall that multiplication by $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$ results in a reflection over $y = x\tan\theta$.

(a) Write a matrix that results in a reflection over the line $y = \frac{\sqrt{3}}{3}x$.

The angle here is

$$\tan^{-1}\frac{\sqrt{3}}{3} = \tan^{-1}\frac{1}{\sqrt{3}} = \tan^{-1}\frac{1/2}{1/\sqrt{3}} = 30°.$$

Thus, the matrix is

$$\begin{bmatrix} \cos 2\cdot 30° & \sin 2\cdot 30° \\ \sin 2\cdot 30° & -\cos 2\cdot 30° \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}.$$

(b) Find the eigenvalues of that matrix, and the corresponding eigenvectors.

We find the eigenvalues:

$$\begin{bmatrix} \frac{1}{2}-\lambda & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left(\frac{1}{2}-\lambda\right)\left(-\frac{1}{2}-\lambda\right)-\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}=\lambda^2-\frac{1}{4}-\frac{3}{4}=\lambda^2-1$$
$$\lambda=\pm 1.$$

We then find the corresponding eigenvectors:
1:

$$\begin{aligned}
\begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} \frac{1}{2}-\lambda & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{3}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\
\Longrightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} -\frac{1}{2}x+\frac{\sqrt{3}}{2}y \\ \frac{\sqrt{3}}{2}x-\frac{3}{2}y \end{bmatrix} \\
\Longrightarrow y &= \frac{x}{\sqrt{3}} \rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = s\begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}.
\end{aligned}$$

-1:

$$\begin{aligned}
\begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} \frac{1}{2}-\lambda & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\
&= \begin{bmatrix} \frac{3}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\
\Longrightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} \frac{3}{2}x+\frac{\sqrt{3}}{2}y \\ \frac{\sqrt{3}}{2}x+\frac{1}{2}y \end{bmatrix} \\
\Longrightarrow y &= -x\sqrt{3} \rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = s\begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}.
\end{aligned}$$

Thus, the eigenvalue-eigenvector pairs are $\left\{1, \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}\right\}$ and $\left\{1, \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}\right\}$.

(c) Do your calculations agree with your answers to the previous problem?

Yes, they do. Here are the graphs of those two eigenvectors:

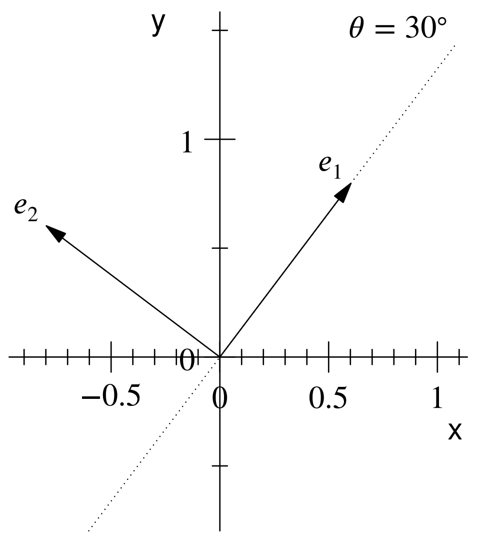

Figure 5: The calculated eigenvectors with the line $\theta = 30°$.

The corresponding eigenvalues also match up.

(d) What are the relationships between the two eigenvectors and between the two eigenvalues?

The two eigenvectors are $90°$ displaced from one another. The two eigenvalues are opposites.

7.

(a) Write a matrix which results in a $60°$ rotation counterclockwise.

This is just $\begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$.

(b) Find the eigenvalues. What do you find strange?

We solve the equation

$$\det \begin{bmatrix} \frac{1}{2} - \lambda & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} - \lambda \end{bmatrix} = 0.$$

$$\begin{aligned}
\left(\frac{1}{2} - \lambda\right)\left(\frac{1}{2} - \lambda\right) - \left(-\frac{\sqrt{3}}{2}\right) \cdot \frac{\sqrt{3}}{2} &= 0 \\
\lambda^2 - \lambda + \frac{1}{4} + \frac{3}{4} &= 0 \\
\lambda^2 - \lambda + 1 &= 0 \\
\lambda &= \frac{1 \pm \sqrt{-3}}{2} \\
\lambda &= \frac{1}{2} + \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i.
\end{aligned}$$

The eigenvalues are complex! They have magnitude 1, however, like the eigenvalues of the reflection.

(c) Find the eigenvectors for those eigenvalues. What's strange about them?

$\frac{1}{2} + \frac{\sqrt{3}}{2}i$:

$$\begin{aligned}
\begin{bmatrix} \frac{1}{2} - \lambda & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} - \lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{3}}{2}i & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2}i \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} -\frac{\sqrt{3}}{2}ix - \frac{\sqrt{3}}{2}y \\ \frac{\sqrt{3}}{2}x - \frac{\sqrt{3}}{2}iy \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\implies y = -ix \to \begin{bmatrix} x \\ y \end{bmatrix} = s\begin{bmatrix} 1 \\ -i \end{bmatrix}.
\end{aligned}$$

Weird!

