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Finding mixed strategy Nash equilibria with decision trees

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ABSTRACT

This paper describes the usefulness of decision tree models for determining mixed strategy Nash equilibria in normal form games, particularly to undergraduate students. The approach is to construct a decision tree for each player, then solve the model via dynamic programming to determine the equations that must be satisfied at Nash equilibrium. This method not only provides a computational device that can be used to calculate the Nash equilibrium, but also serves as a visual aid that helps students understand the Nash equilibrium concept.

1. Introduction

As instructors, we search for alternative methods of explaining and solving problems that are difficult for our students to understand, and for ways to integrate their learning experience across related fields. This is our motivation for using decision trees to teach mixed strategy Nash equilibria.

With practice, most students comprehend techniques for finding pure strategy Nash equilibria in normal form games. However, students find it significantly more difficult to find mixed strategy Nash equilibrium solutions to normal form games. The decision tree method we describe in this paper provides instructors an option for explaining mixed strategies to students, and for training students to find mixed strategy Nash equilibrium (heretofore MSNE) solutions to normal form games.

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A typical undergraduate economics textbook presents normal form games with both pure strategy and mixed strategy equilibrium solutions (see, for example, Frank, 2010; Pindyck and Rubinfeld, 2012; Varian, 2009). These textbook solutions are typically purely algebraic (and not graphical) solutions and are often not intuitive to the student. Other approaches, such as that of taking the derivative of the player's (linear) expectation function (Baldani et al., 2005), seem even less intuitive than the typical algebraic solution. This experience seems not to be limited to our students alone (see, for example, Garrett and Moore, 2008). For this reason, we designed a decision technique to solve for a MSNE that allows students to visualise the solution.

The decision tree methodology we propose has the advantage that it does not require students to have a sophisticated mathematical background. It is intended for those students who are learning about a mixed strategy equilibrium for the first time. These are students who are in an upper-level undergraduate economics or business course (Microeconomic Theory, Managerial Economics, or Operations Management, for instance). Therefore, the only real mathematical tool they need, that of calculating expected values, should be well within their capability.

An additional benefit of this method is that it is an excellent way to integrate the structured approach to decision analysis, typically taught in operations management or management science courses, with the analysis of strategic behavior, introduced in economics courses.

In the next section we describe how to use decision trees to find a mixed strategy Nash equilibrium, and demonstrate the advantages of using this graphical technique. In Section 3, we present the use of the decision tree methodology for solving a game where some strategies are dominated. Finally, we conclude with a discussion of our classroom implementations of the decision tree method and mention potential applications in the decision sciences.

2. Mixed strategy equilibria in normal form games

It is our experience that although our students are able to comprehend and solve games with a pure strategy Nash equilibrium, the mechanics and intuition of finding a MSNE are harder to master. For this reason, we expose them to the decision tree technique as a way to solve for a MSNE that allows them to visualize the solution. By providing a useful visual tool to illustrate the mixed strategy Nash equilibrium concept to students, we believe that the decision tree approach has an advantage over existing methods for solving simple normal form games. Moreover, it can be extended to more complex games.

2.1. Normal form representation

We begin with a normal form representation of a two-player game $G = \{S_1, S_2; u_1, u_2\}$, defined by the ith player's strategy space S_i and payoff function u_i. A Nash equilibrium occurs where there is no incentive for a player to deviate from the chosen strategy. More formally, the strategies $(s_1{}^*, s_2{}^*)$, where $s_i \in S_i$, are a Nash equilibrium if $u_1(s_1{}^*, s_2{}^*) \geq u_1(s_1, s_2{}^*)$ for all $s_1 \in S_1$ and $u_2(s_1{}^*, s_2{}^*) \geq u_2(s_1{}^*, s_2)$ for all $s_2 \in S_2$.

In order to understand the difference in complexity in solving for a pure strategy versus a mixed strategy Nash equilibrium, consider the two-player, normal form game (Gintis, 2000) in Table 1, which represents the payoffs to each player for each possible strategy pair, with Player 1's payoff listed first. In this game, Player 1 chooses from its strategy set $S_1 = \{L, M, R\}$, while Player 2 chooses from its strategy set $S_2 = \{A, B, C\}$. In the example in Table 1, a strategy s_1 can be interpreted as an assignment of

Table 1
Example of a normal form game.

Player 1	Player 2		
	A	B	C
L	0, 0	4, −4	2, −2
M	1, −1	0, 0	2, −2
R	2, −2	1, −1	0, 0

probabilities p_1, p_2, and p_3 to the strategies L, M, and R, with the requirements that $p_1+p_2+p_3=1$ and $0 \le p_i \le 1$ for $i=1, 2, 3$. Similarly, a strategy s_2 is an assignment of probabilities q_1, q_2, and q_3 to the strategies A, B, and C. If the probability assigned to a single strategy is 1, this choice is a pure strategy.

