52nd IEEE Conference on Decision and Control
December 10-13, 2013. Florence, Italy

# Soft-constrained $\ell_{asso}$-MPC for robust LTI tracking: Enlarged feasible region and an ISS gain estimate

Marco Gallieri and Jan M. Maciejowski

***Abstract*— This paper investigates the robustness of a soft constrained LTI MPC for set-point tracking, using an $\ell_1$-regularised cost. The MPC using this type of cost (informally dubbed $\ell_{asso}$-MPC) is suitable, for instance, for redundantly-actuated systems. This is because of its ability to select a set of preferred actuators, leaving the other ones at rest for most of the time. The proposed approach aims to recover from state constraint violation and to track a changing set-point. Nominal stability is guaranteed for all feasible references and robustness to additive uncertainties is formally characterised, under certain assumptions. In particular, sufficient conditions are given for the feasible region to be robustly invariant, this region being larger than the MPC for regulation. The closed-loop system is input-to-state stable (ISS), and a local ISS gain is computed. All results apply to stabilisable LTI systems. Results hold as well for the more common quadratic MPC, a special case of the proposed controller.**

## I. Introduction

The $\ell_{asso}$-MPC, a novel form of MPC, has been proposed in [1], and further refined in [2], [3], [4], [5]. The strategy relies on the theory of $\ell_1$-regularised least squares, or LASSO, [6], [7]. The use of $\ell_{asso}$-MPC results in most of the actuators' commands (or their time differences) being equal to zero. In [3], the authors of the current paper demonstrated the potential of $\ell_{asso}$-MPC for simultaneous regulation and control allocation of redundantly-actuated systems.

It is becoming standard, among researchers, to use a terminal constraint in MPC, which generally depends on the particular set-point we want to regulate to. For changing set-points this means changing the terminal constraint, which could make the problem infeasible. The MPC for tracking, introduced by [8], [9] for LTI systems, and extended to nonlinear systems in [10], overcomes the terminal constraint limitation by means of a common invariant set, valid for a range of feasible targets. A virtual set-point and a penalty function are also included in the MPC cost. These modifications can also significantly enlarge the feasible region of the MPC controller, with respect to the one of standard MPC for regulation. The recent development of [11], combined the approach of [9] with the concept of soft-constraints [12], [13]. The approach of [11] results in an even larger feasible region than [9], and it has also been shown to provide a level of intrinsic robustness to bounded uncertainties, in the form of Input-to-State Stablility (ISS) [14], [15], [16], [17].

In this paper, a modified version of the LTI approach of [11] is formulated. We assume, similar to [9] but in contrast to [11], the terminal state constraint to be inside the set of admissible states. The proposed approach, informally dubbed $\ell_{asso}$-MPC for LTI tracking, features a novel terminal cost, which improves the results of [4]. The formulation allows the terminal set to be computed independently from the cost function. This allows one to tune the MPC cost for performance without modifying the constraints, thus providing a consistent feasible region and a drastic reduction of required computation for online tuning. The control action can be computed by solving, at each time, a positive definite quadratic programme (QP). The resulting closed-loop system is ISS. The main contribution of the present paper is a set of conditions for robust feasibility under bounded additive uncertainties. These conditions require minor modification of the terminal constraint of [9] and, if satisfied, they provide an uncertainty bound for which then the problem is always feasible, the entire feasible region being Robustly Positively Invariant (RPI). This RPI region is allowed to be greater than the state constraints, and, in most cases, is greater than the feasible region of standard MPC, under similar assumptions. Finally, a local ISS gain, which is depends on the system and the cost parameters, is computed. This (conservative) ISS gain is based on worst-case open-loop disturbance propagation through the MPC predictions. The approach is compared to the robust MPC of [18]. The proposed approach presents computational advantages over [18] and, for the considered example, it provides a larger terminal set and a larger set of attainable steady-states.

## II. $\ell_{asso}$-MPC for LTI tracking

This paper concerns the control of uncertain Linear Time-Invariant (LTI) systems of the form

$$x(k+1) = Ax(k) + Bu(k) + Ew(k), \qquad (1)$$

subject to constraints of the form

$$u \in \mathbb{U} \subset \mathbb{R}^{n_u}, \quad x \in \mathbb{X} \subset \mathbb{R}^{n_x}, \quad w \in \mathbb{W} \subset \mathbb{R}^{n_w}. \qquad (2)$$

This paper concerns the asymptotic regulation of $y(k) = Cx(k) + Du(k)$, to an (asymptotically) constant, $y_t \in \mathbb{Y} \subset \mathbb{R}^{n_y}$. A steady target is assumed available, and denoted $z_t = (x_t,\ u_t)$. Denote, $\delta x = x - x_t$. The MPC Predictions from time $k$ to $k+i$, defined as $x_{i|k}$, are simply denoted as $x_i$.

**Assumption 1. (A1)** The sets $\mathbb{X}$, $\mathbb{U}$ are polytopes, with $\mathbb{X} = \{x \in \mathbb{R}^n_x : Lx \leq \underline{1}\}$, $L \in \mathbb{R}^{p_x \times n_x}$, for some $p_x > 0$.

**Remark 1.** [9] If $(A, B)$ is stabilisable, $\exists M_\theta \in \mathbb{R}^{n_x+n_u \times n_\theta}$, $n_\theta \leq n_x + n_u$ such that the steady states of the nominal system satisfy $(x_s, u_s) = M_\theta \theta$, with $M_\theta^T = [M_x^T\ M_u^T]$.

Deparment of Engineering, University of Cambridge, UK. {mg574, jmm1}@cam.ac.uk.

Research supported by the EPSRC grant "Control for Energy and Sustainability", EP/G066477/1.