$\frac{1}{2} - \frac{\sqrt{3}}{2}i$:

$$\begin{bmatrix} \frac{1}{2}-\lambda & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2}i & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2}i \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\sqrt{3}}{2}ix - \frac{\sqrt{3}}{2}y \\ \frac{\sqrt{3}}{2}x + \frac{\sqrt{3}}{2}iy \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies y = ix \rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = s\begin{bmatrix} 1 \\ i \end{bmatrix}.$$

Fascinating! The eigenvalue-eigenvector pairs are $\left\{\frac{1}{2}+\frac{\sqrt{3}}{2}i, \begin{bmatrix} 1 \\ -i \end{bmatrix}\right\}$ and $\left\{\frac{1}{2}-\frac{\sqrt{3}}{2}i, \begin{bmatrix} 1 \\ i \end{bmatrix}\right\}$.

(d) Explain what's going on.

There are no vectors which don't change orientation under a rotation, so the solutions we get can't be real. Nonetheless, a quadratic always has two roots if the discriminant is nonzero, so we get two solutions.

(e) What are the relationships between the two eigenvectors and between the two eigenvalues?

The eigenvalues are each other's complex conjugates. The eigenvectors, graphed in the complex plane, form a $90°$ between each other.

8. The matrix $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is a shear parallel to the x-axis.

(a) What vectors don't change direction when multiplied by this matrix?

Only vectors parallel to the x-axis don't change direction, i.e. $\begin{bmatrix} s \\ 0 \end{bmatrix}$ for any real s.

(b) What would you expect the eigenvectors to be?

We'd expect there to only be one family of eigenvectors, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(c) Find the eigenvectors and eigenvalues of this matrix.

We want $\det \begin{bmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{bmatrix} = 0$, which simplifies to $(\lambda-1)^2 = 0$. Thus, there is only one eigenvalue: 1. This makes sense.

The eigenvector is the solution to $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which gives $\begin{bmatrix} 2y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Thus, $y = 0$, and the eigenvectors are the family $s\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(d) What is different this time?

There is only one eigenvector and eigenvalue!

(e) Can you represent every vector as sums of eigenvectors?

In this case, you cannot represent every vector as a sum of eigenvectors. After all, any sum of the one eigenvector cannot have a nonzero y coordinate.

9. The matrices below result in some stretches. Find the eigenvectors and eigenvalues for both.

(a) $\left[\begin{smallmatrix} 2 & 0 \\ 0 & 5 \end{smallmatrix}\right]$

The characteristic polynomial[26] is just $(2-\lambda)(5-\lambda)$, which gives eigenvalues $\lambda = 2, 5$.
2:

$$\begin{bmatrix} 2-2 & 0 \\ 0 & 5-2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 \\ 3y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus, $y = 0$ and the family of eigenvectors is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
5:

$$\begin{bmatrix} 2-5 & 0 \\ 0 & 5-5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -3x \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus, $x = 0$ and the family of eigenvectors is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(b) $\left[\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right]$

The characteristic polynomial here is $(3-\lambda)(3-\lambda)$, yielding $\lambda = 3$. We find the eigenvector:

$$\begin{bmatrix} 3-3 & 0 \\ 0 & 3-3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus, all $\begin{bmatrix} x \\ y \end{bmatrix}$ are eigenvectors. This makes sense! After all, all vectors are scaled up by a factor of 3, and no vectors change direction.

10. Note that most 2×2 matrices have two eigenvectors. How many would you expect to find for an $n \times n$ matrix?

You'd expect there to be n eigenvectors in an $n \times n$ matrix. One way to rationalize this further is that the characteristic polynomial of an $n \times n$ matrix is degree n, which usually has n roots.

11. Assuming that p, q, r, s, t, u, x, y are real, what conditions would you impose on them in the matrices (i) $\begin{bmatrix} 3 & p \\ q & 4 \end{bmatrix}$, (ii) $\begin{bmatrix} x & -2 \\ 3 & y \end{bmatrix}$, and (iii) $\begin{bmatrix} r & s \\ t & u \end{bmatrix}$ to have...

(a) ... two real eigenvalues?

i. $\begin{bmatrix} 3 & p \\ q & 4 \end{bmatrix}$

The characteristic polynomial here is $(3-\lambda)(4-\lambda) - pq$. Expanded out, this is $\lambda^2 - 7\lambda + 12 - pq$. We want the discriminant to be greater than 0 to have two real eigenvalues, so

$$\begin{aligned} b^2 - 4ac = 7^2 - 4(1)(12 - pq) &> 0 \\ 48 - 4pq &< 49 && \text{Manipulate, flip the inequality} \\ 4pq &> -1 \text{Subtract 48 from both sides} \\ pq &> -\frac{1}{4}. \end{aligned}$$

This is our restriction; we must have $pq > -\frac{1}{4}$.