In the normal form game in Table 1, no pure strategy equilibria exist because for each strategy pair, at least one player can move unilaterally to gain a better payoff. For instance, the strategy pair where Player 1 plays M and Player 2 plays C is not a Nash equilibrium because Player 2 can change to strategy B and earn a better payoff (0 versus −2). Since no pure strategy Nash equilibria exist, a mixed strategy assignment for each player must be found to solve the game.

2.2. Solving for a mixed strategy Nash equilibria

We now illustrate the decision tree approach to finding a MSNE. In our experience, the graphical decision tree representation makes it easier for students to understand and solve these kinds of problems. Please note that decision trees are used here to examine simultaneous move games. Decision trees appear similar to game trees, which are often used to model sequential move games. We recommend the use of this method of solving for MSNE for students who are being exposed to simultaneous move games for the first time, and so have not yet seen a sequential game. Therefore, the use of decision trees to solve simultaneous move games should not be confusing. Furthermore, in simultaneous move games, each player *must* select and implement his or her strategy *without* having observed the choice of the other player. Therefore the student cannot solve for a MSNE as he or she would a sequential move game. However, it may be useful to address this issue explicitly whenever warranted.

The idea behind solving for a MSNE is that each player 'mixes' her strategies so as to make the other player indifferent to his strategies, and vice versa. So, using the numbers in Table 1, Player 1 would mix her strategies so as to make Player 2 indifferent to choices A, B, C. Specifically, Player 1 chooses p_i such that Player 2 receives the same payoff (expected value) from choosing A, B or C, with probabilities q_i, and so is indifferent between these choices. Similarly, Player 2 chooses q_i such that player 1 receives the same payoff (expected value) from choosing L, M or R, with probabilities p_i, with the same goal. As an instructor, one can already see how this process can be intimidating to the student. Students are usually so focused on the mathematics that they do not understand the intuition behind the process.

Since a strategy for each player in a normal form game can be interpreted as an assignment of probabilities to the elements of the strategy space, a decision tree can be constructed to graphically depict how the strategies of each player affect the outcomes of the game. Fig. 1 shows decision tree models for the normal form game with payoffs from Table 1.

The decision tree on the left models Player 1's payoffs. Table 1 shows that Player 1 chooses her strategies L, M, and R with probabilities p_1, p_2, and p_3, respectively. So, when constructing Player 1's tree (on left), the student first draws the tree outlining these choices, placing probabilities on each branch. Once this step is complete, it becomes *visually obvious* that once Player 1 plays L with probability p_1, M with probability p_2, *or* R with probability p_3, we at are a chance node where Player 2 will choose to play A, B or C. It then becomes a matter of reading the table to fill in the rest of the tree: If we are at the upper-most branch where Player 1 has chosen L, and Player 2 has chosen A, Player 1 will receive a payoff of 0. The probability of Player 2 choosing A (q_1) is on the branch that leads to this payoff. Using the same logic, the student then fills in the rest of the tree for Player 1. She then uses the exact same method for creating Player 2's decision tree. The decision tree on the right similarly represents Player 2's payoffs in the game. The chance nodes on the right in each tree are solid to indicate that each player's decision occurs simultaneously with the other player's decision.

Once the decision trees are constructed, they are solved by backward induction (or dynamic programming) to determine the Nash equilibrium strategies in the game. This is done by calculating expected values at each chance node. Fig. 2 shows decision tree models for the normal form game with payoffs in Table 1 rolled back one level. As an example, the expected value calculation using the payoffs and probabilities at the top right chance node in Player 1's decision tree is $EV_1(q_1, q_2, q_3)=0q_1+4q_2+2q_3=4q_2+2q_3$.

Player 2 selects his strategies using the rolled back version of Player 1's decision tree. Solving for a MSNE means that Player 2 must select his strategies such that Player 1 is indifferent to any possible

