Econ 702 Game Theory Problem Set 2 Solution

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Exercise 27.1 (Obsorne).(Prisoner's dilemma with altruistic preferences). Each of two players has two possible actions, Quiet and Fink; each action pair results in the players' receiving amounts of money equal to the numbers correspondng to that action pair in Table 1 (For example, if player 1 chooses Quiet and player 2 chooses Fink, then player 1 receives nothing, whereas player 2 receives $3). The players are not "selfish"; rather, the preferences of each player i are represented by the payoff function $m_i(a) + \alpha m_j(a)$, where $m_i(a)$ is the amount of money received by player i when the action profile is a, j is the other player, and α is a given nonnegative number. Player 1's payoff to the action pair (Quiet, Quiet), for example, is $2 + 2\alpha$.

	Quiet	Fink
Quiet	2,2	0,3
Fink	3,0	1,1

Table 1: The Prisoner's Dilemma

1. Formulate a strategic game that models this situation in the case $\alpha = 1$. Is this game the Prisoner's Dilemma?

2. Find the range of values of α for which the game is not the Prisoner's Dilemma, find its Nash equilibria.

Solution. With altruism, the payoff matrix is shown in Table 2.

	Quiet	Fink
Quiet	$2+2\alpha, 2+2\alpha$	$3\alpha, 3$
Fink	$3, 3\alpha$	$1+\alpha, 1+\alpha$

Table 2: Payoff matrix with altruism in the PD game.

1. When $\alpha = 1$, the payoff matrix is shown in Table 3, which is not the Prisoner's Dilemma.

	Quiet	Fink
Quiet	$4,4$	$3,3$
Fink	$3,3$	$2,2$

Table 3: Payoff matrix with altruism in the PD game when $\alpha = 1$.

2. The conditions for the game to be a PD game is

$$3 > 2 + 2\alpha > 1 + \alpha > 3\alpha \tag{1}$$

   which implies that $\alpha < 0.5$. Therefore, for $\alpha \in [0, 0.5)$, the game is the Prisoner's Dilemma. For $\alpha = 0.5$, all the four outcomes are Nash equilibria. For $\alpha > 0.5$, (Quiet, Quiet) is the NE.

□

Exercise 33.1 (Obsorne). (Contributing to a public good) Each of n people chooses whether or not to contribute a fixed amount toward the provision of a public good. The good is provided if and only if at least k people contribute, where $2 \leq k \leq n$; if it is not provided, contributions are not refunded. Each person ranks outcomes from best to worst as follows: (i) any outcome in which the good is provided and she does not contribute, (ii) any outcome in which the good is provided and she contributes, (iii) any outcome in which the good is not provided and she does not contribute, (iv) any outcome in which the good is not provided and she contributes. Formulate this situation as a strategic game and find its Nash equilibria. (Is there a Nash equilibrium in which more than k people contribute? One in which k people contribute? One in which fewer than k people contribute? (Be careful!))

Solution. The strategic game consists of

Players. $N = \{1, \dots, n\}$.

Actions. For player $i \in N$, the set of action $A_i = \{0, 1\}$ where 1 represents Contribute and 0 represents Not Contribute.

Preferences. The preference of any player can be represented by the following utility function

$$u_i(a) = 2 \cdot \mathbb{1}_{\{a | \sum_i a_i \geq k\}}(a) - a_i \tag{2}$$

where $\mathbb{1}$ is the indicator function. Then the payoffs corresponds to the four cases are $2, 1, 0, -1$.

To find the NE, consider those outcomes in the following cases:

- For any a such that $\sum_i a_i > k$, i.e., there are more than k people contribute. For player i such that $a_i = 1$, $u_i(a) = u_i(1, a_{-i}) = 1$ but $u_i(0, a_{-i}) = 2$. Therefore, i is not doing his best response and a is not NE.
- For any a such that $\sum_i a_i = k$. For player i such that $a_i = 1$, we have $u_i(1, a_{-i}) = 1$ and $u_i(0, a_{-i}) = 0$. For player j such that $a_j = 0$, we have $u_j(0, a_{-j}) = 2$ and $u_j(1, a_{-j}) = 1$. Therefore, all the outcomes in this case are Nash equlibria.
- For any a such that $0 < sum_i a_i < k$. For player i such that $a_i = 1$, then $u_i(1, a_{-i}) = -1 < 0 = u_i(0, a_{-i})$ and therefore a is not NE.
- For $a = \mathbf{0}$, $u_i(0, a_{-i}) = 0 > u_i(1, a_{-i}) = -1$. Therefore, $\mathbf{0}$ is a NE.

