Algebra: What Comes Next?

(In Math 4GR3 and Math 4ET3)

Mike Cummings

Math 3GR3, Tutorial 12
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Slides available at:
math.mcmaster.ca/∼cummim5/teaching/2023/3GR3

Outline

Simple groups

Let G be a group. Recall:

- a subgroup N of G is **normal** if $gN = Ng$ for all g in G
- G is **simple** if it has no nontrivial normal subgroups

Examples of simple groups include:

- the alternating group A_n for $n \geq 5$
- $\mathbb{Z}_p$ for any prime p

Classification of finite simple groups

Theorem

Any finite simple group either is in one of the following inifinite families,

1. $\mathbb{Z}_p$,
2. A_n,
3. *a group of Lie type,*
4. *a derivative of a group of Lie type,*

or is one of 26 "sporadic groups" (such as the Monster)

Timeline of the classification

1832	Galois introduced normal subgroups, finds A_n
1872	Sylow Theorems proved
1892	Hölder asks for a classification of finite simple groups
1893	Cole classifies simple groups up to order 660

Work continued throughout the 1900s and culminated in 2004

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Some mathematicians who worked on this problem include:

Galois, Sylow, Hölder, Cole, Jordan,
Frobenius, Dickson, Burnside, Conway, Gorenstein, Harada

Finitely generated groups

The group $\mathbb{Z} \times \mathbb{Z}_2$ is not cyclic, but it is *finitely generated*,

$$\mathbb{Z} \times \mathbb{Z}_2 = \langle (1,0), (0,1) \rangle$$

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Theorem (Fundamental Theorem of Finitely Generated Groups)

Any finitely generated group is isomorphic to

$$\mathbb{Z}^t \times \mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times \cdots \times \mathbb{Z}_{p_s^{r_s}}$$

for some primes $p_1 \ldots, p_s$ *and positive integer powers* $t, r_1, \ldots, r_s$

How?

- group actions
- Class equation
- Burnside's lemma
- composition series
- p-groups and Sylow theorems

PIDs

An integral domain R is a **principal ideal domain (PID)** if every ideal I of R can be generated by a single element

Example

- $\mathbb{Z}$, $\mathbb{Z}_n$
- any field
- $\mathbb{R}[x]$

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Example

- $\mathbb{Z}$, $\mathbb{Z}_n$
- any field
- $\mathbb{R}[x]$ but not $\mathbb{Z}[x]$

Irreducible and prime elements

Let R be an integral domain

- a **unit** in R is an element with a multiplicative inverse
- an non-zero and non-unit element $a \in R$ is **irreducible** if $a = bc$ implies that either b or c is a unit
- elements a and b in R are **associates** if $a = ub$ for a unit u

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Example

- In $\mathbb{R}[x]$, the elements x and $2x$ are associates and irreducible
- In $\mathbb{Z}$, prime numbers are irreducible

UFDs

Theorem (Fundamental Theorem of Arithmetic)

1. *Any positive integer can be written as a product of primes*
2. *This product is unique up to reordering*

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Definition

An integral domain is a **unique factorization domain (UFD)** if

1. any element can be written as a product of irreducibles
2. this product is unique up to reordering and associates

PIDs and UFDs

Theorem

$PID \implies UFD$

PIDs and UFDs

Theorem

PID $\implies$ UFD, but the converse is false

Example

$$\mathbb{Z}\left[\frac{1+\sqrt{-19}}{2}\right] \quad \text{is a UFD but not a PID}$$

Other classes of rings

- Euclidean domains: integral domains with a division algorithm

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Varieties

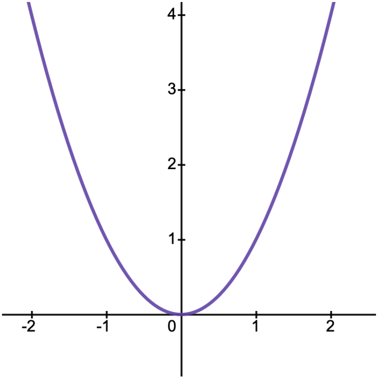
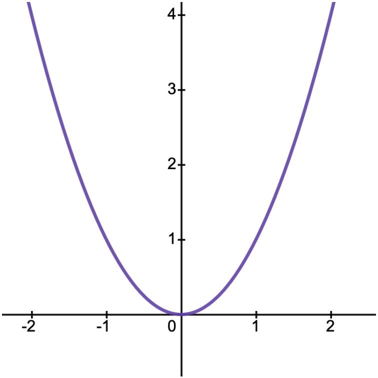