978-1-4673-5716-6/13/$31.00 ©2013 IEEE 

**Definition II.1.** ($\ell_{\mathbf{asso}}$–MPC for robust LTI tracking) Consider the following finite-horizon constrained optimal control problem (FHCOCP)

$$
\begin{aligned}
V_N^o(\delta x) = \min_{\underline{\mathrm{u}},\theta,\zeta,\underline{\mathrm{s}}} &\{V_N(x,\underline{\mathrm{u}},\theta,\zeta,\underline{\mathrm{s}})\} \\
s.t. \quad x_{j+1} = Ax_j + Bu_j,&\ \delta x_j = x_j - M_x\theta, \\
u_j \in \mathbb{U},\ Lx_j \leq \underline{1} + s_j,&\ j = 1,\dots,N-1, \\
\delta u_j = u_j - M_u\theta,&\ Zx_N = X\zeta, \\
s_j \geq 0,\ (\delta x_N,\theta) \in \mathbb{X}_f^t,&\ x_0 = x,
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
V_N(x,\underline{\mathrm{u}},\theta,\underline{\mathrm{s}}) &= F(\zeta,\delta x_N) + \sum_{j=0}^{N-1}\ell(\delta x_j,\delta u_j) \\
&\quad + V_s(\underline{\mathrm{s}}) + V_O(z_s - z_t), \\
V_s(\underline{\mathrm{s}}) &= \sum_{j=0}^{N-1}\ell_s(s_j),\ \ell_s(s_j) = s_j^TQ_cs_j + \gamma_c\|s_j\|_1, \\
V_O(z) &= z^TQ_tz + \gamma_t\|\bar{S}z\|_1,
\end{aligned} \tag{4}
$$

$$\ell(x,u) = x^TQx + u^TRu + \|Su\|_1, \tag{5}$$

$$F(\zeta,x) = \alpha\|\zeta\|_q + x^TPx,\ \alpha,\geq 0, \tag{6}$$

$$Q_c \succ 0,\ Q_t \succ 0,\ \bar{S} \text{ diagonal},\ \gamma_t > 0,\ \gamma_c > 0, \tag{7}$$

with $\underline{\mathrm{s}}^T = \begin{bmatrix} s_0^T, & \cdots, & s_{N-1}^T \end{bmatrix}$. At each iteration $k$, the $\ell_{asso}$-MPC for tracking applies to the plant the first move of the optimal policy, $u(k) = u_0^\star$, obtained by online solution of (3-4), at the current state, $x = x(k)$. The generated implicit control law is referred to as $K_N^t(x,z_t) \equiv u_0^\star$. The closed-loop evolution of (1) is then

$$x(k+1) = Ax(k) + BK_N^t(x(k),z_t) + Ew(k), \tag{8}$$

and its one-step evolution, with $w(k) = 0$ is denoted as $x^+$.

**Remark 2.** All results in this paper are also valid for positive definite costs with cross state-input terms.

The approach of [11] is revised. In particular, an invariant set for tracking is used as terminal set. The terminal set is the only ingredient of the approach upon which robustness depends, and it is computed offline prior to cost function definition, to facilitate tuning. The approach extends the ideas of [19] to LTI tracking using *polytopic Lyapunov functions* [20], [21]. This concept has been clearly addressed in [22], for the case of regulation to the origin. We propose a novel $\ell_{asso}$-MPC for tracking, which is robustly feasible and ISS for a disturbance bound $\mu$, returned by the proposed method.

The slack variable, $\underline{\mathrm{s}}$, allows a feasible region larger than $\mathbb{X}$, so that feasibility could be maintained if $w(k)$ drives the system out of $\mathbb{X}$. Differently from [11], a quadratic slack constraint is not included here. This, besides dispensing the solver from additional complexity, expresses our intention to keep the system's trajectory within constraints, at least under nominal conditions. This choice is taken at the expense of a possibly smaller feasible region than in [11].

**Assumption 2. (A2)** For system (1)

**(H0)** $(A,B)$ is stabilisable, $(C,A)$ is detectable,

**(H1)** $Q \succ 0$, $R \succ 0$, $S \in \mathbb{R}^{n_s \times n_u}$,

**(H2)** $\mathbb{X}$, $\mathbb{U}$ are C-sets (convex, compact sets),

**(H3)** $\mathbb{W}$ is a C-set, with $0 \in \mathrm{int}(\mathbb{W})$.

Define $f_{\underline{\mathrm{u}}}^i(x)$, as the $i$-step evolution of (1), under a policy $\underline{\mathrm{u}}$, with $w(k) = 0, \forall k \geq 0$. For a given $\theta$, we can define

$$\mathbb{X}_f(\theta) = \{\delta x : (\delta x,\theta) \in \mathbb{X}_f^t\}. \tag{9}$$

The feasible region of (3)–(4) is a polytope, given by

$$\mathbb{X}_N^t = \{x : \exists\underline{\mathrm{u}} \in \mathbb{U}^N, \theta \in \mathbb{R}^{n_\theta} \mid f_{\underline{\mathrm{u}}}^N(x) - M_x\theta \in \mathbb{X}_f(\theta)\}. \tag{10}$$

For a fixed $\theta$, the feasible region becomes

$$\mathbb{X}_N(\theta) = \{x : \exists\underline{\mathrm{u}} \in \mathbb{U}^N \mid f_{\underline{\mathrm{u}}}^N(x) - M_x\theta \in \mathbb{X}_f(\theta)\}, \tag{11}$$

which is the typical feasible region for an MPC for regulation, with softened state constraints.

### A. Nominal asymptotic stability

Closed-loop stability under $\ell_{asso}$-MPC has been investigated in [2], [4], and it is further reviewed here.

Define $f_K(\delta x) = (A+BK)\delta x$. It is assumed that

**Assumption 3. (A3)** $\exists K \in \mathbb{R}^{n_u \times n_x} : |\lambda_{\max}(A+BK)| < 1$.

Denote

$$\bar{F}(\delta x) = \min_{\{\zeta, Z\delta x = X\zeta\}} F(\zeta,\delta x). \tag{12}$$

**Assumption 4. (A4)** Assume $Zf_K(\delta x) = X\zeta^+(\delta x)$, for some $\zeta^+(\delta x)$ and that, $\forall(\delta x,\theta) \in \mathbb{X}_f^t$

$$\tilde{\alpha}_1(\|\delta x\|_2) \leq \bar{F}(\delta x) \leq \tilde{\alpha}_2(\|\delta x\|_2), \tag{13}$$

$$F(\zeta^+(\delta x), f_K(\delta x)) - \bar{F}(\delta x) \leq -\ell(\delta x, K\delta x). \tag{14}$$

A method to compute a suitable terminal cost has been proposed in [4], for systems with $n_u \geq n_x$ and $q = 1$. In [4], the terminal set is computed a priori, independently from the cost function. This is useful for re-tuning. A similar approach is presented in Section III, where the ingredients for **(A4)** are computed for any stabilisable plant using a less restrictive technique than [4].