[26]The polynomial involving λ determining the eigenvalues.

ii. $\begin{bmatrix} x & -2 \\ 3 & y \end{bmatrix}$

The characteristic polynomial here is $(x-\lambda)(y-\lambda)-(-2)(3)$. This expands out to $\lambda^2-(x+y)\lambda+xy+6$. Again, we want the discriminant to be greater than 0 to have two real eigenvalues, so

$$\begin{aligned} b^2-4ac = (x+y)^2-4(1)(xy+6) &> 0 \\ x^2+y^2+2xy-4xy-24 &> 0 \\ x^2-2xy+y^2 &> 24 \\ (x-y)^2 &> 24. \end{aligned}$$

Thus, our restriction is $(x-y)^2 > 24$, or equivalently, $|x-y| > \sqrt{24} = 2\sqrt{6}$.
This is always true by the Trivial Inequality[27]. Thus, there are always two real eigenvalues for this matrix.

iii. $\begin{bmatrix} r & s \\ t & u \end{bmatrix}$

The characteristic polynomial here is $(r-\lambda)(u-\lambda)-st$, which expands out to

$$\lambda^2-(u+r)\lambda+ru-st.$$

Again, we want the discriminant to be greater than 0 to have two real eigenvalues, so

$$\begin{aligned} b^2-4ac = (u+r)^2-4(ru-st) &> 0 \\ u^2+r^2+2ru-4ru-4st &> 0 \\ u^2-2ru+r^2 &> 4st \\ (u-r)^2 &> 4st. \end{aligned}$$

There isn't a great way to simplify this, but $(u-r)^2 > 4st$ is a potential answer. The first line of the equations above also gives us a potentially simpler interpretation:

$$(u+r)^2 > 4(ru-st) = 4\det\begin{bmatrix} r & s \\ t & u \end{bmatrix}.$$

Thus, the sum of the top-left to bottom-right diagonal squared must be greater than four times the determinant. Wordy!

(b) ... two complex eigenvalues?

i. $\begin{bmatrix} 3 & p \\ q & 4 \end{bmatrix}$

This is identical to problem (a) part i, but we want the discriminant to be smaller than 0. The proof is identical, just with a flipped inequality sign, so the answer is

$$pq < -\frac{1}{4}.$$

ii. $\begin{bmatrix} x & -2 \\ 3 & y \end{bmatrix}$

This is identical to problem (a) part ii, but we want the discriminant to be smaller than 0. The proof is identical, just with a flipped inequality sign, so the answer is

$$(x-y)^2 < 24.$$

[27]The *Trivial Inequality* states $x^2 \geq 0$ for all real x.

iii. $\begin{bmatrix} r & s \\ t & u \end{bmatrix}$

This is identical to problem (a) part iii, but we want the discriminant to be smaller than 0. The proof is identical, just with a flipped inequality sign, so the answer is

$$(u-r)^2 < 4st \quad \text{or} \quad (u+r)^2 < 4\det\begin{bmatrix} r & s \\ t & u \end{bmatrix}.$$

(c) ... only one eigenvalue?

i. $\begin{bmatrix} 3 & p \\ q & 4 \end{bmatrix}$

This is identical to the previous two iterations of this matrix, but with an equality sign:

$$pq = -\frac{1}{4}.$$

ii. $\begin{bmatrix} x & -2 \\ 3 & y \end{bmatrix}$

This is identical to the previous two iterations of this matrix, but with an equality sign:

$$(x-y)^2 = 24 \to x - y = \pm 2\sqrt{6}.$$

iii. $\begin{bmatrix} r & s \\ t & u \end{bmatrix}$

This is identical to the previous two iterations of this matrix, but with an equality sign:

$$(u-r)^2 = 4st \quad \text{or} \quad (u+r)^2 = 4\det\begin{bmatrix} r & s \\ t & u \end{bmatrix}.$$

12.

(a) Write a 3×3 matrix showing a rotation of θ around the z-axis.

We already did this a couple sections ago. The matrix is

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Name the real eigenvector (this shouldn't require any work).