To conclude, the set of NE is

$$NE = \{a \mid \sum_i a_i = k \text{ or } \sum_i a_i = 0\}$$

□

Exercise 4.16 (MSZ). Find all the equilibria in the following games.

Solution. For Game A, the NE is (α, d) as shown in Table 4. For Game B, β is the dominant strategy for player I and the NE is (β, a). For Game C, the NE are (ϵ, c) and (α, a). □

	a	b	c	d
γ	**7**, 3	**6**,3	5,5	4, **7**
β	4,2	5, **8**	**8**,6	5, **8**
α	6,1	3, 8	2, 4	**6**, **9**

Table 4: Game A

	a	b	c	d
δ	5,2	3,1	2,2	4, 5
γ	0,3	2,2	0,1	-1, 3
β	8,**4**	7,0	6, -1	5,2
α	1,1	0,0	-1, -1	0,0

Table 5: Game B

	a	b	c	d
ϵ	0,0	-1, 1	1, 1	0,-1
δ	1,-1	1,0	0,1	0,0
γ	0,1	-1,-1	1,0	1,-1
β	-1,1	0,-1	-1,1	0,0
α	1,1	0,0	-1-1	0,0

Table 6: Game C

Exercise 4.17 (MSZ). In the following three-player game, Player I chooses a row (A or B), Player II chooses a column (a or b), and Player III chooses a matrix α, β or γ. Find all the equilibria of this game.

α:

	a	b
A	0, 0, 5	0, 0, 0
B	2, 0, 0	0, 0, 0

β:

	a	b
A	1, 2, 3	0, 0, 0
B	0, 0, 0	1, 2, 3

γ:

	a	b
A	0, 0, 0	0, 0, 0
B	0, 5, 0	0, 0, 4

Solution. The NE are (A, b, α), (B, a, α), (A, b, γ) and (B, a, γ) as shown above. □

Exercise 4.18 (MSZ). Find the equilibria of the following three-player game (Player I chooses row T , C, or B, Player II a column L, M, or R, and Player III chooses matrix P or Q).

P:

	L	M	R
T	3, 10, 8	8, 14, 6	4, 12, 7
C	4, 7, 2	5, 5, 2	2, 2, 8
B	3, −5, 0	0, 3, 4	−3, 5, 0

Q:

	L	M	R
T	4, 9, 3	7, 8, 10	5, 7, −1
C	3, 4, 5	17, 3, 12	3, 5, 2
B	9, 7, 2	20, 0, 13	0, 15, 0

Solution. There is no pure-strategy NE. □

Exercise 2. Consider the following game as shown in Table 7.

	A	B
A	9,9	0,8
B	8,0	7,7

Table 7: Payoff matrix

1. List all the Nash equilibria in this game.
2. If you were player 1, what action would you choose? Why?
3. Suppose that you were player 1, and player 2 made the following suggestion before the game was played: "I will play A. Let's both play A"? Would you go along with this suggestion? In answering this question, think about whether you would believe player 2's statement was true. How do your answers bear on the idea that Nash equilibria can be thought of as self-enforcing agreements which players reach before the game is played?

Solution.

1. (A, A) and (B, B)
2. It would be reasonable to say: "I would pick A", because (A, A) is a Nash equilibrium, and also the strategy profile which gives both players the highest payoff in the game. I would expect the other player also to see that this is the way that we could get the highest payoff, and understand that I also understand, and thus we would succeed in coordinating on the best outcome for both of us (A, A). It would also be reasonable to say: "I would pick B." The reason is that the Nash equilibrium (A, A) is much "riskier"[1] than the Nash equilibrium (B, B). If I play A and assume that the other player would play A as well, but I am wrong, this would be very bad for me. Instead of getting 9, the

[1] Note that in order to formally define risk, we need to go beyond the ordinal theory of preferences that we have so far. This will be done when we discuss choice under uncertainty. Nevertheless, notice that in spelling out the intuition that A is "riskier" below, I make only ordinal comparisons.

best payoff in the game, I would get 0, the worst payoff in the game. On the other hand, if I play B, assuming that the opponent will pay B as well, and I am wrong, then this will actually be good for me, and I will get 8 instead of 7, the payoff to (B, B). If I expect the other player to reason similarly and play B, then playing B is a best response to the action that I expect him to take.