Define the set of *admissible steady states*

$$\mathbb{Z}_s = \{z_t = M_\theta\theta : \theta \in \pi_\theta(\mathbb{X}_f^t)\}, \tag{15}$$

where $\pi_\theta(\mathbb{X}_f^t)$ is the projection of $\mathbb{X}_f^t$ on the $\theta$ coordinate. Define also $\bar{r} = \max_r\{r \mid \{\delta x : \|\delta x\|_2 \leq r\} \subseteq \pi_{\delta x}(\mathbb{X}_f^t)\}$, $\bar{V}_N = \max_{x \in \mathbb{X}_N^t}\{V_N^o(\delta x) - V_O(z_s^\star - \bar{z}_t) - V_s(\underline{\mathrm{s}}^\star)\}$, where

$$\bar{z}_t = \arg\min_{z_s \in \mathbb{Z}_s} V_O(z_s - z_t). \tag{16}$$

Denote $\delta z_s^\star = z_s^\star - \bar{z}_t$. The following is obtained

**Lemma II.1.** Assume **(A2)–(A4)**. Then, for $z_t = \bar{z}_t$, there exists two $\mathcal{K}_\infty$-functions, $\alpha_2(r)$, $\alpha_3(r)$, such that

$$\alpha_1(\|\delta x\|_2) \leq V_N^o(\delta x) - V_O(\delta z_s^\star) \leq \alpha_2(\|\delta x\|_2),\ \forall x \in \mathbb{X}_N^t, \tag{17}$$

where, $\alpha_1(r) = \lambda_{\min}(Q)r^2$, and $\alpha_2(r) = \epsilon_2\epsilon_1\tilde{\alpha}_2(r) + \alpha_3(r)$,

$$V_s(\underline{\mathrm{s}}^\star) \leq \alpha_3(\|\delta x\|_2),\ \epsilon_2 = \max\left(1, \frac{\bar{V}_N}{\epsilon_1\tilde{\alpha}_2(\bar{r})}\right),\ \epsilon_1 > 2.$$

Building upon [11], [9], [8], the following can be stated

**Theorem II.2.** Assume **(A1)–(A4)**, $w(k)=0, \forall k \geq 0$. Then

1) If $z_t \in \mathbb{Z}_s$, then $z_t$ is the sole equilibrium for system (8), and it is Asymptotically Stable (AS), with $y(k) \to y_t$ for $k \to \infty$, $\forall x(0) \in \mathbb{X}_N^t$,
2) If $z_t \notin \mathbb{Z}_s$, then $\bar{z}_t$, which is unique, is the sole equilibrium for (8), and it is AS, $\forall x(0) \in \mathbb{X}_N^t$.
3) $\forall x(0) \in \mathbb{X}_N^t$, $x(k) \to \mathbb{X}$, as $k \to \infty$.

## III. $\ell_{asso}$-MPC FOR ROBUST LTI TRACKING

Building upon [8], define the extended state space $(\delta x, \theta)$. Our contribution, for the purpose of robustness, is based the introduction of a $\lambda \in (0,1)$ and of

$$\bar{A}_\lambda = \begin{bmatrix} \frac{1}{\lambda}(A+BK) & 0 \\ 0 & I \end{bmatrix}. \tag{18}$$

**Assumption 5. (A5)** $\lambda \in (|\lambda_{\max}(A+BK)|, 1)$.

Define, similarly to [8], [9],

$$\mathbb{V}_{\alpha_z} = \{(\delta x, \theta) : (\delta x, K\delta x) + M_\theta \theta \in \mathbb{Z},\ M_\theta \theta \in \alpha_z \mathbb{Z} \subset \mathbb{Z}\}, \tag{19}$$

where $\alpha_z \in (0,1)$. Define then, the *$\lambda$-contractive set for tracking*

$$\mathcal{O}^t_{\infty,\lambda,\alpha_z} = \{(\delta x, \theta) : \bar{A}^j_\lambda(\delta x, \theta) \in \mathbb{V}_{\alpha_z},\ \forall j \in \mathbb{I}_{\geq 0}\}. \tag{20}$$

**Assumption 6. (A6)** $\mathbb{X}^t_f = \{(\delta x, \theta) : (\frac{1}{\lambda}\delta x, \theta) \in \mathcal{O}^t_{\infty,\lambda,\alpha_z}\}$.

The following Lemmas are used to prove the main results. Their proofs are omitted for brevity.

**Lemma III.1.** (Contractivity of $\mathbb{X}_f(\theta)$) Assume **(A1)–(A6)**. Then, $\mathcal{O}^t_{\infty,\lambda,\alpha_z}$ is a finitely determined polytope and $x \in \mathbb{X}_f(\theta) \Rightarrow f_K(\delta x) \in \lambda \mathbb{X}_f(\theta)$, $\forall (\delta x, \theta) \in \mathcal{O}^t_{\infty,\lambda,\alpha_z}$.

**Lemma III.2.** (Contractivity of $\pi_{\delta x}(\mathbb{X}^t_f)$) Assume **(A1)–(A6)**. Then,

1) $\pi_{\delta x}(\lambda \mathbb{X}^t_f) \subseteq \lambda \pi_{\delta x}(\mathbb{X}^t_f)$,
2) $(\delta x, \lambda\theta) \in \lambda \mathbb{X}^t_f \Rightarrow \delta x \in \lambda \pi_{\delta x}(\mathbb{X}^t_f)$,
3) $\delta x \in \pi_{\delta x}(\mathbb{X}^t_f) \Rightarrow \exists \theta :\ f_K(\delta x) \in \lambda \pi_{\delta x}(\mathbb{X}^t_f)$, and $(\frac{1}{\lambda} f_K(\delta x), \theta) \in \mathbb{X}^t_f$.

### *A. Main result 1: Computation of the terminal penalty*

For the computation of $F$, an algorithm has been given in [4]. This, unfortunately, can converge only if $n_u \geq n_x$, which is a very strong assumption. In this section, a more general result is provided. To do so, it is assumed (for the sake of brevity) that $\mathbb{X}$ and $\mathbb{U}$ are symmetric. In this case, $\mathcal{X}_f = \pi_{\delta x}(\mathbb{X}^t_f)$ is also symmetric and it admits the irreducible representations $\mathcal{X}_f = \{\delta x : \|G\delta x\|_\infty \leq \lambda\} \equiv \{\delta x = \bar{X}\zeta : \|\zeta\|_1 \leq \lambda\}$ where $\bar{X}$ is a matrix containing the vertices of $\frac{1}{\lambda}\mathcal{X}_f$. This set is $\lambda$-contractive by Lemma III.2, and its Minkowski function [20], $\Psi_{\mathcal{X}_f}(\delta x) = \|G\delta x\|_\infty \equiv \min_{\{\delta x = X\zeta\}} \|\zeta\|_1$, is a control Lyapunov function in this set, with $\Psi_{\mathcal{X}_f}(f_K(\delta x)) \leq \lambda \Psi_{\mathcal{X}_f}(\delta x)$.