Figure: $\mathbb{V}(y - x^2)$

Varieties



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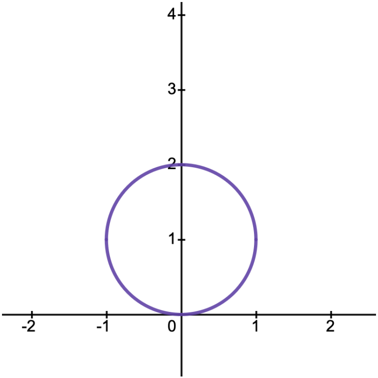

Figure: $\mathbb{V}\big(x^2 + (y-1)^2 - 1\big)$

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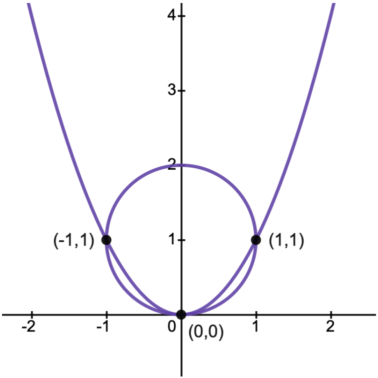

$$\mathbb{V}\left(y - x^2, x^2 + (y-1)^2 - 1\right)$$
$$= \left\{(0,0), (\pm 1, 1)\right\} \subseteq \mathbb{C}^2$$

Varieties $\leftrightarrow$ Ideals

$$V = \mathbb{V}(f_1, \ldots, f_r) \subseteq \mathbb{C}^n$$

$$\Updownarrow$$

$$\mathbb{I}(V) = \langle f_1, \ldots, f_r \rangle \subseteq \mathbb{C}[x_1, \ldots, x_n]$$

Coordinate ring

Let $V \subseteq \mathbb{C}^n$ be a variety

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- $f(x, y) = 0$
- $g(x, y) = y - x^2$
- $h(x, y) = x^2 + y^2 - 1$

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- $g(x, y) = y - x^2 = y + y^2 - 1$ on V
- $h(x, y) = x^2 + y^2 - 1 = 0$ on V, since $h \in \mathbb{I}(V)$

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Let $V \subseteq \mathbb{C}^n$ be a variety

Define a homomorphism of rings

$$\begin{aligned} \varphi : \mathbb{C}[x_1, \ldots, x_n] &\to \mathbb{C}[x_1, \ldots, x_n] \\ \varphi(f) &= f|_V \end{aligned}$$

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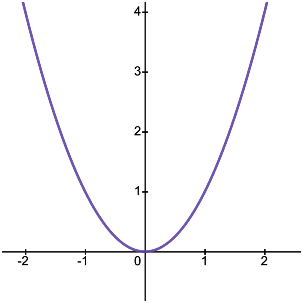
$$\mathbb{C}[V] \cong \mathbb{C}[x_1, \ldots, x_n]/\mathbb{I}(V)$$

Algebra "sees" geometry



$$V = \mathbb{V}(y - x^2) \subseteq \mathbb{C}^2 \qquad \mathbb{C}[V] \cong \mathbb{C}[x,y]/\langle y - x^2 \rangle \cong \mathbb{C}[x]$$

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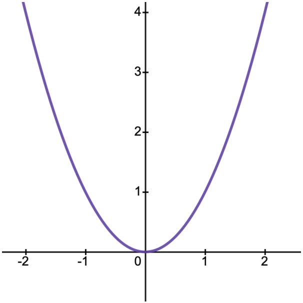

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V "irreducible" $\qquad$ $\mathbb{I}(V)$ prime $\qquad$ $\mathbb{C}[V]$ integral domain