The real eigenvector is the z-axis, since it doesn't move (observe the figure below if you're confused). Explicitly, this is the family of eigenvectors

$$s\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

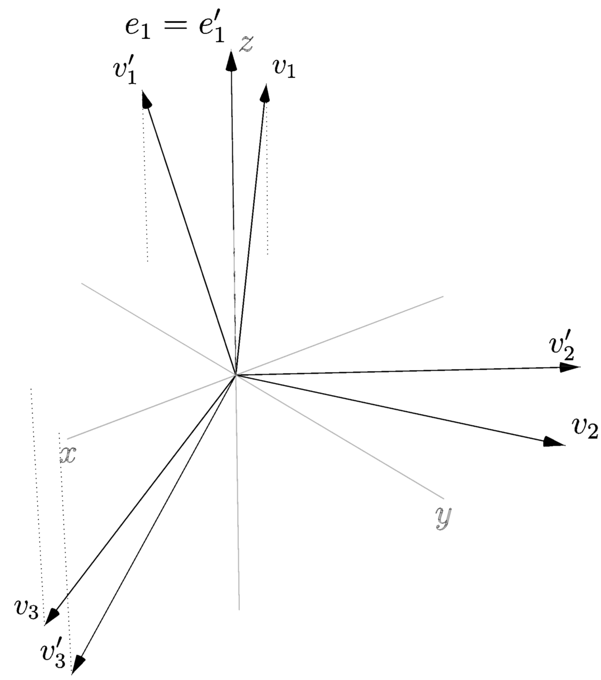

Figure 6: The z-axis remains stationary in a rotation of θ around the z-axis.

(c) Find all three eigenvectors.

The determinant of the eigenvector matrix is, by the minors method:

$$0 = \det \begin{bmatrix} \cos\theta - \lambda & -\sin\theta & 0 \\ \sin\theta & \cos\theta - \lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} = (\cos\theta - \lambda)\det \begin{bmatrix} \cos\theta - \lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} - (-\sin\theta)\det \begin{bmatrix} \sin\theta & 0 \\ 0 & 1-\lambda \end{bmatrix} + 0\,\text{(something)}$$

$$\begin{aligned} &= (\cos\theta - \lambda)(\cos\theta - \lambda)(1-\lambda) + (\sin\theta)(\sin\theta)(1-\lambda) \\ &= (1-\lambda)((\cos\theta - \lambda)^2 + (\sin\theta)(\sin\theta)) \\ &= (1-\lambda)(lambda^2 + \cos^2\theta - 2\lambda\cos\theta + \sin^2\theta) \\ &= (1-\lambda)(\lambda^2 - 2\lambda\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta)}_{\text{diff of squares}} \\ &= (1-\lambda)(\lambda - (\cos\theta - i\sin\theta))(\lambda - (\cos\theta + i\sin\theta)). \end{aligned}$$

This gives eigenvalues 1, $\cos\theta - i\sin\theta$ and $\cos\theta + i\sin\theta$. We already knew about the first one. We now compute the eigenvectors:
1:

$$\begin{bmatrix} \cos\theta - \lambda & -\sin\theta & 0 \\ \sin\theta & \cos\theta - \lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta - 1 & -\sin\theta & 0 \\ \sin\theta & \cos\theta - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (\cos\theta - 1)x - (\sin\theta)y \\ (\sin\theta)x + (\cos\theta - 1)y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus, $(\cos\theta - 1)x - (\sin\theta)y = 0$ and $(\sin\theta)x + (\cos\theta - 1)y = 0$. The first equation yields $x = \frac{\sin\theta}{\cos\theta - 1}y$. Substitution into the second equation yields

$$\begin{aligned} \frac{y\sin^2\theta}{\cos\theta - 1} + (\cos\theta - 1)y &= 0 \\ y\left(\frac{\sin^2\theta + (\cos\theta - 1)^2}{\cos\theta - 1}\right) &= 0 \end{aligned}$$

$$y\left(\frac{\sin^2\theta+\cos^2\theta-2\cos\theta+1}{\cos\theta-1}\right)=0$$
$$y\left(\frac{2-2\cos\theta}{\cos\theta-1}\right)=0$$
$$-2y=0$$
$$y=0.$$

Thus, $x=y=0$, or $\cos\theta-1=0$ (since then the first substitution is invalid). This makes sense! If $\cos\theta=1$, then it's a rotation by 0°, which has all vectors as eigenvectors. Anyway, this otherwise gives the family of eigenvectors $s\begin{bmatrix}0\\0\\1\end{bmatrix}$.

$\cos\theta-i\sin\theta$:

$$\begin{bmatrix}\cos\theta-\lambda & -\sin\theta & 0\\ \sin\theta & \cos\theta-\lambda & 0\\ 0 & 0 & 1-\lambda\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$
$$\begin{bmatrix}i\sin\theta & -\sin\theta & 0\\ \sin\theta & i\sin\theta & 0\\ 0 & 0 & i\sin\theta-\cos\theta\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}(i\sin\theta)x-(\sin\theta)y+1\\(\sin\theta)x+(i\sin\theta)y\\(i\sin\theta-\cos\theta+1)z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Since $i\sin\theta-\cos\theta+1\neq 0$ except for $\cos\theta=1$ (the rotation of 0 again), this yields $z=0$ and $x=iy$, so the family of eigenvectors is $s\begin{bmatrix}i\\1\\0\end{bmatrix}$.