**Assumption 7. (A7)** Given $\bar{n}_u$ columns of $B$, $\bar{B} \in \mathbb{R}^{n_x \times \bar{n}_u}$, with $1 \leq \bar{n}_u \leq n_u$, $(A, \bar{B})$ stabilisable, $K^T = [\bar{K}^T\ 0]$, assume

$$(A+\bar{B}\bar{K})^T P (A+\bar{B}\bar{K}) - P \leq -(\tau I + Q + K^T R K),\ \tau > 0. \tag{21}$$

Define also the scalar

$$\bar{\alpha} = \frac{n_s \|SK(G^T G)^{-1} G^T\|_\infty}{1-\lambda}, \tag{22}$$

where $n_s$ is taken from **(H1)**, and assume the following

**Assumption 8. (A8)** One of the following (at least) holds

1) $S = [0_{n_s \times \bar{n}_u},\ \bullet]$, $K^T = [\bullet,\ 0_{n_x \times (n_u - \bar{n}_u)}]$, $\alpha = Z = X = 0$, $(\zeta = 0\ \forall x)$,
2) $X = I$, $Z = G$, $q = \infty$, $\alpha \geq \bar{\alpha}$, $(\zeta = G\delta x_N\ \forall x)$,
3) $Z = I$, $X = \bar{X}$, $q = 1$, $\alpha \geq \bar{\alpha}$.

The following result is obtained

**Theorem III.3.** Assume **(A1)**, **(A2)**, $\mathbb{X}$, $\mathbb{U}$ symmetric, **(A7)**, **(A8)**, **(A5)**, **(A6)**. Then, **(A3)**, **(A4)** and Theorem II.2 hold.

**Remark 3.** Non-symmetric constraints can be accommodated with minor modifications of (3) and of **(A8)**.

### *B. Enlarged domain of attraction and local optimality*

The approach provides an enlargement of the domain of attraction, with respect to an MPC for regulation. Namely,

**Lemma III.4.** $\mathbb{X}^t_N \cap \mathbb{X} \supseteq \mathbb{X}_N(\theta) \cap \mathbb{X}$, for all $\theta : M_\theta \theta \in \mathbb{Z}_s$.

Local optimality of admissible references can also be characterised, as follows

**Assumption 9. (A9)** [11], [9]. The parameters $\gamma_t, \gamma_c$ are taken to be greater then the supremum over $x$ of the $\infty$-norm of the respective Lagrange multiplier, for (3)–(4) subject to $\|\underline{s}\|_1 = 0$ and $\|z_t - z_s\|_1 = 0$, for $z_t \in \mathbb{Z}^o \subseteq \mathbb{Z}_s$.

**Theorem III.5.** Assume **(A9)**. Then, for $z_t \in \mathbb{Z}^o$, at the optimum of (3)–(4) we have, for $z_t = M_\theta \theta$, that $z^\star_s = z_t, \forall x(0) \in \mathbb{X}_N(\theta)$.

Notice that, for $z_t \in \mathbb{Z}_s \backslash \mathbb{Z}^o$, $\mathbb{X}$ can be also expected to be invariant, and $z^\star_s = z_t$ holds locally [9]. The required tuning procedure can be build, similarly to [11], [9]. Proof of the above statements require minor modification of results in [11], [9], and are omitted.

### *C. Main result 2: Robust feasibility and local ISS gain*

As shown in [11], system (8) is intrinsically ISS. This section provides sufficient conditions for robust feasibility and computes a local worst-case ISS gain.

Assume $G_x$, $G_\theta$ are matrices such that $\mathbb{X}^t_f = \{(\delta x, \theta) \in \mathbb{R}^{n_x + n_\theta} : G_x \delta x + G_\theta \theta \leq \lambda \underline{1}\}$.

**Assumption 10. (A10)** $\mathbb{W} \subseteq \{w \in \mathbb{R}^{n_w} : \|w\|_\infty \leq \mu\}$, with

$$\mu \leq \frac{1-\lambda}{\|GA^{N-1}E\|_\infty}. \tag{23}$$

The following Lemma is instrumental to the main result

**Lemma III.6.** $\exists \kappa_x, \kappa_u > 0 : \|\delta x^\star_j\|_2 \leq \kappa_x \|\delta x\|_2$, $\|\delta u^\star_j\|_2 \leq \kappa_u \|\delta x\|_2, \forall x \in \mathbb{X}^t_N, \forall z_t \in \mathbb{R}^{n_x + n_u}, \forall j \in \mathbb{I}_{N-1}$.

Define $\nabla V^o_N(\delta x) = V^o_N(\delta x^+) - V^o_N(\delta x)$. We are now ready to state the main result

**Theorem III.7.** Assume **(A1)–(A6)**, **(A10)**. Then, $\forall w \in \mathbb{W}$,

1) $\mathbb{X}_N^t$ is RPI, and system (8) is ISS, $\forall x(0) \in \mathbb{X}_N^t$,
2) If **(A9)** holds and $z_t = M_\theta \bar{\theta} \in \mathbb{Z}^o$, then $\forall x(0) \in \mathbb{X}_N(\bar{\theta})$ and for any $\tau_Q \in (0,\ \lambda_{\min}(Q))$ the value function satisfies

$$\nabla V_N^o(\delta x(k)) \leq -\tau_Q \|\delta x(k)\|_2^2 + \sigma(\|w(k)\|_2),$$

with local ISS gain

$$\begin{aligned}
\sigma(r) &= \sigma_Q(r) + \sigma_P(r) + \sigma_G(r) + \sigma_s(r) + \sigma_O(r),\\
\sigma_Q(r) &= \epsilon_Q \left( \sum_{j=1}^{N-1} 2\|Q^{1/2}A^{j-1}E\|_2^2 + \gamma_Q \right) r^2 + \gamma_S r,\\
\sigma_P(r) &= \frac{\epsilon_{P2}\|P\|_2}{\epsilon_{P1}} (\|(A+BK)A^{N-1}E\|_2^2 + \gamma_P) r^2,\\
\sigma_G(r) &= (\|G(A+BK)A^{N-1}E\|_\infty + \gamma_G) r,\\
\sigma_s(r) &= \gamma_s r,\ \ \sigma_O(r) = \gamma_O r.
\end{aligned}$$