3. Whether I would play A or B would depend on whether I expected player 2 to play A or B. However, whatever my belief was initially, the fact that the other player said that he was going to play A would not cause me to change my belief at all. Notice that if player 2 were intending to play A, then it would be in his interest to try to convince me to play A as well. On the other hand, if player 2 were intending to play B, he would also want to persuade me to play A, since if he plays B, he is better off if I play A than if I play B. So no matter what player 2 intended to do, he would have an incentive to try to convince me to play A. Therefore his promise that he would play A is meaningless, and I should disregard it. This bears negatively on the idea that Nash equilibria are self enforcing pre-play agreements, because, for the reasons described above, in this game such agreements are meaningless.

□

Exercise 5. Each student submits a real number between 0 and 100. All students whose numbers are closest to 1/2 of the average evenly split \$1. All other students get \$0. Assume that there are at least 2 students. Find the set of all Nash equilibria in this game. Rigorously prove that it is the set of all equilibira.

Solution. The only NE is $x_i^* = 0$ for all $i = 1, \dots, n$. At any profile $x = (x_1, \dots, x_n)$, the best response for player i is

$$x_i(x_{-i}) = \frac{1}{2n-1} \sum_{j \neq i} x_j$$

since by doing so x_i is exactly 1/2 of the average. This can be verified as below.

$$\sum_j x_j = x_i + \sum_{j \neq i} x_j = x_i + (2n-1)x_i = 2nx_i \Rightarrow x_i = \frac{1}{2n} \sum_j x_j$$

Suppose that $x^* = (x_1^*, \dots, x_n^*)$ is a NE. Then for all i, we have

$$x_i^* = \frac{1}{2n-1} \sum_{j \neq i} x_j^*$$

Then

$$\sum_i x_i^* = \frac{1}{2n-1} \sum_i \sum_{j \neq i} x_j^* = \frac{n-1}{2n-1} \sum_i x_i^*$$

It holds only when $\sum_i x_i^* = 0$, and thus $x_i^* = 0$ for all i since $x_i^* \in [0, 100]$.

□

Exercise 6. Do not assume constant unit cost and linear inverse demand. Assume only that

1. $\frac{\partial \pi_1}{\partial q_2 \partial q_1}(q_1, q_2) < 0$ and $\frac{\partial \pi_2}{\partial q_2 \partial q_1}(q_1, q_2) < 0$ for all (q_1, q_2) and
2. For every strategy choice by one firm, there is a unique best reply for the other firm.

Prove that the best reply functions are downward sloping.

Proof. By Assumption 2, we know that both the first-order condition

$$\frac{\partial \pi_i}{\partial q_i}(q_1, q_2) = 0 \tag{3}$$

and the second-order condition

$$\frac{\partial \pi_i}{\partial q_i^2}(q_1, q_2) < 0 \tag{4}$$

hold for $i = 1, 2$. The best response function is implicitly defined by (3). Total differentiate it on both sides, we have

$$\frac{\partial \pi_i}{\partial q_i^2} dq_i + \frac{\partial \pi_i}{\partial q_i \partial q_j} dq_j = 0 \quad \text{for } i, j = 1, 2 \text{ and } i \neq j$$

and by the Implicit Function Theorem

$$\frac{dq_i}{dq_j} = -\frac{\frac{\partial \pi_i}{\partial q_i \partial q_j}}{\frac{\partial \pi_i}{\partial q_i^2}} < 0$$

by Assumption 1 and (4). □

Exercise 7. Find the equilibrium of the Cournot game with linear inverse demand when the cost function is $C(q) = q^2$.