$\cos\theta+i\sin\theta$:

$$\begin{bmatrix}\cos\theta-\lambda & -\sin\theta & 0\\ \sin\theta & \cos\theta-\lambda & 0\\ 0 & 0 & 1-\lambda\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$
$$\begin{bmatrix}-i\sin\theta & -\sin\theta & 0\\ \sin\theta & -i\sin\theta & 0\\ 0 & 0 & \cos\theta-i\sin\theta+1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-(i\sin\theta)x-(\sin\theta)y\\(\sin\theta)x-(i\sin\theta)y\\(1-i\sin\theta-\cos\theta)z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

For nonzero rotations, this yields $z=0$ and $x=-iy$, giving the family of eigenvectors $s=\begin{bmatrix}-i\\1\\0\end{bmatrix}$.

Overall, the eigenvalue-eigenvector pairs are $\left\{1,\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$, $\left\{\cos\theta-i\sin\theta,\begin{bmatrix}i\\1\\0\end{bmatrix}\right\}$, and $\left\{\cos\theta+i\sin\theta,\begin{bmatrix}-i\\1\\0\end{bmatrix}\right\}$.

13.

(d) What should the absolute value of an eigenvalue of any rotation matrix be?

It should be 1, since rotations don't stretch anything and doesn't change orientation. All distances are preserved. This is true of our eigenvectors.

(e) The complex eigenvalues relate to the angle of rotation. What is that relationship?

The complex eigenvalues are $\cos\theta+i\sin\theta=\operatorname{cis}\theta$ and $\cos\theta-i\sin\theta=\overline{\operatorname{cis}\theta}$, so they make an angle of θ with the real axis[28] in the complex plane. Furthermore, the angle between them is 2θ.

[28]Note that we shouldn't call it the x-axis, because this is a different set of axes than the xyz-axes we're considering in this problem.

14. In a right-handed coordinate system, rotations in three dimensions are performed by combinations of the three matrices

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}, Y = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}, Z = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Each matrix X, Y, Z rotates around the x, y, z axes by α, β, γ, respectively.

In 2D, rotations combine to make other rotations. Similarly, if we combine any number of these rotations, the net result will be a rotation about some axis—though not necessarily a *coordinate* axis. Another way to picture this is that if we operate on an origin-centered sphere with these matrices, there will always be two opposite points on the sphere which have no net movement.

Try computing the following products.

(a) XY

$$XY = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} (1)(\cos\beta) + (0)(0) + (0)(-\sin\beta) & (1)(0) + (0)(1) + (0)(0) & (1)(\sin\beta) + (0)(0) + (0)(\cos\beta) \\ (0)(\cos\beta) + (\cos\alpha)(0) + (-\sin\alpha)(-\sin\beta) & (0)(0) + (\cos\alpha)(1) + (-\sin\alpha)(0) & (0)(\sin\beta) + (\cos\alpha)(0) + (-\sin\alpha)(\cos\beta) \\ (0)(\cos\beta) + (\sin\alpha)(0) + (\cos\alpha)(-\sin\beta) & (0)(0) + (\sin\alpha)(1) + (\cos\alpha)(0) & (0)(\sin\beta) + (\sin\alpha)(0) + (\cos\alpha)(\cos\beta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ \sin\alpha\sin\beta & \cos\alpha & -\sin\alpha\cos\beta \\ -\cos\alpha\sin\beta & \sin\alpha & \cos\alpha\cos\beta \end{bmatrix}.$$

(b) XZ

$$XZ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (1)(\cos\gamma) + (0)(\sin\gamma) + (0)(0) & (1)(-\sin\gamma) + (0)(\cos\gamma) + (0)(0) & (1)(0) + (0)(0) + (0)(1) \\ (0)(\cos\gamma) + (\cos\alpha)(\sin\gamma) + (-\sin\alpha)(0) & (0)(-\sin\gamma) + (\cos\alpha)(\cos\gamma) + (-\sin\alpha)(0) & (0)(0) + (\cos\alpha)(0) + (-\sin\alpha)(1) \\ (0)(\cos\gamma) + (\sin\alpha)(\sin\gamma) + (\cos\alpha)(0) & (0)(-\sin\gamma) + (\sin\alpha)(\cos\gamma) + (\cos\alpha)(0) & (0)(0) + (\sin\alpha)(0) + (\cos\alpha)(1) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \cos\alpha\sin\gamma & \cos\alpha\cos\gamma & -\sin\alpha \\ \sin\alpha\sin\gamma & \sin\alpha\cos\gamma & \cos\alpha \end{bmatrix}.$$

