Gröbner geometry for classes of semisimple Hessenberg varieties

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Outline

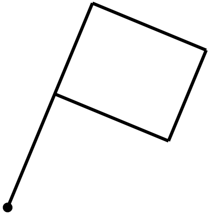
Hessenberg varieties

Hessenberg varieties are subvarieties of the flag variety...

- introduced by works of De Mari, Procesi, and Shayman in the late 1980s, original motivation: **numerical linear algebra**
- connections to **Schubert calculus**, **algebraic combinatorics** (Stanley–Stembridge), **geometric representation theory**...

Their geometry is only well-understood in some cases

Hessenberg varieties



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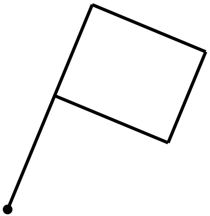

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$$\mathrm{Hess}(A, h) = \left\{ (V_1 \subseteq \cdots \subseteq V_n) \in \mathrm{Flags}(\mathbf{C}^n) \mid AV_i \subseteq V_{h(i)} \right\}$$

where h is a *Hessenberg function*, e.g., $h = (2, 3, 4, 4)$

Local coordinates

$$\mathrm{Flags}(\mathbf{C}^n) \cong {\mathbf{GL}_n(\mathbf{C})}\big/{B}$$

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Patch ideals

Fix a Hessenberg variety $\mathrm{Hess}(A, h)$ and $w \in S_n$. Define

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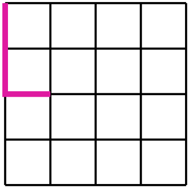

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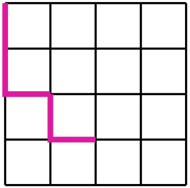

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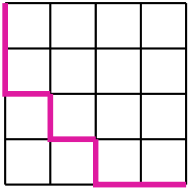

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Why patch ideals?

The use of patch ideals dates to at least the study of Schubert varieties (c1970s)

For Hessenberg varieties:

Year	Authors	Class	Outcome
2012	Insko, Yong	Peterson	Combinatorial description of singular loci
2018	Abe, DeDieu, Galetto, Harada	Regular nilpotent	Local complete intersections; Degree formulae; Newton–Okounkov bodies
2020	Abe, Fujita, Zeng	Regular	Higher cohomology vanishes
2022	Abe, Insko	Regular nilpotent	Singular permutation flags; normality

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What's known for semisimple Hessenberg varieties?

An operator is *semisimple* if it is diagonalizable and *regular* if it has distinct eigenvalues

Theorem (De Mari–Procesi–Shayman '92)

Let S be regular semisimple. Then $\mathrm{Hess}(S, h)$ *is smooth and equidimensional. Moreover,* $\mathrm{Hess}(S, (2, 3, \ldots, n, n))$ *is a toric variety associated to Weyl chambers.*

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- not equidimensional... but we do not have dimension formulae

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Let $S : \mathbf{C}^n \to \mathbf{C}^n$ be semisimple.

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- **Can–Precup–Shareshian–Uğurlu '23+.** Gives a characterization of irreducibility of $\mathrm{Hess}(S, h)$ when S has exactly two eigenvalues and, in the irreducible case, gives a dimension formula.

Semisimple patch ideals

Conjecture (Insko–Precup '19)

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Theorem (Insko–Precup '19)

The conjecture is true when $h = (2, 3, \ldots, n, n)$.

The recursive structure of the generators

Recall that $J_w = \langle f_{k,\ell} \mid k > h(\ell) \rangle$ where $f_{k,\ell} = [(wM)^{-1} S(wM)]_{k,\ell}$. Let

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By induction, $f_{k,\ell}$ is squarefree. So the conjecture is true for $h = (n-1, n, \ldots, n)$.