The (worst case) gains $\gamma_\bullet$ are

$$\begin{aligned}
\gamma_Q = 2(N-1)\max(\gamma_1, \gamma_2),\ \ \gamma_P &= \gamma_1/(2\|Q\|_2),\\
\gamma_1 &= 2\|Q\|_2 \sup \|M_x(\tilde{\theta}-\bar{\theta})\|_2^2/\|w\|_2^2,\\
\gamma_2 &= \|R\|_2 \sup \|M_u(\tilde{\theta}-\bar{\theta})\|_2^2/\|w\|_2^2,\\
\gamma_S &= (N-1)\|S\|_1 \sup \|M_u(\tilde{\theta}-\bar{\theta})\|_1/\|w\|_2,\\
\gamma_G &= \|G\|_\infty \sup \|M_x(\tilde{\theta}-\bar{\theta})\|_\infty/\|w\|_2,\\
\gamma_{O1} &= \gamma_t \sup \|\bar{S}M_\theta(\tilde{\theta}-\bar{\theta})\|_1/\|w\|_2,\\
\gamma_{O2} &= \sup \|Q_t^{1/2}M_\theta(\tilde{\theta}-\bar{\theta})\|_2^2/\|w\|_2^2,
\end{aligned}$$

where the (finite) supremum is taken over $(\delta x_N, w)$ under the following constraints

$$G_x \delta x_N + G_\theta\ \bar{\theta} \leq \lambda \underline{1}, \qquad (24)$$

$$G_x(\delta x_N + A^{N-1}Ew) + G_\theta \tilde{\theta} \leq \underline{1},\ w \in \mathbb{W}, \qquad (25)$$

where, given a pair $(\delta x_N, w)$, the vector $\tilde{\theta}$ minimises $V_O(M_\theta(\tilde{\theta}-\bar{\theta}))$ under (25). The remaining constants are $\epsilon_{P1} = \epsilon_{P2} - 1 = \tau/\|P\|_2$, and

$$\begin{aligned}
\gamma_O &= \sup V_O(M_\theta(\tilde{\theta}-\bar{\theta}))/\|w\|_2,\\
\gamma_s &= \sup |V_s(\tilde{s}) - V_s(\bar{s})|/\|w\|_2,\\
\epsilon_Q &= \frac{(N-1)(\kappa_x^2\|Q\|_2 + \kappa_u^2\|R\|_2)}{\lambda_{\min}(Q) - \tau_Q} + 1,
\end{aligned}$$

with $\kappa_x, \kappa_u > 0$ from Lemma III.6. For $\gamma_s$, the (finite) sup is taken over $(\underline{x}, w)$, $\underline{x} = (x_0, \ldots, x_{N-1})$, under $(x_j, w) \in \mathbb{X}_N^t \times \mathbb{W}$ for $j \in \mathbb{I}_{N-1}$ and where, given a pair $(\underline{x}, w)$, the vectors $\bar{s}$, $\tilde{s}$ minimise $V_s$ subject to, respectively,

$$Lx_j - \underline{1} \leq \bar{s}_j,\ \ L(x_j + A^jEw) - \underline{1} \leq \tilde{s}_j.$$

Condition (23) is only sufficient for robust feasibility. In general, decreasing $\lambda$ is beneficial to $\mu$.

## IV. Illustrative example

The approach is demonstrated for a double integrator, $n_x = 2,\ n_y = n_u = n_\theta = 1,\ E = [1\ 1]^T,\ y = [1\ 0]x$, sampled at 1 Hz. The parameters are $Q = 2I,\ R = 2,\ \tau = 1,\ S = 1,\ \bar{\alpha} = 6.7,\ N = 5$. Constraints are $\|x\|_\infty \leq 10$, $|u| \leq 5$. The candidate terminal controller is an LQR placing the eigenvalues at about 0.4. Figure 1 shows the projections on $\delta x$ of, in order from the largest, the nominal invariant set for tracking, the contractive set computed with the proposed approach for $\lambda = 0.7, \mu = 0.4$, and the RPI set computed with the constraint restriction approach of [18] for the same $\mu$ and the same LQR assumed as disturbance affine feedback.

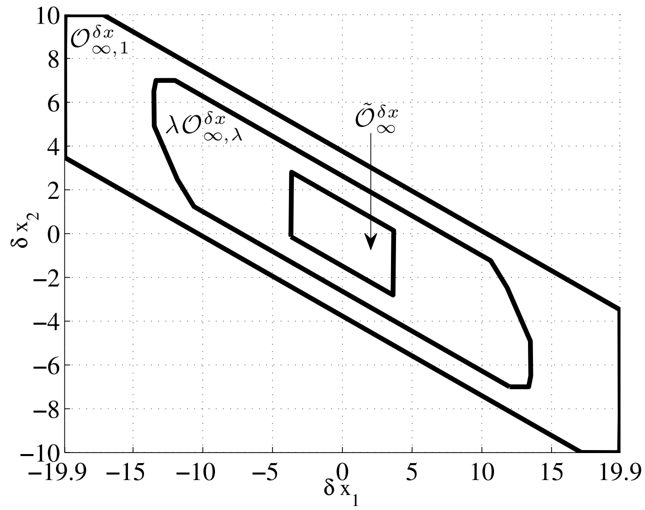


Fig. 1. Terminal sets (on $\delta x$): nominal, proposed, based on [18].

Figure 2 shows the closed loop trajectory of the system using the proposed approach, under admissible disturbances $(w_1, w_2)$. Constraints are violated only for the first 2 samples, since the system has initial state $[5, 5]^T$ and a position disturbance is experienced (in practise the system tolerates much larger disturbances). The target signal is tracked robustly, with minimum offset. For the proposed approach, any $y_t$ with $|y_t| \leq 6.9$ is in fact achievable, while the benchmark approach of [18] is limited to $|y_t| \leq 6.6$. Moreover, for $\lambda = 0.6, \mu = 0.5$, the terminal sets of the two approaches look fairly compatible on $\delta x$, but the proposed approach allows $|y_t| \leq 5.5$ while the benchmark only $|y_t| \leq 1.1$. For decreasing $\lambda$s we can, to a certain extent, expect $\mu$ to increase, until $\mathcal{X}_f$ becomes smaller than the benchmark (as $\lambda$ approaches $|\lambda_{\max}(A+BK)|$). In general, our approach allows a wider range of steady states than the benchmark, which, on the other hand, has the advantage of robust constraints satisfaction. In practise, the two strategies behave quite similarly when constraints are (in most cases inevitably) softened, the proposed one benefiting from lower computation requirements (comparable to the nominal case).

Simulations have been performed, for different $S$. In particular, for $S = 1$, the MPC counteracts disturbances by means of a few control moves, the first being the largest, the following ones being smaller and of opposite sign. For $S = 10$, the smallest moves are eliminated, and the time to reject disturbances increases (0.5 to 1 sec). In other words, the $\ell_{asso}$ cost allows for a tradeoff between “fuel consumption” and speed of convergence.

