Gröbner Geometry for Regular Nilpotent Hessenberg Schubert Cells

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History and introduction

Hessenberg matrices are studied in the field of numerical algorithms, where they correspond to dynamical systems evolving on subvarieties of the full flag variety $\mathbf{GL}_n(\mathbf{C})/B$. In the early 90's, De Mari, Procesi, and Shayman, generalized this notion to Hessenberg varieties: subvarieties of G/B for an arbitrary complex semisimple algebraic group G [3]. Hessenberg varieties arise throughout mathematics, not only in algebraic geometry, but also in combinatorics, cohomology, representation theory, and other areas.

Tymoczko showed in 2007 that Hessenberg varieties associated to a regular nilpotent operator are paved by affines [5]. That a variety is paved by affines is generally used to compute cohomology, in particular, a variety that is paved by affines has no odd cohomology. In this poster, we recover — in type A — this result of Tymoczko via a computational approach.

We translate the results of Da Silva and Harada [2] to the setting of intersections of regular nilpotent Hessenberg varieties and Schubert cells, which allows us to construct Gröbner bases for their local defining ideals. From the Gröbner bases, we compute the Hilbert series of these "Hessenberg Schubert cell ideals", conclude that they are complete intersections, and pave the whole regular nilpotent Hessenberg varieties with these cells.

Hessenberg varieties

Flag varieties are sequences of nested vector subsequences

$$\mathrm{Flags}(\mathbf{C}^n) \coloneqq \left\{(\{0\} = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = \mathbf{C}^n) \mid \dim V_i = i \text{ for all } i\right\},$$

and we make the identification $\mathrm{Flags}(\mathbf{C}^n) \cong \mathbf{GL}_n(\mathbf{C})/B$, where B denotes the subgroup of invertible upper triangular matrices. Hessenberg varieties are defined in three easy steps:

1. An indecomposable **Hessenberg function** is an increasing map $h : \{1,\dots,n\} \to \{1,\dots,n\}$ such that $h(i) \geq i+1$ for all $i = 1,\dots,n-1$.
2. The **Hessenberg space** associated to h is the vector space span of elementary matrices
$$H(h) \coloneqq \left\{E_{i,j} \mid i \geq h(j)\right\}.$$
This gives us a vanishing condition!
3. To h and a linear operator A on $\mathbf{C}^n$ we associate the **Hessenberg variety**, the set of cosets
$$\mathrm{Hess}(A,h) \coloneqq \{MB \in \mathbf{GL}_n(\mathbf{C})/B \mid M^{-1}AM \in H(h)\} \subseteq \mathbf{GL}_n(\mathbf{C})/B \cong \mathrm{Flags}(\mathbf{C}^n).$$

In this poster, we restrict to the case where our linear operator is regular and nilpotent. Explicitly, for our computations, we will take $A = \mathsf{N}$ to be the matrix of 1's on the first superdiagonal and zeros elsewhere.

Hessenberg varieties can defined in other Lie types, but for this poster we restrict to type A.

Local defining equations

For any permutation w, construct a matrix M_w from the permutation matrix of w and placing an indeterminate $x_{i,j}$ whenever the (i,j)-th spot is to the left of a 1, that is, whenever $j < w^{-1}(i)$.

Each M_w is a *coordinate chart* of $\mathrm{Flags}(\mathbf{C}^n)$, and moreover, the collection $\{M_w\}_{w \in S_n}$ forms an open cover of $\mathrm{Flags}(\mathbf{C}^n)$. The corresponding coordinate ring is isomorphic to the polynomial ring $\mathbf{C}[\mathbf{x}_w]$, where $\mathbf{x}_w$ denotes the collection of indeterminates appearing in M_w.

Given a permutation w and Hessenberg function h, we define the **Hessenberg patch ideal** $I_{w,h}$ from the vanishing condition given by the Hessenberg space. That is, if $f^w_{k,\ell}$ denotes the (k,ℓ)-th entry of $M_w^{-1}\mathsf{N}M_w$, then $I_{w,h} \coloneqq \langle f^w_{k,\ell} \mid k > h(\ell)\rangle \subseteq \mathbf{C}[\mathbf{x}_w]$.