(c) YX

$$YX = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} (\cos\beta)(1) + (0)(0) + (\sin\beta)(0) & (\cos\beta)(0) + (0)(\cos\alpha) + (\sin\beta)(\sin\alpha) & (\cos\beta)(0) + (0)(-\sin\alpha) + (\sin\beta)(\cos\alpha) \\ (0)(1) + (1)(0) + (0)(0) & (0)(0) + (1)(\cos\alpha) + (0)(\sin\alpha) & (0)(0) + (1)(-\sin\alpha) + (0)(\cos\alpha) \\ (-\sin\beta)(1) + (0)(0) + (\cos\beta)(0) & (-\sin\beta)(0) + (0)(\cos\alpha) + (\cos\beta)(\sin\alpha) & (-\sin\beta)(0) + (0)(-\sin\alpha) + (\cos\beta)(\cos\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\beta & \sin\beta\sin\alpha & \sin\beta\cos\alpha \\ 0 & \cos\alpha & -\sin\alpha \\ -\sin\beta & \cos\beta\sin\alpha & \cos\beta\cos\alpha \end{bmatrix}.$$

(d) ZX

$$ZX = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} (\cos\gamma)(1) + (-\sin\gamma)(0) + (0)(0) & (\cos\gamma)(0) + (-\sin\gamma)(\cos\alpha) + (0)(\sin\alpha) & (\cos\gamma)(0) + (-\sin\gamma)(-\sin\alpha) + (0)(\cos\alpha) \\ (\sin\gamma)(1) + (\cos\gamma)(0) + (0)(0) & (\sin\gamma)(0) + (\cos\gamma)(\cos\alpha) + (0)(\sin\alpha) & (\sin\gamma)(0) + (\cos\gamma)(-\sin\alpha) + (0)(\cos\alpha) \\ (0)(1) + (0)(0) + (1)(0) & (0)(0) + (0)(\cos\alpha) + (1)(\sin\alpha) & (0)(0) + (0)(-\sin\alpha) + (1)(\cos\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\gamma & -\sin\gamma\cos\alpha & \sin\gamma\sin\alpha \\ \sin\gamma & \cos\gamma\cos\alpha & -\cos\gamma\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}.$$

Interestingly, $ZX \neq XZ$. Indeed, while rotations commute in 2 dimensions, they do not always commute in 3 dimensions.

15.

(a) Without matrices, consider a cube with side length 2 at the origin so its faces are perpendicular to the coordinate axes. Rotate it, first $90°$ counterclockwise about the y-axis, then $90°$ counterclockwise about the x-axis. Note that rotations are done facing from the "positive side" of the coordinate axis. The net result should leave two vertices fixed. Which two?

This requires a good amount of geometric visualization. The answer is the vertices $(1, 1, 1)$ and $(-1, -1, -1)$. Observe the figures below:

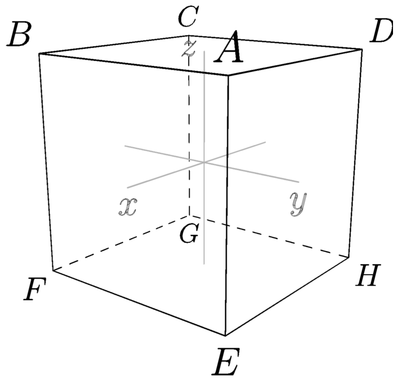

Figure 7: The starting position of the cube.

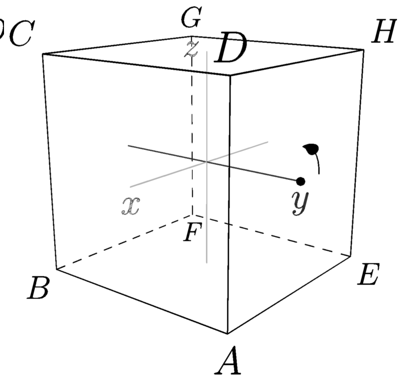

Figure 8: Rotation about the y-axis.

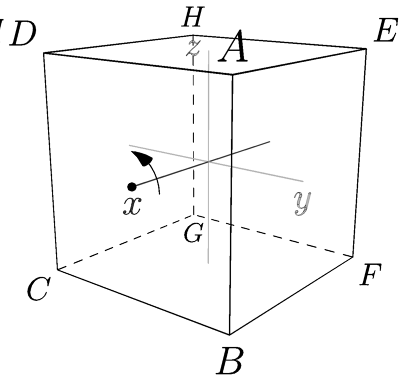

Figure 9: Rotation about the x-axis.

Indeed, A and G remain fixed. These are the vertices $(1, 1, 1)$ and $(-1, -1, -1)$.

(b) Write a vector for the axis of rotation.

The vector is any nonzero multiple of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. In the following figure, the axis of rotation is graphed.

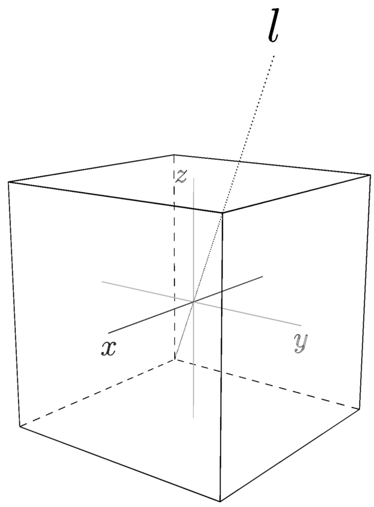

Figure 10: The net axis of rotation is $\langle 1, 1, 1 \rangle$.