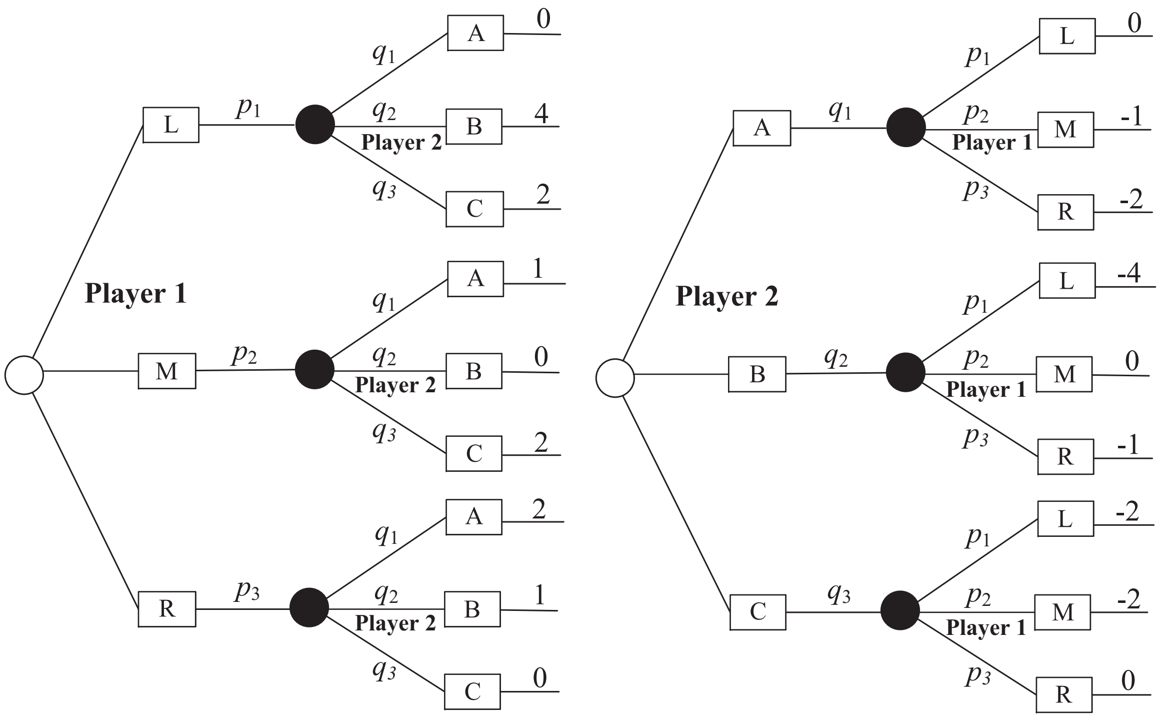

Fig. 1. Decision tree models for the normal form game.

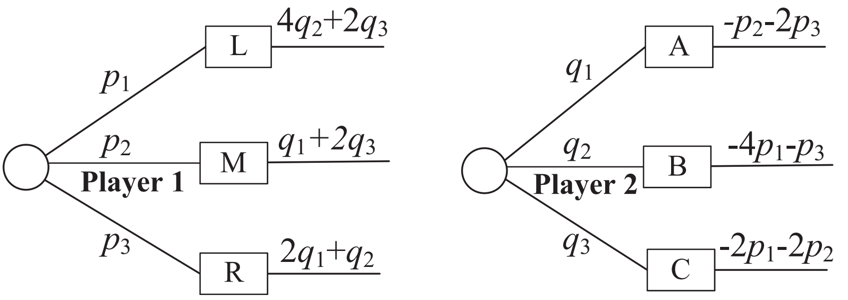

Fig. 2. Decision tree models for the normal form game rolled back one level.

assignment of her strategies. This occurs when the expected values at each endpoint of the decision tree in the left panel of Fig. 2 are equal, or when:

$$4q_2 + 2q_3 = q_1 + 2q_3 = 2q_1 + q_2.$$

This expression can be equivalently represented by the two equations:

$$4q_2 + 2q_3 = q_1 + 2q_3 \textit{ and } 4q_2 + 2q_3 = 2q_1 + q_2.$$

Since these strategies also must meet the requirement that $q_1+q_2+q_3=1$, the Nash equilibrium can be found by solving the following system of equations:

$$4q_2 + 2q_3 = q_1 + 2q_3$$

$$4q_2 + 2q_3 = 2q_1 + q_2$$

$$q_1 + q_2 + q_3 = 1.$$

The solutions are $q_1=8/15$, $q_2=2/15$, and $q_3=1/3$.

Since the expected values to Player 1 from choosing L, M and R are, respectively, $4q_2+2q_3=4(2/15)+2(1/3)=6/5$, $q_1+2q_3=8/15+2(1/3)=6/5$, and $2q_1+q_2=2(8/15)+2/15=6/5$, she is indifferent to playing any particular pure strategy from among L, M, and R. This also means that Player 1 is indifferent between any of the possible assignments to the elements p_1, p_2, and p_3 of its mixed strategy.