Solution. With the linear inverse demand

$$P(Q) = \begin{cases} \alpha - Q, & Q \leq \alpha \\ 0, & Q > \alpha \end{cases}$$

and the cost function $C_i(q_i) = q_i^2$, the firm 1's profit is

$$\pi_1(q_1, q_2) = q_1 P(q_1 + q_2) - C_1(q_1) = \begin{cases} q_1(\alpha - q_1 - q_2) - q_1^2, & q_1 + q_2 \leq \alpha \\ -q_1^2, & q_1 + q_2 > \alpha \end{cases}$$

The conditions for an interior optimum are:

$$\frac{\partial \pi_1}{\partial q_1} = \alpha - q_2 - 4q_1 = 0$$

or equivalently,

$$q_1 = \frac{\alpha - q_2}{4}$$

However, the optimum may not be interior, in which case $q_1 = 0$ and increasing q_1 would decrease profit.

We have an interior solution as long as $\alpha - q_2 > 0$ or $q_2 < \alpha$, which implies that the best response function is given by

$$b_1(q_2) = \begin{cases} \frac{\alpha - q_2}{4}, & q_2 \le \alpha \\ 0, & q_2 > \alpha \end{cases}$$

By $q_1^* = b_1(q_2^*)$ and $q_2^* = b_2(q_1^*)$ we have

$$q_1^* = q_2^* = \frac{\alpha}{5}$$

□

Exercise 69.1 (Osborne).(Bertrand's duopoly game with fixed costs) Consider Bertrand's game under a variant of the assumptions of Section 3.2.2 in which the cost function of each firm i is given by $C_i(q_i) = f + cq_i$ for $q_i > 0$, and $C_i(0) = 0$, where f is positive and less than the maximum of $(p - c)(\alpha - p)$ with respect to p. Denote by $\bar{p}$ the price p that satisfies $(p - c)(\alpha - p) = f$ and is less than the maximizer of $(p - c)(\alpha - p)$ (see Figure 1). Show that if firm 1 gets all the demand when both firms charge the same price then $(\bar{p}, \bar{p})$ is a Nash equilibrium. Show also that no other pair of prices is a Nash equilibrium. (First consider cases in which the firms charge the same price, then cases in which they charge different prices.)

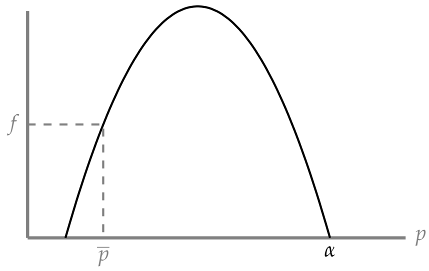

Figure 1: The determination of price $\overline{p}$.

Proof. Solution Observe that if both firms choose $(\overline{p}, \overline{p})$ both make zero profit. If either firm deviates to a lower price, it will make a negative profit. Note that if firm i raises its price, the other firm will get all of the business so firm i will make zero profit. So no firm has an incentive to deviate, and so $(\overline{p}, \overline{p})$ is a Nash equilibrium.

Note that there cannot be any Nash equilibrium in which one firm chooses a price lower than $\overline{p}$ since then some firm will make a negative profit, whereas each firm can guarantee a zero profit by setting $p = \overline{p}$.

Next assume that both firms select the same price p' where $p' > \overline{p}$. Then, by assumption, firm 1 will get all of the demand and make a positive profit, and firm 2 will make zero profit. By selecting a price between $\overline{p}$ and p', firm 2 could make a positive profit. This shows that there cannot be a Nash equilibrium in which both firms select price p' for $p' \neq \overline{p}$.

Next suppose that both firms select different prices. Then the firm that selects the lower price gets all the demand. Above we argued that this price must be at least $\overline{p}$. If the lower price firm selects price $\overline{p}$ it is making zero profit, and then by raising its price slightly it could still get all the demand and make a positive profit, so this is not a Nash equilibrium. On the other hand if the lowest price firm is i and it is selecting $p' > \overline{p}$, then the hight price firm j is making zero profit, but at any price which is lower than both p' and the maximum p such that $(p - c)(\alpha - p) = f$, j would make a positive profit. So this cannot be a Nash equilibrium either. □

Exercise 10. Consider the electoral competition model with three candidates and assume that voters are uniformly distributed on

$[0,1]$. Either find a Nash equilibrium or show that there is no Nash equilibrium.