The “nearly-regular” semisimple case

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Theorem (C. ‘24)

Suppose $S : \mathbf{C}^n \to \mathbf{C}^n$ is semisimple with exactly $n-1$ eigenvalues. Then Insko–Precup’s conjecture is true for $\mathrm{Hess}(S, h)$*: the ideal J_w is radical for every w.*

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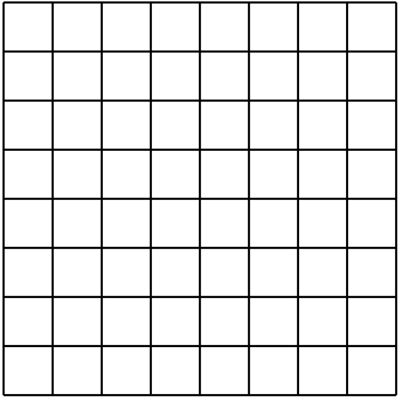
- The initial term of at most one generator $f_{a,b}$ is a product involving the initial term of other generators. Can use the other generators to replace $f_{a,b}$ with a squarefree polynomial whose lead term is in distinct variables. ∎

The initial ideal

Example. $h = (3, 6, 6, 6, 7, 7, 8, 8)$



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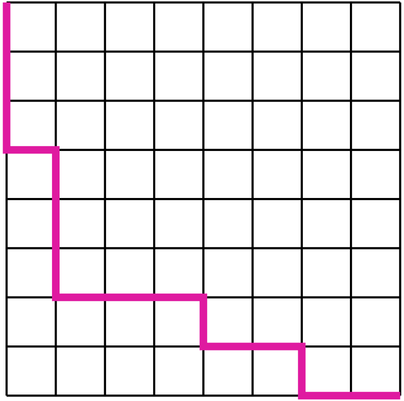

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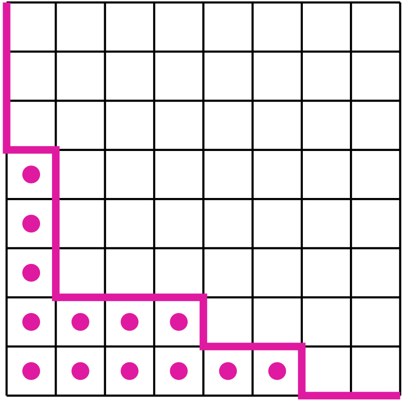

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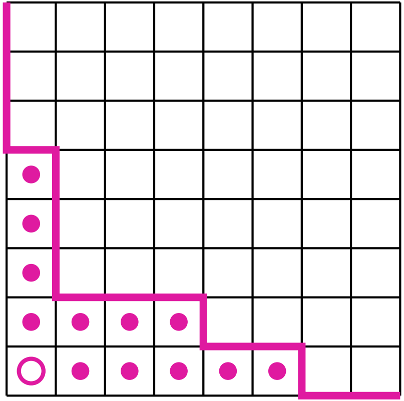

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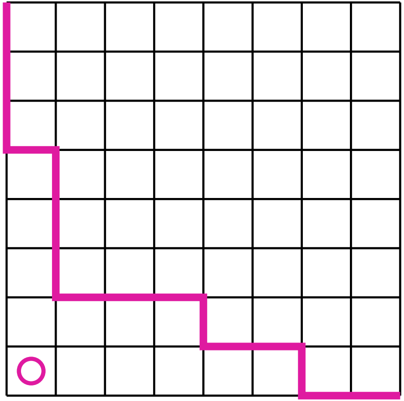

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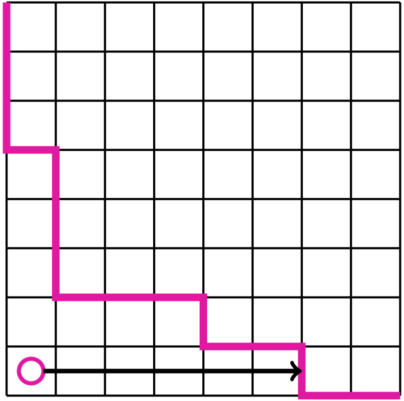