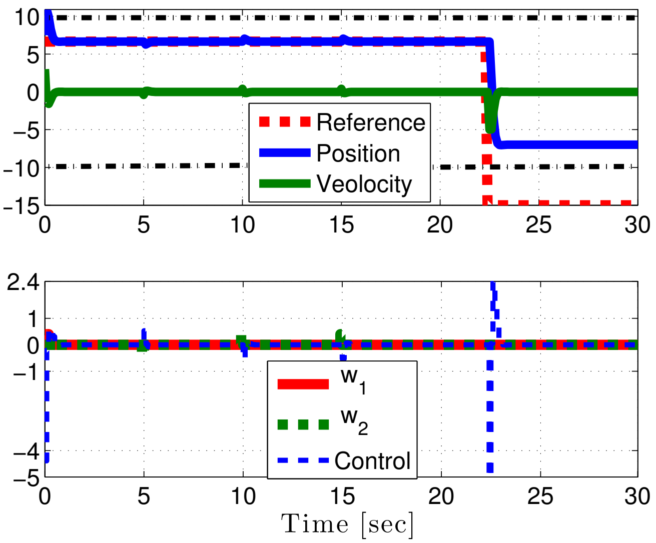


Fig. 2. Closed-loop trajectory with a moving reference and disturbances.

## V. Conclusions

This paper discussed the robustness of a $\ell_{asso}$-MPC for tracking a changing reference, with the use of soft constraints. The $\ell_{asso}$ cost is used to provide input signals that are sparse in time and through actuator channels. This is particularly suitable for redundantly-actuated systems or fuel saving. This soft-constrained MPC for tracking can provide a larger region of attraction than standards robust MPCs, and it has the ability to recover from constraints violation. A new terminal cost and terminal set for tracking have been computed, based on contractive polytopes, providing an $\infty$-norm disturbance bound that allows for robust feasibility. The required terminal set is computed offline, prior to cost definition, so that tuning is facilitated. The approach requires less computation than standard robust MPC, and it's easier to implement. Another advantage of the approach is that the feasible region is by construction always non-empty. A local ISS gain has been given, based on worst-case open loop predictions, providing further insight on the closed-loop behaviour.

## VI. Acknowledgements

The authors are thankful to Prof. D. Limón of University of Seville, for interesting discussions on the paper.

## Appendix

*Proof.* (Lemma II.1) Trivially, $\alpha_1(\|x\|_2) = \sigma_{\min}(Q)\|x\|_2^2$ follows from $\ell(x,u) \geq \sigma_{\min}(Q)\|x\|_2^2, \forall u$, and $V_N^o(x) - V_O(\delta z_s^\star) \geq \ell(x, K_N(x)), \forall x$. To prove the existence of $\alpha_2(\|x\|_2)$, define $\tilde{u} = \{K(A+BK)^i \delta x,\ i = 0, \ldots, N-1\}$, that is a feasible solution for $(\delta x, \theta) \in \mathbb{X}_f^t$. By optimality, by **(A4)** and by recursive application of (14), $\forall (\delta x, \theta) \in \mathbb{X}_f^t$

$$\begin{aligned} V_N^o(\delta x) - V_O(\delta z_s^\star) &\leq \min_{\zeta} V_N(\delta x, \tilde{u}, \theta, \zeta, 0) - V_O(\delta z_s) \\ &\quad + V_O(\delta z_s) - V_O(\delta z_s^\star) \\ &\leq \bar{F}(\delta x) + \alpha_{V_O}(\|\delta x\|_2) \leq \epsilon_1 \tilde{\alpha}_2(\|\delta x\|_2), \end{aligned} \quad (26)$$

where $\epsilon_1 > 2$ and we have used the fact that $\|z_s - z_s^\star\|$ is bounded and it is zero for $\|\delta x\| = 0$. For the 1-norm term of $V_O$, we have used the reverse triangle inequality and, for the quadratic term, we have used the fact that $\|\bar{z}_t - z_s^\star\|$ is bounded above and it is zero if $\|\delta x\| = 0$, together with

$$\|a+b\|_2^2 \leq (1+\epsilon^2)\|a\|_2^2 + (1+1/\epsilon^2)\|b\|_2^2, \forall \epsilon \neq 0. \quad (27)$$

Since $\mathbb{X}$ and $\mathbb{U}$ are C-sets, so is $\mathbb{X}_f^t$. Then, the existence of such $\epsilon_1 \tilde{\alpha}_2(\|x\|_2)$ implies [23] the existence of a $\mathcal{K}_\infty$ function $\epsilon_2 \epsilon_1 \tilde{\alpha}_2(\|\delta x\|_2) \geq V_N^o(\delta x) - V_O(\delta z_s^\star) - V_s(\underline{s}^\star),\ \forall x \in \mathbb{X}_N^t, \forall \theta \in \mathbb{Z}_s$. The same argument applies to $V_s(\underline{s}^\star)$. Since all the sets are bounded, the system is linear, $\delta x = 0 \Rightarrow V_s(\underline{s}^\star) = 0$, and $V_s(\underline{s}^\star) \geq 0$ is bounded above, $\alpha_3$ exists. ■

*Proof.* (Theorem II.2)

1) Convergence to an equilibrium, $\bar{z}_s \in \mathbb{Z}_s$, follows from a direct method Lyapunov argument [8], [9]. In particular, at time $k+1$ the control move $\tilde{u}_{N-1|k+1} = K\delta x_{N|k} + M_u \theta_k^\star$ can be added to the tail of the solution at time $k$ to obtain a new feasible solution. Then,

$$\begin{aligned} \alpha_1(\|\delta x\|_2) \leq V_N^o(\delta x) - V_O(\delta z_s^\star) &\leq \alpha_2(\|\delta x\|_2), \\ V_N^o(\delta x^+) - V_N^o(\delta x) &\leq -\alpha_1(\|\delta x\|_2) - \ell_s(\tilde{s}_1) \\ &\leq -\alpha_1(\|\delta x\|_2), \end{aligned} \quad (28)$$

where $\tilde{s}_1$ comes from the previous solution. Convergence to $z_t$ follows from the contradiction argument of [9], since all assumptions are satisfied.

2) Convergence to $\bar{z}_t$ is discussed in [9]. The cost function is strongly convex, and the optimal solution is unique, so the equilibrium.

3) The claim follows by definition of $\mathbb{Z}_s$ and by part 1). ■

*Proof.* (Theorem III.3) Consider the 3 cases in **(A8)**

1) In the terminal set it is feasible to use just $\bar{n}_u$ actuators, which are non-regularised, and to set the remaining ones to their steady states and $\zeta = 0$. The results follow by application of standard quadratic MPC arguments ([24] Theorem 2.24, p. 123) thanks to **(A7)**.