Similarly, using the expected values in the decision tree in the right panel of Fig. 2, Player 1 solves the following system of equations to find her Nash equilibrium strategy

$$-p_2 - p_3 = -4p_1 - p_3$$

$$-4p_1 - p_3 = -2p_1 - 2p_2$$

$$p_1 + p_2 + p_3 = 1.$$

The solutions are $p_1=1/5$, $p_2=2/5$, and $p_3=2/5$. Gintis (2000) verifies that the mixed strategies found in this example are a MSNE. Obviously, the payoffs in the game must be constructed so that $0<p_i<1$ for $i=1, 2, 3$, and $0<q_j<1$ for $j=1, 2, 3$ or the solution is not a MSNE. In the next section, we will address an example where this condition does not hold.

Note that using the decision tree methodology to solve for a MSNE does not mean that the students avoid the algebra involved. Indeed, it is our goal to make the process more tractable, not less rigorous. As Figs. 1 and 2 illustrate, the decision trees allow students to have confidence that they are calculating the correct equations needed to solve the game. The equations are not written down after simply examining a payoff table, but are actually derived directly from the decision tree solution. In other words, the process of solving the decision tree actually allows students to identify the correct equations. The real contribution of this method, in our experience, is that for the student, the decision tree provides a visual road map of the process that leads to the MSNE.

Garrett and Moore (2008) describe a method to assist students in comprehending MSNE that involves augmenting the normal form of the game with a new column and row with an arbitrarily chosen mixed strategy. Garrett and Moore use the method to train students to understand the concept of a payoff as an expected value, and then teach students to construct and solve the equations by writing them down by directly analyzing the normal form. Our reason for proposing the decision tree method is the same as theirs – the intuition it provides those students who may have difficulty grasping the mixed strategy concept. However, one added advantage of our method is that the process of solving the decision tree actually allows students to correctly identify the equations that yield the MSNE mixed strategies. The equations are developed by the setup of the decision trees, and not by simply examining the table. The direct calculation of the expected values, then, is a "relative advantage" of our paper. An instructor could use the decision tree method to introduce mixed strategies in the classroom. The Garrett and Moore method could then serve as a complement for an instructor who needed to explain the concept further to a struggling student.

3. Games with dominated strategies

In some cases, using the decision tree methodology yields solutions with probabilities that are negative or greater than one. Such results indicate that a MSNE may not exist, or that a strategy for one or both of the players is dominated. In this section, we illustrate the use of decision trees in solving a problem that does not contain a single Nash equilibrium in purely mixed strategies.

Suppose the payoffs in Table 1 for Player 1 playing *M* and Player 2 playing *C* are each changed to zero. The decision trees from Fig. 2 would be replaced by those shown in Fig. 3.

Applying the technique in the previous section and solving the resulting equations yields the solutions $p_1 = 1/5$, $p_2 = 6/5$, and $p_3 = -2/5$ for Player 1 and $q_1 = 2/5$, $q_2 = -2/5$, and $q_3 = 1$ for Player 2. Such invalid probability assignments indicate that one or both of the players has strategies that are dominated. In this situation, the expected values calculated when rolling back the decision trees are useful for identifying these strategies.

Examining the expected values in Fig. 3 reveals that strategy *M* for Player 1 is dominated by strategy *R* because $2q_1 + q_2 \geq q_1$ for any strategy assignment chosen by Player 2. Similarly, for Player 2, strategy *B* is dominated by strategy *C* because $-2p_1 \geq -4p_1 - p_1$ for any strategy assignment selected by Player 1. Thus, we can remove strategies *M* and *B* from consideration when determining the Nash equilibrium. The new decision trees for the problem with these dominated strategies removed are shown in Fig. 4.

Rolling back the trees in Fig. 4 reveals that the Nash equilibrium will meet the conditions $2 - 2q = 2q$ and $-2 + 2p = -2p$. The solutions are $p = 0.5$ and $q = 0.5$. This example shows that the decision tree technique can be used to identify the possible existence of MSNE where only some of the pure strategies are represented.

Please note that finding the solutions to the equations is simply one way to identify the fact that a strategy is dominated. We are not claiming that this is a superior method for finding a dominated

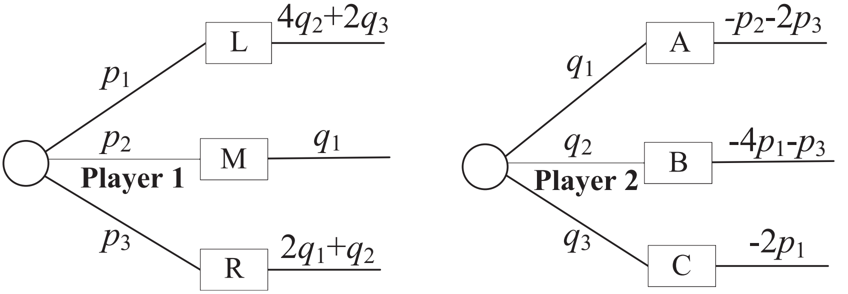

Fig. 3. Decision tree models for the revised game.