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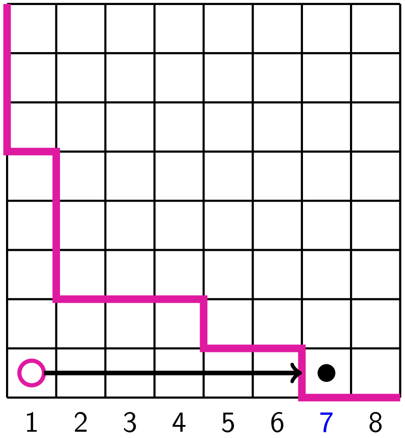

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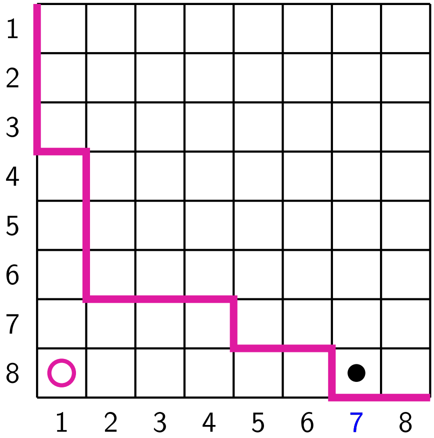

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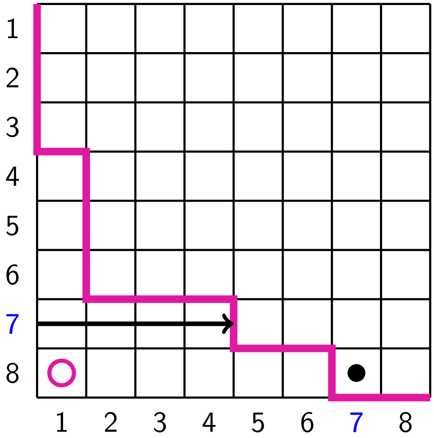

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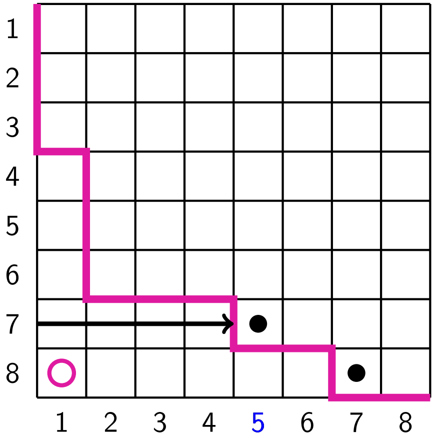

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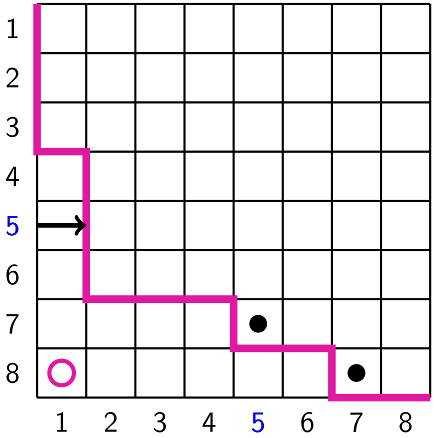

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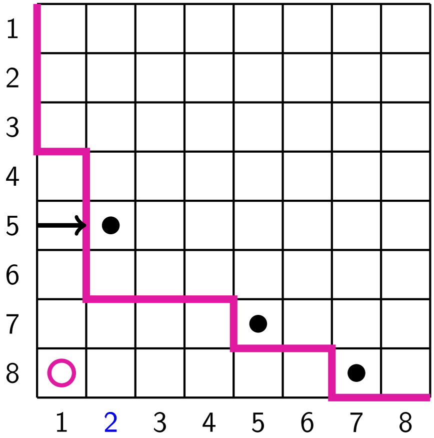
Example. $h = (3, 6, 6, 6, 7, 7, 8, 8)$