(c) How many degrees do you think the net rotation of the cube is? Be careful; the answer is not $180°$.

The rotation is $120°$, because E is going to D, D is going to B and E is going to D, a cycle with period 3.

(d) Let's check our answers using matrices. Write a matrix product that corresponds to a rotation of $90°$ about the y-axis, followed by $90°$ about the x-axis.

Rotation of $90°$ about the y-axis: $Y = \begin{bmatrix} \cos 90° & 0 & \sin 90° \\ 0 & 1 & 0 \\ -\sin 90° & 0 & \cos 90° \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$.

Rotation of $90°$ about the x-axis: $X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 90° & -\sin 90° \\ 0 & \sin 90° & \cos 90° \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$.

As usual, matrix multiplication goes right-to-left, so the product is XY.

(e) Multiply out the matrix product.

$$XY = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (1)(0) + (0)(0) + (0)(-1) & (1)(0) + (0)(1) + (0)(0) & (1)(1) + (0)(0) + (0)(0) \\ (0)(0) + (0)(0) + (-1)(-1) & (0)(0) + (0)(1) + (-1)(0) & (0)(1) + (0)(0) + (-1)(0) \\ (0)(0) + (1)(0) + (0)(-1) & (0)(0) + (1)(1) + (0)(0) & (0)(1) + (1)(0) + (0)(0) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Interesting!

(f) Remember that the real eigenvector in a rotation gives the axis of rotation, and the complex eigenvalues give information about the net rotation. Evaluate these and check your answers for (a) and (b).

We first find the eigenvalues:

$$\det \begin{bmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{bmatrix} = 0$$

$$-\lambda \cdot \det \begin{bmatrix} -\lambda & 0 \\ 1 & -\lambda \end{bmatrix} - 0 \cdot (\text{something}) + 1 \cdot \det \begin{bmatrix} 1 & -\lambda \\ 0 & 1 \end{bmatrix} = 0$$

$$-\lambda^3 + (1 + 0 \cdot -\lambda) = 0$$

$$\lambda^3 = 1.$$

We let $\lambda = \operatorname{cis}\theta$:

$$\operatorname{cis}^3 \theta = 1 \Longrightarrow \theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}.$$

Thus, $\lambda = 1, \operatorname{cis}\frac{2\pi}{3}, \operatorname{cis}\frac{4\pi}{3}$.
We now compute the eigenvector for the axis of rotation, which should correspond to $\lambda = 1$.

$$\begin{bmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -x + z \\ x - y \\ y - z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

These yields $x = y = z$ and the eigenvector family $s\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, confirming our previous result.

We can find the angle of rotation by the angle the complex eigenvalues make with the x-axis. These eigenvalues are $\operatorname{cis}\frac{2\pi}{3}, \operatorname{cis}\frac{4\pi}{3}$, which make a $\frac{2\pi}{3} = 120°$ angle with the x-axis. Thus, the magnitude of the rotation is $120°$, confirming our hypothesis.

16. Here are two rotation matrices:
i. $\begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$, **ii.** $\begin{bmatrix} \frac{7}{9} & \frac{4}{9} & \frac{4}{9} \\ -\frac{4}{9} & -\frac{1}{9} & \frac{8}{9} \\ \frac{4}{9} & -\frac{8}{9} & \frac{1}{9} \end{bmatrix}$.

(a) What is the determinant of each matrix? (Don't work, think!)

The determinant of each matrix is 1, since rotation matrices have determinant 1.

(b) What is true of each row and each column?

The sums of squares of each element in each row and column is 1. Therefore, each row vector and column vector is a unit vector. As an example, consider the top row of (ii):

$$\left(\frac{7}{9}\right)^2 + \left(\frac{4}{9}\right)^2 + \left(\frac{4}{9}\right)^2 = \frac{81}{81} = 1.$$

(c) Find the axis of rotation associated with each matrix.

i. $\begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$

We find the eigenvalues:

$$\det \begin{bmatrix} \frac{2}{3} - \lambda & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} - \lambda & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} - \lambda \end{bmatrix} = 0$$

$$\left(\frac{2}{3} - \lambda\right) \cdot \det \begin{bmatrix} \frac{2}{3} - \lambda & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} - \lambda \end{bmatrix} - \left(-\frac{2}{3}\right) \det \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} - \lambda \end{bmatrix} - \frac{1}{3} \cdot \det \begin{bmatrix} \frac{1}{3} & \frac{2}{3} - \lambda \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} = 0$$

$$-\lambda^3 + 2\lambda^2 - 2\lambda + 1 = 0$$
$$(\lambda - 1)(\lambda^2 - \lambda + 1) = 0.$$