Solution. Assume that candidates care only about winning and consider the following case $x_i = x_j = \frac{1}{4}$ and $x_k = \frac{3}{4}$. For i, if she deviate to some place $x < 0.5$, then k still wins.If she deviates to $x > 0.5$, then j will win and i still loses. Therefore, she has no incentive to deviate. The same holds for j. Also, k has no incentive to deviate. Therefore, it is a Nash equilibrium. Symmetrically, $x_i = x_j = \frac{3}{4}$ and $x_k = \frac{1}{4}$ is another NE. □

Exercise 90.1 (Osborne). Two units of an object are available. There are n bidders. Bidder i values the first unit that she obtains at v_i and the second unit at w_i, where $v_i > w_i > 0$. Each bidder submits two bids; the two highest bids win. Retain the tie-breaking rule in the text. Show that in discriminatory and uniform-price auctions, player i's action of bidding v_i and w_i does not dominate all her other actions, whereas in a Vickrey auction it does. (In the case of a Vickrey auction, consider separately the cases in which the other players' bids are such that player i wins no units, one unit, and two units when her bids are v_i and w_i .)

Discriminatory auction The price paid for each unit is the winning bid for that unit.

Uniform-price auction The price paid for each unit is the same, equal to the highest rejected bid among all the bids for all units.

Vickrey auction A bidder who wins k objects pays the sum of the k highest rejected bids submitted by the *other* bidders.

Solution. Discriminatory auction Suppose that each of the other players submits two bids of 0. Then if player i submits $b^1 \in (0, v_i)$ and $b^2 \in (0, w_i)$ she still wins two units, and pays less than when she bids v_i and w_i

Uniform-price auction Suppose that some bidder other than i submits one bid between w_i and v_i and one bid of 0, and all the remaining bidders submit two bids of 0. Then bidder i wins one unit, and pays the price w_i. If she replaces her bid of w_i with a bid between 0 and w_i then she pays a lower price, and hence is better off

Vickrey auction Suppose that player i bids v_i and w_i. Consider separately the cases in which the bids of the players other than i are such that player i wins 0, 1, and 2 units.

Player i wins 0 units: In this case the second highest of the other players'bids is at least v_i, so that if player i changes her bids so that she wins one or more units, for any unit she wins she pays at least v_i. Thus no change in her bids increases her payoff from its current value of 0 (and some changes lower her payoff).

Player i wins 1 unit: If player i raises her bid of v_i then she still wins one unit and the price remains the same. If she lowers this bid then either she still wins and pays the same price, or she does not win any units. If she raise sher bid of w_i then either the outcome does not change, or she wins a second unit. In the latter case the price she pays is the previously-winning bid she beat, which is at least w_i, so that her payoff either remains zero or becomes negative.

Player i wins 2 units: Player i's raising either of her bids has no effect on the outcome; her lowering a bid either has no effect on the outcome or leads her to lose rather than to win, leading her to obtain the payoff of zero.

□

Exercise 90.2 (Osborne). (Waiting in line) Two hundred people are willing to wait in line to see a movie at a theater whose capacity is one hundred. Denote person i's valuation of the movie in excess of the price of admission, expressed in terms of the amount of time she is willing to wait, by v_i . That is, person i's payoff if she waits for t_i units of time is $v_i - t_i$. Each person attaches no value to a second ticket, and cannot buy tickets for other people. Assume that $v_1 > v_2 > \cdots > v_{200}$. Each person chooses an arrival time. If several people arrive at the same time then their order in line is determined by their index (lower-numbered people go first). If a person arrives to find 100 or more people already in line, her payoff is zero. Model the situation as a variant of a discriminatory multi-unit auction, in which each person submits a bid for only one unit, and find its Nash equilibria. Arrival times for people at movies do not in general seem to conform with a Nash equilibrium. What feature missing from the model could explain the pattern of arrivals?

Solution. In this game, a strategy profile is a list $(t_1, t_2, \ldots, t_{200})$ of waiting times.

Proposition 1. *A strategy profile $t^* = (t_1^*, t_2^*, \ldots, t_{200}^*)$ is a Nash equilibrium if and only if there exists a number r such that (1) $v_{100} \geq r \geq v_{101}$, (2) for all players i with $1 \leq i \leq 100$, $t_i^* = r$, (3) there exists at least one player j with $101 \leq j \leq 200$ such that $t_j^* = r$, and (4) for all players k, with $101 \leq k \leq 200$, $t_k^* \leq r$.*

Proof. Note that given that t^* satisfies (1)-(4), players $1 - 100$ see the movie, and each of them gets a nonnegative utility, since for i between 1 and 100, $v_i - r \geq v_i - v_{100} \geq 0$. Notice that by (3), if any of these movie goers waited less time, they would not get to see the movie. Notice that no player k with $101 \leq k \leq 200$ would like to arrive at the movie earlier, since either it would still not be early enough to see the movie (in which case they would still get zero utility) or else they would see the movie would have to wait more than r, and hence more than v_{101} units of time which would not be worthwhile, because $v_k - v_{101} \leq 0$. So t^* is a Nash equilibrium.