It is computationally difficult to study these ideals as a convenient monomial order likely varies between coordinate charts. We study the intersection of these charts with Hessenberg Schubert cells, allowing us to translate known computational results via an embedding into the intersection.

The w_0 chart

Da Silva and Harada in [2] initiated a study of the Gröbner geometry of Hessenberg patch ideals in the case where $w = w_0$, the longest permutation in Bruhat order. In particular, they showed that, for a particular lexicographic monomial order $<$, the polynomials $\left\{f^{w_0}_{k,\ell} \mid k > h(\ell)\right\}$ have distinct initial terms, each of which is an indeterminate. From this, it follows that:

1. $\left\{f^{w_0}_{k,\ell}\right\}$ is a Gröbner basis for $I_{w_0,h}$ with respect to $<$,
2. the initial ideal $\mathrm{in}_<(I_{w_0,h})$ is generated by indeterminates.

The proof of 1 is a direct application of Buchberger's criterion and 2 follows from 1.

Hessenberg Schubert cells

We define a regular nilpotent Hessenberg Schubert cell to be the intersection of a regular nilpotent Hessenberg variety with a Schubert cell $X^w_\circ$. We will drop the "regular nilpotent" prefix and just write Hessenberg Schubert cell from now on.

Computationally, we have a similar open cover to the Hessenberg variety case. For any permutation w, construct a matrix Ω_w from the permutation matrix of w with indeterminates $z_{i,j}$ in the (i,j)-th spot whenever this entry is to the left of a 1 and not below a 1. Explicitly, this occurs whenever $j < w^{-1}(i)$ and $i < w(j)$.

Similarly to the case of Hessenberg varieties, we denote by $g^w_{k,\ell}$ the (k,ℓ)-th entry of $\Omega_w^{-1}\mathsf{N}\Omega_w$ and define the **Hessenberg Schubert cell ideal** by $J_{w,h} \coloneqq \langle g^w_{k,\ell} \mid k > h(\ell)\rangle \subseteq \mathbf{C}[\mathbf{z}_w]$, where $\mathbf{C}[\mathbf{z}_w]$ is the corresponding coordinate ring. This coordinate ring $\mathbf{C}[\mathbf{z}_w]$ is isomorphic to a polynomial ring in the variables $\mathbf{z}_w$ that appear in the matrix Ω_w.

From w_0 to any Hessenberg Schubert cell

Consider the following example where $w = 3421$.

$$\left.\left(\begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} & 1 \\ x_{2,1} & x_{2,2} & 1 & 0 \\ x_{3,1} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\right|_{x_{3,1}=0} = \begin{bmatrix} x_{1,2} & x_{1,1} & 0 & 1 \\ x_{2,2} & x_{2,1} & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \in X^w_\circ$$

By multiplying M_{w_0} by a convenient choice of permutation, which we will denote by v_w, and setting variables to zero, we get an element of the Schubert cell.

Generalizing this example, define $v_w = w_0 w$ and let D_w denote the set of variables that get sent to 0. We formalize the above phenomenon as a ring homomorphism $\psi_w : \mathbf{C}[\mathbf{x}_{w_0}] \to \mathbf{C}[\mathbf{z}_w]$ given by

$$\psi_w(x_{i,j}) = \begin{cases} 0 & \text{if } x_{i,j} \in D_w, \\ z_{i,v_w^{-1}(j)} & \text{otherwise.} \end{cases}$$