The real eigenvalue is $\lambda = 1$, so we find the corresponding eigenvector:

$$\begin{bmatrix} \frac{2}{3} - \lambda & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} - \lambda & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} - \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{3}x - \frac{2}{3}y - \frac{1}{3}z \\ \frac{1}{3}x - \frac{1}{3}y - \frac{2}{3}z \\ \frac{2}{3}x + \frac{1}{3}y - \frac{1}{3}z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Multiplying by 3 on both sides yields the system of equations

$$\begin{cases} -x - 2y - z = 0 \\ x - y - 2z = 0 \\ 2x + y - z = 0 \end{cases}.$$

The solution to this system of equations is $x = z = -y$. Thus, the eigenvector family is $s\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, and the axis of rotation is the vector $\langle 1, -1, 1 \rangle$.

i. $\begin{bmatrix} \frac{7}{9} & \frac{4}{9} & \frac{4}{9} \\ -\frac{4}{9} & -\frac{1}{9} & \frac{8}{9} \\ \frac{4}{9} & -\frac{8}{9} & \frac{1}{9} \end{bmatrix}$

We find the eigenvalues:

$$\det \begin{bmatrix} \frac{7}{9} - \lambda & \frac{4}{9} & \frac{4}{9} \\ -\frac{4}{9} & -\frac{1}{9} - \lambda & \frac{8}{9} \\ \frac{4}{9} & -\frac{8}{9} & \frac{1}{9} - \lambda \end{bmatrix} = 0$$

$$\left(\frac{7}{9} - \lambda\right) \cdot \det \begin{bmatrix} -\frac{1}{9} - \lambda & \frac{8}{9} \\ -\frac{8}{9} & \frac{1}{9} - \lambda \end{bmatrix} - \left(\frac{4}{9}\right) \cdot \det \begin{bmatrix} -\frac{4}{9} & \frac{8}{9} \\ \frac{4}{9} & \frac{1}{9} - \lambda \end{bmatrix} + \left(\frac{4}{9}\right) \cdot \det \begin{bmatrix} -\frac{4}{9} & -\frac{1}{9} - \lambda \\ \frac{4}{9} & -\frac{8}{9} \end{bmatrix} = 0$$

$$-\lambda^3 + \frac{7\lambda^2}{9} - \frac{7\lambda}{9} + 1 = 0$$
$$-\frac{1}{9}(\lambda - 1)(9\lambda^2 + 2\lambda + 9) = 0.$$

The real eigenvalue is $\lambda = 1$, so we find the corresponding eigenvector:

$$\begin{bmatrix} \frac{7}{9} - \lambda & \frac{4}{9} & \frac{4}{9} \\ -\frac{4}{9} & -\frac{1}{9} - \lambda & \frac{8}{9} \\ \frac{4}{9} & -\frac{8}{9} & \frac{1}{9} - \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} -\frac{2}{9} & \frac{4}{9} & \frac{4}{9} \\ -\frac{4}{9} & -\frac{10}{9} & \frac{8}{9} \\ \frac{4}{9} & -\frac{8}{9} & \frac{8}{9} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{9} \begin{bmatrix} -2x + 4y + 4z \\ -4x - 10y + 8z \\ 4x - 8y + 8z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ . \end{bmatrix}$$

The solution to this system of equations is $\langle x, y, z \rangle = s\langle -2, 0, 1 \rangle$. This is the axis of rotation: $\langle -2, 0, 1 \rangle$.

(d) Find the angle of rotation associated with each matrix.

i. $\begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$

From the last time we dealt with this matrix, we found that the complex eigenvalues satisfy $\lambda^2 - \lambda + 1 = 0$. By the quadratic formula, the solutions to this quadratic are $\lambda = \frac{1 \pm i\sqrt{3}}{2}$.

Since $\operatorname{cis} \pm 60° = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i = \lambda$, the rotation is $60°$.

i. $\begin{bmatrix} \frac{7}{9} & \frac{4}{9} & \frac{4}{9} \\ -\frac{4}{9} & -\frac{1}{9} & \frac{8}{9} \\ \frac{4}{9} & -\frac{8}{9} & \frac{1}{9} \end{bmatrix}$

Previously, we found that the complex eigenvalues of this matrix satisfy the polynomial equation $9\lambda^2 + 2\lambda + 9 = 0$. By the quadratic formula, the roots of this equation are

$$\frac{-2 \pm \sqrt{2^2 - 4 \cdot 9^2}}{18} = -\frac{1}{9} \pm \frac{4i\sqrt{5}}{9}.$$

The angle of rotation is given by

$$\tan^{-1} \frac{y}{x} = \frac{\pm \frac{4\sqrt{5}}{9}}{-\frac{1}{9}} = \pm \tan^{-1} 4 \cdot \sqrt{5},$$

which has magnitude $\tan^{-1}(4\sqrt{5})$.