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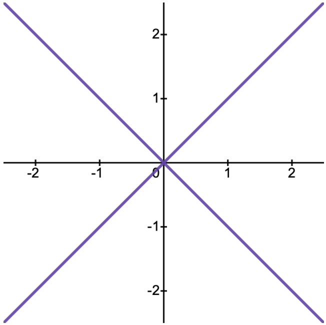

$$V = \mathbb{V}\big((y-x)(y+x)\big) \qquad \mathbb{C}[V] \cong \mathbb{C}[x,y]/\langle y^2 - x^2\rangle$$

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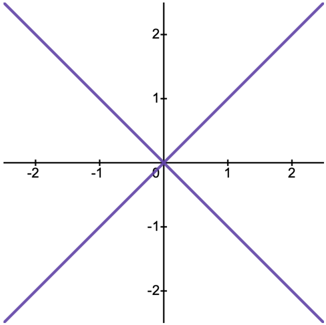

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	$\mathbb{I}(V)$ not prime	$\mathbb{C}[V]$ not an integral domain
V "reducible"	$y - x \notin \mathbb{I}(V)$	
	$y + x \notin \mathbb{I}(V)$	$(y-x)(y+x) \in \mathbb{I}(X)$

Morphisms and isomorphisms

Let $V \subseteq \mathbb{C}^n$ and $W \subseteq \mathbb{C}^m$ be varieties

A **map of varieties** $\varphi : V \to W$ is of the form

$$\varphi(a_1, \dots, a_n) = \big(f_1(a_1, \dots, a_n), \dots, f_m(a_1, \dots, a_n)\big)$$

where each f_i is a polynomial in n variables

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An **isomorphism** is a bijective map that admits an inverse; we say that V and W are **isomorphic**

Example of an isomorphism

$$V = \mathbb{V}(0) = \mathbb{C} \qquad\qquad W = \mathbb{V}(y - x^2) \subseteq \mathbb{C}^2$$

$$\varphi : V \to W \qquad\qquad \psi : W \to V$$

$$\varphi(t) = (t, t^2) \qquad\qquad \psi(u, v) = u$$

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$$\mathbb{C}[V] \cong \frac{\mathbb{C}[t]}{\langle 0 \rangle} = \mathbb{C}[t] \qquad \mathbb{C}[W] \cong \frac{\mathbb{C}[x, y]}{\langle y - x^2 \rangle} \cong \mathbb{C}[x]$$

Isomorphisms of varieties and coordinate rings

Theorem

Let $V \subseteq \mathbb{C}^n$ *and* $W \subseteq \mathbb{C}^m$ *be varieties*

$$V \cong W \quad \textit{if and only if} \quad \mathbb{C}[V] \cong \mathbb{C}[W]$$

Nodal cubic

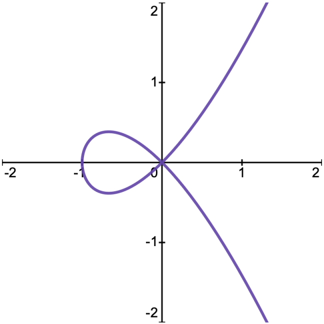

Figure: $V = \mathbb{V}(y^2 - x^3 - x^2)$

Question: $V \cong \mathbb{C}$?

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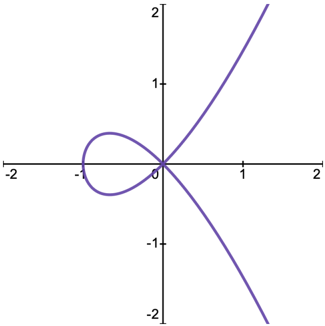

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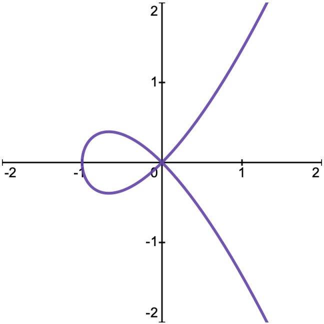