Next notice that at any Nash equilibrium, all winners must wait the same amount of time, since otherwise the winner who waits longest could reduce their wait time and still see the movie. Let r be the the common waiting time of all winners. Next observe that the winners must be players $1 - 100$. Suppose that some player k with $k > 100$ wins. Then some player $i \leq 100$ must lose. But assuming we are in equilibrium, it follows that $v_k - r \geq 0$. But since

$v_i > v_k$, $v_i - r > 0$. If player i arrived r units of time before the movie, he would win, getting a positive utility. Since player 100 is a winner, it must be worthwhile for him to win, which implies that $v_{100} \geq r$. Since player 101 is a loser, it must not be worthwhile for him to arrive earlier in order to win, which implies that $r \geq v_{101}$. Note that in equilibrium all $k \geq 101$ must choose $t_k \leq r$ since otherwise, they would get into the movie, but would have to wait longer than it was worth to them. The above argument establishes that every equilibrium must have properties (1)-(4). □

Arrival times do not conform to the above Nash equilibrium because all people who get into the movie do not arrive at once. The missing ingredient in the model could be uncertainty about the value of the movie to others. If people are uncertain of the value to others, then those who value it more would be less willing to take a risk that many others who may also highly value the movie would arrive before them, and hence would arrive earlier than those who value it less. □

Exercise 91.1 (Osborne). (Lobbying as an auction) A government can pursue three policies, x, y, and z. The monetary values attached to these policies by two interest groups, A and B, are given in Table 8. The government chooses a policy in response to the payments the interest groups make to it. Consider the following two mechanisms.

	x	y	z
A	0	3	-100
B	0	-100	3

Table 8: The values of the interest groups for the policies x, y, and z.

First-price auction Each interest group chooses a policy and an amount of money it is willing to pay. The government chooses the policy proposed by the group willing to pay the most. This group makes its payment to the government, and the losing group makes no payment.

Menu auction Each interest group states, for each policy, the amount it is willing to pay to have the government implement that policy. The government chooses the policy for which the sum of the payments the groups are willing to make is the highest, and each group pays the government the amount of money it is willing to pay for that policy.

In each case each interest group's payoff is the value it attaches to the policy implemented minus the payment it makes. Assume that a tie is broken by the government's choosing the policy, among those tied, whose name is first in the alphabet.

Show that the first-price auction has a Nash equilibrium in which lobby A says it will pay 103 for y, lobby B says it will pay 103 for z, and the government's revenue is 103. Show that the menu auction has a Nash equilibrium in which lobby A announces that it will pay 3 for x, 6 for y, and 0 for z, and lobby B announces that it will pay 3 for x, 0 for y, and 6 for z, and the government chooses x, obtaining a revenue of 6. (In each case the pair of actions given is in fact the unique equilibrium.)

Solution.

First-price auction In the action pair, each interest group's payoff is -100. Consider group A. If it raises the price it will pay for y, then the government still chooses y, and A is worse off. If it lowers the price it will pay for y, then the government chooses z and A's payoff remains -100. Now suppose it changes its bid from y to x and bids p. If $p < 103$, then the government chooses z and A's payoff remains -100. If $p \geq 103$, then the government chooses x and A's payoff is at most -103. Group A cannot increase its payoff by changing its bid from y to z, for similar reasons. A similar argument applies to group B's bid.

Menu auction In the action pair, each group's payoff is -3. Consider group A. If it changes its bids then either the outcome remains x and it pays at least 3, so that its payoff is at most -3, or the outcome becomes y and it pays at least 6, in which case its payoff is at most -3, or the outcome becomes z and it pays at least 0, in which case its payoff is at most -100. (Note that if it reduces its bids for both x and y then z is chosen.) Thus no change in its bids increases its payoff. Similar considerations apply to group B's bid.

□