2) We have $F(\zeta, \delta x_N) = \bar{F}(\delta x_N) = \alpha\|G\delta x_N\|_\infty + x_N^T P x_N$. The quadratic part is accommodated by **(A7)**. To satisfy **(A4)**, we need $\alpha\|G(A+BK)x\|_\infty - \alpha\|Gx\|_\infty + \|SKx\|_1 \leq 0,\ \forall x \in \mathcal{X}_f$. Since $\|Gx\|_\infty$ is the Minkowski function of (the $\lambda$-contractive) $\mathcal{X}_f$, and $G$ has full column rank, from $\alpha$ satsfing (22) we have $0 \geq \alpha(\lambda - 1) + n_s\|SK(G^TG)^{-1}G^T\|_\infty$. Multiplying for $\|Gx\|_\infty$ provides $0 \geq \alpha(\lambda - 1)\|Gx\|_\infty + n_s\|SK(G^TG)^{-1}G^T\|_\infty\|Gx\|_\infty \geq \alpha(\|G(A+BK)x\|_\infty - \|Gx\|_\infty) + n_s\|SKx\|_\infty \geq \alpha(\|G(A+BK)x\|_\infty - \|Gx\|_\infty) + \|SKx\|_1$, that is the desired result. Upper and lower bounding $\mathcal{K}_\infty$-functions are obtained trivially.

3) We have $F(\zeta, \delta x_N) = \alpha\|\zeta\|_1 + x_N^T P x_N$, $Gx = \bar{X}\zeta$. The quadratic part is accommodated by **(A7)**, and $\min_{\{x = \bar{X}\zeta\}} \|\zeta\|_1 \equiv \|Gx\|_\infty$. From contractivity, $\zeta^+$ from **(A4)** exists, and we can proceed as in 2). ■

*Proof.* (Theorem III.7)

1) Suppose $w(k) \neq 0$, for some $k \geq 0$. Define a candidate $N-1$ steps MPC prediction, from time $k+1$, as $\tilde{x}_{N-1|k+1} = x^\star_{N|k} + A^{N-2}Ew(k)$, where $x^\star_{N|k}$ denotes $x^\star_N$ at time $k$. If $\delta x_{N|k+1} \in \mathbb{X}^t_f(\theta)$, with $M_\theta \theta \in \mathbb{Z}_s$, it is feasible from $\tilde{x}_{N-1|k+1}$, $\forall w(k) \in \mathbb{W}$, then $\mathbb{X}^t_N$ is RPI and the system is ISS [11]. Define the Minkowski function of $\mathcal{X}_f$ as $\psi(x) = \max_i G_i x$. Then, $\psi(\delta\tilde{x}_{N-1|k+1}) = \psi(\delta x^\star_{N|k} + A^{N-1}Ew) \leq \psi(\delta x^\star_{N|k}) + \psi(A^{N-1}Ew) \leq \lambda + \|GA^{N-1}Ew\|_\infty \leq \lambda + \|GA^{N-1}E\|_\infty \|w\|_\infty \leq \lambda + \|GA^{N-1}E\|_\infty \mu$. Hence, taking $\mu$ from (23) provides $\psi(\tilde{x}_{N-1|k+1}) \leq 1$, and $\tilde{x}_{N-1|k+1} \in \frac{1}{\lambda}\mathcal{X}_f$, from which, by Lemma III.1, $\exists \bar{\theta}$ such that $u_{\bar{\theta}} = K\delta\tilde{x}_{N-1|k+1} + M_u\bar{\theta}$ is feasible, $u_{\bar{\theta}}$ steering $\tilde{x}_{N-1|k+1}$ to a feasible $\delta x_{N|k+1} \in \mathcal{X}_f$.
2) Take $\bar{u}, \bar{s}$ as the tails of the previous optimal control and slacks sequences, plus the terminal control law and zero. Recall $\tilde{x}_{i|k} = x^\star_{i|k} + A^{i-1}Ew$, and $z_t = M_\theta\bar{\theta}$. Because of Theorem III.5, we have, $\theta^\star = \bar{\theta}$, $\forall x \in \tilde{X}_N(\bar{\theta})$ and, $\forall x \in X_N(\bar{\theta})$. Another $\tilde{\theta}$, providing feasibility at time $k+1$, exists because of part 1). Denote and $\tilde{u}$ as $\bar{u}$ with terminal move replaced by the feasible $\tilde{u}_{N|k} = K(\tilde{x}_{N|k} - M_x\tilde{\theta}) + M_u\tilde{\theta}$. Denote $\tilde{s}$ as the new vector of slack variables, for $\tilde{x}_{\bullet|k}$. Then, by the principle of optimality,

$$
\begin{aligned}
\nabla V^o_N(\delta x) \leq V_N(\tilde{x}_{1|k}, \tilde{u}, \tilde{\theta}, \tilde{s}) - V_N(x^\star_{1|k}, \bar{u}, \bar{\theta}, \bar{s}) \\
+ V_N(x^\star_{1|k}, \bar{u}, \bar{\theta}, \bar{s}) - V^o_N(\delta x) \leq \\
\sum_{i=1}^{N-1} \|Q^{1/2}(\delta\tilde{x}_{i|k} + M_x(\bar{\theta} - \tilde{\theta}))\|_2^2 - \|Q^{1/2}\delta x_{i|k}\|_2^2 \quad (29) \\
+ \|R^{1/2}(\delta u_{i|k} + M_u(\bar{\theta} - \tilde{\theta}))\|_2^2 - \|R^{1/2}\delta u_{i|k}\|_2^2 \quad (30) \\
+ (N-1)\|SM_u(\tilde{\theta} - \bar{\theta})\|_1 + \|GM_u(\tilde{\theta} - \bar{\theta})\|_\infty \quad (31) \\
+ \|G(A+BK)A^{N-1}E\|_\infty \|w(k)\|_2 \quad (32) \\
+ |\|P^{1/2}(\delta\tilde{x}_{N|k} + M_x(\bar{\theta} - \tilde{\theta}))\|_2^2 - \|P^{1/2}\delta x_{N|k}\|_2^2 \quad (33) \\
+ V_O(M_\theta(\bar{\theta} - \tilde{\theta})) + V_s(\tilde{s}) - V_s(\underline{s}^\star) \quad (34) \\
- (\lambda_{\min}(Q) - \tau_Q + \tau_Q)\|\delta x\|_2^2 - \ell(s^\star_{1|k}), \quad (35)
\end{aligned}
$$

where for simplicity we have minimised over $\zeta$ and, for (31–32), we have used the triangle inequality. Since $\|\bar{\theta} - \tilde{\theta}\|$ is bounded above and zero when $w = 0$, then $\gamma_S, \gamma_G, \gamma_s, \gamma_O$ can be computed. Repeated use of (27) provides the following upper bound for (29–30)

$$
\begin{aligned}
\frac{1}{\epsilon^2 - 1} \sum_{i=1}^{N-1} \|Q\|_2 \|\delta x_{i|k}\|_2^2 + \|R\|_2 \|\delta u_{i|k}\|_2^2 + \\
2\epsilon^2 \sum_{i=1}^{N-1} \|Q^{1/2}A^{i-1}E\|_2^2 \|w(k)\|_2^2 + \\
\epsilon^2 \left( 2\|Q^{1/2}M_x(\bar{\theta} - \tilde{\theta}))\|_2^2 + \|R^{1/2}M_u(\bar{\theta} - \tilde{\theta}))\|_2^2 \right).
\end{aligned}
$$