By construction we have that $\Omega_w = \psi_w(M_{w_0}v_w)$, where we take the convention that we are applying ψ_w entry-by-entry, and that ψ_w is injective on the domain $\mathbf{C}[\mathbf{z}_w] \setminus D_w$. This translation allows us to express the generators $g^w_{k,\ell}$ for the Hessenberg Schubert cell ideal $J_{w,h}$ in terms of the generators $f^{w_0}_{k,\ell}$ studied by Da Silva and Harada. Indeed,

$$\begin{aligned} g^w_{k,\ell} &= [\Omega_w^{-1}\mathsf{N}\Omega_w]_{k,\ell} = \left[(\psi_w(M_{w_0}v_w))^{-1}\,\mathsf{N}\,(\psi_w(M_{w_0}v_w))\right]_{k,\ell} \\ &= \psi_w\Big(\left[v_w^{-1}(M_{w_0}\mathsf{N}M_{w_0})v_w\right]_{k,\ell}\Big) = \psi_w\Big(f^{w_0}_{v_w(k),v_w(\ell)}\Big). \end{aligned}$$

We define a monomial order $\prec_w$ on $\mathbf{C}[\mathbf{z}_w]$ by $z_{i,j} \succ_w z_{a,b}$ if and only if $\psi_w^{-1}(z_{i,j}) > \psi_w^{-1}(z_{a,b})$, where $<$ is Da Silva and Harada's monomial order on $\mathbf{C}[\mathbf{x}_{w_0}]$.

The natural generators $g^w_{k,\ell}$ are Gröbner

Harnessing the results of Da Silva and Harada via the ψ_w map yields the following result. For any $w \in S_n$, the polynomials $\left\{g^w_{k,\ell} \mid k > h(\ell)\right\}$ have distinct initial terms and form a Gröbner basis for $J_{w,h}$ with respect to $\prec_w$. Hence $\mathrm{in}_{\prec_w}(J_{w,h})$ is an ideal of indeterminates.

Proof: That $g^w_{k,\ell} = \psi_w(f^{w_0}_{v_w(k),v_w(\ell)})$ implies that $\mathrm{in}_{\prec_w}(g^w_{k,\ell}) = \psi_w(\mathrm{in}_<(f^{w_0}_{v_w(k),v_w(\ell)}))$. Then since ψ_w is injective on $\mathbf{C}[\mathbf{x}_{w_0}] \setminus D_w$, the first claim follows upon comparing the explicit definition of D_w to the description of $\mathrm{in}_<(f^{w_0}_{a,b})$ given in [2]. Applying Buchberger's criterion completes the proof. □

Applications

Complete intersections

The initial ideal $\mathrm{in}_{\prec_w}(J_{w,h})$ is generated by indeterminates and is hence a complete intersection. Since these indeterminates each correspond to a distinct generator $g^w_{k,\ell}$ it follows that $J_{w,h}$ is also a complete intersection [4, Proposition 19.3.8].

A little more can be said: the ideals $J_{w,h}$ are a *triangular* complete intersection, meaning that the initial terms are (scalar multiples) of indeterminates and satisfy a divisibility relation.

Paving by affines

An affine paving of an algebraic variety X is an ordered partition $X_0, X_1, X_2 \dots$ such that $X = \bigcup_i X_i$, each X_i is homeomorphic to some $\mathbf{A}^s$, and $\bigcup_{i=0}^r X_i$ is Zariski-closed in X for all r.

That the ideals $J_{w,h}$ are triangular complete intersections implies that Hessenberg Schubert cells are not misnomers, they are isomorphic to affine space. The affine paving of the flag variety $\mathbf{GL}_n(\mathbf{C})/B$ then induces an affine paving of $\mathrm{Hess}(\mathsf{N},h)$ by the set of all Hessenberg Schubert cells.

Hilbert series

Equip $R = \mathbf{C}[\mathbf{z}_w]$ with a nonstandard grading that arises from the circle action on the Hessenberg Schubert cell. Explicitly, $\deg(z_{i,j}) = w(j) - i$ for all $z_{i,j} \in \mathbf{z}_w$. Then the Hilbert series of $R/J_{w,h}$ is given by

$$H_{R/J_{w,h}}(t) = \frac{\displaystyle\prod_{k>h(\ell)} \left(1 - t^{v_w(k)-v_w(\ell)-1}\right)}{\displaystyle\prod_{\substack{i<w(j)\\ j\leq w^{-1}(i)}} \left(1 - t^{w(j)-i}\right)}.$$

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