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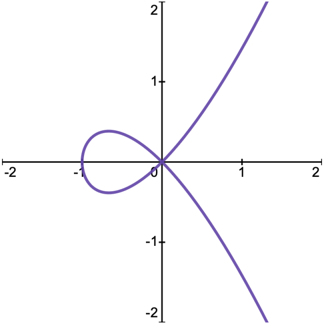


Figure: $V = \mathbb{V}(y^2 - x^3 - x^2)$

Question: $V \cong \mathbb{C}$?

$$\mathbb{C}[V] \cong \frac{\mathbb{C}[x,y]}{\langle y^2 - x^3 - x^2 \rangle} \ncong \mathbb{C}[t]$$

In $\mathbb{C}[V]$,

$$\begin{aligned} y^2 &= y \cdot y \\ &= x^2(x+1) \end{aligned}$$

$\mathbb{C}[V]$ is not a UFD, but $\mathbb{C}[t]$ is

So $V \ncong \mathbb{C}$

Twisted cubic

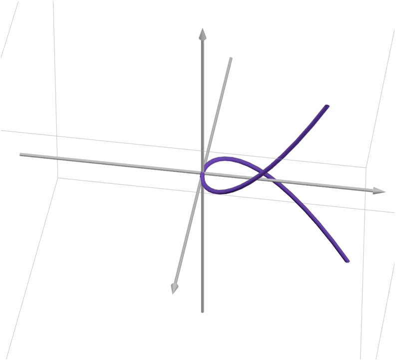
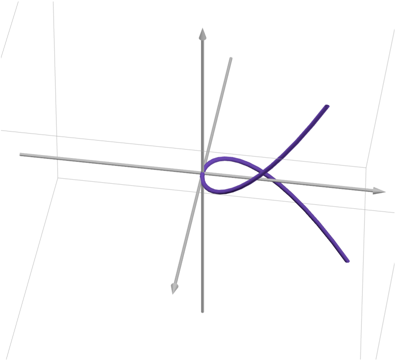


Figure: $V = \mathbb{V}(y - x^2, z - x^3)$

Desmos 3D Link

Twisted cubic



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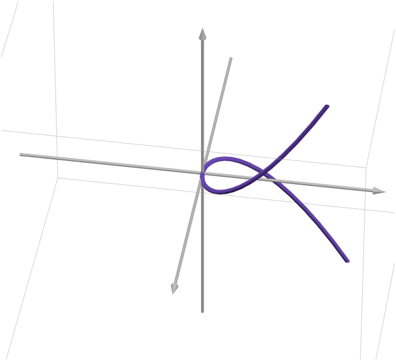

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$$V \cong \mathbb{C}$$

$$\begin{aligned}\mathbb{C}[V] &\cong \frac{\mathbb{C}[x,y,z]}{\langle y - x^2, z - x^3 \rangle} \\ &\cong \mathbb{C}[x, x^2, x^3] \\ &\cong \mathbb{C}[x]\end{aligned}$$

Cuspidal cubic

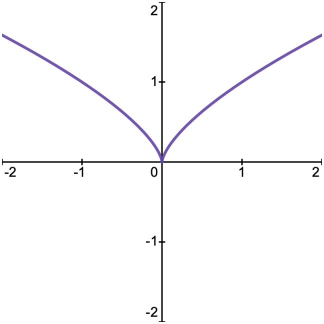

Figure: $V = \mathbb{V}(y^3 - x^2)$

$$V \ncong \mathbb{C}$$

$$\mathbb{C}[V] \cong \frac{\mathbb{C}[x,y]}{\langle y^3 - x^2 \rangle}$$

Not a UFD:

$$y^3 = y \cdot y \cdot y = x \cdot x$$

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- Schubert calculus

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Connections to number theory

- Arithmetic and elliptic curves

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Computational algebraic geometry (Math 4ET3)

- Gröbner bases
- degeneration (initial ideals)

Reading

Groups and Rings

- Judson, Chapters 13–15, 17, 18, 21
- Dummit and Foote. Abstract Algebra

Algebraic Geometry

- Karen Smith et al. Invitation to Algebraic Geometry
- Cox, Little, and O'Shea. Ideals, Varieties, and Algorithms