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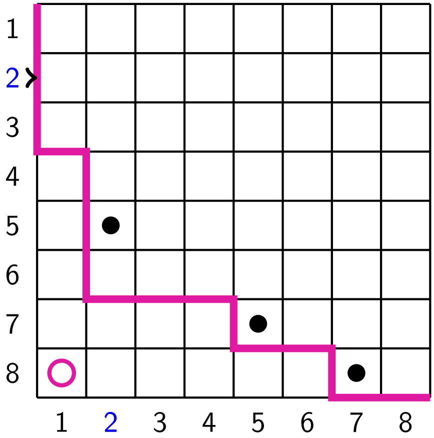

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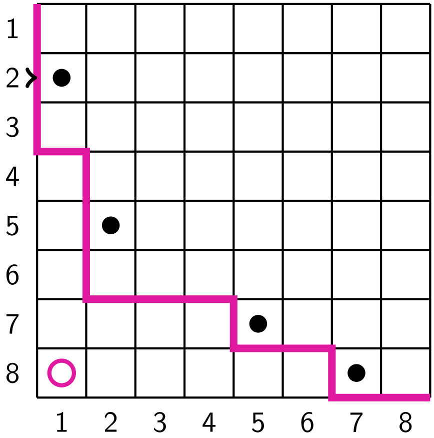

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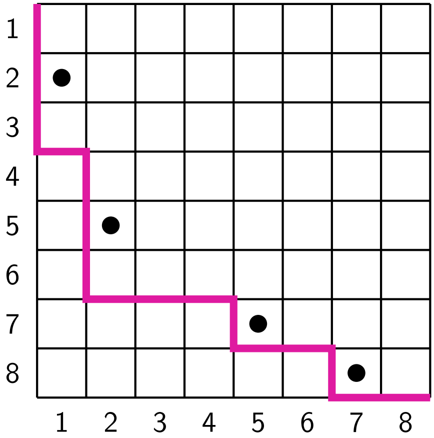

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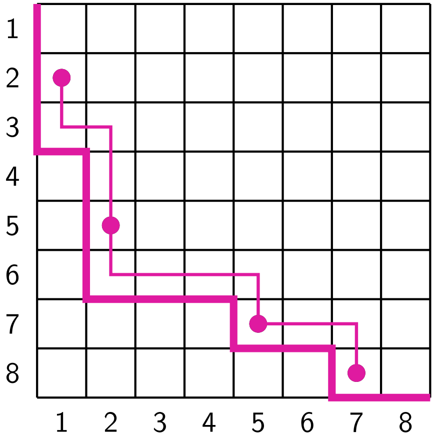

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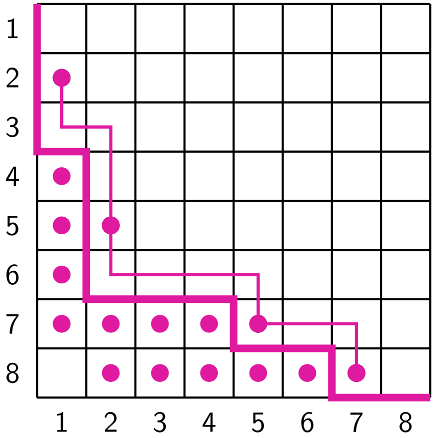

Further consequences & questions

In the “nearly-regular” semisimple case:

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- conjectured characterization of irreducibility—and give a framework to prove this

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We also conjecture a strengthening of Insko–Precup's conjecture.

Conjecture (C. '24)

Consider the semisimple Hessenberg variety $\mathrm{Hess}(S, h)$. *For each* w, *there is a Frobenius splitting that compatibly splits* J_w.

Summary

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