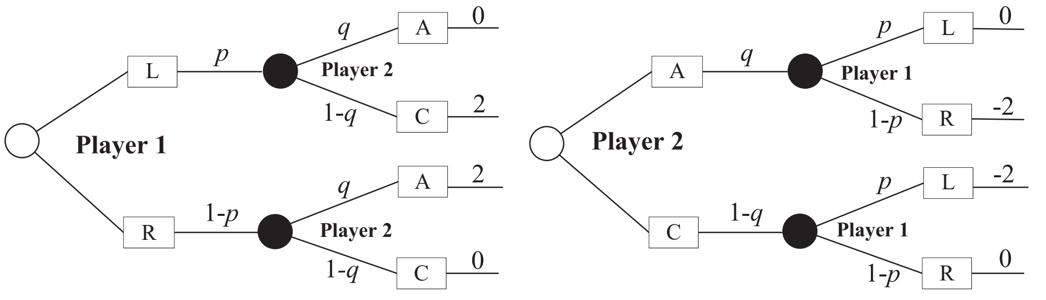

Fig. 4. Decision tree models with dominated strategies removed.

strategy. Any method can be used to find the dominated strategy, after which the decision trees can be set up to solve the game as previously discussed.

The decision tree method also works well for problems in which the two players have different numbers of choices in their strategy sets. For example, we have applied our decision tree technique to solve each of the examples used by Magirou et al. (2008) to demonstrate their graphical technique for solving normal form games.

4. Discussion

In this paper, we introduce the use of decision tree models for determining mixed strategy Nash equilibria in normal form games. The approach is to construct decision trees for each player. A player then rolls back the opponent's decision tree to determine the equations required for the MSNE solutions. Our motivation for applying this technique is to provide students with an alternative approach to formulating the required equations needed to determine mixed strategy equilibria. As instructors, we find that when students understand alternative approaches to solving the same problem, they gain confidence and a better understanding of the underlying concepts.

We utilized this technique in junior and senior-level Intermediate Microeconomics and Managerial Economics courses. We typically introduce a 2×2 normal-form game with only a MSNE and have students verify that no pure strategy equilibria exist. We then use the decision tree technique to demonstrate how to calculate the MSNE. We assign another problem for the students to work on their own prior to the next class period. Between 70% and 80% of students in each course solved the assigned problem correctly using the decision tree technique. We have only a few sections for each course with small class sizes, so we have not formed a control group or conducted a more formal experiment; however, the results validate our intuition that the decision tree approach can complement the traditional method of constructing the equations directly from the normal form.

From a pedagogical standpoint, the value of introducing the decision tree approach is that it provides a visual model that explains the process for solving for a mixed strategy Nash equilibrium, without requiring a sophisticated quantitative background. If students are having difficulty analysing the strategic form of the game and formulating the correct equations, the instructor can employ the decision tree approach to facilitate the explanation. This may be particularly true if the students are familiar with decision tree models.

Understanding mixed strategy Nash equilibrium is important since it allows students to model and understand the extent to which people with different preferences can cooperate. Our technique allows a more thorough understanding of this concept because students can observe that the Nash equilibrium strategies make each player indifferent between all his or her possible strategies. In other words, they can graphically visualize the process by which we arrive at the Nash equilibrium.

Additionally, since game theory concepts are traditionally taught in economics courses, our decision tree methodology shows that decision analysis tools can be successfully utilized cross-functionally to complement the teaching of other analytical techniques. Recent research shows that decision trees can be used to incorporate game-theoretic thinking into decision analysis (van Binsbergen and Marx, 2007; Cobb and Basuchoudhary, 2009). For example, when modelling a firm's production decision, variables such as the production quantity of a rival firm would ordinarily be modelled as a random variable. However, in some industries, the decision of a firm may affect the production decision of another firm, and vice versa. Game-theoretic concepts can be used to model such interactions, but decision trees can still be used to perform the analysis and find optimal outcomes. To the manager making a decision in an environment where strategic interaction affects outcomes, making a choice that is consistent with a Nash equilibrium solution often means avoiding undesirable outcomes in favor of a solution that is strategically stable.

The decision tree technique, in combination with other methodology, facilitates an increased understanding of a game-theoretic problem. Further, it allows the integration of the structured approach of decision analysis into economics.

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