The arbitrary $\epsilon > 1$ and $\tau_Q$ are used to eliminate the positive terms (36). In particular, from Lemma III.6 $\exists \kappa : \|\delta u^\star_{i|k}\|_2^2 \leq \kappa_u^2 \|\delta x(k)\|_2^2$ and $\|\delta x^\star_{i|k}\|_2^2 \leq \kappa_x^2 \|\delta x(k)\|_2^2, \forall i \geq 1$. Sufficiency of $\epsilon^2 \geq \epsilon_Q$ is easily verified. Given $\tau > 0$ from **(A7)**, then $\epsilon_{P2}$ is computed similarly to $\epsilon_Q$, and $\gamma_Q, \gamma_P$ similarly to $\gamma_S$. ■

## References


[1] H. Ohlsson, F. Gustafsson, L. Ljung, and S. Boyd, "Trajectory generation using sum-of-norms regularization," in *Proc. of IEEE Conf. Decision and Control (CDC)*, 2010.

[2] M. Nagahara and D. E. Quevedo, "Sparse representations for packetized predictive networked control," in *IFAC 18th World Congress*, Milan, Italy, 2011, pp. 84–89.

[3] M. Gallieri and J. Maciejowski, "$\ell_{asso}$ MPC: Smart regulation of overactuated systems," in *Proc. of the American Control Conference*, Montréal, Canada, June 2012, pp. 1217 – 1222.

[4] ——, "Stabilising terminal cost and terminal controller for $\ell_{asso}$-MPC: Enhanced optimality and region of attraction," in *Proc. European Control Conf. (ECC)*, 2013, pp. 524 – 529.

[5] M. R. Jovanović and F. Lin, "Sparse quadratic regulator," in *Proc. European control conf. (ECC)*, 2013, pp. 1047 – 1052.

[6] R. Tibshirani, "Regression shrinkage and selection via the LASSO," *J. R. Statist. Soc.*, vol. 58, no. 1, pp. 267 – 288, 1996.

[7] M. Osborne and B. Presnell, "A new approach to variable selection in least squares problems," *IMA J. Num. Analysis*, vol. 20, no. 3, pp. 389 – 403, 2000.

[8] D. Limon, I. Alvarado, T. Alamo, and E. Camacho, "MPC for tracking piecewise constant references for constrained linear systems." *Automatica*, vol. 44, no. 9, pp. 2382 – 2387, 2008.

[9] A. Ferramosca, D. Limon, I. Alvarado, T. Alamo, and E. Camacho, "MPC for tracking with optimal closed-loop performance," in *Proc. of IEEE Conf. Decision and Control (CDC)*, 2008, pp. 4055 –4060.

[10] ——, "MPC for tracking of constrained nonlinear systems," in *Conf. on Decision and Control*, 2009, pp. 7978 – 7983.

[11] M. Zeilinger, C. Jones, and M. Morari, "Robust stability properties of soft constrained MPC," in *Proc. of IEEE Conf. Decision and Control (CDC)*, 2010, pp. 5276 –5282.

[12] E. C. Kerrigan and J. M. Maciejowski, "Soft constraints and exact penalty functions in model predictive control," in *Proc. UKACC International Conference (Control 2000)*, Cambridge, UK, September 2000.

[13] J. Maciejowski, *Predictive Control with Constraints*, P. Hall, Ed. Prentice Hall, 2002.

[14] Z.-P. Jiang and Y. Wang, "Input-to-state stability for discrete-time nonlinear systems," *Automatica*, vol. 37, no. 6, pp. 857 – 869, 2001.

[15] G. Grimm, M. J. Messina, S. E. Tuna, and A. R. Teel, "Examples when nonlinear model predictive control is nonrobust," *Automatica*, vol. 40, no. 10, pp. 1729 – 1738, 2004.

[16] D. Limon, T. Alamo, D. M. Raimondo, D. Muñoz de la Peña, J. M. Bravo, A. Ferramosca, and E. F. Camacho, "Input-to-state stability: A unifying framework for robust model predictive control," *Nonlinear Model Predictive Control: Lect. Notes in Control and Informa. Sciences*, vol. 384, pp. 1 – 26, 2009.

[17] M. Lazar, "Model predictive control of hybrid systems: Stability and robustness," Ph.D. dissertation, Tech. Universiteit Eindhoven, 2006.

[18] A. Ferramosca, D. Limon, I. González, I. Alvarado, and E. F. Camacho, "Robust MPC for tracking zone regions based on nominal predicitons," *Jour. Process Control*, vol. 22, pp. 1966 – 1974, 2012.

[19] D. Limon Marruedo, T. Alamo, and E. Camacho, "Input-to-state stable MPC for constrained discrete-time nonlinear systems with bounded additive uncertainties," in *Decision and Control, 2002, Proceedings of the 41st IEEE Conference on*, vol. 4, 2002, pp. 4619 – 4624.

[20] F. Blanchini and S. Miani, *Set-theoretic methods in control*. Birkhäuser, 2008.

[21] F. Blanchini, "Ultimate boundedness control for uncertain discrete-time systems via set-induced Lyapunov functions," *IEEE Trans. Auto. Contr.*, vol. 39, pp. 428 – 433, 1994.

[22] S. Grammatico and G. Pannocchia, "Achieving a large domain of attraction with short-horizon linear MPC via polyhedral lyapunov functions," in *Proc. European Control Conf. (ECC)*, 2013, pp. 1059 – 1064.

[23] D. Limón, T. Alamo, F. Salas, and E. F. Camacho, "Input to state stability of min-max MPC controllers for nonlinear systems with bounded uncertainties," *Automatica*, vol. 42, no. 5, pp. 797 – 803, 2006.

[24] J. B. Rawlings and D. Q. Mayne, *Model Predictive Control Theory and Design*, L. Nob Hill Pub, Ed. Nob Hill Pub, Llc, 2010.